OPERATIONS RESEARCH

[MTH2C10]

STUDY MATERIAL

II SEMESTER

CORE COURSE

M.Sc. Mathematics

(2019 Admission onwards)

UNIVERSITY OF CALICUT

SCHOOL OF DISTANCE EDUCATION

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SCHOOL OF DISTANCE EDUCATION

UNIVERSITY OF CALICUT

STUDY MATERIAL
SECOND SEMESTER

M.Sc. Mathematics (2019 ADMISSION ONWARDS)

CORE COURSE:

MTH2C10-OPERATIONS RESEARCH

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PRE-REQUISITES

A collection of n real numbers, taken in order such that the first number is the value of x_1, the second of x_2, and so on, is called an ordered n-tuple of real numbers. We may denote an ordered n-tuple by a single symbol X, so that $x=(x_1, x_2,\ldots,x_n)$ and a set of such n-tuples by R_n, so that $R_n=\{X|X=(x_1,x_2,\ldots\ldots,x_n)\}$.

Definitions

1. Vector space:

   Let V be a set such that if X, Y, Z $\in$ V and a, b$\in$ R, then the following postulates (defining the binary operation of sum and the operation of product with a real number) hold.

   Sum:

   i. $X + Y \in V$;

   ii. $X + Y = Y + X$

   iii. $(X + Y) + Z = X + (Y + Z)$;

   iv. There exists an element $0\in V$, called the null or zero vector, such that $X + 0 = X$;

   v. There exists an element $-X\in V$, called th additive inverse of X, such that $X + (-X) = 0$;

   Product:

   vi. $a X \in V$;

   vii. $a (X + Y) = a X + a Y$;

   viii. $(a+b) X = a X + b X$;

   ix. $(ab) X = a (bX)$;

   x. $IX = X$.

   Then V is called a vector space and its elements are called vectors.

2. Subspace of a vector space

   A subset W of V is called a subspace of the vector space V if W is itself a vector space with respect to the operations of sum and product defined in V.

3. Linear combination

   Let X_i, i=1, 2, ……., m, be vectors of V. Then X is called a linear combination of the vectors X_i, if X = $\sum_{i=1}^{m} a_i X_i, a_i \in R.$

4. Linearly dependent and independent vectors

   The vectors X_i, i=1,2,…., m, of a vector space V are said to be linearly dependent if there exist real numbers a_i, not all zero, such that $\sum_{i=1}^{m} a_i X_i = 0.$

   If, however, this is so only if $a_i = 0$ for all *i*, then the vectors are said to be linearly independent.

   Let X_1 = [2–1 3 2]’, X_2 = [1 2 2–4]’, X_3 = [4 3 7–6]’. Since $X_1+2 X_2-X_3 = 0$, the vectors are linearly dependent. But x_1, x_2 linearly independent. So are X_2, X_3.

5. Dimension of a vector space V.

   V is said to be of dimension m if there exists at least one set of m linearly independent vectors in V, while every set of m+1 vectors in V is linearly dependent. The linearly independent set is called a basis of V.

6. Inner product and orthogonality.

   The inner product $<X, Y>$ of any two vectors X and Y of V is a real number satisfying the following properties.

   i. $<X, Y> = <Y, X>$;

   ii. $<X + Z, Y> = <X, Y> + <Z, Y>, Z \in V$;

   iii. $<aX, Y> = a<X, Y>, a \in R$;

   iv. $<X, X>$ 0 if $X \neq 0$, $<X, X> = 0$ if $X = 0$.

   Two nonzero vectors are said to be orthogonal if their inner product is zero.

7. Euclidean space

A vector space with an inner product defined on it is called a Euclidean space.

For vectors of the vector space R_n, the expression $X^1Y = \sum_{i=1}^{n} x_i\, y_i$

satisfies the definition of inner product. With this definition R_n becomes a Euclidean space. This Euclidean space, if of dimension n, shall be denoted by E_n. On account of its importance in the present work we give afresh the definition of E_n which may be understood without reference to general definition of vectors and vector spaces given above.

8. R_n as a Euclidean space.

Let R_n be a set of ordered n-tuples of real numbers. For every pair of n-tuples X, Y $\in R_n$, let.

i. Sum: $X + Y = Y + X = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \in R_n$;

ii. Product: $aX=(ax_1, ax_2, \ldots., ax_n) \in R_n, a\in R$;

iii. Inner product : $X;'Y = Y'X = x_1y_1 + x_2y_2 + \ldots + x_n\, y_n \in R$;

be defined. Then the n-tuples are called vectors and R_n is called a Euclidean space.

iv. Also let there be at least one set of n linearly independent vectors in R_n. Then R_n is a Euclidean space of dimension n which we shall denote as E_n.

The set of column vectors $[1\ 0\ 0]^1$, $[0\ 1\ 0]^1$, $[0\ 0\ 1]^1$ is a basis of R_3. For, these vectors are linearly independent, and any vector $[x_1\ \ x_2\ \ x_3]^1$ of R_3 can be expressed as a linear combination of these vectors as follows.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This basis is called the canonical or the natural basis of R_3. Another basis of R_3 is $[1\ 0\ 0]^1$, $[1\ 1\ 0]^1$, $[1\ 1\ 1]^1$. For

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

where $c = x_3$, $b = x_2 - x_3$, $a = x_1 - x_3$.

9. Norm of a vector

Let X, Y be vectors. Then any real number $\| X \|$ such that

i. $\| X \| \geq 0$, $\| X \| = 0 \Leftrightarrow X = 0$;

ii. $\|aX\| = |a|\ \|X\|$, $a \in R$;

iii. $\|X+Y\| \leq \|X\| + \|Y\|$;

defines a norm of X.

The last condition is known as the triangular inequality because for E_2 the Euclidean norm (defined below) reduces to the familiar triangular inequality of plane geometry.

Example: $\|X\| = |x_1| + |x_2| + |x_2| + \ldots + |x_n|$ is a norm of n-vector X.

For,

i. When $X \neq 0$, at least one of the $x_j = 1, 2, \ldots, n$, is nonzero and therefore $\|X\| > 0$; if $X = 0$, then $\| X \| = 0$;

ii. $\| a X \| = |ax_1| + |ax_2| + \ldots + |ax_n|$

$= |a|\ |x_1| + |a|\ |x_2| + \ldots + |a|\ |x_n| = |a|\ \|X\|$;

iii. $\|X + Y\| = |x_1 + y_1| + |x_2 + y_2| + \ldots + |x_n + y_n|$

$\leq |x_1| + |y_1| + |x_2| + |y_2| + \ldots + |x_n| + |y_n|$

$= \| X \| + \| Y \|$.

10. Euclidean norm of a vector

The Euclidean norm of an n-vector X is defined as

$\|X\| = (X^1X)^{1/2} = \left(x_1^2 + x_2^2 + \ldots + x_n^2\right)^{1/2}$. It is usually denoted as $| X |$.

