You are *not* allowed to work with your peers. Show all work and derivations to receive credit. Write clearly and legibly for your own benefit.

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Attention!!: You will have to upload the final Python notebook to Canvas in the midterm exam assignment to get credit in addition to attaching a print out pdf to the exam you submit!

Problem 1 (Linear Independence, Matrices) [6pts].

Consider three vectors $u, v, w \in \mathbb{R}^3$ that are linearly independent, and let $a, b, c \in \mathbb{R}^3$ be defined as

$$a = u + v,\ b = v + w,\ c = u + w,\ d = u - v.$$

a. Suppose vectors u and v have unit length. What is the angle between the vectors a and d?

b. Again suppose vectors u and v have unit length. Give an expression for a left-inverse of the matrix $A = \begin{bmatrix} a & d \end{bmatrix}$ (your expression can depend on a and d).

c. Show that nullspace of the matrix $B = \begin{bmatrix} a & b & c \end{bmatrix}$ contains only the zero vector $\mathbf{0} \in \mathbb{R}^3$.

Solution.

a. $90°$

b. $XA = I$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} a & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} u+v & u-v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left.\begin{aligned} x_1(u+v) &= 1 \\ x_2(u+v) &= 0 \\ x_1(u-v) &= 0 \\ x_2(u-v) &= 1 \end{aligned}\right\} \quad x_1 = \frac{1}{u+v},\ x_2 = \frac{1}{u-v}$$

$$\boxed{X = \begin{bmatrix} 1/a \\ 1/d \end{bmatrix}}$$

C. If the nullspace of the matrix only contains the zero vector, then this means a,b,c are linear independent. We see that a = u + v, b = v + w, and c = u + w and it is given that u,v,w are linear independent so that means a, b, c are linear independent. Thus the nullspace only contains the zero vector by definition.

Problem 2 (Matrix inverse & properties) [6pts]. Consider the following $n \times n$ matrix,

$$S = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix}$$

What does the matrix S do when applied to a vector? Find the inverse of S: Solve $SX = I$ for the unknown matrix X by writing the linear equations each column of X should satisfy. What is the interpretation of S^{-1}?

Considering SX=B where S is a lower triangular matrix containing only 1's, then when S is applied to some vector X, the result B is a matrix with the ith value equivalent to the sum of the first i indices of X.

finding the inverse :

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{I}$$

X looks like a matrix w/ ones along the diagonal and negative ones to the left of the diagonal.

$s_{11} x_{11} + \dots + s_{1n} x_{n1} = 1$ ← column 1 of x has to satisfy

$s_{21} x_{12} + \dots + s_{2n} x_{n2} = 1$ ← column 2 of x has to satisfy

$\vdots \qquad \vdots$

$s_{n1} x_{1n} + \dots + s_{nn} x_{nn} = 1$ ← column n of x has to satisfy

all other equations set to 0

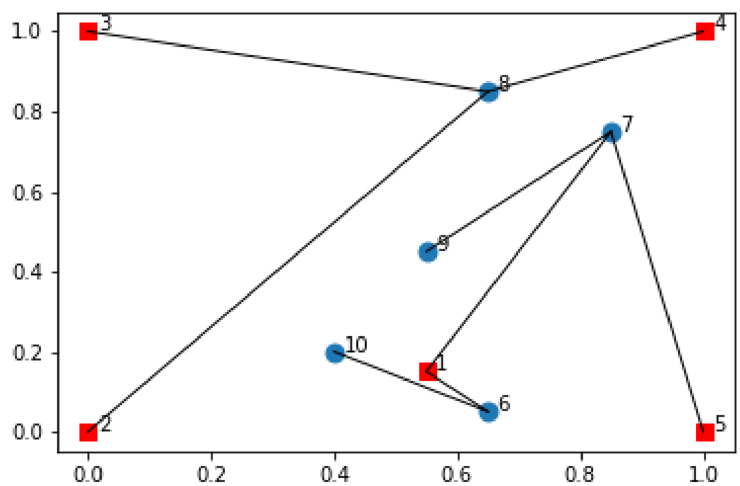
Problem 3 (Least Squares Placement) [18pts]. This problem has parts a–e. We have provided space below each problem for the solution.

