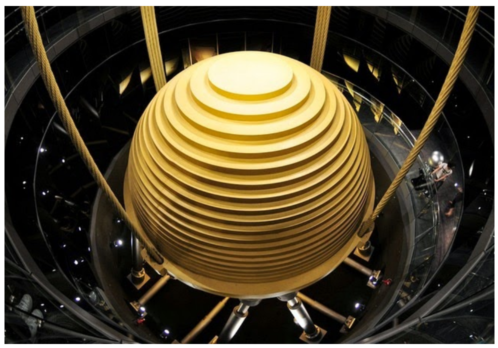
An investigation into the relationship between the length and oscillation period of a simple pendulum.

Introduction:

Pendulums can be found everywhere, but they are often overlooked for the important role they play in our lives. One example of a pendulum application would be the tuned mass damper (TMD). The tuned mass damper is an innovative device that engineers have created to reduce structural vibration, in the event of a natural disaster, or any other type of violent motion. Before its invention, the only method to stabilize large structures was to make them excessively stiff and strong, which often proved to be very expensive. Today, many structures around the world utilize TMD's, of which the most famous one, in Taipei 101, is displayed to the public.



The Tuned Mass Damper in Taipei 101

I was first introduced to TMD's when I came across one of my professor's lessons on pendulums. The fact that a small addition in the tuned mass damper was able to make such a significant impact, piqued my curiosity. After some further research, I was able to learn that a tuned mass damper operates based on the idea of resonant frequency. In order to reduce, or "dampen" the motion of the structures, pendulum dampers are tuned to match the natural frequency of the structure. Systems are able to oscillate at a greater amplitude when a force is applied to it periodically at the same frequency. To simplify, the pendulum will be able to oscillate at a greater amplitude which would "absorb" more kinetic energy from the structure. Engineers are able to adjust the frequency of the damper by calculating the length that the pendulum needs to be suspended from. This idea of adjusting the period of a pendulum by changing the length is not just common to the tuned mass damper. As someone who has played the piano for over 10 years, I often use the metronome to help me keep the intended musical time. Similar to the tuned mass damper, I can adjust the period at which the metronome makes a sound by sliding a weight up and down the pendulum rod. As such, I wondered if the relationship between the pendulum length and the time period was one that can be modelled and expressed as an equation. To define the goal of this experimental investigation, the **research question** presented is:

How does the change in suspension length (meters) of a simple pendulum affect its oscillation time period (seconds)?

The simple pendulum is a theoretical model that can be used to represent the motion of a real pendulum. In this model, it is assumed that a point mass is attached to the end of a taut massless rod, suspended from a frictionless pivot. A pendulum can also be described as an object that oscillates in harmonic motion, and because so, we can use the formula of simple harmonic motion to develop the theoretical equation of the simple pendulum.

Simple Harmonic Motion: An object is said to oscillate in simple harmonic motion if it experiences a constant restoring force proportional to its displacement from the equilibrium position. With this definition, we arrive at the simple harmonic motion formula of $F = -kx$, also known as Hooke's Law; where F is the force in newtons, k is the spring constant in newtons per meter and x is the displacement from the equilibrium point in meters. We can then equate this with Newton's Second Law formula, which states that: $F = ma$; where m is the mass in kilograms and a is the acceleration in meters per second squared.

$$-kx = ma$$

We can represent the displacement of an object in simple harmonic motion as a sinusoidal wave in the form of $A\sin(\omega t + \phi)$; where A is the amplitude, ω is the angular frequency in radians per second, and ϕ is the phase shift in seconds (Homer & Bowen-James, 2014). As velocity is the change in displacement over the change in time, we can use the mathematical process of differentiation to express velocity as a derivative of displacement with respect to time:

$$v = \frac{dx}{dt} = \frac{d}{dt}\left[A\sin(\omega t + \phi)\right]$$

↳ $$A\frac{d}{dt}\left[\sin(\omega t + \phi)\right] = A\left[\cos(\omega t + \phi)\right]\left[\frac{d}{dt}(\omega t + \phi)\right]$$