Note:

1. Cauchy – Schwarz inequality.

For any pair of n-vectors X, Y, $|X^1Y| \leq |X|\ |Y|$

2. A particular n-vector $X \in E_n$ may be said to define a point X in space E_n with the origin corresponding to the zero vector. Then $|X|$ may be regarded as the length of the vector X or the distance of the point X from the origin 0. A vector with norm 1 is called a unit vector. A unit vector in the direction of X is $X/|X|$.

   $|X-Y| = [(x_1-y_1)^2 + \ldots + (x_n-y_n)^2]^{1/2}$ may be regarded as the distance between the points X and Y, or the length of the vector from X to Y or Y to X.

11. Consider a system of m linear equation in n unknowns $x_1, x_2, \ldots, x_n$ of the form

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \qquad (1)$$

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$, where a_w and b_i, i=1, 2…, m, j=1, 2, …., n are real numbers. In matrix notation we may write (1)

$$\text{as } AX=B \qquad (2)$$

where A is the m x n matrix $\begin{bmatrix} a_{11}\, a_{12} \ldots . a_{1n} \\ a_{21}\, a_{22} \ldots a_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}\, a_{m2} \ldots a_{mn} \end{bmatrix}$

B is the column vector $B = [b_1\ b_2\ \ldots..\ b_m]^1$, and X is the column vector $X = [x_1, x_2, \ldots\ldots x_n]^1$

A vector "X which satisfies, the system of equations (1) is said to be a solution of (1).

The system of equations (1) is said to be inconsistent if it has no solution, otherwise it is consistent.

If B=0, equation (2) reduces to the homogeneous form AX=0.

12. δ-neighbourhood

The δ-neighbourhood of X in E_n is defined as the set of all points Y in E_n such that $|Y - X| < \delta$, where $\delta > 0$, $\delta \in R$.

13. Boundary point

X is a boundary point of the set S if every δ-neighbourhood of X contains some which are in S and some which are not in S.

For example, in $S_1 = \{X| \mid X \mid \leq 1\}$, $S_2 = \{X \mid \mid X \mid < 1\}$, $X \in E_2$, the points on the circumference of the circle $x_1^2 + x_2^2 = 1$ are the boundary points. S_1 contains all its boundary points while S_2 contains none of them. Thus the boundary points of a set may belong to the set or may not. This leads to the concept of closed and open sets.

14. Complement in E_n

If $S \subseteq E_n$, the set $E_n - S$ of all points which are in E_n but not in S is called the complement of S.

15. Open and closed sets

A set is said to be closed if it contains all its boundary points, and is said to be open if its complement is closed.

16. Bounded from below and above

A set S is bounded from below if there exist a Y in E_n with each component finite, such that for every X in S, $Y \leq X$.

[Note: $Y \leq X \Leftrightarrow y_j \leq x_j$, $j = 1, 2, \ldots, n$.]

If the inequality in the above definition is reversed we get the definition of a set bounded from above.

17. Bounded set.

A set S is bounded if there exists a finite real number $M \geq 0$ such that for all X in S, $|X| \leq M$.

It can be proved that a bounded set $\Leftrightarrow$ a set bounded from below and above.

Example: the set $S = \{x|(x,-1)^z + x_2^z \leq 4\}$, $S \subseteq E_2$

is bounded, since for every X, $|X| \leq 3$. It is bounded from above as for every (x_1, x_2) in S, $(x_1, x_2) \leq (3, 2)$. It is bounded from below as $(x_1, x_2) \geq (-1, -2)$.

$S = \{X| (x_1 \leq x_2, x_1 \geq 0, x_2 \geq 0\}$, $S \subseteq E_2$, is bounded from below as for every X in S, $X \geq 0$. But it is not bounded from above. So it is not bounded.

18. Line, half line and line segment

Let X_1, X_2 be two points in $S \subseteq E_n$.

The set $L = \{X | X = (1-\lambda)X_1 + \lambda X_2\}$, $\lambda \in R$, is defined as a line in E_n passing through X_1 and X_2.

If $\lambda \geq 0$, the set $L^+ = \{X | X = (1-\lambda)X_1 + \lambda X_2\}$ is called the half line originating from the point X_1.

By restricting λ to $0 \leq \lambda \leq 1$, we get the set of points lying on the line segment between X_1 and X_2.

19. Convex linear combination of two points.

The point $X = (1-\lambda)X_1 + \lambda X_2$, $0 \leq \lambda \leq 1$, is called the convex linear combination of X_1 and X_2.

It is any point lying on the line segment joining X_1 and X_2.

20. Convex Sets

A set $K \subseteq E_n$ is said to be convex if the convex linear combination of any two points in K belongs to K. In other words, K is convex if $X_1, X_2 \in K \Rightarrow X \in K$ where $X = (I-\lambda)X_1 + \lambda X_2$, $0 \leq \lambda \leq 1$.

Some simple examples of convex and nonconvex sets in E_2 are shown in Fig. 2 to give and intuitive idea. In each convex set every point P on the line segment joining any two points in the set which does not satisfy this condition, the set is not convex.

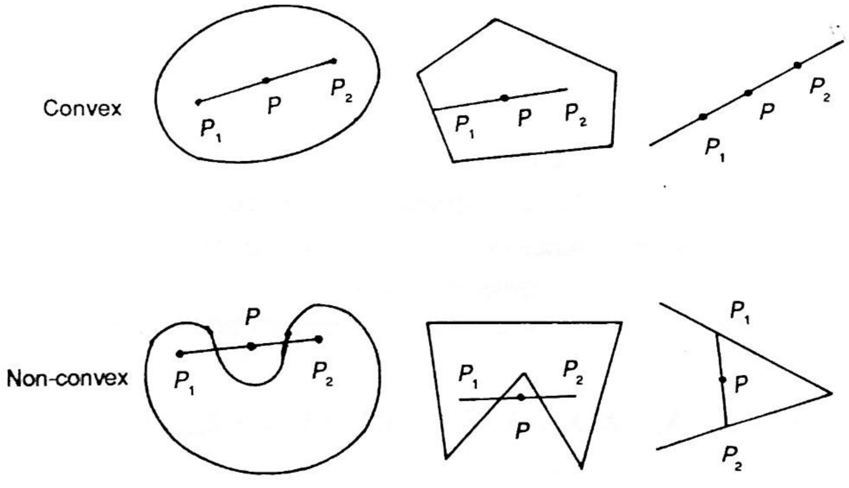


Fig. 2

To take an example in E_n, consider $S_1 = \{X|\ |X|\leq 1\}$.

