



UNIVERSITY OF CALICUT

SCHOOL OF DISTANCE EDUCATION

M.COM

(II SEMESTER)

Course Code: MCM2C10

MANAGEMENT SCIENCE

(2019 Admission)

190610

SCHOOL OF DISTANCE EDUCATION

M.COM

(II SEMESTER)

STUDY MATERIAL

PREPARED BY :

***SRI. VINEETHAN. T
ASSISTANT PROFESSOR
GOVT. COLLEGE, MADAPPALLY***

SCRUTINIZED BY:

***ABOOBACKER SIDDEEQ K.C
ASSISTANT PROFESSOR (COMMERCE)
EMEA COLLEGE OF ARTS & SCIENCE,
KONDOTTY***

UNIVERSITY OF CALICUT

School of Distance Education

M. Com (CBCSS) Semester II

MCM2C10 : MANAGEMENT SCIENCE

CONTENTS

Chapter No.	Name of Chapter	Page No.
1	Introduction to Management Science	
2	Linear Programming and Graphical Method	
3	Linear Programming : Simplex Method	
4	Transportation Problems	
5	Assignment Problems	
6	Inventory Management	
7	Game Theory	
8	Queuing Theory	
9	Markov Analysis	
10	Network Analysis	

CHAPTER 1

INTRODUCTION TO MANAGEMENT SCIENCE

INTRODUCTION

Management Science is an approach of a manager to solve management problems. This includes identifying, stating, and modelling the problems; and finding, testing and implementing the solution with the help of knowledge in Statistics, Optimisation techniques and computer programming. Operations Research is another name of Management Science. The term Management Science is preferred by Americans while the Europeans prefer to use Operations Research.

Operation Research is a relatively new discipline. The contents and the boundaries of the OR are not yet fixed. Therefore, to give a formal definition of the term Operations Research is a difficult task. The OR starts when mathematical and quantitative techniques are used to substantiate the decision being taken. The main activity of a manager is the decision making. In our daily life we make the decisions even without noticing them. The decisions are taken simply by common sense, judgment and expertise without using any mathematical or any other model in simple situations. But the decisions we are concerned here with are complex and heavily responsible. Example: (1) Planning of public transportation network in a city having its own hospitals, factories, residential blocks, etc. (2) Finding the appropriate product mix when there are large number of products with different profit contributions and production requirement etc.

Operations Research tools are not from any one discipline. Operations Research takes tools from different discipline such as mathematics, statistics, economics, psychology, engineering etc. and combines these tools to make a new set of knowledge for decision making. Today, O.R. became a professional discipline which deals with the application of scientific methods for making decision, and especially to the allocation of scarce resources. The main purpose of O.R. is to provide a rational basis for decisions making in the absence of complete information, because the systems composed of human, machine, and procedures may do not have complete information.

The emphasis on analysis of operations as a whole distinguishes the O.R. from other research and engineering. O.R. is an interdisciplinary discipline which provided solutions to problems of military operations during World War II, and also successful in other operations. Today business applications are primarily concerned with O.R. analysis for the possible alternative actions. The business and industry benefitted from O.R. in the areas of inventory, reorder policies, optimum location and size of warehouses, advertising policies, etc.

DEFINITION OF MANAGEMENT SCIENCE

As stated earlier, defining Operations Research is a difficult task. The definitions stressed by various experts and Societies on the subject together enable us to know what O.R. is, and what it does. They are as follows:

According to the **Operational Research Society of Great Britain**, Operational Research is the attack of modern science on complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business, government and defence. Its distinctive approach is to develop a scientific model of the system, incorporating measurements of factors such as change and risk, with which to predict and compare the outcomes of alternative decisions, strategies or controls. The purpose is to help management determine its policy and actions scientifically.

Randy Robinson stresses that Operations Research is the application of scientific methods to improve the effectiveness of operations, decisions and management. By means such as analyzing data, creating mathematical models and proposing innovative approaches, Operations Research professionals develop scientifically based information that gives insight and guides decision-making. They also develop related software, systems, services and products.

Morse and Kimball have stressed O.R. is a quantitative approach and described it as “a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control”.

Saaty considers O.R. as tool of improving quality of answers. He says, “O.R. is the art of giving bad answers to problems which otherwise have worse answers”.

Miller and Starr state, “O.R. is applied decision theory, which uses any scientific, mathematical or logical means to attempt to cope with the problems that confront the executive, when he tries to achieve a thorough-going rationality in dealing with his decision problem”.

Pocock stresses that O.R. is an applied Science. He states “O.R. is scientific methodology (analytical, mathematical, and quantitative) which by assessing the overall implication of various alternative courses of action in a management system provides an improved basis for management decisions”.

HISTORY OF OPERATIONS RESEARCH (MANAGEMENT SCIENCE)

Operation Research is a relatively new discipline. It was only in the late 1930's that operational research began in a systematic fashion, and it started in the UK. As such it would be interesting to give a short history of O.R.

1936 : Early in 1936 the British Air Ministry established Bawdsey Research Station, on the east coast, near Felixstowe, Suffolk, as the centre where all pre-war radar experiments for both the Air Force and the Army would be carried out. Experimental radar equipment was brought up to a high state of reliability and ranges of over 100 miles on aircraft were obtained.

It was also in 1936 that Royal Air Force (RAF) Fighter Command, charged specifically with the air defence of Britain, was first created. It lacked however any effective fighter aircraft - no Hurricanes or Spitfires had come into service - and no radar data was yet fed into its very elementary warning and control system.

It had become clear that radar would create a whole new series of problems in fighter direction and control so in late 1936 some experiments started at Biggin Hill in Kent into the effective use of such data. This early work, attempting to integrate radar data with ground based observer data for fighter interception, was the start of OR.

1937 : The first of three major pre-war air-defence exercises was carried out in the summer of 1937. The experimental radar station at Bawdsey Research Station was brought into operation and the information derived from it was fed into the general air-defence warning and control system. From the early warning point of view this exercise was encouraging, but the tracking information obtained from radar, after filtering and transmission through the control and display network, was not very satisfactory.

1938 : In July 1938 a second major air-defence exercise was carried out. Four additional radar stations had been installed along the coast and it was hoped that Britain now had an aircraft location and control system greatly improved both in coverage and effectiveness. Not so! The exercise revealed, rather, that a new and serious problem had arisen. This was the need to coordinate and correlate the additional, and often conflicting, information received from the additional radar stations. With the outbreak of war apparently imminent, it was obvious that something new - drastic if necessary - had to be attempted. Some new approach was needed.

Accordingly, on the termination of the exercise, the Superintendent of Bawdsey Research Station, A.P. Rowe, announced that although the exercise had again demonstrated the technical feasibility of the radar system for detecting aircraft, its operational achievements still fell far short of requirements. He, therefore, proposed that a crash program of research into the operational - as opposed to the technical - aspects of the system should begin immediately. The term "*operational research*" [RESEARCH into OPERATIONS (military)] was coined as a suitable description of this new branch of applied science.

1939 : In the summer of 1939 Britain held what was to be its last pre-war air defence exercise. It involved some 33,000 men, 1,300 aircraft, 110 antiaircraft guns, 700 searchlights, and 100 barrage balloons. This exercise showed a great improvement in the operation of the air defence warning and control system. The contribution made by the OR team was so apparent that the Air Officer Commander-in-Chief RAF Fighter Command (Air Chief Marshal Sir Hugh Dowding) requested that, on the outbreak of war, they should be attached to his headquarters at Stanmore in north London.

Initially, they were designated the "Stanmore Research Section". In 1941 they were redesignated the "Operational Research Section" when the term was formalised and officially accepted, and similar sections set up at other RAF commands.

1940 : On May 15th 1940, with German forces advancing rapidly in France, Stanmore Research Section was asked to analyse a French request for ten additional fighter squadrons (12 aircraft a squadron - so 120 aircraft in all) when losses were running at some three squadrons every two days (i.e. 36 aircraft every 2 days). They prepared graphs for Winston Churchill (the British Prime Minister of the time), based upon a study of current daily losses and replacement rates, indicating how rapidly such a move would deplete fighter strength. No aircraft were sent and most of those currently in France were recalled.

This is held by some to be the most strategic contribution to the course of the war made by OR (as the aircraft and pilots saved were consequently available for the successful air defense of Britain, the Battle of Britain).

1941 onward : In 1941, an Operational Research Section (ORS) was established in Coastal Command which was to carry out some of the most well-known OR work in World War II. The responsibility of Coastal Command was, to a large extent, the flying of long-range sorties by single aircraft with the object of sighting and attacking surfaced U-boats (German submarines). The technology of the time meant that (unlike modern day submarines) surfacing was necessary to recharge batteries, vent the boat of fumes and recharge air tanks. Moreover U-boats were much faster on the surface than underwater as well as being less easily detected by sonar.

Thus the Operation Research started just before World War II in Britain with the establishment of teams of scientists to study the strategic and tactical problems involved in military operations. The *objective* was to find the most effective utilization of limited military resources by the use of quantitative techniques. Following the end of the war OR spread, although it spread in different ways in the UK and USA.

In 1951 a committee on Operations Research formed by the National Research Council of USA, and the first book on “Methods of Operations Research”, by Morse and Kimball, was published. In 1952 the Operations Research Society of America came into being.

Success of Operations Research in army attracted the attention of the industrial managers who were seeking solutions to their complex business problems. Now a days, almost every organization in all countries has staff applying operations research, and the use of operations research in government has spread from military to wide variety of departments at all levels. The growth of operations research has not limited to the U.S.A. and U.K., it has reached many countries of the world.

MANAGEMENT SCIENCE: INDIAN CONTEXT

India was one the few first countries who started using operations research. In India, Regional Research Laboratory located at Hyderabad was the first Operations Research unit established during 1949. At the same time another unit was set up in Defence Science Laboratory to solve the Stores, Purchase and Planning Problems. In 1953, Operations Research unit was established in Indian Statistical Institute, Calcutta, with the objective of using Operations Research methods in National Planning and Survey. In 1955, Operations Research Society of India (ORSI) was formed. In 1959, ORSI became a member of the International Federation of Operations Research Societies. Today Operations Research is a popular subject in management institutes and schools of mathematics.

METHODOLOGY OF MANAGEMENT SCIENCE

The process of O.R consists of six important steps. They are:

Step I: Observe the problem environment

Step II: Analyze and define the problem

Step III: Develop a model

Step IV: Select appropriate data input

Step V: Provide a solution and test its reasonableness

Step VI: Implement the solution

Step I: Observe the problem environment

The first step in the process of O.R. development is the problem environment observation. This step includes different activities; they are conferences, site visit, research, observations etc. These activities provide sufficient information to the O.R. specialists to formulate the problem.

Step II: Analyze and define the problem

This step is analyzing and defining the problem. In this step in addition to the problem definition the objectives, uses and limitations of O.R. study of the problem also defined. The outputs of this step are clear grasp of need for a solution and its nature understanding.

Step III: Develop a model

This step develops a model; a model is a representation of some abstract or real situation. The models are basically mathematical models, which describes systems, processes in the form of equations, formula/relationships. The different activities in this step are variables definition, formulating equations etc. The model is tested in the field under different environmental constraints and modified in order to work. Sometimes the model is modified to satisfy the management with the results.

Step IV: Select appropriate data input

A model works appropriately when there is appropriate data input. Hence, selecting appropriate input data is important step in the O.R. development stage or process. The activities in this step include internal/external data analysis, fact analysis, and collection of opinions and use of computer data banks.

The objective of this step is to provide sufficient data input to operate and test the model developed in Step III.

Step V: Provide a solution and test its reasonableness

This step is to get a solution with the help of model and input data. This solution is not implemented immediately, instead the solution is used to test the model and to find there is any limitations. Suppose if the solution is not reasonable or the behaviour of the model is not proper, the model is updated and modified at this stage. The output of this stage is the solution(s) that supports the current organizational objectives.

Step VI: Implement the solution

At this step the solution obtained from the previous step is implemented. The implementation of the solution involves many behavioural issues. Therefore, before implementation the implementation authority has to resolve the issues. A properly implemented solution results in quality of work and gains the support from the management.

TOOLS AND TECHNIQUES OF MANAGEMENT SCIENCE

Operations Research uses any suitable tools or techniques available. The common frequently used tools/techniques are mathematical procedures, cost analysis, electronic computation. However, operations researchers given special importance to the development and the use of techniques like linear programming, game theory, decision theory, queuing theory, inventory models and simulation. In addition to the above techniques, some other common tools are non-linear programming, integer programming, dynamic programming, sequencing theory, Markov process, network scheduling (PERT/CPM), symbolic Model, information theory, and value theory. The brief explanations of some of the above techniques/tools are given below:

Linear Programming:

This is a constrained optimization technique, which optimize some criterion within some constraints. In Linear programming the objective function and constraints are linear. There are different methods available to solve linear programming.

Game Theory:

This is used for making decisions under conflicting situations where there are one or more players/opponents. In this the motive of the players are dichotomized. The success of one player tends to be at the cost of other players and hence they are in conflict.

Decision Theory:

Decision theory is concerned with making decisions under conditions of complete certainty about the future outcomes and under conditions such that we can make some probability about what will happen in future.

Queuing Theory:

This is used in situations where the queue is formed (for example customers waiting for service, aircrafts waiting for landing, jobs waiting for processing in the computer system, etc). The objective here is minimizing the cost of waiting without increasing the cost of servicing.

Inventory Models:

Inventory model make a decisions that minimize total inventory cost. This model successfully reduces the total cost of purchasing, carrying, and out of stock inventory.

Simulation:

Simulation is a procedure that studies a problem by creating a model of the process involved in the problem and then through a series of organized trials and error solutions attempt to determine the best solution. Sometimes this is a difficult/time consuming procedure. Simulation is used when actual experimentation is not feasible or solution of model is not possible.

Non-linear Programming:

This is used when the objective function and the constraints are not linear in nature. Linear relationships may be applied to approximate non-linear constraints but limited to some range, because approximation becomes poorer as the range is extended. Thus, the non-linear programming is used to determine the approximation in which a solution lies and then the solution is obtained using linear methods.

Dynamic Programming:

Dynamic programming is a method of analyzing multistage decision processes. In this each elementary decision depends on those preceding decisions and as well as external factors.

Integer Programming:

If one or more variables of the problem take integral values only then dynamic programming method is used. For example number of motor in an organization, number of passenger in an aircraft, number of generators in a power generating plant, etc.

Markov Process:

Markov process permits to predict changes over time information about the behaviour of a system is known. This is used in decision making in situations where the various states are defined. The probability from one state to another state is known and depends on the current state and is independent of how we have arrived at that particular state.

Network Scheduling:

This technique is used extensively to plan, schedule, and monitor large projects. The aim of this technique is minimize trouble spots (such as delays, interruption, production bottlenecks, etc.) by identifying the critical factors. The different activities and their relationships of the entire project are represented diagrammatically with the help of networks and arrows, which is used for identifying critical activities and path. There are two main types of technique in network scheduling. They are:

Program Evaluation and Review Technique (PERT) – This is used when activities time is not known accurately/ only probabilistic estimate of time is available.

Critical Path Method (CPM) – This is used when activities time is known accurately.

Information Theory:

This analytical process is transferred from the electrical communication field to O.R. field. The objective of this theory is to evaluate the effectiveness of flow of information with a given system. This is used mainly in communication networks but also has indirect influence in simulating the examination of business organizational structure with a view of enhancing flow of information.

APPLICATIONS OF OPERATIONS RESEARCH

Today, almost all fields of business and government utilizing the benefits of Operations Research. Operations Research has a lot of applications. But it is not feasible to cover all applications of O.R. in brief. The following are the abbreviated set of typical operations research applications to show how widely these techniques are used today:

Accounting:

- Assigning audit teams effectively
- Credit policy analysis
- Cash flow planning
- Developing standard costs
- Establishing costs for by products
- Planning of delinquent account strategy

Construction:

- Project scheduling, monitoring and control
- Determination of proper work force
- Deployment of work force
- Allocation of resources to projects

Facilities Planning:

- Factory location and size decision
- Estimation of number of facilities required
- Hospital planning
- International logistic system design
- Transportation loading and unloading
- Warehouse location decision

Finance:

- Building cash management models
- Allocating capital among various alternatives
- Building financial planning models
- Investment analysis
- Portfolio analysis
- Dividend policy making

Manufacturing:

- Inventory control
- Marketing balance projection
- Production scheduling
- Production smoothing

Marketing:

- Advertising budget allocation
- Product introduction timing
- Selection of Product mix
- Deciding most effective packaging alternative

Organizational Behaviour / Human Resources:

- Personnel planning
- Recruitment of employees
- Skill balancing
- Training program scheduling
- Designing organizational structure more effectively

Purchasing:

- Optimal buying
- Optimal reordering
- Materials transfer

Research and Development:

- R & D Projects control
- R & D Budget allocation
- Planning of Product introduction

LIMITATIONS OF MANAGEMENT SCIENCE

Management Science has number of applications. At the same time it is not free from limitations. These limitations are mostly related with the problem of model building, money and time factors involved, etc. Some of them are as given below:

- a) **Distance between M.S. Specialist and Manager:** Operations Researchers job needs a mathematician or statistician, who might not be aware of the business problems. Similarly, a manager is unable to understand the complex nature of Operations Research. Thus there is a big gap between the two personnel.
- b) **Magnitude of Calculations:** The aim of M.S is to find out optimal solution taking into consideration all the factors. In this modern world these factors are enormous and expressing them in quantitative model and establishing relationships among these require voluminous calculations, which can be handled only by machines.
- c) **Money and Time Costs:** The basic data are subjected to frequent changes, incorporating these changes into the operations research models is very expensive. However, a fairly good solution at present may be more desirable than a perfect operations research solution available in future or after some time.
- d) **Non-quantifiable Factors:** When all the factors related to a problem can be quantifiable only then operations research provides solution otherwise not. The non-quantifiable factors are not incorporated in M.S models. Importantly M.S models do not take into account emotional factors or qualitative factors.
- e) **Implementation:** Once the decision has been taken it should be implemented. The implementation of decisions is a delicate task. This task must take into account the complexities of human relations and behaviour.

REVIEW QUESTIONS:

1. Define Management Science.
2. Give a brief account about the origin of Management Science.
3. Briefly explain the Indian context of Management Science.
4. What are the various steps in the methodology of Management Science?
5. What are the important tools and techniques of Management Science?
6. Explain the various applications of Management Science.
7. What are the important limitations of Management Science?

CHAPTER 2

LINEAR PROGRAMMING

Introduction to Linear Programming

Linear Programming is a special and versatile technique which can be applied to a variety of management problems viz. Advertising, Distribution, Investment, Production, Refinery Operations, and Transportation analysis. The linear programming is useful not only in industry and business but also in non-profit sectors such as Education, Government, Hospital, and Libraries.

The linear programming method is applicable in problems characterized by the presence of decision variables. The objective function and the constraints can be expressed as linear functions of the decision variables. The decision variables represent quantities that are, in some sense, controllable inputs to the system being modelled. An objective function represents some principal objective criterion or goal that measures the effectiveness of the system such as maximizing profits or productivity, or minimizing cost or consumption. There is always some practical limitation on the availability of resources viz. man, material, machine, or time for the system. These constraints are expressed as linear equations involving the decision variables. Solving a linear programming problem means determining actual values of the decision variables that optimize the objective function subject to the limitation imposed by the constraints.

The main important feature of linear programming model is the presence of linearity in the problem. The use of linear programming model arises in a wide variety of applications. Some model may not be strictly linear, but can be made linear by applying appropriate mathematical transformations. Still some applications are not at all linear, but can be effectively approximated by linear models. The ease with which linear programming models can usually be solved makes an attractive means of dealing with otherwise intractable nonlinear models.

George B Dantzig is recognized as the pioneer of linear programming. In 1947 he published the first paper on simplex method. Since 1947, many other researchers joined him in developing and exploring new applications of linear programming. Now Linear programming problem is being applied to many areas of human activity. Its application has increased with the development of computer technology.

Definition of LPP

Linear programming may be defined as a method of determining an optimum programme of independent activities in view of available resources. The objective of Linear Programming Problem is to maximize profit or minimize cost, as the case may be subject to number of limitations known as constraints. For this, an objective function is constructed which represents total profit or total cost as the case may be. The constraints are expressed in the form of inequalities or equations. Both the objective function and constraint are linear relationship between the variables. The solution to a Linear Programming Problem shows how much should be produced (or sold or purchased) which will optimize the objective function and satisfy the constraints.

Steps in Formulation of LPP

1. Identify the nature of the problem (maximization/minimization problem).
2. Identify the number of variables
3. Establish the objective function.
4. Formulate the constraints.
5. Develop non-negativity constraints.

Basic Assumptions of LPP

Following are the basic assumptions of linear programming technique:

- (a) **Certainty:** A very basic assumption is that the various parameters namely objective is known with certainty. Thus the profit or cost per unit of product, requirements of material and labour per unit and availability of material and labour, are to be given in the problem and they do not change with the passage of time.
- (b) **Proportionality:** The assumption of linear programming model proportionality exists in the objective function and constraints inequalities. For example, if one unit product contributes Rs. 5 towards profit, then 10 units of product, it would be Rs 50 and for 20 units it would be Rs 100 and so on. If the output and sale is doubled, the profit will also be doubled.
- (c) **Additivity:** The assumption of additivity underlying the linear programming model is that in both, the objective function and constraints inequalities, the total of all the activities is given by the sum total of each activity conducted separately. Thus, the total profit in the objective function is determined by the sum of the profits contributed by such of the products separately. Similarly, the total amount of the resources used is equal to the sum of the resources used by various activities.
- (d) **Divisibility:** The assumption of divisibility of the linear programming model is that the solution need not be in whole number. Instead, they are divisible and may take any fractional value.
- (e) **Finiteness:** A linear programming also assumes that a finite number of choices are available to a decision maker to find out optimum solution.
- (f) **Optimality:** The solution to a Linear Programming Problem is to be optimum.

Applications of Linear Programming in Industry and Management

Linear programming is exclusively used to solve variety of industrial and management problems. Few examples where linear programming can be applied are given bellow:

1. **Product Mix:** This problem essentially deals with determining the quantum of different product to be manufactured knowing the managerial contribution of each product and amount of available used by each product. The objective is to maximize the total contribution subject to constraints formed by available resources.
2. **Product Smoothing:** This problem in which the manufacturer has to determine the best plan for producing a product with a fluctuating demand.
3. **Media Selection:** The problem is to select advertising mix that will maximize the number of effective exposure subject to constraints like total advertising budget, usage of various media etc.
4. **Travelling Salesman Problem:** The problem is to find the shortest route for a salesman starting from a given city, visiting of each specified cities and returning to the original point of departure.
5. **Capital Investment:** The problem is to find the allocation to which maximize the total return when a fixed amount of capital is allocated to a number of activities.
6. **Transportation Problem:** Using transportation technique of linear programming we can determine the distribution system that will minimize total shipping cost from several warehouses to various market locations.
7. **Assignment and Transportation problems:** The problem of assigning the given number of personnel to different jobs can be tackled with the help of assignment model. The objective of may be to minimize the total time taken or total cost.
8. **Staffing Problem:** Linear Programming technique can be used to minimize the total number of employees in restaurant, hospital, police station etc. meeting the staff need at all hours.
9. **Blending Problem:** These problems are likely to arise when product can be made from variety of available raw materials of various composition and prices. The problem is to find the number of unit of each raw material to be blended to make one unit of product.
10. **Communication Industry:** Linear programming methods are used in solving problems involving facility for transmission, switching, relaying etc.
11. **Rail Road Industry:** Linear Programming technique can be used to minimize the total crew and engine expenses subject to restriction on hiring and paying the trainmen, scheduling of shipment, capacities of rail, roads etc.

Question:1

Suppose an industry is manufacturing two types of products P1 and P2. The profits per Kg of the two products are Rs.30 and Rs.40 respectively. These two products require processing in three types of machines. The following table shows the available machine hours per day and the time required on each machine to produce one Kg of P1 and P2. Formulate the problem in the form of linear programming model.

Profit/Kg	P1 Rs.30	P2 Rs.40	Total available Machine hours/day
Machine 1	3	2	600
Machine 2	3	5	800
Machine 3	5	6	1100

Solution:

The procedure for linear programming problem formulation is as follows:

Introduce the decision variable as follows:

Let x_1 = No. of units of P1

x_2 = No. of units of P2

In order to maximize profits, we establish the objective function as:

$$30x_1 + 40x_2$$

Since one Kg of P1 requires 3 hours of processing time in machine 1 while the corresponding requirement of P2 is 2 hours. So, the first constraint can be expressed as:

$$3x_1 + 2x_2 \leq 600$$

Similarly, corresponding to machine 2 and 3 the constraints are:

$$3x_1 + 5x_2 \leq 800$$

$$5x_1 + 6x_2 \leq 1100$$

In addition to the above there is no negative production, which may be represented algebraically as:

$$x_1 \geq 0 \text{ and } x_2 \geq 0$$

Thus, the product mix problem in the linear programming model is as follows:

Maximize

$$30x_1 + 40x_2$$

Subject to:

$$3x_1 + 2x_2 \leq 600$$

$$3x_1 + 5x_2 \leq 800$$

$$5x_1 + 6x_2 \leq 1100$$

$$x_1 \geq 0, x_2 \geq 0$$

Question 2:

A company owns two flour mills viz. A and B, which have different production capacities for high, medium and low quality flour. The company has entered a contract to supply flour to a firm every month with at least 8, 12 and 24 quintals of high, medium and low quality respectively. It costs the company Rs.2000 and Rs.1500 per day to run mill A and B respectively. On a day, Mill A produces 6, 2 and 4

quintals of high, medium and low quality flour, Mill B produces 2, 4 and 12 quintals of high, medium and low quality flour respectively. How many days per month should each mill be operated in order to meet the contract order most economically.

Solution:

Let us define x_1 and x_2 are the mills A and B.

Here the objective is to minimize the cost of the machine runs and to satisfy the contract order.

The linear programming problem is:

Minimize

$$2000x_1 + 1500x_2$$

Subject to:

$$6x_1 + 2x_2 \geq 8$$

$$2x_1 + 4x_2 \geq 12$$

$$4x_1 + 12x_2 \geq 24$$

$$x_1 \geq 0, x_2 \geq 0$$

Question: 3

The manager of an oil refinery must decide on the optimal mix of two possible blending processes of which the inputs and outputs per production run are as follows:

Production Run	Crude A	Crude B	Diesel X	Diesel B
1	6	3	6	9
2	5	6	5	5

Maximum availability of crude A and B are 250 units and 200 units respectively. The market requirement shows that at least 150 units of Diesel X and 130 units of Diesel Y must be produced. The profits per production run from process 1 and 2 are Rs 40 and Rs 50 respectively. Formulate the problem for maximizing the profit

Let x, y be the number of production runs of the two processes respectively.

The objective is to maximize profit $Z = 40x + 80y$

The constraints of the problems are :

$$6x + 5y \leq 250 \quad (\text{Crude material constraint})$$

$$3x + 6y \leq 200 \quad (\text{Crude material constraint})$$

$$6x + 5y \geq 150 \quad (\text{demand constraint})$$

$$9x + 5y \geq 130 \quad (\text{demand constraint})$$

Now the formulated problem is:

$$\begin{array}{ll} \text{Max} & Z = 40x + 80y \\ \text{Subject to} & 6x + 5y \leq 250 \\ & 3x + 6y \leq 200 \\ & 6x + 5y \geq 150 \\ & 9x + 5y \geq 130 \\ & x, y \geq 0 \end{array}$$

=====

SOLUTION TO LPP

A Linear Programming Problem is formulated in order to obtain a solution, ie, to find the unknown variables, say x and y . An optimal solution to an LPP is obtained by choosing from several values of decision variable. An optimal solution will be one set of values that satisfies the given set of constraint simultaneously and also provides maximum profit or minimum cost, as per given objective function.

LPPs can be solved by two methods - Graphic method or Simplex method. LPP with two variables only can be solved by graphical method. In the graphical method, the constraints are plotted as straight lines. The two variables are represented on x and y axis. A feasible area is identified and solution obtained by trial and error method.

Graphic method

Graphic method applies the method of two dimensional graph, consisting of x and y axis. Linear programming problems involving two variables can be solved by Graphical method. This method is simple and easy to apply. A layman can easily apply this method to solve a LPP.

But Linear programming problems involving more than two variables cannot be solved by this method. Each constraint is represented by a line. If there are many constraints, many lines are to be drawn. This will make the graph difficult to read.

The procedure for solving a LPP by graphic method is:

1. Formulate the problem into a Linear Programming Problem.
2. Each inequality in the constraint may be written as equality.
3. Draw straight lines corresponding to the equations obtained in step 2. So there will be as many straight lines as there are equations.
4. Identify the feasible region. Feasible region is the area which satisfies all the constraints simultaneously.
5. The vertices of the feasible region are to be located and their co-ordinates are to be measured.

How to draw Constraint Lines

For each constraint equation, there will be a straight line. A straight line connects two points. These points are obtained as below.

For example, an equation is $20x + 30y = 120$

First take $y = 0$,

Then $20x + 0 = 120$. And $x = 120/20 = 6$. The point is $(6,0)$.

Then take $x = 0$, $0 + 30y = 120$, and $y = 120/30 = 4$. The point is $(0,4)$.

Now draw straight line connecting 6 and 4 on the two axes. Similarly all constraints equations are plotted on the graph as separate straight lines.

Graphs of equation

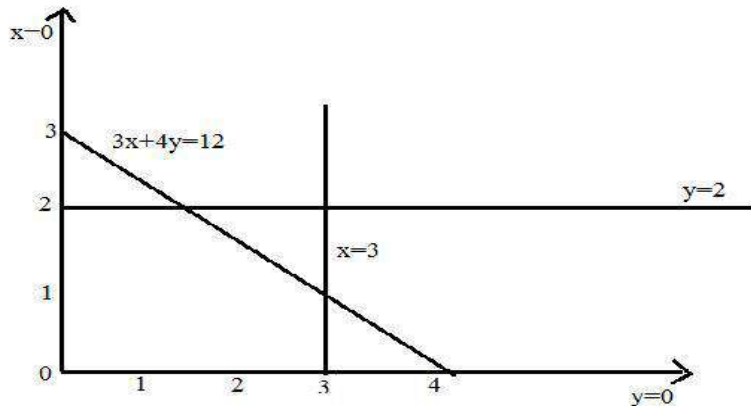
Each line represents an equation. A line consists of two points, which are derived from equations. For example the equation $3x + 4y = 12$ can be drawn as a line on a graph.

First Put 0 for x , then $3x = 0$, $4y = 12$. So, $y = 3$

Then put 0 for y , $4y = 0$. Now, $3x = 12$. So $x = 4$

Thus we get the two points - $x = 4$ and $y = 3$. Plot them and join them, and we get the line.

Similarly, two other equations $x = 3$, and $y = 2$, can be drawn as straight lines. These three lines are depicted below:



Optimality of graphical solution

While obtaining optimal solution to LPP by the graphical method, following rules are relevant:

1. The collection of all constraint equations constitute convex set whose extreme points correspond to basic feasible solution
2. There are finite number of basic feasible solutions within feasible solution base
3. If a convex set of feasible solutions form a polygon, at least one of the corner points gives optimal solution.
4. If the optimal solution occurs at more than one point, one of the solutions can be accepted at optimal combination point.

Ex.6. 1: Indicate on a graph paper, the region satisfying the following restraints.

$$X \geq 0, y \geq 0, 2x+y \leq 20, 2y+x=20$$

Under the above conditions maximize the function $x+3y$

Ans:

I step: write all the constraints in the form of equation:

then they are $x=0$; $y=0$

$$2x+y=20$$

$$x+2y=20$$

II step: Draw these lines:

- 1) For $x=0$ draw the y-axis
- 2) For $y=0$ draw the x-axis
- 3) For $2x+y=20$, find two points on the line. Put any value for x, say 0 then $y=20$ therefore, one the point is (0, 20)
Put any value for y, say 0 then $2x=20$ or $x=10$ Therefore, Another point is (10, 0).
Plot points (0, 20) and (10, 0) and join them we get the straight line DC
- 4) For $x+2y=20$ also find two points:

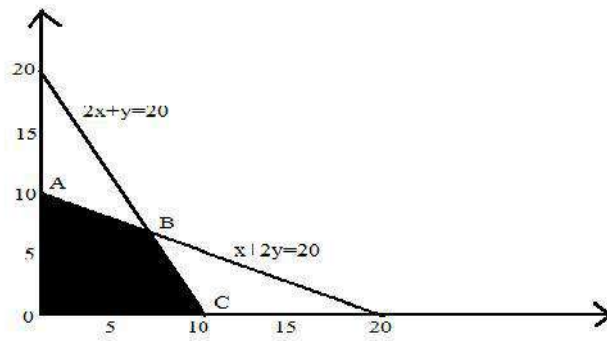
Put $x=0$, we get $y=10$ therefore, (0, 10) is a point

Put $y=0$, we get $x=20$ therefore, (20, 0) is also a point

Plot (0, 10) and (20, 0) and join them we get this line, AE.

Draw all the four lines. They provide the boundaries of the feasible region.

The feasible region is OABC (shaded).



Co-ordinates of O,A,B,C are (0,0),(0,10),(6.7,6.7),(10,0).[Coordinates of B is obtained by solving the equations $2x+y=20$ and $x+2y=20$]

Substituting these values in the function $x+3y$ as shown below:

Point	X	Y	$Z = x + 3y$
O	0	0	$0 + 0 = 0$
A	0	10	$0 + 30 = 30$
B	6.7	6.7	$6.7 + 20.1 = 26.8$
C	10	0	$10 + 0 = 10$

The maximum value of Z is 30. Therefore, the solution is:

Question: Solve the following problem graphically:

Max:

$$Z = 60x_1 + 40x_2$$

$$2x_1 + x_2 \leq 60$$

$$x_1 \leq 25$$

$$x_2 \leq 35$$

$$x_1 \geq 0, x_2 \geq 0$$

Answer: Convert the constraints as equations:

$$2x_1 + x_2 = 60 \dots\dots\dots(1)$$

$$x_1 = 25 \dots\dots\dots(2)$$

$$x_2 = 35 \dots\dots\dots(3)$$

$$x_1 = 0 \dots\dots\dots(4) \text{ and } x_2 = 0 \dots\dots\dots(5)$$

1) $2x_1 + x_2 = 60$

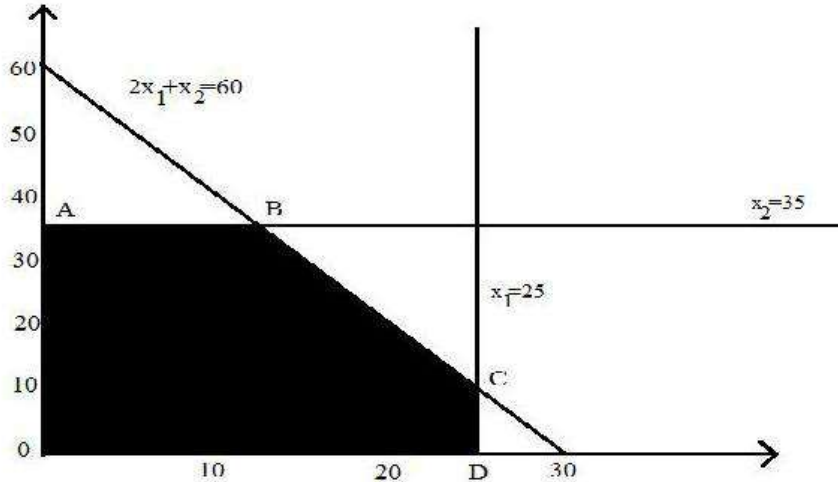
Put $x_1 = 0$, then $x_2 = 60$

So one point is (0, 60)

Put $x_2 = 0$, then $2x_1 = 60$. Therefore, $x_1 = 30$. So another point is (30, 0)

Plot (0, 60) and (30, 0) and join them, we get the line $2x_1 + x_2 = 60$

- 2) $x_1 = 25$ is a line parallel to x_2 –axis
- 3) $x_2 = 35$ is a line parallel to x_1 –axis
- 4) $x_1 = 0$ and $x_2 = 0$ are two axis.



The feasible region is OABCD (shaded).

Coordinates of A,B,C and D are respectively (0,35),(12.5,25),(25,10),(25,0). Coordinates of B can be obtained by solving the equations.

When we solve the equations we get $x_1 = 12.5$ and $x_2 = 35$

Point	x_1	x_2	$Z = 60x_1 + 40x_2$
O	0	0	$(60 * 0) + (40 * 0) = 0$
A	0	35	$(60 * 0) + (40 * 35) = 1400$
B	12.5	35	$(60 * 12.5) + (40 * 35) = 2150$
C	25	10	$(60 * 25) + (40 * 10) = 1900$
D	25	0	$(60 * 25) + (40 * 0) = 1500$

The value of Z is maximum at the point B. Therefore, the solution is:

$x_1 = 12.5$ and $x_2 = 35$, which maximises Z , i.e; 2150.

Question: A toy company manufacture two types of dolls a basic version doll A and a deluxe Version doll B. Each doll of type B takes twice as long as to produce as one type A and the company would have time to make maximum of 2,000 per day if it produced only the basic version. The supply of plastic is sufficient to produce 1,500 dolls per day (both A and B combined). The deluxe version requires a fancy dress of which there are 600 per day available. If the company makes profit of Rs.3.00 per doll and Rs.5.00 per doll respectively on doll A and B, how many of each should be produced per day in order to maximize profit?

Ans: Let x_1 dolls of type A and x_2 dolls of type B be produced per day.

therefore, Total profit, $Z = 3x_1 + 5x_2$ (Rs.)

Total time per day consumed to prepare x_1 and x_2 dolls of type A and B is $x_1(t) + x_2(2t)$ which should be less than 2000t where 't' is the time required for one doll of type.

Therefore, $x_1 + 2x_2 \leq 2000$ and $x_1 + 2x_2 \leq 2000$

Since plastic is available to produce 1500 doll only, $x_1 + x_2 \leq 1500$

Fancy dress is available for 600 dolls only therefore, $x_2 \leq 600$

Hence the Linear programming problem is as follows.

$$\begin{array}{ll}
 \text{Maximize} & Z = 3x_1 + 5x_2 \\
 & x_1 + 2x_2 \leq 2000 \\
 \text{Subject to} & \\
 & x_1 + x_2 \leq 1500 \\
 & x_2 \leq 600 \\
 & x_1 \geq 0, x_2 \geq 0
 \end{array}$$

First we consider the constraint as equation.

$$\begin{array}{ll}
 & x_1 + 2x_2 = 2000 \\
 \text{Therefore,} &
 \end{array}$$

$$x_1 + x_2 = 1500$$

$$x_2 = 600$$

$$x_1 = 0, x_2 = 0$$

Putting $x_1 = 0$, $x_2 = 1000$ and putting $x_2 = 0$, $x_1 = 2000$

Therefore, (0, 1000) and (2000, 0) are the two points on the first line.