↳ $$A\left[\cos(\omega t + \phi)\right](\omega) = A\omega\cos(\omega t + \phi)$$

We also know that acceleration is the rate of change of velocity with respect to time. As such, a similar differentiation process is used once more:

$$a = \frac{dv}{dt} = \frac{d}{dt}\left[A\omega\cos(\omega t + \phi)\right]$$

↳ $$A\omega\frac{d}{dt}\left[\cos(\omega t + \phi)\right] = -A\omega\left[\sin(\omega t + \phi)\right]\left[\frac{d}{dt}(\omega t + \phi)\right]$$

↳ $$-A\omega\left[\sin(\omega t + \phi)\right](\omega) = -A\omega^2\sin(\omega t + \phi)$$

If we substitute the acceleration and displacement equations into our original force formula, we are able to express angular frequency in terms of mass and the spring constant:

$$m\left[-A\omega^2\sin(\omega t+\phi)\right] = -k\left[A\sin(\omega t+\phi)\right]$$

$$\frac{-m\omega^2 A\sin(\omega t+\phi)}{-mA\sin(\omega t+\phi)} = \frac{-kA\sin(\omega t+\phi)}{-mA\sin(\omega t+\phi)}$$

↳ $\omega^2 = \dfrac{k}{m}$

The definition of angular frequency is the change in angle over the change in time: $\omega = \dfrac{2\pi}{T}$; where T is the time for one period in seconds. Therefore, we can take our previous expression for ω^2 and substitute it into the angular frequency formula:

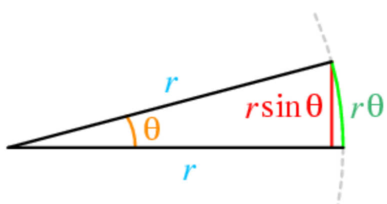
$$\sqrt{\frac{k}{m}} = \frac{2\pi}{T}$$

After rearrangement, it will give us the simple harmonic motion equation of:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Simple Pendulum: It was mentioned that the simple pendulum can be represented as an object in harmonic motion. However, the simple pendulum is not, in fact, a "true" harmonic oscillator. If we go back to the definition of simple harmonic motion, it emphasizes the fact that the force is directly proportional to the displacement, in reference to linear periodic motion. This is not applicable to a simple pendulum, as it oscillates in circular motion. In the figure below, we can see that there is a difference in length between the circular motion path and what would be the supposed linear motion path.

Figure 1: Linear Motion Vs. Circular Motion



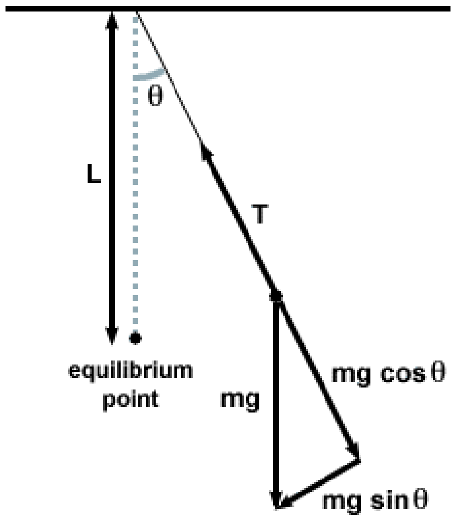
The reason why the simple pendulum can be classified as a harmonic oscillator is because of a simplification known as "small angle approximation". In mathematics, the Taylor Series is a representation of a function, as an infinite sum of terms. Below is the expansion of sine function as a Taylor Series, where x is the angle in radians:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

(*Wolfram Alpha, n.d*)

It can be observed that as the angle increases, the sum of sine function starts to deviate from the true value of the angle. However, if the angle is small, the error can be described as infinitesimal. For example, if an angle of 15° is used, the difference between the value of $\sin(x)$ and (x) is an error of 0.0029 radians (0.17°). As such, we can use small angle approximation to help simplify and solve rather complex problems such as the simple pendulum**.**

Figure 2: Pendulum Restoring Force



In a simple pendulum, there are two forces, tension and gravity, that act upon the system. From *Figure 2*, we see that the cosine of the gravitational force counteracts the tension force, which means that the string will not travel in the direction of either vector. This matches the assumption that the rod of the simple pendulum does not slack nor extend. The sine of the gravitational force is the only force left and as such, the restoring force acting upon a pendulum is: $-mg\sin\theta$