The vectors $p_1, \ldots, p_N$, each in $\mathbb{R}^2$ represent the locations of N objects. There are two types of objects: factories and warehouses. The first K objects are factories, whose locations are fixed and given. Our goal in the placement problem to choose the locations of the last $N-K$ objects i.e the warehouses.

Our choice of the locations is guided by an undirected graph; an edge between two objects means we would like them to be close to each other. In least squares placement, we choose the locations $p_{K+1}, \ldots, p_N$ of warehouses so as to minimize the sum of the squares of the distances between objects connected by an edge, where the L edges of the graph are given by the set E. For a specific location of factories $p_1, \ldots, p_K$, we can frame our task as solving the following optimization:

$$g(p_1, \ldots p_K) = \min_{p_{K+1}, \cdots p_N} \sum_{(i,j)\in E} \|p_i - p_j\|^2$$

For illustration, see the figure below. The factories are denoted by red squares, and warehouses by blue circles; we want to move the blue circles so that the objective is minimized.



a. In each of the simplified cases below, state what the optimal objective value would be. Further, what assignment to $\{p_{K+1}, \ldots p_N\}$ would achieve this optimal objective? If there are multiple optimal assignments, state any one.

   i. The factories are fixed at $p_{N-K+1} = \cdots = p_{N-1} = p_N = \mathbf{0}$.

   ii. There is $K = 1$ factory. Write your answer in terms of the fixed location p_1.

   iii. All edges are from one warehouse to another. In other words, for any $(i, j) \in E$, both (p_i, p_j) are warehouses. Write your answer in terms of E.

Solution part a.

i. if all the factories are fixed at zero, then the optimal assignment will just be placed at zero also since any other placement would not minimize the distance. Optimal objective value is zero.

ii. If there is only one factory, then the optimal assignment should be placed at that factory since that is the only distance to consider. Optimal objective value is 0. The optimal location is at P1.

iii. The optimal assignment would be Pi = Pj in E because then the optimal objective value would be 0.

b. Now, we move towards solving the problem in the more general case.

Let $\mathcal{D}$ be the Dirichlet energy of the graph, which is defined as follows (more details are given in §7.3 of VMLS, and specifically page 135; you may want to take a look):

> Consider a graph $\mathcal{G} = (\{1, \ldots, N\}, E)$ composed of a set of edges E and N nodes $\{1, \ldots, N\}$ such that nodes $i, j \in \{1, \ldots, N\}$ are connected if and only if there exists an edge $(i, j) \in E$. Let B be the incidence matrix of the graph, and let $w \in \mathbb{R}^N$ be the *potential* for the graph—i.e., w_i is the value of some quantity at node $i \in \{1, \ldots, N\}$. The Dirichlet energy is $\mathcal{D}(w) = \|B^\top w\|^2$ and can be equivalently expressed as
>
> $$\mathcal{D}(v) = \sum_{(i,j) \in E} (w_j - w_j)^2$$
>
> which is the sum of the squares of the potential differences of w across all edges in the graph.

Show that the sum of the squared distances between the N factories can be expressed as $\mathcal{D}(u) + \mathcal{D}(v)$, where $u = ((p_1)_1, \ldots, (p_N)_1)$ and $v = ((p_1)_2, \ldots, (p_N)_2)$ are N-vectors containing the first and second coordinates of the factories, respectively.

Solution part b.

$D(u) + D(v)$ for first and second coordinates

$$= (u_{i_1} - u_{j_1})^2 + \ldots + (u_{i_L} - u_{j_L})^2$$
$$+ (v_{i_1} - v_{j_1})^2 + \ldots + (v_{i_L} - v_{j_L})^2$$

$$= \left\| \begin{bmatrix} u_{i_1} - u_{j_1} \\ v_{i_1} - v_{j_1} \end{bmatrix} \right\|^2 + \ldots + \left\| \begin{bmatrix} u_{i_L} - u_{j_L} \\ v_{i_L} - v_{j_L} \end{bmatrix} \right\|^2$$

$$= \|p_{i_1} - p_{j_1}\|^2 + \ldots + \|p_{i_L} - p_{j_L}\|^2$$

c. Express the least squares placement problem as a least squares problem, with variable $x = (u_{1:(N-K)}, v_{1:(N-K)})$. In other words, express the objective above (the sum of squares of the distances across edges) as $\|Ax - b\|^2$, for an appropriate $m \times n$ matrix A and m-vector b. You will find that $m = 2L$.