Let $X_1, X_2 \in S_1$. Then $|X_1| \leq 1$, $|X_2| \leq 1$, and for $0 \leq \lambda \leq 1$,

$$|\lambda X_1 + (-\lambda)X_2| \leq |\ \lambda X_1| + |\ (1-\lambda)X_2|$$

$$= \lambda|X_1| + (1-\lambda)X_2 \in S_1 \Rightarrow S_1 \text{ is convex.}$$

21. Convex linear combination of points

Let $X_i \in E_n$, and let λ_i be non-negative real numbers such that $\sum_{i=1}^{m} \lambda_i = 1.$ Then

$X = \lambda_1 X_1 + \lambda_2 X_2 + \ldots + \lambda_m X_m$

is called the convex linear combination of points X_i, $i = 1, 2, \ldots, m$.

22. Convex hull of a set

The convex hull of a set S is the intersection of all convex sets of which S is a subset. We shall denote by [S] the convex hull of S.

Let $S \subseteq E_2$, $S=\{X|X = X_i,\ i = 1, 2, 3, 4\}$, where X_i are the vertices of a quadrilateral in a plane. Then either [S] is the convex quadrilateral or the convex triangle formed as shown in Fig. 4. Any point in [S] can be expressed as a convex linear combination of at the most three vertices of S.

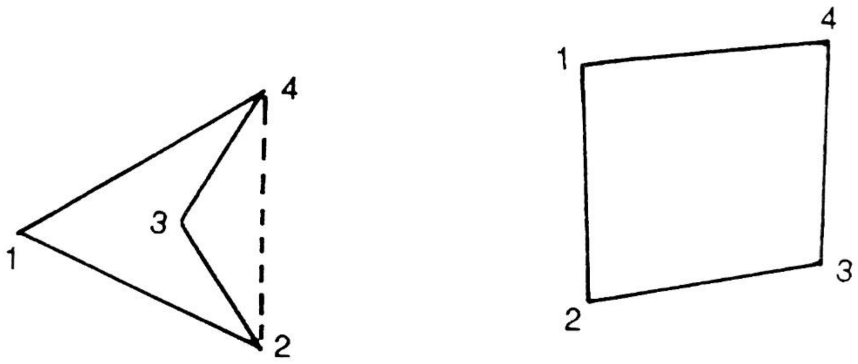


Fig. 4

23. Vertices or extreme points of a convex set.

A point X of a convex set K is an extreme point or vertex of K if it is not possible to find two points X_1, X_2, in K such that $X = (1-\lambda)X_1 + \lambda X_2$, $0<\lambda<1$. A point of K which is not a vertex of K is an internal point of K.

24. Convex polyhedron

The set of all convex combinations of a finite number of points X_1, i=1,2,….,m, is the convex polyhedron spanned by these points.

25. Hyperplanes, half-spaces

Let $X \in E_n$, C ($\neq 0$) a constant row n-vector and $\alpha \in R$. Then we define

i. a hyperplane as $\{X|CX=\alpha\}$;

ii. a closed half-space as $\{X|CX \leq \alpha\}$;

iii. an open half-space as $\{X|CX<\alpha\}$.

26. Polytope

The intersection of a finite number of closed half-spaces is called a polytope. The hyperplanes producing the half-spaces are called the generating hyperplanes of the polytope.

27. Cone

The polytope generated by hyperplanes all of which intersect in one and only one point is a cone.

28. Edge of a polytope

A line in a polytope is said to be an edge of the polytope if the line is the only intersection of those generating hyperplanes which contain the line.

29. Supporting hyperplane

Let S be a nonempty set (not necessarily convex) and let $CX=\alpha$ be a hyperplane which meets S such that S is contained either in the half-space $CX\leq\alpha$ or in $CX\geq\alpha$. Then $CX = \alpha$ is called a supporting hyperplane to the set S.

30. Quadratic forms and definiteness

Let $X\in E_n$, A homogeneous expression of the second degree of the form

$$f(X)=c_{11}x_1^2+c_{22}x_2^2+\ldots+c_{nn}x_n^2+c_{12}x_1x_2+c_{13}x_1x_3+c_{23}x_2x_3+\ldots$$

Where $c_{ij} \in R;i,j = 1, 2, \ldots, n$,

is called a quadratic form in the n variables $x_1, x_2, \ldots, x_n$.

A quadratic from X^1AX is said to be positive definite if $X^1AX>0$ for all $X\neq0$. It is said to be positive semi definite if $X^1AX \geq 0$ for al $X\neq0$ and there is at least one nonzero vector for which $X^1AX=0$.

Negative definite and negative semi definite forms are defined by reversing the inequality signs in the above definition.

Examples: The quadratic form $x_1^2+2x_2^2+3x_3^2$ is positive definite as it is positive for every vector $[x_1\ x_2\ x_3]^1$ except $x_1 = x_2 = x_3 = 0$.

The expression $(x_1- x_2)^2 + 4x_3^2$ is positive semi definite as it is either positive or zero (never negative) for all values of x_1, x_2, x_3, and is zero for at least one value, say $x_1 = x_2 = 1, x_3=0$, other than $x_1 = x_2 = x_3 = 0$.

The quadratic form $x_1^2-2x_2^2$ is positive for some values of x_1, x_2 and negative for others. It is neither definite nor semi definite. It may be called indefinite.

DEFINITION

31.Real valued function

Let X be a vector in S, a subset of E_n.

If for every X there exists a unique real number f(X), then f(X) is said to be a real valued function of X. In other words, it is mapping of S into R, the set of real numbers. S is the domain of the function.

32.Continuous function

F(X) is said to be continuous at X_0 if for each real number m>0 there exists a real number $\delta>0$ such that

$|X - X_0| < \delta \Rightarrow f(X) - f(X_0)| < \eta$.

It can be proved that f(X) is continuous at X_0 if and only if $X \to X_0 \Rightarrow f(X) \to f(X_0)$.

If f(X) is continuous at every point in S, it is said to be continuous in S, S being a subset of E_n.

33.Partial derivatives & gradient vector.

Let $\triangle x_j = [0\ 0 \ldots \delta x_j\ 0\ 0]^1$ be an n-vector in E_n with all its components zero except the *j*th which is δx_j.

The partial derivative of f(X) with respect to any component x_j of X is defined as

$$\lim_{\delta x_j \to 0} \frac{f(X + \Delta x_j) - f(X)}{\delta x_j},$$

and is denoted as $\partial f / \partial x_j$.

The n-vector whose components are the n partial derivatives of f(X) is called the gradient of f(X):

$$\text{grad } f = \nabla f = \left[\frac{\partial f}{\partial x_1} \ \frac{\partial f}{\partial x_2} \ \cdots \ \frac{\partial f}{\partial x_n} \right].$$

34.Differentiability.

If f(X) has continuous partial derivatives with respect to each of its variables, it is said to be differentiable.

