Summary of Optimal Transportation Theory - Part II

Applications of Optimal Transportation Theory

Using the powerful tools of optimal transportation theory, we proved the following inequalities

Theorem 1 (Results that can be obtained using optimal transportation). *The following holds*

i) (Isopermetric ineq.) Let $\Omega \subset \mathbb{R}^n$ be open and bounded with $\partial\Omega \in C^\infty$ and $|\Omega| = |B_1(0)|$. Then $|\partial\Omega| \geq |\partial B_1(0)|$ with equality iff Ω is isometrically isomorph to $B_1(0)$.

ii) (Gaussian log-Sobolev ineq.) For all $f : \mathbb{R}^n \to \mathbb{R}_+$ we have $Ent_{\gamma_n}(f) \leq \frac{1}{2} I_{\gamma_n}(f)$ with equality iff $f(x)d\gamma_n(x) = d\gamma_n(x + \vec{a})$.

iii) (Brunn-Minkowski ineq.) Let $X, Y \subset \mathbb{R}^n$ be compact. Then $|X + Y|^{\frac{1}{n}} \geq |X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}}$ with equality iff X is a translation of Y (case when both, X and Y have positive volume).

Displacement Interpolation in the Space of Probability Measures

Let $M^\pm = \mathbb{R}^n$ and $\mathcal{P}_{p,\text{ac}}(\mathbb{R}^n)$ the space of probability measures with finite p^{th} moment and which are absolutely continuous with respect to the Lebesgue-measure. We equip this space with the so called p-Wasserstein-distance defined by

$$d_{W,p}(\mu^+, \mu^-) := \left(\min_{\gamma \in \Gamma(\mu^+,\mu^-)} \int_{M^+} \int_{M^-} |x - y|^p \gamma(dx, dy) \right)^{\frac{1}{p}}.$$

The space $(\mathcal{P}_{p,\text{ac}}(\mathbb{R}^n), d_{W,p})$ is a metric space. We restricted us to the case where $p = 2$. The following theorem summarizes our results on the description of geodesics in this space.

Theorem 2 (McCahn: displacement interpolation). *Let $\mu^\pm \in \mathcal{P}_2(\mathbb{R}^n), \gamma_0 \in \Gamma_{op}(\mu^+, \mu^-)$ and define for $s \in [0, 1]$:*

$$\begin{aligned} \Pi_s(x, y) &:= (1 - s)x + sy \\ \mu_s &:= (\Pi_s)_\sharp \gamma_0. \qquad \textit{(displacement interpolation)} \end{aligned}$$

Then $s \mapsto \mu_s$ is a $d_{W,2}$-minimizing geodesic from μ^+ to μ^- and $\gamma_s := (\Pi_0 \times \Pi_s)_\sharp \gamma_0 \in \Gamma_{op}(\mu^+, \mu^-)$. If $\mu^+ \in \mathcal{P}_{2,ac}(\mathbb{R}^n)$, then $\mu_s \ll dx$ for all $0 \leq s \leq 1$, i.p. $\mathcal{P}_{2,ac}(\mathbb{R}^n)$ is geodesically convex.

Considering displacement interpolations can be used to define a meaningful interpretation of convexity, namely displacement convexity. This is nothing but geodesic convexity in $\mathcal{P}_{2,\text{ac}}(\mathbb{R}^n)$.

Theorem 3 (displacement convexity for functionals of mass distribution). *For $\rho \in \mathcal{P}_{2,ac}(\mathbb{R}^n)$ let $\rho(x)$ denote it's density with respect to the Lebesgue-measure. For $V, W : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ and $U : \mathbb{R}_{\geq 0} \to \mathbb{R}$ define the functionals of self-interacting, potential and internal energy as*

$$\mathcal{W}(\rho) := \frac{1}{2} \int \int W(x - y) d\rho(x) d\rho(y), \quad \mathcal{V}(\rho) := \int V(x) d\rho(x), \quad \mathcal{U}(\rho) := \int U(\rho(x)) dx.$$

Then the following holds

i) If W is convex (resp. strictly, resp. λ-uniformly) then $\mathcal{W}$ is displacement convex (resp. strictly, resp. λ-uniformly).

ii) If V is convex (resp. strictly, resp. λ-uniformly) then $\mathcal{V}$ is displacement convex (resp. strictly, resp. λ-uniformly).

iii) If $U(0) = 0$ and $\lambda \to U(\frac{r}{\lambda^n})\lambda^n$ is non-increasing and convex for $r > 0$, then $\mathcal{U}$ is displacement convex.

Differential Geometry on $\mathcal{P}_{2,\mathbf{ac}}(\mathbb{R}^n)$

In $\mathbb{R}^n$ the distance between two points is a so called *length distance* given by the formula

$$d(x,y)^2 = |x-y|^2 = \inf_{\gamma}\left\{\int_0^1 |\dot{\gamma}|^2 dt \quad \middle| \gamma(0)=x, \gamma(1)=y\right\}.$$

We want to show that the Wasserstein distance underlies a similar structure, i.e. $d_W^2(\rho_0,\rho_1) = \inf_{\text{curves } \rho_t}\left\{\int_0^1 \|\frac{d\rho_t}{dt}\|^2_{\rho_t} dt\right\}$. Therefore, we have to make find an suitable metric structure. An important result is a different understanding of Wasserstein distance which is given in the following theorem by Bernamon and Brenier

$$\inf_{\gamma\in\Gamma(\mu,\nu)}\left\{\int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^2 d\gamma\right\} =: \boxed{d_W(\mu,\nu) = \tilde{d}_W(\mu,\nu)} := \inf_{\substack{(\rho,V)\ \text{admissible} \\ \rho_0=\mu,\,\rho_1=\nu}} \left\{\int_0^1\int_{\mathbb{R}^n} |V_t|^2 d\rho_t dt\right\}.$$

where $(\rho,V) = (\rho_t,V_t)_{0\le t\le 1}$, ρ_t denotes a curve and V_t it's velocity and admissible means that (ρ,V) satisfies a set of *nice* properties, mainly a weak form of the continuity equation $\frac{d\rho_t}{dt} + \operatorname{div}(\rho_t\vec{V_t}) = 0$. We want to define $\|\frac{d\rho_t}{dt}\|^2 := \int_{\mathbb{R}^n} V d\rho$ but this definition is ambiguous since the V in above equation is not unique which can be seen by adding divergence-free terms.

Theorem 4. *Let $V \in L^2_\rho(\mathbb{R}^n\times\mathbb{R}^n)$. Then there exists the unique decomposition $V = W + \nabla u$ where $W \in L^2_\rho$ is divergence-free and $u \in W^{1,2}_\rho$. Furthermore, ∇u minimizes $\int |V|^2 d\rho$ and thus the norm $\|\frac{d\rho_t}{dt}\|^2_\rho := \int |\nabla u|^2 d\rho$ is well defined.*