Compressed Sensing

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Outline

The Problem

Setting and main Theorem

The Problem

discrete-time signal $f \in \mathbb{C}^N$
observe $\hat{f}$ on set of frequencies Ω

Can we reconstruct f from values of $\hat{f}$ on Ω?

Of course, not in general but for example if f is of bounded frequency and Ω is large enough. (Shanon Sampling)

In many cases possible with way smaler Ω! If f is *sparse* and if we choose Ω of size $O(|T| \log N)$ *uniform at random* then *with high probability* it is possible!

Wait a moment. Isn't that cheating???

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NO!

It is a reasonable assumption.

Sparsity Assumption

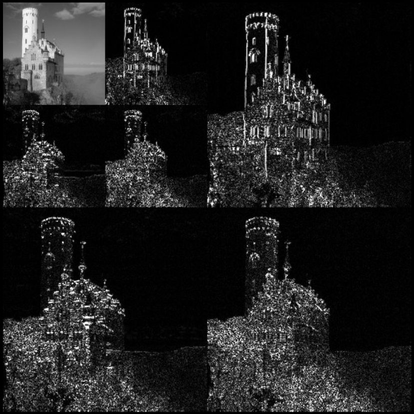
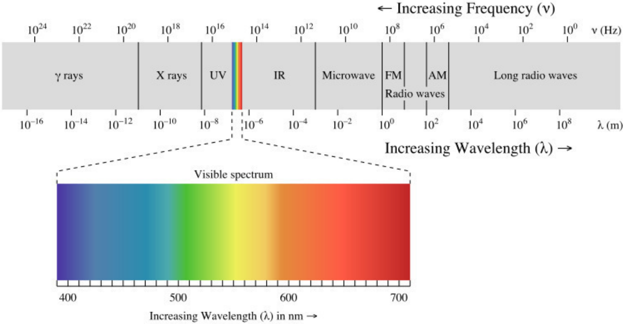


Figure 1: wavelet tranformation

(Wikipedia wavelet article)

Sparsity Assumption



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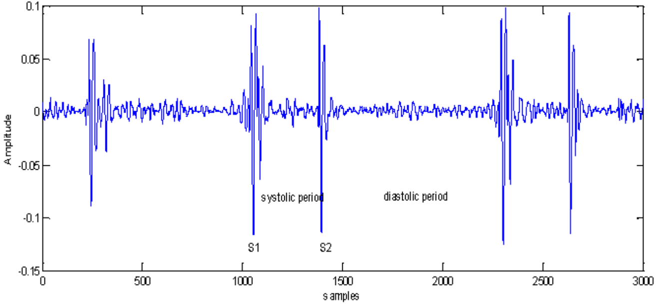
Sparsity Assumption



Adaptive Analysis of Diastolic Murmurs for Coronary Artery Disease Based on Empirical Mode Decomposition. By Zhidong Zhao, Yi Luo, Fangqin Ren, Li Zhang and Changchun Shi

Figure 2: Terence Tao, Emmanuel Candès and Justin Romberg

The main Theorem

Theorem (Candès, Romberg, Tao - 2004)

Let $f \in \mathbb{C}^N$ be a discrete signal supported on an unknown set T, and choose Ω of size $|\Omega|$ uniformly at random. For a given accuracy parameter M, if

$$|T| \leqslant C_M (\log N)^{-1} |\Omega|$$

then with probability at least $1 - O(N^{-M})$, the minimizer of the problem

$$\min_{g \in \mathbb{C}^N \|g\|_{l^1}} := \sum_{t \in \mathbb{Z}_N} |g(t)|, \quad \hat{g}|_\Omega = \hat{f}|_\Omega$$

is unique and is equal to f.

The Proof

Why does it work?

Duality

For a vector $f \in \mathbb{C}^N$ with $T := \text{supp}(f)$ define

$$\text{sgn}(f) := \begin{cases} \frac{f(t)}{|f(t)|}, & t \in T \\ 0, & \textit{else}. \end{cases}$$

Lemma (Candès, Romberg, Tao - 2004)

Suppose there exists a vector P such that

i) $\hat{P}$ is supported in Ω,

ii) $P(t) = \text{sgn}(f)(t)$ for all $t \in T$,

iii) $|P(t)| < 1$ for all $t \notin T$.

