Comprehensive Analysis of a 4R Serial Spatial Manipulator

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A Final Project Report for
MEEN 612 - Mechanics of Robotic Manipulators

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May 4, 2023

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1 Abstract

This report provides a thorough analysis of a 4R serial spatial manipulator, covering its frame assignment, kinematics, dynamics, and practical application. To illustrate the frame assignment, CAD screenshots have been included. The D-H table has been built using these frames as a basis for computing the transforms from the end effector to the ground.

To study the dynamics of the manipulator, the Jacobian matrix has been computed. This matrix is essential for analyzing the manipulator's dynamics. Additionally, a step-by-step guide has been provided to facilitate the practical application of the manipulator. This guide demonstrates how to solve forward and inverse kinematics using MATLAB.

Overall, this report presents a comprehensive analysis of the 4R serial spatial manipulator, providing detailed insights into its frame assignment, kinematics, dynamics, and practical application.

2 Introduction

Robotic manipulators find applications in places where the operation gets repeated for a multitude of iterations, thereby reducing the chances of human error and increasing efficiency. In industries such as manufacturing, assembly, and material handling, robotic manipulators are extensively used to perform repetitive tasks with high precision and accuracy.

A 4R serial spatial robot is a type of robotic manipulator that consists of four revolute joints, allowing it to move in three-dimensional space. This robot is designed to perform complex tasks in a variety of industries, including automotive, aerospace, and pharmaceuticals. In the automotive industry, 4R serial spatial robots are used for tasks such as welding, painting, and assembly. In the aerospace industry, these robots are used for drilling, riveting, and inspection of aircraft components. In the pharmaceutical industry, they are used for precise handling of materials and mixing of compounds.

The benefits of using 4R serial spatial robots include increased productivity, reduced cycle times, and improved product quality. These robots can operate continuously and accurately for extended periods, reducing the need for human intervention and increasing production rates. Additionally, the use of robotic manipulators reduces the risk of workplace injuries and improves overall safety in the workplace.

In conclusion, 4R serial spatial robots have become an integral part of many industries due to their ability to perform complex tasks with high precision and accuracy. These robots have revolutionized the manufacturing, assembly, and material handling processes, leading to increased productivity and improved product quality.

3 CAD Model

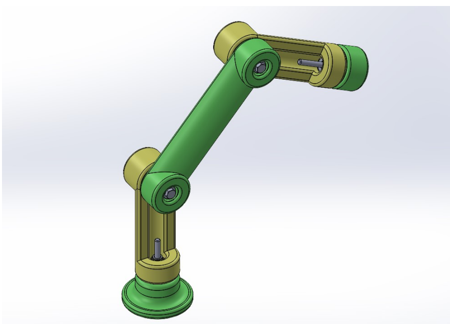

Figure 1: Solidworks Model of 4R-Serial-Spatial Manipulator.

4 Assembly of 3D Printed Links

The following is an assemblage of a 4R serial spatial manipulator, constructed using links produced via 3D printing in a Rapid Prototyping Laboratory.

Figure 2: 3D printed 4R Serial Spatial Manipulator

5 Forward Kinematics

The forward kinematics problem involves determining the position of the robotic manipulator's end-effector by using the kinematic equations with the specified joint values. Kinematically describing a robotic manipulator requires providing values for four quantities for each link, two describing the link itself and two for its connection to a neighboring link.

In the case of a revolute joint, the joint variable is denoted by θ_i, while the other three quantities represent fixed link parameters. For prismatic joints, d_i represents the joint variable, and the other three quantities are fixed link parameters. The convention used to describe mechanisms through these quantities is known as the Denavit-Hartenberg notation or DH notation.

For a four-jointed robot, 12 numbers would be required to completely describe the fixed portion of its kinematics. In the case of a four-jointed robot with all revolute joints, the 18 numbers are in the form of four sets of $(a_i \ \alpha_i \ \theta_i)$

5.1 Convention for Affixing Frames to Links

In order to describe the location of each link relative to its neighbors, we define a frame attached to each link. The link frames are named by number according to the link to which they are attached. That is, frame i is attached rigidly to link i. The following procedure is followed to place the Frames to the links:

1. Identify the joint axes and imagine (or draw) infinite lines along them. For steps 2 through 5 below, consider two of these neighboring lines.

2. Identify the common perpendicular between i and $i{+}1$, or point of intersection. At the point of intersection, or at the point where the common perpendicular meets the ith axis, assign the ith link-frame origin.

