Probability meets PDEs: The Numerics of Stochastic Heat Equation

Tim Jaschek
The University of British Columbia

Abstract

The stochastic heat equation models a heat flow in a material disturbed by a space-time white noise.
To make sense of the stochastic heat equation, the Q-Wiener Process and a generalization of the Itô-Integral with respect to this process will be introduced. Those will lead us to a beautiful relation of weak solutions to the stochastic heat equation in one spacial dimension and the Ornstein-Uhlenbeck Process, a well known Itô-Diffusion process.
Finally, this poster presents numerical solutions to the stochastic heat equation in one and two spacial dimensions using both, finite elements and finite difference method.

The Problem and Definition of Solutions

Let U be an bounded domain in $\mathbb{R}^n$. We will consider a modification of the well known heat equation with homogeneous Dirichlet boundary conditions and given initial condition u_0 and random force term W.

$$\begin{cases} \partial_t u(t) - \Delta u = W(t,x,\omega), & \text{in } U \times (0,T) \\ u(0,x) = u_0(x), & \text{in } U \\ u(t) = 0, & \text{on } (0,T) \times \partial U. \end{cases} \tag{1}$$

Since the force term is a stochastic process, a solution to the stochastic heat equation will be a stochastic process as well.

Notation for Solutions of Stochastic Heat

The lack of regularity of many stochastic processes will require very weak notations of solutions. A predictable H-valued Process $\{u_t : t \in [0,T]\}$ with $u_t \in L^2(\Omega, H)$ is called a *strong solution* of the stochastic heat equation if

$$u_t = u_0 + \int_0^t \Delta u_s ds + \int_0^t \sigma dW_s, \quad \mathbf{P} - a.s., \tag{2}$$

a *weak solution* of the stochastic heat equation if for all $v \in \mathcal{D}(\Delta)$:

$$\langle u_t, v\rangle = \langle u_0, v\rangle + \int_0^t -\langle u_s, \Delta v\rangle ds + \int_0^t \langle \sigma dW_s, v\rangle, \quad \mathbf{P} - a.s., \tag{3}$$

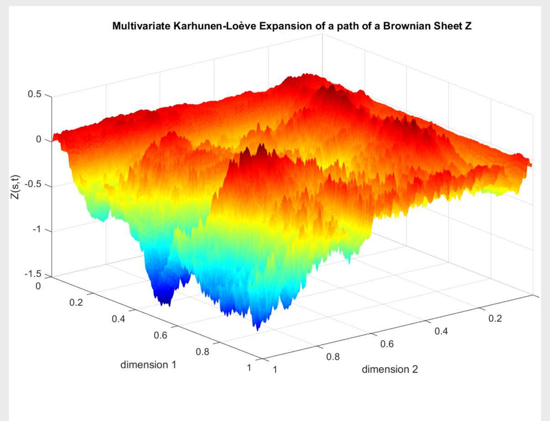
and a *mild solution* of the stochastic heat equation if

$$u_t = e^{-t\Delta} u_0 + \int_0^t e^{-(t-s)\Delta} \sigma dW_s, \quad \mathbf{P} - a.s.. \tag{4}$$