35. Taylor series.

Let f(X) be a real-valued differentiable function of X in E_n. Let X + $\triangle$X be a neighbouring point of X in E_n such that

$\triangle X = [\delta x_1\ \delta x_2\ \ldots\ \delta x_n]^1$

and $$X + \triangle X = [x_1 + \delta x_1\ x_2 + \delta x_2\ \ldots\ x_n + \delta x_n]^1.$$

The Taylor series for a function of n variables may be written as

$$f(x_1+\delta x_1, x_2+\delta x_2,\ldots x_n + \delta x_n) = f(x_1, x_2, \ldots, x_n) + \left(\delta x_1 \frac{\partial f}{\partial x_1} + \delta x_2 \frac{\partial f}{\partial x_2} + \ldots + \delta x_n \frac{\partial f}{\partial x_n}\right)$$

$$+ \frac{1}{2}\left(\delta x_1 \frac{\partial}{\partial x_1} + \delta x_2 \frac{\partial}{\partial x_2} + \ldots + \delta x_n \frac{\partial}{\partial x_n}\right)^2 f + \text{terms in third and higher powers of } \delta x_j,$$

We have $\nabla f(x) = \left[\frac{\partial f}{\partial x_1}\ \frac{\partial f}{\partial x_2}\ \ldots\ \frac{\partial f}{\partial x_n}\right]^1$.

Also we define the Hessian of f(X) as the n x n matrix

$$H(X) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

We may then write the Taylor series in the form

$$F(X+\triangle X)=f(X)+(\triangle X)^1\nabla f(X)+\frac{1}{2}(\triangle X)^1 H(X)(\triangle X)+e(X,\triangle X) \mid \triangle X \mid^2 \ldots\ldots(1)$$

where $e(X,\triangle X) \to 0$ as $\mid \triangle X \mid \to 0$.

36. Directional derivative, direction of steepest descent

Let $X \in E_n$ and f(X) be a differentiable real–valued function. Let Y be a unit vector in E_n. Then tY and X+tY are also vectors in E_n, where $t \in R$.

The directional derivative of f(X) in the direction of Y is defined as $\lim\limits_{t \to 0} \frac{f(X + tY) - f(X)}{t}$.

37. The directions of ∇f and $-\nabla f$ are respectively called the directions of steepest ascent and steepest descent.

38. Global minimum and maximum.

f(x) is said to have a global minimum at X_0 in S if for all X in S, $f(X) \geq f(X_0)$. By reversing the inequality sign we get the definition of a global maximum.

39. F(X) is said to have a local or relative minimum at X_0 in S if there exists a δ-neighbourhood of X_0, such that $f(X) \geq f(X_0)$ for all X in the neighbourhood. Again, by reversing the sign of inequality we get the definition of a local or relative maximum.

The word extremum (plural extrema) is used to indicate either maximum or minimum.

40. Saddle point

The function f(Y,Z) is said to have a saddle point at (Y_0, Z_0)if $f(Y_0, Z) \leq f(Y_0, Z_0) \leq f(Y, Z_0)$ for all (Y,Z) in the neighbourhood of (Y_0, Z_0).

41. Relative extremum and constrained extremum

f(X) is said to have a relative extremum at X_0 subject to constraints $g_i(X)=0$ 1=1,2,…….m, where each g;(X) is a real valued function if there is a δ-neighbourhood N of X_0 such that $f(X) \leq f(X_0)$ (maximum) or $f(X) \geq f(X_0)$ (minimum) for every X in $S_1 \cap N$. In either case X_0 is a point of constrained extremum of f(X).

42. Implicit function theorem

Let $X \in E_n$. Let $g_i(X)$, i=1, 2, …, m, be m real-valued differentiable functions in some neighbourhood of X_0 in $S \subseteq E_n$, and let,

$$g_i(X_0) = 0, \quad i=1,2,\ldots,m. \tag{6}$$

Also let the Jacobian

$$\frac{\partial(g_1, g_2 \ldots, g_m)}{\partial(x_1, x_2, \ldots, x_m)} \neq 0$$

at X_0. Then there exists a neighbourhood N of $(x_{m+10}, X_{m+20}, \ldots, x_{n0})$ and functions $\phi_1, \phi_2, \ldots. \phi_m$ differentiable in N such that

$$x_k = \phi_k (x_{m+1}, x_{m+2}, \ldots., x_{n0}), k=1,2,\ldots.,m,$$

is a solution of (6) in N giving x_{k0} at $(x_{m+10}, x_{m+20}, \ldots. X_{n0})$

43.Method of Lagrange multipliers

Let $X \in S \subseteq E_n$ and f(X) be a real-valued differentiable function. Also let

$$g_i(X)=0, i=1, 2, \ldots, m, \tag{7}$$

where each $g_i(X)$ is a real-valued differentiable function and for each X in S the m x n matrix

$$\left[\frac{\partial g_1}{\partial x_j}\right], i=1, 2, \ldots, m, j = 1, 2, \ldots., n, \tag{8}$$

is of rank m. Further let f(X) have a constrained relative extremum at X_0 subject to constraints (7). Then there exist real numbers λ_i, i=1, 2, .., m, such that X_0 is a stationary point of the function.

$$F(X)=f(X) + \sum_{i=1}^{m} \lambda_i g_i(X). \tag{9}$$

MODULE – I

CHAPTER – 1
CONVEX FUNCTIONS

Convex functions

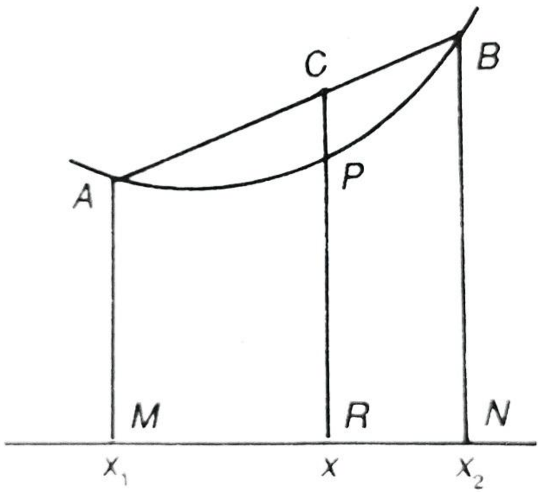
Let $X \in K \subseteq E_n$ where K is a convex set. A function f(X) is said to be convex if for any two points X_1 and X_2 in K,

$f(X) \leq (1-\lambda) f(X_1) + \lambda f(X_2)$ $0 \leq \lambda \leq 1$,

for every $X = (1-\lambda) X_1 + \lambda X_2$.

The function is said to be concave if the inequality sign is reversed or if $-f(X)$ is convex.

Interpreting in E_1, let x_1, x_2 be two points M and N respectively on the real line (Plot 1) and let a convex linear combination of x_1 and x_2 be x which is any point R on the segment MN. With f(x) as the curve shown, $MA=f(x_1)$, $NB = f(x_2)$, $RP=f(x)$. If $x=(1-\lambda) x_1 + \lambda x_2$, it is easy to see that $RC = (1-\lambda) f(x_1) + \lambda f(x_2)$. Now $RP \leq RC$ for all R lying in MN means that the curve is bulging out or convex towards the real line. If this happen for all x_1, x_2 in a convex domain [a, b] we say the curve f(x) is convex. If $RP \geq RC$ f(x) is concave.



