APPENDIX for Testing the Predictability of U.S. Housing Price Index Returns Based on an IVX-AR Model

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This appendix presents the supplemental lemmas and proofs of the main results in the main text with the same equation numbers.

First, we introduce some notations. Define the short-run covariance matrices associated with the error terms v_t and $e_t - \mathbf{e}_{t-1,q}\phi$ as

$$\sigma_v^2 = E(v_t^2), \quad \sigma_{ve,\phi} = E(v_t(e_t - \mathbf{e}_{t-1,q}\phi)^\intercal), \quad \sigma_{ee,\phi} = E((e_t - \mathbf{e}_{t-1,q}\phi)(e_t - \mathbf{e}_{t-1,q}\phi)^\intercal),$$

and their estimators

$$\check{\sigma}_v^2 = \frac{1}{T}\sum_{t=1}^{T}\check{v}_t^2, \quad \check{\sigma}_{ve,\phi} = \frac{1}{T}\sum_{t=1}^{T}\check{v}_t(\check{e}_t - \check{\mathbf{e}}_{t-1,q}\check{\phi})^\intercal,$$

$$\check{\sigma}_{ee,\phi} = \frac{1}{T}\sum_{t=1}^{T}(\check{e}_t - \check{\mathbf{e}}_{t-1,q}\check{\phi})(\check{e}_t - \check{\mathbf{e}}_{t-1,q}\check{\phi})^\intercal,$$

where $\check{e}_t$ is the OLS residuals from the model (5), and $\check{v}_t$ and $\check{\phi}$ are respectively the OLS residuals and coefficients from model (3) with $\check{u}_t$ being the OLS residuals from model (2). We use the OLS estimators rather than the IVX-based estimators from equations (9) and (10) due to the superconsistency properties of the former when the regressor belongs to one of the classes (II)∼(VI) and efficiency when the regressor is stationary (class (I)).

Meanwhile, we assume that the bandwidth $M_T \to \infty$ and $M_T/\sqrt{T} \to 0$ as $T \to \infty$. Then,

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we can define their long-run covariance matrices as

$$\Omega_{ee,\phi} = \sum_{\ell=-\infty}^{\infty} E\left[(e_t - \mathbf{e}_{t-1,q}\phi)(e_{t-\ell} - \mathbf{e}_{t-1-\ell,q}\phi)^\intercal\right], \ \Omega_{ve,\phi} = \sigma_{ve,\phi} + \Lambda_{ev,\phi}^\intercal,$$

$$\Lambda_{ev,\phi} = \sum_{\ell=1}^{\infty} E\left[(e_t - \mathbf{e}_{t-1,q}\phi)v_{t-\ell}\right],$$

and their estimators

$$\check{\Omega}_{ee,\phi} = \check{\sigma}_{ee,\phi} + \check{\Lambda}_{ee,\phi} + \check{\Lambda}_{ee,\phi}^\intercal, \quad \check{\Omega}_{ve,\phi} = \check{\sigma}_{ve,\phi} + \check{\Lambda}_{ev,\phi}^\intercal,$$

where

$$\check{\Lambda}_{ee,\phi} = \frac{1}{T}\sum_{\ell=1}^{M_T}(1 - \ell/(M_T+1)) \sum_{t=\ell+1}^{T} (\check{e}_t - \check{\mathbf{e}}_{t-1,q}\check{\phi})(\check{e}_{t-\ell} - \check{\mathbf{e}}_{t-1-\ell,q}\check{\phi})^\intercal,$$

and

$$\check{\Lambda}_{ev,\phi} = \frac{1}{T}\sum_{\ell=1}^{M_T}(1 - \ell/(M_T+1)) \sum_{t=\ell+1}^{T} (\check{e}_t - \check{\mathbf{e}}_{t-1,q}\check{\phi})\check{v}_{t-\ell}.$$

Clearly, $\Omega_{ve,\phi}$ is a one-sided long-run covariance matrix because $\{v_t\}_{t=1}^T$ is a martingale difference sequence. And the above estimators $\check{\Omega}_{ee,\phi}$ and $\check{\Omega}_{ve,\phi}$ are the commonly employed Newey-West type estimators of $\Omega_{ee,\phi}$ and $\Omega_{ve,\phi}$, respectively.

Further, let $B_{e,\phi}(t)$ be a d-dimensional Brownian motion with covariance matrix $\Omega_{ee,\phi}$ and $J_{C,\phi}(t) = \int_0^t e^{C(t-s)}dB_{e,\phi}(s)$ an Ornstein-Uhlenbeck process, and their demeaned versions $\underline{B}_{e,\phi}(t) = B_{e,\phi}(t) - \int_0^1 B_{e,\phi}(t)dt$ and $\underline{J}_{C,\phi}(t) = J_{C,\phi}(t) - \int_0^1 J_{C,\phi}(t)dt$. Then we define

$$\Sigma_{xx,\phi} = E\{(x_{t-1} - \sum_{j=1}^{q}\phi_j x_{t-j-1})(x_{t-1} - \sum_{j=1}^{q}\phi_j x_{t-j-1})^\intercal\}, \quad \Sigma_{uu,q} = E(\mathbf{u}_{t-1,q}\mathbf{u}_{t-1,q}^\intercal),$$

$$V_{C,\phi} = \int_0^\infty e^{rC}\Omega_{ee,\phi}e^{rC}dr, \quad V_{C_z,\phi} = \int_0^\infty e^{rC_z}\Omega_{ee,\phi}e^{rC_z}dr, \quad \mathbb{V}_\phi = \int_0^\infty e^{rC}V_{C,\phi}e^{rC_z}dr,$$

$$\mathcal{V}_{C,\phi} \equiv N\left(0, \int_0^\infty e^{-rC}\Omega_{ee,\phi}e^{-rC}dr\right), \quad \eta_x \wedge \eta_z = min(\eta_x, \eta_z),$$

$$C_z^* = \begin{cases} -C_z^{-1} & \text{if } \eta_z < \eta_x, \\ C^{-1} & \text{if } \eta_z > \eta_x, \\ (C - C_z)^{-1} & \text{if } \eta_z = \eta_x, \end{cases} \quad \Psi_{ee,\phi} = \begin{cases} V_{C,\phi}C + \Omega_{ee,\phi} & \text{under (II)}, \\ \int_0^1 \underline{J}_{C,\phi}dJ_{C,\phi}^\intercal + \Omega_{ee,\phi} & \text{under (III), (V)}, \\ \int_0^1 \underline{B}_{e,\phi}dB_{e,\phi}^\intercal + \Omega_{ee,\phi} & \text{under (IV)}, \\ \int_0^\infty e^{-rC}\mathcal{V}_{C,\phi}\mathcal{V}_{C,\phi}^\intercal e^{-rC}dr & \text{under (VI)}. \end{cases}$$

Finally, we introduce the following regularity conditions:

(A1) Assume that the error terms u_t follow an AR(q)+GARCH(m,n) process as in equations (3) and (4), in which the parameters in the AR part satisfy that all roots of the polynomial $1-\sum_{j=1}^{q}\phi_j L^j$ are outside the unit circle and the parameters in the GARCH part satisfy $\omega_0>0$, $a_i\geq 0$, $b_i\geq 0$, and $\sum_{i=1}^{max(m,n)}(a_i+b_i)<1$.