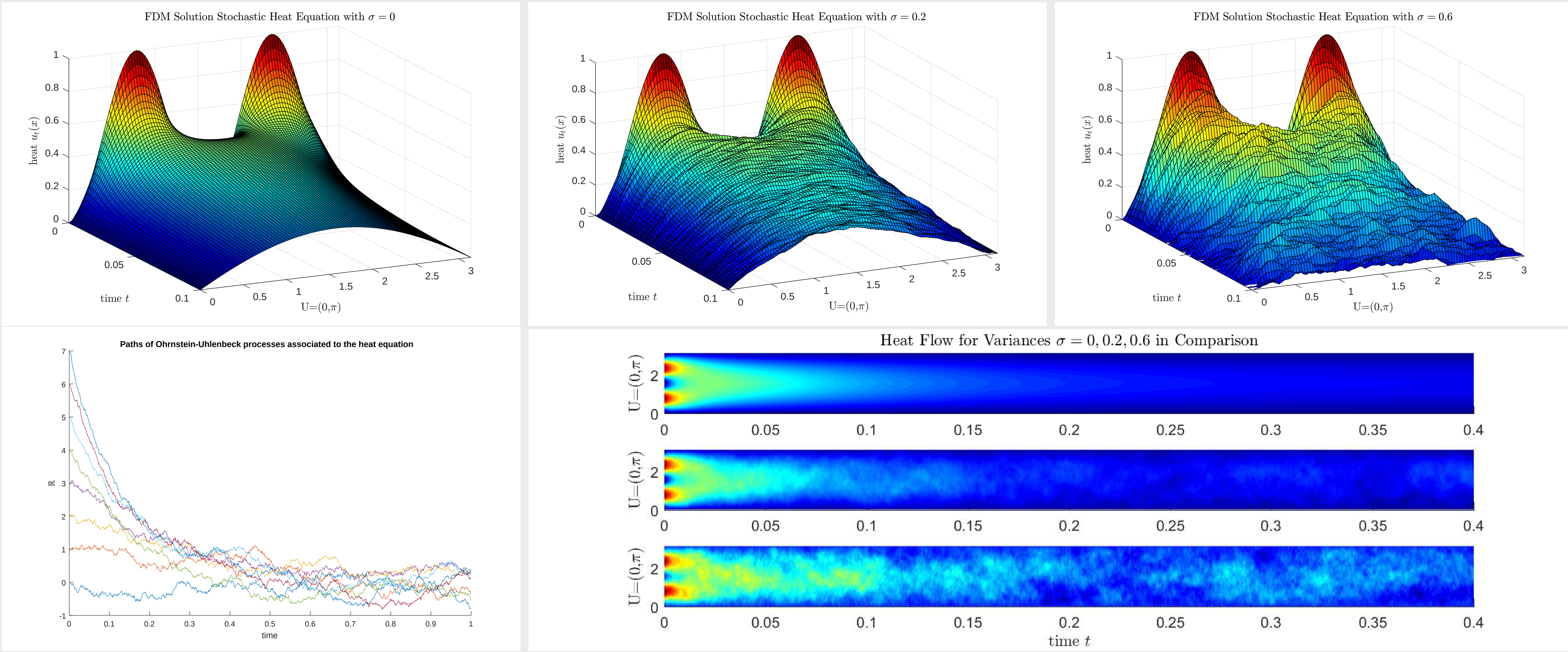
Where $e^{-t\Delta}$ denotes the semigroup generated by Δ.

The Random Force Term

To model the noise we will use a so called Q-Wiener Process. This is an $L^2(D)$-valued stochastic process. This figure shows a simulation of the process in a very illustrative way. Imagine dimension 2 is the time dimension. For each fixed time step you will obtain the graph of an element in $L^2(D)$. In this case, the operator Q is an integraloperator associated to the covariance function of Brownian motion.



Simulations in One Spacial Dimension using Finite Difference Method



Important Definitions and Theorems

Rather than seeing u as a function of time and space we want to see it as a $L^2(D)$-valued stochastic process. The random force term will defined as follows

Assumption on Q Let $Q \in \mathcal{L}(H,H)$ be non-negative definite, symmetric and such that there exists an orthonormal basis $\{\varphi_i : i \in \mathbb{N}\}$ of eigenfunctions with corresponding eigenvalues $\lambda_i \geq 0$ such that $\sum_{i\in\mathbb{N}} \lambda_i < \infty$.

Q-Wiener Process A H-valued stochastic process $\{W_t : t \geq 0\}$ is called Q-Wiener Process if

i) $W_0 = 0$ a.s.,
ii) W_t is a continious function $\mathbb{R}_+ \to H$ for each $\omega \in \Omega$,
iii) W_t is $\mathcal{F}_t$-adapted and $W_t - W_s$ is independent of $\mathcal{F}_s$ for $s \leq t$,
iv) $W_t - W_s \sim \mathcal{N}(0, (t-s)Q)$ for all $0 \leq s \leq t$.