Plot : 1

It is important to note that the convexity or concavity of a function is defined only when its domain is a convex set. The use of convex sets in this context gives us regions in E_n which are in a sense 'unbroken' in each variable x_j, j = 1, 2, …, n.

The following theorems are proved for convex functions. The corresponding theorems for concave functions can be easily enunciated.

Theorem 3. Let $X \in E_n$ and let $f(X) = X'AX$ be a quadratic form. If f(X) is positive semi definte, then f(X) is a convex function.

Proof. Let X_1, X_2 be any two points in E_n, and let $X = (1-\lambda) X_1 + \lambda X_2$, $0 \leq \lambda \leq 1$. Also let $f(X) = X^1AX$ be positive semidefine, that is, $X^1AX \geq 0$ for any $X \in E_n$. Then

$(1-\lambda) f(X_1) + \lambda f(X_2) - f(X)$

$= (1-\lambda) X^1{}_1AX_1 + \lambda X^1{}_2AX_2-((1-\lambda) X_1 + \lambda X_2)^1 A((1-\lambda)X_1+\lambda X_2)$

$= (1-\lambda) X^1{}_1AX_1 + \lambda X^1{}_2AX_2 - (1-\lambda)^2 X^1{}_1AX_1-\lambda^2 X^1{}_2AX_2 - 2\lambda(1-\lambda) X^1{}_1AX_2$

$= \lambda(1-\lambda) (X^1{}_1AX_1+ X^1{}_2AX_2 - 2X^1{}_1AX_2)$

$= \lambda(1-\lambda) (X_1-X_2)^1A (X_1-X_2) \geq 0$

because $0 \leq \lambda \leq 1$ and X_1-X_2 is any vector in E_n. Hence

$f(X) \leq (1-\lambda) f(X_1) + \lambda f(X_2)$

which means f(X) is a convex function. Proved.

Theorem 4. Let $K \subseteq E_n$ be a convex set, $X \in K$, and f(X) a convex function. Then if f(X) has a relative minimum, it is also a global minimum. Also if this minimum is attained at more than one point, the minimum is attained at the convex linear combination of all such points.

Proof: Let f(X) have a relative minimum at X_0, Let $X_1 \in K$. The for any $\delta>0$ it is possible to choose λ_n, $0 < \lambda < 1$, such that there exists $X=\lambda X_0 + (1-\lambda)X_1$ lying in the δ-neighbourhood of X_0. By the definition of relative minimum, with X in this neighbourhood.

$f(X_0) \leq f(X)$

$\Rightarrow f(X_0) \leq f(\lambda X_0+(1-\lambda)X_1) \leq \lambda\, f(X_0) + (1-\lambda)\, f\,(X_1)$, since f(X) is convex

$\Rightarrow (1-\lambda)\, f(X_0) \leq (1-\lambda)\, f(X_1)$

$\Rightarrow f(X_0) \leq f(X_1)$, since $1-\lambda$ is positive,

$\Rightarrow f(X_0)$ is a global minimum.

Let Y_0 be another point where the minimum is attained. Then

$f(X_0) = f(Y_0)$.

Since Y_0 is a point in K, what is true of X_1 is also true of Y_0, and so

$f(X_0) \leq f(\lambda X_0 + (1-\lambda)Y_0)$

$\leq \lambda f(_0) + (1-\lambda)f(Y_0) = f(Y_0)$

$\Rightarrow f(X_0) = f(\lambda X_0 + (1-\lambda)Y_0)$

Which means minimum is also attained at the convex linear combination of X_0 and Y_0. Thus the set of points where f(X) is minimum is a convex set and is therefore a convex linear combination of points (not necessarily only two) in it.

Proved.

Theorem 5. Let f(X) be defined in a convex domain $K \subseteq E_n$ and be differentiable. Then f(X) is a convex function if and only if

$f(X_2) - f(X_1) \geq (X_2-X_1)^1 \nabla f(X_1)$

for all X_1, X_2 in K.

Proof: First, for any X_1, X_2 in K let

$f(X_2)-f(X_1) \geq (X_2-X_1)^1 \nabla f(X_1)$

Let X_3 be any point in K such that

$X_1 = \lambda X_2 + (1-\lambda)X_3,\ 0 \leq \lambda \leq 1$.

Then, from hypothesis,

$f(X_3) - f(X_1) \geq (X_3-X_1)^1 \nabla f(X_1)$.

From the above two inequalities

$\lambda f(X_2) - \lambda f(X_1) + (1-\lambda)f(X_3) - (1-\lambda)\, f(x_1)$

$\geq [\lambda(X_2-X_1)^1 + (1-\lambda)\,(X_3-X_1)^1]\nabla f(X_1)$

$\Rightarrow \lambda f(X_2) + (1-\lambda) f(X_3) - f(X_1) \geq [\lambda X_2 + (1-\lambda)X_3 - X_1]^1 \nabla f(X_1) = 0$

$\Rightarrow \lambda f(X_2) + (1-\lambda) f(X_3) \geq f(X_1) = f(\lambda X_2 + (1-\lambda)X_3)$

which means f(X) is a convex function.

To prove the converse, let f(X) be a convex function. Then for X_1, X_2 in K and $0 < \lambda < 1$,

$(1-\lambda) f(X_1) + \lambda f(X_2) \geq f((1-\lambda) X_1 + \lambda X_2)$

$\Rightarrow \lambda f(X_2) - \lambda f(X_1) \geq f((1-\lambda)X_1 + \lambda X_2) - f(X_1)$

$\Rightarrow f(X_2) - f(X_1) \geq \dfrac{f(X_1 + \lambda(X_2 - X_1)) - f(X_1)}{\lambda}$

Taking limit as $\lambda \to 0$,

$f(X_2) - f(X_1) \geq (X_2 - X_1)^1 \nabla f(X_1)$. Proved.

Theorem 6. Let f(X) be a convex differentiable function defined in a convex domain $K \subseteq E_n$. Then $f(X_0)$, $X_0 \in K$, is a global minimum if and only if $(X-X_0)'\nabla f(X_0) \geq 0$ for all X in K.

Proof: First, let $f(X_0)$ be a global minimum. Then for all X in K.

$f(X) \geq f(X_0)$.

Also, since for any X in K, $\lambda X + (1-\lambda)X_0$ is also in K.

$f(\lambda X + (1-\lambda)X_0) \geq f(X_0)$, $0 < \lambda < 1$,

$\Rightarrow f(X_0 + \lambda(X - X_0)) \geq f(X_0)$

$\Rightarrow f(X_0 + \lambda(X - X_0)) - f(X_0) \geq 0$.

Dividing by λ and taking limit as $\lambda \to 0$,

$(X - X_0)^1 \nabla f(X_0) \geq 0$.