3. Assign the $\hat{Z}_i$ axis pointing along the ith joint axis.

4. Assign the $\hat{X}_i$ axis pointing along the common perpendicular, or, if the axes intersect, assign $\hat{X}_i$ to be normal to the plane containing the two axes.

5. Assign the $\hat{Y}_i$ axis to complete a right-hand coordinate system.

6. Assign 0 to match 1 when the first joint variable is zero. For N, choose an origin location and $\hat{X_N}$ direction freely, but generally.

5.2 Affixing Frames on CAD Model

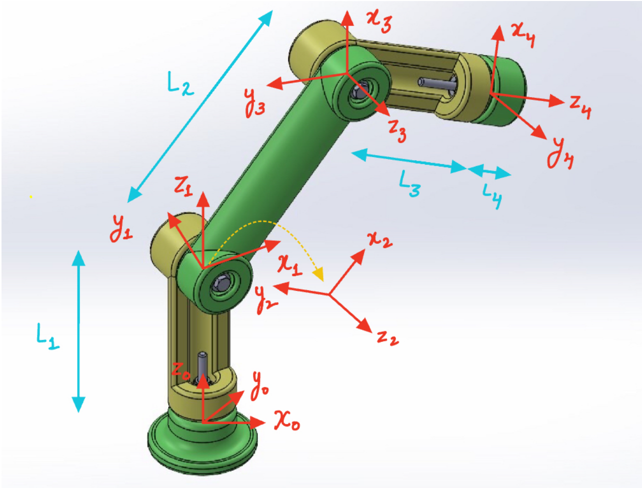

Figure 3: Frames attached to CAD Model

5.3 Link Parameters

If the link frames have been attached to the links according to our convention, the following definitions of the link parameters are valid:

- a_i = the distance from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured along $\hat{X}_i$
- α_i = the angle from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured about $\hat{X}_i$
- d_i = the distance from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured along $\hat{Z}_i$
- θ_i = the angle from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured about $\hat{Z}_i$

5.4 Denavit–Hartenberg (DH) parameters

Reference to the frames defined above and the definition of Link Parameters, following DH Table is defined for our 4R Serial Spatial Manipulator:

Table 1: DH Parameters for 4R serial spatial manipulator

Link(i)	$\boldsymbol{\alpha_{i-1}}$	$\boldsymbol{a_{i-1}}$	$\boldsymbol{d_i}$	$\boldsymbol{\theta_i}$
1	0	0	L_1	θ_1
2	$\frac{\pi}{2}$	0	0	θ_2
3	0	L_2	0	θ_3
4	$\frac{\pi}{2}$	0	L_3	θ_4

If we wish to write the transformation that transforms vectors defined in **i** to their description in **i − 1**, we use the general form of ${}^{i-1}\mathbf{T}_i$:

$$
{}^{i-1}\mathbf{T}_i = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & a \\ \sin(\theta)\cos(\alpha) & \cos(\theta)\cos(\alpha) & -\sin(\alpha) & -d\sin(\alpha) \\ \sin(\theta)\sin(\alpha) & \cos(\theta)\sin(\alpha) & \cos(\alpha) & d\cos(\alpha) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}
$$

6 Transformation Matrices

After substituting link parameters in equation 1 row by row we get the following transformations:

$$
{}^{0}\mathbf{T}_1 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}
$$

$$
{}^{1}\mathbf{T}_2 = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin(\theta_2) & \cos(\theta_2) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}
$$

$$
{}^{2}\mathbf{T}_3 = \begin{bmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 & L_2 \\ \sin(\theta_3) & \cos(\theta_3) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}
$$

$$
{}^{3}\mathbf{T}_4 = \begin{bmatrix} \cos(\theta_4) & -\sin(\theta_4) & 0 & 0 \\ 0 & 0 & -1 & -L_3 \\ \sin(\theta_4) & \cos(\theta_4) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}
$$