Karhunen-Loève Expansion for Q-Wiener Process Let Q satisfy our basic assumptions. Then W_t is a Q-Wiener process if and only if

$$W_t = \sum_{i=1}^{\infty} \sqrt{\lambda_i} \varphi_i \beta_t^{(i)} \quad a.s. \tag{5}$$

where $\beta^{(i)}$ are i.i.d. $\mathcal{F}_t$-Brownian motions and the series converges in $L^2(\Omega, H)$. Moreover it converges in $L^2(\Omega, \mathcal{C}([0,T], H))$.

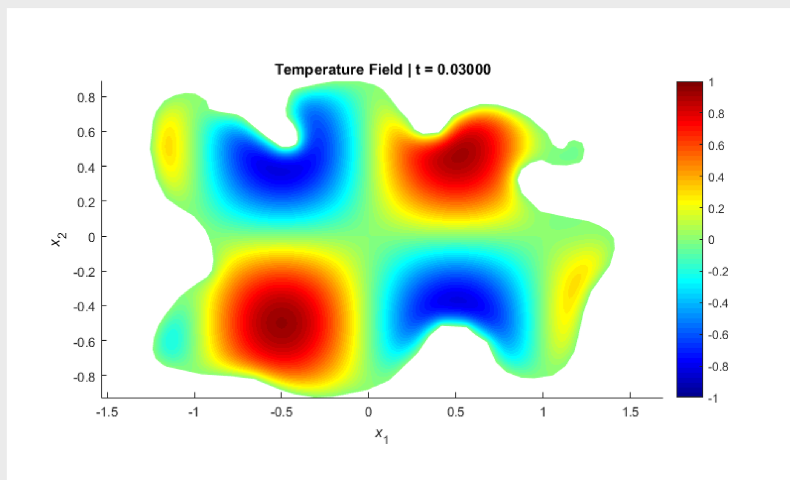
Stochastic Integral with respect to a Q-Wiener Process Using the Karhunen-Loève Expansion we define

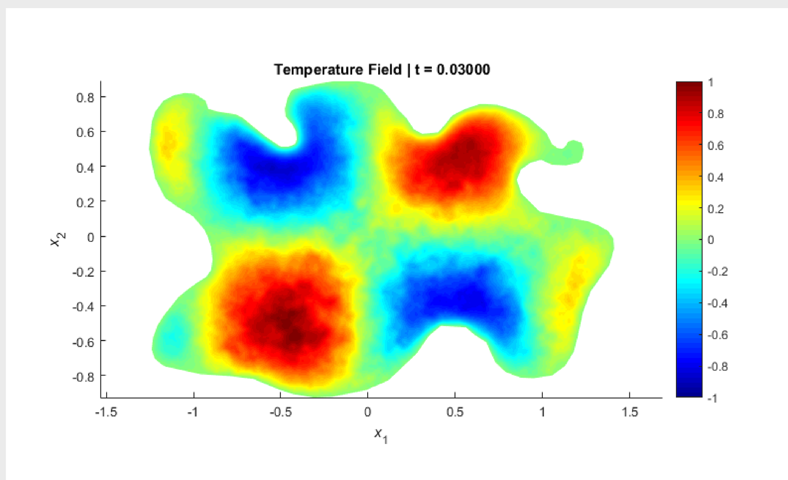
$$\int_0^t X_s dW_s := \sum_{i=1}^{\infty} \sqrt{\lambda_i} \varphi_i \int_0^t X_s d\beta_s^{(i)}, \tag{6}$$

where the integrals with respect to Brownian motion are so called Itô integrals.

Simulations in Two Spacial Dimensions using Finite Element Method

The theoretical section to the stochastic heat equation in 2D shows, that there does not exist a weak solution when the force term is space-time white noise. However, since our numerical method interpolates in between discretization steps we can still obtain an interesting result. Unfortunately, we can not present a time-changing simulation of the numerical solution on a poster. Let us at least have a look on a fixed time step. Whereas the plot on the left is obtained by solving the equation with $\sigma = 0$, the on the right is result when we let the force term be random.





