Proposal for Independent Research Project

Numerical Approximations of solutions to the Stochastic Heat Equation

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In my independent research project I would like to consider the stochastic heat equation, which is a generalization of the well known heat equation. More precisely, we will consider the initial-boundary-value problem

$$\begin{cases} u_t(t,x) = u_{xx}(x,t) + \dot{W} & \text{in } \Omega = (0,\infty) \times (0,1) \\ u(0,x) = u_0(x) & \text{if } 0 < x < 1 \\ u(t,0) = u(t,1) = 0 & \text{if } t \geqslant 0, \end{cases} \tag{1}$$

where $\dot{W}$ represents a space-time white noise and u_0 a boundary condition with certain regularity.

The Project can be devised in three different sections:

- The first step will be to rigorously define the above problem since the white noise will certainly contradict usual regularity assumptions. In particular I will have to give a brief summary on stochastic processes and Itô integration since not every student will be familiar with these topics.
- Second, I would like to answer some theoretical questions on existence and uniqueness. Maybe we can even give an explicit formula for an analytic solution.
- The final aim of the project is to implement a solver for the stochastic heat equation using either a form of finite elements or finite difference method.

My personal experience with stochastic differential equations restricts to ordinary stochastic differential equations which characterize Itô diffusion processes. Such an equation has the form

$$dX_t = a(t, X_t)dt + \sigma(t, X_t)dB_t. \tag{2}$$

Under certain Lipschitz assumptions to a and σ solutions to equation 2 can be simulated using a modification of the Euler-scheme for ODEs. Analogously, I would like to extend a numerical method that we studied in class to the stochastic heat equation.

I promise myself a way better understanding of stochastic differential equations and a first deeper insight into stochastic partial differential equations.

The numerical analysis will hopefully develop an intuition that helps me in theoretical questions. My supervisor told me that there will arise SPDEs in my main research project, so this is an exceptional chance to get used to them. In particular he told me that the SPDE

$$\frac{dX}{dt} = \frac{\sigma^2 \triangle X}{2} + gX + \sqrt{2bX}\dot{W} \tag{3}$$

will be of interest. It looks very similar to the stochastic heat equation. Additionally, I think that presenting basics about stochastic processes and stochastic integration in class will help me to understand it even better.

The most promising book covering the topic seems to be [LPS14]. It has a big section on theoretical backround and another big section on numerical methods for SPDEs.

A very comprehensive book covering numerics of SDEs is [KP92]. It will be useful for foundations but it does not cover a lot about stochastic *partial* differential equations.

The lecture notes [Hai09] of Fields medalist Martin Hairer might be more useful for the theory. Another promising source seems to be the second chapter of [Tud13], where solutions to stochastic heat equation are discussed.

Numerical methods are discussed in [Gun08] and in [CW11]. The first one focuses on a finite element approach whereas a finite difference scheme is used in the second one.

As a personal objective for my project I would like to set a focus on creating a poster that summarizes theory and results. Creating a high quality scientific poster is a very useful skill and I never did that before. The skill is measurable by criteria like sufficient content, clear structure, beautiful design and an balanced ration from text to images.

References

[CW11] Yuxiang Chong and John B. Walsh. The roughness and smoothness of numerical ssolution to the stochastic heat equation, July 2011. www.math.ubc.ca/ walsh/yuxiang.pdf.

[Gun08] Max Gunzburger. Numerical methods for stochastic partial differential equations and their control, July 2008. http://www.maths.dur.ac.uk/lms/088/Talks/gunzburger1.pdf.

[Hai09] Martin Hairer. An introduction to stochastic pdes. The University of Warwick / Courant Institute, July 2009. Lecture Nores.

[KP92] Peter E. Kloeden and Eckhard Platen. *Numerical Solution of Stochastic Differential Equations.* Springer Verlag Berlin Heidelberg GmbH, 1992.

[LPS14] Gabriel J. Lord, Catherine E. Powell, and Tony Shardlow. *An Introduction to Computational Stochastic PDEs.* Cambridge University Press, 2014.

[Tud13] Ciprian A. Tudor. *Analysis of Variations for Self-similar Processes - A stochastic calculus approach.* Springer Verlag Berlin Heidelberg GmbH, 2013.