Proposal for Independent Research Project

Description of Fibers along Geodesics in $\mathcal{P}_2(E)$ under the Canonical Projections from $\mathcal{P}(D([0,T],E))$

Abstract: We will consider projections from the space of laws of stochastic processes with *càdlàg* paths to laws of individual random variables and try to understand which stochastic processes induce geodesics with their laws. More general we want to understand properties of the projection operator between those spaces.

Intruction of Notation

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space and let further (E, d_E) be a polish space with associated borel sigma field $\mathcal{E}$. We consider the space of E valued, $(\mathcal{F}_t)$-adapted stochastic processes on $[0,T]$ that are right-continuous and obtain left limits (french *càdlàg*). Equip the space $D([0,T],E)$ of càdlàg paths with the Skohorod J1-topology induced by the Skorokhod J1 metric d_{J1} defined by

$$d_{J1}(x,y) := \inf\left\{ \varepsilon > 0 \middle| \exists\lambda \in \Lambda \text{ s.t. } \sup_{t\in[0,T]} |\lambda(t) - t| \leq \varepsilon \text{ and } \sup_{t\in[0,T]} |x_t - y_{\lambda(t)}| \leq \varepsilon \right\}, \quad (1)$$

where $\Lambda = \{\lambda : [0,T] \to [0,T] |$ strictely increasing, cts. and with $\lambda(0) = 0, \lambda(T) = T\}$. We can see $\lambda \in \Lambda$ as time scaling that will effect that two paths with jumps close to each other will be close in the Skorokhod metric. The space $(D([0,T],E), d_{J1})$ is a separable metric space. It is not complete but we can construct an equivalent metric $\hat{d}_{J1}$ such that $(D([0,T],E), \hat{d}_{J1})$ is complete. Let $\mathcal{D}$ be it's borel sigma field. More to the Skorokhod space can be found in the excellent book [Bil68].

A stochastic process with càdlàg paths can be interpreted as a random variable taking values in the Skorokhod space, i.e. a measurable mapping

$$\begin{aligned} X: \quad (\Omega, \mathcal{F}) &\longrightarrow (D([0,T],E), \mathcal{D}) \\ \omega &\longmapsto (x_t)_{t\in[0,T]}. \end{aligned}$$

It's law $\mathbb{P}_X \in \mathcal{P}(D(E))$ is defined by $\mathbb{P}_X := \mathbb{P}(X^{-1}) = X_\sharp \mathbb{P}$. Let $\mathcal{P}(D([0,T],E))$ and $\mathcal{P}(E)$ be the spaces of probability measures on $(D([0,T],E), \mathcal{D})$ and $(E, \mathcal{E})$ respectively. We equip those spaces with the Prokhorov metric d_P that measures the distance between two probabilities $\mu, \nu \in \mathcal{P}(D([0,T],E))$ in the following way:

$$d_P^{\mathcal{P}(D([0,T],E))}(\mu,\nu) := \inf\left\{ \varepsilon > 0 \middle| \mu(A) \leq \nu(A^\varepsilon) \text{ and } \nu(A) \leq \mu(A^\varepsilon), \quad \text{for all } A \in \mathcal{D} \right\}$$

where $A^\varepsilon := \{x \in D([0,T],E) | d_{J1}(x,A) \leq \varepsilon\}$. And analogously for $\mathcal{P}(E)$ with d_E instead of d_{J1}. The Prokhorov metric is well described in [EK05]. It is natural to consider the projections of the path of a stochastic process to the coordinates and the induced projection of the law from the process to individual random variables.

$$\begin{aligned} \Pi_t: \quad D(E) &\longrightarrow E \\ x &\longrightarrow x_t \end{aligned} \qquad \text{and} \qquad \begin{aligned} \hat{\Pi}_t : \mathcal{P}(D(E)) &\longrightarrow \mathcal{P}(E) \\ \mu &\longrightarrow \mu(\Pi_t^{-1}) =: \mu_t. \end{aligned}$$

It is $\Pi_t X = X_t$ with law $\mathbb{P}_{X_t} = \mathbb{P}(X_t^{-1})$. Now we can introduce our problem.