Then, if $\mathcal{F}_{T \to \Omega}$ is injective, the l^1-minimizer is unique and equal to f. Conversely, if f is the unique l^1-minimizer, then there exists a vector P with the properties i)-iii).

...there are many trigonometric polynomials supported on Ω in the Fourier domain which satisfy ii). We choose, with the hope that its magnitude on T^c is small, the one with minimum energy

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$$P = \mathcal{F}_{\Omega}^{*} \mathcal{F}_{T \to \Omega} \left(\mathcal{F}_{T \to \Omega}^{*} \mathcal{F}_{T \to \Omega} \right)^{-1} \iota^{*} \mathrm{sgn}(f).$$

Alperen's Epiphany

$$
\begin{tikzcd}
\text{``}P\text{''} \arrow[ddd, dashed] & & & \arrow[lll, dashed, dash, "B"'] \\
& \ell_2(\mathbb{Z}_N) \arrow[r, shift right, "\mathcal{F}"'] \arrow[d, shift right, "\iota^*_{T\to\mathbb{Z}_N}"'] & \ell_2(\mathbb{Z}_N) \arrow[l, shift right, "\mathcal{F}^* = N\mathcal{F}^{-1}"'] \arrow[d, shift right, "\iota^*_{\Omega\to\mathbb{Z}_N}"'] & \\
& \ell_2(T) \arrow[r, shift left, "\mathcal{F}_{T\to\Omega}"] \arrow[u, shift right, "\iota_{T\to\mathbb{Z}_N}"'] & \ell_2(\Omega) \arrow[l, shift left, "\mathcal{F}^*_{T\to\Omega}"] \arrow[u, shift right, "\iota_{\Omega\to\mathbb{Z}_N}"'] & \\
\iota^*_{T\to\mathbb{Z}_N}\operatorname{sgn}(f) & & & v \arrow[lll, dashed, mapsto, "A"] \arrow[uuu, dashed, dash]
\end{tikzcd}
$$

Observe that A $(= \mathcal{F}^*_{T\to\Omega})$ and B commute since A^* and B^* do so.
Suppose we can find a vector $v \in \ell_2(\Omega)$ such that $Av = (\iota^*_{T\to\mathbb{Z}_N}\operatorname{sgn}(f))$.
Then letting $P = \mathcal{F}^*(\iota_{\Omega\to\mathbb{Z}_N})v \in \ell_2(\mathbb{Z}_N) = \mathbb{C}^N$, we get the followings.

(i) $\hat{P} = \mathcal{F}P = \mathcal{F}\mathcal{F}^*(\iota_{\Omega\to\mathbb{Z}_N})v = \mathcal{F}(N\mathcal{F}^{-1})(\iota_{\Omega\to\mathbb{Z}_N})v = N(\iota_{\Omega\to\mathbb{Z}_N})v$ is supported in Ω since $\iota_{\Omega\to\mathbb{Z}_N}$ is the embedding of $\ell_2(\Omega)$ in $\ell_2(\mathbb{Z}_N)$.

(ii) $\iota^*_{T\to\mathbb{Z}_N}P = (\iota^*_{T\to\mathbb{Z}_N})\mathcal{F}^*(\iota_{\Omega\to\mathbb{Z}_N})v = Bv = Av = \iota^*_{T\to\mathbb{Z}_N}\operatorname{sgn}(f)$, so $P(t) = \operatorname{sgn}(f)(t)$ for all $t \in T$.

Construction of the Dual Polynomial P

So, if we can find a vector $v \in \ell_2(\Omega)$ such that

$$\mathcal{F}^*_{T\to\Omega} v = Av = (\iota^*_{T\to\mathbb{Z}_N} \operatorname{sgn}(f)),$$

the signal $P = \mathcal{F}^*(\iota_{\Omega\to\mathbb{Z}_N})v \in \ell_2(\mathbb{Z}_N)$ satisfies the conditions (i) and (ii). What remains is the condition (iii), i.e., $|P(t)| < 1$ for all $t \notin T$.

We hope that among all such P, <u>one</u> with "minimum" ℓ^2-norm should most likely work. (Careful! No sign of existence or uniqueness yet.)

Observe that $\iota_{\Omega\to\mathbb{Z}_N}$ preserves the ℓ_2-norm while $\mathcal{F}^*$ multiplies it with a constant, $\sqrt{N}$, which does not depend on v.