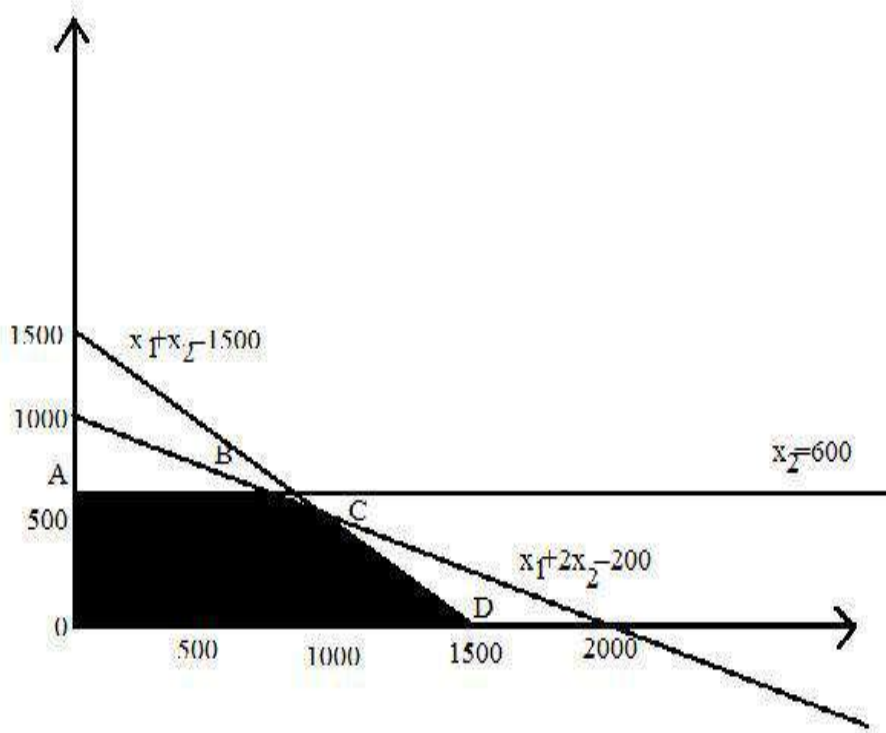
Putting $x_1 = 0$, $x_2 = 1500$ and putting $x_2 = 0$, $x_1 = 1500$

Therefore, (0, 1500) and (1500, 0) are the two points on the second line.

$x_2 = 600$ is parallel to x_1 axis.

$x_1 = 0$ and $x_2 = 0$ are the two axes.

Draw all the lines



The feasible region is OABCD (shaded)

Point	x_1	x_2	$Z = 3x_1 + 5x_2$
O	0	0	0
A	0	600	3000
B	800	600	5400
C	1000	500	5500
D	1500	0	4500

Z is maximum at the point C. So the solution is:

$x_1 = 1000$ and $x_2 = 500$ which maximises Z, i.e; 5500.

Therefore, The company should manufacture 1000 units of doll A and 500 units of doll B in order to have maximum profit of Rs.5500.

Review Questions:

1. What is meant by linear programming?
2. What are the requirements of linear programming?
3. State the assumptions in linear programming
4. What is meant by objective function?
5. Explain proportionality.
6. Explain additivity.
7. What is divisibility?
8. Explain optimality.
9. What is meant by non negativity?
10. State the characteristics of LPP.
11. State steps in formulation of LPP.
12. Explain merits of LP model.

13. What are the disadvantages of LPP?
14. What do you understand by 'Graphic method' of solving a LP problem?
15. What are methods of solutions of solving LPP?
16. How is to draw line of equation?
17. Explain the various steps involved in solving LPP by graphic method.
18. What do you mean by feasible region?
19. What are the merits of graphical method of solving a LPP?
20. What are the limitations of graphical method of solving LPP?
21. What are constraints?
22. A trader wants to purchase a number of fans and sewing machines. He has only Rs 5760 to invest and has space at most for 20 items. A fan costs him Rs 360 and a sewing machine Rs 240. His expectation is that he can sell a fan at a profit of Rs 22 and a sewing machine at a profit of Rs 18. Assuming that he can sell all the items he purchase, how should he invest his money in order to maximize his profit.
23. A firm has two types of pens – A and B . Pen A is a superior quality and pen B is lower quality. Profits on pen A and Pen B are Rs 5 and Rs 3 respectively. Raw materials required for each pen A is twice as that of pen B. the supply of raw material is sufficient only for 1000 pen B per day. Pen A requires a special clip and only 400 such clips are available per day. For pen B, only 700 clips are available per day. Formulate the problem into a LPP.
24. A toy company manufacture two types of dolls a basic version doll 'A' and a deluxe version doll 'B'. Each doll of type 'B' takes twice as long as to produce as one type 'A' and the company would have time to make maximum of 2,000 per day if it produced only the basic version. The supply of plastic is sufficient to produce 1,500 dolls per day (both 'A' and 'B' combined). The deluxe version requires a fancy dress of which there are 600 per day available. If the company makes profit of Rs.3.00 per doll and Rs.5.00 per doll respectively on doll 'A' and 'B', how many of each should be produced per day in order to maximize profit?
25. Solve the following LPP graphically:

Maximize	$Z = 5x_1 + 8x_2$
Subject to	$3x_1 + 2x_2 \leq 36$
	$x_1 + 2x_2 \leq 20$
	$3x_1 + 4x_2 \leq 42$

CHAPTER 3

SIMPLEX METHOD OF SOLVING LPP

Graphic method of LPP is limited to two variables. We have to look to other procedure which offers an efficient means of solving more complex LPP. Although the graphical method of solving LPP is an invaluable aid to understand the basic structure, the method is of limited application in industrial problems as the number of variables occurring there, is substantially large. So another method known as Simplex method is suitable for solving LPP with a large number of variables. The method though an iterative process, progressively approaches and ultimately reaches to the maximum or minimum value of the objective function. The method also helps the decision maker to identify an unbounded solution, multiple solution and infeasible solution.

The Simplex method was originally developed by George .B Dantzig, an American Mathematician. It has the advantage of being universal, ie, any linear model for which the solution exist, can be solved by it. In principle, it consists of starting with a certain solution of which all that we know is that, it is feasible,, ie, it satisfies non negativity conditions. We improve this solution at consecutive stages, until after a certain finite number of stages we arrive at optimal solution.

For arriving at the solution of LPP by this method the constraint and the objective function are presented in table known as simplex table. Then following a set procedure and rules, the optimal solution is obtained making step by step improvement.

Thus Simplex method is an iterative (step by step) procedure in which in systematic step from an initial Basic Feasible solution to another Basic Feasible solution and finally, in a finite number of steps to an optimal Basic Feasible solution, in such a way that value of the objective function at each step is better (or at least not worst) than that at preceding steps. In other words simplex algorithm consists of the following main steps.

- (1) Find a trial Basic Feasible Solution of Linear Programming Problem.
- (2) Test whether it is an optimal solution or not.
- (3) If not optimal, improve the first trial Basic Feasible Solution by a set of rules.
- (4) Repeat step (2) and step (3) till optimal solution is obtained.

How to construct simplex table?

Simplex table consists of rows and columns. If there are ‘m’ original variables and ‘n’ introduced variables, then there will be $3+m+n$ columns in the Simplex table [n’ introduced variables are slack, surplus or artificial variables].

First column (B) contains the basic variables. Second column (c) shows the coefficient of basic variables in the objective function. Third column (x_B) gives the value of the basic variables. Each of the next ‘m + n’ columns contain coefficient of the variables in the constraints, when they are converted into equations.

In a simplex table there is a vector associated with every variable. The vector associated with the basic variables is unit vectors.

Basic Concepts:

The Simplex method makes use of certain mathematical terms and basic concepts, as described below:

- (1) **Feasible Solution:** A feasible solution in a Linear Programming Problem is a set of values of the variables which satisfy all the constraints and non-negative restriction of the problem.
- (2) **Optimal Solution:** A feasible solution to a Linear Programming Problem is said to be optimum if it's optimizes the objective function, Z , of the problem. It should either maximize profit or minimize loss.
- (3) **Basic Feasible Solution:** A feasible solution is a Linear Programming Solution in which the vectors associated to non-zero variables are linearly independent is called a basic feasible solution.
- (4) **Slack Variables:** If a constraint has \leq sign, then in order to make it equality, we have to add a variable to the left hand side. This is called slack variable.
- (5) **Surplus Variable:** If a constraint has \geq sign, then in order to make it equality, we have to subtract a variable from the left hand side. This is called slack variable.

Basics of Simplex Method

The basic of simplex method is explained with the following linear programming problem.

Example 3.1:

Maximize $Z = 60x_1 + 70x_2$

Subject to:

$$2x_1 + x_2 \leq 300$$

$$3x_1 + 4x_2 \leq 509$$

$$4x_1 + 7x_2 \leq 812$$

$$x_1, x_2 \geq 0$$

Solution

First we introduce the variables

$$s_1, s_2, s_3 \geq 0$$

So that the constraints becomes equations, thus

$$2x_1 + x_2 + s_1 = 300$$

$$3x_1 + 4x_2 + s_2 = 509$$

$$4x_1 + 7x_2 + s_3 = 812$$

Corresponding to the three constraints, the variables s_3, s_4, s_5 are called as slack variables. Now, the system of equation has three equations and five variables.

There are two types of solutions they are basic and basic feasible, which are discussed as follows:

Basic Solution

We may equate any two variables to zero in the above system of equations, and then the system will have three variables. Thus, if this system of three equations with three variables is solvable such a solution is called as basic solution.

For example suppose we take $x_1=0$ and $x_2=0$, the solution of the system with remaining three variables is $s_3=300, s_4=509$ and $s_5=812$, this is a basic solution and the variables s_3, s_4 , and s_5 are known as basic variables where as the variables x_1, x_2 are known as non-basic variables.

The number of basic solution of a linear programming problem is depends on the presence of the number of constraints and variables. For example if the number of constraints is m and the number of variables including the slack variables is n then there are at most ${}^nC_{n-m} = {}^nC_m$ basic solutions.

Basic Feasible Solution

A basic solution of a linear programming problem is called as basic feasible solutions if it is feasible it means all the variables are non-negative. The solution $s_3=300$, $s_4=509$ and $s_5=812$ is a basic feasible solution.

The number of basic feasible solution of a linear programming problem is depends on the presence of the number of constraints and variables. For example if the number of constraints is m and the number of variables including the slack variables is n then there are at most ${}^nC_{n-m} = {}^nC_m$ basic feasible solutions.

Every basic feasible solution is an extreme point of the convex set of feasible solutions and every extreme point is a basic feasible solution of the set of given constraints. It is impossible to identify the extreme points geometrically if the problem has several variables but the extreme points can be identified using basic feasible solutions. Since one the basic feasible solution will maximize or minimize the objective function, the searching of extreme points can be carry out starting from one basic feasible solution to another.

The Simplex Method provides a systematic search so that the objective function increases in the cases of maximization progressively until the basic feasible solution has been identified where the objective function is maximized.

Simplex Method Computation This section describes the computational aspect of simplex method. Consider the following linear programming problem

$$\begin{aligned} \text{Maximize } Z &= 10x_1 + 6x_2 \\ \text{Subject to: } &2x_1 + 2x_2 \leq 4 \\ &10x_1 + 4x_2 \leq 20 \\ &6x_1 + 16x_2 \leq 24 \\ &x_1, x_2 \geq 0 \end{aligned}$$

Procedure for solving the LPP is:

Step 1: Introduce slack variables and convert the structural constraints into equations.
Now we get the constraints as:

$$\begin{aligned} 2x_1 + 2x_2 + s_1 &= 4 \\ 10x_1 + 4x_2 + s_2 &= 20 \\ 6x_1 + 16x_2 + s_3 &= 24 \end{aligned}$$

The above can be written in an orderly manner:

$$\begin{aligned} 2x_1 + 2x_2 + 1s_1 + 0s_2 + 0s_3 &= 4 \\ 10x_1 + 4x_2 + 0s_1 + 1s_2 + 0s_3 &= 20 \\ 6x_1 + 16x_2 + 0s_1 + 0s_2 + 1s_3 &= 24 \end{aligned}$$

Step 2: Express the constraint equations in the form of matrix:

x_1	x_2	s_1	s_2	s_3	X_B
2	2	1	0	0	4

$$10 \quad 4 \quad 0 \quad 1 \quad 0 \quad = \quad 20$$

$$6 \quad 16 \quad 0 \quad 0 \quad 1 \quad = \quad 24$$

Step 3: Identify the basic variables. Basic variables are the variables with unit vectors in the constraint matrix. Here, the basic variables are s_1 , s_2 and s_3 . The first basic variable is s_1 , the second basic variable is s_2 and the third is s_3 .

Step 4: Rewrite the objective function and express it in matrix form:

$$\text{Maximize } Z = 10x_1 + 6x_2 + 0s_1 + 0s_2 + 0s_3$$

$$[10 \quad 6 \quad 0 \quad 0 \quad 0]$$

Step 5: Draw initial simplex table and find net evaluation(Δ_j):

Initial Simplex Table								
B	C	x_B	x_1	x_2	s_1	s_2	s_3	Replacement Ratio $\theta = x_B/x_1$
s_1	0	4	2	2	1	0	0	$4/2 = 2 \rightarrow$
s_2	0	20	10	4	0	1	0	$20/4 = 5$
s_3	0	24	6	16	0	0	1	$24/6 = 4$
		z_j	0	0	0	0	0	
		c_j	10	6	0	0	0	
		$\Delta_j (z_i - c_j)$	-10	-6	0	0	0	



Step 6: Test the optimality of the solution.

Here the solution is $S_1 = 0$, $S_2 = 0$ and $S_3 = 0$

Since some of the Δ_j values are negative, the solution is not optimal. Therefore, some revision is to be made in this solution. The variable having the highest negative value (x_1) should be included in the table as basic variable, and one variable from the existing basic variable should be gone out.

Here the variable, x_1 is called incoming variable and it is marked by \uparrow . The outgoing variable is to be determined by the replacement ratio (θ). The basic variable having the minimum non-negative replacement ratio is to be gone out.

Now find the replacement ratio of all the existing basic variables. The basic variable, S_1 has the minimum replacement ratio. Therefore, S_1 is the outgoing variable and is marked by \rightarrow . The incoming and outgoing vectors are intersecting at the value 2 and it is called key value or pivot value.

Step 7: Draw the second simplex table with revisions in the existing solutions using pivot value.

B	C	x_B	x_1	x_2	s_1	s_2	s_3
x_1	10	$4/2=2$	$2/2 = 1$	$2/2 = 1$	$1/2 = 0.5$	$0/2 = 0$	$0/2 = 0$
s_2	0	$(-10*2)+20 = 0$	$(-10*1)+10 = 0$	$(-10*1)+4 = -6$	$(-10*0.5)+0 = -5$	$(-10*0)+1 = 1$	$(-10*0)+0 = 0$
s_3	0	$(-6*2)+24 = 12$	$(-6*1) + 6 = 0$	$(-6*1) + 16 = 10$	$(-6*0.5) + 0 = -3$	$(-6*0) + 0 = 0$	$(-6*0) + 1 = 1$

	Z _j	10	10	5	0	0
	C _j	10	6	0	0	0
	Δ _j (z _i – c _j)	0	4	5	0	0

Since all the net evaluation (Δ_j) values are either positive or zero, the solution is optimal.

The solution is $x_1 = 2$ and $x_2 = 0$, which maximises $Z = (10 \times 2) + (6 \times 0) = 20$.

Question: Solve the LPP using simplex method:

$$\begin{aligned}
 &\text{Maximize} && Z = 7x_1 + 5x_2 \\
 &\text{Subject to :} && x_1 + 2x_2 \leq 6 \\
 &&& 4x_1 + 3x_2 \leq 12 \\
 &&& x_1, x_2 \geq 0.
 \end{aligned}$$

Solution:

Introduce slack variables and convert the structural constraints into equations.

$$x_1 + 2x_2 + s_1 = 6$$

$$4x_1 + 3x_2 + s_2 = 12$$

The above can be written in an orderly manner:

$$x_1 + 2x_2 + 1s_1 + 0s_2 = 6$$

$$4x_1 + 3x_2 + 0s_1 + 1s_2 = 12$$

Express the constraint equations in the form of matrix:

$$\begin{array}{cccc|c}
 x_1 & x_2 & s_1 & s_2 & X_B \\
 1 & 2 & 1 & 0 & 6 \\
 4 & 3 & 0 & 1 & 12
 \end{array}$$

Here, the basic variables are s_1 , and s_2 . The first basic variable is s_1 , the second is s_2 .

Rewrite the objective function and express it in matrix form:

$$\begin{aligned}
 &\text{Maximize } Z = 7x_1 + 5x_2 + 0s_1 + 0s_2 \\
 &\quad \quad \quad [7 \quad 5 \quad 0 \quad 0]
 \end{aligned}$$

Draw initial simplex table and find net evaluation(Δ_j):

B	C	x _B	x ₁	x ₂	s ₁	s ₂	Replacement Ratio $\theta = X_B/x_1$
s ₁	0	6	1	2	1	0	6/1 = 6
s ₂	0	12	4	3	0	1	12/4 = 3 →
		z _j	0	0	0	0	
		c _j	7	5	0	0	
		Δ _j (z _i – c _j)	-7	-5	0	0	

Here the solution is $S_1 = 0$ and $S_2 = 0$.

Since some of the Δ_j values are negative, the solution is not optimal. Therefore, some revision is to be made in this solution. The variable having the highest negative value (x_1) should be included in the table as basic variable, and one variable from the existing basic variable should be gone out.

Here the variable, x_1 is called incoming variable and it is marked by \uparrow . The outgoing variable is to be determined by the replacement ratio (θ). The basic variable having the minimum non-negative replacement ratio is to be gone out.

Now find the replacement ratio of all the existing basic variables. The basic variable, S_1 has the minimum replacement ratio. Therefore, S_1 is the outgoing variable and is marked by \rightarrow . The incoming and outgoing vectors are intersecting at the value 4 and it is called key value or pivot value.

Draw the second simplex table with revisions in the existing solutions using pivot value.

B	C	x_B	x_1	x_2	s_1	s_2
S_1	0	$(3*-1)+6=3$	$(1*-1) + 1 = 0$	$(0.75*-1)+2 = 1.25$	$(0*-1) + 1 = 1$	$(0.25*-1) + 0 = -0.25$
x_1	7	$12/4 = 3$	$4/4 = 1$	$3/4 = 0.75$	$0/4 = 4$	$1/4 = 0.25$
		Z_j	7	6.5	0	1.75
		C_j	7	5	0	0
		$\Delta_j (z_i - c_j)$	0	0.25	0	1.75

Since all the net evaluation (Δ_j) values are either positive or zero, the solution is optimal.

The solution is $x_1 = 2$ and $x_2 = 0$, which maximises $Z = (7*3) + (5*0) = 21$.

Question:

Solve the LPP using simplex method:

$$\begin{aligned} \text{Maximize } Z &= 6x_1 + 4x_2 \\ \text{Subject to: } &-2x_1 + x_2 \leq 2 \\ &x_1 - x_2 \leq 2 \\ &3x_1 + 2x_2 \leq 9 \\ &x_1, x_2 \geq 0 \end{aligned}$$

Solution:

Introduce slack variables and convert the structural constraints into equations.

$$-2x_1 + x_2 + s_1 = 4$$

$$x_1 - x_2 + s_2 = 2$$

$$3x_1 + 2x_2 + s_3 = 9$$

The above can be written in an orderly manner:

$$-2x_1 + x_2 + 1s_1 + 0s_2 + 0s_3 = 2$$

$$1x_1 - 1x_2 + 0s_1 + 1s_2 + 0s_3 = 2$$

$$3x_1 + 2x_2 + 0s_1 + 0s_2 + 1s_3 = 24$$

Express the constraint equations in the form of matrix:

x_1	x_2	s_1	s_2	s_3	x_B
-2	1	1	0	0	2

$$\begin{array}{cccccc} 1 & -1 & 0 & 1 & 0 & = & 2 \\ 3 & 2 & 0 & 0 & 1 & & 9 \end{array}$$

Identify the basic variables. Here, the basic variables are s_1 , s_2 and s_3 . The first basic variable is s_1 , the second basic variable is s_2 and the third is s_3 .

Rewrite the objective function and express it in matrix form:

$$\begin{array}{l} \text{Maximize } Z = 6x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 \\ \quad \quad \quad [6 \quad 4 \quad 0 \quad 0 \quad 0] \end{array}$$

Draw initial simplex table and find net evaluation(Δ_j):

Initial Simplex Table								
B	C	x_B	x_1	x_2	s_1	s_2	s_3	Replacement Ratio $\theta = x_B/x_1$
s_1	0	2	-2	1	1	0	0	$2 \div -2 = -1$
s_2	0	2	1	-1	0	1	0	$2/1 = 2 \rightarrow$
s_3	0	9	3	2	0	0	1	$9/3 = 3$
		z_j	0	0	0	0	0	
		c_j	6	4	0	0	0	
		$\Delta_j (z_i - c_j)$	-6 ↑	-4	0	0	0	

Test the optimality of the solution.

Here the solution is $S_1 = 0$, $S_2 = 0$ and $S_3 = 0$

Since some of the Δ_j values are negative, the solution is not optimal. Therefore, some revision is to be made in this solution. The variable having the highest negative value (x_1) should be included in the table as basic variable, and one variable from the existing basic variable should be gone out.

Here the variable, x_1 is called incoming variable and it is marked by ↑. The outgoing variable is to be determined by the replacement ratio (θ). The basic variable having the minimum non-negative replacement ratio is to be gone out.

Now find the replacement ratio of all the existing basic variables. The basic variable, S_1 has the minimum replacement ratio. Therefore, S_1 is the outgoing variable and is marked by →. The incoming and outgoing vectors are intersecting at the value 1 and it is called key value or pivot value.

Draw second simplex table with revisions in the existing solutions using pivot value.

Second Simplex Table								$\theta = X_B/x_2$
B	C	x_B	x_1	x_2	s_1	s_2	s_3	
s_1	0	$(2*2) + 2 = 6$	$(2*1) + -2 = 0$	$(2*-1)+1 = -1$	$(2*0) + 1 = 1$	$(2*1) + 0 = 2$	$(2*0) + 0 = 0$	$6/-1 = -6$
x_1	6	$2/2 = 1$	$1/1 = 1$	$-1/1 = -1$	$0/1 = 0$	$1/1 = 1$	$2/-1 = -2$	$2/-1 = -2$
s_3	0	$(-3*2)+9 = 3$	$(-3*1) + 3 = 0$	$(-3*-1) + 2 = 5$	$(-3*0) + 0 = 0$	$(-3*1) + 0 = -3$	$(-3*0) + 1 = 1$	$3/5 = 0.6 \rightarrow$
Z_j		6	6	-6	0	6	0	
C_j		6	6	4	0	0	0	
$\Delta_j (z_i - c_j)$		0	0	-10	0	6	0	
				\uparrow				

Since one of the net evaluation (Δ_j) values is negative, the solution is not optimal.

Here the variable, x_2 is called incoming variable and it is marked by \uparrow . The outgoing variable is to be determined by the replacement ratio (θ). The basic variable having the minimum non-negative replacement ratio is to be gone out.

Now find the replacement ratio of all the existing basic variables. The basic variable, s_1 has the minimum replacement ratio. Therefore, s_3 is the outgoing variable and is marked by \rightarrow . The incoming and outgoing vectors are intersecting at the value 5 and it is called key value or pivot value.

Draw second simplex table with revisions in the existing solutions using pivot value.

Third Simplex Table							
B	C	x_B	x_1	x_2	s_1	s_2	s_3
s_1	0	$(1*0.6) + 6 = 6.6$	$(1*0) + 0 = 0$	$(1*1)+-1 = 0$	$(1*0) + 1 = 1$	$(1*-0.6) + 2 = 1$	$(1*0.2) + 0 = 0.2$
x_1	6	$(1*0.6) + 2 = 2.6$	$(1*0)+1 = 1$	$(1*1)+-1 = 0$	$(1*0)+0 = 0$	$(1*-0.6)+1 = 0.4$	$(1*0.2) + 0 = 0.2$
x_2	4	$(3/5) = 0.6$	$0/5 = 0$	$5/5 = 1$	$0/5 = 0$	$-3/5 = -0.6$	$1/5 = 0.2$
Z_j		6	6	4	0	0	2
C_j		6	6	4	0	0	0
$\Delta_j (z_i - c_j)$		0	0	0	0	0	2
			\uparrow				

Since all the Δ_j values are either zero or positive, the solution is optimal.

The solution is $x_1 = 2.6$ and $x_2 = 0.6$, which maximises $Z = (6 \times 2.6) + (4 \times 0.6) = 18$.

Minimisation Problems

In minimisation problems, if all the values of net evaluation (Δ_j) are either zero or negative, the solution is optimal. If any one of the Δ_j values is positive, the solution is not optimal. The variable with highest positive Δ_j value is taken as incoming variable.

Question:

Solve the LPP using Simplex method:

$$\begin{aligned} \text{Minimize } Z &= x_1 - 3x_2 + 2x_3 \\ \text{Subject to: } 3x_1 - x_2 + 3x_3 &\leq 7 \\ -2x_1 + 4x_2 &\leq 12 \\ -4x_1 + 3x_2 + 8x_3 &\leq 9 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Solution:

Introduce slack variables and convert the structural constraints into equations.

$$\begin{aligned} 3x_1 - x_2 + 3x_3 + s_1 &= 7 \\ -2x_1 + 4x_2 + s_2 &= 12 \\ -4x_1 + 3x_2 + 8x_3 + s_3 &= 10 \end{aligned}$$

The above can be written in an orderly manner:

$$\begin{aligned} 3x_1 - x_2 + 3x_3 + 1s_1 + 0s_2 + 0s_3 &= 7 \\ -2x_1 + 4x_2 + 0x_3 + 0s_1 + 1s_2 + 0s_3 &= 12 \\ -4x_1 + 3x_2 + 8x_3 + 0s_1 + 0s_2 + 1s_3 &= 10 \end{aligned}$$

Express the constraint equations in the form of matrix:

$$\begin{array}{cccccc|c} \underline{x_1} & \underline{x_2} & \underline{x_3} & \underline{s_1} & \underline{s_2} & \underline{s_3} & \underline{X_B} \\ 3 & -1 & 3 & 1 & 0 & 0 & 7 \\ -2 & 4 & 0 & 0 & 1 & 0 & 12 \\ -4 & 3 & 8 & 0 & 0 & 1 & 10 \end{array}$$

Identify the basic variables. Here, the basic variables are s_1 , s_2 and s_3 . The first basic variable is s_1 , the second basic variable is s_2 and the third is s_3 .

Rewrite the objective function and express it in matrix form:

$$\begin{aligned} \text{Maximize } Z &= x_1 - 3x_2 + 2x_3 + 0s_1 + 0s_2 + 0s_3 \\ [1 \quad -3 \quad 2 \quad 0 \quad 0 \quad 0] \end{aligned}$$

Draw initial simplex table and find net evaluation (Δ_j):

Initial Simplex Table									
B	C	x _B	x ₁	x ₂	x ₃	s ₁	s ₂	s ₃	Replacement Ratio θ = x_B / x₂
s ₁	0	7	3	-1	3	1	0	0	7/-1 = -7
s ₂	0	12	-2	<div>4</div>	0	0	1	0	12/4 = 3 →
s ₃	0	10	-4	3	8	0	0	1	10/3 = 3.33
		Z _j	0	0	0	0	0	0	
		C _j	1	-3	2	0	0	0	
		Δ _j (z _i – c _j)	-1	3	-2	0	0	0	
		↑							

Now Test the optimality of the solution.

Here the solution is $S_1 = 0$, $S_2 = 0$ and $S_3 = 0$

Since one of the Δ_j values is positive, the solution is not optimal. Therefore, some revision is to be made in this solution. The variable having the highest positive value (x_2) should be included in the table as basic variable, and one variable from the existing basic variable should be gone out.

Here the variable, x_2 is called incoming variable and it is marked by \uparrow . The outgoing variable is to be determined by the replacement ratio (θ). The basic variable having the minimum non-negative replacement ratio is to be gone out.

Now find the replacement ratio of all the existing basic variables. The basic variable, S_1 has the minimum replacement ratio. Therefore, S_1 is the outgoing variable and is marked by \rightarrow . The incoming and outgoing vectors are intersecting at the value 4 and it is called key value or pivot value.

Draw second simplex table with revisions in the existing solutions using pivot value.

Second Simplex Table									$\theta = X_B/x_2$
B	C	x_B	x_1	x_2	x_3	s_1	s_2	s_3	
s_1	0	10	5/2	0	3	1	1/4	0	$10 * 2/5 = 4 \rightarrow$
x_2	-3	3	-1/2	1	0	0	1/4	0	$3 * -2/1 = -6$
s_3	0	1	-5/2	0	8	0	-3/4	1	$1 * -2/5 = -0.4$
		Z_j	3/2	-3	0	0	-3/4	0	
		C_j	1	-3	2	0	0	0	
		$\Delta_j (z_j - c_j)$	1/2	0	-2	0	-3/4	0	
			\uparrow						

Since one of the net evaluation (Δ_j) values is positive, the solution is not optimal.

Here the variable, x_1 is called incoming variable and it is marked by \uparrow . The outgoing variable is to be determined by the replacement ratio (θ). The basic variable having the minimum non-negative replacement ratio is to be gone out.

Now find the replacement ratio of all the existing basic variables. The basic variable, s_1 has the minimum replacement ratio. Therefore, s_1 is the outgoing variable and is marked by \rightarrow . The incoming and outgoing vectors are intersecting at the value 5/2 and it is called key value or pivot value.

Draw third simplex table with revisions in the existing solutions using pivot value.

Third Simplex Table								
B	C	x_B	x_1	x_2	x_3	s_1	s_2	s_3
x_1	1	4	1	0	6/5	2/5	1/10	0
x_2	-3	5	0	1	3/5	1/5	3/10	0
s_3	0	11	0	0	11	1	-1/2	0
		Z_j	1	-3	-3/5	-1/5	-8/10	0
		C_j	1	-3	2	0	0	0

	$\Delta_j (z_i - c_j)$	0	0	-13/5	-1-5	-8/10	0
		↑					

Since all the Δ_j values are either zero or negative, the solution is optimal.

The solution is $x_1 = 4$, $x_2 = 5$ and $x_3 = 0$, which minimizes $Z = 4 - (3 \times 5) + (2 \times 0) = -11$.

ARTIFICIAL VARIABLE TECHNIQUE

In a linear Programming Problem, if any of the constraints have \leq (less than or equal to) sign, we have to subtract surplus variable from LHS of that constraint. But we do not get a starting basic feasible solution by incorporating surplus variable. So, in such cases some other variables have to be introduced to obtain starting basic feasible solution. These variables which are incorporated only for computational purposes are called artificial variables.

When artificial variables are introduced, the LPP can be solved in two ways. They are:

- (a) Big M Method, and
- (b) Two Phase Method

Big M Method for solving LPP

When artificial variables are introduced in any one of the constraints, the same have to be incorporated in the objective function also. Generally, a large value (i.e; M) is assigned as coefficient to each of the artificial variable in the objective function. In cases of maximisation objective function, “-M” is assigned and in minimisation cases “+M” is assigned. Since “M” so assigned is called penalty, this method is also called penalty method.

Question: Solve the following LPP using Big M method:

$$\begin{aligned} \text{Minimize } Z &= 5x_1 + 6x_2 \\ \text{Subject to: } &2x_1 + 5x_2 \geq 1500 \\ &3x_1 + x_2 \geq 1200 \\ &x_1, x_2 \geq 0 \end{aligned}$$

Solution:

Introduce surplus variables and convert the structural constraints into equations.

$$2x_1 + 5x_2 - s_1 + A_1 = 1500$$

$$3x_1 + x_2 - s_2 + A_2 = 1200$$

The above equations can be written in an orderly manner:

$$2x_1 + 5x_2 - 1s_1 + 0s_2 + 1A_1 + 0A_2 = 1500$$

$$3x_1 + x_2 + 0s_1 - 1s_2 + 0A_1 + 1A_2 = 1200$$

Express the constraint equations in the form of matrix:

x_1	x_2	s_1	s_2	A_1	A_2		X_B
2	5	-1	0	1	0	=	1500
3	1	0	-1	0	1		1200

Here, the basic variables are A_1 , and A_2 . The first basic variable is A_1 , the second basic variable is A_2 .

Rewrite the objective function and express it in matrix form:

$$\begin{aligned} \text{Minimize } Z &= 5x_1 + 6x_2 + 0s_1 + 0s_2 + MA_1 + MA_2 \\ &[5 \quad 6 \quad 0 \quad 0 \quad M \quad M] \end{aligned}$$

Draw initial simplex table and find net evaluation(Δ_j).

Initial Simplex Table									
B	C	x_B	x_1	x_2	s_1	s_2	A_1	A_2	Replacement Ratio $\theta = X_B/x_2$
A_1	M	1500	2	5	-1	0	1	0	$1500/5 = 300 \rightarrow$
A_2	M	1200	3	1	0	-1	0	1	$1200/1 = 1200$
		z_j	5M	6M	-M	-M	M	M	
		c_j	5	6	0	0	M	M	
		$\Delta_j (z_i - c_j)$	5M-5	6M-6	-M	-M	0	0	
				↑					

Here the solution is $A_1 = 0$ and $A_2 = 0$.

Since some of the Δ_j values are positive, the solution is not optimal. Therefore, some revision is to be made in this solution. The variable having the highest positive value (x_2) should be included in the simplex table as basic variable, and one variable from the existing basic variable should be gone out.

Here the variable, x_2 is called incoming variable and it is marked by \uparrow . The outgoing variable is to be determined by the replacement ratio (θ). The basic variable having the minimum non-negative replacement ratio is to be gone out.

Now find the replacement ratio of all the existing basic variables. The basic variable, A_1 has the minimum replacement ratio. Therefore, A_1 is the outgoing variable and is marked by \rightarrow . The incoming and outgoing vectors are intersecting at the value 5 and it is called key value or pivot value (shown in a rectangle).

Draw the second simplex table with revisions in the existing solutions using pivot value.

Second Simple Table									
B	C	x_B	x_1	x_2	s_1	s_2	A_2	$\theta = X_B/x_1$	
x_2	6	300	2/5	5/5=1	-1/5	0	0	$300*5/2 = 750$	
A_2	M	$(-1*300) + 1200 = 900$	$(-1*2/5) + 3 = 13/5$	$(-1*1) + 1 = 0$	$(-1*-1/5) + 0 = 1/5$	$(-1*0) + -1 = -1$	$(-1*0) + 1 = 1$	$900*5/13 = 346.15 \rightarrow$	
		z_j	$13M/5 + 12/5$	6	$M/5 - 6/5$	-M	M		
		C_j	5	6	0	0	M		
		$\Delta_j (z_i - c_j)$	$13M/5 - 13/5$	0	$M/5 - 6/5$	-M	0		
			↑						

Here the solution is $x_2 = 300$ and $A_2 = 900$.

Since some of the Δ_j values are positive, the solution is not optimal. Therefore, some revision is to be made in this solution. The variable having the highest positive value (x_1) should be included in the simplex table as basic variable, and one variable from the existing basic variable should be gone out.

Here the variable, x_1 is called incoming variable and it is marked by \uparrow . The outgoing variable is to be determined by the replacement ratio (θ). The basic variable having the minimum non-negative replacement ratio is to be gone out.

Now find the replacement ratio of all the existing basic variables. The basic variable, A_1 has the minimum replacement ratio. Therefore, A_1 is the outgoing variable and is marked by \rightarrow . The incoming and outgoing vectors are intersecting at the value 5 and it is called key value or pivot value which is shown in a rectangle.

Draw the third simplex table by dropping A_2 with revisions in the existing solutions using pivot value.

B	C	x_B	x_1	x_2	s_1	s_2
x_2	6	2100/13	$(-2/5*1)$ $+2/5 = 0$	$(-2/5*0)$ $+1 = 1$	$(-2/5*1/13)$ $+ -1/5 = -3/13$	$(-2/5*-5/13)$ $+ 0 = 2/13$
x_1	5	4500/13	1	0	1/13	-5/13
		z_j	5	6	-1	-1
		C_j	5	6	0	0
		$\Delta_j (z_i - c_j)$	0	0	-1	-1

Since all the Δ_j values are either zero or negative, the solution is optimal.
The solution is $x_1 = 4500/13$ and $x_2 = 2100/13$, which maximises $Z = 27000$.

LPP with Unbounded Solution

While testing optimality of a solution to LPP, if the replacement ratios (θ) of all the basic variables are either negative or void, then the LPP is an LPP with unbounded solution.

Question: Solve the following LPP:

$$\begin{aligned} \text{Maximize } Z &= 10x_1 + 8x_2 \\ \text{Subject to : } 2x_1 &\leq 14 \\ 2x_1 - 2x_2 &\leq 16 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution:

Introduce slack variables and convert the structural constraints into equations.

$$\begin{aligned} 2x_1 + s_1 &= 14 \\ 2x_1 - 2x_2 + s_2 &= 16 \end{aligned}$$

The above can be written in an orderly manner:

$$\begin{aligned} 2x_1 + 0x_2 + 1s_1 + 0s_2 &= 14 \\ 2x_1 - 2x_2 + 0s_1 + 1s_2 &= 16 \end{aligned}$$

Express the constraint equations in the form of matrix:

$$\begin{array}{cccc} x_1 & x_2 & s_1 & s_2 \\ 2 & 0 & 1 & 0 \\ 2 & -2 & 0 & 1 \end{array} = \begin{array}{c} x_B \\ 14 \\ 16 \end{array}$$

Here, the basic variables are s_1 , and s_2 . The first basic variable is s_1 , the second is s_2 .

Rewrite the objective function and express it in matrix form:

$$\text{Maximize } Z = 10x_1 + 8x_2 + 0s_1 + 0s_2$$

$$[10 \quad 8 \quad 0 \quad 0]$$

Draw initial simplex table and find net evaluation (Δ_j):

Initial Simplex Table							
B	C	x_B	x_1	x_2	s_1	s_2	$\theta = x_B/x_1$
s_1	0	14	2	0	1	0	$14/2 = 7 \rightarrow$
s_2	0	16	2	-2	0	1	$16/2 = 8$
		z_j	0	0	0	0	
		c_j	10	8	0	0	
		$\Delta_j (z_i - c_j)$	-10	-8	0	0	

Since some of the net evaluation values are negative, the solution is not optimal. Here x_2 is the incoming variable and s_1 is outgoing variable. Pivot value is 2.