The Problem

For simplification let $E = \mathbb{R}$ and $[0, T] = [0, 1]$. Clearly, a stochastic process induces a mapping from $[0, T]$ to $\mathcal{P}(\mathbb{R})$ by $t \mapsto \mathbb{P}_{X_t}$. It is a priori not even clear, that $t \mapsto \mathbb{P}_{X_t}$ is a continuous path. We want to go even further and ask the question, for which processes this mapping is a geodesic. Geodesics in the space of probability measures can be described in a beautiful way. The procedure is explained by Robert J. McCann in [McC97] with tools of optimal transportation theory.

Theorem 1 (McCahn: displacement interpolation). *Let $\mu^{\pm} \in \mathcal{P}_2(\mathbb{R}^n), \gamma_0 \in \Gamma_{op}(\mu^+, \mu^-)$ and define for $s \in [0, 1]$:*

$$\tilde{\Pi}_s(x, y) := (1 - s)x + sy \tag{2}$$

$$\mu_s := \left(\tilde{\Pi}_s\right)_{\sharp} \gamma_0. \qquad \textit{(displacement interpolation)} \tag{3}$$

Then $s \mapsto \mu_s$ is a $d_{W,2}$-minimizing geodesic from μ^+ to μ^- and $\gamma_s := \left(\tilde{\Pi}_0 \times \tilde{\Pi}_s\right)_{\sharp} \gamma_0 \in \Gamma_{op}(\mu^+, \mu^-)$. If $\mu^+ \in \mathcal{P}_{2,ac}(\mathbb{R}^n)$, then $\mu_s \ll dx$ for all $0 \le s \le 1$, i.p. $\mathcal{P}_{2,ac}(\mathbb{R}^n)$ is geodesically convex.

More general we are asking about the properties of $\hat{\Pi}_t$ since they encode the pre-image of a geodesic in the following way:

$$\left\{ \mu \in \mathcal{P}\left(D([0,1], \mathbb{R})\right) \,\middle|\, \begin{array}{c} \mu \text{ is law of a processes X} \\ \text{inducing a geodesic} \\ \text{between } \mathbb{P} \circ X_0^{-1} \text{ and } \mathbb{P} \circ X_1^{-1} \end{array} \right\} = \bigcap_{t \in [0,1]} \hat{\Pi}_t^{-1}\{\rho_t\} \tag{4}$$

Where ρ_t is given by the displacement interpolation between initial and terminal law. Define the t-fiber of $\mu \in \mathcal{P}(E)$ as $\hat{\Pi}_t^{-1}(\mu)$. Can we describe those fibers and their intersections more explicitly?

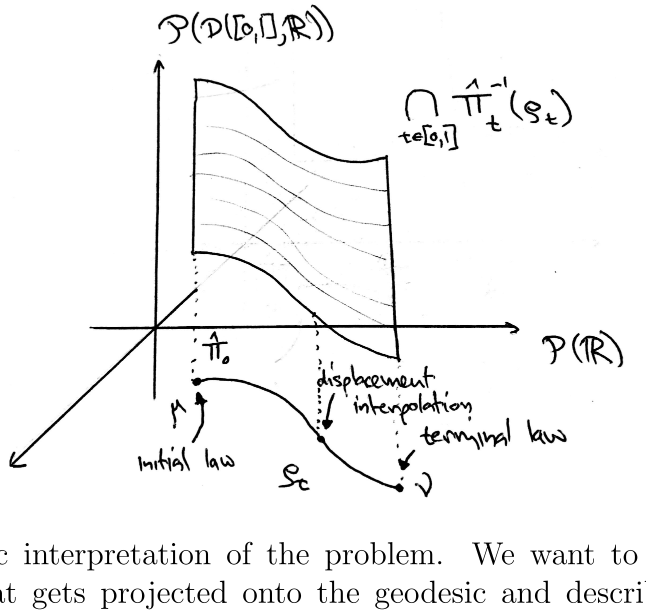

Figure 1: Geometric interpretation of the problem. We want to identify the surface in $\mathcal{P}(D[0, 1], \mathbb{R}))$ that gets projected onto the geodesic and describe properties of the projection maps.