(A2) The vector moving average process $e_t=\sum_{\ell=0}^{\infty}\psi_\ell\varepsilon_{t-\ell}$ is stationary satisfying $\psi_0=I_d$, $\sum_{\ell=0}^{\infty}\psi_\ell$ being full rank and $\sum_{\ell=0}^{\infty}\ell\|\psi_\ell\|<\infty$, where $\|A\|$ is the spectral norm of a given vector/matrix A, i.e., the square root of the maximal eigenvalue of the matrix $AA^\intercal$.

(A3) The matrices $\Sigma_{xx,\phi}$, $\Sigma_{uu,q}$, $V_{C,\phi}$, $V_{C_z,\phi}$, $\mathbb{V}_\phi$ and $\Psi_{ee,\phi}$ are all positive definite.

(A4) $E\|v_t\|^4<\infty$, $E\|\varepsilon_t\|^4<\infty$ and $\{(\epsilon_t,\varepsilon_t)^\intercal\}$ is a sequence of independent and identically distributed random vectors with zero means.

The condition (A1) is commonly employed in financial time series analysis; see Tsay (2010). The condition (A2) imposes short-memory in the error term e_t as in (5). The condition (A3) is introduced so that their inverses are well defined. Finite fourth moments are imposed in (A4) so that the GARCH specification does not have any effect on the convergence rate of estimators.

Theorem A. Let $\{(x_{t-1},y_t)\}_{t=1}^{T}$ be a time series sequence following equations (2)$\sim$(5). Under Conditions (A1)-(A4), as $T\to\infty$, we have

(i) under class (I) and $0=\eta_x<\eta_z<1$,

$$\sqrt{T}(\hat{\beta}-\beta)\Rightarrow N\left(0,\Sigma_{xx,\phi}^{-1}\Xi_{xx,\phi}\Sigma_{xx,\phi}^{-1}\right);$$

(ii) under class (II) and $0<\eta_x=\eta_z<1$,

$$T^{(1+\eta_x)/2}(\hat{\beta}-\beta)\Rightarrow N\left(0,\mathbb{V}_\phi^{-1}C^{-1}V_{C,\phi}C^{-1}(\mathbb{V}_\phi^\intercal)^{-1}\sigma_v^2\right);$$

(iii) under classes (II) and $0<\eta_x<\eta_z<1$,

$$T^{(1+\eta_x)/2}(\hat{\beta}-\beta)\Rightarrow N\left(0,V_{C,\phi}^{-1}\sigma_v^2\right);$$

(iv) under class (II) $\sim$ (V) and $0<\eta_z<\eta_x\leq 1$,

$$T^{(1+\eta_z)/2}(\hat{\beta}-\beta)\Rightarrow MN\left(0,(\Psi_{ee,\phi}^{-1})^\intercal C_z V_{C_z,\phi}C_z\Psi_{ee,\phi}^{-1}\sigma_v^2\right);$$

(v) under class (VI) and $1/2 < \eta_x < 1$,

$$T^{\eta_x}\Pi_x^T(\hat{\beta}-\beta) \Rightarrow MN\left(0, \Psi_{ee,\phi}^{-1}\sigma_v^2\right);$$

(vi) under classes (I)$\sim$ (VI),

$$\sqrt{T}(\hat{\phi}-\phi_0) \Rightarrow N\left(0, \Sigma_{uu,q}^{-1}\Xi_{uu,v}\Sigma_{uu,q}^{-1}\right),$$

where " $\Rightarrow$ " denotes convergence in distribution, "$MN(\cdot)$" represents mixed normal distribution, $\Xi_{xx,\phi} = E\left[(x_{t-1}-\mathbf{x}_{t-2,q}\phi)(x_{t-1}-\mathbf{x}_{t-2,q}\phi)^\intercal v_t^2\right]$ and $\Xi_{uu,v} = E(\mathbf{u}_{t-1,q}\mathbf{u}_{t-1,q}^\intercal v_t^2)$.

The resulting IVX-AR estimator of β, calculated by equations (9) and (10), has a slightly slower convergence rate than that of the OLS estimator when the predictive variable is highly persistent; see case (iv). And asymptotically it has mixed Gaussian in the limit in the presence of serially correlated errors. Meanwhile, it has the same convergence rate as the OLS estimator in Magdalinos and Phillips (2009) when the regressors belong to mildly explosive root; see case (v). Further, the asymptotic distribution of autoregressive coefficients $\hat{\phi}$ in the error terms is not affected by the estimators $\hat{\beta}$ in the linear predictive regression, see case (vi).

To prove Theorem A, we present some lemmas as below.

Lemma 1. *Let $\{(x_{t-1}, y_t)\}_{t=1}^T$ be a time series sequence following equations (2)$\sim$(5). Under Conditions (A1)$\sim$(A4), the following approximations hold as $T \to \infty$.*

(i) $\sum_{t=1}^T(\underline{\boldsymbol{y}}_{t-1,q} - \underline{\boldsymbol{x}}_{t-2,q}^\intercal\hat{\beta})(\underline{y}_t - \underline{x}_{t-1}^\intercal\hat{\beta}) = \{1+o_p(1)\}\sum_{t=1}^T \underline{\boldsymbol{u}}_{t-1,q}\underline{u}_t$*;*

(ii) $\sum_{t=1}^T(\underline{\boldsymbol{y}}_{t-1,q} - \underline{\boldsymbol{x}}_{t-2,q}^\intercal\hat{\beta})(\underline{\boldsymbol{y}}_{t-1,q} - \underline{\boldsymbol{x}}_{t-2,q}^\intercal\hat{\beta})^\intercal = \{1+o_p(1)\}\sum_{t=1}^T(\underline{\boldsymbol{y}}_{t-1,q} - \underline{\boldsymbol{x}}_{t-2,q}^\intercal\beta)(\underline{\boldsymbol{y}}_{t-1,q} - \underline{\boldsymbol{x}}_{t-2,q}^\intercal\beta)^\intercal$*;*

(iii) $\sum_{t=1}^T(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\hat{\phi})(\underline{y}_t - \underline{\boldsymbol{y}}_{t-1,q}^\intercal\hat{\phi}) = \{1+o_p(1)\}\sum_{t=1}^T(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)(\underline{y}_t - \underline{\boldsymbol{y}}_{t-1,q}^\intercal\phi)$*;*

(iv) $\sum_{t=1}^T(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\hat{\phi})(\underline{x}_{t-1} - \underline{\boldsymbol{x}}_{t-2,q}\hat{\phi})^\intercal = \{1+o_p(1)\}\sum_{t=1}^T(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)(\underline{x}_{t-1} - \underline{\boldsymbol{x}}_{t-2,q}\phi)^\intercal$.