It should be noticed that if X_0 is an interior point in K, $f(X_0)$ is also a local minimum and so $\nabla f(X_0) = 0$ and then necessarily $(X - X_0)^1 \nabla f(X_0) = 0$. It is only when X_0 is a boundary point that $\nabla f(X_0)$ may not be zero, but even then necessarily $(X - X_0)^1 \nabla f(X_0) \geq 0$.

To prove the converse, let for every X in K.

$(X - X_0)^1 \nabla f(X_0) \geq 0$.

Since f(X) is convex, from theorem 5,

$f(X) - f(X_0) \geq (X - X_0)^1 \nabla f(X_0) \geq 0.$

which means $f(X_0)$ is a global minimum.

CHAPTER – II

GENERAL PROBLEM OF MATHEMATICAL PROGRAMMING

In the mathematical model of the system occur variables $X = (x_1, x_2, \ldots., x_n)$ which can be controlled and varied, and parameters over which there is no control. The latter are to be regarded as given constants. The limitation on X, when put in mathematical terms, take the form of constraints of the type

$g_i(X) \leq 0$, $i=1, 2, \ldots, p$; $g_i(X) \geq 0$, $i=p+1, \ldots r$; $g_i(X) = 0$, $i=r+1, \ldots, m$; where $g_i(X)$ are real-valued functions of X. In general the constraints can always be put as

$g_i(X) \leq 0$, $i=1, 2, \ldots,m$,

because

$g_i(X) \geq 0 \Leftrightarrow g_i(X) \leq 0$,

$g_i(X) = 0 \Leftrightarrow g_i(X) \geq 0, g_i(X) \leq 0$.

It is also possible to convert an inequality into an equation by introducing an extra variable with a constraint imposed on it. Thus

$$g_i(X) \geq 0 \Leftrightarrow g_i(X) + x_{n+i} = 0,\ x_{n+i} \geq 0, \qquad \text{(i)}$$

and $$g_i(X) \geq 0 \Leftrightarrow g_i(X) - x_{n+i} = 0,\ x_{n+i} \geq 0 \qquad \text{(ii)}$$

The variable x_{n+i} so introduced is called a slack variable. Many authors distinguish between cases (i) and (ii) by calling x_{n+i} in case (i) the slack variable and in case (ii) the surplus variable. We, however, propose to use the term slack to cover both cases.

The constraints $X \geq 0$, usually referred to as nonnegativity conditions, often occur in mathematical models of systems either because negative values of variables do not make any sense and are therefore excluded from consideration or because it is mathematically convenient to introduce some slack variables with this constraint.

The performance, return, utility or whatever other objective is sought to be achieved through the system is generally measured by a real-valued function f(X). It is given various names in various problems, such as performance index, utility measure, return function, objective function, gain, etc. We shall throughout this book call it the objective function. The most advantageous or optimum value of the objective function is sought to be achieved through a mathematical solution. The value X_0 of the variable X which makes f(X) optimum is called the optimal value of X or the optimal solution. Usually the optimum value of f(X) is the maximum or the minimum of f(X) under constraints.

Mathematical programming is the general term used for such problems. The general problem may be stated as follows.

Mathematical programming is the general term used for such problems. The general problem may be stated as follows.

Let f(X), $g_i(X)$, i=1, 2, ….,m, be real-valued functions of X in E_n, and let S be the subset of E_n containing all points satisfying the constraints.

$g_i(X) \leq 0$, i=1, 2, ….., m; $X \geq 0$.

To find X_0 in S such that $f(X_0)$ is a global minimum in S.

As explained above, we may introduce slack variables x_{n+i} to put the constraints as

$g_i(X)+x_{n+i} = 0$, $x_{n+i} \geq 0$.

Then the total number of variables becomes n+m. We can state the general problem in the following alternative form also.

To find $X_0 \in S \subseteq E_n$ such that $f(X_0)$ is a global minimum in S and for all X in S

$g_i(X) = 0$, i=1, 2, …,m; $X \geq 0$.

It should however be remembered that n in the above statement is n+m of the earlier statement and neither X nor g_i are identical in the two cases.

If S is a convex set and $f(X)$ and $g_i(X)$ are convex functions, the problem is said to be of convex programming. The following theorem is significant in this connection.

Theorem 7. Let $X \in E_n$ and let $g_i(X)$, $i=1, 2, \ldots, m$, be convex functions in E_n. Let $S \subseteq E_n$ be the set of points satisfying the constraints $g_i(X) \leq 0$, $i=1, 2, \ldots, m$. Then S is a convex set.

Proof: Let X_1, X_2 be in S, and let $X_3 = \lambda X_1+(1-\lambda)X_2$, $0 \leq \lambda \leq 1$. Since $g_i(X)$ is a convex function and $g_i(X_1) \leq 0$, $g_i(X_2) \leq 0$,

$$
\begin{aligned}
g_i(X_3) &= g_i\,(\lambda X_1+ (1-\lambda)X_2) \\
&\leq \lambda g_i(X_1) + (1-\lambda)g_i(X_2) \leq 0.
\end{aligned}
$$

Hence X_3 is in S and so S is a convex set.

CHAPTER – III

LINEAR PROGRAMMING

III.1 Introduction

The general problem of mathematical programming, described in chapter 2, reduces to linear programming (LP) when the functions $f(X)$, $g_i(X)$, $i=1,2,\ldots,m$, all are linear. So the problem of LP is to find a minimum (or maximum) of a linear function subject to linear constraints.

III.2 Linear programming in two dimensional space.

Since the basic features of LP can be illustrated in two-dimensional space, we first consider an LP problem in two variables.

Let $X \in E_2$, and

$$f(X)=4x_1 + 5x_2 \qquad (1)$$

Also let

$$x_1 - 2x_2 \leq 2, \qquad \text{(i)}$$
$$2x_1 + x_2 \leq 6, \qquad \text{(ii)}$$
$$x_1 + 2x_2 \leq 5, \qquad \text{(iii)} \qquad (2)$$
$$-x_1 + x_2 \leq 2, \qquad \text{(iv)}$$
$$x_1 + x_2 \geq 1, \qquad \text{(v)}$$
$$x_1, x_2 \geq 0. \qquad (3)$$

The problem is to find $X_0=(x_{10}, x_{20})$ which maximizes $f(X)$ and satisfies the constraints (2) and (3).

Let us try a graphical approach. The non-negativity conditions (3) restrict the point X_0 to the first quadrant of the E_2 space (x_1 x_2 plane). Constraint (i) gives the half-space bounded by the straight line $x_1-2x_2=2$. If (i) is to be satisfied, the point X_0 is either on this line or on the same side of it as the origin. Similarly (ii) is the half-space of points lying on the line $2x_1+x_2=6$ or on the same side of it as the origin. So with (iii) and (iv). Constraint (v) however is the half-space of points either on the line $x_1 + x_2 = 1$ or on that side of it which is opposite to the origin. The intersection of all constraints (2) and (3) is the convex polygon ABCDEFG (see Fig. 1). Any point X within the polygon or on its boundary satisfies constraints (2) and (3). An infinity of such points exist. Our problem is to find that point (or points) X_0 within the polygon or on its boundary which makes $f(X) = 4x_1 + 5x_2$ the maximum.