Hence, we would like to find a solution to $Av = (\iota^*_{T\to\mathbb{Z}_N} \operatorname{sgn}(f))$ with ℓ_2-norm as small as possible.

Construction of the Dual Polynomial P

Enter linear algebra and the method of least squares.

Suppose $\mathcal{F}_{T\to\Omega}$ is injective, i.e., a "tall" matrix with full column rank. So, $\mathcal{F}^*_{T\to\Omega} = A$ is a "wide" matrix having full row rank.

Then, the square matrix (AA^*) is necessarily invertible and the method of least squares gives us that the solution with minimum ℓ_2-norm to the underdetermined system $Av = (\iota^*_{T\to\mathbb{Z}_N}\operatorname{sgn}(f))$ is unique and is equal to

$$v_0 = A^*(AA^*)^{-1}(\iota^*_{T\to\mathbb{Z}_N}\operatorname{sgn}(f)).$$

Substituting $A = \mathcal{F}^*_{T\to\Omega}$ and $P = \mathcal{F}^*(\iota_{\Omega\to\mathbb{Z}_N})v_0$, we get the glorious expression

$$P = \mathcal{F}^*(\iota_{\Omega\to\mathbb{Z}_N})\mathcal{F}_{T\to\Omega}(\mathcal{F}^*_{T\to\Omega}\mathcal{F}_{T\to\Omega})^{-1}(\iota^*_{T\to\mathbb{Z}_N})\operatorname{sgn}(f).$$

Change the Probability model

From now on, for convenience, we will use the notation $\mathcal{F}_\Omega^* := \mathcal{F}^* \iota_{\Omega \to \mathbb{Z}_N}$ and $\iota := \iota_{T \to \mathbb{Z}_N}$ so that

$$P = \mathcal{F}_\Omega^* \mathcal{F}_{T\to\Omega} \left(\mathcal{F}_{T\to\Omega}^* \mathcal{F}_{T\to\Omega}\right)^{-1} \iota^* \mathrm{sgn}(f).$$

Let T be fixed subset and choose Ω according to the Bernoulli model with parameter τ.

Theorem 2.2 (Candès, Romberg, Tao - 2004)

Suppose $|T| \leqslant C_M (\log N)^{-1} \tau N$. Then $\mathcal{F}_{T\to\Omega}^* \mathcal{F}_{T\to\Omega}$ is invertible with probability at least $1 - O(N^{-M})$.

Lemma 2.3 (Candès, Romberg, Tao - 2004)

Under these assumptions P obeys $|P(t)| < 1$ for $t \notin T$ with probability at least $1 - O(N^{-M})$.

A clever trick: sampling with the Bernoulli model

Before: * Ω choosen uniformly at random fixed size
(each of the $\binom{N}{N_\omega}$ possible subsets have same probability)

Now: Ω' sampled by Bernoulli model with parameter $\tau \in (0,1)$:
- Introduce for each ω random variables $I_\omega \sim Ber(\tau)$
- $\Omega' = \{\omega : I_\omega = 1\}$ size random
- Note that $|\Omega'|$ is random! $|\Omega'| \sim Bin(N, \tau)$ with $\mathbb{E}|\Omega'| = \tau N$ and by LLN $\frac{|\Omega'|}{N} \approx \tau$ if N large (weak law)

With this probability model we will show that

$$P = F_\Omega^* F_{T\to\Omega} (F_{T\to\Omega}^* F_{T\to\Omega})^{-1} i^* \operatorname{sgn}(f)$$

obeyes all conditions with high probability and exists.

Theorem 2.2: T fixed, Ω chosen with Bernoulli model with $0<\tau<1$. If $|T| \leq C_M \cdot (\log N)^{-1} \tau N$, then $F_{T\to\Omega}^* F_{T\to\Omega}$ is invertible with probability at least $1 - O(N^{-M})$.

$\tau \geq \frac{\log N}{C_M} \frac{|T|}{N}$ or $\tau N \geq \frac{\log N}{C_M} |T|$

Lemma 2.3: Under these assumptions $|P(t)| < 1$ $\forall t \notin T$ with probability at least $1 - O(N^{-M})$.

Why does this give our main theorem? There Ω is obtained from the uniform model?