We know that, the individual link transformations can be multiplied together to find the single transformation that relates frame 4 to frame 0:

$$
{}^{0}\mathbf{T}_4 = {}^{0}\mathbf{T}_1 {}^{1}\mathbf{T}_2 {}^{2}\mathbf{T}_3 {}^{3}\mathbf{T}_4 \tag{6}
$$

$$
{}^{0}\mathbf{T}_4 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & P_x \\ r_{21} & r_{22} & r_{23} & P_y \\ r_{31} & r_{32} & r_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}
$$

where,

$$
\begin{aligned}
r_{11} &= c_4(c_1c_2c_3 - c_1s_2s_3) + s_1s_4 \\
r_{12} &= c_4s_1 - s_4(c_1c_2c_3 - c_1s_2s_3) \\
r_{13} &= c_1s_{23} \\
r_{21} &= -c_1s_4 + c_4(c_2c_3s_1 - s_1s_2s_3) \\
r_{22} &= -c_1c_4 - s_4(c_2c_3s_1 - s_1s_2s_3) \\
r_{23} &= s_1s_{23} \\
r_{31} &= c_4s_{23} \\
r_{32} &= -s_4s_{23} \\
r_{33} &= -c_{23} \\
P_x &= L_3c_1s_{23} + L_2c_1c_2 \\
P_y &= L_3s_1s_{23} + L_2s_1c_2 \\
P_z &= L_1 - L_3c_{23} + L_2s_2
\end{aligned}
$$

The transformation matrix ${}^{0}T_4$ for the 4R serial spatial manipulator is dependent on all four joint variables. With input from the robot's joint-position sensors, this matrix can be utilized to compute the Cartesian position and orientation of the final link.

Equations 7 provide the kinematics of the manipulator and define the process of determining the position and orientation of frame 4 relative to frame 0 of the robot. These equations are fundamental for all kinematic analyses related to this manipulator.

For the calculation of the transformation matrix from frame 4 to frame 0, which represents the position and orientation of the Wrist Center relative to frame 0, the MATLAB file 'ForwardKinematics.m' can be employed.

7 Space Jacobian

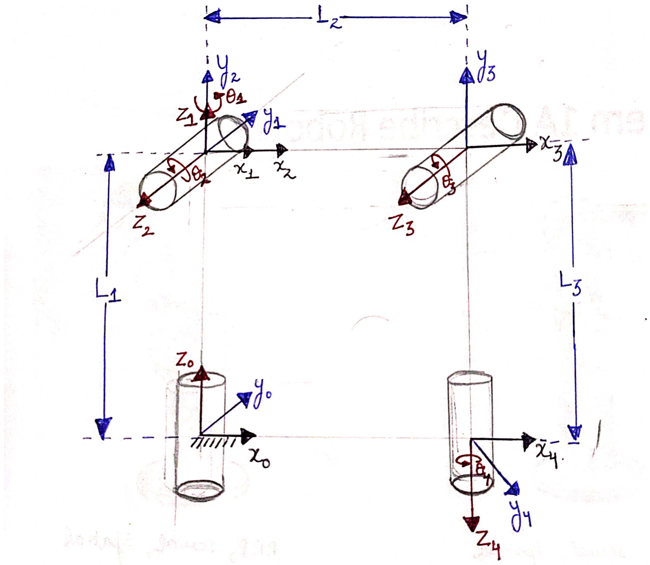

Figure 4: Home position (All $\theta's$ at zero) of 4R Serial Spatial Manipulator

Here,

$$q_i = \text{A point on the Screw Axis}$$
$$\omega_{si} = \text{direction of Screw Axis}$$
$$v_{si} = \text{linear velocity}$$

- Observe that first screw axis is in the direction of $\hat{\boldsymbol{z}}_{\boldsymbol{s}}$, hence $\boldsymbol{\omega_{s1}} = (0; 0; 1)$. Choosing q_1 as the origin $\boldsymbol{v_{s1}} = (0; 0; 0)$
- The second screw axis is in the direction of $\boldsymbol{\omega_{s2}} = (\sin(\theta_1); -cos(\theta_1); 0)$. Choosing q_2 as $(0; 0; L_1)$ we get, $\boldsymbol{v_{s2}} = -\omega_{s2} \times q_2 = (L_1 \cos(\theta_1); L_1 \sin(\theta_1); 0)$
- The third screw axis is in the direction of $(0; -1; 0)$ at the home position. After rotation of screw axis 1 and screw axis 2 by θ_1 and θ_2 respectively, we get

  $$w_{s3} = Rot(\hat{z}, \theta_1)Rot(\hat{y}, -\theta_2)(0; -1; 0)$$

  Similarly, q_3 in the home configuration of manipulator was at $(L2, 0, 0)$ and after rotation, we get :

  $$\boldsymbol{q_3} = \boldsymbol{q_2} + \mathrm{Rot}(\hat{z}, \theta_1)\,\mathrm{Rot}(\hat{y}, -\theta_2)(L_2; 0; 0)$$
  $$\boldsymbol{v_{s3}} = -\boldsymbol{\omega_{s3}} \times \boldsymbol{q_3}$$