Hint: You can use the fact that $\mathcal{D}(y) = \left\|B^\top y\right\|^2$, where B is the incidence matrix of the graph.

Solution part c.

$$D(u) + D(v) = \|B^T u\|^2 + \|B^T v\|^2$$

$$\begin{bmatrix} B^T & 0 \\ 0 & B^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \text{where } x_1 = u_{1:N-K}, \; x_2 = v_{1:N-K}$$

$$\begin{bmatrix} B^T & 0 \\ 0 & B^T \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0$$

d. Solve the least squares placement problem for the specific problem with $N = 10$, $K = 5, L = 13$, fixed locations

$$p_1 = (0.55, 0.15), \quad p_2 = (0, 0), \quad p_3 = (0, 1), \quad p_4 = (1, 1), \quad p_5 = (1, 0),$$

The edges are:

$$(1,6), \quad (2,6), \quad (5,6), \quad (1,7), \quad (4,7), \quad (2,8),$$
$$(3,8), \quad (3,9), \quad (5,9), \quad (5,10), \quad (7,9), \quad (6,10), \quad (7,10).$$

Using the provided Python notebook, plot the locations, showing the graph edges as lines connecting the locations. Specifically, solve **Python-P3d** in the provided notebook and print the pdf and attach it at the end of your exam. **You will have to upload the final notebook to Canvas to get credit in addition to attaching a print out pdf to the exam you submit!**

Problem 4 (Recursive Least Squares) [12pts]. Suppose we have data $(x^{(1)}, \ldots, x^{(m)})$ and $(y^{(1)}, \ldots, y^{(m)})$ and we believe that $y^{(i)} \approx (x^{(i)})^\top \theta$—that is, $y^{(i)}$ is approximately a linear function of $x^{(i)}$. The $x^{(i)} \in \mathbb{R}^n$ are n-dimensional vectors and $y^{(i)} \in \mathbb{R}$ are scalars. The least squares method seeks to minimize

$$J(\theta) = \sum_{k=1}^{m} (y^{(k)} - (x^{(k)})^\top \theta)^2$$

We saw in lecture that

$$\hat{\theta}_m := \left(\sum_{k=1}^{m} x^{(k)} (x^{(k)})^\top \right)^{-1} \sum_{k=1}^{m} x^{(k)} y^{(k)}$$

minimizes $J(\theta)$. There are three parts to the problem and on the next page put your solutions to parts a and b. For part c, print your pdf and attach it at the end of your exam. You will also need to provide the ipynb of your solutions.

a. Define $X_m^\top = [x^{(1)}\ x^{(2)}\ \cdots\ x^{(m)}]$ and $Y_m = [y^{(1)}\ y^{(2)}\ \cdots\ y^{(m)}]^\top$. Show that

$$\hat{\theta}_m := (X_m^\top X_m)^{-1} X_m^\top Y_m$$

minimizes $\|Y_m - X_m \theta\|^2$ by taking the derivative of $\|Y_m - X_m \theta\|^2$ and setting it to zero.

b. Now suppose one more data pair $(x^{(m+1)}, y^{(m+1)})$ becomes available and define

$$X_{m+1}^\top = [X_m^\top\ x^{(m+1)}] \quad \text{and} \quad Y_m = [Y_m^\top\ y^{(m+1)}]^\top$$

Hence, the new least squares solution is

$$\hat{\theta}_{m+1} := (X_{m+1}^\top X_{m+1})^{-1} X_{m+1}^\top Y_{m+1} = R_{m+1}^{-1} \sum_{k=1}^{m+1} x^{(k)} y^{(k)} \tag{1}$$

where

$$R_{m+1} := \sum_{k=1}^{m+1} x^{(k)} (x^{(k)})^\top.$$

Show that the least squares solution can be obtained by the following recursive procedure:

$$\hat{\theta}_{m+1} = \hat{\theta}_m + R_{m+1}^{-1} x^{(m+1)} (y^{(m+1)} - (x^{(m+1)})^\top \hat{\theta}_m) \tag{2}$$

$$R_{m+1} = R_m + x^{(m+1)} (x^{(m+1)})^\top \tag{3}$$

That is, show that these equations hold given the definition of R_m and the expression for the least squares solution.
Hint: First, verify (3) holds given the definition of R_{m+1}. Then, to verify (2), take the expression in (1) for $\hat{\theta}_{m+1}$, and expand the sum $\sum_{k=1}^{m+1} x^{(k)} y^{(k)} = x^{(m+1)} y^{(m+1)} + \sum_{k=1}^{m} x^{(k)} y^{(k)}$. Now, use (3) and the expression you have for $\hat{\theta}_m$ to show (2) holds.