Second Simplex Table							
B	C	x_B	x_1	x_2	s_1	s_2	$\theta = x_B/x_2$
x_1	10	14	$2/2 = 1$	$0/2 = 0$	$1/2$	$0/2 = 0$	$14/0 = \text{infinity}$
s_2	0	12	$(-2*1) + 2 = 0$	$(-2*0) + -2 = -2$	$(-2*1/2) + 0 = -1$	$(-2*0) + 1 = 1$	$12/-2 = -6$
		Z_j	10	0	5	0	
		C_j	10	8	0	0	
		$\Delta_j (z_i - c_j)$	0	-8	5	0	

Here, the net evaluation of x_2 is negative and hence the solution is not optimal. x_2 is the incoming variable. The replacement ratio of x_1 and s_2 shows either a negative value or infinity. So is not possible to determine the outgoing variable. The given LPP is a case of LPP with unbounded solution.

LPP with Infeasible Solution

If an artificial variable appears in the final simplex table with a positive value, then it is a case of LPP with infeasible solution.

Question: Solve the LPP using Simplex method:

$$\begin{aligned}
 &\text{Maximize } Z = 10x_1 + 15x_2 \\
 &\text{Subject to: } 2x_1 + x_2 \leq 40 \\
 &\quad \quad \quad 4x_1 - x_2 \leq 20 \\
 &\quad \quad \quad x_1 \geq 30 \\
 &\quad \quad \quad x_1, x_2 \geq 0
 \end{aligned}$$

Solution:

Introduce slack and surplus variables and convert the structural constraints into equations.

$$\begin{aligned}
 2x_1 + x_2 + s_1 &= 40 \\
 4x_1 - x_2 + s_2 &= 20
 \end{aligned}$$

$$x_1 - s_3 = 30$$

The above can be written in an orderly manner:

$$2x_1 + x_2 + 1s_1 + 0s_2 + 0s_3 + 0A_1 = 40$$

$$4x_1 - 1x_2 + 0s_1 + 1s_2 + 0s_3 + 0A_1 = 20$$

$$0x_1 + 0x_2 + 0s_1 + 0s_2 - 1s_3 + 1A_1 = 30$$

Express the constraint equations in the form of matrix:

$$\begin{array}{cccccc|c} \underline{x_1} & \underline{x_2} & \underline{s_1} & \underline{s_2} & \underline{s_3} & \underline{A_1} & \underline{x_B} \\ -2 & 1 & 1 & 0 & 0 & 0 & 2 \\ 1 & -1 & 0 & 1 & 0 & 0 & 2 \\ 3 & 2 & 0 & 0 & -1 & 1 & 9 \end{array}$$

Identify the basic variables. Here, the basic variables are s_1 , s_2 and A_1 . The first basic variable is s_1 , the second basic variable is s_2 and the third is A_1 .

Rewrite the objective function and express it in matrix form:

$$\text{Maximize } Z = 10x_1 + 15x_2 + 0s_1 + 0s_2 + 0s_3 - MA_1$$

$$[10 \quad 15 \quad 0 \quad 0 \quad 0 \quad -M]$$

Draw initial simplex table and find values of net evaluation (Δ_j):

Initial Simplex Table									
B	C	x_B	x_1	x_2	s_1	s_2	s_3	A_1	$\theta = x_B/x_1$
s_1	0	40	2	1	1	0	0	0	$40 \div 2 = 20$
s_2	0	20	4	-1	0	1	0	0	$20/4 = 5 \rightarrow$
A_1	-M	30	1	0	0	0	-1	1	$30/1 = 30$
		Z_j	-M	0	0	0	M	-M	
		C_j	10	15	0	0	0	-M	
		$\Delta_j (z_i - c_j)$	-M-10 ↑	-15	0	0	M	0	

Here incoming variable = x_1 , Outgoing Variable = s_2 and Pivot Value = 4.

Second Simplex Table									
B	C	x_B	x_1	x_2	s_1	s_2	s_3	A_1	$\theta = x_B/x_2$
s_1	0	$(-2*5) + 40 = 30$	$(-2*1) + 2 = 0$	$(-2*-1/4) + 1 = 1.5$	$(-2*0) + 1 = 1$	$(-2*1/4) + 0 = -1/2$	$(-2*0) + 0 = 0$	$(-2*0) + 0 = 0$	$30 \div 1.5 = 20 \rightarrow$
x_1	10	$20/4 = 5$	$4/4 = 1$	-1/4	$0/4 = 0$	1/4	$0/4 = 0$	$0/4 = 0$	$5*4/-1 = -20$
A_1	-M	$(-1*5) + 25 = 25$	$(-1*1) + 1 = 0$	$(-1*-1/4) + 0 = 1/4$	$(-1*0) + 0 = 0$	$(-1*1/4) + 0 = -1/4$	$(-1*0) + -1 = -1$	$(-1*0) + 1 = 1$	$25*4/1 = 100$
		Z_j	10	$-10/4 - M/4$	0	$2.5 + M/4$	M	-M	
		C_j	10	15	0	0	0	-M	
		$\Delta_j (z_i - c_j)$	0	-17.25 - 0.25M ↑	0	$2.5 + 0.25M$	M	0	

Here incoming variable = x_2 , Outgoing Variable = s_1 and Pivot Value = 1.5

B	C	x_B	x_1	x_2	s_1	s_2	s_3	A_1
x_2	15	20	0	1	$2/3$	$-1/3$	0	0
x_1	10	10	1	0	$1/6$	$1/6$	0	0
A_1	-M	20	0	0	$-1/6$	$-1/6$	-1	1
		z_j	10	15	$70/6 + M/6$	$-20/6 + M/6$	M	-M
		c_j	10	15	0	0	0	-M
		$\Delta_j (z_j - c_j)$	0	0	$70/6 + M/6$	$-20/6 + M/6$	M	0

Here all the Δ_j values are either positive or zero. But an artificial variable is present in the basic variable column. Therefore, the LPP has no feasible solution.

Two Phase Method

LPP with artificial variable can also be solved by another method, called two-phase method. Under this method, the optimal solution is obtained in two different phases.

Phase I (Steps):

1. Convert all the structural constraints into equations by incorporating slack, surplus and artificial variables.
2. Express them in matrix form
3. Identify the basic variables
4. Rewrite the objective function by incorporating slack, surplus and artificial variables. But assign '0' as the coefficient values of all the decision variables, slack variables and surplus variables. Also assign '-1' as the coefficient values of all the artificial variables if the objective function is maximization and assign '+1' as the coefficient values of all the artificial variables if the objective function is minimization.
5. Draw the simplex tables one by one until an optimal solution is obtained.

Phase II (Steps):

1. Express the objective function of LPP incorporating slack and surplus variables and not incorporating artificial variables. Here, the coefficient values of decision variables should be their original values and that of slack and surplus variables should be zero.
2. Draw the initial simplex table with all the values (except the values in the 2nd column), of the final simplex table in the first phase. In the second column, the values should be their original values in the objective function, and not zeros.
3. Continue the simplex iteration until an optimal solution is arrived.

Question:

Solve the LPP using two-phase method:

Maximize $Z = 5x_1 + 8x_2$

Subject to:

$$\begin{aligned} 3x_1 + 2x_2 &\geq 3 \\ x_1 + 4x_2 &\geq 4 \end{aligned}$$

$$\begin{aligned} x_1 + x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution:

Convert the structural constraints into equations:

$$3x_1 + 2x_2 - S_1 + A_1 = 3$$

$$x_1 + 4x_2 - S_2 + A_2 = 4$$

$$x_1 + x_2 + S_3 = 5$$

The above equations can be written in matrix form as follows:

$$\begin{array}{ccccccc|c} \underline{x_1} & \underline{x_2} & \underline{s_1} & \underline{s_2} & \underline{s_3} & \underline{A_1} & \underline{A_2} & \underline{x_B} \\ 3 & 2 & -1 & 0 & 0 & 1 & 0 & 3 \\ 1 & 4 & 0 & -1 & 0 & 0 & 1 & 4 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 5 \end{array} =$$

The basic variables are A1, A2 and S3

The objective function may be written as:

$$\text{Maximize } Z = 0x_1 + 0x_2 + 0s_1 + 0s_2 + 0s_3 - A_1 - A_2$$

$$[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -1 \quad -1]$$

The Cj values are (0, 0, 0, 0, 0, -1, -1)

Initial Simplex Table										
B	C	x _B	x ₁	x ₂	s ₁	s ₂	s ₃	A ₁	A ₂	$\theta = x_B/x_2$
A ₁	-1	3	3	2	-1	0	0	1	0	$3 \div 2 = 1.5$
A ₂	-1	4	1	4	0	-1	0	0	1	$4/4 = 1 \rightarrow$
S ₃	0	5	1	1	0	0	1	0	0	$5/1 = 5$
		Z _j	-4	-6	1	1	0	-1	-1	
		C _j	0	0	0	0	0	-1	-1	
		$\Delta_j (z_i - c_j)$	-4	-6 ↑	1	1	0	0	0	

Since some of the (z_j – c_j) values are negative, the solution is optimal. The highest negative value is -6. Therefore, x₂ is the incoming variable and A₂ is the outgoing variable

Second Simplex Table										
B	C	x _B	x ₁	x ₂	s ₁	s ₂	s ₃	A ₁	$\theta = x_B/x_1$	
A ₁	-1	1	5/2	0	-1	1/2	0	1	$1 * 2/5 = 0.4$	\rightarrow
X ₂	0	1	1/4	1	0	-1/4	0	0	$1 * 4/1 = 4$	
S ₃	0	4	3/4	0	0	1/4	1	0	$4 * 4/3 = 5.330$	
		Z _j	-10/4	0	1	-1/2	0	-1		
		C _j	0	0	0	0	0	-1		
		$\Delta_j (z_i - c_j)$	-10/4 ↑	0	1	-1/2	0	0		

The solution is not optimal.

Third Simplex Table							
B	C	x_B	x_1	x_2	s_1	s_2	s_3
x_1	0	2/5	1	0	-2/5	1/5	0
x_2	0	9/10	0	1	1/10	-3/10	0
s_3	0	37/10	0	0	3/10	1/10	1
		Z_j	0	0	0	0	0
		C_j	0	0	0	0	0
		$\Delta_j (z_i - c_j)$	0	0	0	0	0

Since all the net evaluation values are zero, the solution is optimal.
Now, the first phase is over.

Phase 2:

The Objective function is:

$$\text{Maximize } Z = 5x_1 + 8x_2 + 0s_1 + 0s_2 + 0s_3$$

Now, the c_j values are (5, 8, 0, 0, 0).

Draw the initial simplex table with all the values of the final simplex table at the first stage. But in the second column (C), the coefficient values of the Basic variable must be their actual values and not zero as seen in the final simplex table.

First Simplex Table								
B	C	x _B	x ₁	x ₂	s ₁	s ₂	s ₃	θ = x _B /s ₂
x ₁	5	2/5	1	0	-2/5	1/5	0	2/5 * 5/1 = 2 →
x ₂	8	9/10	0	1	1/10	-3/10	0	9/10 * -10/3 = -3
s ₃	0	37/10	0	0	3/10	1/10	1	37/10 * 10/1 = 3.7
		Z _j	5	8	-6/5	-7/5	0	
		C _j	5	8	0	0	0	
		Δ _j (z _i – c _j)	0	0	-6/5	-7/5	0	
↑								

Since net evaluation values involve negative values, the solution is not optimal.
 s_2 and x_1 are incoming and outgoing variables respectively.

Second Simplex Table								
B	C	X _B	x ₁	x ₂	s ₁	s ₂	s ₃	$\theta = X_B/S_1$
S ₁	0	2	5	0	-2	1	0	$2/-2 = -1$
x ₂	8	3/2	3/2	1	-1/2	0	0	$3/2 * -2/1 = -3$
S ₃	0	7/2	-1/2	0	1/2	0	1	$7/2 * 2/1 = 7 \rightarrow$
		Z _j	12	8	-4	0	0	
		C _j	5	8	0	0	0	
		$\Delta_j (z_i - c_j)$	7	0	-4	0	0	
					↑			

Since net evaluation values involve a negative value, the solution is not optimal.

S₁ and S₃ are incoming and outgoing variables respectively.

Third Simplex Table								
B	C	X _B	x ₁	x ₂	s ₁	s ₂	s ₃	$\theta = X_B/S_1$
S ₂	0	16	3	0	0	1	4	$2/-2 = -1$
x ₂	8	5	1	1	0	0	1	$3/2 * -2/1 = -3$
S ₁	0	7	-1	0	1	0	2	$7/2 * 2/1 = 7 \rightarrow$
		Z _j	8	8	0	0	8	
		C _j	5	8	0	0	0	
		$\Delta_j (z_i - c_j)$	3	0	0	0	8	
					↑			

Since all the net evaluation values are either zero or positive, the solution is optimal solution.

The solution is: $x_1 = 0$, $x_2 = 5$, which maximizes $Z = (5*0) + (8*5) = 40$

DUALITY IN LPP

Every LPP is associated with another LPP, called Dual. The original LPP is called Primal. As per duality principle, for every maximization problem in LPP, there is a unique similar problem of minimization involving the same data. Similarly, for every minimization problem in LPP, there is a unique similar problem of maximization involving the same data. Therefore, the optimal solution for primal LPP and dual LPP would be the same, because they originate from the same data.

Procedure for finding Dual LPP

1. If the primal is maximization, the dual is minimization and vice versa; and is denoted by Z'
2. The coefficients of objective function of the primal will become the constants at RHS of constraints of the dual.
3. The constants at the RHS of constraints of the primal will become the coefficients of the objective function of the dual.
4. The transpose of coefficients of the constraints of primal will become the coefficients of the constraints of the dual.
5. The symbol, ' \leq ' in the constraints of primal will become ' \geq ' symbol in the constraints of the dual.
6. The symbol, ' \geq ' in the constraints of primal will become ' \leq ' symbol in the constraints of the dual.

7. Take y_1, y_2, y_3 , etc, as the decision variables of the Dual.

Pre requisite for conversion of a Primal into Dual

- ✓ If the primal is with maximization objective function, then all the structural constraints must be with ' \leq ' symbol.
- ✓ If the primal is with minimization objective function, then all the structural constraints must be with ' \geq ' symbol.

Note: If the constraints do not possess the required symbol, they may be multiplied by -1 to make them as constraints with required symbol.

Question: Convert the following LPP into a dual:

$$\begin{aligned} \text{Maximize } Z &= 3x_1 + x_2 + 2x_3 \\ \text{Subject to: } &x_1 + x_2 + x_3 \leq 5 \\ &2x_2 + x_3 \leq 10 \\ &x_2 + 3x_3 \leq 15 \\ &x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution:

$$\begin{aligned} \text{Minimize } Z' &= 5y_1 + 10y_2 + 15y_3 \\ \text{Subject to: } &y_1 + 2y_2 \geq 3 \\ &y_1 + y_3 \geq 1 \\ &y_1 + y_2 + 3y_3 \geq 2 \\ &y_1, y_2, y_3 \geq 0 \end{aligned}$$

REVIEW QUESTIONS:

Q1. A soft drinks company has a two products viz. Coco-cola and Pepsi with profit of \$2 and \$1 per unit. The following table illustrates the labour, equipment and materials to produce per unit of each product. Determine suitable product mix which maximizes the profit using simplex method.

	Pepsi Cola	Coco Cola	Total Resources
Labour	3	2	12
Equipment	1	2.3	6.9
Material	1	1.4	4.9

Q2. A factory produces three using three types of ingredients viz. A, B and C in different proportions. The following table shows the requirements of various ingredients as inputs per kg of the products:

Products	Ingredients		
	A	B	C
1	4	8	8
2	4	6	4
3	8	4	0

The three profits coefficients are 20, 20 and 30 respectively. The factory has 800 kg of ingredients A, 1800 kg of ingredients B and 500 kg of ingredient C.

Determine the product mix which will maximize the profit and also find out maximum profit.

Q3. Solve the following linear programming problem using two phase and M method.

$$\text{Maximize } Z = 12x_1 + 15x_2 + 9x_3$$

Subject to:

$$8x_1 + 16x_2 + 12x_3 \leq 250$$

$$4x_1 + 8x_2 + 10x_3 \geq 80$$

$$7x_1 + 9x_2 + 8x_3 = 105$$

$$x_1, x_2, x_3 \geq 0$$

Q4. Solve the following linear programming problem using simplex method.

$$\text{Maximize } 3x_1 + 2x_2$$

Subject to:

$$x_1 - x_2 \leq 1$$

$$x_1 + x_2 \geq 3$$

$$x_1, x_2 \geq 0$$

Q5. Solve the following linear programming problem using simplex method.

Maximize

$$x_1 + x_2$$

Subject to:

$$-2x_1 + x_2 \leq 1$$

$$x_1 \leq 2$$

$$x_1 + x_2 \leq 3$$

$$x_1, x_2, x_3 \geq 0$$

Q6. Maximize $Z = 3x_1 + 4x_2 + x_3$

Subject to: $x_1 + 2x_2 + x_3 \leq 6$

$$2x_1 + 2x_3 \leq 4$$

$$3x_1 + x_2 + x_3 \leq 9$$

$$x_1, x_2, x_3 \geq 0$$

CHAPTER 4

TRANSPORTATION PROBLEM

Introduction

Transportation Problem is a special class of LPP, where the objective is to minimize the cost of distributing a product from a number of Sources (e.g. factories) to a number of Destinations (e.g. warehouses) while satisfying both the supply limits and the demand requirement. Because of the special structure of the Transportation Problem the Simplex Method of solving is unsuitable for the Transportation Problem. The model assumes that the distributing cost on a given route is directly proportional to the number of units distributed on that route. Generally, the transportation model can be extended to areas other than the direct transportation of a commodity, including among others, inventory control, employment scheduling, and personnel assignment.

The transportation problem special feature is illustrated here with the help of following example:
Suppose a manufacturing company owns three factories (sources) and distribute his products to five different retail agencies (destinations). The following table shows the capacities of the three factories, the quantity of products required by the various retail agencies and the cost of shipping one unit of the product from each of the three factories to each of the five retail agencies.

Factories	Retail Agency					Capacity
	1	2	3	4	5	
1	1	9	13	36	51	50
2	24	12	16	20	1	100
3	14	33	1	23	26	150
Requirement	100	60	50	50	40	300

Usually the above table is referred as Transportation Table, which provides the basic information regarding the transportation problem. The quantities inside the table are known as transportation cost per unit of product. The capacity of the factories 1, 2, 3 is 50, 100 and 150 respectively. The requirement of the retail agency 1, 2, 3, 4, 5 is 100,60,50,50, and 40 respectively.

In this case, the transportation cost of one unit
 from factory 1 to retail agency 1 is 1,
 from factory 1 to retail agency 2 is 9,
 from factory 1 to retail agency 3 is 13, and so on.

A transportation problem can be formulated as linear programming problem using variables with two subscripts.

Let

x_{11} =Amount to be transported from factory 1 to retail agency 1

x_{12} = Amount to be transported from factory 1 to retail agency 2

.....

.....

.....

.....

x_{35} = Amount to be transported from factory 3 to retail agency 5

Let the transportation cost per unit be represented by $C_{11}, C_{12}, \dots, C_{35}$ that is $C_{11}=1, C_{12}=9$, and so on.

Let the capacities of the three factories be represented by $a_1=50, a_2=100, a_3=150$.

Let the requirement of the retail agencies are $b_1=100, b_2=60, b_3=50, b_4=50$, and $b_5=40$.

Thus, the problem can be formulated as:

$$\text{Minimize } Z = C_{11}x_{11} + C_{12}x_{12} + \dots + C_{35}x_{35}$$

Subject to:

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = a_1$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = a_2$$

$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = a_3$$

$$x_{11} + x_{21} + x_{31} = b_1$$

$$x_{12} + x_{22} + x_{32} = b_2$$

$$x_{13} + x_{23} + x_{33} = b_3$$

$$x_{14} + x_{24} + x_{34} = b_4$$

$$x_{15} + x_{25} + x_{35} = b_5$$

$$x_{11}, x_{12}, \dots, x_{35} \geq 0.$$

Thus, the problem has 8 constraints and 15 variables. So, it is not possible to solve such a problem using the Simplex method. This is the reason for the need of special computational procedure to solve transportation problem. There are varieties of procedures, which are described in the next section.

Transportation Algorithm

The steps of the transportation algorithm are exact parallels of the simplex algorithm, they are:

Step 1: Determine a starting basic feasible solution, using any one of the following three methods

North West Corner Method

Least Cost Method

Vogel Approximation Method

Step 2: Determine the optimal solution using the following method 1.
MODI (Modified Distribution Method) or UV Method.

Basic Feasible Solution of a Transportation Problem

The computation of an initial feasible solution is illustrated in this section with the help of the example discussed in the previous section. The problem in the example has 8 constraints and 15 variables we can eliminate one of the constraints since $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 + b_4 + b_5$. Thus now the problem contains 7 constraints and 15 variables. Note that any initial (basic) feasible solution has at most 7 non-zero X_{ij} . Generally, any basic feasible solution with m sources (such as factories) and n destination (such as retail agency) has at most $m + n - 1$ non-zero X_{ij} .

The special structure of the transportation problem allows securing a non artificial basic feasible solution using one the following three methods.

North West Corner Method

Least Cost Method

Vogel Approximation Method

The difference among these three methods is the quality of the initial basic feasible solution they produce, in the sense that a better that a better initial solution yields a smaller objective value. Generally the Vogel Approximation Method produces the best initial basic feasible solution, and the North West Corner Method produces the worst, but the North West Corner Method involves least computations.

North West Corner Method:

The method starts at the North West (upper left) corner cell of the tableau (variable x_{11}).

Step 1: Allocate as much quantity as possible to the north-west corner cell, which is equal to the minimum of row total and column total. So either a row total or column total gets exhausted. So cross off that row or column as the case may be.

Step 2: Draw the reduced matrix. In hat allocate much quantity as possible to the north-west corner cell, which is equal to the minimum of row total and column total. So either a row total or column total gets exhausted. So cross off that row or column as the case may be.

Step 3: Repeat the process until all the quantities are exhausted.

Consider the problem discussed in above example to illustrate the North West Corner Method of determining basic feasible solution.

Factories	Retail Agency					Capacity
	1	2	3	4	5	
1	1	9	13	36	51	50
2	24	12	16	20	1	100
3	14	33	1	23	26	150
Requirement	100	60	50	50	40	300

Table 1

	1	2	3	4	5	Capacity
1	1 50	9	13	36	51	50
2	24	12	16	20	1	100

3	14	33	1	23	26	150
Requirement	100	60	50	50	40	300

Table 2

	1	2	3	4	5	Capacity
2	24 50	12	16	20	1	100
3	14	33	1	23	26	150
Requirement	50	60	50	50	40	250

Table 3

	2	3	4	5	Capacity
2	12 50	16	20	1	50
3	33	1	23	26	150
Requirement	60	50	50	40	200

Table 4

	2	3	4	5	Capacity
3	33 10	1	23	26	150
Requirement	10	50	50	40	150

Table 5

	3	4	5	Capacity
3	1 50	23	26	140
Requirement	50	50	40	140

Table 6

	4	5	Capacity
3	23 50	26	90
Requirement	50	40	90

Table 8

	5	Capacity
3	26 40	40
Requirement	40	40

Consolidated Table 1

	1	2	3	4	5	Capacity
1	1 50	9	13	36	51	50
2	24 50	12 50	16	20	1	100
3	14	33 10	1 50	23 50	26 40	150
Requirement	100	60	50	50	40	300

The solution is :

Transport 50 Units from Factory 1 to Retail Agency 1, Cost = 50 * Rs:1 = 50

Transport 50 Units from Factory 2 to Retail Agency 1, Cost = 50 * Rs:24 = 1200

Transport 50 Units from Factory 2 to Retail Agency 2, Cost = 50 * Rs:12 = 600

Transport 10 Units from Factory 3 to Retail Agency 2, Cost = 10 * Rs:33 = 330

Transport 50 Units from Factory 3 to Retail Agency 3, Cost = 50 * Rs:1 = 50

Transport 50 Units from Factory 3 to Retail Agency 4, Cost = 50 * Rs:23 = 1150

Transport 40 Units from Factory 3 to Retail Agency 5, Cost = 40 * Rs:26 = 1040

Total Transportation Cost is = 4420

Least Cost Method

The least cost method is also known as matrix minimum method in the sense we look for the row and the column corresponding to which C_{ij} is minimum. This method finds a better initial basic feasible solution by concentrating on the cheapest routes. Under this method, instead of allocating at the northwest cell as in the North West Corner Method, we allocate as much quantity as possible to that cell which has the smallest unit cost. All other steps are same as in the case of NWCM.

The least cost method of determining initial basic feasible solution is illustrated with the help of problem presented in the above section:

Factories	Retail Agency					Capacity
	1	2	3	4	5	
1	1	9	13	36	51	50
2	24	12	16	20	1	100
3	14	33	1	23	26	150
Requirement	100	60	50	50	40	300

Table 1

	1	2	3	4	5	Capacity
1	1 50	9	13	36	51	50
2	24	12	16	20	1	100
3	14	33	1	23	26	150
Requirement	100	60	50	50	40	300

Table 2

	1	2	3	4	5	Capacity
2	24	12	16	20	1	100
3	14	33	1 50	23	26	150
Requirement	50	60	50	50	40	250

Table 3

	1	2	4	5	Capacity
2	24	12	20	1 40	100
3	14	33	23	26	100
Requirement	50	60	50	40	200

Table 4

	1	2	4	Capacity
2	24	12 60	20	60
3	14	33	23	100
Requirement	50	60	50	160

Table 5

	1	4	Capacity
3	14 50	23	100
Requirement	50	50	100

Table 6

	4	Capacity
3	23 50	50
Requirement	50	50

Consolidated Table

	1	2	3	4	5	Capacity
1	1 50	9	13	36	51	50
2	24	12 60	16	20	1 40	100
3	14 50	33	1 50	23 50	26	150
Requirement	100	60	50	50	40	300

So the solution is:

Transport 50 Units from Factory 1 to Retail Agency 1, Cost = $50 * \text{Rs:}1 = 50$

Transport 60 Units from Factory 2 to Retail Agency 2, Cost = $60 * \text{Rs:}12 = 720$

Transport 40 Units from Factory 2 to Retail Agency 5, Cost = $40 * \text{Rs:}1 = 40$

Transport 50 Units from Factory 3 to Retail Agency 1, Cost = $50 * \text{Rs:}14 = 700$

Transport 50 Units from Factory 3 to Retail Agency 3, Cost = $50 * \text{Rs:}1 = 50$

Transport 50 Units from Factory 3 to Retail Agency 4, Cost = $50 * \text{Rs:}23 = 1150$

Total Transportation Cost is $= \underline{2710}$

Vogel Approximation Method (VAM):

VAM is an improved version of the least cost method that generally produces better solutions. The steps involved in this method are:

1. Find penalty in respect of each row and column. Penalty is the difference between the lowest and the next lowest costs
2. Select the row or column having the largest penalty and allocate as many quantity as possible to the cell with lowest cost in that row or column as the case may be. So either a row total or column total gets exhausted. So cross off that row or column as the case may be.
3. Draw the reduced matrix and follow the above steps until the total of all rows and columns are exhausted.

Example: Consider the following transportation problem:

Origin	Destination				A _i
	1	2	3	4	
1	20	22	17	4	120
2	24	37	9	7	70
3	32	37	20	15	50
b _j	60	40	30	110	240

Solution:

Table 1

Origin	1	2	3	4	A _i	
1	20	22 40	17	4	120	(13)
2	24	37	9	7	70	(2)
3	32	37	20	15	50	(5)
b _j	60	40	30	110	240	
P	(4)	(15)	(8)	(3)		

Table 2

Origin	1	3	4	A _i	
			4		(13)
1	20	17	80	80	
2	24	9	7	70	(2)
3	32	20	15	50	(5)
b _j	60	30	110	200	
P	(4)	(8)	(3)		

Table 3

Origin	1	3	4	A _i	
		9			(2)
2	24	30	7	70	
3	32	20	15	50	(5)
b _j	60	30	30	120	
P	(8)	(11)	(8)		

Table 4

Origin	1	4	A _i	
		7		(17)
2	24	30	40	
3	32	15	50	(17)
b _j	60	30	90	
P	(8)	(8)		

Table 5

Origin	1	A _i	
2	24	10	(24)
3	32 50	50	(32)
b _j	60	60	
P	(8)		

Table 5

Origin	1	A_i	
2	24 10	10	(24)
b_j	10	10	
P	(24)		

Consolidated Table

Origin	Destination				A_i
	1	2	3	4	
1	20 24	22 40	17	4 80	120
2	10	37	9 30	7 30	70
3	50	32 37	20	15	50
b_j	60	40	30	110	240

$$\begin{aligned} \text{Total Transportation Cost} &= (40 \times 22) + (80 \times 4) + (30 \times 9) + (30 \times 7) + (50 \times 32) + (10 \times 24) \\ &= \underline{\underline{\text{Rs. 3,520}}} \end{aligned}$$

Modified Distribution Method (MODI Method)

The Modified Distribution Method, also known as MODI method or u-v method, which provides a minimum cost solution (optimal solution) to the transportation problem. The following are the steps involved in this method.

Step 1: Find out the basic feasible solution of the transportation problem using any one of the three methods discussed in the previous section.

Step 2: Introduce dual variables corresponding to the row constraints and the column constraints. If there are m origins and n destinations then there will be $m+n$ dual variables. The dual variables corresponding to the row constraints are represented by u_i , $i=1,2,\dots,m$ where as the dual variables corresponding to the column constraints are represented by v_j , $j=1,2,\dots,n$. The values of the dual variables are calculated from the equation given below $u_i + v_j = c_{ij}$ if $x_{ij} > 0$

Step 3: Any basic feasible solution has $m + n - 1$ $x_{ij} > 0$. Thus, there will be $m + n - 1$ equation to determine $m + n$ dual variables. One of the dual variables can be chosen arbitrarily. It is also to be noted that as the primal constraints are equations, the dual variables are unrestricted in sign.

Step 4: If $x_{ij}=0$, the dual variables calculated in Step 3 are compared with the c_{ij} values of this allocation as $c_{ij} - u_i - v_j$. If $\text{all } c_{ij} - u_i - v_j \geq 0$, then by the *theorem of complementary slackness* it can be shown that the corresponding solution of the transportation problem is optimum. If one or more $c_{ij} - u_i - v_j < 0$, we select the cell with the least value of $c_{ij} - u_i - v_j$ and allocate as much as possible subject to the row and column constraints. The allocations of the number of adjacent cell are adjusted so that a basic variable becomes non-basic.

Step 5: A fresh set of dual variables are calculated and repeat the entire procedure from Step 1 to Step 5.

Example: Consider the transportation problem given below:

						Supply
	1	9	13	36	51	50
	24	12	16	20	1	100
	14	33	1	23	26	150
Demand	100	70	50	40	40	300

Step 1: First we have to determine the basic feasible solution. The basic feasible solution using least cost method is

$$x_{11}=50, x_{22}=60, x_{25}=40, x_{31}=50, x_{32}=10, x_{33}=50 \text{ and } x_{34}=40$$

Step 2: The dual variables u_1, u_2, u_3 and v_1, v_2, v_3, v_4, v_5 can be calculated from the corresponding c_{ij} values, that is

$$\begin{array}{llll} u_1+v_1=1 & u_2+v_2=12 & u_2+v_5=1 & u_3+v_1=14 \\ u_3+v_2=33 & u_3+v_3=1 & u_3+v_4=23 & \end{array}$$

Step 3: Choose one of the dual variables arbitrarily is zero that is $u_3=0$ as it occurs most often in the above equations. The values of the variables calculated are

$$\begin{array}{l} u_1 = -13, u_2 = -21, u_3 = 0 \\ v_1 = 14, v_2 = 33, v_3 = 1, v_4 = 23, v_5 = 22 \end{array}$$

Step 4: Now we calculate $c_{ij} - u_i - v_j$ values for all the cells where $x_{ij}=0$ (.e. unallocated cell by the basic feasible solution)

That is

$$\begin{array}{l} \text{Cell}(1,2) = c_{12} - u_1 - v_2 = 9 + 13 - 33 = -11 \\ \text{Cell}(1,3) = c_{13} - u_1 - v_3 = 13 + 13 - 1 = 25 \\ \text{Cell}(1,4) = c_{14} - u_1 - v_4 = 36 + 13 - 23 = 26 \\ \text{Cell}(1,5) = c_{15} - u_1 - v_5 = 51 + 13 - 22 = 42 \\ \text{Cell}(2,1) = c_{21} - u_2 - v_1 = 24 + 21 - 14 = 31 \\ \text{Cell}(2,3) = c_{23} - u_2 - v_3 = 16 + 21 - 1 = 36 \\ \text{Cell}(2,4) = c_{24} - u_2 - v_4 = 20 + 21 - 23 = 18 \\ \text{Cell}(3,5) = c_{35} - u_3 - v_5 = 26 - 0 - 22 = 4 \end{array}$$

Note that in the above calculation all the $c_{ij} - u_i - v_j \geq 0$ except for cell (1, 2) where $c_{12} - u_1 - v_2 = 9 + 13 - 33 = -11$.

Thus in the next iteration x_{12} will be a basic variable changing one of the present basic variables non-basic. We also observe that for allocating one unit in cell (1, 2) we have to reduce one unit in cells (3, 2) and (1, 1) and increase one unit in cell (3, 1). The net transportation cost for each unit of such reallocation is:

$$-33 - 1 + 9 + 14 = -11$$

The maximum that can be allocated to cell (1, 2) is 10 otherwise the allocation in the cell (3, 2) will be negative. Thus, the revised basic feasible solution is

$$x_{11}=40, x_{12}=10, x_{22}=60, x_{25}=40, x_{31}=60, x_{33}=50, x_{34}=40$$

Unbalanced Transportation Problem

In the previous section we discussed about the balanced transportation problem i.e. the total supply (capacity) at the origins is equal to the total demand (requirement) at the destination. In this section we are going to discuss about the unbalanced transportation problems i.e. when the total supply is not equal to the total demand, which are called as **unbalanced transportation problem**.

In the unbalanced transportation problem if the total supply is more than the total demand then we introduce an additional column which will indicate the surplus supply with transportation cost zero. Similarly, if the total demand is more than the total supply an additional row is introduced in the transportation table which indicates unsatisfied demand with zero transportation cost.

Example:

Consider the following unbalanced transportation problem

Plant	Warehouses			Supply
	W1	W2	W3	
X	20	17	25	400
Y	10	10	20	500
Demand	400	400	500	

In this problem the demand is 1300 whereas the total supply is 900. Thus, we now introduce an additional row with zero transportation cost denoting the unsatisfied demand. So that the modified transportation problem table is as follows:

Plant	Warehouses			Supply
	W1	W2	W3	
X	20	17	25	400
Y	10	10	20	500
	0	0	0	400
Demand	400	400	500	1300

Now we can solve as balanced problem discussed as in the previous sections.

Degenerate Transportation Problem

In a T P, if a basic feasible solution with m origins and n destinations has less than $m+n-1$, occupied cells, then the problem is said to be a **degenerate transportation problem**. The degeneracy problem does not cause any serious difficulty, but it can cause computational problem while determining the optimal minimum solution. Therefore, it is important to identify a degenerate problem as early as beginning and take the necessary action to avoid any computational difficulty. The degeneracy can be identified through the following results:

“In a transportation problem, a degenerate basic feasible solution exists if and only if some partial sum of supply (row) is equal to a partial sum of demand (column). For example the following transportation problem is degenerate. Because in this problem

$$a_1 = 400 = b_1$$

$$a_2 + a_3 = 900 = b_2 + b_3$$

Plant	Warehouses			Supply
	W1	W2	W3	
X	20	17	25	400
Y	10	10	20	500
Unsatisfied Demand	0	0	0	400
Demand	400	400	500	1300

There is a technique called perturbation, which helps to solve the degenerate problems.

Perturbation Technique:

The degeneracy of the transportation problem can be avoided if we ensure that no partial sum of a_i (supply) and b_j (demand) is equal. We set up a new problem where

$$\begin{aligned} a_i &= a_i + d & i &= 1, 2, \dots, m \\ b_j &= b_j & j &= 1, 2, \dots, n-1 \\ b_n &= b_n + m_d & d &> 0 \end{aligned}$$

This modified problem is constructed in such a way that no partial sum of a_i is equal to the b_j . Once the problem is solved, we substitute $d = 0$ leading to optimum solution of the original problem.

Example:

Consider the above problem

Plant	Warehouses			Supply
	W1	W2	W3	
X	20	17	25	$400 + d$
Y	10	10	20	$500 + d$
Unsatisfied Demand	0	0	0	$400 + d$
Demand	400	400	$500 + 3d$	$1300 + 3d$

Now this modified problem can be solved by using any of the three methods viz. North-west Corner, Least Cost, or VAM.

Transportation Problem with Maximization Objective Function

There are certain types of transportation problem where the objective function is to be maximized instead of minimized. These kinds of problems can be solved by converting the maximization problem into minimization problem. The conversion of maximization into minimization is done by subtracting the unit costs from the highest unit cost of the table. The maximization of transportation problem is illustrated with the following example:

Example: A company has three factories located in three cities viz. X, Y, Z. These factories supplies consignments to four dealers viz. A, B, C and D. The dealers are spread all over the country. The production capacity of these factories is 1000, 700 and 900 units per month respectively. The net return per unit product is given in the following table:

Factory	Dealers				Capacity
	A	B	C	D	
X	6	6	6	4	1000
Y	4	2	4	5	700
Z	5	6	7	8	900
Requirement	900	800	500	400	2600

Determine a suitable allocation to maximize the total return.

This is a maximization problem. Hence first we have to convert this in to minimization problem. The conversion of maximization into minimization is done by subtracting the unit cost of the table from the highest unit cost.

Look the table, here 8 is the highest unit cost. So, subtract all the unit cost from the 8, and then we get the revised minimization transportation table, which is given below:

Factory	Dealers				Capacity
	A	B	C	D	
X	2	2	2	4	1000 = a ₁
Y	4	6	4	3	700 = a ₂
Z	3	2	1	0	900 = a ₃
Requirement	900=b ₁	800=b ₂	500=b ₃	400=b ₄	2600

Now we can solve the problem as a minimization problem.