Take $\tau = N_\omega / N$

Ω uniformly with $|\Omega| = N_\omega$

Ω' Bernoulli with τ

To see this let us define the events

$$\text{Failure}(\Omega_0) = \{\text{no } \hat{P} \text{ supported on } \Omega_0 \text{ satisfying ii) \& iii)}\}$$

Then

$$\sum_{\text{TI}} \frac{P(A \cap B)}{P(B)} \cdot P(B)$$

$$\mathbb{P}(\text{Failure}(\Omega')) = \sum_{k=0}^{N-1} \mathbb{P}(\text{Failure}(\Omega') \mid |\Omega'| = k)\, \mathbb{P}(|\Omega'| = k)$$

law of total probability

$$= \sum_{k=0}^{N-1} \mathbb{P}(\text{Failure}(\Omega_k))\, \mathbb{P}(|\Omega'| = k)$$

Ω_k picked uniformly $|\Omega_k| = k$

sum just till N_D and the

$$\geq \mathbb{P}(\text{Failure}(\Omega)) \sum_{k=0}^{N_\omega} \mathbb{P}(|\Omega'| = k)$$

Observation: the larger Ω, the easier it is to construct P, i.e. small Failure probability

$$= \mathbb{P}(\text{Failure}(\Omega))\, \mathbb{P}(|\Omega'| \leq N_\omega = N\tau)$$

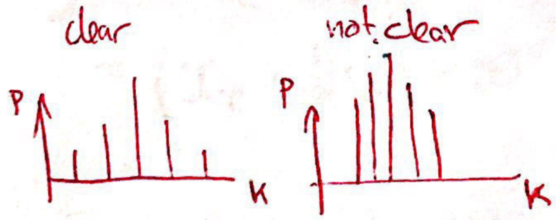
$$\mathbb{P}(|\Omega'| \leq \mathbb{E}|\Omega'|)$$

$$\geq \frac{1}{2} \mathbb{P}(\text{Failure}(\Omega))$$

since $N\tau = \mathbb{E}|\Omega'|$ is integer, it is the mean

mean m satisfies

$\lfloor N\tau \rfloor \leq m \leq \lceil N\tau \rceil$



Thus we can bound the Failure in the Bernoulli model, the Failure in the uniform model will be no more than twice as large.

We now want to prove 2.2. Therefore we will need a more tangible form of P and a key estimate which we will show later on.

Tame The Dual Polynomial

We want to find a more tangible expression for

$$P = \mathcal{F}_{\Omega}^{*}\mathcal{F}_{T\to\Omega}\left(\mathcal{F}_{T\to\Omega}^{*}\mathcal{F}_{T\to\Omega}\right)^{-1}\iota^{*}\mathrm{sgn}(f).$$

Tame the dual polynomial

$\mathcal{F}^*_{T\to\Omega}$ ← transpose + konjugate (adjoint)

Recap that we want to set:

$$\mathcal{P} = \mathcal{F}^*_{\Omega}\mathcal{F}_{T\to\Omega}(\mathcal{F}^*_{T\to\Omega}\mathcal{F}_{T\to\Omega})^{-1} i^* \operatorname{sgn}(f)$$

To under-stand this better let's look at $\mathcal{F}^*_{T\to\Omega}\mathcal{F}_{T\to\Omega}$

$$\frac{1}{|\Omega|}\mathcal{F}^*_{T\to\Omega}\mathcal{F}_{T\to\Omega} f \quad (t) = \frac{1}{|\Omega|}\sum_{\omega\in\Omega} e^{2\pi i t\omega/N}\left(\mathcal{F}_{T\to\Omega} f(\omega)\right)$$

$\mathcal{F}_{T\to\Omega} f = \hat{f}|_{\Omega}$

$$= \frac{1}{|\Omega|}\sum_{\omega\in\Omega} e^{2\pi i t\omega/N}\sum_{t'\in T} e^{-2\pi i t'\omega/N} f(t')$$

$$= \frac{1}{|\Omega|}\sum_{\omega\in\Omega}\sum_{t'\in T} e^{2\pi i (t-t')\omega/N} f(t')$$

$$= \underbrace{f(t)}_{\text{constant diagonal}} + \underbrace{\frac{1}{|\Omega|}\sum_{\omega\in\Omega}\sum_{\substack{t'\in T\\ t\neq t'}} e^{2\pi i (t-t')\omega/N} f(t')}_{\text{wild fourier noice}}$$