Proof. For part (i), we have

$$
\begin{aligned}
& \sum_{t=1}^{T}(\underline{\mathbf{y}}_{t-1,q} - \underline{\mathbf{x}}_{t-2,q}^{\intercal}\hat{\beta})(\underline{y}_t - \underline{x}_{t-1}^{\intercal}\hat{\beta}) \\
= \; & \sum_{t=1}^{T}(\underline{\mathbf{y}}_{t-1,q} - \underline{\mathbf{x}}_{t-2,q}^{\intercal}\hat{\beta})(\underline{y}_t - \underline{x}_{t-1}^{\intercal}\hat{\beta}) - \sum_{t=1}^{T}(\underline{\mathbf{y}}_{t-1,q} - \underline{\mathbf{x}}_{t-2,q}^{\intercal}\hat{\beta})(\underline{y}_t - \underline{x}_{t-1}^{\intercal}\beta) \\
& + \sum_{t=1}^{T}(\underline{\mathbf{y}}_{t-1,q} - \underline{\mathbf{x}}_{t-2,q}^{\intercal}\hat{\beta})(\underline{y}_t - \underline{x}_{t-1}^{\intercal}\beta) - \sum_{t=1}^{T}(\underline{\mathbf{y}}_{t-1,q} - \underline{\mathbf{x}}_{t-2,q}^{\intercal}\beta)(\underline{y}_t - \underline{x}_{t-1}^{\intercal}\beta) \\
& + \sum_{t=1}^{T}(\underline{\mathbf{y}}_{t-1,q} - \underline{\mathbf{x}}_{t-2,q}^{\intercal}\beta)(\underline{y}_t - \underline{x}_{t-1}^{\intercal}\beta) \\
= \; & -\sum_{t=1}^{T}(\underline{\mathbf{y}}_{t-1,q} - \underline{\mathbf{x}}_{t-2,q}^{\intercal}\hat{\beta})\underline{x}_{t-1}^{\intercal}(\hat{\beta} - \beta) - \sum_{t=1}^{T}\underline{\mathbf{x}}_{t-2,q}^{\intercal}(\hat{\beta} - \beta)(\underline{y}_t - \underline{x}_{t-1}^{\intercal}\beta) \\
& + \sum_{t=1}^{T}(\underline{\mathbf{y}}_{t-1,q} - \underline{\mathbf{x}}_{t-2,q}^{\intercal}\beta)(\underline{y}_t - \underline{x}_{t-1}^{\intercal}\beta) \\
= \; & \sum_{t=1}^{T}\underline{\mathbf{x}}_{t-2,q}^{\intercal}(\hat{\beta} - \beta)(\hat{\beta} - \beta)^{\intercal}\underline{x}_{t-1} - \sum_{t=1}^{T}\underline{\mathbf{u}}_{t-1,q}\underline{x}_{t-1}^{\intercal}(\hat{\beta} - \beta) \\
& - \sum_{t=1}^{T}\underline{\mathbf{x}}_{t-2,q}^{\intercal}(\hat{\beta} - \beta)\underline{u}_t + \sum_{t=1}^{T}\underline{\mathbf{u}}_{t-1,q}\underline{u}_t,
\end{aligned}
$$

which implies (i) because the term $\sum_{t=1}^{T}\underline{\mathbf{u}}_{t-1,q}\underline{u}_t$ dominates others because $(\hat{\beta} - \beta)$ is of the order $O_p(T^{-(1+\eta_x\wedge\eta_z)/2})$ for classes (I)∼(V), and $o_p(T^{-\eta_x})$ for class (VI). Together with the fact that $(\hat{\phi} - \phi)$ is of order $T^{-1/2}$, Lemma 1 (ii)∼(iv) can be shown in a similar way. □

Lemma 2. *Let* $\{(x_{t-1}, y_t)\}_{t=1}^{T}$ *be a time series sequence following equations (2)∼(5). Under Conditions (A1)∼ (A4), the following approximations hold as* $T \to \infty$.

(i) Under classes (I) *and* (II)*, if* $0 \le \eta_x \le \eta_z < 1$,

$$\sum_{t=1}^{T}(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi) = O_p(T^{\eta_x+\eta_z/2});$$

(ii) Under class (II)*, if* $0 < \eta_z < \eta_x < 1$,

$$T^{-\eta_z}\sum_{t=1}^{T}(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi) = -C_z^{-1}(x_{T-1} - \boldsymbol{x}_{T-2,q}\phi) + o_p(1);$$

(iii) Under classes (III) ∼ (V)*, if* $0 < \eta_z < \eta_x = 1$,

$$T^{-(1/2+\eta_z)}\sum_{t=1}^{T}(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi) = -C_z^{-1}T^{-1/2}(x_{T-1} - \boldsymbol{x}_{T-2,q}\phi) + o_p(1);$$

(iv) Under class (VI),

$$\sum_{t=1}^{T}(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi) = O_p(T^{\eta_x/2+\eta_z}\Pi_x^T).$$

Proof. Parts (i)$\sim$(iii) follow from the same arguments in proving Lemma 2 in the Online Appendix of Kostakis et al. (2015).

For part (iv), by the equation (6) in the Online Appendix of Kostakis et al. (2015) and the fact that $\tilde{z}_t = z_t + \frac{C}{T^{\eta_x}}\psi_{Tt}$, we have

$$\sum_{t=1}^{T}\tilde{z}_{t-1} = C_z^{-1}T^{\eta_z}(z_T + \frac{C}{T^{\eta_x}}\psi_{TT} - x_T + x_0)$$

where $z_T = O_p(T^{\eta_z/2})$, $\frac{C}{T^{\eta_x}}\psi_{TT} = O_p(T^{\eta_x\wedge\eta_y-\eta_x/2}\Pi_x^T)$ implied by Equation (2.15) in Phillips and Lee (2016), and $x_T = O_p(T^{\eta_x/2}\Pi_x^T)$ implied by Lemma 2.1 in Phillips and Lee (2016). It follows that

$$\sum_{t=1}^{T}\tilde{z}_{t-1} = O_p(T^{\eta_x/2+\eta_z}\Pi_x^T) \quad \text{and} \quad \sum_{t=1}^{T}(\tilde{z}_{t-1} - \tilde{\mathbf{z}}_{t-2,q}\phi) = O_p(T^{\eta_x/2+\eta_z}\Pi_x^T).$$

$\square$

Lemma 3. *Let $\{(x_{t-1}, y_t)\}_{t=1}^T$ be a time series sequence following equations (2)$\sim$(5). Under Conditions (A1)$\sim$ (A4), the following approximations hold as $T \to \infty$:*

(i) Under class (I), we have

$$\frac{1}{T}\sum_{t=1}^{T}(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)^\intercal = \frac{1}{T}\sum_{t=1}^{T}(x_{t-1} - \boldsymbol{x}_{t-2,q}\phi)(x_{t-1} - \boldsymbol{x}_{t-2,q}\phi)^\intercal + o_p(1);$$

(ii) Under classes (II)$\sim$(V), if $0 < \eta_z < \eta_x \le 1$, we have

$$\frac{1}{T^{1+\eta_z}}\sum_{t=1}^{T}(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)^\intercal = \frac{1}{T^{1+\eta_z}}\sum_{t=1}^{T}(z_{t-1} - \boldsymbol{z}_{t-2,q}\phi)(z_{t-1} - \boldsymbol{z}_{t-2,q}\phi)^\intercal + o_p(1);$$