The problem here is degenerate, since the partial sum of $a_1=b_2+b_3$ or $a_3=b_3$. So consider the corresponding perturbed problem, which is shown below:

Factory	Dealers				Capacity
	A	B	C	D	
X	2	2	2	4	1000 + d
Y	4	6	4	3	700 + d
Z	3	2	1	0	900 + d
Requirement	900	800	500	400 + 3d	2600 + 3d

First we have to find out the basic feasible solution. The basic feasible solution by lowest cost method is $x_{11}=100+d$, $x_{22}=700-d$, $x_{23}=2d$, $x_{33}=500-2d$ and $x_{34}=400+3d$.

Once if the basic feasible solution is found, next we have to determine the optimum solution using MODI (Modified Distribution Method) method. By using this method we obtain

$$\begin{array}{lll} u_1+v_1=2 & u_1+v_2=2 & u_2+v_2=6 \\ u_2+v_3=4 & u_3+v_3=1 & u_3+v_4=0 \end{array}$$

Taking $u_1=0$ arbitrarily we obtain

$$\begin{array}{l} u_1=0, u_2=4, u_3=1 \text{ and} \\ v_1=2, v_2=3, v_3=0 \end{array}$$

On verifying the condition of optimality, we know that

$$C_{12}-u_1-v_2 < 0 \quad \text{and} \quad C_{32}-u_3-v_2 < 0$$

So, we allocate $x_{12}=700-d$ and make readjustment in some of the other basic variables.

The revised values are:

$$x_{11}=200+d, x_{12}=800, x_{21}=700-d, x_{23}=2d, x_{33}=500-3d, \text{ and } x_{34}=400+3d$$

$$u_1+v_1=2 \quad u_1+v_2=2 \quad u_2+v_1=4$$

$$u_2 + v_3 = 4 \quad u_3 + v_3 = 1 \quad u_3 + v_4 = 0$$

Taking $u_1 = 0$ arbitrarily we obtain

$$u_1 = 0, u_2 = 2, u_3 = -1$$

$$v_1 = 2, v_2 = 2, v_3 = 2, v_4 = 1$$

Now, the optimality condition is satisfied.

Finally, taking $d = 0$ the optimum solution of the transportation problem is

$$x_{11} = 200, x_{12} = 800, x_{21} = 700, x_{33} = 500 \text{ and } x_{34} = 400$$

Thus, the maximum return is:

$$(6 \times 200) + (6 \times 800) + (4 \times 700) + (7 \times 500) + (8 \times 400) = \underline{15500}$$

REVIEW QUESTIONS:

1. What do you mean by Transportation Problem?
2. Explain the algorithm of solving Transportation Problem.
3. What do you mean by Basic Feasible Solution to a Transportation Problem?
4. What are the different methods to find a BFS to TP?
5. What is NWCM? Explain the procedure of finding initial feasible solution.
6. What is Lowest Cost Entry Method? Explain the procedure of finding initial feasible solution.
7. What is VAM? Explain the procedure of finding initial feasible solution.
8. What are the different methods to find optimal solution to TP?
9. What is MODI Method? Explain the procedure.
10. What is Stepping Stone Method? Explain the procedure.
11. What do you mean by unbalanced TP?
12. What do you mean by degeneracy in TP?
13. Four companies viz. W, X, Y and Z supply the requirements of three warehouses viz; A, B and C respectively. The companies' availability, warehouses requirements and the unit cost of transportation are given in the following table. Find an initial basic feasible solution using:
North West Corner Method
Least Cost Method
Vogel Approximation Method (VAM)

Company	Warehouses			Supply
	A	B	C	
W	10	8	9	15
X	5	2	3	20
Y	6	7	4	30
Z	7	6	9	35
Requirement	25	26	49	100

14. Find the optimum Solution of the following Problem using MODI method.

Source	Destination			Capacity
	1	2	3	
A 42	8	9	10	30
B				
C	9	11	11	28

	10	12	9	
Demand	35	40	25	100

15. The ABT transport company ships truckloads of food grains from three sources viz. X, Y, Z to four mills viz. A, B, C, D respectively. The supply and the demand together with the unit transportation cost per truckload on the different routes are described in the following transportation table. Assume that the unit transportation costs are in hundreds of dollars. Determine the optimum minimum shipment cost of transportation using MODI method.

Source	Mill				Supply
	A	B	C	D	
X	10	2	20	11	15
Y	12	7	9	20	25
Z	4	14	16	18	10
Demand	5	15	15	15	

16. An organization has three plants at X, Y, Z which supply to warehouses located at A, B, C, D, and E respectively. The capacity of the plants is 800, 500 and 900 per month and the requirement of the warehouses is 400, 400, 500, 400 and 800 units respectively. The following table shows the unit transportation cost.

	A	B	C	D	E
X	Rs:5	Rs:8	Rs:6	Rs:6	Rs:3
Y	Rs:4	Rs:7	Rs:7	Rs:6	Rs:6
Z	Rs:8	Rs:4	Rs:6	Rs:6	Rs:3

Determine an optimum distribution for the organization in order to minimize the total cost of transportation.

17. Solve the following transshipment problem

Consider a transportation problem has two sources and three depots. The availability, requirements and unit cost are as follows:

Source	Depot			Availability
	D1	D2	D3	
S1	9	8	1	30

S2	1	7	8	30
Requirement	20	20	20	60

In addition to the above, suppose that the unit cost of transportation from source to source and from depot to depot are as:

		Source	
		S1	S2
Source	S1	0	1
	S2	2	0

		Depot		
		D1	D2	D3
Depot	D1	0	2	1
	D2	2	0	9
	D3	1	9	0

Find out minimum transshipment cost of the problem and also compare this cost with the corresponding minimum transportation cost.

18. Saravana Store, T.Nagar, Chennai interested to purchase the following type and quantities of dresses

Dress Type	V	W	X	Y	Z
Quantity	150	100	75	250	200

Four different dress makers are submitted the tenders, who undertake to supply not more than the quantities indicated below:

Dress Maker	A	B	C	D
Dress Quantity	300	250	150	200

Saravana Store estimates that its profit per dress will vary according to the dress maker as indicates in the following table:

V	W	X	Y	Z
---	---	---	---	---

A	2.75	3.5	4.25	2.25	1.5
B	3	3.25	4.5	1.75	1
C	2.5	3.5	4.75	2	1.25
D	3.25	2.75	4	2.5	1.75

Determine how should the orders to be places for the dresses so as to maximize the profit.

CHAPTER 5

ASSIGNMENT PROBLEMS

Assignment model deals in allocating the various resources or items to various activities on one to one basis to each way that the time or cost involved is minimized and sale or profit is maximized. Such types of problem can also be solved with the help of simplex method or by transportation method, but simpler and more efficient methods for getting the solution is available through assignment models.

Several problems of management have a structure identical with the assignment Transportation problem. A department head may have six people available for assignment and six jobs to assign. He may like to know which job should be assigned to which person so that all these jobs can be completed in the shortest possible time.

Assignment problem is a special case of the transportation problem in which the objective is to assign a number of origins (or persons) to the equal number of destinations (or tasks) at a minimum cost.

Assumptions in assignment problem

1. There are finite number of persons and jobs
2. Number of persons must be equal to number of jobs.
3. All the persons are capable of taking up all the jobs, with different time or cost.
4. Number of columns is always equal to number of rows.
5. There is exactly one occupied cell in each row and each column of the table.

Difference between Transportation Problem and Assignment Problem

Both Transportation problem and Assignment problems are special type of linear programming problems. They deal in allocating various resources to various activities, so as to minimize time or cost. However there are following differences between them:

- (1) Transportation problem is one of the sub classes of linear programming problems in which the objective is to transport various quantum of a commodity that are initially stored at various origins to different destinations in such a way that the transportation cost is minimum. The assignment problem is a special case of Transportation problem in which the objective is to assign a number of origins to the equal number of destinations at a minimum cost.
- (2) In Transportation problems number of rows and number of columns need not be equal. In Assignment problems the number of persons and number of tasks are equal so that number of rows and number of columns are equal.
- (3) Transportation problems are said to be unbalanced if the total demand and total supply are not equal while Assignment problems are unbalanced when the number of rows are not equal to number of columns.
- (4) In Transportation problems a positive quantity is allocated from a source (origin) to a destination. In Assignment problems a source (job) is assigned to a destination (a man)

Methods of finding solution to assignment problem

An assignment problem can be solved by using any one of the following four methods. However, the Hungarian method is popularly used for finding solution to the assignment problems:

1. Enumeration Method
2. Simplex Method
3. Transportation Method
4. Hungarian Method

Hungarian Method

The Hungarian method is developed by a Mathematician, D Konig. He stated a theorem for the method of modifying the rows and columns of matrix until there is at least one zero component in each row and column so that a complete assignment corresponding to the zero can be made. So, the complete method is concentrated on reducing the given matrix. If one could reduce the matrix to the extent of having at least one zero in each row and column, it is possible to have optimal assignments.

Procedure:

1. Develop the cost matrix of the given problem and ensure that it is a square matrix (i.e; the number of sources is equal to the number of destinations).
2. If cost matrix is not a square matrix, add a dummy source/destination for making it a square matrix.
3. Select the smallest element in each row of given cost matrix and then subtract it from each element of that row.
4. In the reduced matrix of step 3, select the smallest element of each column and then subtract it from each element of that column. Each row and column will have at least one zero.
5. In the modified matrix of step 4, search an optimal assignment as follows:
 - (a) Examine the rows until a row with a single zero is found. Highlight this zero by using a square symbol and cross (X) all the other zeros in its column. Continue this process for all the rows.
 - (b) Repeat the above process for each column.
 - (c) If a row or column has more than one zeros, any one of the zero could be selected or highlighted and the remaining would be crossed out.
 - (d) Repeat the steps (a) to (c) until the chain of assignments (marked by “square symbol”) or cross (marked by “X” symbol) ends.
6. An optimal solution is considered to be reached, if the numbers of assignments are equal to the order of matrix. Else if the numbers of assignments are less than the order of the matrix, go to the next step.
7. Draw the minimum number of horizontal and/or vertical lines to cover all the zeros of matrix, by the following steps:
 - (a) Mark “√” on the rows that do not have any assigned zero
 - (b) Mark “√” on the columns that have zeros in marked rows.
 - (c) Mark “√” on the rows that have assigned zeros in marked columns
 - (d) Repeat (b) and (c) until the chain of marking is completed.
 - (e) Draw straight lines through all the unmarked rows and marked columns.
8. Generate a new matrix as follows:
 - (a) Select the smallest element of matrix not covered by any of the lines.
 - (b) Subtract this element from all the uncovered elements and add the same to all the elements lying at the intersection of any two lines.
9. Go to step 6 and repeat the process until an optimum solution is obtained.

Question: Find the optimal solution for the following assignment problem:

Workers	Job		
	P	Q	R
A	18	17	16
B	15	13	14
C	19	20	21

Solution:

Subtract the smallest element of every row from all the elements of that row:

Workers	Job		
	P	Q	R
A	2	1	0
B	2	0	1
C	0	1	2

Subtract the smallest element of every column from all the elements of that column:

Workers	Job		
	P	Q	R
A	2	1	0
B	2	0	1
C	0	1	2

The solution is:

Assign worker A to Job R

Assign worker B to Job Q, and

Assign worker C to job P

Total assignment cost = $16 + 13 + 19 = \text{Rs: } 48$

Question:

Assign the workers in such a way that the total time (days) for the completion of the project is minimised:

Task	Workers			
	A	B	C	D
P	12	10	8	9
Q	8	9	11	7
R	11	14	12	10
S	9	9	8	9

Solution:

Subtract the smallest element of every row from all the elements of that row:

Table I

Task	Workers			
	A	B	C	D
P	4	2	0	1
Q	1	2	4	0
R	1	4	2	0
S	1	1	0	1

Subtract the smallest element of every column from all the elements of that column:

Table II

Task	Workers			
	A	B	C	D
P	3	1	0	1
Q	0	1	4	0
R	0	3	2	0
S	0	0	0	1

Make assignments in one zero rows and then one zero columns:

Table III

Task	Workers			
	A	B	C	D
P	3	1	0	1
Q	0	1	4	0
R	0	3	2	0
S	0	0	0	1

Select arbitrarily the first zero of the second row of Table III and complete the assignment:

Table IV

Task	Workers			
	A	B	C	D
P	3	1	0	1
Q	0	1	4	0
R	0	3	2	0
S	0	0	0	1

The solution is:

P to C, Q to A, R to D and S to B. Total Time = $8 + 8 + 10 + 9 = 35$ days

We can have an alternative solution if we had selected last zero of second row in Table IV.

Table IV

Task	Workers			
	A	B	C	D
P	3	1	0	1
Q	0	1	4	0
R	0	3	2	0
S	0	0	0	1

Now the solution is:

P to C, Q to D, R to A and S to B. Total Time = $8 + 7 + 11 + 9 = 35$ days

Question:

Solve the following assignment problem:

	Worker I	Worker II	Worker III	Worker IV	Worker V
Job : I	1	3	2	3	6
Job : II	2	4	3	1	5
Job : III	5	6	3	4	6
Job : IV	3	1	4	2	2
Job : V	1	5	6	5	4

Solution:

Subtract the smallest element of every row from all the elements of that row:

Table I

	Worker I	Worker II	Worker III	Worker IV	Worker V
Job : I	0	2	1	2	5
Job : II	1	3	2	0	4
Job : III	2	3	0	1	3
Job : IV	2	0	3	1	1
Job : V	0	4	5	4	3

Subtract the smallest element of every column from all the elements of that column:

Table II

	Worker I	Worker II	Worker III	Worker IV	Worker V
Job : I	0	2	1	2	4
Job : II	1	3	2	0	3
Job : III	2	3	0	1	2
Job : IV	2	0	3	1	0
Job : V	0	4	5	4	2

Make assignments in one zero rows and then one zero columns:

Table III

	Worker I	Worker II	Worker III	Worker IV	Worker V
Job : I	0	2	1	2	4
Job : II	1	3	2	0	3
Job : III	2	3	0	1	2
Job : IV	2	0	3	1	0
Job : V	0	4	5	4	2

Row of Job : E and column of Worker : V have no assignment.

Now we have to proceed to the next step. At this step, minimum numbers of lines are to be drawn covering all zeros. For this purpose, marking of “√” is to be done to the relevant rows and columns.

Table IV

	Worker I	Worker II	Worker III	Worker IV	Worker V	
Job : I	0	2	1	2	4	√
Job : II	1	3	2	0	3	
Job : III	2	3	0	1	2	
Job : IV	2	0	3	1	0	
Job : V	0	4	5	4	2	√
	√					

In the above table, the smallest of the elements uncovered by the lines is 1. Subtract this element (i.e, 1) from the elements uncovered by the lines and add that element (1) to every element that lie at the intersection of the two lines. Then make zero assignments.

Table V

	Worker I	Worker II	Worker III	Worker IV	Worker V
Job : I	0	1	0	1	3
Job : II	2	3	2	0	3
Job : III	3	3	0	1	2
Job : IV	3	0	3	1	0
Job : V	0	3	4	3	1

Now, the row of Job : I and the column of Worker : V do not have any assignment.

So draw minimum number of lines passing through all zeros after marking √ to the respective rows and columns.

Table VI

	Worker I	Worker II	Worker III	Worker IV	Worker V	
Job : I	0	1	0	1	3	✓
Job : II	2	3	2	0	3	
Job : III	3	3	0	1	2	✓
Job : IV	3	0	3	1	0	
Job : V	0	3	4	3	1	✓
	✓		✓			

In the above table, the smallest of the elements uncovered by the lines is 1. Subtract this element (i.e., 1) from the elements uncovered by the lines and add that element (1) to every element that lie at the intersection of the two lines. Then make zero assignments.

Table VII

	Worker I	Worker II	Worker III	Worker IV	Worker V
Job : I	0	0	0	0	2
Job : II	3	3	3	0	3
Job : III	3	2	0	0	1
Job : IV	4	0	4	1	0
Job : V	0	2	4	2	0

The solution is:

Job I to Worker I, Job II to Worker IV, Job III to Worker III, Job IV to Worker II and Job V to Worker V. The minimum cost of the assignment is $1 + 1 + 3 + 1 + 4 = \underline{10}$

We can have an alternative solution if we had selected last zero of fourth row (Job: IV) in Table VII:

Table VII

	Worker I	Worker II	Worker III	Worker IV	Worker V
Job : I	0	0	0	0	2
Job : II	3	3	3	0	3
Job : III	3	2	0	0	1
Job : IV	4	0	4	1	0
Job : V	0	2	4	2	0

The alternative solution is:

Job I to Worker II, Job II to Worker IV, Job III to Worker III, Job IV to Worker V and Job V to Worker I. The minimum cost of the assignment is $3 + 1 + 3 + 2 + 1 = \underline{10}$

Both the solutions have the same minimum cost of Rs: 10.

Maximisation in Assignment Problems

In general, the objective function of an assignment problem is to minimise the cost of assignment. But sometimes, the objective function may be to maximize the effectiveness like maximising profit. In such a case, the problem is to be converted into minimisation problems. For this, each element is to be subtracted from the highest element of the matrix. Now the problem has become one with objective function of minimization.

Question: Find an optimal solution which maximizes total profit:

	Worker I	Worker II	Worker III	Worker IV
Work I	51	53	54	50
Work II	47	50	48	50
Work III	49	50	60	61
Work IV	63	64	60	61

Solution:

The highest element is 64. Subtract each element from 64. Now, we get the table as follows:

	Worker I	Worker II	Worker III	Worker IV
Work I	13	11	10	14
Work II	17	14	16	14
Work III	15	14	4	3
Work IV	1	0	4	3

Subtract the smallest element of every row from all the elements of that row:

Table I

	Worker I	Worker II	Worker III	Worker IV
Work I	3	1	0	4
Work II	3	0	2	0
Work III	12	11	1	0
Work IV	1	0	4	3

Subtract the smallest element of every column from all the elements of that column:

Table II

	Worker I	Worker II	Worker III	Worker IV
Work I	2	1	0	4
Work II	2	0	2	0
Work III	11	11	1	0
Work IV	0	0	4	3

The solution is:

Work I to Worker III, Work II to Worker II, Work III to Worker IV and Work IV to Worker II.

Total Profit is $54 + 50 + 61 + 63 = \underline{228}$

Unbalanced Assignment Problems

In an assignment problem, when the number of tasks is not equal to number of persons, then the assignment problem is called unbalanced assignment problem. Here, the cost matrix is not a square matrix. So, for finding solution to the problem, we have to make it as a square matrix. This is done by adding dummy row or column, as the case may be.

Question: A company has 4 workers to do 3 jobs. Each job can be assigned to one and only one worker. The cost of each job on each worker is given in the following matrix:

	Worker 1	Worker 2	Worker 3	Worker 4
Job 1	18	24	28	32
Job 2	8	13	17	19
Job 3	10	15	19	22

Find the job assignments which minimises the cost.

Solution:

Here the number of rows is less than the number of columns. The given problem is an unbalanced assignment problem. Therefore, a dummy row has to be introduced to make it as a balanced problem. Then we get:

	Worker 1	Worker 2	Worker 3	Worker 4
Job 1	18	24	28	32
Job 2	8	13	17	19
Job 3	10	15	19	22
Job 4	0	0	0	0

Subtract the smallest element of every row from all the elements of that row:

Table I

	Worker 1	Worker 2	Worker 3	Worker 4
Job 1	0	6	10	14
Job 2	0	5	9	11
Job 3	0	5	9	12
Job 4	0	0	0	0

Subtract the smallest element of every column from all the elements of that column. As there is zero in each column, we get the same table:

Table II

	Worker 1	Worker 2	Worker 3	Worker 4
Job 1	0	6	10	14
Job 2	0	5	9	11
Job 3	0	5	9	12
Job 4	0	0	0	0

Now make assignments in the cells with zero values. Then we get:

Table III

	Worker 1	Worker 2	Worker 3	Worker 4
Job 1	0	6	10	14
Job 2	0	5	9	11
Job 3	0	5	9	12
Job 4	0	0	0	0

Since assignments are not complete, we have to move to the next step. Draw minimum number of lines, covering all zeros, using \sqrt marks:

Table IV

	Worker 1	Worker 2	Worker 3	Worker 4	
Job 1	0	6	10	14	\sqrt
Job 2	0	5	9	11	\sqrt
Job 3	0	5	9	12	\sqrt
Job 4	0	0	0	0	
	\sqrt				

In the above table, the smallest of the elements uncovered by the lines is 5. Subtract this element (i.e., 5) from the elements uncovered by the lines and add that element (5) to every element that lie at the intersection of the two lines. Then make zero assignments. Now we get:

TABLE V

	Worker 1	Worker 2	Worker 3	Worker 4
Job 1	0	1	5	9
Job 2	0	0	4	6
Job 3	0	0	4	7
Job 4	5	0	0	0

Since assignment is not complete, we have to go for the next step. Draw minimum number of lines covering all zeros using “√” marks:

TABLE VI

	Worker 1	Worker 2	Worker 3	Worker 4	
Job 1	0	1	5	9	√
Job 2	0	0	4	6	√
Job 3	0	0	4	7	√
Job 4	5	0	0	0	
	√	√			

In the above table, the smallest of the elements uncovered by the lines is 4. Subtract this element (i.e., 4) from the elements uncovered by the lines and add that element (4) to every element that lie at the intersection of the two lines. Then make zero assignments. Now we get:

TABLE VII

	Worker 1	Worker 2	Worker 3	Worker 4
Job 1	0	1	1	5
Job 2	0	0	0	2
Job 3	0	0	0	3
Job 4	9	4	0	0

The solution is:

Job I to Worker I, Job II to Worker II, Job III to Worker III.

The total cost = $18 + 13 + 19 = \underline{50}$

An alternative solution is:

TABLE VII

	Worker 1	Worker 2	Worker 3	Worker 4
Job 1	0	1	1	5
Job 2	0	0	0	2
Job 3	0	0	0	3
Job 4	9	4	0	0

Job I to Worker I, Job II to Worker III, Job III to Worker II.

The total cost = $18 + 17 + 15 = \underline{50}$

REVIEW QUESTIONS:

1. What do you mean by assignment problem?
2. What are the important assumptions of assignment problem?
3. Distinguish between Transportation Problem and Assignment Problem.
4. What are the different methods to find solution to assignment problems?
5. What do you mean by Hungarian method? Explain the procedure.
6. A tourist company owns a one car in each of the five locations viz. L1, L2, L3, L4, L5 and a passengers in each of the five cities C1, C2, C3, C4, C5 respectively. The following table shows the distant between the locations and cities in kilometer. How should be cars be assigned to the passengers so as to minimize the total distance covered:

Locations	Cities				
	C1	C2	C3	C4	C5
L1	120	110	115	30	36
L2	125	100	95	30	16
L3	155	90	135	60	50
L4	160	140	150	60	60
L5	190	155	165	90	85

7. Solve the following assignment problem:

	1	2	3	4	5
1	Rs: 3	Rs: 8	Rs: 2	Rs: 10	Rs: 3
2	Rs: 8	Rs: 7	Rs: 2	Rs: 9	Rs: 7
3	Rs: 6	Rs: 4	Rs: 2	Rs: 7	Rs: 5
4	Rs: 8	Rs: 4	Rs: 2	Rs: 3	Rs: 5
5	Rs: 9	Rs: 10	Rs: 6	Rs: 9	Rs: 10

CHAPTER 6

INVENTORY MANAGEMENT

Introduction

Simply inventory is a stock of physical assets. The physical assets have some economic value, which can be either in the form of material, men or money. Inventory is also called as an idle resource as long as it is not utilized. Inventory may be regarded as those goods which are procured, stored and used for day to day functioning of the organization.

Inventory can be in the form of physical resource such as raw materials, semi-finished goods used in the process of production, finished goods which are ready for delivery to the consumers, human resources, or financial resources such as working capital etc.

Inventories means measures of power and wealth of a nation or of an individual during centuries ago. That is a business man or a nation's wealth and power were assessed in terms of grams of gold, heads of cattle, quintals of rice etc.

In recent past, inventories mean measure of business failure. Therefore, businessmen have started to put more emphasis on the liquidity of assets as inventories, until fast turnover has become a goal to be pursued for its own sake.

Today inventories are viewed as a large potential risk rather than as a measure of wealth due to the fast developments and changes in product life. The concept of inventories at present has necessitated the use of scientific techniques in the inventory management called as inventory control.

Thus, inventory control is the technique of maintaining stock items at desired levels. In other words, inventory control is the means by which material of the correct quality and quantity is made available as and when it is needed with due regard to economy in the holding cost, ordering costs, setup costs, production costs, purchase costs and working capital.

Objectives of Carrying Inventory

Carrying of inventory has the following main objectives:

- (1) To supply the raw material, sub-assemblies, semi-finished goods, finished goods, etc. to its users as per their requirements at right time and at right price.
- (2) To maintain the minimum level of waste, surplus, inactive, scrap and obsolete items.
- (3) To minimize the inventory costs such as holding cost, replacement cost, breakdown cost and shortage cost.
- (4) To maximize the efficiency in production and distribution.

- (5) To maintain the overall inventory investment at the lowest level.
- (6) To treat inventory as investment which is risky? For some items, investment may lead to higher profits and for others less profit.

Inventory is an Essential Requirement

Inventory is a part and parcel of every facet of business life. Without inventory no business activity can be performed, whether it being a manufacturing organization or service organization such as libraries, banks, hospitals etc. Irrespective of the specific organization, inventories are reflected by way of a conversion process of inputs to outputs.

The stocks at input are called raw materials whereas the stocks at the output are called products. The stocks at the conversion process may be called finished or semi-finished goods or sometimes may be raw material depending on the requirement of the product at conversion process, where the input and output are based on the market situations of uncertainty, it becomes physically impossible and economically impractical for each stock item to arrive exactly where it is required and when it is required.

Even it is physically possible to deliver the stock when it is required, it costs more expensive. This is the basic reason for carrying the inventory. Thus, inventories play an essential and pervasive role in any organization because they make it possible:

- To meet unexpected demand

- To achieve return on investment

- To order largest quantities of goods, components or materials from the suppliers at advantageous prices

- To provide reasonable customer service through supplying most of the requirements from stock without delay

- To avoid economically impractical and physically impossible delivering/getting right amount of stock at right time of required

- To maintain more work force levels

- To facilitate economic production runs

- To advantage of shipping economies

- To smooth seasonal or critical demand

- To facilitate the intermittent production of several products on the same facility

- To make effective utilization of space and capital

- To meet variations in customer demand

- To take the advantage of price discount

- To hedge against price increases

- To discount quantity

Basic Functions of Inventory

I. The important basic function of inventory is:
Increase the profitability- through manufacturing and marketing support. But zero inventory manufacturing- distribution system is not practically possible, so it is important to remember that each rupee invested in inventory should achieve a specific goal.

II. The other basic inventory functions are:

- (a) Geographical Specialization
- (b) Decoupling
- (c) Balancing supply and demand and
- (d) Safety stock

A. Geographical Specialization

Another basic inventory function is to allow the geographical specialization individual operating units. There is a considerable distance between the economical manufacturing location and demand areas due to factors of production such as raw material, labour, water, power. So that the goods from various manufacturing locations are collected at a simple warehouse or plant to assemble in final product or to offer consumers a single mixed product shipment. This also provides economic specialization between manufacturing and distribution units/locations of an organization.

B. Decoupling

The provision of maximum efficiency of operations within a single facility is also one of the important basic functions of the inventory. This is achieved by decoupling, which is done by breaking operations apart so that one operation(s) supply is independent of another(s) supply.

The decoupling function serves in two ways of purposes, they are:

Inventories are needed to reduce the dependencies among successive stages of operations so that shortage of materials, breakdowns or other production fluctuations at one stage do not cause later stage to shut down.

One organizational unit schedules its operations independently of another organizational unit. For example: Consider an automobile organization, here assembly process can be schedule separately from engine built up operation, and each can be decoupled from final automobile assembly operations through in process inventories.

C. Supply and Demand Balancing

The function of Balancing concerns elapsed time between manufacturing and using the product. Balancing inventories exist to reconcile supply with demand. The most noticeable example of balancing is seasonal production and year round usage like sugar, rice, woolen textiles, etc. Thus the investment of balancing inventories links the economies of manufacturing with variations of usage.

D. Safety Stock

The safety stock also called as buffer stock. The function of safety stock concerns short range variations in either replacement or demand. Determination of the safety stock size requires a great deal of inventory planning. Safety stock provides protection against two types of uncertainty. They are:

- (1) Sales in excess of forecast during the replenishment period
- (2) Delays in replenishment

Thus, the inventories committed to safety stocks denote the greatest potential for improved performances. There are different techniques are available to develop safety stocks.

Types of Inventory

Inventory may be classified into manufacturing, service and control aspects. Each inventory type is discussed below:

A. Raw Material/Manufacturing Inventory

There are five types of Manufacturing Inventory, they are

(1) Production Inventory

Items going to final product such as raw materials, sub-assemblies purchased from outside are called production inventory.

(2) Work-in-Process Inventory

The items in the form of semi-finished or products at different stage of production process are known as work-in-process inventory.

(3) M.R.O. Inventory

Maintenance, Repair and Operating supplies such as spare parts and consumable stores, which do not go into final product but are consumed during the production process.

(4) Finished Goods Inventory

Finished Goods Inventory includes the products ready for dispatch to the consumers or distributors/retailers.

(5) Miscellaneous Inventory

Items excluding those mentioned above, such as waste, scrap, obsolete, and un-saleable items arising from the main production process, stationery used in the office and other items required by office, factory and other departments, etc. are called miscellaneous inventory.

B. Service Inventory

The service inventory can be classified into four types, they are:

(1) Lot Size Stocks

Lot size means purchasing in lots. The reasons for this is to

- Obtain quantity discounts

- Minimize receiving and handling costs

- Reduce purchase and transport costs

For example: It would be uneconomical for a textile factory to buy cotton everyday rather than in bulk during the cotton season.

(2) Anticipation Stocks

Anticipation stocks are kept to meet predictable changes in demand or in availability of raw materials. For example: The purchase of potatoes in the potato season for sale of roots preservation products throughout the year.

(3) Fluctuation Stocks

Fluctuation stocks are carried to ensure ready supplies to consumers in the face of irregular fluctuations in their demands

(4) Risk Stocks

Risk stocks are the items required to ensure that there is no risk of complete production breakdown. Risk stocks are critical and important for production.

C. Control Inventory

A good way of examining an inventory control is: to make ABC classification, which is also known as ABC analysis. ABC analysis means the “control” will be “Always Better” if we start with the ABC classification of inventory.

The 3 groups of inventory items are called A-items group, B-items group, C-items group, which are explained as follows:

A-items Group: This constitutes 10% of the total number of inventory items and 70% of total money value for all the items.

B-items Group: This constitutes 20% of the total number of inventory items and 20 % of total money value for all the items.

C-items Group: This constitutes 70% of the total number of inventory items and 10 % of total money value for all the items. This is just opposite of A-items group.

The ABC classification provides us clear indication for setting properties of control to the items, and A-class item receive the importance first in every respect such as tight control, more security, and high operating doctrine of the inventory control.

The coupling of ABC classification with VED classification enhances the inventory control efficiency. VED classification means Vital, Essential and Desirable Classification. From the above description, it may be noted that ABC classification is based on the logic of proportionate value while VED classification based on experience, judgment, etc. The ABC /VED classification is presented in the following figure:

	V	E	D	Total
A	3	5	2	10
B	5	7	8	20
C	10	40	20	70
Total	18	52	30	100

ABC/VED Classifications

This is an example of a particular case. The values are expressed in percentage. Note that the total number of categories becomes nine.

Costs Associated with Inventories

1. Purchase Cost (or Production Cost)

The cost of the item is the direct manufacturing cost if it is produced in in-house or the cost paid to the supplier for the item received. This cost usually equal to the purchase price. When the marketing price goes on fluctuating, inventory planning is based on the average price mostly it is called as a fixed price. When price discounts can be secured or when large production runs may result in a decrease in the production cost, the price factor is of special interest.

2. Procurement Costs (Ordering Cost)

The costs of placing a purchase order is known as ordering costs and the costs of initial preparation of a production system (if in-house manufacturing) is called as set up cost. These costs are called as procurement cost, but these costs vary directly with each purchase order placed or with set up made and are normally assumed independent of the quantity ordered or produced.

Procurement costs include costs of transportation of items ordered, expediting and follow up, goods receiving and inspection, administration (includes telephone bills, computer cost, postage, salaries

of the persons working for tendering, purchasing, paper work, etc.), payment processing etc. This cost is expressed as the cost per order/setup.

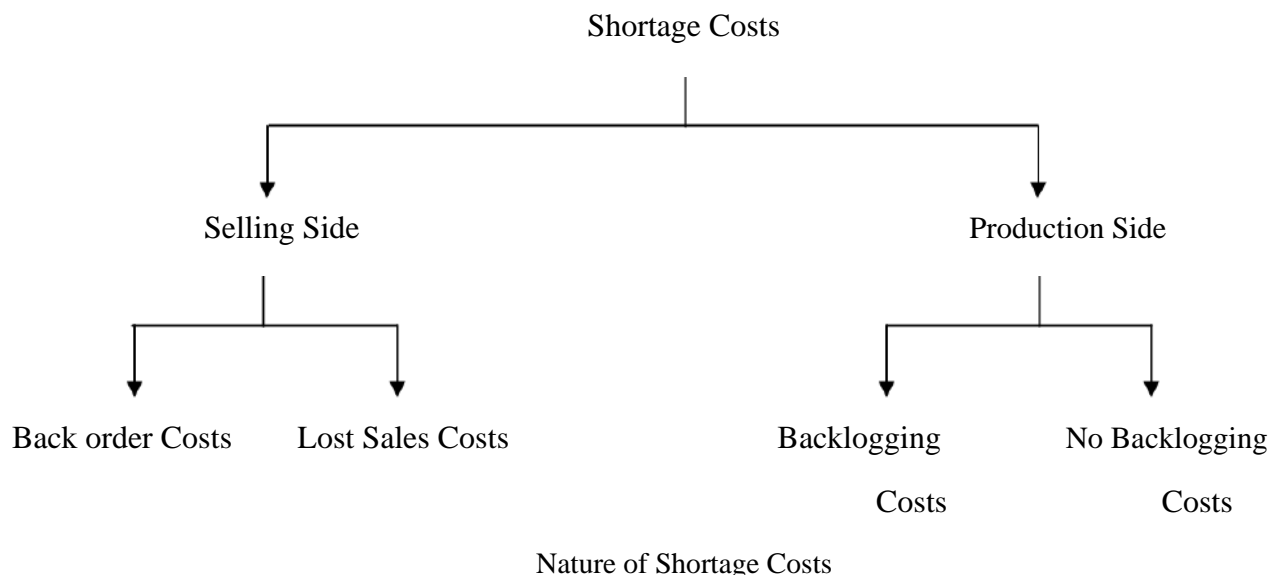
3. Holding Costs (or Holding Cost)

The holding costs also called as carrying costs. The cost associated with holding/carrying of stocks is called holding cost or carrying cost or possession cost. Holding costs includes handling/carrying cost, maintenance cost, insurance, safety measures, warehouse rent, depreciation, theft, obsolescence, salaries, interest on the locked money, etc. Thus, by considering all these elements the storage cost is expressed either as per unit of item held per unit of time or as a percentage of average money value of investment held. Therefore the size of all these holding costs usually increases or decreases in proportion to the amount of inventory that is carried.

4. Shortage Costs (or Stock out Cost)

These costs are penalty costs as a result of running out of stock at the time of item is required. There are different forms of shortage cost, which is illustrated in the following figure 3.8. One form of the shortage costs is called as back order on the selling side or backlogging cost on the production side when the unsatisfied demand can be satisfied at later stage that is consumers has to wait till they gets the supply.

The second form of shortage costs is called as lost sales costs on the selling side or no backlogging costs on the production side, when the unsatisfied demand is lost or the consumers goes somewhere else instead of waiting for the supply.



These includes the costs of production stoppage, overtime payments, idle machine, loss of goodwill, loss of sales opportunity, special order at higher price, loss of profits etc.

Factors Affecting Inventory Control

Following factors play an important role in the study of inventory control:

1. **Demand:** The number of units required per period is called demand. A commodity's demand pattern may be deterministic or probabilistic.

- (a) **Deterministic Demand:** In this case, the demand is assumed that the quantities of commodity needed over subsequent periods of time are known with certainty. This is expressed over equal time periods in terms of known constant demands or in terms of variable demands. The two cases are called as static and dynamic demands.
 - (b) **Probabilistic Demand:** This occurs when requirements over a certain time period are not known with certainty but their pattern can be denoted by a known probability distribution. In this case, the probability distribution is said to be stationary or non-stationary over time periods. The terms stationary and non-stationary are equivalent to the terms static and dynamic in the deterministic demand. For a given time period the demand may be instantaneously satisfied at the beginning of the time period or uniformly during that time period. The effect of uniform and instantaneous demand directly reflects on the total cost of carrying inventory.
2. **Ordering Cycle:** The ordering cost is related with the inventory situation time measurement. An ordering cycle can be identified by the time period between two successive placements of orders. The later may be initiated in one of two ways as:
- (a) **Periodic Review:** In this case, the orders are placed at equally intervals of time.
 - (b) **Continuous Review:** In this case, an inventory record is updated continuously until a certain lower limit is reached at this point a new order is placed. Sometimes this is referred as the two-bin system.
3. **Delivery Lag:** The requirement of the inventory is felt and an order is placed, it may be delivered instantaneously or sometimes it may be needed before delivery if affected. The time period between the placement of the requisition for an item and its receipt for actual use is called as delivery lag. The delivery lags also known as lead time.
- There are four different types of lead time, they are:
- (a) Administrative Lead
 - (b) Time Transportation
 - (c) Lead Time Suppliers
 - (d) Lead Time
 - (e) Inspection Lead Time

The Inspection Lead Time and Administrative Lead Time can be fixed in nature, where as the Transportation Lead Time and Suppliers Lead Time can never be fixed. It means generally the lead time may be deterministic or probabilistic.

4. **Time Horizon:** Time horizon is, the planning period over which inventory is to be controlled. The planning period may be finite or infinite in nature. Generally, inventory planning is done on annual basis in most of the organizations.

INVENTORY MODELLING

In general, the inventory models can be classified into two types, viz. deterministic models and stochastic models. In Deterministic models, variables are known with certainty, but in stochastic models, the variables are probabilistic.