$$= \left(I_T - \frac{1}{|\Omega|}H_0\right) f(t)$$

$$\frac{1}{|\Omega|}\mathcal{F}^*_{\Omega}\mathcal{F}_{T\to\Omega} f \ (t) = \frac{1}{|\Omega|}\sum_{\omega\in\Omega} e^{2\pi i t\omega/N}\left(\mathcal{F}_{T\to\Omega} f(\omega)\right)$$

$$\vdots$$

$$= f(t) + \frac{1}{|\Omega|}\sum_{\omega\in\Omega}\sum_{\substack{t'\in T\\ t'\neq t}} e^{2\pi i (t-t')\omega/N} f(t')$$

$$= \left(i - \frac{1}{|\Omega|}H\right) f(t)$$

Rather do 2^{nd} one first and then use $i^* H = H_0$, $i^* i = I_T$ to get 1^{st}

Tame The Dual Polynomial

We want to find a more tangible expression for

$$P = \mathcal{F}_{\Omega}^{*}\mathcal{F}_{T\to\Omega}\left(\mathcal{F}_{T\to\Omega}^{*}\mathcal{F}_{T\to\Omega}\right)^{-1}\iota^{*}\mathrm{sgn}(f).$$

Use matrix notation to obtain

$$P = \left(\iota - \frac{1}{|\Omega|}H\right)\left(I_T - \frac{1}{|\Omega|}H_0\right)^{-1}\iota^{*}\mathrm{sgn}(f),$$

where

$$H(f)(t) := -\sum_{\omega\in\Omega}\sum_{t'\in T:t'\neq t} e^{2\pi i(t-t')\omega/N} f(t') \quad \text{and} \quad H_0 = \iota^{*}H.$$

Proof of Theorem 2.2

- Neumann Series (need $\|H_0\|_{op} \leqslant |\Omega|$)
- Change of Norm (op $\to$ Forb)
- High Moments ($\mathrm{Tr}(H_0^{2n})$)
- Markov
- Key Estimate

$$\mathbf{E}(\mathrm{Tr}(H_0^{2n})) \leqslant 2\left(\frac{4}{e(1-\tau)}\right)^n n^{n+1}|\tau N|^n |T|^{n+1}$$

- Concentration

This new form of P enables us to proof theorem 2.2

Proof of 2.2

- $\frac{1}{|\Omega|} \mathcal{F}^*_{T \to \Omega} \mathcal{F}_{T \to \Omega} = \left(I_T - \frac{1}{|\Omega|} H_0 \right)$. When is this invertible?

$\Rightarrow$ Neumann Series: If $\left\| \sum_{k=0}^{\infty} T^k \right\|_{op} < \infty$ then $(I-T)$ is invertible with $(I-T)^{-1} = \sum_{k=0}^{\infty} T^k$

That means we have to show $\|H_0\|_{op} < |\Omega|$ with high probability.

- Change of norms:

$= \inf \{ c \in \mathbb{R} : \|Hf\| \leq c\|f\| \}$

$$\|H_0\|_{op} = \sup_{f \neq 0} \frac{\|H_0 f\|_{\ell^2(T)}}{\|f\|_{\ell^2(T)}} = \|H_0\|_{\ell^2} \quad \text{(induced norm)}$$

$$= \sqrt{\lambda_{max}(H_0 H_0^*)}$$

$$\|H_0\|_{op}^2 \leq \|H_0\|_F^2 = \mathrm{Tr}(H_0 H_0^*)$$

Before: $\|H_0\|_{op}^2 \leq \|H_0\|_F^2 = \mathrm{Tr}(H_0 H_0^*) = \sum_{t_1, t_2} \underbrace{|(H_0)_{t_1, t_2}|^2}_{\sim |\Omega|}$

Hence invertibility if $|T| \sim \sqrt{|\Omega|}$

- Now: look at high powers to use cancellation

$$\|H_0\|^{2n} = \|H_0^n\|^2 \leq \|H_0^n\|_F^2 = \mathrm{Tr}(H_0^{2n})$$

• Markov Inequality: For the Forbenius norm

$$\mathbb{P}\left(\|H_0^n\|_F \geq \frac{|\tau N|^n}{\sqrt{2}^n}\right) \leq \frac{\mathbb{E}\|H_0^n\|_F^2}{\left(\frac{|\tau N|}{\sqrt{2}}\right)^{2n}}$$