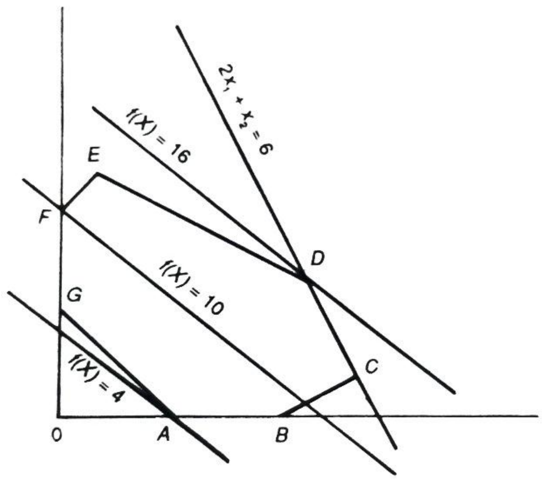


Fig. 1

Consider the parallel straight lines that correspond to different values of f(X). The lines f(X) = 8, 10, 12 cut through the polygon and f(X) = 16 just goes through the point D. The line for values of f(X) greater than 16 will not intersect the polygon at all. Therefore the maximum value, subject to the constraints, that f(X) can have is 16, and it is attained at the point $x_1 = 7/3$, $x_2 = 4/3$, which is one of the vertices of the polygon.

Suppose the problem were to find the minimum value of f(X) subject to the same constraints. It can be seen that the line f(X) =4 just passes through the pint A(1,0) and lines for smaller values of f(X) do not intersect the polygon. The solution therefore would be min f(X) = 4 for X = (1, 0) which again is a vertex of the polygon.

In both cases the point where f(X) attains its extreme value is unique and is a vertex of the polygon.

Let us consider a slightly different case. Let f(X) be $2x_1+x_2$, and suppose the maximum of f(X) subject to the same constraints has to be determined. The answer in this case is f(X) = 6 with any point on the side DC of the polygon giving this value. f(X) thus attains its maximum value at the vertex C(14/5, 2/5) and also at D(7/3, 4/3) and also at every pint on CD, that is, any convex linear combination of C and D.

The following features of the problem which are of fundamental significance deserve notice.

(i) The set of solutions of (2) and (3) is a convex set with vertices.

(ii) f(X) is optimum (maximum or minimum) at a vertex of this convex set, and if there are two such vertices, every convex linear combination of the vertices is also a point where f(X) is optimum.

Constraints (2) are in the form of inequalities. Mathematically it is easier to deal with equations, and theoretical discussion would be simpler if constraints coud be written as equations. It can be done by introducing one slack variable in each of the constraints. The resulting system of equations and constraints is.

$$\begin{aligned}
x_1 - 2x_2 + x_3 \qquad\qquad\qquad &= 2,\\
2x_1 + x_2 \quad + x_4 \qquad\qquad &= 6,\\
x_1 + 2x_2 \qquad + x_5 \qquad &= 5,\\
-x_1 + x_2 \qquad\qquad + x_6 &= 2,\\
x_1 + x_2 \qquad\qquad\qquad - x_7 &= 1;
\end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0.$$

Note that we introduce the slack variables in such a way that all of them, along with the original variables, satisfy the non-negativity conditions (5).

We have replaced (2) and (3) with the equivalent conditions (4) and (5) but only by increasing the number of variables. But any complication caused by this increase is more than compensated by the fact that we have to deal with equations now.

Let us examine equations (4) closely. These are five equations in seven unknowns and therefore can have an infinity of solutions. Putting any two variables zero, we get unique values for the rest of the variables, thus getting a basic solution. There are ${}^{7}C_{2}$(=21) ways of choosing two variables as zero. Table 1 gives all the 21 basic solutions of (4). The significant thing about these solutions is that there are exactly seven solutions which are non-negative, and these, so far as the values of x_1 and x_2 are concerned, correspond to the seven vertices of the convex polygon we have obtained graphically. One can suspect some theoretical relationship between such solutions of equations of constraints and the solution to the LP problem. We shall establish such a relationship in the following sections.

TABLE 1

No.	X_1	X_2	X_3	X_4	X_5	X_6	X_7	Vertices of Polygon
1	0	0	2	6	5	2	–1	
2	0	–1	0	7	7	3	–2	
3	0	6	14	0	–7	–4	5	
4	0	5/2	7	7/2	0	–1/2	3/2	
5	0	2	6	4	1	0	1	F
6	0	1	4	5	3	1	0	G
7	2	0	0	2	3	4	1	B
8	3	0	–1	0	2	5	2	
9	5	0	–3	–4	0	7	4	
10	–2	0	4	10	7	0	–3	
11	1	0	1	4	4	3	0	A
12	14/5	2/5	0	0	7/5	22/5	11/5	C
13	7/2	¾	0	–7/4	0	19/4	–13/4	

14	–6	–4	0	22	19	0	11	
15	4/3	–1/3	0	11/3	13/3	11/6	0	
16	7/3	4/3	7/3	0	0	3	8/3	D
17	4/3	10/3	22/3	0	–3	0	–11/3	
18	5	–4	–11	0	8	11	0	
19	1/3	7/3	19/3	3	0	0	5/3	E
20	–3	4	13	8	0	–5	0	
21	–1/2	3/2	11/2	11/2	5/2	0	0	

III.3 General LP Problem

We can now enunciate the general LP problem as follows.

Let $X \in E_n$ and f(X) and $g_i(X)$ be linear functions defined as $f(X)=\sum_{j=1}^{n} c_j x_j$,

$g_i(X) = \sum_{j=1}^{n} a_{ij} x_j - b_i$; $c_j, a_{ij}, b_i \in R$; i=1, 2, …, m, j=1, 2, …, n.

To find X_0 such that

$$f(X_0) \leq f(X)$$

for all X satisfying the constraints

$$g_i(X) = 0, \ i=1, 2, \ldots, m,$$

$$\text{and } x \geq 0$$

It is more customary to state this LP problem, in matrix, notation, as

Minimize $\quad f(X) = CX, \quad (6)$

Subject to $\quad AX = B, \quad (7)$

$\quad X \geq 0 \quad (8)$

where C is a row vector and X and B are column vectors.