c. Implement recursive least squares. In the provided Python notebook, solve **Python-P4**. **You will have to upload the final notebook to Canvas to get credit in addition to attaching a print out pdf to the exam you submit!**

Solution.

a. **Provide your solution to part a. here.**

$$\frac{d}{d\theta}\|Y_m - X_m\theta\|^2 = -2(Y_m - X_m\theta)X_m$$

set derivative equal to zero

$$-2(Y_m - X_m\hat{\theta})X_m = 0$$

$$-2X_m^T(Y_m - X_m\hat{\theta}) = 0$$

$$-2X_m^T Y_m + 2X_m^T X_m\hat{\theta} = 0$$

$$X_m^T Y_m = X_m^T X_m\hat{\theta}$$

$$\Rightarrow \hat{\theta} = (X_m^T X_m)^{-1}(X_m^T Y_m)$$

b. **Provide your solution to part b. here.**

$$\hat{\theta}_{m+1} = R_{m+1}^{-1} \sum_{k=1}^{m+1} x^k y^k$$
$$= R_{m+1}^{-1} \left(x^{m+1} y^{m+1} + \sum_{k=1}^{m} x^k y^k \right)$$
$$= \hat{\theta}_m + R_{m+1}^{-1} \left(x^{m+1} y^{m+1} - (R_{m+1} - R_m)\hat{\theta}_m \right)$$
$$= \hat{\theta}_m + R_{m+1}^{-1} \left(x^{m+1} y^{m+1} - (R_{m+1} - R_m) R_m^{-1} \sum_{k=1}^{m} x^k y^k \right)$$
$$= \hat{\theta}_m + R_{m+1}^{-1} x^{m+1} \left(y^{m+1} - (x^{m+1})^T \theta_m \right)$$

Problem 5 (Least Norm Solution) [6pts]. In class we saw how to obtain the least squares approximate solution to $Ax = b$ corresponding to an over-determined set of equations—that is, where $A \in \mathbb{R}^{m\times n}$ with $m > n$ ("tall matrix"). In this case, there is rarely an exact solution to $Ax = b$, and instead we find an approximate solution by minimizing $\|Ax - b\|^2$.

If on the other hand A is a "wide matrix", meaning $n > m$, then the set of equations is under-determined. This means there are more unknowns than equations and hence, potentially many solutions. In this case we select, amongst the solutions, the one with the smallest norm—i.e., the least norm solution. That is, we seek $\hat{x}$ such that $A\hat{x} = b$ and $\|\hat{x}\| \le \|x\|$ for all other x such that $Ax = b$.

Consider an under-determined system of equations $Ax = b$ where $b \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m\times n}$ are given with $n > m$. Suppose that A is full rank and $b \in \text{range}(A)$—this ensures there is at least one solution. Show that $\hat{x} = A^\top (AA^\top)^{-1} b$ is the least norm solution to $Ax = b$. In particular, letting $\mathcal{S} = \{x \in \mathbb{R}^n | \; Ax = b\}$ be the set of solutions to $Ax = b$, you need to show that

$$\|\hat{x}\| \le \|x\| \quad \forall \, x \in \mathcal{S}.$$

Solution.

Let the least squares solution be $\hat{x} = A^T (AA^T)^{-1} b$

We can say AA^T is invertible because A is full rank.

Goal: minimize $\|x\|$ considering $Ax = b$

assume $A(x - \hat{x}) = 0$ then

$$\hat{x} = (x - \hat{x})^T A^T (AA^T)^{-1} b$$

$$(x-\hat{x})^T \hat{x} = (x-\hat{x})^T A^T (AA^T)^{-1} b$$
$$= \underbrace{A(x-\hat{x})^T}_{=0} (AA^T)^{-1} b$$

this means $(x-\hat{x})^T \hat{x} = 0$ and thus orthogonal

In class we talked about the orthogonality principle.