• Key Estimate: If $\tau \leq (1+e)^{-1}$ and $n \leq \frac{\tau N}{4(1-\tau)|T|}$

$$\boxed{\mathbb{E}(\mathrm{Tr}(H_0^{2n})) \leq 2\left(\frac{4}{e(1-\tau)}\right)^n n^{n+1}|\tau N|^n |T|^{n+1}}$$

Together with previous step this gives us

$$\mathbb{P}\left(\|H_0^n\|_F \geq \frac{|\tau N|^n}{\sqrt{2}^n}\right) \leq 2\left(\frac{4n\cdot 2}{(1-\tau)}\right)^n \left(\frac{|T|}{|\tau N|}\right)^n |T| e^{-n} n$$

C_M $$= 2\left(\frac{8n}{(1-\tau)}\right)^n \left(\frac{|T|}{\tau N}\right)^n |T| e^{-n} n$$

If $|T| \leq \frac{(1-\tau)}{8}\frac{|\tau N|}{n}$ and $n = (M+1)\log N$ (corresponds to our assumptions in the theorem)

$$\mathbb{P}\left(\|H_0^n\|_F \geq \frac{|\tau N|^n}{\sqrt{2}^n}\right) \leq 2e^{-n}|T|n \leq \frac{1}{4}e^{-n}|\tau N|$$

A_M' $$= \frac{1}{4}N^{-(M+1)}|\tau N| \leq \frac{1}{4}N^{-M}$$

• Concentration: large deviations

$$\mathbb{P}(|\Omega| < \mathbb{E}|\Omega| - t) \leq \exp\left(-t^2/2\mathbb{E}|\Omega|\right)$$

$$\mathbb{P}\left(|\Omega| < |\tau N| - \sqrt{\frac{2M\log N}{|\tau N|}}\,|\tau N|\right) \leq \exp(-\cancel{2}M\log N) = N^{-M}$$

B_M $\{\|H_0\| \geq \frac{|\tau N|}{\sqrt{2}}\} \Rightarrow \{\frac{|\tau N|^n}{\sqrt{2}^n} \leq \|H_0\|^M \leq \|H_0^n\|_F\}$

$A_M = \{\|H_0^n\|_F \geq \frac{|\tau N|}{\sqrt{2}}\}$ $\Rightarrow$ on $(A_M \cup B_M)^c$

$$\|H_0\| < \frac{\tau N}{\sqrt{2}} \leq \frac{|\Omega|}{\sqrt{2}\left(1-\sqrt{\frac{2M\log N}{|\tau N|}}\right)}.$$

$$\mathbb{P}(A_M \cup B_M)^c = 1-\mathbb{P}(A_M \cup B_M) \geq 1-(\mathbb{P}(A_M)+\mathbb{P}(B_M))$$

The Key Estimate

Theorem 3.3 (Candès, Romberg, Tao - 2004)

Let $\tau \leqslant 1/(1+e)$ and $n_0 = \frac{\tau N}{4(1-\tau)|T|}$. With the Bernoulli model, if $n \leqslant n_0$, then

$$\mathbf{E}(\mathrm{Tr}(H_0^{2n})) \leqslant 2\left(\frac{4}{e(1-\tau)}\right)^n n^{n+1}|\tau N|^n|T|^{n+1}$$

and if $n > n_0$,

$$\mathbf{E}(\mathrm{Tr}(H_0^{2n})) \leqslant 2\left(\frac{4}{e(1-\tau)}\right)^n n^{n+1}|\tau N|^n|T|^{n+1}.$$

Sketch of Proof of the Key Estimate

- find formula for $H_0^{2n}(t_1, t_1)$
- take expectation
- equivalence relation
- Inclusion-Exclusion Formulae
- Stirling numbers

First formula for $\mathbf{E}(\mathrm{Tr}(H_0^{2n}))$

$$H_0(t,t') = \begin{cases} 0, & t = t' \\ c(t-t'), & t \neq t' \end{cases}, \text{ where } \quad c(u) = \sum_{\omega \in \Omega} e^{2\pi i u\omega/N}$$

$$H_0^{2n}(t_1,t_1) = \sum_{t_2,\dots,t_{2n}:t:j\neq t_{j+1}} c(t_1-t_2)\cdots c(t_{2n}-t_1)$$

$$\mathbf{E}(\mathrm{Tr}(H_0^{2n})) = \sum_{t_1,\dots,t_{2n}:t:j\neq t_{j+1}} \mathbf{E}\left[\sum_{\omega_1,\dots,\omega_{2n}\in\Omega} e^{2\pi i/N \sum_{j=1}^{2n}\omega_j(t_j-t_{j+1})}\right]$$

$$= \sum_{t_1,\dots,t_{2n}:t:j\neq t_{j+1}} \sum_{0\leqslant\omega_1,\dots,\omega_{2n}\leqslant N-1} e^{2\pi i/N \sum_{j=1}^{2n}\omega_j(t_j-t_{j+1})}\mathbf{E}\left[\prod_{j=1}^{2n} I_{\{\omega_j\in\Omega\}}\right]$$