The Stochastic Heat Equation in 1D

We take $U = (0,\pi)$. Then $-\Delta$ has eigenfunctions and eigenvalues

$$\varphi_i(x) = \sqrt{2/\pi} \sin(ix), \qquad \lambda_i = i^2.$$

Let now W be a Q-Wiener process be such that Q has the same eigenfunctions as $-\Delta$ with corresponding eigenvalues ξ_i. Then for $v \in \mathcal{D}(\Delta)$ a weak solution statisfies:

$$\langle u_t, v\rangle_{L^2(U)} = \langle u_0, v\rangle_{L^2(U)} + \int_0^t \langle -u_s, \Delta v\rangle_{L^2(U)} ds + \sum_{i=1}^{\infty} \int_0^t \sigma \sqrt{\xi_i} \langle \varphi_i, v\rangle dB_s^{(i)}.$$

Expand $u_t = \sum_{i=1}^{\infty} \hat{u}_t^{(i)} \varphi_i$ for $\hat{u}_t^{(i)} := \langle u_t, \varphi_i\rangle_{L^2(U)}$ and take $v = \varphi_i$ to see

$$\hat{u}_t^{(i)} = \hat{u}_0^{(i)} - \int_0^t \lambda_i \hat{u}_s^{(i)} ds + \int_0^t \sigma \sqrt{\xi_i} dB_s^{(i)}. \tag{7}$$

Hence $\hat{u}^{(i)}$ is an Ornstein-Uhlenbeck Process. To simulate a weak solution to the stochastic heat equation we can thus simulate Ornstein Uhlenbeck Processes and compute the truncated sum in the above equation. One can show that $\mathrm{Var}(\hat{u}_t^{(i)}) = \frac{\sigma^2 \xi_i}{2\lambda_i}(1 - e^{-2\lambda_i t})$ and thus by Parseval's identity

$$\|u_t\|^2_{L^2(\Omega, L^2(0,\pi))} = \mathbb{E}\left[\sum_{i=1}^{\infty} |\hat{u}_t^{(i)}|^2\right] = \sum_{i=1}^{\infty} \frac{\sigma^2 \xi_i}{2\lambda_i}(1 - e^{-2\lambda_i t}) \tag{8}$$

which converges if $\sum_{i=1}^{\infty} \frac{\xi_i}{\lambda_i} < \infty$, which is the case since Q is trace class.

The Stochastic Heat Equation in 2D

Now let $U = (0,\pi) \times (0, pi)$. One can show that $-\Delta$ as the eigenvalues $\lambda_{i,j} = i^2 + j^2$. Again, let us assume that Q has the same eigenfunctions but with corresponding eigenvalues $\xi_{i,j}$. Expand and substitute to see

$$d\hat{u}^{(i,j)} = -\lambda_{i,j} \hat{u}^{(i,j)} dt + \sigma \sqrt{\xi_{i,j}} dB_t^{(i,j)}. \tag{9}$$

Once again, let us apply Parseval's identity to obtain

$$\|u_t\|^2_{L^2(\Omega, L^2((0,\pi)) \times (0,\pi)} = \mathbb{E}\left[\sum_{i,j=1}^{\infty} |\hat{u}_t^{(i,j)}|^2\right] = \sum_{i,j=1}^{\infty} \frac{\sigma^2 \xi_{i,j}}{2\lambda_{i,j}}(1 - e^{-2\lambda_{i,j} t}). \tag{10}$$

This justifies that u is in $L^2(\Omega, L^2((0,\pi) \times (0,\pi))$ since Q is trace class. However, for a cylindrical Wiener Process there is no solution since

$$\sum_{i,j=1}^{\infty} \frac{1}{\lambda_{i,j}} = \sum_{i,j=1}^{\infty} \frac{1}{i^2 + j^2} = \infty. \tag{11}$$

References

An Introduction to Computational Stochastic PDEs, J. Lord, E. Powell, T. Shardlow, August 2014, Camebridge University Press.
An Introduction to SPDEs, Martin Hairer, July 2014, Lecture notes, The University of Warwick.
Numerical Solution of Stochas- tic Differential Equations, Peter E. Kloeden and Eckhard Platen, 1992, Springer Verlag Berlin Heidelberg GmbH
The Roughness and Smoothness of Numerical Solutions to the Stochastic Heat Equation, Yuxiang Chong and John B. Walsh , July 2011, Potential Analysis, Volume 37.
Numerical methods for stochastic par- tial differential equations and their control, Max Gunzburger, July 2008. http://www.maths.dur.ac.uk/lms/088/Talks/gunzburger1.pdf.