(iii) Under classes (II)$\sim$(V), if $0 < \eta_x < \eta_z < 1$, we have

$$\frac{1}{T^{1+\eta_x}}\sum_{t=1}^{T}(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)^\intercal = \frac{1}{T^{1+\eta_x}}\sum_{t=1}^{T}(x_{t-1} - \boldsymbol{x}_{t-2,q}\phi)(x_{t-1} - \boldsymbol{x}_{t-2,q}\phi)^\intercal + o_p(1);$$

(iv) Under class (VI), we have

$$\frac{1}{T^{2(\eta_x\wedge\eta_z)}}\sum_{t=1}^{T}\Pi_x^{-T}(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)^\intercal\Pi_x^{-T} \Rightarrow CC_z^*\Psi_{ee,\phi}CC_z^*.$$

Proof. Part (i) follows from the same arguments in proving Lemma B2 (ii) in the Online Appendix of Kostakis et al. (2015). Parts (ii) and (iii) respectively follow the equations (13) and (14) of Lemma B3 in the Online Appendix of Kostakis et al. (2015). Part (iv) follows Lemma A.2 in Phillips and Lee (2016). □

Lemma 4. *Let $\{(x_{t-1}, y_t)\}_{t=1}^{T}$ be a time series sequence following equations (2)∼(5). Under Conditions (A1)∼ (A4), the following approximations hold as $T \to \infty$:*
(i) Under classes (I) and (II), if $0 \le \eta_x < \eta_z < 1$, we have

$$\frac{1}{T^{1+\eta_x}} \sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)(x_{t-1} - \boldsymbol{x}_{t-2,q}\phi)^{\intercal} = \frac{1}{T^{1+\eta_x}} \sum_{t=1}^{T} (x_{t-1} - \boldsymbol{x}_{t-2,q}\phi)(x_{t-1} - \boldsymbol{x}_{t-2,q}\phi)^{\intercal} + o_p(1);$$

(ii) Under classes (II)∼(V), if $0 < \eta_z < \eta_x \le 1$, we have

$$\begin{aligned}\tfrac{1}{T^{1+\eta_z}} \textstyle\sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)(x_{t-1} & - \boldsymbol{x}_{t-2,q}\phi)^{\intercal} = \tfrac{1}{T^{1+\eta_z}} \textstyle\sum_{t=1}^{T} (z_{t-1} - \boldsymbol{z}_{t-2,q}\phi)(x_{t-1} - \boldsymbol{x}_{t-2,q}\phi)^{\intercal} \\ & - \tfrac{1}{T^{1+\eta_x}} \textstyle\sum_{t=1}^{T} (x_{t-1} - \boldsymbol{x}_{t-2,q}\phi)(x_{t-1} - \boldsymbol{x}_{t-2,q}\phi)^{\intercal} C C_z^{-1} + o_p(1).\end{aligned}$$

Proof. It follows from the same arguments in proving the equations (11) and (12) of Lemma B3 in the Online Appendix of Kostakis et al. (2015) with x_{t-1}, z_{t-1}, and $\tilde{z}_{t-1}$ replaced by $x_{t-1} - \mathbf{x}_{t-2,q}\phi$, $z_{t-1} - \mathbf{z}_{t-2,q}\phi$ and $\tilde{z}_{t-1} - \tilde{\mathbf{z}}_{t-2,q}\phi$, respectively. □

Lemma 5. *Let $\{(x_{t-1}, y_t)\}_{t=1}^{T}$ be a time series sequence following equations (2)∼(5). Under Conditions (A1)∼ (A4), the following limits hold as $T \to \infty$.*
(i) Under class (I)*, if $0 = \eta_x < \eta_z < 1$, we have*

$$\frac{1}{T} \sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)(\underline{x}_{t-1} - \underline{\boldsymbol{x}}_{t-2,q}\phi)^{\intercal} = \Sigma_{xx,\phi} + o_p(1);$$

(ii) Under class (II)*, if $0 < \eta_z < \eta_x < 1$, we have*

$$\frac{1}{T^{1+\eta_z}} \sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)(\underline{x}_{t-1} - \underline{\boldsymbol{x}}_{t-2,q}\phi)^{\intercal} = -(\Omega_{ee,\phi} + V_{C,\phi} C) C_z^{-1} + o_p(1);$$

(iii) Under class (II)*, if $0 < \eta_x < \eta_z < 1$, we have*

$$\frac{1}{T^{1+\eta_x}} \sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)(\underline{x}_{t-1} - \underline{\boldsymbol{x}}_{t-2,q}\phi)^{\intercal} = V_{C,\phi} + o_p(1);$$

(iv) Under class (II)*, if $0 < \eta_x = \eta_z < 1$, we have*

$$\frac{1}{T^{1+\eta_x}} \sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)(\underline{x}_{t-1} - \underline{\boldsymbol{x}}_{t-2,q}\phi)^{\intercal} = -C\mathbb{V}_{\phi} + o_p(1);$$

(v) Under classes (III) *and* (V), *if* $0 < \eta_z < \eta_x = 1$, *we have*

$$\frac{1}{T^{1+\eta_z}} \sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)(\underline{x}_{t-1} - \underline{\boldsymbol{x}}_{t-2,q}\phi)^\intercal \Rightarrow -\big(\int_0^1 \underline{J}_{C,\phi}(t) dJ_{C,\phi}^\intercal(t) + \Omega_{ee,\phi}\big) C_z^{-1};$$

(vi) Under class (IV), *if* $0 < \eta_z < \eta_x = 1$, *we have*

$$\frac{1}{T^{1+\eta_z}} \sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)(\underline{x}_{t-1} - \underline{\boldsymbol{x}}_{t-2,q}\phi)^\intercal \Rightarrow -\big(\int_0^1 \underline{B}_{e,\phi}(t) dB_{e,\phi}^\intercal(t) + \Omega_{ee,\phi}\big) C_z^{-1}.$$

(vii) Under class (VI), *we have*

$$\frac{1}{T^{\eta_x + (\eta_x \wedge \eta_z)}} \sum_{t=1}^{T} \Pi_x^{-T} (\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)(\underline{x}_{t-1} - \underline{\boldsymbol{x}}_{t-2,q}\phi)^\intercal \Pi_x^{-T} \Rightarrow C C_z^* \times \Psi_{ee,\phi}.$$