Deterministic Inventory Models:

There are different deterministic inventory models, they are:

I. Deterministic single item Inventory Models:

EOQ – Economic Order Quantity Model – I
 EOQ – Economic Order Quantity Model – II (instantaneous supply when shortages are allowed)
 EPQ – Economic Production Model (Gradual supply case and shortage not allowed)
 Price Discounts Model (instantaneous supply with no shortages)
 Dynamic Demand Models

II. Deterministic multi-item Inventory Models:

Unknown cost structure Model
 Known cost structure Model

Some important models only are discussed below:

Deterministic Single Item inventory Models Economic Lot Size Models or Economic Order Quantity models (EOQ models) - with uniform rate of demand

F. Harries first developed the Economic Order Quantity concept in the year 1916. The idea behind the concept is that the management is confronted with a set of opposing costs like ordering cost and inventory carrying costs. As the lot size ' q ' increases, the carrying cost ' C_1 ' also increases while the ordering cost ' C_3 ' decreases and vice versa. Hence, Economic Ordering Quantity – *EOQ* - is that size of order that minimizes the total annual (or desired time period) cost of carrying inventory and cost of ordering under the assumed conditions of certainty and that annual demands are known.

Economic Order Quantity by Trial and Error Method

Let us try to work out Economic Order Quantity formula by trial and error method to understand the average inventory concept. The steps involved are:

Select the number of possible lot sizes to purchase.

Determine total cost for each lot size chosen.

Calculate and select the order quantity that minimizes total cost.

While working the problems, we will consider *Average inventory* concept. This is because, the inventory carrying cost which is the cost of holding the inventory in the stock, cannot be calculated day to day as and when the inventory level goes on decreasing due to consumption or

increases due to replenishment. For example, let us say the rent for the storeroom is Rs.500/- and we have an inventory worth Rs. 1000/-. Due to daily demand or periodical demand the level may vary and it is practically difficult to calculate the rent depending on the level of inventory of the day. Hence what we do is we use average inventory concept. This means that at the beginning of the cycle the level of inventory is Worth Rs. 1000/- and at the end of the cycle, the level is zero. Hence we can take the average of this two *i.e.* $(0 + 1000) / 2 = 500$. Let us take a simple example and see how this will work out.

Demand for the item: 8000 units. (q)

Unit cost is Re.1/- (p)

Ordering cost is Rs. 12.50 per order, (C_3)

Carrying cost is 20% of average inventory cost. (C_1)

<i>Number of Orders Per year</i>	<i>Lot size Q</i>	<i>Average Inventory $q / 2$</i>	<i>Carrying Charges $C_1 = 0.20 (Rs)$</i>	<i>Ordering Cost $C_3 (Rs)$</i>	<i>Total cost (Rs.)</i>
1	8000	4000	800	12.50	812.50
2	4000	2000	400	25.00	425.00
4	2000	1000	200	50.00	250.00
8	1000	500	100	100.00	200.00
12	667	323	66	150.00	216.00
16	500	250	50	200.00	250.00
32	50	125	25	400.00	425.00

Observe the last column. The total cost goes on reducing and reaches the minimum of Rs. 200/- and then it increases. Also as lot size goes on decreasing, the carrying cost decreases and the ordering cost goes on increasing. Hence we can say the optimal order quantity is 1000 units and optimal number of orders is 8. See at the optimal order quantity of 1000 units, both ordering cost and inventory costs are same. Hence we can say *that the optimal order quantity occurs when ordering cost is equal to the inventory carrying cost*. This we can prove mathematically and illustrate by a graph. This will be shown in the coming discussion.

It is not always easy to work for economic order quantity by trial and error method as it is difficult to get exact quantity and hence we may not get that ordering cost and inventory carrying costs equal. Hence it is better to go for mathematical approach.

Economic Lot Size (for manufacturing model) or Economic Order Quantity (EOQ for purchase models) without shortage and deterministic Uniform demand

When we consider a manufacturing problem, we call the formula as **Economic Lot Size (ELS) or Economic Batch Quantity (EBQ)**. Here the quantity manufactured per batch is lot size (order quantity in manufacturing model), fixed charges or set up cost per batch, which is shared by all the components manufactured in that batch is known as **Set up cost** (similar to ordering cost, as the cost of order is shared by the items purchased in that order), the cost of maintaining the in process inventory is the inventory carrying charges. Here a formula for economic lot size ' q ' per cycle (production run) of a single product is derived so as to minimize the total average variable cost per unit time.

Assumptions made:

Demand is uniform at a rate of ' r ' quantity units per unit of time.

Lead time or time of replenishment is zero (some times known exactly).

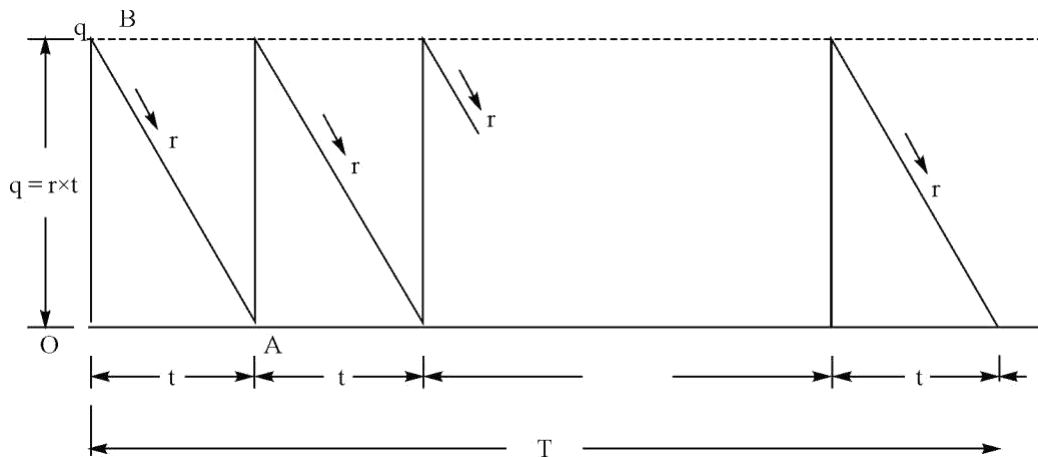
Production rate is infinite, *i.e.* production is instantaneous.

Shortages are not allowed. (*i.e.* stock out cost is zero).

Holding cost is Rs. C_1 per quantity unit per unit of time.

Set up cost is Rs. C_3 per run or per set up.

By trial and error method we have seen that economic quantity exists at a point where both ordering cost and inventory carrying cost are equal. This is the basis of algebraic method of derivation of formula. The following figure shows the lot size ' q ', uniform demand ' r ' and the pattern of inventory cycle.



Deterministic uniform demand with no shortages.

Total inventory in one cycle *i.e.* for one unit of time = Area of triangle $OAB = \frac{1}{2} \times \text{base } (t) \times \text{altitude } (q) =$

$$\frac{1}{2} \times q \times t = \frac{1}{2} qt$$

Total cost = Inventory carrying cost + material cost.

Graphical representation of Total cost curve:

- (i) Behaviour of inventory carrying cost: As the level of inventory goes on increasing, the inventory carrying cost goes on increasing as it solely depends on the size of the inventory.
- (ii) The ordering cost or set up cost per unit reduces with the increase in the number of orders.
- (iii) Total cost first goes on reducing and after reaching the minimum it goes on increasing. In the first part, *i.e.* while it decreases, it has the influence of ordering cost and in the latter part, *i.e.* while it is increasing, it has the influence of inventory carrying cost.
- (iv) When curves are drawn, both carrying cost curve and ordering cost curve will intersect at a point. This point lies exactly where the lowest total cost appears on the graph. This is

Figure 8.9 Cost curves.

2. Optimal inventory cost = $2 \times C_1 \times C_3 \times r =$

(2 × carrying cost × ordering cost × demand rate.)

3. Optimal order time = $(2 \times C_3) / C_1 \times r = (2 \times \text{ordering cost}) / (\text{carrying cost} \times \text{demand rate}).$

4. Optimum number of orders: $N_0 = \lambda / q_0 = \text{Annual demand} / \text{optimal order quantity}.$

The number of day's supply per optimum order is obtained by $= d = 365 / N_0$
 $= 365 / \text{optimum number of orders}.$

Problem:

The demand for an item is 8000 units per annum and the unit cost is Re.1/-. Inventory carrying charges of 20% of average inventory cost and ordering cost is Rs. 12.50 per order. Calculate optimal order quantity, optimal order time, optimal inventory cost and number of orders.

Solution:

Data: $\lambda = 8000$ units, $p = \text{Re.1/-}$, $C_1 = 20\%$ of average inventory or 0.20, Ordering cost = Rs. 12.50 per order.

$$q_0 = \sqrt{[(2 \times 12.5 \times 8,000) / (1 \times 0.20)]} = \sqrt{(2,00,000 / 0.2)} = \sqrt{10,00,000} = 1,000$$

$$c_0 = \sqrt{(2 \times 0.2 \times 1 \times 12.5 \times 8,000)} = \sqrt{40,000} = \text{Rs: } 200$$

$$\text{Inventory carrying cost} = (q/2) \times p \times C_1 = (1000/2) \times 1.00 \times 0.20 = \text{Rs. } 100/-$$

Total ordering cost = Number of orders \times ordering cost = (Demand / q_0) \times C_3 = (8000 / 1000) \times 12.50 = Rs. 100/-

Total inventory cost = Carrying cost + ordering cost = Rs. 100 + Rs. 100 = Rs. 200/- (This is same as obtained by application of formula for total cost.

Optimal number of orders = Annual demand / optimal order quantity = λ / q_0 = 8000 / 1000 = 8 orders.

Optimal order period = $t_0 = q_0 / r$ = Optimal order quantity / demand rate. = 1000 / 8000 = 1/8 of a year.

$$= 365 / 8 = 45.6 \text{ days} = \text{app } 46 \text{ days.}$$

Total cost including material cost = Inventory cost + material cost = Rs. 200 + Rs. 8000 = Rs. 8200/-

Problem:

For an item the production is instantaneous. The storage cost of one item is Re.1/- per month and the set up cost is Rs. 25/- per run. If the demand for the item is 200 units per month, find the optimal size of the batch and the best time for the replenishment of inventory.

Solution:

Here we take one month as one unit of time. (Note: Care must be taken to see that all the data given in the problem must have same time base *i.e.* year / month/week etc. If they are different, *e.g.* the carrying cost is given per year and the demand is given per month, then both of them should be taken on same time base.). Hence it is better to write data given in the problem first with units and then proceed to solve.

Data: Storage cost: Re.1/- per month = C_1 , Set up cost per run = Rs. 25/- per run, Demand = 200 units per month.

$$\text{Optima batch quantity} = \text{Economic Batch Quantity} = EBQ = \sqrt{[(2 \times 25 \times 200) / 1]} = \sqrt{10,000} = 100$$

$$\text{Optimal time of replenishment} = T_0 = q_0 / r = 100 / 200 \text{ } \frac{1}{2} \text{ month} = 15 \text{ days}$$

$$\text{Optimal cost} = C_0 = \sqrt{(2C_3 \times C_1 \times r)} = \sqrt{(2 \times 1 \times 25 \times 200)} = \sqrt{10000} = \text{Rs:}100$$

OR

$$\begin{aligned} \text{It can also be found by Total cost} &= \text{Carrying cost} + \text{Ordering cost} = (q/2) \times C_1 + C_3 \times r/q \\ &= (100/2) \times 1 + 25 \times 200 / 100 = 50 + 50 \\ &= \text{Rs: } 100 \end{aligned}$$

Problem:

A producer has to supply 12,000 units of a product per year to his customer. The demand is fixed and known and backlogs are not allowed. The inventory holding cost is Rs.0.20 per unit per month and the set up cost per run is Rs. 350/- per run. Determine (a) the optimal lot size, (b) Optimum scheduling period,

(c) Minimum total expected yearly cost.

Solution:

Data: λ = Demand per year 12,000 units, C_1 Rs. 0.20 per unit per month, C_3 = Rs. 350/- per run.

λ = demand per month 12,000/12 = 1000 units.

$$q_0 = \sqrt{(2C_3 \times \lambda) / C_1} = \sqrt{(2 \times 350 \times 1000) / 0.2} = \sqrt{(700000 / 0.2)} = \sqrt{3500000} = 1870 \text{ units per batch.}$$

$$t_0 = q_0 / \lambda = 1870 / 1000 = 1.87 \text{ month.} = 8.1 \text{ weeks.}$$

$$C_0 = \sqrt{(2C_3 \times C_1 \lambda)} = \sqrt{(2 \times 350 \times 0.2 \times 1,000)} = \sqrt{140000} = \text{Rs.} 374 \text{ approx.}$$

Problem:

A particular item has a demand of 9,000 units per year. The cost of one procurement is Rs. 100/- and the holding cost per unit is Rs. 2.40 per year. The replacement is instantaneous and no shortages are allowed. Determine: (a) Economic lot size, (b) The number of orders per year, (c) The time between orders, and (d) the total cost per year if the cost of one unit is Re.1/-.

Solution:

Data: λ = 9,000 units per year, C_1 = Rs. 2.40 per year per unit, C_3 = Rs. 100/- per procurement.

$$\begin{aligned} \text{(a) } q_0 &= \sqrt{(2C_3 \lambda) / C_1} = \sqrt{(2 \times 100 \times 9000) / 2.4} = \sqrt{1800000 / 2.4} = \sqrt{750000} \\ &= 866 \text{ units per procurement} \end{aligned}$$

$$\begin{aligned} \text{(c) } N &= \sqrt{(1/t_0)} = \sqrt{(C_1 \times \lambda) / 2C_3} = \sqrt{(2.4 \times 9000) / 2 \times 100} = \sqrt{108} \\ &= 10.4 \text{ orders per year.} \end{aligned}$$

This can also found out by $\lambda / q_0 = 9000 / 866 = 10.4$ orders per year.

$$\text{(c) } t_0 = 1/N = 1/10.4 = 0.0962 \text{ years between orders. (= 35.12 days = App. 35 days.)}$$

OR

$$t_0 = q_0 / \lambda = 866 / 9000 = 0.0962 \text{ years between orders.}$$

$$(\text{= } 35.12 \text{ days} = \text{App. } 35 \text{ days.})$$

$$(d) C_0 = \sqrt{2C_1C_3\lambda} = \sqrt{2 \times 2.40 \times 100 \times 9000} = \sqrt{4320000} = \text{Rs: } 2078.$$

$$\text{Total cost including material cost} = 9000 \times 1 + 2,078 = \text{Rs. } 11,078 \text{ per year.}$$

Problem:

A precision engineering company consumes 50,000 units of a component per year. The ordering, receiving and handling costs are Rs.3/- per order, while the trucking cost are Rs. 12/- per order. Further details are as follows:

Interest cost Rs. 0.06 per units per year. Deterioration and obsolescence cost Rs.0.004 per unit

per year. Storage cost Rs. 1000/- per year for 50,000 units. Calculate the economic order quantity, Total inventory carrying cost and optimal replacement period.

Solution:

Data: $\lambda = 50,000$ units per year.

$$C_3 = \text{Rs. } 3/- + \text{Rs. } 12/- = \text{Rs. } 15/- \text{ per order.}$$

$$C_1 = \text{Rs. } 0.06 + 0.004 + 1000 / 50,000 \text{ per unit} = \text{Rs. } 0.084 / \text{unit.}$$

$$q_0 = \sqrt{(2C_3\lambda)/C_1} = \sqrt{(2 \times 15 \times 50000)/0.084} = \sqrt{17857143} = 4,226 \text{ units.}$$

$$t_0 = \lambda / q_0 = 50,000 / 4226 = 11.83 \text{ years.}$$

$$C_0 = \sqrt{2C_3C_1\lambda} = \sqrt{(2 \times 15 \times 0.084 \times 50,000)} = \sqrt{1,26,000} = \text{Rs: } 355$$

Problem:

You have to supply your customer 100 units of certain product every Monday and only on Monday. You obtain the product from a local supplier at Rs/ 60/- per units. The cost of ordering and transportation from the supplier are Rs. 150/- per order. The cost of carrying inventory is estimated at 15% per year of the cost of the product carried. Determine the economic lot size and the optimal cost.

Solution:

$$\lambda = 100 \text{ units per week,}$$

$$C_3 = \text{Rs. } 150/- \text{ per order,}$$

$$C_1 = (15/100) \times 60 \text{ per year} = \text{Rs. } 9/\text{year i.e.; Rs} = 9/52 \text{ per week.} = \text{Rs: } 0.1731$$

$$q_0 = \sqrt{(2C_3 \lambda)/C_1} = \sqrt{(2 \times 150 \times 100)/0.1731} = \sqrt{(30,000/0.1731)} = \sqrt{1,73,310}$$

$$= 416 \text{ units.}$$

$$C_0 = \sqrt{2C_3 C_1 \lambda} = \sqrt{(2 \times 150 \times 0.1731 \times 100)} = \sqrt{5,193} = \text{Rs: } 72$$

Including material cost $(60 \times 100) + 72 = \text{Rs. } 6072 \text{ per year.}$

Problem:

A stockiest has to supply 400 units of a product every Monday to his customers. He gets the product at Rs. 50/- per unit from the manufacturer. The cost of ordering and transportation from the manufacturer is Rs. 75 per order. The cost of carrying inventory is 7.5% per year of the cost of the product. Find

(i) Economic lot size, (ii) The total optimal cost (including the capital cost).

Solution:

Data: $\lambda = 400$ units per week,

$C_3 = \text{Rs. } 75/\text{per order,}$

$p = \text{Rs. } 50 \text{ per unit.}$

$C_1 = 7.5\% \text{ per year of the cost of the product.} = \text{Rs. } (7.5/100) \times 50 \text{ per unit per year.}$

$= \text{Rs. } (7.5/100) \times (50/52) \text{ per week.} = \text{Rs. } 3.75/52 \text{ per week} = \text{Rs. } 0.072 \text{ per week.}$

$$q_0 = \sqrt{(2C_3 \lambda)/C_1} = \sqrt{(2 \times 75 \times 400)/0.072} = \sqrt{(60,000/0.072)} = \sqrt{8,33,333}$$

$$= 913 \text{ units per order.}$$

$$C_0 = \sqrt{2C_3 C_1 \lambda} = \sqrt{(2 \times 75 \times 0.072 \times 400)} = \sqrt{4,320} = \text{Rs: } 65.73$$

Total cost including material cost $= (400 \times 50) + 65.80$

$$= 20,000 + 65.73$$

$$= \text{Rs. } 20,065.73 \text{ per week.}$$

Economic order Quantity for purchase model

All the assumptions made in the Economic Batch Quantity model will remain same but we will take annual demand (λ) and price per unit (*i.e.* Material cost) $= p$ and inventory carrying charges are expressed as 'i' % of average inventory value. Hence, we take inventory carrying charges $C_1 = ip$.

Let us work Economic Order Quantity (EOQ) formula for purchase model.

$$\text{Average inventory} = q/2$$

$$\text{Inventory carrying charges} = ip$$

$$\text{Therefore inventory carrying charge} = (q/2) \times ip \quad \text{OR} = ipq/2$$

$$\text{As the annual demand} = \lambda \quad \text{and the order quantity} = q,$$

$$\text{Number of orders (N)} = \lambda/q$$

$$\text{Hence Ordering Cost} = C_3 \times (\lambda/q)$$

$$\text{Total Cost} = Cq = (ip) \times (q/2) + (\lambda/q) \times C_3$$

The minimum of Cq will get when first derivative is equal to zero.

$$DC/dq = \frac{1}{2} ip - C_3 \lambda / q^2 = 0, \text{ i.e.; } \frac{1}{2} ip = C_3 \lambda / q^2$$

Simplifying, we get

$$q_0 = \sqrt{[(2C_3 \lambda)/ip]}. \text{ This is known as Economic Order Quantity (EOQ).}$$

$$\text{Similarly, } t_0 = \text{Optimal order time} = 2C_3 / (ip \times \lambda)$$

$$\text{Optimal Cost} = C_{q_0} = \sqrt{[2C_3 \times ip \times \lambda]}$$

Total Cost including material cost is given by:

$$\text{Ordering Cost} + \text{Carrying Cost} + \text{Material Cost}$$

$$= (\lambda/q) \times C_3 + (q/2) \times ip + p \lambda \quad \text{OR} = \sqrt{(2C_3 \times ip \times \lambda)} + \lambda p$$

Problem:

A company uses annually 24,000 units of raw material, which costs Rs. 1.25 per units. Placing each order cost Rs. 22.50, and the carrying cost is 5.4% of the average inventory. Find the economic lot size and the total inventory cost including material cost.

Solution:

$$\text{Annual Demand } (\lambda) = 24,000$$

$$C_3 = \text{Rs. 22.50 per order,}$$

$$i = 5.4\% \text{ of average inventory,}$$

$$p = \text{Rs. 1.25 per unit.}$$

$$q_0 = \sqrt{[(2C_3 \lambda)/ip]}$$

$$= \sqrt{[(2 \times 22.50 \times 24,000)/(0.054 \times 1.25)]}$$

$$= \sqrt{[10,80,000/0.0675]} = \sqrt{1,60,00,000} = 4,000 \text{ units}$$

Total cost: Total cost can be found in two ways.

$$(1) \text{ Total Cost} = \text{Ordering Cost} + \text{Carrying Cost} + \text{Material Cost}$$

$$\begin{aligned}
&= [(\lambda / q) \times C_3] + [(q / 2) \times ip] + p \lambda \\
&= [(24,000/4000) \times 22.5] + [(4,000/2) \times 0.054 \times 1.25] + [24,000 \times 1.25] \\
&= 135 + 135 + 30,000 \\
&= \underline{30,270}
\end{aligned}$$

$$\begin{aligned}
(2) \text{ Total Cost} &= \sqrt{(2C_3 \times ip \times \lambda)} + (\lambda p) \\
&= \sqrt{[(2C_3 \times ip \times \lambda)]} + (\lambda p) \\
&= \sqrt{[(2C_3 \times ip \times \lambda)]} + (\lambda p) \\
&= \sqrt{[(2 \times 22.5 \times 0.054 \times 1.25 \times 24,000)]} + (24,000 \times 1.25) \\
&= \sqrt{72900} + 30,000 \\
&= 270 + 30,000 = \underline{30,270}
\end{aligned}$$

Quantity Discount Model

Sometimes, the seller may offer discount to the purchaser, if he purchases larger amount of items. Say for example, if the unit price is Rs. 10/-, when customer purchase 10 or more than 10 items, he may be given 1% discount on unit price of the item. That means the purchaser, may get the item at the rate of Rs. 9/- per item. This may save the material cost. But, as he purchases more than the required quantity his inventory carrying charges will increase, and as he purchases more items per order, his ordering cost will reduce. When he wants to work out the optimal order quantity, he has to take above factors into consideration. The savings part of discount model is: (a) lower unit price, (b) lower ordering cost. The losing part of the model is (a) inventory carrying charges. The discount will be accepted when the savings part is greater than the increase in the carrying cost.

There are two types of discounts. They are: (a) **All units discount**: Here the customer is offered discount on all the items he purchase irrespective of quantity.

(b) **Incremental discount**: Here, the discount is offered to the customer on every extra item he purchases beyond some fixed quantity, say 'q'. Up to 'q' units the customer pays usual unit price and over and above 'q' he is offered discount on the unit price.

Problem:

A shopkeeper has a uniform demand of an item at the rate of 50 items per month. He buys from a supplier at a cost of Rs.6/- per item and the cost of ordering is Rs. 10/- per order. If the stock holding costs are 20% of stock value, how frequently should he replenish his stock? Suppose the supplier offers 5% discount on orders between 200 and 999 items and a 10% discount on orders exceeding or equal to 1000 units. Can the shopkeeper reduce his costs by taking advantage of either of these discounts?

Solution:

$C_1 = 20\%$ per year of stock value

$C_3 = \text{Rs. } 10$

$\lambda = 50$ items per month, $\lambda = 12 \times 50 = 600$ units per year

$p = \text{Rs. } 6$ per item

Discounted price a) $\text{Rs. } 6 - 0.05 \times 6 = \text{Rs. } 5.70$, from 200 to 999 items

(d) $\text{Rs. } 6 - 0.10 \times 6 = \text{Rs. } 5.40$, for 1000 units and above and $i = 0.20$

$q_0 = 2C_3 \lambda / ip = (2 \times 10 \times 600) / (0.2 \times 6) = 100$ units

$t_0 = q_0 / \lambda = 100/600 = 1/6$ of a year = 2 months

$C_0 = 3720$. This may be worked out as below: Material cost + carrying cost + ordering cost
 $= 600 \times 6 + (100/2) \times 0.20 \times 6 + 600 / 6 \times 10 = 3600 + 60 + 60 = \text{Rs. } 3720/-$

(a) To get a discount of 5% the minimum quantity to be purchased is 200. Hence, let us take $q_0 = 200$

Savings: Savings in cost of material. Now the unit price is Rs. 5.70. Hence the savings is $600 \times \text{Rs. } 6 - 600 \times \text{Rs. } 5.70 = \text{Rs. } 3600 - \text{Rs. } 3420 = \text{Rs. } 180/-$

Savings in ordering cost. Number of orders $= \lambda / q_0 = 600 / 200 = 3$ orders. Hence ordering cost

$3 \times \text{Rs. } 10/- = \text{Rs. } 30$. Hence the savings = Ordering cost of EOQ – present ordering cost = $\text{Rs. } 60 - \text{Rs. } 30 = \text{Rs. } 30$.

Hence Total savings = $\text{Rs. } 180 + 30 = \text{Rs. } 210/-$

Additional cost due to increased inventory = present carrying cost – Carrying cost of $EOQ = (200 / 2) \times 0.20 \times \text{Rs. } 5.70 - (100/2) \times 0.20 \times \text{Rs. } 6/- = 100 \times 1.14 - 50 \times \text{Rs. } 1.2 = 114 - 60 =$

$\text{Rs. } 54/-$

Therefore, **by accepting 5% discount, the company can save Rs. 210 – Rs. 54 = Rs. 156/-per year.**

(b) 10% discount on q_0 &

1000. Savings: Ordering cost:

Since 1000 items will be useful for $1000 / 600 = 5/3$ years, the number of orders $= 1 / (5/3) = 3 / 5$ times in a year. Hence number of orders $= 6 - 3/5 = 5.4$ orders. Hence ordering cost $= 5.4 \times 10 = \text{Rs. } 54/-$.

Savings in material cost: $(10/100) \times 6 \times 600 = \text{Rs. } 360/-$

Hence total savings = $\text{Rs. } 360 + \text{Rs. } 54 = \text{Rs. } 414/-$

Increase in the holding cost : $(1000/2) \times 0.20 \times 0.90 \times \text{Rs. } 6/- = \text{Rs. } 480/-$ As the savings is less than the increase in the total cost the discount offer of 10% cannot be accepted.

REVIEW QUESTIONS:

1. What do you mean by inventory management?
2. What are the important objectives of carrying inventory?
3. What are the basic functions of inventory?
4. What are the different types of inventories?
5. Explain the various costs associated with inventory.
6. Briefly explain the factors affecting inventory control.
7. What are the important deterministic inventory models?
8. Define EOQ.
9. What do you mean by Quantity Discount Model?
10. An aircraft company uses rivets at an approximate customer rate of 2,500 Kg. per year. Each unit costs Rs. 30/- per Kg. The company personnel estimate that it costs Rs. 130 to place an order, and that the carrying costs of inventory is 10% per year. How frequently should orders for rivets be placed? Also determine the optimum size of each order.
11. The data given below pertains to a component used by Engineering India (P) Ltd. in 20 different assemblies

Purchase price = p = Rs. 15 per 100 units,

Annual usage = 1,00,000 units,

Cost of buying office = Rs. 15,575 per annum, (fixed),

Variable cost = Rs. 12/- per order,

Rent of component = Rs. 3000/- per annum

Heating cost = Rs. 700/- per annum

Interest = Rs. 25/- per annum,

Insurance = 0.05% per annum based on total purchases,

Depreciation = 1% per annum of all items purchased.

(i) Calculate *EOQ* of the component.

(ii) The percentage changes in total annual variable costs relating to component if the annual usage happens to be (a) 125,000 and (b) 75,000.

*****.

CHAPTER 7

GAME THEORY

Introduction to game theory

Game theory seeks to analyse competing situations which arise out of conflicts of interest. Abraham Maslow's hierarchical model of human needs lays emphasis on fulfilling the basic needs such as food, water, clothes, shelter, air, safety and security. There is conflict of interest between animals and plants in the consumption of natural resources. Animals compete among themselves for securing food. Man competes with animals to earn his food. A man also competes with another man. In the past, nations waged wars to expand the territory of their rule. In the present day world, business organizations compete with each other in getting the market share. The conflicts of interests of human beings are not confined to the basic needs alone. Again considering Abraham Maslow's model of human needs, one can realize that conflicts also arise due to the higher levels of human needs such as love, affection, affiliation, recognition, status, dominance, power, esteem, ego, self-respect, etc. Sometimes one witnesses clashes of ideas of intellectuals also. Every intelligent and rational participant in a conflict wants to be a winner but not all participants can be the winners at a time. The situations of conflict gave birth to Darwin's theory of the 'survival of the fittest'. Nowadays the concepts of conciliation, co-existence, co-operation, coalition and consensus are gaining ground. Game theory is another tool to examine situations of conflict so as to identify the courses of action to be followed and to take appropriate decisions in the long run. Thus this theory assumes importance from managerial perspectives. The pioneering work on the theory of games was done by von Neumann and Morgenstern through their publication entitled 'The Theory of Games and Economic Behaviour' and subsequently the subject was developed by several experts. This theory can offer valuable guidelines to a manager in 'strategic management' which can be used in the decision making process for merger, take-over, joint venture, etc. The results obtained by the application of this theory can serve as an early warning to the top level management in meeting the threats from the competing business organizations and for the conversion of the internal weaknesses and external threats into opportunities and strengths, thereby achieving the goal of maximization of profits. While this theory does not describe any procedure to play a game, it will enable a participant to select the appropriate strategies to be followed in the pursuit of his goals. The situation of failure in a game would activate a participant in the analysis of the relevance of the existing strategies and lead him to identify better, novel strategies for the future occasions.

Definitions of game theory

There are several definitions of game theory. A few standard definitions are presented below. In the perception of Robert Mockler, "Game theory is a mathematical technique helpful in making decisions in situations of conflicts, where the success of one part depends at the expense of others, and where the individual decision maker is not in complete control of the factors influencing the outcome".

The definition given by William G. Nelson runs as follows: "Game theory, more properly **the theory of games of strategy**, is a mathematical method of analyzing a conflict. The

alternative is not between this decision or that decision, but between this strategy or that strategy to be used against the conflicting interest”.

In the opinion of Martin Shubik, “Game theory is a method of the study of decision making in situation of conflict. It deals with human processes in which the individual decision-unit is not in complete control of other decision-units entering into the environment”.

According to von Neumann and Morgenstern, “The ‘Game’ is simply the totality of the rules which describe it. Every particular instance at which the game is played – in a particular way – from beginning to end is a ‘play’. The game consists of a sequence of moves, and the play of a sequence of choices”.

J.C.C McKinsey points out a valid distinction between two words, namely ‘game’ and ‘play’. According to him, “game refers to a particular realization of the rules”.

In the words of O.T. Bartos, “The theory of games can be used for ‘prescribing’ how an intelligent person should go about resolving social conflicts, ranging all the way from open warfare between nations to disagreements between husband and wife”.

Martin K Starr gave the following definition: “Management models in the competitive sphere are usually termed game models. By studying game theory, we can obtain substantial information into management’s role under competitive conditions, even though much of the game theory is neither directly operational nor implementable”.

According to Edwin Mansfield, “A game is a competitive situation where two or more persons pursue their own interests and no person can dictate the outcome. Each player, an entity with the same interests, make his own decisions. A player can be an individual or a group”.

Assumptions for a Competitive Game

Game theory helps in finding out the best course of action for a firm in view of the anticipated countermoves from the competing organizations. A competitive situation is a competitive game if the following properties hold:

1. The number of competitors is finite, say N .
2. A finite set of possible courses of action is available to each of the N competitors.
3. A play of the game results when each competitor selects a course of action from the set of courses available to him. In game theory we make an important assumption that all the players select their courses of action simultaneously. As a result, no competitor will be in a position to know the choices of his competitors.
4. The outcome of a play consists of the particular courses of action chosen by the individual players. Each outcome leads to a set of payments, one to each player, which may be either positive, or negative, or zero.

Managerial Applications of the Theory of Games

The techniques of game theory can be effectively applied to various managerial problems as detailed below:

- 1) Analysis of the market strategies of a business organization in the long run.
- 2) Evaluation of the responses of the consumers to a new product.

- 3) Resolving the conflict between two groups in a business organization.
- 4) Decision making on the techniques to increase market share.
- 5) Material procurement process.
- 6) Decision making for transportation problem.
- 7) Evaluation of the distribution system.
- 8) Evaluation of the location of the facilities.
- 9) Examination of new business ventures and
- 10) Competitive economic environment.

Key concepts in the Theory of Games

Several of the key concepts used in the theory of games are described below:

Players:

The competitors or decision makers in a game are called the players of the game.

Strategies:

The alternative courses of action available to a player are referred to as his strategies.

Pay off:

The outcome of playing a game is called the pay off to the concerned player.

Optimal Strategy:

A strategy by which a player can achieve the best pay off is called the optimal strategy for him.

Zero-sum game:

A game in which the total payoffs to all the players at the end of the game is zero is referred to as a zero-sum game.

Non-zero sum game:

Games with “less than complete conflict of interest” are called non-zero sum games. The problems faced by a large number of business organizations come under this category. In such games, the gain of one player in terms of his success need not be completely at the expense of the other player.

Payoff matrix:

The tabular display of the payoffs to players under various alternatives is called the payoff matrix of the game.

Pure strategy:

If the game is such that each player can identify one and only one strategy as the optimal strategy in each play of the game, then that strategy is referred to as the best strategy for that player and the game is referred to as a game of pure strategy or a pure game.

Mixed strategy:

If there is no one specific strategy as the 'best strategy' for any player in a game, then the game is referred to as a game of mixed strategy or a mixed game. In such a game, each player has to choose different alternative courses of action from time to time.

N-person game:

A game in which N-players take part is called an N-person game.

Maximin-Minimax Principle :

The maximum of the minimum gains is called the maximin value of the game and the corresponding strategy is called the maximin strategy. Similarly the minimum of the maximum losses is called the minimax value of the game and the corresponding strategy is called the minimax strategy. If both the values are equal, then that would guarantee the best of the worst results.

Negotiable or cooperative game:

If the game is such that the players are taken to cooperate on any or every action which may increase the payoff of either player, then we call it a negotiable or cooperative game.

Non-negotiable or non-cooperative game:

If the players are not permitted for coalition then we refer to the game as a non-negotiable or non-cooperative game.

Saddle point:

A saddle point of a game is that place in the payoff matrix where the maximum of the row minima is equal to the minimum of the column maxima. The payoff at the saddle point is called **the value of the game** and the corresponding strategies are called the **pure strategies**.

Dominance:

One of the strategies of either player may be inferior to at least one of the remaining ones. The superior strategies are said to dominate the inferior ones.

Types of Games:

There are several classifications of a game. The classification may be based on various factors such as the number of participants, the gain or loss to each participant, the number of strategies available to each participant, etc. Some of the important types of games are enumerated below.

Two person games and n-person games:

In two person games, there are exactly two players and each competitor will have a finite number of strategies. If the number of players in a game exceeds two, then we refer to the game as n-person game.

Zero sum game and non-zero sum game:

If the sum of the payments to all the players in a game is zero for every possible outcome of the game, then we refer to the game as a zero sum game. If the sum of the payoffs from any play of the game is either positive or negative but not zero, then the game is called a non-zero sum game.

Games of perfect information and games of imperfect information:

A game of perfect information is the one in which each player can find out the strategy that would be followed by his opponent. On the other hand, a game of imperfect information is the one in which no player can know in advance what strategy would be adopted by the competitor and a player has to proceed in his game with his guess works only.

Games with finite number of moves / players and games with unlimited number of moves:

A game with a finite number of moves is the one in which the number of moves for each player is limited before the start of the play. On the other hand, if the game can be continued over an extended period of time and the number of moves for any player has no restriction, then we call it a game with unlimited number of moves.

Constant-sum games:

If the sum of the game is not zero but the sum of the payoffs to both players in each case is constant, then we call it a constant sum game. It is possible to reduce such a game to a zero-sum game.

2x2 two person game and 2xn and mx2 games:

When the number of players in a game is two and each player has exactly two strategies, the game is referred to as 2x2 two person game.

A game in which the first player has precisely two strategies and the second player has three or more strategies is called an 2xn game. A game in which the first player has three or more strategies and the second player has exactly two strategies is called an mx2 game.

3x3 and large games:

When the number of players in a game is two and each player has exactly three strategies, we call it a 3x3 two person game.

Two-person zero sum games are said to be larger if each of the two players has 3 or more choices.

The examination of 3x3 and larger games involves difficulties. For such games, the technique of linear programming can be used as a method of solution to identify the optimum strategies for the two players.

Non-constant games :

Consider a game with two players. If the sum of the payoffs to the two players is not constant in all the plays of the game, then we call it a non-constant game.

Such games are divided into negotiable or cooperative games and non-negotiable or non-cooperative games.

TWO-PERSON ZERO SUM GAMES

Definition of two-person zero sum game

A game with only two players, say player A and player B, is called a two-person zero sum game if the gain of the player A is equal to the loss of the player B, so that the total sum is zero.

Payoff matrix:

When players select their particular strategies, the payoffs (gains or losses) can be represented in the form of a payoff matrix.

Since the game is zero sum, the gain of one player is equal to the loss of other and vice-versa. Suppose A has m strategies and B has n strategies. Consider the following payoff matrix:

Strategies of A	Strategies of B		
	B ₁	B ₂	N
A ₁	a ₁₁	a ₁₂	
A ₂	a ₂₁	a ₂₂	
A _m			

Assumptions for two-person zero sum game:

For building any model, certain reasonable assumptions are quite necessary. Some assumptions for building a model of two-person zero sum game are listed below.

- (i) Each player has available to him a finite number of possible courses of action. Sometimes the set of courses of action may be the same for each player. Or, certain courses of action may be available to both players while each player may have certain specific courses of action which are not available to the other player.
- (ii) Player A attempts to maximize gains to himself. Player B tries to minimize losses to himself.
- (iii) The decisions of both players are made individually prior to the play with no communication between them.
- (iv) The decisions are made and announced simultaneously so that neither player has an advantage resulting from direct knowledge of the other player's decision.
- (v) Both players know the possible payoffs of themselves and their opponents.

