Summary of Optimal Transportation Theory

For $M^{\pm} \subseteq \mathbb{R}^n$, $\mu^{\pm} \in \mathcal{P}(M^{\pm})$ and a given cost function $c : M^+ \times M^- \to \mathbb{R}$ define the set of transport plans from M^+ to M^- as $\Gamma(M^+, M^-) := \{\gamma \in \mathcal{P}(M^+ \times M^-) | \prod_{\sharp}^{\pm} \gamma = \mu^{\pm}\}$, where $\prod^{\pm}$ denotes the projections operator on $M^{\pm}$ respectively. We want to consider the so called Monge-Kantorovich problem:

$$\min_{\gamma \in \Gamma(M^+, M^-)} \int_{M^+} \int_{M^-} c(x, y)\gamma(dx, dy). \qquad (\text{MKP})^c_{\mu^{\pm}}$$

This problem can be interpreted as finding a transport plan such that the cost of mass transport between the measures $\mu^{\pm}$ is minimal. In the first part of the lectures we discussed questions of uniqueness and existence of solutions to $(\text{MKP})^c_{\mu^{\pm}}$.
Using weak$-*$ topology, a first existence theorem can be obtained if we set some assumptions on $M^{\pm}$ and the cost function.

Theorem 1 (Existence of optimal plan)**.** *Let $M^{\pm} \subseteq \mathbb{R}^n$ compact and c be lower semi-continuous. Then there exists $\gamma \in \Gamma_{op}(M^+, M^-)$.*

To analyze uniqueness we introduced some important notation and results of convex analysis.

Theorem 2 (Results of convex analysis on c-cyclical monotonicity and subdifferentials)**.** *Let $\gamma_0 \in \Gamma_{op}(M^+, M^-)$, $c \in C(M^+ \times M^-)$,$\mu^{\pm} \in \mathcal{P}(M^{\pm})$,E be a top. vec. space and $S \subseteq E \times E^*$.*

i) supp(γ_0) is c-cyclical monotone.

ii) $\{(x_i, y_i)\}$ cyclical monotone $\Leftrightarrow \exists$ convex $\varphi : \mathbb{R}^n \to \mathbb{R}$, $y_i \in \partial\varphi(x_i)$.

iii) S is c-cyclical monotone $\Leftrightarrow \exists$ a proper c-convex function φ E, s.t. $S \subseteq \partial^c\varphi$.

iv) It exists c-cyclical monotone $S \subseteq M^+ \times M^-$ s.t. $\forall \gamma \in \Gamma_{op}(M^+, M^-)$: supp$(\gamma) \subseteq S$.

Using this, we were able to prove the following theorem which gives a result on uniqueness to $(\text{MKP})^c_{\mu^{\pm}}$ and a link between optimal maps and optimal plans.

Theorem 3 (Existence and uniqueness of Monge Sol'n)**.** *Let $M^{\pm} \subseteq \mathbb{R}^n$ compact, φ c-convex s.t. supp$(\gamma) \subseteq \partial^c\varphi \quad \forall\gamma \in \Gamma_{op}(M^+, M^-)$ (existence by previous results). Assume $\partial^c\varphi$ is single valued μ^+-almost everywhere. Then*

i) γ_{op} is unique.

ii) $\exists$ optimal map T s.t. $T_\sharp\mu^+ = \mu^-$.

iii) $\gamma_{op} = (Id \times T)_\sharp\mu^+$ and $T(x) = \partial^c\varphi(x) \quad \mu^+-$almost everywhere.

To show the assumption of this theorem it is often useful to apply Rademacher's theorem, which states that Lipschitz functions are almost everywhere differentialble. Alexandrov's theorem justifies the use of even second order differentiability of φ in the following theorem:

Theorem 4 (Brenier - PDE version)**.** *Let $M^{\pm} \subseteq \mathbb{R}^n$ be bounded and open, $\mu^{\pm} \in \mathcal{P}(M^{\pm})$ s.t. $\mu^{\pm} \ll dx$ and define $f^{\pm}(x)dx = d\mu^{\pm}(x)$. Then there exists a unique convex φ (up to an additive constant) on M^+ such that in weak sense*

$$\det(D^2\varphi(x))f^-(\nabla\varphi(x)) = f^+(x).$$

Furthermore, $\nabla\varphi$ is the unique optimal map.