Proof. For part (i), Lemma B2 (i) in the Online Appendix of Kostakis et al. (2015) implies $\frac{1}{T}\sum_{t=1}^{T} \tilde{z}_{t-1-j} \underline{x}_{t-1-k}^\intercal = \frac{1}{T}\sum_{t=1}^{T} x_{t-1-j} x_{t-1-k}^\intercal + o_p(1)$ for $j = 0, 1, \cdots, q$ and $k = 0, 1, \cdots, q$. It follows that

$$\begin{aligned}
& \frac{1}{T}\sum_{t=1}^{T} (z_{t-1} - \mathbf{z}_{t-2,q}\phi)(x_{t-1} - \mathbf{x}_{t-2,q}\phi)^\intercal \\
= \quad & \frac{1}{T}\sum_{t=1}^{T} x_{t-1}x_{t-1}^\intercal - \sum_{j=1}^{q} \phi_j \Big(\frac{1}{T}\sum_{t=1}^{T} x_{t-1}x_{t-1-j}^\intercal\Big) - \sum_{k=1}^{q} \phi_k \Big(\frac{1}{T}\sum_{t=1}^{T} x_{t-1-k}x_{t-1}^\intercal\Big) \\
& + \sum_{j=1}^{q}\sum_{k=1}^{q} \phi_j\phi_k \Big(\frac{1}{T}\sum_{t=1}^{T} x_{t-1-j}x_{t-1-k}^\intercal\Big) + o_p(1) \\
= \quad & \frac{1}{T}\sum_{t=1}^{T} (x_{t-1} - \mathbf{x}_{t-2,q}\phi)(x_{t-1} - \mathbf{x}_{t-2,q}\phi)^\intercal + o_p(1) \\
= \quad & \Sigma_{xx,\phi} + o_p(1).
\end{aligned}$$

For part (ii), under class (II), if $0 < \eta_z < \eta_x < 1$, we have

$$
\begin{aligned}
& \frac{1}{T^{1+\eta_z}} \sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\mathbf{z}}_{t-2,q}\phi)(\underline{x}_{t-1} - \underline{\mathbf{x}}_{t-2,q}\phi)^\intercal \\
= \quad & \frac{1}{T^{1+\eta_z}} \sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\mathbf{z}}_{t-2,q}\phi)(x_{t-1} - \mathbf{x}_{t-2,q}\phi)^\intercal \\
& - \left(\frac{1}{T^{1/2+\eta_z}} \sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\mathbf{z}}_{t-2,q}\phi) \right) \left(\frac{1}{T^{3/2}} \sum_{t=1}^{T} (x_{t-1} - \mathbf{x}_{t-2,q}\phi) \right)^\intercal \\
= \quad & \frac{1}{T^{1+\eta_z}} \sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\mathbf{z}}_{t-2,q}\phi)(x_{t-1} - \mathbf{x}_{t-2,q}\phi)^\intercal \\
& + \frac{1}{T^{1/2}} C_z^{-1} (x_{T-1} - \mathbf{x}_{T-2,q}\phi) \left(\frac{1}{T^{3/2}} \sum_{t=1}^{T} (x_{t-1} - \mathbf{x}_{t-2,q}\phi) \right)^\intercal + o_p(1) \\
= \quad & \frac{1}{T^{1+\eta_z}} \sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\mathbf{z}}_{t-2,q}\phi)(x_{t-1} - \mathbf{x}_{t-2,q}\phi)^\intercal + o_p(1),
\end{aligned}
$$

where the second equation follows Lemma 2 (ii) and the third equation follows the facts that $x_{T-1} - \mathbf{x}_{T-2,q}\phi = O_p(T^{\eta_x/2})$ and $\sum_{t=1}^{T}(x_{t-1} - \mathbf{x}_{t-2,q}\phi) = O_p(T^{1/2+\eta_x})$.

Further, Lemma 4 (ii) implies that

$$
\begin{aligned}
& \frac{1}{T^{1+\eta_z}} \sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\mathbf{z}}_{t-2,q}\phi)(x_{t-1} - \mathbf{x}_{t-2,q}\phi)^\intercal \\
= \quad & \frac{1}{T^{1+\eta_z}} \sum_{t=1}^{T} (z_{t-1} - \mathbf{z}_{t-2,q}\phi)(x_{t-1} - \mathbf{x}_{t-2,q}\phi)^\intercal \\
& - \frac{1}{T^{1+\eta_x}} \sum_{t=1}^{T} (x_{t-1} - \mathbf{x}_{t-2,q}\phi)(x_{t-1} - \mathbf{x}_{t-2,q}\phi)^\intercal C C_z^{-1} + o_p(1) \\
= \quad & -(\Omega_{ee,\phi} + V_{C,\phi} C) C_z^{-1} + o_p(1),
\end{aligned}
$$

where $\frac{1}{T^{1+\eta_z}} \sum_{t=1}^{T}(z_{t-1} - \mathbf{z}_{t-2,q}\phi)(x_{t-1} - \mathbf{x}_{t-2,q}\phi)^\intercal = -\Omega_{ee,\phi} C_z^{-1} + o_p(1)$ follows from equation (20) in Phillips and Magdalinos (2009), and $\frac{1}{T^{1+\eta_x}} \sum_{t=1}^{T}(x_{t-1} - \mathbf{x}_{t-2,q}\phi)(x_{t-1} - \mathbf{x}_{t-2,q}\phi)^\intercal = \int_0^\infty e^{rC} \Omega_{ee,\phi} e^{rC} dr + o_p(1) = V_{C,\phi} + o_p(1)$.

For part (iii), under class (II), if $0 < \eta_x < \eta_z < 1$, we have

$$\begin{aligned}
& \frac{1}{T^{1+\eta_x}} \sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\mathbf{z}}_{t-2,q}\phi)(\underline{x}_{t-1} - \underline{\mathbf{x}}_{t-2,q}\phi)^{\intercal} \\
= \quad & \frac{1}{T^{1+\eta_x}} \sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\mathbf{z}}_{t-2,q}\phi)(x_{t-1} - \mathbf{x}_{t-2,q}\phi)^{\intercal} \\
& - \left(\frac{1}{T^{1/2+\eta_x}} \sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\mathbf{z}}_{t-2,q}\phi) \right) \left(\frac{1}{T^{3/2}} \sum_{t=1}^{T} (x_{t-1} - \mathbf{x}_{t-2,q}\phi) \right)^{\intercal} \\
= \quad & \frac{1}{T^{1+\eta_x}} \sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\mathbf{z}}_{t-2,q}\phi)(x_{t-1} - \mathbf{x}_{t-2,q}\phi)^{\intercal} + o_p(1) \\
= \quad & \frac{1}{T^{1+\eta_x}} \sum_{t=1}^{T} (x_{t-1} - \mathbf{x}_{t-2,q}\phi)(x_{t-1} - \mathbf{x}_{t-2,q}\phi)^{\intercal} + o_p(1) \\
= \quad & V_{C,\phi} + o_p(1),
\end{aligned}$$

where the second equation follows from Lemma 2 (i) and $\sum_{t=1}^{T}(x_{t-1} - \mathbf{x}_{t-2,q}\phi) = O_p(T^{1/2+\eta_x})$, and the third equation follows from Lemma 4 (i).