Minimax and Maximin Principles

The selection of an optimal strategy by each player without the knowledge of the competitor's strategy is the basic problem of playing games.

The objective of game theory is to know how these players must select their respective strategies, so that they may optimize their payoffs. Such a criterion of decision making is referred to as minimax-maximin principle. This principle in games of pure strategies leads to the best possible selection of a strategy for both players.

For example, if player A chooses his i^{th} strategy, then he gains at least the payoff $\min_{j} a_{ij}$, which is minimum of the i^{th} row elements in the payoff matrix. Since his objective is to maximize his payoff, he can choose strategy i so as to make his payoff as large as possible, i.e., a payoff which is not less than $\max_i \min_j a_{ij}$. Similarly player B can choose j^{th} column elements so as to make his loss not greater than $\min_j \max_i a_{ij}$.

If the maximin value for a player is equal to the minimax value for another player, then the game is said to have a saddle point (equilibrium point) and the corresponding strategies are called optimal strategies. If there are two or more saddle points, they must be equal. The amount of pay off, i.e; V at an equilibrium point is known as the Value of game.

The optimal strategies can be identified by the players in the long run.

The game is said to be fair if the value of the game $V = 0$.

Problem: Solve the game with the following pay-off matrix.

		Player B Strategies				
		I	II	III	IV	V
Player A Strategies	1	-2	5	-3	6	7
	2	4	6	8	-1	6
	3	8	2	3	5	4
	4	15	14	18	12	20

Solution:

First consider the minimum of each row:

Row	Minimum Value
1	-3
2	-1
3	2
4	12

Maximum of $\{-3, -1, 2, 12\} = 12$

Next consider the maximum of each column:

Column	Maximum Value
1	15
2	14
3	18
4	12
5	20

Minimum of $\{15, 14, 18, 12, 20\} = 12$

We see that the maximum of row minima = the minimum of the column maxima. So the game has a saddle point. The common value is 12. Therefore the value V of the game = 12.

Interpretation:

In the long run, the following best strategies will be identified by the two players:

The best strategy for player A is strategy 4. The best strategy for player B is strategy IV.

The game is favourable to player A.

Problem 2:

Solve the game with the following pay-off matrix:

Strategies of Player X	Strategies of Player Y				
	<i>I</i>	<i>II</i>	<i>III</i>	<i>IV</i>	<i>V</i>
1	9	12	7	14	26
2	25	35	20	28	30
3	7	6	-8	3	2
4	8	11	13	-2	1

Solution:

First consider the minimum of each row.

Row	Minimum Value
1	7
2	20
3	-8
4	-2

Maximum of $\{7, 20, -8, -2\} = 20$

Next consider the maximum of each column.

Column	Maximum Value
1	25
2	35
3	20
4	28
5	30

Minimum of $\{25, 35, 20, 28, 30\} = 20$

It is observed that the maximum of row minima and the minimum of the column maxima are equal. Hence the given the game has a saddle point. The common value is 20. This indicates that the value V of the game is 20.

Interpretation: The best strategy for player X is strategy 2. The best strategy for player Y is strategy III. The game is favourable to player A.

Problem 3:

Solve the following game:

Player A (Strategies)	Player B (strategies)			
	<i>I</i>	<i>II</i>	<i>III</i>	<i>IV</i>
1	1	-6	8	4
2	3	-7	2	-8
3	5	-5	-1	0
4	3	-4	5	7

Solution:

First consider the minimum of each row:

Row	Minimum Value
1	-6
2	-8
3	-5
4	-4

Maximum of $\{-6, -8, -5, -4\} = -4$

Next consider the maximum of each column.

Column	Maximum Value
1	5
2	-4
3	8
4	7

Minimum of $\{5, -4, 8, 7\} = -4$

Since the $\max \{\text{row minima}\} = \min \{\text{column maxima}\}$, the game under consideration has a saddle point. The common value is -4 . Hence the value of the game is -4 .

Interpretation:

The best strategy for player A is strategy 4. The best strategy for player B is strategy II. Since the value of the game is negative, it is concluded that the game is favourable to player B.

REVIEW QUESTIONS:

1. Explain the concept of a game.
2. Define a game.

3. State the assumptions for a competitive game.
4. State the managerial applications of the theory of games.
5. Explain the following terms: strategy, pay-off matrix, saddle point, pure strategy and mixed strategy.
6. Explain the following terms: two person game, two person zero sum game, value of a game, $2 \times n$ game and $m \times 2$ game.
7. What is meant by a two-person zero sum game? Explain.
8. State the assumptions for a two-person zero sum game.
9. Explain Minimax and Maximin principles.
10. How will you interpret the results from the payoff matrix of a two-person zero sum game? Explain.
11. What is a fair game? Explain.
12. Solve the game with the following pay-off matrix.

		Player B Strategies				
		<i>I</i>	<i>II</i>	<i>III</i>	<i>IV</i>	<i>V</i>
Player A Strategies	1	7	5	2	3	9
	2	10	8	7	4	5
	3	9	12	0	2	1
	4	11	2	--1	3	4

13. Solve the game with the following pay-off matrix.

		Player B Strategies				
		<i>I</i>	<i>II</i>	<i>III</i>	<i>IV</i>	<i>V</i>
Player A Strategies	1	--2	-- 3	8	7	0
	2	1	-- 7	--5	--2	3
	3	4	-- 2	3	5	--1
	4	6	-- 4	5	4	7

CHAPTER 8

QUEUEING THEORY

Introduction:

A flow of customers from finite or infinite population towards the service facility forms a **queue (waiting line)** an account of lack of capability to serve them all at a time. In the absence of a perfect balance between the service facilities and the customers, **waiting time** is required either for the service facilities or for the customers arrival. In general, the **queueing system** consists of one or more queues and one or more servers and operates under a set of procedures. Depending upon the server status, the incoming customer either waits at the queue or gets the turn to be served. If the server is free at the time of arrival of a customer, the customer can directly enter into the counter for getting service and then leave the system. In this process, over a period of time, the system may experience “Customer waiting” and/or “Server idle time”

Queueing System:

A queueing system can be completely described by

- (1) the input (arrival pattern)
- (2) the service mechanism (service pattern)
- (3) The queue discipline and
- (4) Customer's behaviour

The input (arrival pattern)

The input described the way in which the customers arrive and join the system. Generally, customers arrive in a more or less random manner which is not possible for prediction. Thus the arrival pattern can be described in terms of probabilities and consequently the probability distribution for **inter-arrival** times (the time between two successive arrivals) must be defined. We deal with those Queueing system in which the customers arrive in poisson process. The mean arrival rate is denoted by λ .

The Service Mechanism:

This means the arrangement of service facility to serve customers. If there is infinite number of servers, then all the customers are served instantaneously or arrival and there will be no queue. If the number of servers is finite then the customers are served according to a specific order with service time a constant or a random variable. Distribution of service time follows '**Exponential distribution**' defined by

$$f(t) = \lambda e^{-\lambda t}, t > 0$$

The mean Service rate is $E(t) = 1/\lambda$

Queueing Discipline:

It is a rule according to which the customers are selected for service when a queue has been formed. The most common disciplines are

1. First come first served – (FCFS)
2. First in first out – (FIFO)
3. Last in first out – (LIFO)
4. Selection for service in random order (SIRO)

Customer's behaviour

1. Generally, it is assumed that the customers arrive into the system one by one. But in some cases, customers may arrive in groups. Such arrival is called **Bulk arrival**.
2. If there is more than one queue, the customers from one queue may be tempted to join another queue because of its smaller size. This behaviour of customers is known as **jockeying**.
3. If the queue length appears very large to a customer, he/she may not join the queue. This property is known as **Balking** of customers.
4. Sometimes, a customer who is already in a queue will leave the queue in anticipation of longer waiting line. This kind of departure is known as **reneging**.

List of Variables

The list of variable used in queuing models is give below:

- N - No of customers in the system
- C - No of servers in the system
- $P_n(t)$ - Probability of having n customers in the system at time t .
- P_n - Steady state probability having customers in the system
- P_0 - Probability of having zero customer in the system
- L_q - Average number of customers waiting in the queue.
- L_s - Average number of customers waiting in the system (in the queue and in the service counters)
- W_q - Average waiting time of customers in the queue.
- W_s - Average waiting time of customers in the system (in the queue and in the service counters)
- λ - Arrival rate of customers
- μ - Service rate of server
- \emptyset - Utilization factor of the server
- λ - Effective rate of arrival of customers
- M - Poisson distribution
- N - Maximum numbers of customers permitted in the system. Also, it denotes the size of the calling source of the customers.
- GD - General discipline for service. This may be first-in-first serve (FIFS), last-in-first-serve (LIFS), random order (Ro), etc.

Traffic Intensity (or Utilization Factor)

An important measure of a simple queue is its traffic intensity given by

$$\begin{aligned} \text{Traffic intensity } \rho &= \text{Mean arrival time/Mean service time} \\ &= \lambda/\mu \quad (< 1) \end{aligned}$$

1. Classification of Queuing Models

Generally, queuing models can be classified into six categories using Kendall's notation with six parameters to define a model. The parameters of this notation are:

P- Arrival rate distribution ie probability law for the arrival /inter – arrival time.

Q - Service rate distribution, ie probability law according to which the customers are being served.

R - Number of Servers (ie number of service stations)

X - Service discipline

Y - Maximum number of customers permitted in the system. Z - Size of the calling source of the customers.

A queuing model with the above parameters is written as: (P/Q/R : X/Y/Z)

Model 1 : (M/M/1) : (GD/ ∞ / ∞) Model

In this model:

- (i) the arrival rate follows poisson (M) distribution.
- (ii) Service rate follows poisson distribution (M)
- (iii) Number of servers is 1
- (iv) Service discipline is general discipline (ie GD)
- (v) Maximum number of customers permitted in the system is infinite (∞)
- (vi) Size of the calling source is infinite (∞)

The steady state equations to obtain, P_n the probability of having customers in the system and the values for L_s , L_q , W_s and W_q are given below:

$$n = 0, 1, 2, \dots, \infty \text{ where } \rho = \lambda / \mu < 1$$

L_s – Average number of customers waiting in the system (i.e; waiting in the queue and in the service station)

$$P_n = \rho^n (1 - \rho)$$

$$L_s = \rho / (1 - \rho)$$

$$W_q = \rho / (\mu - \lambda)$$

Question 1:

At one-man barber shop, customers arrive according to Poisson distribution with mean arrival rate of 5 per hour and the hair cutting time was exponentially distributed with an average hair cut taking 10 minutes. It is assumed that because of his excellent reputation, customers were always willing to wait. Calculate the following:

- (i) Average number of customers in the shop and the average numbers waiting for a haircut.
- (ii) The percentage of time arrival can walk in straight without having to wait.
- (iii) The percentage of customers who have to wait before getting into the barber's chair.

Solution:

$$\begin{aligned} \text{Given mean arrival of customer } \lambda &= 5/60 = 1/12 \\ \text{and mean time for server } \mu &= 1/10 \\ \therefore \phi = \lambda / \mu &= [1/12] \times 10 = 10/12 \\ &= \mathbf{0.833} \end{aligned}$$

- (i) Average number of customers in the system (numbers in the queue and in the service station) L_s

$$\begin{aligned} &= \phi / 1 - \phi = 0.83 / 1 - 0.83 \\ &= 0.83 / 0.17 \\ &= 4.88 \\ &= \mathbf{5 \text{ Customers}} \end{aligned}$$

- (ii) The percentage of time arrival can walk straight into barber's chair without waiting is Service utilization $= \phi$ %

$$\begin{aligned} &= \lambda / \mu \% \\ &= 0.833 \times 100 \\ &= \mathbf{83.3} \end{aligned}$$

Question: 2

Vehicles are passing through a toll gate at the rate of 70 per hour. The average time to pass through the gate is 45 seconds. The arrival rate and service rate follow poisson distribution. There is a complaint that the vehicles wait for a long duration. The authorities are willing to install one more gate to reduce the average time to pass through the toll gate to 35 seconds if the idle time of the toll gate is less than 9% and the average queue length at the gate is more than 8 vehicle, check whether the installation of the second gate is justified?

Solution:

$$\begin{aligned} \text{Arrival rate of vehicles at the toll gate } \lambda &= 70 \text{ per hour} \\ \text{Time taken to pass through the gate} &= 45 \text{ Seconds} \end{aligned}$$

$$\begin{aligned}\text{Service Rate } \mu &= 1 \text{ hr}/45 \text{ seconds} = 3600 / 45 = 80 \\ &= 80 \text{ vehicles per hour}\end{aligned}$$

$$\begin{aligned}\therefore \text{Utilization factor } \phi &= \lambda / \mu \\ &= 70 / 80 \\ &= \mathbf{0.875}\end{aligned}$$

$$\begin{aligned}\text{(a) Waiting no. of vehicles in the queue is } L_q \\ L_q &= \phi^2 / (1 - \phi) = (0.875)^2 / (1 - 0.875) \\ &= 0.7656 / 0.125 = 6.125 \\ &= 6 \text{ vehicles}\end{aligned}$$

$$\begin{aligned}\text{(b) Revised time taken to pass through the gate} &= 30 \text{ seconds} \\ \therefore \text{The new service rate after installation of an additional gate} &= 1 \text{ hour}/35 \text{ Seconds} \\ &= 3600/35\end{aligned}$$

$$= \mathbf{102.68 \text{ Vehicles / hour}}$$

$$\begin{aligned}\therefore \text{Utilization factor } \phi &= \lambda / \mu = 70 / 102.86 \\ &= \mathbf{0.681}\end{aligned}$$

$$\begin{aligned}\text{Percentage of idle time of the gate} &= (1 - \phi)\% \\ &= (1 - 0.681)\% \\ &= 0.319\% \\ &= 31.9 \\ &= \mathbf{32\%}\end{aligned}$$

This idle time is not less than 9% which is expected.

Therefore the installation of the second gate is not justified since the average waiting number of vehicles in the queue is more than 8 but the idle time is not less than 32%. Hence idle time is far greater than the number of vehicles waiting in the queue.

Second Model (M/M/C) : (GD/ ∞ / ∞) Model

The parameters of this model are as follows:

- (i) Arrival rate follows poisson distribution
- (ii) Service rate follows poisson distribution
- (iii) No of servers is C'.
- (iv) Service discipline is general discipline.
- (v) Maximum number of customers permitted in the system is infinite

Then the steady state equation to obtain the probability of having n customers in the system is $P_n = \phi^n P_0$, $0 \leq n \leq C$.

REVIEW QUESTIONS:

1. What is Queuing Theory?
2. What do you mean by arrival pattern?
3. What is meant by service mechanism?
4. What do you mean by Queuing Discipline?
5. What are the different types of customers based on queuing behaviour?
6. Explain about any one of the Queuing Models.

CHAPTER 9

MARKOV ANALYSIS

Introduction

Markov analysis is a method of analysing the current movement (behaviour) of some variable in order to predict the future movement (behaviour) of that variable. This was first used by the Russian Mathematician, A Markov to predict the behaviour of gas particles in a closed container. That is why this analysis is known as Markov analysis. As a management tool, Markov analysis has been used during the last several years. Today, as a marketing aid, it is used for examining and predicting the behaviour of customers in terms of their brand loyalty and their switching pattern to other brands. In the field of accounting, it can be applied to the behaviour of accounts receivable. It can also be used for determining the future manpower requirements of an organisation.

Markov Process

Markov process is also called Markov chain or Markov sequence. Markov Process is a stochastic process (random process) which is used to analyse the decision problems in which the occurrence of a specific event depends on the occurrence of the event immediately preceding to the current event. Basically, two things are very important in Markov process. They are:

- (a) A specific state of the system being studied
- (b) The state-transition relationship

State and Transition Probabilities

The occurrence of an event at a specified point of time puts the system in a given situation or condition or status. This particular situation of a system resulted by the occurrence of an event is called “*state*.” If another event occurs after the passage of one time unit, it would put the system in another situation (called another “*state*”).

The probability of moving from one state to another or remain in the same state in a single time period is called “transition probability.” Since the probability of moving from one state to another depends on the probability of the preceding state, transition probability is a conditional probability.

Characteristics (Assumptions) of a Markov Process

The Markov chain analysis is based on the following assumptions:

- (1) The given system has a finite number of states, none of which “absorbing” in nature (a state is said to be absorbing if a customer would never switch to a particular brand)
- (2) The states are both collectively exhaustive and mutually exclusive.
- (3) The probability of moving from one state to another depends only on the immediately preceding state.
- (4) Transition probabilities are stationary, i.e, they are constant.
- (5) The process has a set of initial probabilities which may be given or determined.
- (6) The transition probabilities of moving to alternative states in the next time period, given a state in the current time period must sum to unity.

State Transition Matrix

A state-transition matrix is a rectangular array which summarises the transition probabilities for a given Markov process. In this matrix the rows show the current state of the system being studied and the columns show the alternative states to which the system can move.

Let S_i = state i of a stochastic process; ($i = 1, 2, 3, \dots, m$)

P_{ij} = transition probability of moving from state S_i to state S_j in one step

Then, one stage state-transition matrix \mathbf{P} can be described as given below:

$$\mathbf{P} = \begin{matrix} & \begin{matrix} S_1 & S_2 & \dots & S_m \end{matrix} \\ \begin{pmatrix} S_1 & P_{11} & P_{12} & \dots & P_{1m} \\ S_2 & P_{21} & P_{22} & \dots & P_{2m} \\ \dots & \dots & \dots & \dots & \dots \\ S_m & P_{m1} & P_{m2} & \dots & P_{mn} \end{pmatrix} \end{matrix}$$

In the transition matrix of the Markov chain, $P_{ij} = 0$ when no transition occurs from state i to state j ; and $P_{ij} = 1$ when the system is in state i , it can move only to state j at the next transition.

Each row of the transition matrix represents a one-step transition probability distribution over all states. This means:

$$P_{i1} + P_{i2} + P_{i3} + \dots + P_{im} = 1 \quad \text{for all } i \text{ and } 0 \leq P_{ij} \leq 1.$$

REVIEW QUESTIONS:

1. What is Markov process?
2. What do you mean by state?
3. What do you mean by transition probability?

4. What are the important assumptions of a Markov process?
5. What do you mean by a state-transition matrix?

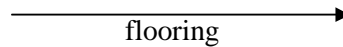
CHAPTER 10

PROJECT NETWORK

Certain key concepts pertaining to a project network are described below:

1. Activity

An activity means a work. A project consists of several activities. An activity takes time. It is represented by an arrow in a diagram of the network. For example, an activity in house construction can be flooring. This is represented as follows:



Construction of a house involves various activities. Flooring is an activity in this project. We can say that a project is completed only when all the activities in the project are completed.

2. Event

It is the beginning or the end of an activity. Events are represented by circles in a project network diagram. The events in a network are called the **nodes**. **Example:**



Starting a punching machine is an activity. Stopping the punching machine is another activity.

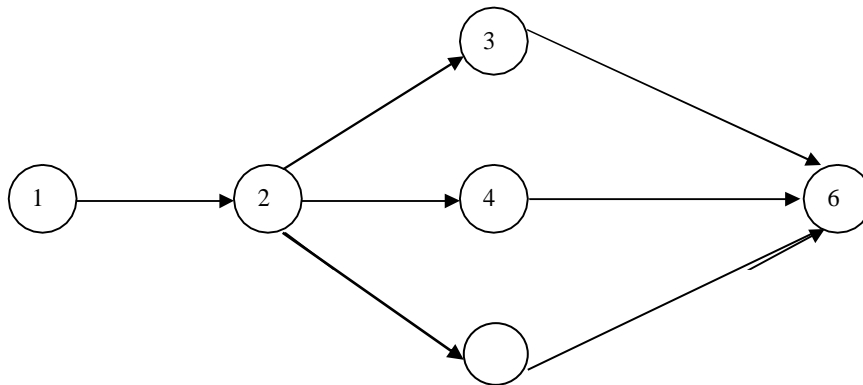
3. Predecessor Event

The event just before another event is called the predecessor event.

4. Successor Event

The event just following another event is called the successor event.

Example: Consider the following.



In this diagram, event 1 is predecessor for the event 2.

Event 2 is successor to event 1.

Event 2 is predecessor for the events 3, 4 and 5.

Event 4 is predecessor for the event 6.

Event 6 is successor to events 3, 4 and 5.

5. Network

A network is a series of related activities and events which result in an end product or service. The activities shall follow a prescribed sequence. For example, while constructing a house, laying the foundation should take place before the construction of walls. Fitting water tapes will be done towards the completion of the construction. Such a sequence cannot be altered.

6. Dummy Activity

A dummy activity is an activity which does not consume any time. Sometimes, it may be necessary to introduce a dummy activity in order to provide connectivity to a network or for the preservation of the logical sequence of the nodes and edges.

7. Construction of a Project Network

A project network consists of a finite number of events and activities, by adhering to a certain specified sequence. There shall be a **start event** and an **end event (or stop event)**. All the other events shall be between the start and the end events. The activities shall be marked by directed arrows. An activity takes the project from one event to another event.

An event takes place at a point of time whereas an activity takes place from one point of time to another point of time.

CONSTRUCTION OF PROJECT NETWORK DIAGRAMS

Problem 1:

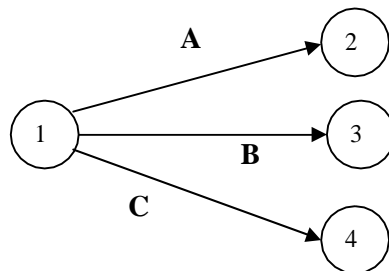
Construct the network diagram for a project with the following activities:

Activity Event→Event	Name of Activity	Immediate Predecessor Activity
1→2	A	-
1→3	B	-
1→4	C	-
2→5	D	A
3→6	E	B
4→6	F	C
5→6	G	D

Solution:

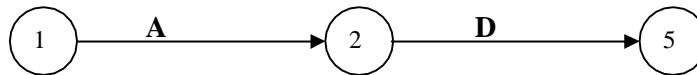
The start event is node 1.

The activities A, B, C start from node 1 and none of them has a predecessor activity. A joins nodes 1 and 2; B joins nodes 1 and 3; C joins nodes 1 and 4. So we get the following:

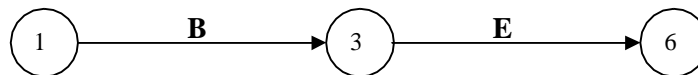


This is a part of the network diagram that is being constructed.

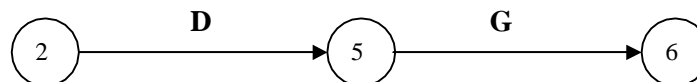
Next, activity D has A as the predecessor activity. D joins nodes 2 and 5. So we get



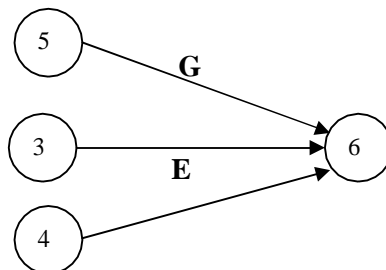
Next, activity E has B as the predecessor activity. E joins nodes 3 and 6. So we get



Next, activity G has D as the predecessor activity. G joins nodes 5 and 6. Thus we obtain

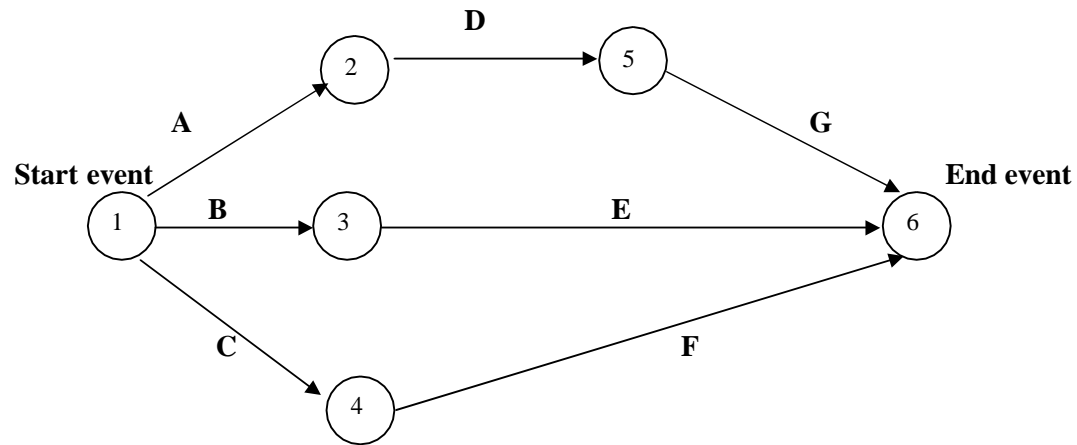


Since activities E, F, G terminate in node 6, we get 6 is the end event.



F

Combining all the pieces together, the following network diagram is obtained for the given project:



We validate the diagram by checking with the given data.

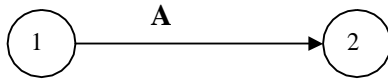
Problem 2:

Develop a network diagram for the project specified below:

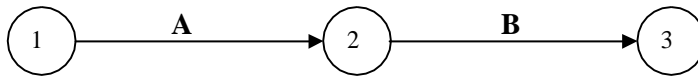
Activity	Immediate Predecessor Activity
A	-
B	A
C, D	B
E	C
F	D
G	E, F

Solution:

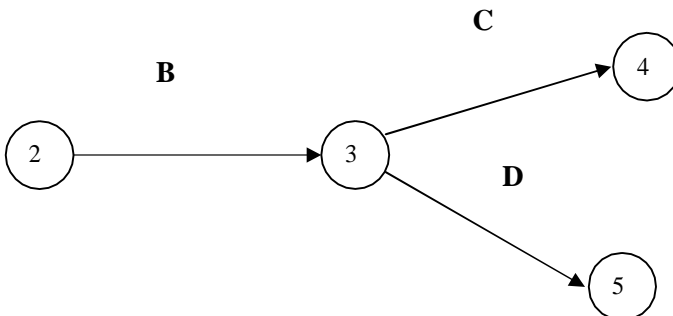
Activity A has no predecessor activity. i.e., It is the first activity. Let us suppose that activity A takes the project from event 1 to event 2. Then we have the following representation for A:



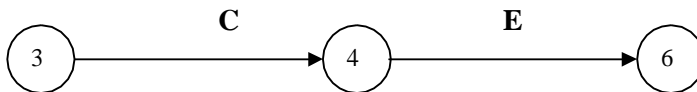
For activity B, the predecessor activity is A. Let us suppose that B joins nodes 2 and 3. Thus we get



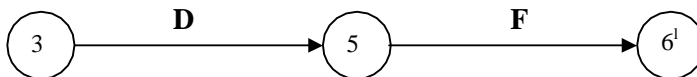
Activities C and D have B as the predecessor activity. Therefore we obtain the following:



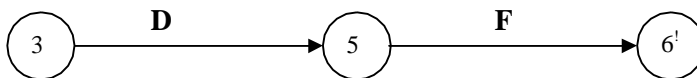
Activity E has D as the predecessor activity. So we get



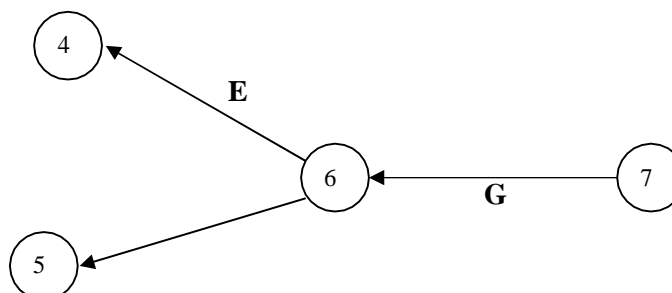
Activity F has D as the predecessor activity. So we get



Activity G has E and F as predecessor activities. This is possible only if nodes 6 and 6' are one and the same. So, rename node 6' as node 6. Then we get



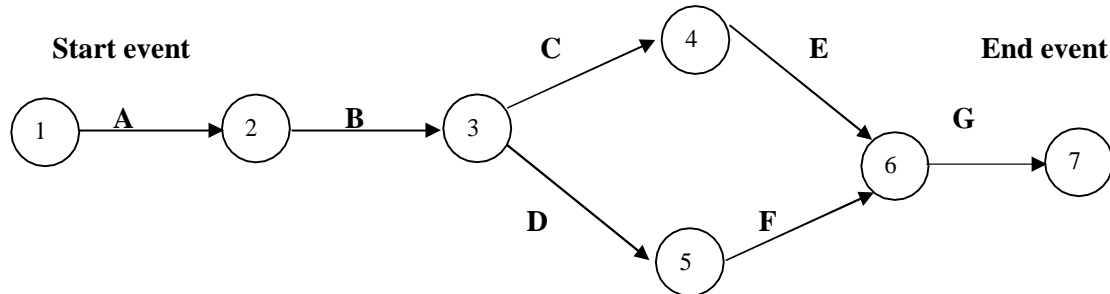
and



F

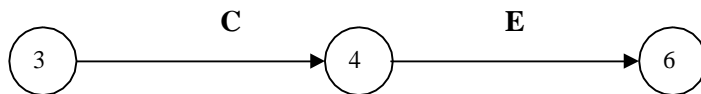
G is the last activity.

Putting all the pieces together, we obtain the following diagram the project network:

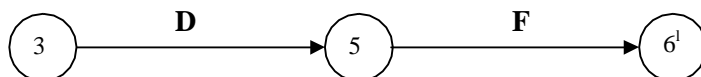


The diagram is validated by referring to the given data.

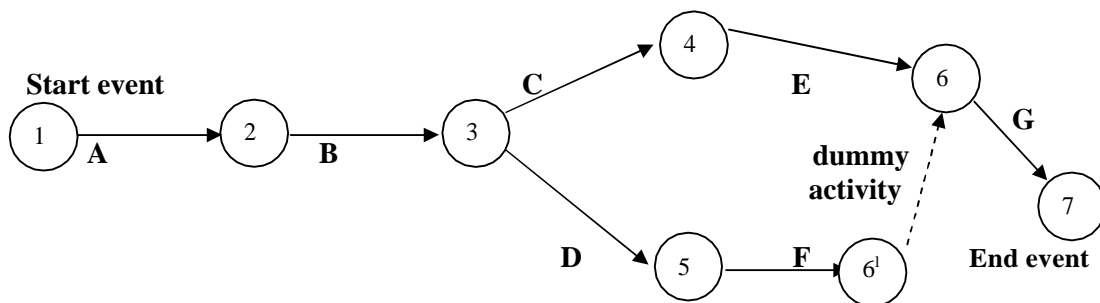
Note: An important point may be observed for the above diagram. Consider the following parts in the diagram



and



We took nodes 6 and 6' as one and the same. Instead, we can retain them as different nodes. Then, in order to provide connectivity to the network, we join nodes 6' and 6 by a dummy activity. Then we arrive at the following diagram for the project network:



REVIEW QUESTIONS:

1. Explain the terms: event, predecessor event, successor event, activity, dummy activity, net work.
2. Construct the network diagram for the following project:

Activity	Immediate Predecessor Activity
A	-
B	-
C	A
D	B
E	A
F	C, D
G	E
H	E
I	F, G
J	H, I

CRITICAL PATH METHOD (CPM)

The critical path method (CPM) aims at the determination of the time to complete a project and the important activities on which a manager shall focus attention.

ASSUMPTION FOR CPM

In CPM, it is assumed that precise time estimate is available for each activity.

PROJECT COMPLETION TIME

From the start event to the end event, the time required to complete all the activities of the project in the specified sequence is known as the project completion time.

PATH IN A PROJECT

A continuous sequence, consisting of nodes and activities alternatively, beginning with the start event and stopping at the end event of a network is called a path in the network.

CRITICAL PATH AND CRITICAL ACTIVITIES

Consider all the paths in a project, beginning with the start event and stopping at the end event. For each path, calculate the time of execution, by adding the time for the individual activities in that path.

The path with the largest time is called the **critical path** and the activities along this path are called the **critical activities** or **bottleneck activities**. The activities are called critical

because they cannot be delayed. However, a non-critical activity may be delayed to a certain extent. Any delay in a critical activity will delay the completion of the whole project. However, a certain permissible delay in a non –critical activity will not delay the completion of the whole project. It shall be noted that delay in a non-critical activity beyond a limit would certainly delay the completion the whole project. Sometimes, there may be several critical paths for a project. A project manager shall pay special attention to critical activities.

Problem 1:

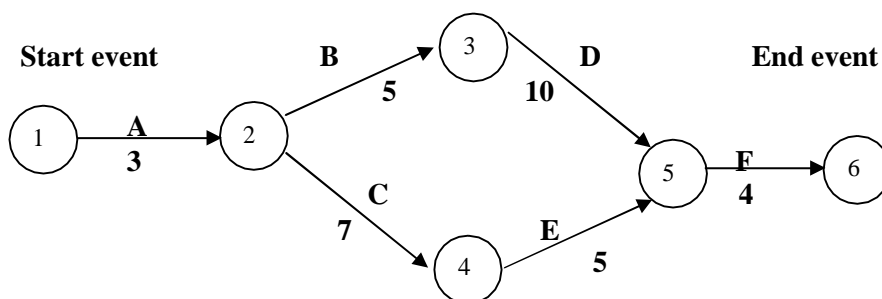
The following details are available regarding a project:

Activity	Predecessor Activity	Duration (Weeks)
A	-	3
B	A	5
C	A	7
D	B	10
E	C	5
F	D,E	4

Determine the critical path, the critical activities and the project completion time.

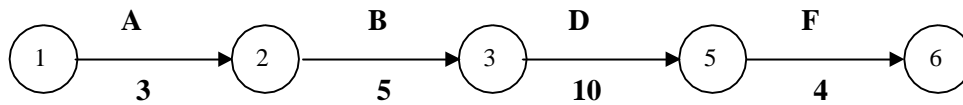
Solution:

First let us construct the network diagram for the given project. We mark the time estimates along the arrows representing the activities. We obtain the following diagram:



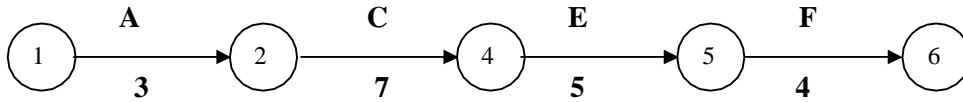
Consider the paths, beginning with the start node and stopping with the end node. There are two such paths for the given project. They are as follows:

Path I



with a time of $3 + 5 + 10 + 4 = 22$ weeks.

Path II



with a time of $3 + 7 + 5 + 4 = 19$ weeks.

Compare the times for the two paths. Maximum of $\{22, 19\} = 22$. We see that path I has the maximum time of 22 weeks. Therefore, path I is the critical path. The critical activities are A, B, D and F. The project completion time is 22 weeks.

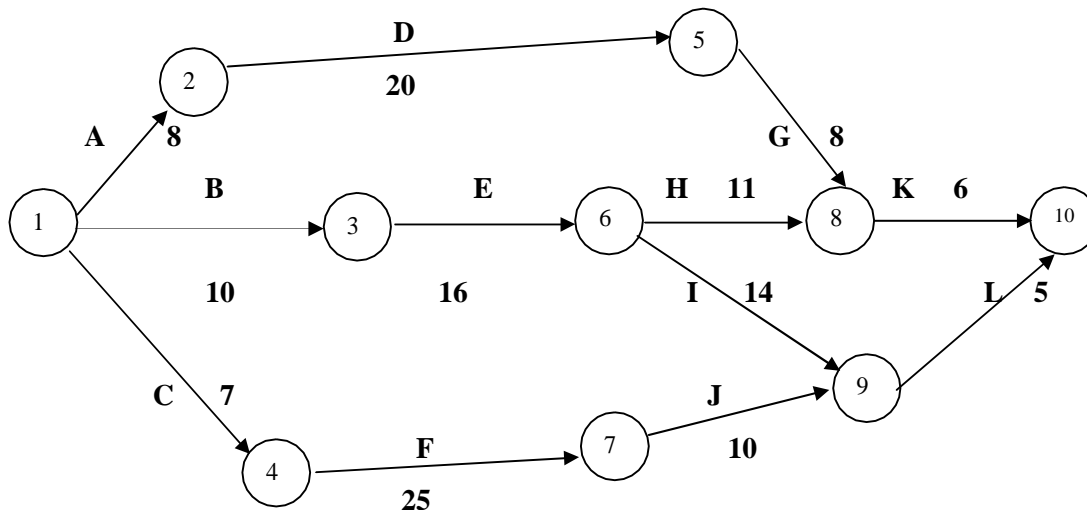
We notice that C and E are non-critical activities.

Time for path I - Time for path II = $22 - 19 = 3$ weeks.

Therefore, together the non-critical activities can be delayed up to a maximum of 3 weeks, without delaying the completion of the whole project.

Problem 2:

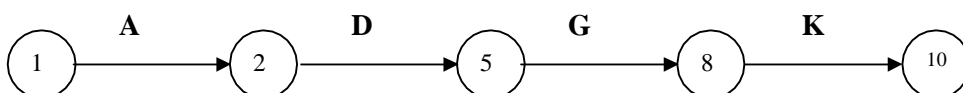
Find out the completion time and the critical activities for the following project:



Solution:

In all, we identify 4 paths, beginning with the start node of 1 and terminating at the end node of 10. They are as follows:

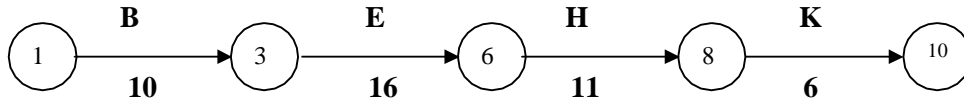
Path I



8 20 8 6

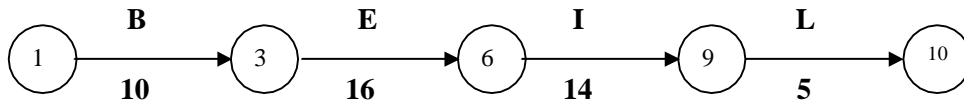
Time for the path = $8 + 20 + 8 + 6 = 42$ units of time.

Path II



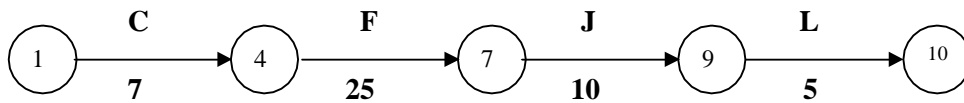
Time for the path = $10 + 16 + 11 + 6 = 43$ units of time.