With small angle approximation, the force can be simplified into:

$$-mg\sin\theta \approx -mg\theta$$

The original simple harmonic motion formula can then be equated to the restoring force of the pendulum, where x for displacement can be represented as arc distance:

$$F_{Restoring} = -kx = -k(L\theta)$$

$$-k(L\theta) = -mg\theta$$

and when rearranged:

$$\frac{m}{k} = \frac{L}{g}$$

The length and gravitational force are then substituted into the simple harmonic motion formula to obtain the equation:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

and if we input the constant values for 2π and g, the simple pendulum equation becomes:

$$T = \frac{(6.28)}{(\sqrt{9.81})}(\sqrt{L})$$

$$= 2.01\sqrt{L}$$

Hypothesis: With the assumption that a small angle displacement is used, if the suspension length of the pendulum were to increase, then the period of oscillation will also be expected to increase. There is a logical explanation behind this hypothesis. As gravity is constant, we know that two pendulums, even with different lengths, would have the same acceleration vector. However, the pendulum with the longer length will have a further arc distance to travel, therefore, it will take more time to complete a full period. Furthermore, the theoretical equation supports this statement as it illustrates a relationship where the square root of the pendulum length is proportional to the time period:

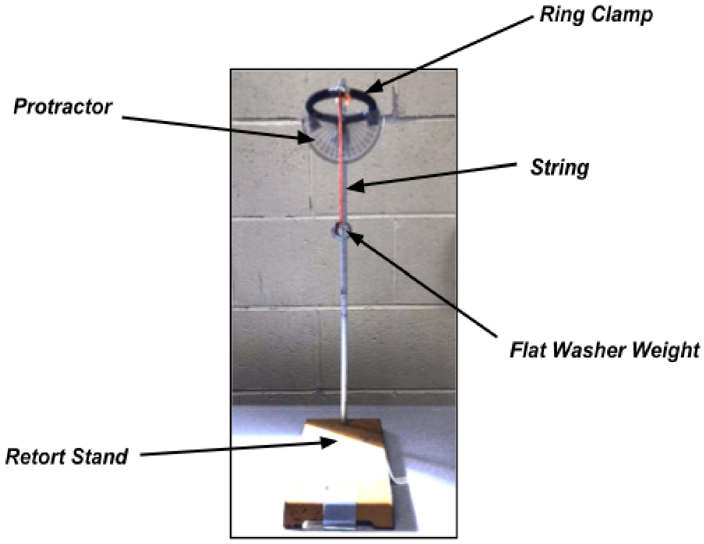
$$T = 2\pi\sqrt{\frac{L}{g}} \Rightarrow \sqrt{L} \propto T$$

<u>Apparatus & Procedure</u>:

- ➢ Flat Washers (SAE 3/8) OD: 1 in ID: 0.436 in
- ➢ Retort Stand
- ➢ Ring Clamp
- ➢ String (Medium Thickness)
- ➢ Meter Stick ($\pm$0.001 m)
- ➢ Painter's Tape
- ➢ Laptop (Installed with Computer Program Tracker)