Decomposition by Equivalence Relations

$\mathbb{Z}_N = \{0, \ldots, N-1\}$ set of all frequencies
$A := \{1, \ldots, 2n\}$ finite set
For all $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_{2n})$ we define the equivalence relation $\sim_{\boldsymbol{\omega}}$ on A by saying

$$j \sim_{\boldsymbol{\omega}} j' \Leftrightarrow \omega_j = \omega_{j'}$$

Then define $\mathcal{P}(A)$ as the set of all equivalence relations on A and note that this comes with a natural ordering

$$\sim_1 \leqslant \sim_2 \Leftrightarrow a \sim_2 b \text{ implies } a \sim_1 b \text{ for all } a, b \in A$$

Finally, define the two sets

$$\Omega(\sim) := \{\boldsymbol{\omega} \in \mathbb{Z}_N^{2n} : \sim_{\boldsymbol{\omega}} = \sim\}$$
$$\Omega_{\leqslant}(\sim) := \bigcup_{\sim' \in \mathcal{P} : \sim' \leqslant \sim} \Omega(\sim')$$

$$\mathbf{E}(\mathrm{Tr}(H_0^{2n})) = \sum_{t_1,\ldots,t_{2n}: t{:}j \neq t_{j+1}} \sum_{0 \leqslant \omega_1,\ldots,\omega_{2n} \leqslant N-1} e^{2\pi i/N \sum_{j=1}^{2n} \omega_j(t_j - t_{j+1})} \mathbf{E}\left[\prod_{j=1}^{2n} I_{\{\omega_j \in \Omega\}}\right]$$

$$= \sum_{t_1,\ldots,t_{2n}: t{:}j \neq t_{j+1}} \sum_{\sim \in \mathcal{P}(A)} \tau^{|A/\sim|} \sum_{\boldsymbol{\omega} \in \Omega(\sim)} e^{2\pi i/N \sum_{j=1}^{2n} \omega_j(t_j - t_{j+1})}$$

Stirling numbers: $S(n,k) := \#\{\sim \in \mathcal{P}(A) : |A/\sim| = k\}$.
Inclusion exclusion formula:

$$\sum_{\boldsymbol{\omega} \in \Omega(\sim)} f(\boldsymbol{\omega}) = \sum_{\sim_1 \in \mathcal{P}: \sim_1 \leqslant \sim} (-1)^{|A/\sim| - |A/\sim_1|}$$

$$\times \left(\prod_{A' \in A/\sim_1} (|A'/\sim| - 1)! \right) \sum_{\boldsymbol{\omega} \in \Omega_{\leqslant}(\sim_1)} f(\boldsymbol{\omega})$$

Cool Pictures

Why compressed sensing is so important

Magnetic Resonance Imaging

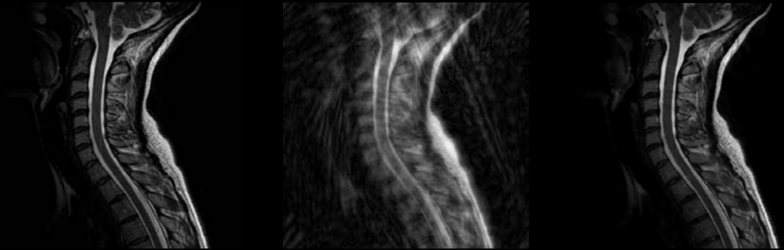
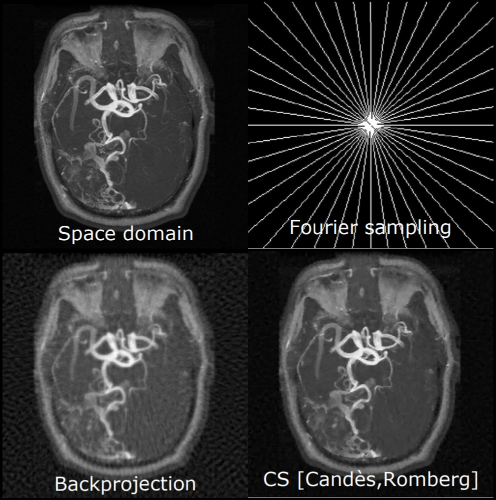