For part (iv), under class (II), if $0 < \eta_x = \eta_z < 1$, using the similar arguments in part (iii), we can show that
$\frac{1}{T^{1+\eta_x}} \sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\mathbf{z}}_{t-2,q}\phi)(\underline{x}_{t-1} - \underline{\mathbf{x}}_{t-2,q}\phi)^{\intercal} = \frac{1}{T^{1+\eta_x}} \sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\mathbf{z}}_{t-2,q}\phi)(x_{t-1} - \mathbf{x}_{t-2,q}\phi)^{\intercal} + o_p(1)$.
Moreover, following the similar arguments in proving Lemma 3.6 in Phillips and Magdalinos (2009), we can show that

$$\frac{1}{T^{1+\eta_x}} \sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\mathbf{z}}_{t-2,q}\phi)(x_{t-1} - \mathbf{x}_{t-2,q}\phi)^{\intercal} = -C\mathbb{V}_{\phi} + o_p(1).$$

For part (v), Lemma 4 (ii) and Lemma 2 (iii) imply that

$$\begin{aligned}
& \tfrac{1}{T^{1+\eta_z}} \textstyle\sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\mathbf{z}}_{t-2,q}\phi)(\underline{x}_{t-1} - \underline{\mathbf{x}}_{t-2,q}\phi)^{\intercal} \\
= \quad & \tfrac{1}{T^{1+\eta_z}} \textstyle\sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\mathbf{z}}_{t-2,q}\phi)(x_{t-1} - \mathbf{x}_{t-2,q}\phi)^{\intercal} \\
& - \left(\tfrac{1}{T^{1/2+\eta_z}} \textstyle\sum_{t=1}^{T} (\tilde{z}_{t-1} - \tilde{\mathbf{z}}_{t-2,q}\phi) \right) \left(\tfrac{1}{T^{3/2}} \textstyle\sum_{t=1}^{T} (x_{t-1} - \mathbf{x}_{t-2,q}\phi) \right)^{\intercal} \\
= \quad & \tfrac{1}{T^{1+\eta_z}} \textstyle\sum_{t=1}^{T} (z_{t-1} - \mathbf{z}_{t-2,q}\phi)(x_{t-1} - \mathbf{x}_{t-2,q}\phi)^{\intercal} \\
& - \tfrac{1}{T^{1+\eta_x}} \textstyle\sum_{t=1}^{T} (x_{t-1} - \mathbf{x}_{t-2,q}\phi)(x_{t-1} - \mathbf{x}_{t-2,q}\phi)^{\intercal} C C_z^{-1} \\
& + \tfrac{1}{T^{1/2}} (x_{t-1} - \mathbf{x}_{t-2,q}\phi) C_z^{-1} \left(\tfrac{1}{T^{3/2}} \textstyle\sum_{t=1}^{T} (x_{t-1} - \mathbf{x}_{t-2,q}\phi) \right)^{\intercal} + o_p(1).
\end{aligned} \tag{1}$$

By equation (20) in Phillips and Magdalinos (2009), the first term on the right hand side of

(1) follows

$$\frac{1}{T^{1+\eta_z}}\sum_{t=1}^{T}(z_{t-1}-\mathbf{z}_{t-2,q}\phi)(x_{t-1}-\mathbf{x}_{t-2,q}\phi)^{\intercal}\Rightarrow -\Omega_{ee,\phi}C_z^{-1}-\int_0^1 J_{C,\phi}(t)dB_{e,\phi}^{\intercal}(t)C_z^{-1}.$$

Under Class (III), the second term on the right hand side of (1) follows

$$\frac{1}{T^{1+\eta_x}}\sum_{t=1}^{T}(x_{t-1}-\mathbf{x}_{t-2,q}\phi)(x_{t-1}-\mathbf{x}_{t-2,q}\phi)^{\intercal}CC_z^{-1}\Rightarrow \int_0^1 J_{C,\phi}(t)J_{C,\phi}^{\intercal}(t)dtCC_z^{-1}.$$

The third term on the right hand side of (1) follows

$$\frac{1}{T^{1/2}}(x_{t-1}-\mathbf{x}_{t-2,q}\phi)C_z^{-1}\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}(x_{t-1}-\mathbf{x}_{t-2,q}\phi)\right)^{\intercal}\Rightarrow J_{C,\phi}(1)C_z^{-1}\left(\int_0^1 J_{C,\phi}^{\intercal}(t)dt\right).$$

Therefore,

$$\begin{aligned}
& \frac{1}{T^{1+\eta_z}}\sum_{t=1}^{T}(\tilde{z}_{t-1}-\tilde{\mathbf{z}}_{t-2,q}\phi)(\underline{x}_{t-1}-\underline{\mathbf{x}}_{t-2,q}\phi)^{\intercal} \\
\Rightarrow\quad & -\Omega_{ee,\phi}C_z^{-1}-\int_0^1 J_{C,\phi}(t)dB_{e,\phi}^{\intercal}(t)C_z^{-1} \\
& -\int_0^1 J_{C,\phi}(t)J_{C,\phi}^{\intercal}(t)dtCC_z^{-1}+J_{C,\phi}(1)C_z^{-1}\left(\int_0^1 J_{C,\phi}^{\intercal}(t)dt\right) \\
=\quad & -\Omega_{ee,\phi}C_z^{-1}-\int_0^1 J_{C,\phi}(t)dJ_{C,\phi}^{\intercal}(t)C_z^{-1}+J_{C,\phi}(1)C_z^{-1}\left(\int_0^1 J_{C,\phi}^{\intercal}(t)dt\right) \\
=\quad & -\left(\Omega_{ee,\phi}+\int_0^1 \underline{J}_{C,\phi}(t)dJ_{C,\phi}^{\intercal}(t)\right)C_z^{-1}.
\end{aligned}$$

For part (vi), when $c_j=0$, the above equation becomes

$$\frac{1}{T^{1+\eta_z}}\sum_{t=1}^{T}(\tilde{z}_{t-1}-\tilde{\mathbf{z}}_{t-2,q}\phi)(\underline{x}_{t-1}-\underline{\mathbf{x}}_{t-2,q}\phi)^{\intercal}\Rightarrow -\left(\Omega_{ee,\phi}+\int_0^1 \underline{B}_{e,\phi}(t)dB_{e,\phi}^{\intercal}(t)\right)C_z^{-1}.$$

For part (vii), it follows by Lemma 2.4 (2) in Phillips and Lee (2016).

□

Lemma 6. *Let $\{(x_{t-1},y_t)\}_{t=1}^{T}$ be a time series sequence following equations (2)∼(5). Under Conditions (A1)∼ (A4), the following approximations hold as $T\to\infty$.*
(i) Under class (I), if $0=\eta_x<\eta_z<1$,

$$T^{-1/2}\sum_{t=1}^{T}(\tilde{z}_{t-1}-\tilde{\mathbf{z}}_{t-2,q}\phi)\underline{v}_t=T^{-1/2}\sum_{t=1}^{T}(\tilde{z}_{t-1}-\tilde{\mathbf{z}}_{t-2,q}\phi)v_t+o_p(1);$$

(ii) Under class (II), if $0 < \eta_x \le \eta_z < 1$,

$$T^{-(1+\eta_x)/2}\sum_{t=1}^{T}(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)\underline{v}_t = T^{-(1+\eta_x)/2}\sum_{t=1}^{T}(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)v_t + o_p(1);$$

(iii) Under classes (II)∼(V), if $0 < \eta_z < \eta_x \le 1$,

$$T^{-(1+\eta_z)/2}\sum_{t=1}^{T}(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)\underline{v}_t = T^{-(1+\eta_z)/2}\sum_{t=1}^{T}(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)v_t + o_p(1).$$

(iv) Under class (VI),

$$T^{-(\eta_x\wedge\eta_z)}\sum_{t=1}^{T}\Pi_x^{-T}(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)\underline{v}_t = T^{-(\eta_x\wedge\eta_z)}\sum_{t=1}^{T}\Pi_x^{-T}(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)v_t + o_p(1).$$

Proof. The proof is straightforward, and we skip details. □

Lemma 7. *Let* $\{(x_{t-1}, y_t)\}_{t=1}^{T}$ *be a time series sequence following equations (2)∼(5). Under Conditions (A1)∼ (A4), the following approximations hold as* $T \to \infty$.