The Conjectures

Conjecture 1. *The maps $\hat{\Pi}_t$ are contractions, i.e. for some $c < 1$ it holds*

$$d_P^{\mathcal{P}(\mathbb{R})}(\hat{\Pi}_t\mu, \hat{\Pi}_t\nu) \leq c d_P^{\mathcal{P}(D([0,1],\mathbb{R})}(\mu, \nu) \quad \forall \mu, \nu \in \mathcal{P}(D([0,1], \mathbb{R}^n)) \quad \forall t \in [0,1] \tag{5}$$

Ideas for Proof. It seems to be very hard to get bounds for the Prokhorov metric. One can show that for a separable metric space the Prokhorov metric is equivalent to weak convergence of measure. And indeed $\mathbb{R}$ is separable metric space. Now, note that we characterized Wasserstein distance by weak convergence and an additional requirement. I could imagine that for a proof we could use that weak convergence in the big space $\mathcal{P}(D([0,1], \mathbb{R}^n))$ implies weak convergence of the marginals in $\mathcal{P}(\mathbb{R})$! Maybe there is a way to prove our conjecture like this. During my research on this, I found the paper [GS02] which relates many different probability metrics by upper and lower bounds. It looks very interesting and might be helpful for a rigorous proof!
Interesting related questions: Knowing weather the conjecture is true or false would be really helpful. We would obtain Lipschitz continuity and hence unifom continuity of the projections. If the result with the inverse inequality was true, we could restrict our search for the pre-image of a geodesic on balls in $\mathcal{P}(D([0,1], \mathbb{R}))$ to closed balls of radius 1 around $\rho_0 = \mu$.

□

Conjecture 2. *Let $g : [0,1] \to \mathcal{P}(\mathbb{R}^n)$ be defined by $g(t) = \rho_t$ where ρ_t is obtained by displacement interpolation between μ, ν. Then there exists a map $G : [0,1] \to \mathcal{P}(D([0,1], \mathbb{R}^n))$ such that the following diagram commutes.*

$$\begin{tikzcd} & \mathcal{P}(D([0,1],\mathbb{R}^n)) \arrow[d, "\hat{\Pi}_t"] \\ {[0,1]} \arrow[ur, dashed, "G"] \arrow[r, "g"'] & \mathcal{P}(\mathbb{R}^n) \end{tikzcd}$$

If we require G to be continuous, then the mapping is unique.

Ideas for Proof. We recall the well known result for product typologies: Let X_i be topological spaces, I be an index set, $X = \prod_{i \in I} X_i$ their product space and Y another topological space. If for every $i \in I$ there are continuous maps $f_i : Y \to X_i$ then there exists precisely one continuous map $f : Y \to X$ such that the following diagram commutes.

$$\begin{tikzcd} & X \arrow[d, "p_i"] \\ Y \arrow[ur, dashed, "f"] \arrow[r, "f_i"'] & X_i \end{tikzcd}$$

If we show that $\mathcal{P}(D([0,1], \mathbb{R}^n))$ was the product space of $\mathcal{P}(\mathbb{R}^n)$, then this would finish the proof. Unfortunately, this is not the case. $\mathcal{P}(D([0,1], \mathbb{R}^n))$ must be a smaller space then the product space. However, I think one can fix this but I don't know how.
Interesting related questions: It would be very interesting if a continuous G would indeed exist. Would it be a geodesic from $G(0)$ to $G(1)$ in $\mathcal{P}(D([0,1], \mathbb{R}^n))$? Are there even (unique) geodesics in this bigger space? Can we use optimal transportation to describe the geometry of this huge space? If so, can we extend our theory from the lecture cannonically to even more general spaces like $\mathcal{P}(E)$ where E is polish?

□

References

[Bil68] Patrick Billingsley. *Convergence of Probability Measures.* John Wiley and Sons, Inc., 1968.

[EK05] Stewart N. Ethier and Thomas G. Kurtz. *Markov Processes: Characterization and Convergence.* John Wiley and Sons, Inc., 2005.

[GS02] Alison L. Gibbs and Francis Edward Su. On choosing and bounding probability metrics. *International Statistical Review*, 70(3):419–435, December 2002.

[McC97] Robert J. McCann. A convexity principle for interacting gases. *Advances in Mathematics*, 128(AI971634):153–179, 1997.