Figure 3: fully sampled, 6x undersampled classical, 6x undersampled CS

Trzasko, Manduca, Borisch (Mayo Clinic)

Magnetic Resonance Imaging



Magnetic Resonance Angiography

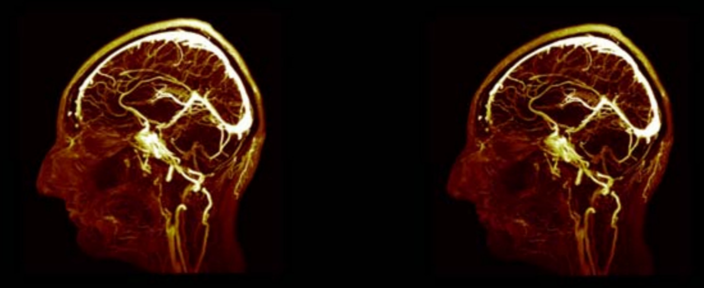


Figure 4: fully sampled, 6x undersampled CS

Trzasko, Manduca, Borisch (Mayo Clinic)

Single Pixel Camera

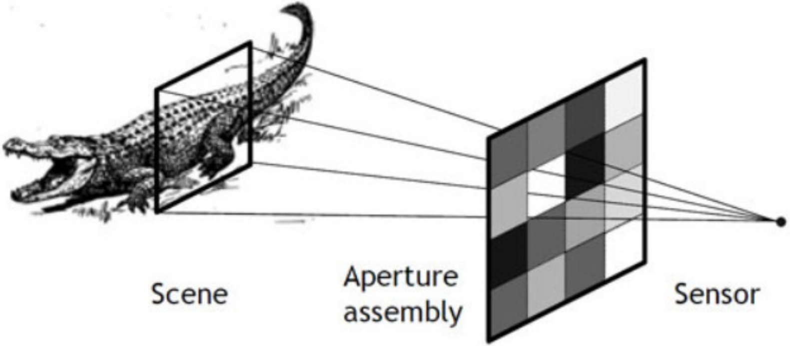


Figure 5: Idea of Single Pixel Camera

Gang Huang, Hong Jiang, Kim Matthews, Paul Wilford

Single Pixel Camera

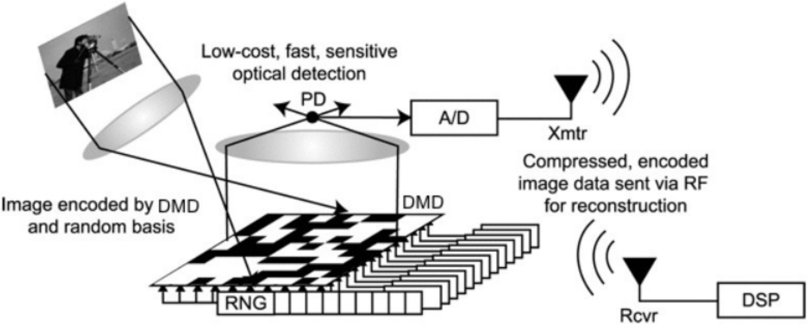


Figure 6: More detailed Idea

Duke University

Single Pixel Camera

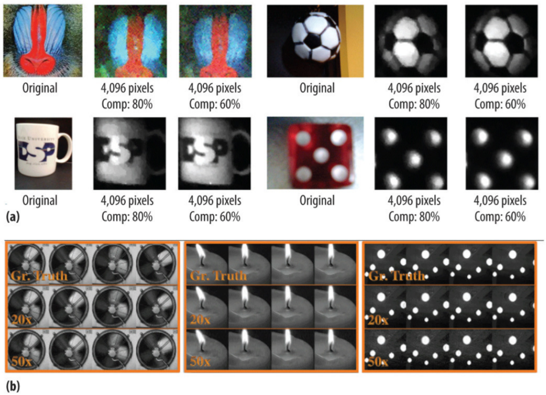


Figure 7: Some Pictures taken with a SPC

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Thank you.