- The fourth screw axis is in the direction of $(0; 0; -1)$ at the home configuration. After rotation of screw axis 1, screw axis 2 and screw axis 3 by θ_1, θ_2 and θ_3 respectively, we get

  $$\boldsymbol{w_{s4}} = \mathrm{Rot}(\hat{z}, \theta_1)\,\mathrm{Rot}(\hat{y}, -\theta_2)\,\mathrm{Rot}(\hat{y}, -\theta_3)(0; 0; -1)$$

  Similarly, q_4 in the home configuration of manipulator was at $(0; 0; -L_3)$ and after rotation, we get :

  $$\boldsymbol{q_4} = \boldsymbol{q_3} + \mathrm{Rot}(\hat{z}, \theta_1)\,\mathrm{Rot}(\hat{y}, -\theta_2)\,\mathrm{Rot}(\hat{y}, -\theta_3)(0; 0; -L_3)$$
  $$\boldsymbol{v_{s4}} = -\boldsymbol{\omega_{s4}} \times \boldsymbol{q_4}$$

.

The space Jacobian can now be computed and written in matrix form as follows:

$$Jacobian = \begin{bmatrix} \omega_{s1} & \omega_{s2} & \omega_{s3} & \omega_{s4} \\ \boldsymbol{v}_{s1} & \boldsymbol{v}_{s2} & \boldsymbol{v}_{s3} & \boldsymbol{v}_{s4} \end{bmatrix}$$

$$Jacobian = \begin{bmatrix} \omega_{s1} & \omega_{s2} & \omega_{s3} & \omega_{s4} \\ -\omega_{s1} \times q_1 & -\omega_{s2} \times q_2 & -\omega_{s3} \times q_3 & -\omega_{s4} \times q_4 \end{bmatrix}$$

The MATLAB file "Jacobian.m" can be utilized to calculate the Space Jacobian Matrix. For our 4R Serial Spatial Manipulator, the Jacobian Matrix will have a size of 6x4 with w_{si} and v_{si} being 3x1 vector.

8 Inverse Kinematics

In this section we'll work out the inverse kinematics of our 4R serial spatial manipulator.

Following parameters are given:

- Lengths of the manipulator i.e. L_1, L_2, L_3 and L_4
- Wrist center's position by P_x, P_y and P_z
- Roll, Pitch and yaw for the orientation of wrist center w.r.t. the ground frame

With wrist center's position and its orientation we can calculate ${}^0T_4$ which can be represented by:

$$
{}^0\mathbf{T}_4 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & P_x \\ r_{21} & r_{22} & r_{23} & P_y \\ r_{31} & r_{32} & r_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

We wish to solve for θ_1, θ_2, θ_3 and θ_4.

Projecting wrist center point P (P_x, P_y, P_z) onto $(X_0 - Y_0)$ plane, θ_1, which is first spatial rotation of the manipulator can be calculated. This can be visualized by the below given figure:

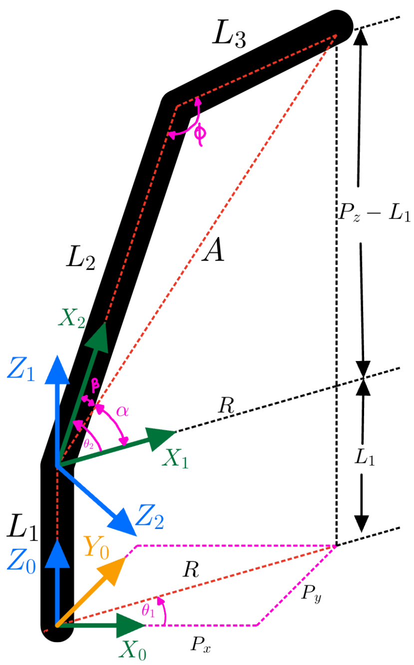

Figure 5: 4R Serial Spatial Manipulator

It is clearly visible from above figure that:

$$
\theta_1 = \tan^{-1} \frac{P_y}{P_x} \tag{8}
$$

Another solution for θ_1 can be:

$$
\theta_1 = \tan^{-1} \frac{P_y}{P_x} + \pi \tag{9}
$$

Also,

$$
R = \sqrt{P_x{}^2 + P_y{}^2} \tag{10}
$$

And,

$$
A = \sqrt{R^2 + (P_z - L_1)^2} \tag{11}
$$

Determining angles θ_2 and θ_3 for the robotic arm now reduces to the inverse kinematics problem for a planar two-link chain with length L_2 and L_3 which is clearly visible in below given figure:

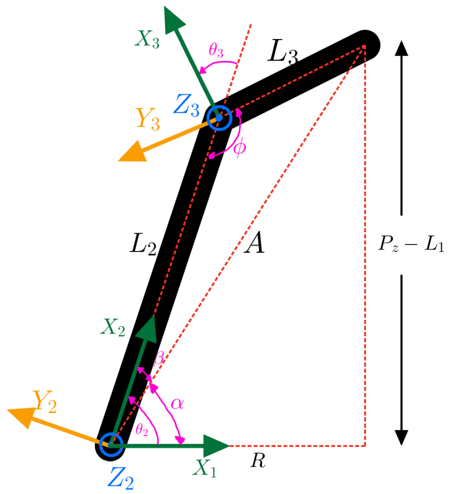

Figure 6: Elbow-up Solution for 4R Serial Spatial Manipulator

From Figure 6, it can be deduced quite easily that:

$$\alpha = \tan^{-1} \frac{P_z - L_1}{R} \tag{12}$$

$$\beta = \cos^{-1} \frac{(L_2)^2 + (A)^2 - (L_3)^2}{2AL_2} \tag{13}$$

Elbow-up solution for θ_2 is:

$$\theta_2 = \alpha + \beta \tag{14}$$

Similarly, we can calculate ϕ, using which we can calculate θ_3

$$\phi = \cos^{-1} \frac{(L_2)^2 + (L_3)^2 - (A)^2}{2L_2L_3} \tag{15}$$

Elbow-up solution for $theta_3$ is:

$$\theta_3 = \phi - \frac{\pi}{2} \tag{16}$$

Now, Elbow-down solution for the manipulator can be visualized by the below given figure

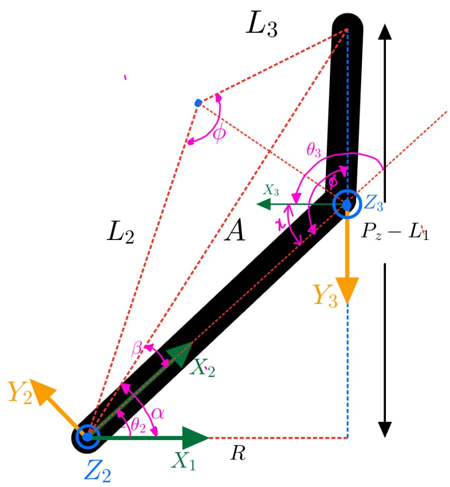

Figure 7: Elbow-down Solution for 4R Serial Spatial Manipulator

Elbow-down solution for θ_2 and θ_3 are:

$$\theta_2 = \alpha - \beta \tag{17}$$

$$\theta_3 = \frac{\pi}{2} - \phi \tag{18}$$

Now that we have all the angles from θ_1 to θ_3, we can calculate ${}^0T_3$ using Equation 2, Equation 3 and Equation 4 by:

$$^0\mathbf{T}_3 = {}^0\mathbf{T}_1{}^1\mathbf{T}_2{}^2\mathbf{T}_3 \tag{19}$$

Now, ${}^3T_{Wrist}$ can be calculated using:

$$^3\mathbf{T}_{Wrist} = ({}^0\mathbf{T}_3)^{-1} \times {}^0\mathbf{T}_4 \tag{20}$$

We know that, ${}^3T_{Wrist}$ is nothing but should be equivalent to ${}^3T_4$ which is equation 5. Therefore,

$$\theta_4 = \cos^{-1}({}^3\mathbf{T}_{Wrist}(1,1)) \tag{21}$$

The MATLAB file "InverseKinematics.m" can be utilized to calculate the Space Inverse Kinematic for our 4R Serial Spatial Manipulator. It takes in inputs of Link Lengths, Position and Orientation of Wrist Center w.r.t. Frame 0 and throws out Joint Angles from θ_1 to θ_4

References

1. Lynch, Kevin M., & Park, Frank C. (2017). *Modern Robotics* (1st ed.). Cambridge University Press.

2. Craig, John C. (2016). *Introduction to Robotics - Mechanics and Control* (4th ed.). Pearson India Education Services Pvt. Ltd.