(i) Under class (I), if $0 = \eta_x < \eta_z < 1$,

$$T^{-1/2}\sum_{t=1}^{T}(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)v_t = T^{-1/2}\sum_{t=1}^{T}(x_{t-1} - \boldsymbol{x}_{t-2,q}\phi)v_t + o_p(1);$$

(ii) Under class (I), if $0 = \eta_x < \eta_z < 1$,

$$T^{-1/2}\sum_{t=1}^{T}(x_{t-1} - \boldsymbol{x}_{t-2,q}\phi)v_t \Rightarrow N\Big(0, \Xi_{xx,\phi}\Big);$$

(iii) Under class (II), if $0 < \eta_x \le \eta_z < 1$,

$$T^{-(1+\eta_x)/2}\sum_{t=1}^{T}(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)v_t = T^{-(1+\eta_x)/2}\sum_{t=1}^{T}(x_{t-1} - \boldsymbol{x}_{t-2,q}\phi)v_t + o_p(1);$$

(iv) Under class (II), if $0 < \eta_x \le \eta_z < 1$,

$$T^{-(1+\eta_x)/2}\sum_{t=1}^{T}(x_{t-1} - \boldsymbol{x}_{t-2,q}\phi)v_t \Rightarrow N\Big(0, V_{C,\phi}\sigma_v^2\Big);$$

(v) Under classes (II)∼(V), if $0 < \eta_z < \eta_x \le 1$,

$$T^{-(1+\eta_z)/2}\sum_{t=1}^{T}(\tilde{z}_{t-1} - \tilde{\boldsymbol{z}}_{t-2,q}\phi)v_t = T^{-(1+\eta_z)/2}\sum_{t=1}^{T}(z_{t-1} - \boldsymbol{z}_{t-2,q}\phi)v_t + o_p(1);$$

(vi) Under classes (II)∼(V), if $0 < \eta_z < \eta_x \le 1$,

$$T^{-(1+\eta_z)/2}\sum_{t=1}^{T}(z_{t-1}-\boldsymbol{z}_{t-2,q}\phi)v_t \Rightarrow N\Big(0, V_{C_z,\phi}\sigma_v^2\Big).$$

(vii) Under class (VI),

$$T^{-(\eta_x\wedge\eta_z)}\sum_{t=1}^{T}\Pi_x^{-T}(\tilde{z}_{t-1}-\tilde{\boldsymbol{z}}_{t-2,q}\phi)v_t \Rightarrow CC_z^* \times MN\Big(0, \Psi_{ee,\phi}\sigma_v^2\Big).$$

Proof. Parts (i) and (ii) respectively follow from Lemma B2 (iv) and Lemma B4 (iii), and parts (iv) and (vi) follow from Lemma B4 (i) and (ii) in the Appendix of Kostakis et al. (2015). The proof for parts (iii) and (v) are similar to Lemma 3.5 (i) and Lemma 3.1 (i) in Phillips and Magdalinos (2009). Further, part (vii) follows from Lemma 2.4 (1) in Phillips and Lee (2016). □

Proof of Theorem A. We write equation (10) as

$$\sum_{t=1}^{T}(\underline{\mathbf{y}}_{t-1,q}-\underline{\mathbf{x}}_{t-2,q}^{\intercal}\hat{\beta})(\underline{y}_t-\underline{x}_{t-1}^{\intercal}\hat{\beta})-\Bigg(\sum_{t=1}^{T}(\underline{\mathbf{y}}_{t-1,q}-\underline{\mathbf{x}}_{t-2,q}^{\intercal}\hat{\beta})(\underline{\mathbf{y}}_{t-1,q}-\underline{\mathbf{x}}_{t-2,q}^{\intercal}\hat{\beta})^{\intercal}\Bigg)\hat{\phi}=0.$$

Lemma 1 (i) and (ii) imply that

$$\sum_{t=1}^{T}(\underline{\mathbf{y}}_{t-1,q}-\underline{\mathbf{x}}_{t-2,q}^{\intercal}\beta)(\underline{y}_t-\underline{x}_{t-1}^{\intercal}\beta)-\Bigg(\sum_{t=1}^{T}(\underline{\mathbf{y}}_{t-1,q}-\underline{\mathbf{x}}_{t-2,q}^{\intercal}\beta)(\underline{\mathbf{y}}_{t-1,q}-\underline{\mathbf{x}}_{t-2,q}^{\intercal}\beta)^{\intercal}\Bigg)\hat{\phi}=0,$$

which leads to

$$\sum_{t=1}^{T}\underline{\mathbf{u}}_{t-1,q}\underline{u}_t-\Bigg(\sum_{t=1}^{T}\underline{\mathbf{u}}_{t-1,q}\underline{\mathbf{u}}_{t-1,q}^{\intercal}\Bigg)\hat{\phi}=0$$

and

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\underline{\mathbf{u}}_{t-1,q}\underline{v}_t-\Bigg(\frac{1}{T}\sum_{t=1}^{T}\underline{\mathbf{u}}_{t-1,q}\underline{\mathbf{u}}_{t-1,q}^{\intercal}\Bigg)\sqrt{T}(\hat{\phi}-\phi_0)=0.$$

By the definitions of $\underline{\mathbf{u}}_{t-1,q}$ and $\underline{v}_t$, we have

$$\{1+o_p(1)\}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbf{u}_{t-1,q}v_t-\{1+o_p(1)\}\Bigg(\frac{1}{T}\sum_{t=1}^{T}\mathbf{u}_{t-1,q}\mathbf{u}_{t-1,q}^{\intercal}\Bigg)\sqrt{T}(\hat{\phi}-\phi_0)=0.$$

Hence, under classes (I)∼(VI), part (vi) holds with

$$\sqrt{T}(\hat{\phi}-\phi_0)\xrightarrow{d}N(0,\Sigma_{uu,q}^{-1}\Xi_{uu,v}\Sigma_{uu,q}^{-1}),$$

where $\Sigma_{uu,q}=E(\mathbf{u}_{t-1,q}\mathbf{u}_{t-1,q}^{\intercal})$ and $\Xi_{uu,v}=E(\mathbf{u}_{t-1,q}\mathbf{u}_{t-1,q}^{\intercal}v_t^2)$.