In this investigation, I designed and conducted an experiment where I measured the time period of a pendulum with different lengths starting from 0.2 m and increasing by 0.1 m each time (0.2 m, 0.3 m, 0.4 m, 0.5 m, 0.6 m). To start the experiment, the list of materials above was gathered. Then, the retort stand was set up on a firm table. It was important to make sure the table would not experience any shaky motion during the experiment, or else an additional force may be added to the pendulum. For further stability, the base of the retort stand was taped onto the table. Next, the ring clamp was attached near the top of the retort stand. To create the pendulum, a string with medium thickness was cut at a length of 0.45 m. The length of the string was doubled to account for the fact that it would loop around the pivot to hold the weight. The additional 0.05 m was added because some length will be used to tie the string around the clamp. After the string was cut, the pendulum weight was formed by taping 10 flat washers together. A pendulum bob would have been more preferable but due to the lack of access to resources, flat washers were used instead. The string was then looped around the hole of the flat washer and over the center of the ring clamp. Putting the meter stick beside the retort stand, the knot of the string was adjusted so that the suspension length would be accurate to 0.2 m. It was imperative that the string was pulled tight when measured, based on the assumption that the pendulum rod should always remain taut. Next, a circular protractor was attached by taping it to the bottom of the ring clamp such that it was aligned with the pendulum and its face parallel to the plane formed by the movement of the pendulum. Then, the pendulum was displaced at an angle of 15°, based on the visual reference provided by the protractor. When adjusting the initial angle of the pendulum, the adjuster should use only one eye as their line of vision to reduce parallax error. The computer camera was then opened to capture the motion of the pendulum, making sure that it was on the same level as the retort stand. Subsequently, the pendulum was dropped from its initial angle displacement. Afterwards, the video of the pendulum was loaded on the computer program Tracker, where the function autotracker was used to model the motion on a displacement-time graph. Two more trials were then completed for a total of 3 trials. The same process was then repeated for the lengths of 0.3 m, 0.4 m, 0.5 m and 0.6 m, making sure that an additional length of 0.05 m was added for each string.

Set-Up Diagram:

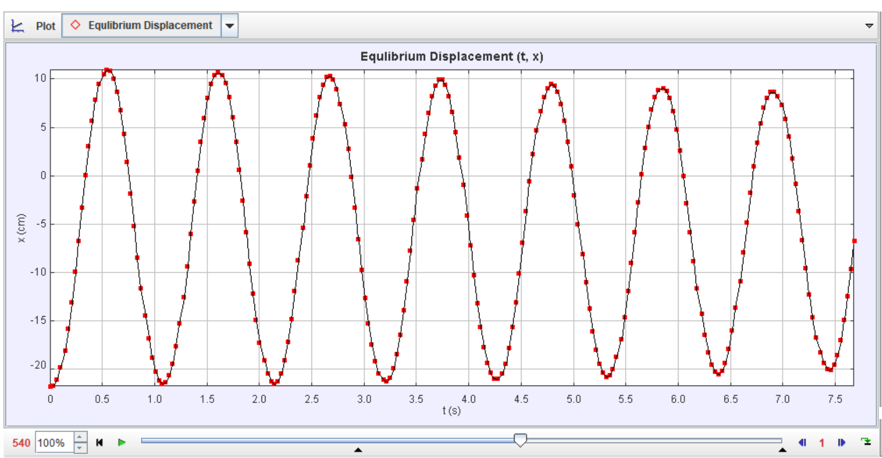


Risk Assessment:
There are no safety, environmental, or ethical concerns in this investigation.

<u>Data Analysis</u>:

Raw Data:

Data Process Exemplar (0.30m Trial 3):



t (s)	x (cm)
0.576	10.81
0.608	10.07
0.640	8.716
0.672	6.772
0.704	4.290
0.736	1.410
0.768	-1.804
0.800	-5.148
0.832	-8.457
0.864	-11.61
0.912	-14.40
1.600	10.66

*The data table and the displacement table above were created on Tracker from the motion capture of the camera. It is indicated that the first wave peak occurs at 0.576 seconds, and the second peak at 1.600 seconds. If we subtract the two, a time period of 1.024 seconds is formulated.

Table 1: The Measured Time Period of Pendulum Relative to its Suspension Length

Pendulum Suspension Length $\pm$0.001 (m)	Oscillation Time Period of Pendulum (s)*			Range of Time Period (s)
	Trial 1	Trial 2	Trial 3	
0.20	0.832	0.800	0.896	0.096
0.30	0.992	1.088	1.024	0.096
0.40	1.184	1.248	1.184	0.064
0.50	1.376	1.344	1.312	0.064
0.60	1.472	1.568	1.440	0.128

*Note: The elapsed time between each frame in Tracker was 0.032 seconds. This meant that a data point of the tracked motion was plotted at an interval of 0.032 seconds. Therefore, the smallest decimal place of the device was 0.001 seconds. However, I felt that the size of the uncertainty did not reflect the errors that affected the time period. As a result, the range of the time period divided in half was used instead.