Path III



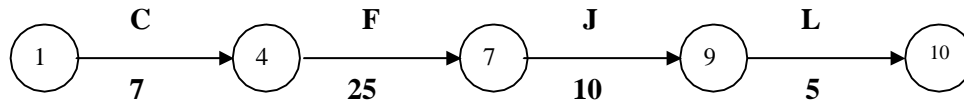
Time for the path = $10 + 16 + 14 + 5 = 45$ units of time.

Path IV



Time for the path = $7 + 25 + 10 + 5 = 47$ units of time.

Compare the times for the four paths. Maximum of $\{42, 43, 45, 47\} = 47$. We see that the following path has the maximum time and so it is the critical path:



The critical activities are C, F, J and L. The non-critical activities are A, B, D, E, G, H, I and K. The project completion time is 47 units of time.

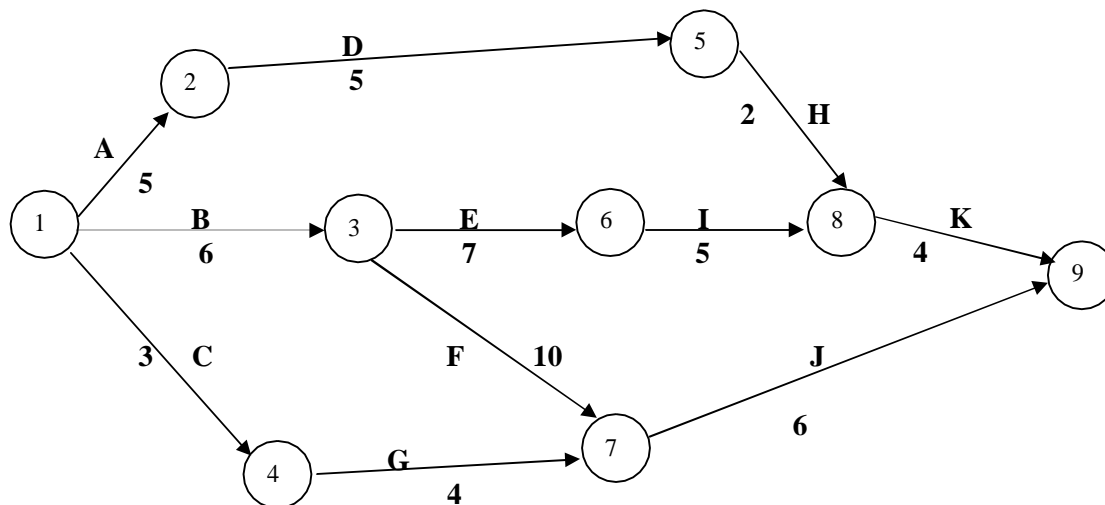
Problem 3:

Draw the network diagram and determine the critical path for the following project:

Activity	Time estimate (Weeks)
1- 2	5
1- 3	6

1-4	3
2-5	5
3-6	7
3-7	10
4-7	4
5-8	2
6-8	5
7-9	6
8-9	4

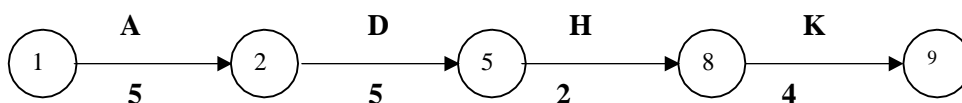
Solution: We have the following network diagram for the project:



Solution:

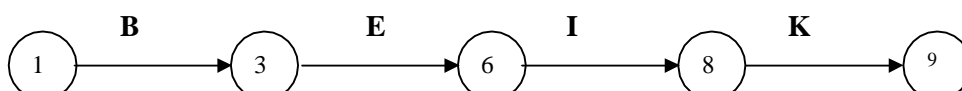
We assert that there are 4 paths, beginning with the start node of 1 and terminating at the end node of 9. They are as follows:

Path I



Time for the path = $5 + 5 + 2 + 4 = 16$ weeks.

Path II



6

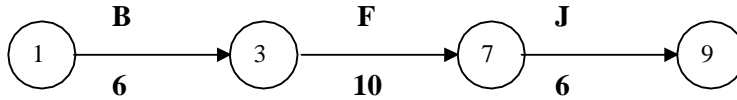
7

5

4

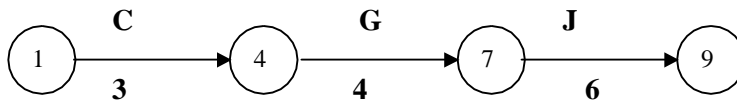
Time for the path = $6 + 7 + 5 + 4 = 22$ weeks.

Path III



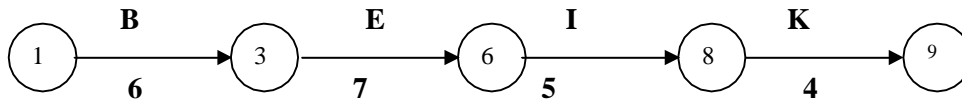
Time for the path = $6 + 10 + 6 = 16$ weeks.

Path IV



Time for the path = $3 + 4 + 6 = 13$ weeks.

Compare the times for the four paths. Maximum of $\{16, 22, 16, 13\} = 22$. We see that the following path has the maximum time and so it is the critical path:



The critical activities are B, E, I and K. The non-critical activities are A, C, D, F, G, H and J.

The project completion time is 22 weeks.

REVIEW QUESTIONS:

1. Explain the terms: critical path, critical activities.
2. The following are the time estimates and the precedence relationships of the activities in a project network:

Activity	IMMEDIATE Predecessor Activity	time estimate (weeks)
A	-	4
B	-	7
C	-	3
D	A	6
E	B	4
F	B	7
G	C	6
H	E	10
I	D	3
J	F, G	4
K	H, I	2

Draw the project network diagram. Determine the critical path and the project completion time.

PERT

Programme Evaluation and Review Technique (PERT) is a tool that would help a project manager in project planning and control. It would enable him in continuously monitoring a project and taking corrective measures wherever necessary. This technique involves statistical methods.

ASSUMPTIONS FOR PERT

Note that in CPM, the assumption is that precise time estimate is available for each activity in a project. However, one finds most of the times that this is not practically possible.

In PERT, we assume that it is not possible to have precise time estimate for each activity and instead, probabilistic estimates of time alone are possible. A multiple time estimate approach is followed here. In probabilistic time estimate, the following 3 types of estimate are possible:

1. Pessimistic time estimate (t_p)
2. Optimistic time estimate (t_o)
3. Most likely time estimate (t_m)

The optimistic estimate of time is based on the assumption that an activity will not involve any difficulty during execution and it can be completed within a short period. On the other hand, a pessimistic estimate is made on the assumption that there would be unexpected

problems during the execution of an activity and hence it would consume more time. The most likely time estimate is made in between the optimistic and the pessimistic estimates of time. Thus the three estimates of time have the relationship

$$t_o \leq t_m \leq t_p.$$

Practically speaking, neither the pessimistic nor the optimistic estimate may hold in reality and it is the most likely time estimate that is expected to prevail in almost all cases. Therefore, it is preferable to give more weight to the most likely time estimate.

We give a weight of 4 to most likely time estimate and a weight of 1 each to the pessimistic and optimistic time estimates. We arrive at a time estimate (t_e) as the weighted average of these estimates as follows:

$$t_e = \frac{t_o + 4 t_m + t_p}{6}$$

Since we have taken 6 units (1 for t_p , 4 for t_m and 1 for t_o), we divide the sum by 6. With this time estimate, we can determine the project completion time as applicable for CPM.

Since PERT involves the average of three estimates of time for each activity, this method is very practical and the results from PERT will be have a reasonable amount of reliability.

MEASURE OF CERTAINTY

The 3 estimates of time are such that

$$t_o \leq t_m \leq t_p.$$

Therefore the range for the time estimate is $t_p - t_o$.

The time taken by an activity in a project network follows a distribution with a standard deviation of one sixth of the range, approximately.

i.e., The standard deviation = $\sigma = \frac{t_p - t_o}{6}$

and the variance = $\sigma^2 = \left(\frac{t_p - t_o}{6} \right)^2$

The certainty of the time estimate of an activity can be analysed with the help of the variance. The greater the variance, the more uncertainty in the time estimate of an activity.

Problem 1:

Two experts A and B examined an activity and arrived at the following time estimates.

Expert	Time Estimate
--------	---------------

	t_o	T_m	t_p
A	4	6	8
B	4	7	10

Determine which expert is more certain about his estimates of time:

Solution:

$$\text{Variance (} \square^2 \text{) in time estimates} = \left(\frac{t_p - t_o}{6} \right)^2$$

$$\text{In the case of expert A, the variance} = \left(\frac{8 - 4}{6} \right)^2 = \frac{4}{9}$$

$$\text{As regards expert B, the variance} = \left(\frac{10 - 4}{6} \right)^2 = 1$$

So, the variance is less in the case of A. Hence, it is concluded that the expert A is more certain about his estimates of time.

Determination of Project Completion Time in PERT

Problem 2:

Find out the time required to complete the following project and the critical activities:

Activity	Predecessor Activity	Optimistic time estimate (t_o days)	Most likely time estimate (t_m days)	Pessimistic time estimate (t_p days)
A	-	2	4	6
B	A	3	6	9
C	A	8	10	12
D	B	9	12	15
E	C	8	9	10
F	D, E	16	21	26
G	D, E	19	22	25
H	F	2	5	8
I	G	1	3	5

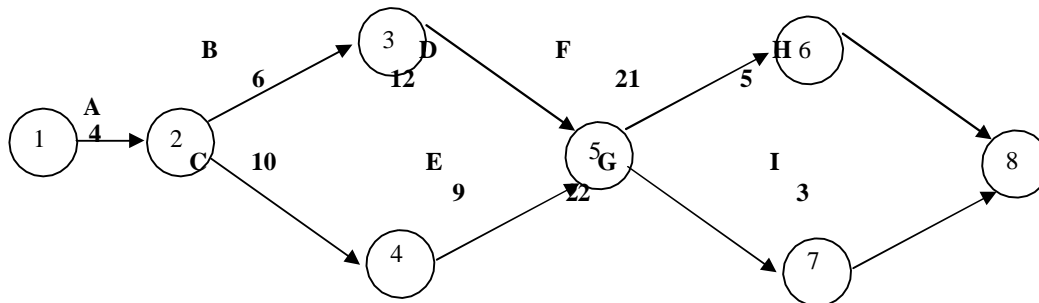
Solution:

From the three time estimates t_p , t_m and t_o , calculate t_e for each activity. We obtain the following table:

Activity	Optimistic time estimate (t_o)	4 x Most likely time estimate	Pessimistic time estimate (t_p)	$t_o + 4t_m + t_p$	Time estimate $t_e = \frac{t_o + 4t_m + t_p}{6}$
A	2	16	6	24	4
B	3	24	9	36	6
C	8	40	12	60	10
D	9	48	15	72	12
E	8	36	10	54	9
F	16	84	26	126	21

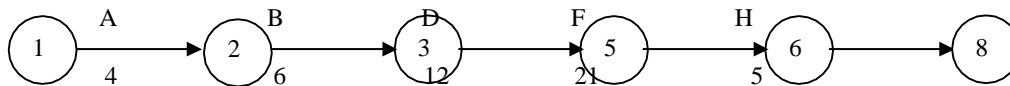
G	19	88	25	132	22
H	2	20	8	30	5
I	1	12	5	18	3

Using the single time estimates of the activities, we get the following network diagram for the project.



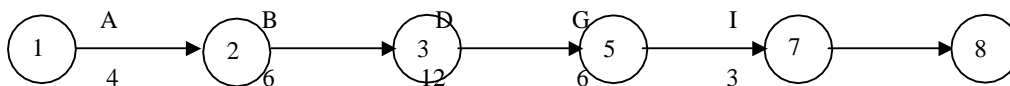
Consider the paths, beginning with the start node and stopping with the end node. There are four such paths for the given project. They are as follows:

Path I



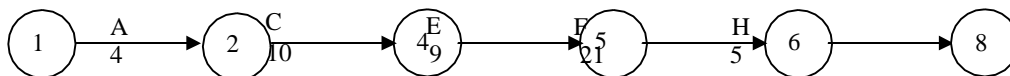
Time for the path: $4+6+12+21+5 = 48$ days.

Path II



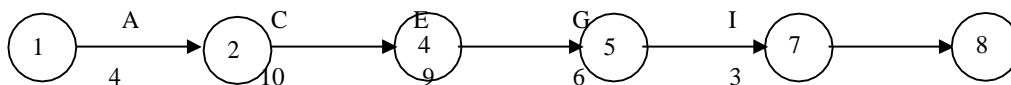
Time for the path: $4+6+12+6+3 = 31$ days.

Path III



Time for the path: $4+10+9+21+5 = 49$ days.

Path IV



Time for the path: $4+10+9+6+3 = 32$ days.

Compare the times for the four paths.

Maximum of $\{48, 31, 49, 32\} = 49$.

We see that Path III has the maximum time.

Therefore the critical path is Path III. i.e., $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 8$.

The critical activities are A, C, E, F and H.

The non-critical activities are B, D, G and I. Project time (Also called project length) = 49 days.

Problem 3:

Find out the time, variance and standard deviation of the project with the following time estimates in weeks:

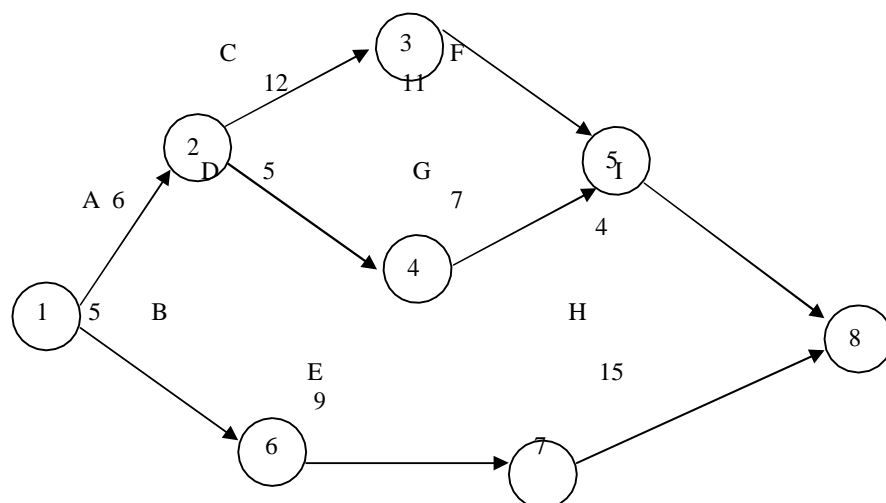
Activity	Optimistic time estimate (t_o)	Most likely time estimate (t_m)	Pessimistic time estimate (t_p)
1-2	3	6	9
1-6	2	5	8
2-3	6	12	18
2-4	4	5	6
3-5	8	11	14
4-5	3	7	11
6-7	3	9	15
5-8	2	4	6
7-8	8	16	18

Solution:

From the three time estimates t_p , t_m and t_o , calculate t_e for each activity. We obtain the following table:

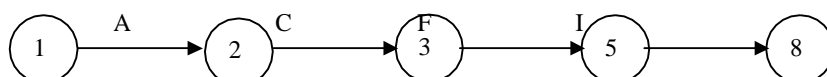
Activity	Optimistic time estimate (t_o)	4 x Most likely time estimate	Pessimistic time estimate (t_p)	$t_o + 4t_m + t_p$	Time estimate $t_e = \frac{t_o + 4t_m + t_p}{6}$
1-2	3	24	9	36	6
1-6	2	20	8	30	5
2-3	6	48	18	72	12
2-4	4	20	6	30	5
3-5	8	44	14	66	11
4-5	3	28	11	42	7
6-7	3	36	15	54	9
5-8	2	16	6	24	4
7-8	8	64	18	90	15

With the single time estimates of the activities, we get the following network diagram for the project.



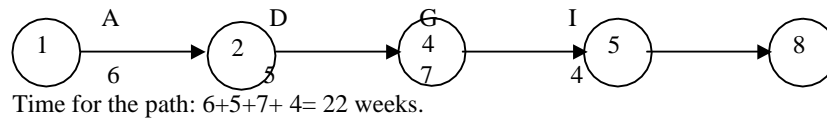
Consider the paths, beginning with the start node and stopping with the end node. There are three such paths for the given project. They are as follows:

Path I

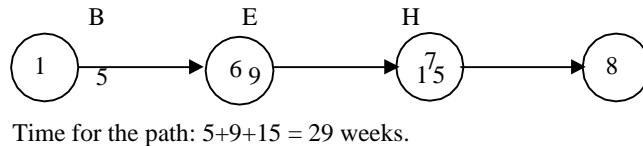


6 12 11 4
 Time for the path: $6+12+11+4 = 33$ weeks.

Path II



Path III



Compare the times for the three paths.
 Maximum of $\{33, 22, 29\} = 33$.
 It is noticed that Path I has the maximum time.
 Therefore the critical path is Path I. i.e., $1 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 8$
 The critical activities are A, C, F and I.
 The non-critical activities are B, D, G and H.
 Project time = 33 weeks.

Calculation of Standard Deviation and Variance for the Critical Activities:

Critical Activity	Optimistic time estimate (t_o)	Most likely time estimate (t_m)	Pessimistic time estimate (t_p)	Range ($t_p - t_o$)	Standard deviation = $\sigma = \frac{t_p - t_o}{6}$	Variance $\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$
A: $1 \rightarrow 2$	3	6	9	6	1	1
C: $2 \rightarrow 3$	6	12	18	12	2	4
F: $3 \rightarrow 5$	8	11	14	6	1	1
I: $5 \rightarrow 8$	2	4	6	4	$2/3$	$4/9$

Variance of project time (Also called Variance of project length) =
 Sum of the variances for the critical activities = $1+4+1+ 4/9 = 58/9$ Weeks.
 Standard deviation of project time = $\sqrt{\text{Variance}} = \sqrt{58/9} = 2.54$ weeks.

Problem 4

A project consists of seven activities with the following time estimates. Find the probability that the project will be completed in 30 weeks or less.

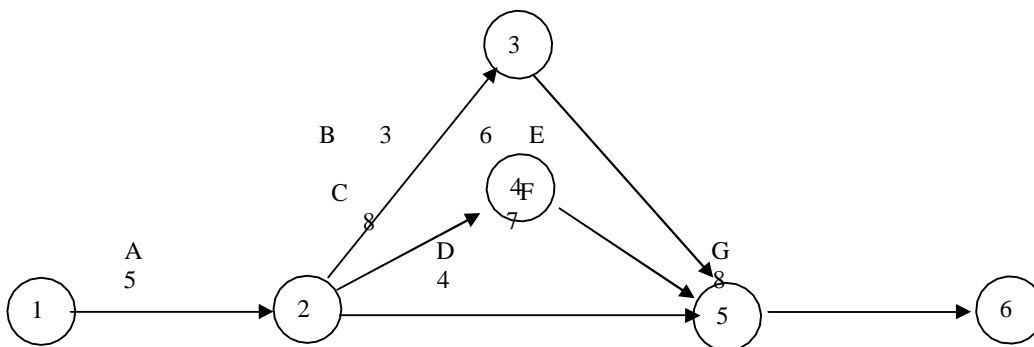
Activity	Predecessor Activity	Optimistic time estimate (t_o days)	Most likely time estimate (t_m days)	Pessimistic time estimate (t_p days)
A	-	2	5	8
B	A	2	3	4
C	A	6	8	10
D	A	2	4	6
E	B	2	6	10
F	C	6	7	8
G	D, E, F	6	8	10

Solution:

From the three time estimates t_p , t_m and t_o , calculate t_e for each activity. The results are furnished in the following table:

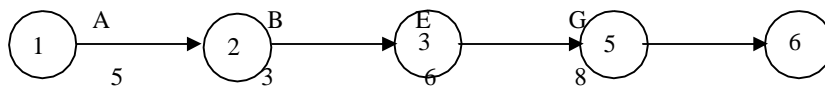
Activity	Optimistic time estimate (t_o)	4 x Most likely time estimate	Pessimistic time estimate (t_p)	$t_o + 4t_m + t_p$	Time estimate $t_e = \frac{t_o + 4t_m + t_p}{6}$
A	2	20	8	30	5
B	2	12	4	18	3
C	6	32	10	48	8
D	2	16	6	24	4
E	2	24	10	36	6
F	6	28	8	42	7
G	6	32	10	48	8

With the single time estimates of the activities, the following network diagram is constructed for the project.



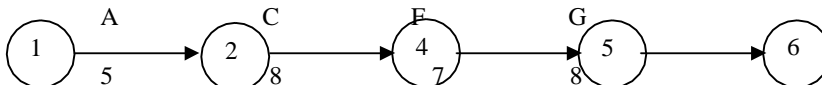
Consider the paths, beginning with the start node and stopping with the end node. There are three such paths for the given project. They are as follows:

Path I



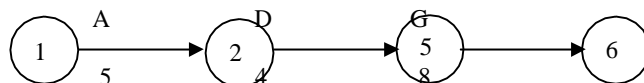
Time for the path: $5+3+6+8 = 22$ weeks.

Path II



Time for the path: $5+8+7+8 = 28$ weeks.

Path III



Time for the path: $5+4+8 = 17$ weeks.

Compare the times for the three paths.

Maximum of $\{22, 28, 17\} = 28$.

It is noticed that Path II has the maximum time.

Therefore the critical path is Path II. i.e., 1 → 2 → 4 → 5 → 6.
The critical activities are A, C, F and G.
The non-critical activities are B, D and E.
Project time = 28 weeks.

Calculation of Standard Deviation and Variance for the Critical Activities:

Critical Activity	Optimistic time estimate (t_o)	Most likely time estimate (t_m)	Pessimistic time estimate (t_p)	Range ($t_p - t_o$)	Standard deviation = $\sigma = \frac{t_p - t_o}{6}$	Variance $\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$
A: 1→2	2	5	8	6	1	1
C: 2→4	6	8	10	4	$\frac{2}{3}$	$\frac{4}{9}$
F: 4→5	6	7	8	2	$\frac{1}{3}$	$\frac{1}{9}$
G: 5→6	6	8	10	4	$\frac{2}{3}$	$\frac{4}{9}$

Standard deviation of the critical path = $\sqrt{2} = 1.414$

The standard normal variate is given by the formula

$$Z = \frac{\text{Given value of } t - \text{Expected value of } t \text{ in the critical path}}{\text{SD for the critical path}}$$

$$\text{So we get } Z = \frac{30 - 28}{1.414} = 1.414$$

We refer to the Normal Probability Distribution Table.

Corresponding to $Z = 1.414$, we obtain the value of 0.4207

We get $0.5 + 0.4207 = 0.9207$

Therefore the required probability is 0.92

i.e., There is 92% chance that the project will be completed before 30 weeks. In other words, the chance that it will be delayed beyond 30 weeks is 8%

QUESTIONS:

1. Explain how time of an activity is estimated in PERT.
2. Explain the measure of certainty in PERT.
3. The estimates of time in weeks of the activities of a project are as follows:

Activity	Predecessor Activity	Optimistic estimate of time	Most likely estimate of time	Pessimistic estimate of time
A	-	2	4	6
B	A	8	11	20
C	A	10	15	20
D	B	12	18	24

E	C	8	13	24
F	C	4	7	16
G	D,F	14	18	28
H	E	10	12	14
I	G,H	7	10	19

Determine the critical activities and the project completion time.

4. Draw the network diagram for the following project. Determine the time, variance and standard deviation of the project.:

Activity	Predecessor Activity	Optimistic estimate of time	Most likely estimate of time	Pessimistic estimate of time
A	-	12	14	22
B	-	16	17	24
C	A	14	15	16
D	A	13	18	23
E	B	16	18	20
F	D,E	13	14	21
G	C,F	6	8	10

5. Consider the following project with the estimates of time in weeks:

Activity	Predecessor Activity	Optimistic estimate of time	Most likely estimate of time	Pessimistic estimate of time
A	-	2	4	6
B	-	3	5	7
C	A	5	6	13
D	A	4	8	12
E	B,C	5	6	13
F	D,E	6	8	14

Find the probability that the project will be completed in 27 weeks.

EARLIST AND LATEST TIME

A project manager has the responsibility to see that a project is completed by the stipulated date, without delay. Attention is focused on this aspect in what follows.

EARLIEST TIMES OF AN ACTIVITY

We can consider (i) Earliest Start Time of an activity and (ii) Earliest Finish Time of an activity.

Earliest Start Time of an activity is the earliest possible time of starting that activity on the condition that all the other activities preceding to it were began at the earliest possible times.

Earliest Finish Time of an activity is the earliest possible time of completing that activity. It is given by the formula.

The Earliest Finish Time of an activity = The Earliest Start Time of the activity + The estimated duration to carry out that activity.

LATEST TIMES OF AN ACTIVITY

We can consider (i) Latest Finish Time of an activity and (ii) Latest Start Time of an activity.

Latest Finish Time of an activity is the latest possible time of completing that activity on the condition that all the other activities succeeding it are carried out as per the plan of the management and without delaying the project beyond the stipulated time.

Latest Start Time of an activity is the latest possible time of beginning that activity. It is given by the formula

Latest Start Time of an activity = The Latest Finish Time of the activity - The estimated duration to carry out that activity.

TOTAL FLOAT OF AN ACTIVITY

Float seeks to measure how much delay is acceptable. It sets up a control limit for delay.

The total float of an activity is the time by which that activity can be delayed without delaying the whole project. It is given by the formula

$$\text{Total Float of an Activity} = \text{Latest Finish Time of the activity} - \text{Earliest Finish Time of that activity.}$$

It is also given by the formula

$$\text{Total Float of an Activity} = \text{Latest Start Time of the activity} - \text{Earliest Start Time of that activity.}$$

Since a delay in a critical activity will delay the execution of the whole project, the total float of a critical activity must be zero.

EXPECTED TIMES OF AN EVENT

An event occurs at a point of time. We can consider (i) Earliest Expected Time of Occurrence of an event and (ii) Latest Allowable Time of Occurrence an event.

The Earliest Expected Time of Occurrence of an event is the earliest possible time of expecting that event to happen on the condition that all the preceding activities have been completed.

The Latest Allowable Time of Occurrence of an event is the latest possible time of expecting that event to happen without delaying the project beyond the stipulated time.

PROCEDURE TO FIND THE EARLIEST EXPECTED TIME OF AN EVENT

Step 1. Take the Earliest Expected Time of Occurrence of the Start Event as zero.

Step 2. For an event other than the Start Event, find out all paths in the network which connect the Start node with the node representing the event under consideration.

Step 3. In the “**Forward Pass**” (i.e., movement in the network from left to right), find out the sum of the time durations of the activities in each path identified in Step 2.

Step 4. The path with the longest time in Step 3 gives the Earliest Expected Time of Occurrence of the event

Working Rule for finding the earliest expected time of an event:

For an event under consideration, locate all the predecessor events and identify their earliest expected times. With the earliest expected time of each event, add the time duration of the activity connecting that event to the event under consideration. The maximum among all these values gives the Earliest Expected Time of Occurrence of the event.

PROCEDURE TO FIND THE LATEST ALLOWABLE TIME OF AN EVENT

We consider the “**Backward Pass**” (i.e., movement in the network from right to left).

The latest allowable time of occurrence of the End Node must be the time of completion of the project. Therefore it shall be equal to the time of the critical path of the project.

Step 1. Identify the latest allowable time of occurrence of the End Node.

Step 2. For an event other than the End Event, find out all paths in the network which connect the End node with the node representing the event under consideration.

Step 3. In the “**Backward Pass**” (i.e., movement in the network from right to left), subtract the time durations of the activities along each such path.

Step 4. The Latest Allowable Time of Occurrence of the event is determined by the path with the longest time in Step 3. In other words, the smallest value of time obtained in Step 3 gives the Latest Allowable Time of Occurrence of the event.

Working Rule for finding the latest allowable time of an event:

For an event under consideration, locate all the successor events and identify their latest allowable times. From the latest allowable time of each successor event, subtract the time duration of the activity that begins with the event under consideration. The minimum among all these values gives the Latest Allowable Time of Occurrence of the event.

SLACK OF AN EVENT

The allowable time gap for the occurrence of an event is known as the slack of that event. It is given by the formula

Slack of an event = Latest Allowable Time of Occurrence of the event - Earliest Expected Time of Occurrence of that event.

SLACK OF AN ACTIVITY

The slack of an activity is the float of the activity.

Problem 1:

The following details are available regarding a project:

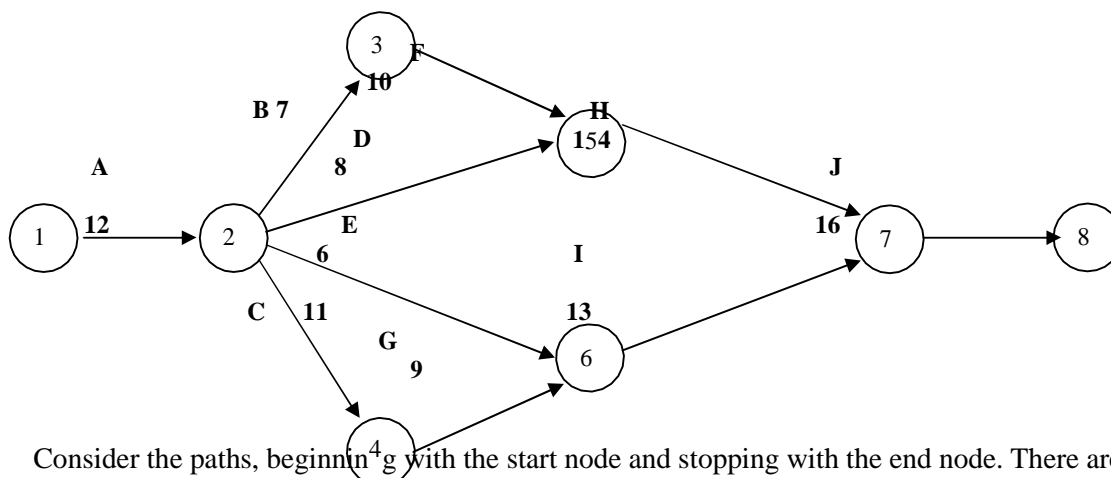
Activity	Predecessor Activity	Duration (Weeks)
A	-	12

B	A	7
C	A	11
D	A	8
E	A	6
F	B	10
G	C	9
H	D, F	14
I	E, G	13
J	H, I	16

Determine the earliest and latest times, the total float for each activity, the critical activities and the project completion time.

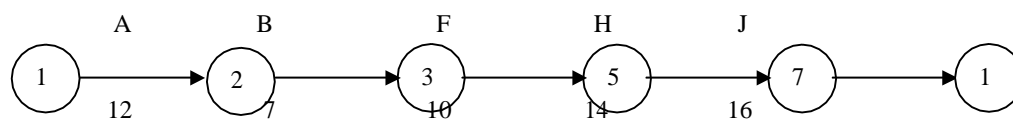
Solution:

With the given data, we construct the following network diagram for the project.



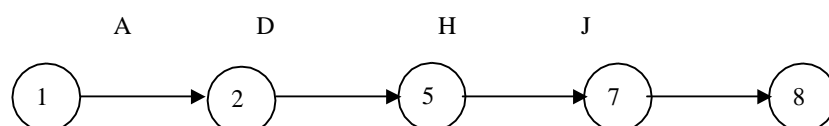
Consider the paths, beginning with the start node and stopping with the end node. There are four such paths for the given project. They are as follows:

Path I



Time of the path = $12 + 7 + 10 + 14 + 16 = 59$ weeks.

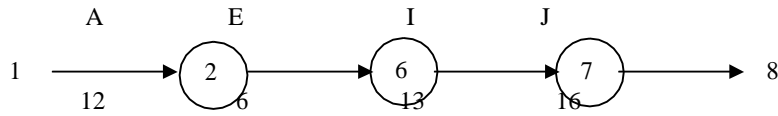
Path II



12 8 14 16

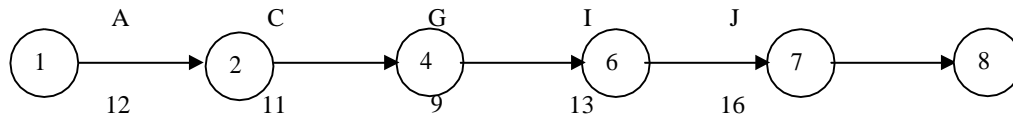
Time of the path = $12 + 8 + 14 + 16 = 50$ weeks.

Path III



Time of the path = $12 + 6 + 13 + 16 = 47$ weeks.

Path IV



Time of the path = $12 + 11 + 9 + 13 + 16 = 61$ weeks.

Compare the times for the four paths. Maximum of $\{51, 50, 47, 61\} = 61$. We see that the maximum time of a path is 61 weeks.

Forward pass:

Calculation of Earliest Expected Time of Occurrence of Events

Node	Earliest Time of Occurrence of Node
1	0
2	Time for Node 1 + Time for Activity A = $0 + 12 = 12$
3	Time for Node 2 + Time for Activity B = $12 + 7 = 19$
4	Time for Node 2 + Time for Activity C = $12 + 11 = 23$
5	Max {Time for Node 2 + Time for Activity D, Time for Node 3 + Time for Activity F} = Max $\{12 + 8, 19 + 10\} = \text{Max } \{20, 29\} = 29$
6	Max {Time for Node 2 + Time for Activity E, Time for Node 4 + Time for Activity G} = Max $\{12 + 6, 23 + 9\} = \text{Max } \{18, 32\} = 32$
7	Max {Time for Node 5 + Time for Activity H, Time for Node 6 + Time for Activity I} = Max $\{29 + 14, 32 + 13\} = \text{Max } \{43, 45\} = 45$
8	Time for Node 7 + Time for Activity J = $45 + 16 = 61$

Using the above values, we obtain the Earliest Start Times of the activities as follows:

Activity	Earliest Start Time (Weeks)
----------	--------------------------------

A	0
B	12
C	12
D	12
E	12
F	19
G	23
H	29
I	32
J	45

Backward pass:

Calculation of Latest Allowable Time of Occurrence of Events

Node	Latest Allowable Time of Occurrence of Node
8	Maximum time of a path in the network = 61
7	Time for Node 8 - Time for Activity J = $61 - 16 = 45$
6	Time for Node 7 - Time for Activity I = $45 - 13 = 32$
5	Time for Node 7 - Time for Activity H = $45 - 14 = 31$
4	Time for Node 6 - Time for Activity G = $32 - 9 = 23$
3	Time for Node 5 - Time for Activity F = $31 - 10 = 21$
2	Min {Time for Node 3 - Time for Activity B, Time for Node 4 - Time for Activity C, Time for Node 5 - Time for Activity D, Time for Node 6 - Time for Activity E} = Min {21 - 7, 23 - 11, 31 - 8, 32 - 6} = Min {14, 12, 23, 26} = 12
1	Time for Node 2 - Time for Activity A = $12 - 12 = 0$

Using the above values, we obtain the Latest Finish Times of the activities as follows:

Activity	Latest Finish Time (Weeks)
J	61
I	45
H	45
G	32
F	31

E	32
D	31
C	23
B	21
A	12

Calculation of Total Float for each activity:

Activity	Duration (Weeks)	Earliest Start Time	Earliest Finish Time	Latest Start Time	Latest Finish Time	Total Float = Latest Finish Time - Earliest Finish Time
A	12	0	12	0	12	0
B	7	12	19	14	21	2
C	11	12	23	12	23	0
D	8	12	20	23	31	11
E	6	12	18	26	32	14
F	10	19	29	21	31	2
G	9	23	32	23	32	0
H	14	29	43	31	45	2
I	13	32	45	32	45	0
J	16	45	61	45	61	0

The activities with total float = 0 are A, C, G, I and J. They are the critical activities.
Project completion time = 61 weeks.

Problem 2:

The following are the details of the activities in a project:

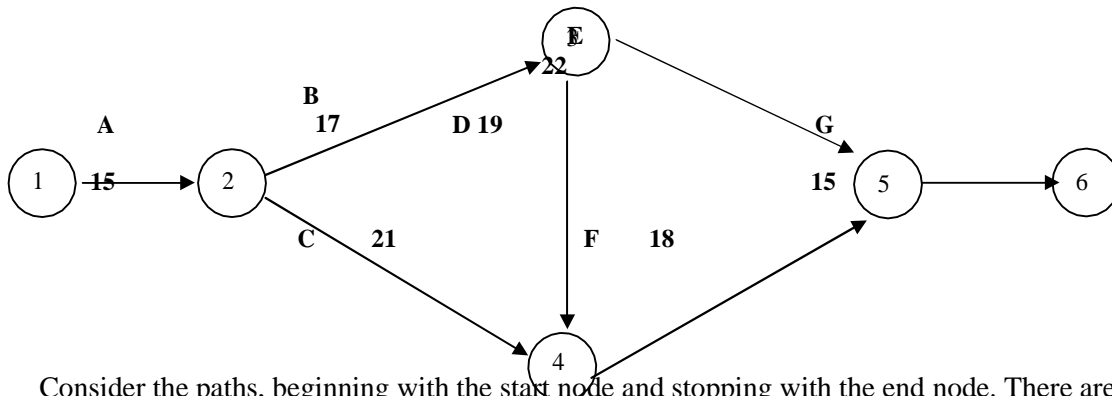
Activity	Predecessor Activity	Duration (Weeks)
A	-	15
B	A	17
C	A	21
D	B	19
E	B	22

F	C, D	18
G	E, F	15

Calculate the earliest and latest times, the total float for each activity and the project completion time.

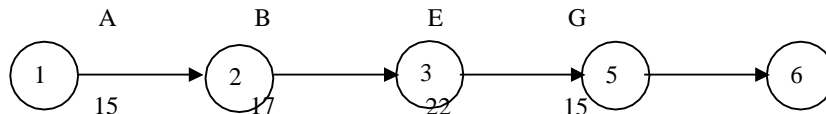
Solution:

The following network diagram is obtained for the given project.



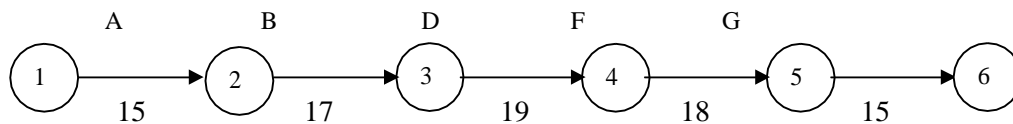
Consider the paths, beginning with the start node and stopping with the end node. There are three such paths for the given project. They are as follows:

Path I



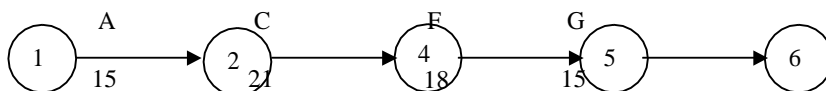
Time of the path = $15 + 17 + 22 + 15 = 69$ weeks.