Write equation (9) as

$$\sum_{t=1}^{T}(\tilde{z}_{t-1}-\tilde{\mathbf{z}}_{t-2,q}\hat{\phi})(\underline{y}_t-\underline{\mathbf{y}}_{t-1,q}^{\intercal}\hat{\phi})-\left(\sum_{t=1}^{T}(\tilde{z}_{t-1}-\tilde{\mathbf{z}}_{t-2,q}\hat{\phi})(\underline{x}_{t-1}-\underline{\mathbf{x}}_{t-2,q}\hat{\phi})^{\intercal}\right)\hat{\beta}=0.$$

Then Lemma 1 (iii) and (iv) imply that

$$\sum_{t=1}^{T}(\tilde{z}_{t-1}-\tilde{\mathbf{z}}_{t-2,q}\phi)(\underline{y}_t-\underline{\mathbf{y}}_{t-1,q}^{\intercal}\phi)-\left(\sum_{t=1}^{T}(\tilde{z}_{t-1}-\tilde{\mathbf{z}}_{t-2,q}\phi)(\underline{x}_{t-1}-\underline{\mathbf{x}}_{t-2,q}\phi)^{\intercal}\right)\hat{\beta}=0,$$

which leads to

$$\sum_{t=1}^{T}(\tilde{z}_{t-1}-\tilde{\mathbf{z}}_{t-2,q}\phi)\underline{v}_t-\left(\sum_{t=1}^{T}(\tilde{z}_{t-1}-\tilde{\mathbf{z}}_{t-2,q}\phi)(\underline{x}_{t-1}-\underline{\mathbf{x}}_{t-2,q}\phi)^{\intercal}\right)(\hat{\beta}-\beta)=0. \tag{2}$$

For part (i), under class (I) with $0=\eta_x<\eta_z<1$, using Lemma 4 (i) and Lemma 7 (i), equation (2) becomes

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(x_{t-1}-\mathbf{x}_{t-2,q}\phi)v_t-\Sigma_{xx,\phi}\sqrt{T}(\hat{\beta}-\beta)+o_p(1)=0.$$

Therefore, it follows from Lemma 7 (ii) that $\sqrt{T}(\hat{\beta}-\beta)\Rightarrow N(0,\Sigma_{xx,\phi}^{-1}\Xi_{xx,\phi}\Sigma_{xx,\phi}^{-1})$, where $\Xi_{xx,\phi}=E\left[(x_{t-1}-\mathbf{x}_{t-2,q}\phi)(x_{t-1}-\mathbf{x}_{t-2,q}\phi)^{\intercal}v_t^2\right]$.

Under class (II) with $0<\eta_x=\eta_z<1$, it follows from Lemma 6 (ii) and Lemma 7 (iii)∼(iv) that

$$\begin{aligned}T^{-(1+\eta_x)/2}\sum_{t=1}^{T}(\tilde{z}_{t-1}-\tilde{\mathbf{z}}_{t-2,q}\phi)\underline{v}_t= &\quad T^{-(1+\eta_x)/2}\sum_{t=1}^{T}(\tilde{z}_{t-1}-\tilde{\mathbf{z}}_{t-2,q}\phi)v_t+o_p(1)\\ = &\quad T^{-(1+\eta_x)/2}\sum_{t=1}^{T}(x_{t-1}-\mathbf{x}_{t-2,q}\phi)v_t+o_p(1)\\ \Rightarrow &\quad N\left(0,V_{C,\phi}\sigma_v^2\right).\end{aligned}$$

Because Lemma 5 (iv) implies that $\frac{1}{T^{1+\eta_x}}\sum_{t=1}^{T}(\tilde{z}_{t-1}-\tilde{\mathbf{z}}_{t-2,q}\phi)(\underline{x}_{t-1}-\underline{\mathbf{x}}_{t-2,q}\phi)^{\intercal}=-C\mathbb{V}_{\phi}+o_p(1)$, part (ii) in Theorem A follows.

Under class (II) with $0<\eta_x<\eta_z<1$, using Lemma 5 (iii) and Lemma 7 (iii)∼(iv), we can show that part (iii) holds.

For part (iv), under classes (II)∼(V), if $0<\eta_z<\eta_x\leq 1$, Lemma 6 (iii) and 7 (v) implies that

$$\frac{1}{T^{(1+\eta_z)/2}}\sum_{t=1}^{T}(z_{t-1}-\mathbf{z}_{t-2,q}\phi)v_t$$
$$-\frac{1}{T^{(1+\eta_z)}}\left(\sum_{t=1}^{T}(\tilde{z}_{t-1}-\tilde{\mathbf{z}}_{t-2,q}\phi)(\underline{x}_{t-1}-\underline{\mathbf{x}}_{t-2,q}\phi)^\intercal\right)T^{(1+\eta_z)/2}(\hat{\beta}-\beta)+o_p(1)=0.$$

Thus, it follows from Lemma 5 (ii) (v)(vi) and Lemma 7 (vi) that

$$T^{(1+\eta_z)/2}(\hat{\beta}-\beta)\Rightarrow MN\left(0,(\Psi_{ee,\phi}^{-1})^\intercal C_z V_{C_z,\phi}C_z\Psi_{ee,\phi}^{-1}\sigma_v^2\right).$$

For part (v), by Lemma 6 (iv), equation (2) becomes

$$T^{-(\eta_x\wedge\eta_z)}\sum_{t=1}^{T}\Pi_x^{-T}(\tilde{z}_{t-1}-\tilde{\mathbf{z}}_{t-2,q}\phi)v_t$$
$$\left(T^{-\eta_x-(\eta_x\wedge\eta_z)}\sum_{t=1}^{T}\Pi_x^{-T}(\tilde{z}_{t-1}-\tilde{\mathbf{z}}_{t-2,q}\phi)(\underline{x}_{t-1}-\underline{\mathbf{x}}_{t-2,q}\phi)^\intercal\Pi_x^{-T}\right)T^{\eta_x}\Pi_x^T(\hat{\beta}-\beta)+o_p(1)=0.$$

It follows Lemma 5 (vii) and Lemma 7 (vii) that

$$T^{\eta_x}\Pi_x^T(\hat{\beta}-\beta)\Rightarrow MN\left(0,\Psi_{ee,\phi}^{-1}\sigma_v^2\right).$$

□

Proof of Theorem 1. By Lemma 2 and the fact that $x_{T-1}-\mathbf{x}_{T-2,q}\phi=O_p(T^{\eta_x/2})$ when the regressor belongs to one of the classes (I)∼(V) and $x_{T-1}-\mathbf{x}_{T-2,q}\phi=O_p(T^{\eta_x/2}\Pi_x^T)$ for the class (VI), one can easily verify that $\bar{z}_{T-1}\bar{z}_{T-1}^\intercal\check{\sigma}_{FM,\phi}$ is of smaller order than $\hat{\Upsilon}_T$.

By Theorem A and Lemma 3, it is straightforward to conclude that Theorem 1 holds whenever the regressor x_{t-1} belongs to any class of (I)∼(VI), and we skip details. □