Processed Data:

Table 2: The Average Time Period of Pendulum Relative to its Suspension Length

Pendulum Suspension Length $\pm$0.001 (m)	Average Oscillation Time Period of Pendulum (s)	*Uncertainty of Time Period (s)
0.20	0.843	0.048
0.30	1.035	0.048
0.40	1.205	0.032
0.50	1.344	0.032
0.60	1.493	0.064

(Sample Calculation) Average Time Period: $\frac{Trial\ 1 + Trial\ 2 + \ Trial\ 3}{3}$

$$T = \frac{0.832 + 0.800 + 0.896}{3}$$

$$= 0.843$$

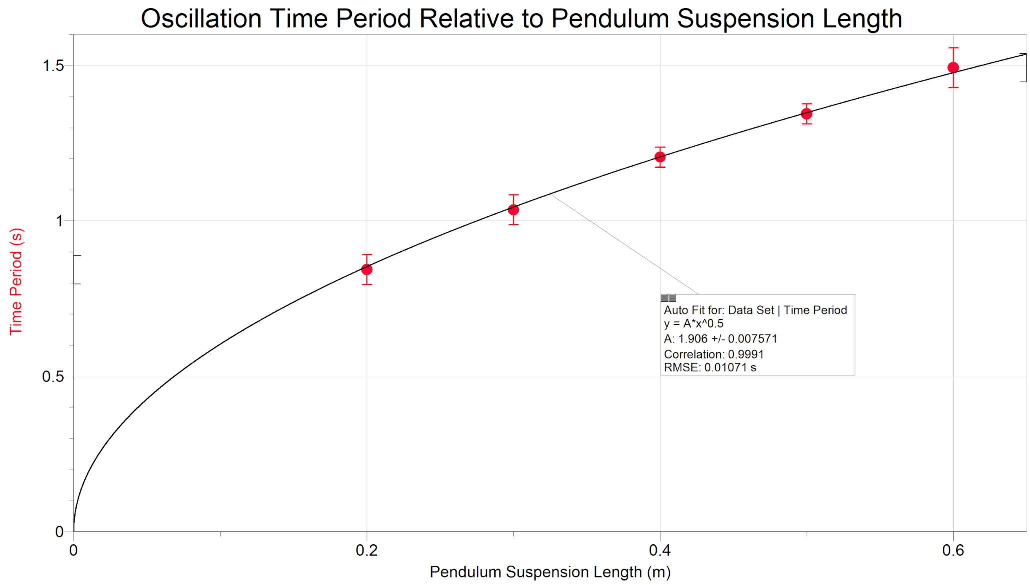
(Sample Calculation) Time Period Uncertainty: $\frac{Max\ Period - Min\ Period}{2}$

$$\frac{0.896 - 0.800}{2}$$

$$= 0.048$$

The computer program *"Logger Pro"* was used to create the following graphs in this experiment

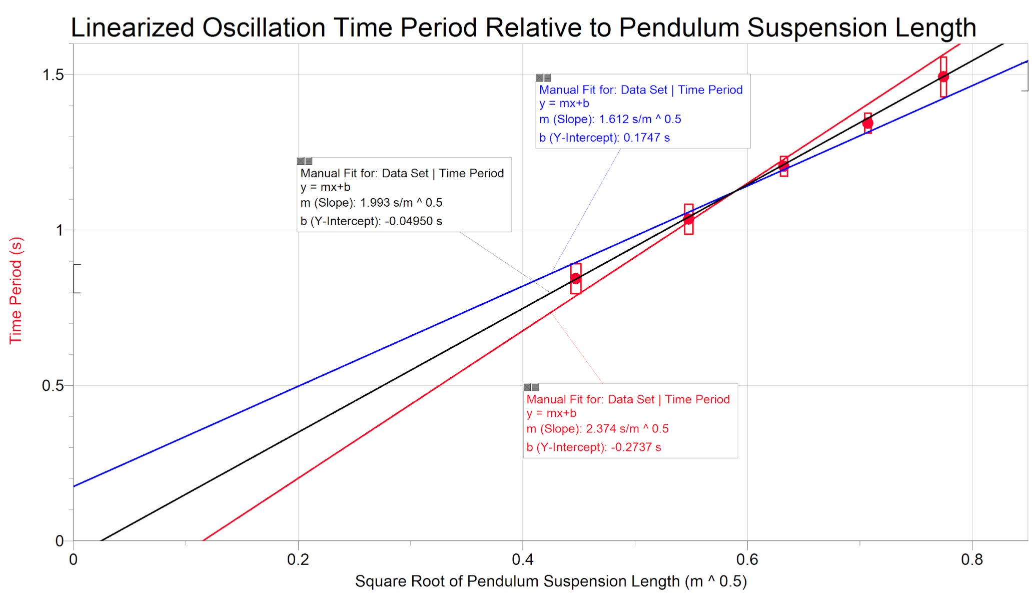
Figure 3:



*Note: As a result of the scale, the horizontal uncertainty bars may not be visible

After plotting the data points, the equation for the line of best fit was found to be $y = 1.906x^{0.5}$. No outliers were observed as the curve passed through the uncertainty bars of every point. The correlation in *Figure 2* was very strong, indicated by the correlation coefficient (R) of 0.9991, and a determination coefficient (R^2) of 0.9982. Therefore, we can justify the presumed square root relationship between the pendulum length and the time period. To further analyze the relationship, the data was linearized according to the found square root correlation.

Figure 4:



*Note: As a result of the scale, the horizontal uncertainty bars may not be visible

Linearization Calculation: To linearize the data, the independent variable (pendulum suspension length) was raised to the power of ½, and the y-axis remained the same as the time period.

$$X_{Linearized} = \sqrt{X}_{Non-Linearized}$$

The uncertainty for the independent variable was found from multiplying the relative uncertainty of the non-linearized value by the linearized value. It was then multiplied by ½ as the non-linearized measurement was raised to the power of ½.

$$Uncertainty_{Lineaerized} = \%\ Uncertainty_{Non-Linearized} \cdot \frac{1}{2} \cdot X_{Linearized}$$

Average Line of Best Fit: The average line of best fit for the linearized data was found through the use of two other lines of best fit. One of the lines, would have the smallest slope that goes through all of the uncertainty bars and the other line would have the greatest slope that goes through the uncertainty bars. These two lines were found on Logger Pro using the line drag function. Then, their slopes and y-intercepts were averaged out for the average line of best fit.

$$T = \frac{min\ slope + max\ slope}{2}\sqrt{L} + \frac{min\ y \sim int + max\ y \sim int}{2}$$

$$T = \frac{2.374 + 1.612}{2}\sqrt{L} + \frac{-0.2737 + 0.1747}{2}$$

$$T = 1.993\sqrt{L} - 0.0495$$

Uncertainty of Average Slope: The uncertainty of the average slope was calculated from the range of the two slopes divided in half: $\frac{max\ slope - min\ slope}{2}$

$$= \frac{2.374 - 1.612}{2}$$

$$= 0.381$$

Uncertainty of Average Y~Intercept: The same procedure was repeated with the uncertainty of the y-intercept: $\frac{max\ y \sim intercept - min\ y \sim intercept}{2}$

$$= \frac{0.1747 - (-0.2737)}{2}$$

$$= 0.2242$$

Final Experimental Equation: After the addition of uncertainties, the equation can now be written as:

$$T = (1.993 \pm 0.381)\sqrt{L} - (0.0495 \pm 0.2242)$$

When compared to the experimental line of best fit, it can be seen that the slope of the theoretical equation $T = 2.01\sqrt{x}$, falls within the range of the experimental slope and uncertainty (1.993 ± 0.381) . Therefore, it can be concluded that the slope does follow what theory has proven. The theoretical equation also predicted that the line would pass through the origin (0,0), which was supported by the data obtained. The experimental line of best fit intercepted the y-axis at -0.00495 seconds, however the error was accounted for in the uncertainty of the y-intercept. The accuracy of the experiment did not come as a surprise, because I knew that errors such as small angle approximation would not have a large influence on the period. Nevertheless, I thought it would be interesting to examine just how much the angle approximation affected my experiment. To calculate the percent error, we need to go back to the simple pendulum calculation, and substitute $\sin\theta$ for θ:

$$\frac{m}{k} = \frac{L\theta}{g\sin\theta}$$

The expression can then be expressed as the inverse of gravitational force over the length:

$$\frac{m}{k} = \frac{1}{\frac{g\sin\theta}{L\theta}}$$

If we substitute the expression back into the simple harmonic equation for time period and input an angle of 0.262 radians (15°), an equation is formed: :

$$T = 2\pi\sqrt{\frac{1}{\frac{g}{L}\frac{(\sin\theta)}{\theta}}}$$

$$T = 1.012\left(2\pi\sqrt{\frac{L}{g}}\right)$$

If we compare the equation above to the theoretical equation, we see that the calculated value for the period would have an error of 1.2%. Then, if multiplied with the theoretical slope value of 2.01, we would obtain an error of 0.0241 seconds. We can also use this information to deduce the sum of all of the other errors, by subtracting the error of angle approximation from the y-intercept, which would become -0.0736 seconds. With this value, we now know how errors in the methodology were able to affect the results. To improve upon the accuracy and reduce the errors in the experiment, possible suggestions that can be performed in the future are discussed in the table below.

Systematic Errors:

Error Description	Significance to Experiment	Improvement Suggestions
Angle Approximation	The error for the time period associated with an angle of 15° was 1.2%	To reduce the effect of angle approximation, the initial angle displacement can be lowered to 10°.
Pivot Friction	One of the assumptions made for the simple pendulum was that the pendulum is to be suspended from a frictionless pivot. However, it was evident that there were energy losses due to friction from the pivot during the pendulum's motion. This may have reduced the amplitude of the motion as well as caused the pendulum to change direction before its maximum displacement, leading to an underestimation of the period.	An improvement that can be made would be to use a stainless steel ring clamp or another clamp with a smoother surface. Furthermore, the ring clamp should have been wiped clean to clear off any residue that may have added friction to the system.
Elliptical Motion	An ideal pendulum would move in one field of motion, such as in a two-dimensional plane. However, for some trials in the experiment, the pendulum may have moved in an ellipse pattern instead of in a linear line. This is significant because the pendulum would appear to spend more time around the circular edges at maximum displacement, which would result in skewed data.	When energy is consumed in three dimensional motion, it means that torque was created. A cause that may have contributed to this error was the fact that the pivot was not entirely fixed. Instead of taping the string to the ring clamp, a split cork can be used to hold the pendulum in place.

Random Errors:

Error Description	Significance to Experiment	Improvement Suggestions
Tracker Uncertainty	The software application “Tracker” was used to capture and record the motion of the pendulum. The elapsed time between each plotted point was	Given the access of resources, Tracker was the most reliable option. The usage of a software application as opposed to

	0.032 seconds. The interval of time in between may have affected the accuracy of the results. For example, the pendulum may have not reached its maximum displacement when a data point was plotted.	human reaction time reduced the amount of random errors. However, the default delta (elapsed) time could have been set to a lower value, so that the error of the would have been lowered.

Conclusion

The intended goal to model the motion of a pendulum was achieved in this investigation. It can be concluded that the experimental results proved the theoretical equation on how the suspension length would affect the period time, to be correct. As discussed in *Figure 3*, the relationship was found to be a strong positive square root relationship. In addition, this can be reinforced with the linearized data in *Figure 4*, where both the theoretical slope and y-intercept were within the domain and range of the uncertainties. While there were some limitations in the process, overall, the experiment was a success. I think a reason that can be attributed to the success of the experiment was the fact that the variables that could have affected the time period were well observed and accounted for. For example, adding 0.05 m of string ensured that there would be sufficient length for the string to achieve the intended value as the independent variable.

Further Exploration

Through this exploration, I was able to gain a whole new understanding of the pendulum and how they function. With the findings in this investigation, I can use it to help explore my interest in the tuned mass damper. For example, I can investigate how changing the length of the damper to be closer to the resonant frequency, would dampen the motion of a structure. Furthermore, another topic that can be explored in the future would be the factors that affect the period of other simple harmonic oscillators such as the mass-spring system, and see how they compare to a simple pendulum.

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