Path II



Time of the path = $15 + 17 + 19 + 18 + 15 = 84$ weeks.

Path III



Time of the path = $15 + 21 + 18 + 15 = 69$ weeks.

Compare the times for the three paths. Maximum of $\{69, 84, 69\} = 84$. We see that the maximum time of a path is 84 weeks.

Forward pass:

Calculation of Earliest Time of Occurrence of Events

Node	Earliest Time of Occurrence of Node
1	0
2	Time for Node 1 + Time for Activity A = $0 + 15 = 15$
3	Time for Node 2 + Time for Activity B = $15 + 17 = 32$
4	Max {Time for Node 2 + Time for Activity C, Time for Node 3 + Time for Activity D} = Max { $15 + 21, 32 + 19$ } = Max {36, 51} = 51
5	Max {Time for Node 3 + Time for Activity E, Time for Node 4 + Time for Activity F} = Max { $32 + 22, 51 + 18$ } = Max {54, 69} = 69
6	Time for Node 5 + Time for Activity G = $69 + 15 = 84$

Calculation of Earliest Time for Activities

Activity	Earliest Start Time (Weeks)
A	0
B	15
C	15
D	32
E	32
F	51
G	69

Backward pass:

Calculation of the Latest Allowable Time of Occurrence of Events

Node	Latest Allowable Time of Occurrence of Node
6	Maximum time of a path in the network = 84
5	Time for Node 6 - Time for Activity G = $84 - 15 = 69$
4	Time for Node 5 - Time for Activity F = $69 - 18 = 51$
3	Min {Time for Node 4 - Time for Activity D, Time for Node 5 - Time for Activity E} = Min { $51 - 19, 69 - 22$ } = Min {32, 47} = 32
2	Min {Time for Node 3 - Time for Activity B, Time for Node 4 - Time for Activity C} = Min { $32 - 17, 51 - 21$ } = Min {15, 30} = 15
1	Time for Node 2 - Time for Activity A = $15 - 15 = 0$

Calculation of the Latest Finish Times of the activities

Activity	Latest Finish Time (Weeks)
G	84

F	69
E	69
D	51
C	51
B	32
A	15

Calculation of Total Float for each activity:

Activity	Duration (Weeks)	Earliest Start Time	Earliest Finish Time	Latest Start Time	Latest Finish Time	Total Float = Latest Finish Time - Earliest Finish Time
A	15	0	15	0	15	0
B	17	15	32	15	32	0
C	21	15	36	30	51	15
D	19	32	51	32	51	0
E	22	32	54	47	69	15
F	18	51	69	51	69	0
G	15	69	84	69	84	0

The activities with total float = 0 are A, B, D, F and G. They are the critical activities.
Project completion time = 84 weeks.

Problem 3:

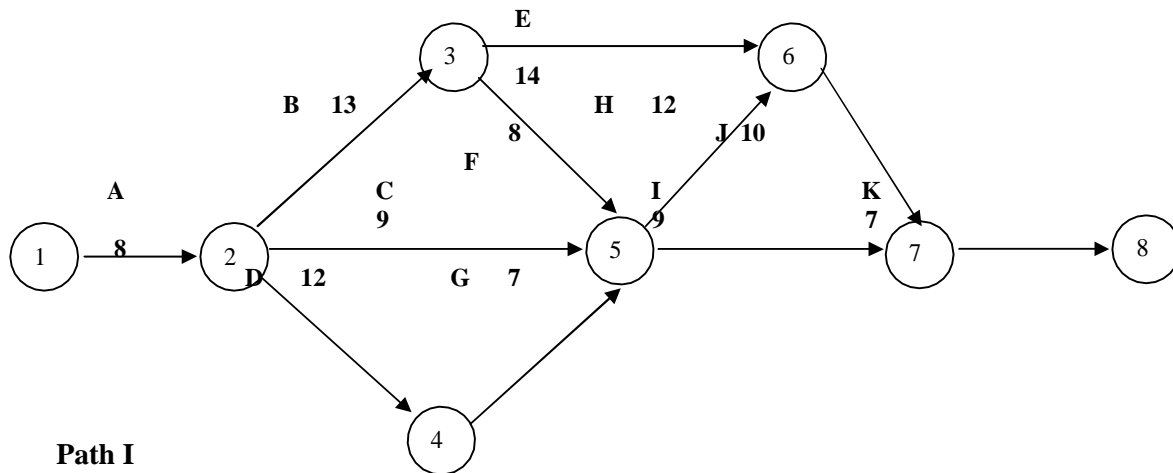
Consider a project with the following details:

Name of Activity	Predecessor Activity	Duration (Weeks)
A	-	8
B	A	13
C	A	9
D	A	12
E	B	14
F	B	8
G	D	7
H	C, F, G	12
I	C, F, G	9
J	E, H	10
K	I, J	7

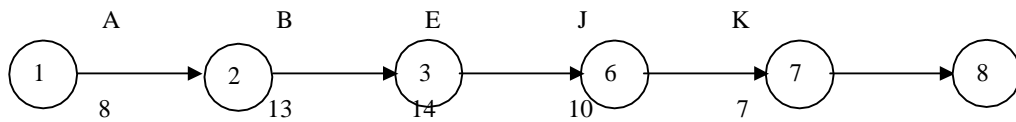
Determine the earliest and latest times, the total float for each activity, the critical activities, the slacks of the events and the project completion time.

Solution:

The following network diagram is got for the given project:

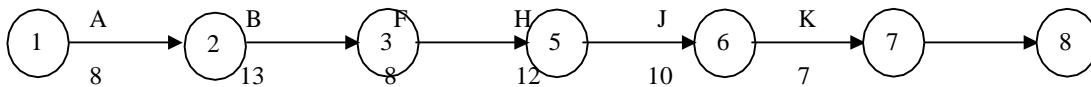


Path I



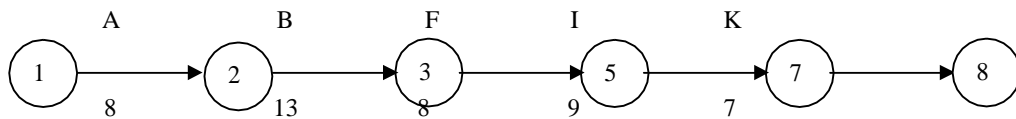
Time of the path = $8 + 13 + 14 + 10 + 7 = 52$ weeks.

Path II



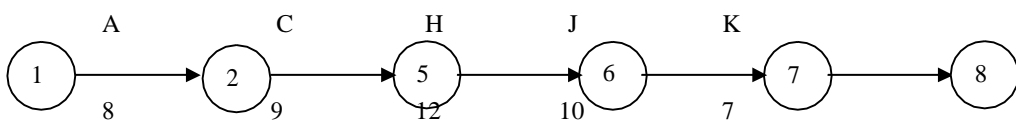
Time of the path = $8 + 13 + 8 + 12 + 10 + 7 = 58$ weeks.

Path III

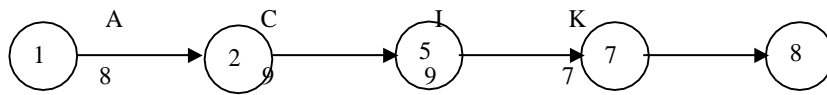


Time of the path = $8 + 13 + 8 + 9 + 7 = 45$ weeks.

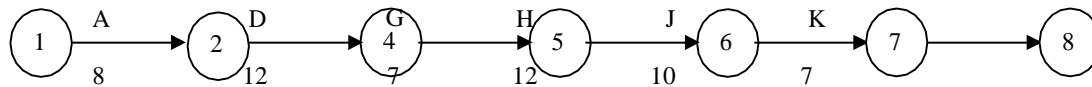
Path IV



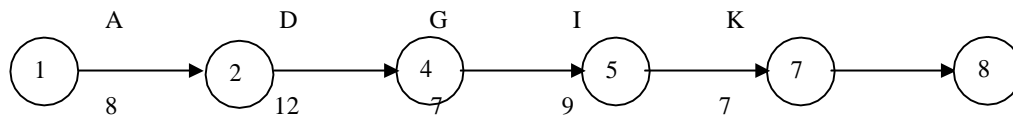
Time of the path = $8 + 9 + 12 + 10 + 7 = 46$ weeks.

Path V

Time of the path = $8 + 9 + 9 + 7 = 33$ weeks.

Path VI

Time of the path = $8 + 12 + 7 + 12 + 10 + 7 = 56$ weeks.

Path VII

Time of the path = $8 + 12 + 7 + 9 + 7 = 43$ weeks.

Compare the times for the three paths. Maximum of $\{52, 58, 45, 46, 33, 56, 43\} = 58$.

We see that the maximum time of a path is 58 weeks.

Forward pass:

Calculation of Earliest Time of Occurrence of Events

Node	Earliest Time of Occurrence of Node
1	0
2	Time for Node 1 + Time for Activity A = $0 + 8 = 8$
3	Time for Node 2 + Time for Activity B = $8 + 13 = 21$
4	Time for Node 2 + Time for Activity D = $8 + 12 = 20$
5	Max {Time for Node 2 + Time for Activity C, Time for Node 3 + Time for Activity F, Time for Node 4 + Time for Activity G} = Max { $8 + 9$, $21 + 8$, $20 + 7$ } = Max {17, 29, 27} = 29
6	Max {Time for Node 3 + Time for Activity E, Time for Node 5 + Time for Activity H} = Max { $21 + 14$, $29 + 12$ } = Max {35, 41} = 41
7	Max {Time for Node 5 + Time for Activity I, Time for Node 6 + Time for Activity J} = Max { $29 + 9$, $41 + 10$ } = Max {38, 51} = 51
8	Time for Node 7 + Time for Activity J = $51 + 7 = 58$

Earliest Start Times of the activities

Activity	Earliest Start Time (Weeks)
A	0
B	8
C	8
D	8
E	21
F	21
G	20
H	29
I	29
J	41
K	51

Backward pass:

Calculation of Latest Allowable Time of Occurrence of Events

Node	Latest Allowable Time of Occurrence of Node
8	Maximum time of a path in the network = 58
7	Time for Node 8 - Time for Activity K = $58 - 7 = 51$
6	Time for Node 7 - Time for Activity J = $51 - 10 = 41$

5	Min {Time for Node 6 - Time for Activity H, Time for Node 7 - Time for Activity I} = Min {41 - 12, 51 - 9} = Min {29, 42} = 29
4	Time for Node 5 - Time for Activity G = 29 - 7 = 22
3	Min {Time for Node 5 - Time for Activity F, Time for Node 6 - Time for Activity E} = Min {29 - 8, 41 - 14} = Min {21, 27} = 21
2	Min {Time for Node 3 - Time for Activity B, Time for Node 4 - Time for Activity D, Time for Node 5 - Time for Activity C} = Min {21 - 13, 22 - 12, 29 - 9} = Min {8, 10, 20} = 8
1	Time for Node 2 - Time for Activity A = 8 - 8 = 0

Latest Finish Times of the activities

Activity	Latest Finish Time (Weeks)
K	58
J	51
I	51
H	41
G	29
F	29
E	41
D	22
C	29
B	21
A	8

Calculation of Total Float for each activity:

Activity	Duration (Weeks)	Earliest Start Time	Earliest Finish Time	Latest Start Time	Latest Finish Time	Total Float = Latest Finish Time - Earliest Finish Time
A	8	0	8	0	8	0
B	13	8	21	8	21	0
C	9	8	17	20	29	12
D	12	8	20	10	22	2
E	14	21	35	27	41	6
F	8	21	29	21	29	0

G	7	20	27	22	29	2
H	12	29	41	29	41	0
I	9	29	38	42	51	13
J	10	41	51	41	51	0
K	7	51	58	51	58	0

The activities with total float = 0 are A, B, F, H, J and K. They are the critical activities.
Project completion time = 58 weeks.

Calculation of slacks of the events

Slack of an event = Latest Allowable Time of Occurrence of the event - Earliest Expected Time of Occurrence of that event.

Event (Node)	Earliest Expected Time of Occurrence of Event	Latest Allowable Time of Occurrence of Event	Slack of the Event
1	0	0	0
2	8	8	0
3	21	21	0
4	20	22	2
5	29	29	0
6	41	41	0
7	51	51	0
8	58	58	0

Interpretation:

On the basis of the slacks of the events, it is concluded that the occurrence of event 4 may be delayed upto a maximum period of 2 weeks while no other event cannot be delayed.

QUESTIONS

1. Explain the terms: The earliest and latest times of the activities of a project.
2. Explain the procedure to find the earliest expected time of an event.
3. Explain the procedure to find the latest allowable time of an event.
4. What is meant by the slack of an activity? How will you determine it?
5. Consider the project with the following details:

activity	Duration (weeks)
1→2	1
2→3	3

2→4	7
3→4	5
3→5	8
4→5	4
5→6	1

Determine the earliest and the latest times of the activities. Calculate the total float for each activity and the slacks of the events.

CRASHING OF A PROJECT

THE MEANING OF CRASHING:

The process of shortening the time to complete a project is called crashing and is usually achieved by putting into service additional labour or machines to one activity or more activities. Crashing involves more costs. A project manager would like to speed up a project by spending as minimum extra cost as possible. Project crashing seeks to minimize the extra cost for completion of a project before the stipulated time.

STEPS IN PROJECT CRASHING:

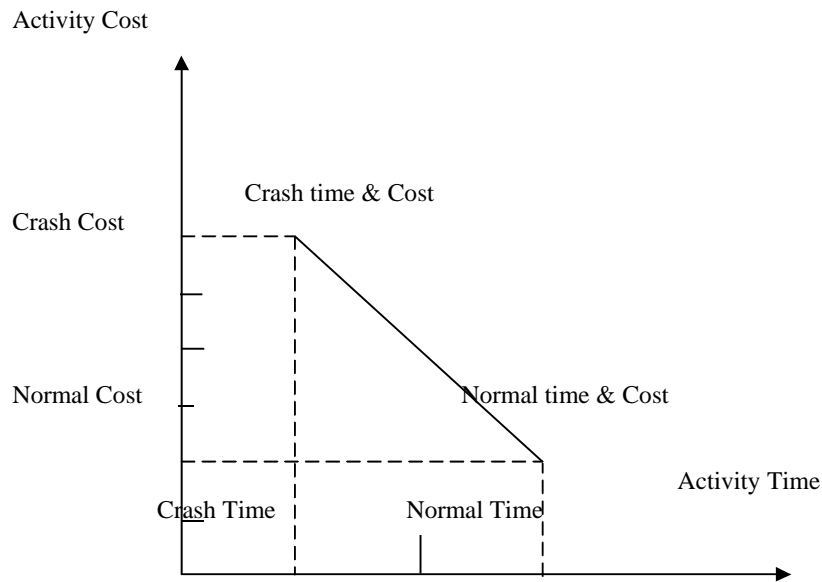
Assumption: It is assumed that there is a linear relationship between time and cost.

Let us consider project crashing by the critical path method. The following four-step procedure is adopted.

Step 1: Find the critical path with the normal times and normal costs for the activities and identify the critical activities.

Step 2: Find out the crash cost per unit time for each activity in the network. This is calculated by means of the following formula.

$$\frac{\text{Crash cost} - \text{Normal cost}}{\text{Normal time} - \text{Crash time}} = \frac{\text{Crash cost} - \text{Normal cost}}{\text{Normal time} - \text{Crash time}}$$



Step 3: Select an activity for crashing. The **criteria for the selection** is as follows:

Select the activity on the critical path with the smallest crash cost per unit time. Crash this activity to the maximum units of time as may be permissible by the given data.

Crashing an activity requires extra amount to be spent. However, even if the company is prepared to spend extra money, the activity time cannot be reduced beyond a certain limit in view of several other factors.

In step 1, we have to note that reducing the time of an activity along the critical path alone will reduce the completion time of a project. Because of this reason, we select an activity along the critical path for crashing.

In step 3, we have to consider the following question:

If we want to reduce the project completion time by one unit, which critical activity will involve the least additional cost?

On the basis of the least additional cost, a critical activity is chosen for crashing. If there is a tie between two critical activities, the tie can be resolved arbitrarily.

Step 4: After crashing an activity, find out which is the critical path with the changed conditions. Sometimes, a reduction in the time of an activity in the critical path may cause a non-critical path to become critical. If the critical path with which we started is still the longest path, then go to Step 3. Otherwise, determine the new critical path and then go to Step 3.

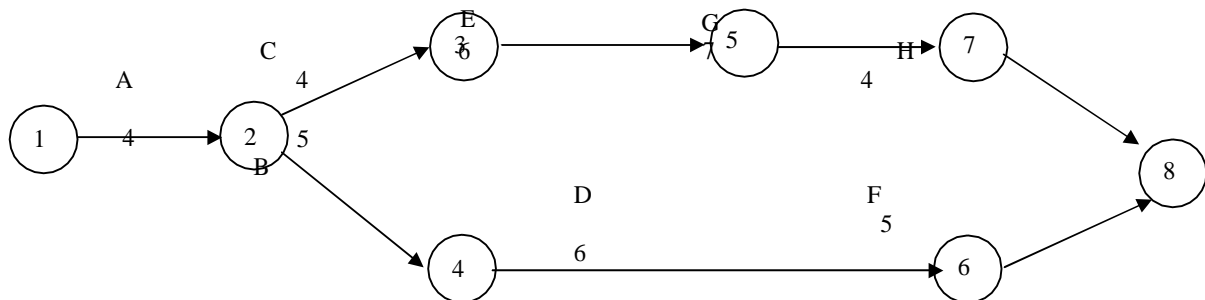
Problem 1: A project has activities with the following normal and crash times and cost:

Activity	Predecessor Activity	Normal Time (Weeks)	Crash Time (Weeks)	Normal Cost (Rs.)	Crash Cost (Rs.)
A	-	4	3	8,000	9,000
B	A	5	3	16,000	20,000
C	A	4	3	12,000	13,000
D	B	6	5	34,000	35,000
E	C	6	4	42,000	44,000
F	D	5	4	16,000	16,500
G	E	7	4	66,000	72,000
H	G	4	3	2,000	5,000

Determine a crashing scheme for the above project so that the total project time is reduced by 3 weeks.

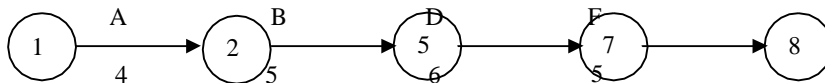
Solution:

We have the following network diagram for the given project with **normal costs**:



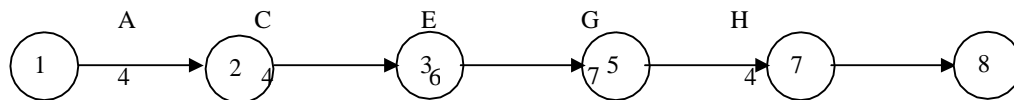
Beginning from the Start Node and terminating with the End Node, there are two paths for the network as detailed below:

Path I:



The time for the path = $4 + 5 + 6 + 5 = 20$ weeks.

Path II:



The time for the path = $4 + 4 + 6 + 7 + 4 = 25$ weeks.

Maximum of $\{20, 25\} = 25$.

Therefore Path II is the critical path and the critical activities are A, C, E, G and H. The non-critical activities are B, D and F.

Given that the normal time of activity A is 4 weeks while its crash time is 3 weeks. Hence the time of this activity can be reduced by one week if the management is prepared to spend an additional amount. However, the time cannot be reduced by more than one week even if the management may be prepared to spend more money. The normal cost of this activity is Rs. 8,000 whereas the crash cost is Rs. 9,000. From this, we see that crashing of activity A by one week will cost the management an extra amount of Rs. 1,000. In a similar fashion, we can work out the crash cost per unit time for the other activities also. The results are provided in the following table.

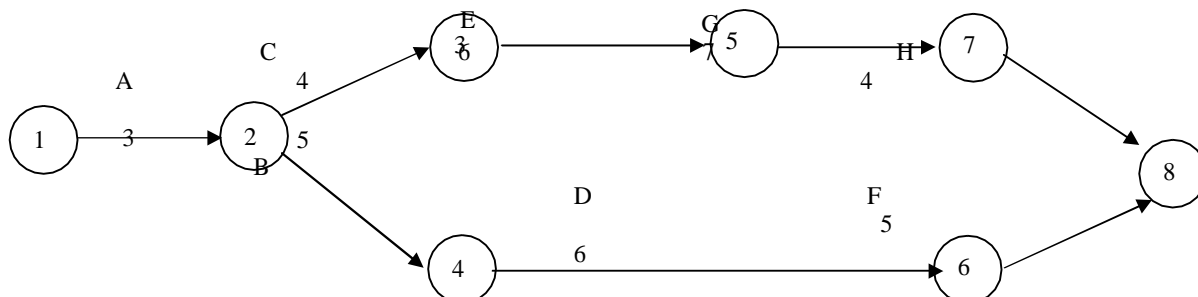
Activity	Normal Time	Crash Time	Normal Cost	Crash Cost	Crash cost - Normal Cost	Normal Time - Crash Time	Crash Cost per unit time
A	4	3	8,000	9,000	1,000	1	1,000
B	5	3	16,000	20,000	4,000	2	2,000
C	4	3	12,000	13,000	1,000	1	1,000
D	6	5	34,000	35,000	1,000	1	1,000
E	6	4	42,000	44,000	2,000	2	1,000
F	5	4	16,000	16,500	500	1	500
G	7	4	66,000	72,000	6,000	3	2,000
H	4	3	2,000	5,000	3,000	1	3,000

A non-critical activity can be delayed without delaying the execution of the whole project. But, if a critical activity is delayed, it will delay the whole project. Because of this reason, we have to select a critical activity for crashing. Here we have to choose one of the activities A, C, E, G and H. The crash cost per unit time works out as follows:

Rs. 1,000 for A; Rs. 1,000 for C; Rs. 1,000 for E; Rs. 6,000 for G; Rs. 3,000 for H.

The maximum among them is Rs. 1,000. So we have to choose an activity with Rs.1,000 as the crash cost per unit time. However, there is a tie among A, C and E. The tie can be resolved arbitrarily. Let us select A for crashing. We reduce the time of A by one week by spending an extra amount of Rs. 1,000.

After this step, we have the following network with the revised times for the activities:



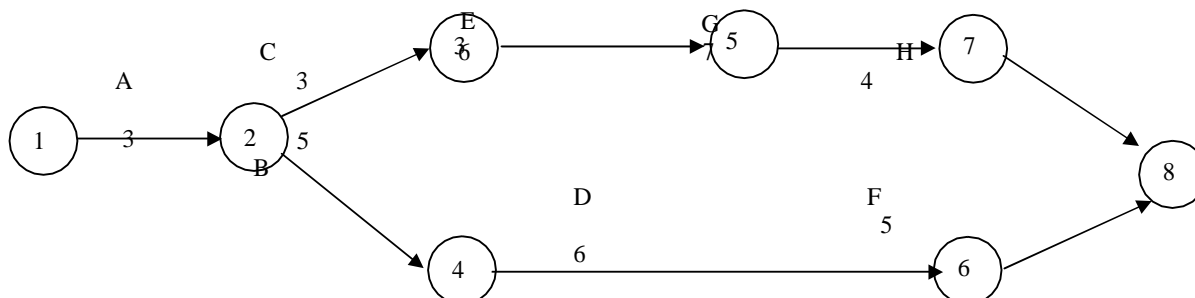
The revised time for Path I = 3 + 5 + 6 + 5 = 19 weeks.

The time for Path II = 3 + 4 + 6 + 7 + 4 = 24 weeks.

Maximum of {19, 24} = 24.

Therefore Path II is the critical path and the critical activities are A, C, E, G and H. However, the time for A cannot be reduced further. Therefore, we have to consider C, E, G and H for crashing. Among them, C and E have the least crash cost per unit time. The tie between C and E can be resolved arbitrarily. Suppose we reduce the time of C by one week with an extra cost of Rs. 1,000.

After this step, we have the following network with the revised times for the activities:



The time for Path I = $3 + 5 + 6 + 5 = 19$ weeks.

The time for Path II = $3 + 3 + 6 + 7 + 4 = 23$ weeks.

Maximum of $\{19, 23\} = 23$.

Therefore Path II is the critical path and the critical activities are A, C, E, G and H. Now the time for A or C cannot be reduced further. Therefore, we have to consider E, G and H for crashing. Among them, E has the least crash cost per unit time. Hence we reduce the time of E by one week with an extra cost of Rs. 1,000.

By the given condition, we have to reduce the project time by 3 weeks. Since this has been accomplished, we stop with this step.

Result: We have arrived at the following crashing scheme for the given project:

Reduce the time of A, C and E by one week each.

Project time after crashing is 22 weeks.

Extra amount required = $1,000 + 1,000 + 1,000 = \text{Rs. } 3,000$.

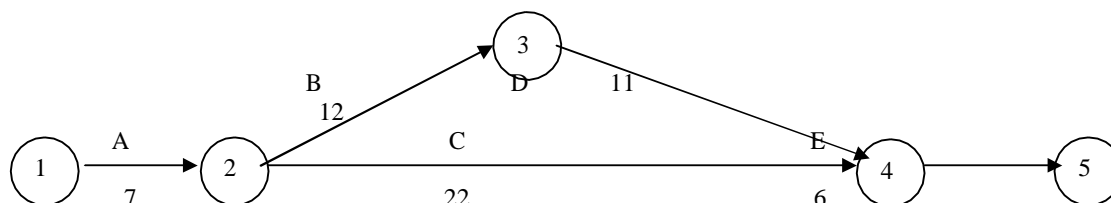
Problem 2:

The management of a company is interested in crashing of the following project by spending an additional amount not exceeding Rs. 2,000. Suggest how this can be accomplished.

Activity	Predecessor Activity	Normal Time (Weeks)	Crash Time (Weeks)	Normal Cost (Rs.)	Crash Cost (Rs.)
A	-	7	6	15,000	18,000
B	A	12	9	11,000	14,000
C	A	22	21	18,500	19,000
D	B	11	10	8,000	9,000
E	C, D	6	5	4,000	4,500

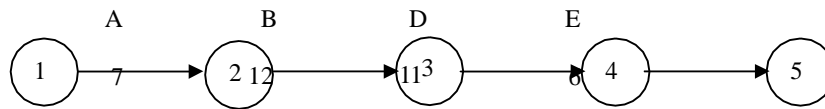
Solution:

We have the following network diagram for the given project with **normal costs**:



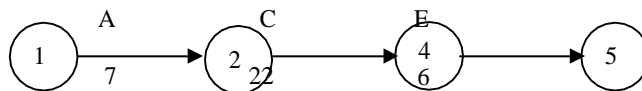
There are two paths for this project as detailed below:

Path I:



The time for the path = $7 + 12 + 11 + 6 = 36$ weeks.

Path II:



The time for the path = $7 + 22 + 6 = 35$ weeks.

Maximum of $\{36, 35\} = 36$.

Therefore Path I is the critical path and the critical activities are A, B, D and E. The non-critical activity is C.
The crash cost per unit time for the activities in the project are provided in the following table.

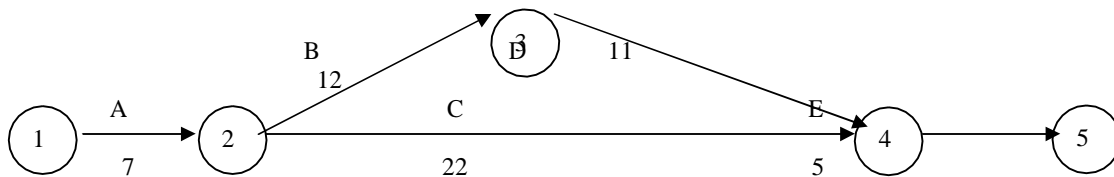
Activity	Normal Time	Crash Time	Normal Cost	Crash Cost	Crash cost - Normal Cost	Normal Time - Crash Time	Crash Cost per unit time
A	7	6	15,000	18,000	3,000	1	3,000
B	12	9	11,000	14,000	3,000	3	1,000
C	22	21	18,500	19,000	500	1	500
D	11	10	8,000	9,000	1,000	1	1,000
E	6	5	4,000	4,500	500	1	500

We have to choose one of the activities A, B, D and E for crashing. The crash cost per unit time is as follows:

Rs. 3,000 for A; Rs. 1,000 for B; Rs. 1,000 for D; Rs. 500 for E.

The least among them is Rs. 500. So we have to choose the activity E for crashing. We reduce the time of E by one week by spending an extra amount of Rs. 500.

After this step, we have the following network with the revised times for the activities:



The revised time for Path I = $7 + 12 + 11 + 5 = 35$ weeks.

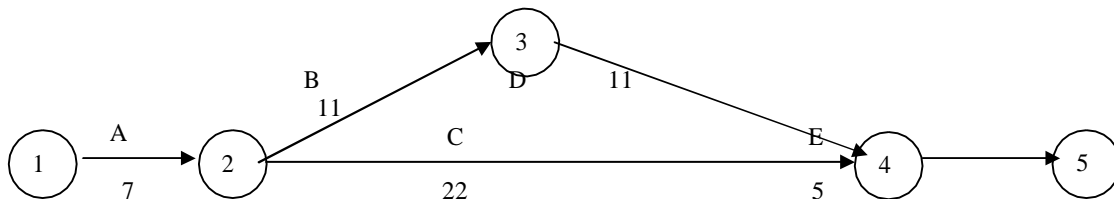
The time for Path II = $7 + 22 + 5 = 34$ weeks.

Maximum of $\{35, 34\} = 35$.

Therefore Path I is the critical path and the critical activities are A, B, D and E. The non-critical activity is C.

The time of E cannot be reduced further. So we cannot select it for crashing. Next B and have the smallest crash cost per unit time. Let us select B for crashing. Let us reduce the time of E by one week at an extra cost of Rs. 1,000.

After this step, we have the following network with the revised times for the activities:



The revised time for Path I = $7 + 11 + 11 + 5 = 34$ weeks.

The time for Path II = $7 + 22 + 5 = 34$ weeks.

Maximum of $\{34, 34\} = 34$.

Since both paths have equal times, both are critical paths. So, we can choose an activity for crashing from either of them depending on the least crash cost per unit time. In path I, the activities are A, B, D and E. In path II, the activities are A, C and E.

The crash cost per unit time is the least for activity C. So we select C for crashing. Reduce the time of C by one week at an extra cost of Rs. 500.

By the given condition, the extra amount cannot exceed Rs. 2,000. Since this state has been met, we stop with this step.

Result: The following crashing scheme is suggested for the given project:
Reduce the time of E, B and C by one week each.

Project time after crashing is 33 weeks.

Extra amount required = $500 + 1,000 + 500 = \text{Rs. } 2,000$.

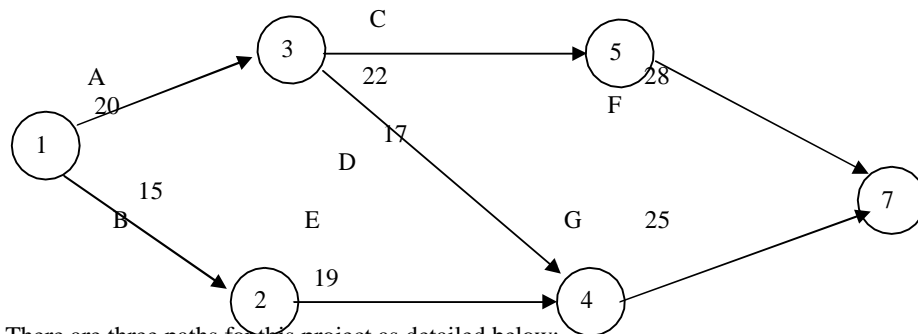
Problem 3:

The manager of a company wants to apply crashing for the following project by spending an additional amount not exceeding Rs. 2,000. Offer your suggestion to the manager.

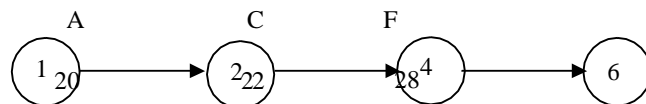
Activity	Predecessor Activity	Normal Time (Weeks)	Crash Time (Weeks)	Normal Cost (Rs.)	Crash Cost (Rs.)
A	-	20	19	8,000	10,000
B	-	15	14	16,000	19,000
C	A	22	20	13,000	14,000
D	A	17	15	7,500	9,000
E	B	19	18	4,000	5,000
F	C	28	27	3,000	4,000
G	D, E	25	24	12,000	13,000

Solution:

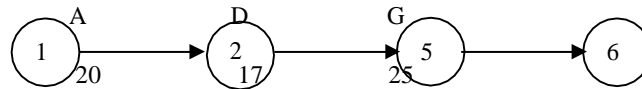
We have the following network diagram for the given project with **normal costs**:



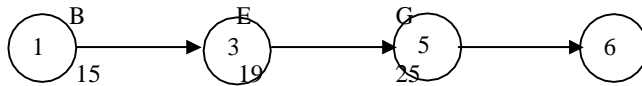
There are three paths for this project as detailed below:

Path I:

The time for the path = 20 + 22 + 28 = 70 weeks.

Path II:

The time for the path = $20 + 17 + 25 = 62$ weeks.

Path III:

The time for the path = $15 + 19 + 25 = 69$ weeks.

Maximum of $\{70, 62, 69\} = 70$.

Therefore Path I is the critical path and the critical activities are A, C and F. The non-critical activities are B, D, E and G.

The crash cost per unit time for the activities in the project are provided in the following table

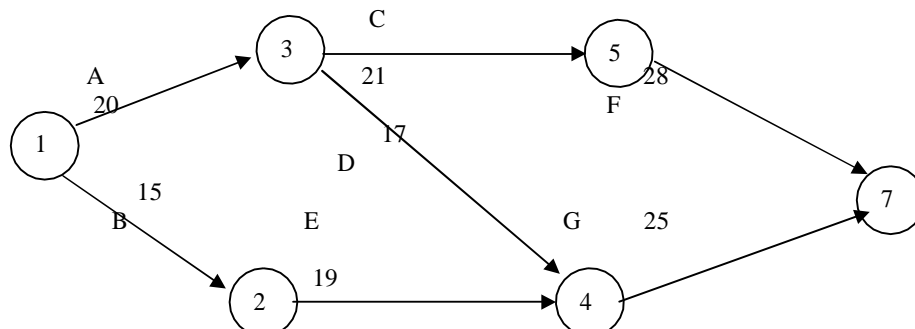
Activity	Normal Time	Crash Time	Normal Cost	Crash Cost	Crash cost - Normal Cost	Normal Time - Crash Time	Crash Cost per unit time
A	20	19	8,000	10,000	2,000	1	2,000
B	15	14	16,000	19,000	3,000	1	3,000
C	22	20	13,000	14,000	1,000	2	500
D	17	15	7,500	9,000	1,500	2	750
E	19	18	4,000	5,000	1,000	1	1,000
F	28	27	3,000	4,000	1,000	1	1,000
G	25	24	12,000	13,000	1,000	1	1,000

We have to choose one of the activities A, C and F for crashing. The crash cost per unit time is as follows:

Rs. 2,000 for A; Rs. 500 for C; Rs. 1,000 for F.

The least among them is Rs. 500. So we have to choose the activity C for crashing. We reduce the time of C by one week by spending an extra amount of Rs. 500.

After this step, we have the following network with the revised times for the activities:



The revised time for Path I = $20 + 21 + 28 = 69$ weeks.

The time for Path II = $20 + 17 + 25 = 62$ weeks.

The time for Path III = $15 + 19 + 25 = 69$ weeks.

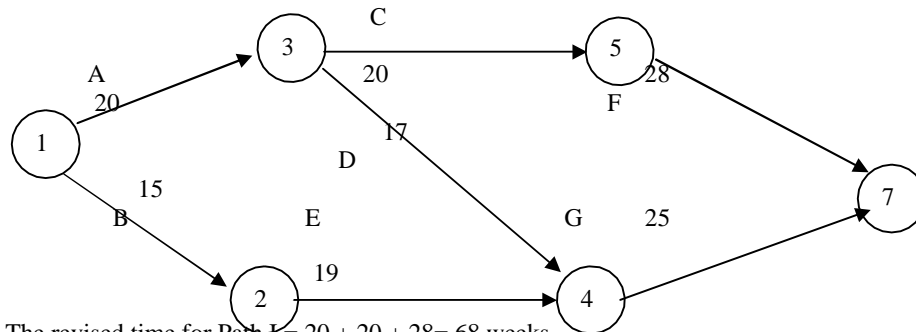
Maximum of $\{69, 62, 69\} = 69$.

Since paths I and III have equal times, both are critical paths. So, we can choose an activity for crashing from either of them depending on the least crash cost per unit time.

In path I, the activities are A, C and F. In path III, the activities are B, E and G.

The crash cost per unit time is the least for activity C. So we select C for crashing. Reduce the time of C by one week at an extra cost of Rs. 500.

After this step, we have the following network with the revised times for the activities:



The revised time for Path I = $20 + 20 + 28 = 68$ weeks.

The time for Path II = $20 + 17 + 25 = 62$ weeks.

The time for Path III = $15 + 19 + 25 = 69$ weeks.

Maximum of $\{68, 62, 69\} = 69$.

Therefore path III is the critical activities. Hence we have to select an activity from Path III for crashing. We see that the crash cost per unit time is as follows:

Rs. 3,000 for B; Rs. 1,000 for E; Rs. 1,000 for G.

The least among them is Rs. 1,000. So we can select either E or G for crashing. Let us select E for crashing. We reduce the time of E by one week by spending an extra amount of Rs. 1,000.

By the given condition, the extra amount cannot exceed Rs. 2,000. Since this condition has been reached, we stop with this step.

Result: The following crashing scheme is suggested for the given project:

Reduce the time of C by 2 weeks and that of E by one week.

Project time after crashing is 67 weeks.

Extra amount required = $2 \times 500 + 1,000 = \text{Rs. } 2,000$.

REVIEW QUESTIONS:

1. What is network diagram?
2. What is event?
3. Define an activity.
4. What is Dummy Activity?
5. What is CPM?

6. What do you mean by PERT?
7. Distinguish between CPM and PERT.
8. What are the similarities of CPM and PERT.
9. What is Forward pass?
10. What is meant by Backward pass?
11. What do you mean by EST, EFT, LST and LFT?