$C=[c_1\, c_2 \ldots\, c_n]$, $X=[x_1\, x_2 \ldots\, x_n]'$, $B=[b_1\, b_2 \ldots\, b_m]'$,

and A is an m x n matrix.

$$A=\begin{bmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ . & . \;\ldots\; . \\ a_{m1} & a_{m2} \ldots a_{mn} \end{bmatrix}$$

The equivalent form of the problem in ordinary notation is

Minimize $\quad f=\sum_{j=1}^{n} c_j x_j$, $\quad$ (6)

subject to $\quad \sum_{j=1}^{n} a_{ij} x_j = b_i$; $i=1, 2, \ldots, m$, $\quad$ (7)

$x_j \geq 0$, $j=1,2,\ldots,n$ $\quad$ (8)

Expression (6) is the objective function to be minimized, (7) are the constraints and (8) the non-negativity conditions. Equations (8) are also constraints but, because of their simplicity, are treated separately form (7). The coefficients c_j are usually called the cost coefficients.

The form (6), (7), (8) of an LP problem is general. If given in any other form it can always be converted to this form. If f(X) is to be maximized, then we may put $-f(X) = \Psi(X)$ which is to be minimized. If a constraint is an inequality, then it can be converted to an equation by introducing a slack variable. If a variable x_j is unconstrained, that is, if it may vary from $-\infty$ to $+\infty$, then we may replace x_j by two other variables x_{j1} and x_{j2} such that $x_j=x_{j1}-x_{j2}$ where $x_{j1} \geq 0$, $x_{j2} \geq 0$.

Example: Write the following LP in the above standard form.

Maximize $\quad f=2x_1 + x_2 - x_3$,

subject to $\quad 2x_1 - 5x_2 + 3x_3 \leq 4$,

$3x_1 + 6x_2 - x_3 \geq 2$,

$X_1 + x_2 + x_3 = 4$,

$X_1 \geq 0$, $x_3 \geq 0$, x_2 unrestricted.

It has equality as well as inequality constraints, and one variable x_2 is unrestricted.

Replacing x_2 by two variables x_{21}, x_{22}, such that $x_2 = x_{21}-x_{22}$, $x_{21} \geq 0$, $x_{22} \geq 0$, putting all the constraints as equations by introducing slack variables x_4 and x_5, and changing the sign of the objective function, the problem takes the following standard form.

Minimize $\quad \Psi=(-f) = -2x_1-x_{21} + x_{22} + x_3,$

subject to $\quad 2x_1-5x_{21} + 5x_{22} + 3x_3 + x_4 = 4,$

$$3x_1 + 6x_{21} - 6x_{22} - x_3 = 4,$$

$$x_1, x_{21}, x_{22}, x_3, x_4, x_5 \geq 0.$$

III.4 Feasible solutions

Definition 1. A solution of (7) and (8) is called a feasible solution.

We shall denote by S_F the set of feasible solutions. It is possible that there may be no feasible solution. In that case S_F is empty.

Theorem 1. The set S_F of feasible solutions, if not empty, is a closed convex set (polytope) bounded from below and so has at least one vertex.

Proof: S_F is the intersection of the hyperplanes $g_i(X)=0$, $i=1,2,\ldots.., m$, and the set $H=\{X|X\geq 0\}$. All these are closed convex sets and H is bounded from below. Hence S_F is a closed convex set (polytope) bounded from below, and so it has a vertex.

Alternatively, we can give a more direct proof of the convexity of S_F as follows.

Let X_1 and X_2 be two feasible solutions. Then

$$X_1 \geq 0, X_2 \geq 0; \tag{9}$$

and $\quad AX_1 = B, AX_2 = B. \quad (10)$

Let X be any convex linear combination of X_1 and X_2. Then

$$X=(1-\lambda) X_1 + \lambda X_2, 0\leq\lambda\leq 1,$$

$$\geq 0, \text{ from (9)}, \tag{11}$$

Further $\quad AX = A (1-\lambda) X_1+\lambda X_2]$

$$= (1-\lambda)AX_1 + \lambda AX_2 = B, \text{ from } (10) \tag{12}$$

(11) and (12) mean that X is a feasible solution. Thus the convex linear combination of every two feasible solutions is a feasible solution. Therefore the set of feasible solutions is a convex set.

III.5 Basic solutions

Equations (7), namely

$$AX = B \tag{7}$$

are m equations in n unknowns. We shall assume that m<n and the equations are linearly independent. Generally constraints appear as inequalities in mathematical models and the introduction of slack variables makes m<n. Therefore the assumption is justified.

If any of the n−m variables x_j are given t he values zero, the remaining system of m equations in m unknowns may have a unique solution. This solution along with the assumed zeros is a solution of (7). It is called a basic solution. The m variables remaining in the system after n−m variables have been put equal to zero are called the basic variables or simply the basis. The rest of the variables may be called non basic. Since the unique solution of m equations in m variables may also contain zeros, a basic solution must contain at least n−m zeros. (The case when the number of zeros is more than n−m is called degenerate, and will be discussed later.)

Equations (7) may be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \dots a_{1m} \\ a_{21} & a_{22} & a_{23} \dots a_{2m} \\ . & . & . \quad . \quad . \\ a_{m1} & a_{m2} & a_{m3} \dots a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ b_m \end{bmatrix} \tag{13}$$

or as $x_1P_1 + x_2P_2 + x_3P_3 + \dots + x_nP_n = B$ (14)

where P_j, j=1,2,…,n, is the m-vector in the *j*th column of A.

Since P_j is a vector in E_m, not more than m of the vectors $P_1, P_2, \ldots. P_n$ can be linearly independent. Since the equations are assumed linearly independent, exactly m of the vectors are linearly independent. Let these m vectors (suffixes rearranged, if necessary).

$P_1, P_2, P_3, \ldots, P_m$.

That they are linearly independent means that there do not exist $\propto_j \in R$, j=1,2,…,m, not all zero, such that

$\propto_1 P_1 + \propto_2 P_2 + \ldots + \propto_m P_m = 0$.

On the other hand it is possible to find $\propto_j$, not all zero, such that

$\propto_1 P_1 + \propto_2 P_2 + \ldots + \propto_m P_m + \propto_{m+r} P_{m+r} = 0$.

where P_{m+r} is any of the remaining vectors of the set. Thus the vectors P_{m+1}, $P_{m+2}, \ldots, P_n$ can be separately expressed as linear combinations of $P_1, P_2, \ldots.$, P_m, and s (14) can be rewritten as $y_1P_1 + y_2P_2 + \ldots. + y_mP_m = B$

Suppose that the m-vector $[\xi_1\ \xi_2\ \xi_3\ \ldots\ \xi_m]^1$ is the solution of the above equations. Then the n-vector $[\xi_1\ \xi_2 \ldots \xi_m 0\ 0 \ldots 0]^1$ is a solution of (14) or (7).

This is a basic solution of AX = B. The corresponding linearly independent vectors $P_1, P_2, \ldots, P_m$ are a basis and the variables $x_1, x_2, \ldots, x_m$ are the basic variables.

III. 6. Basic feasible solutions.

Definition 2. A basic solution of (7) satisfying (8) is called a basic feasible solution (b.f.s).

Theorem 2. A basic feasible solution of the LP problem is a vertex of the convex set of feasible solutions. Or, equivalently, if a set of vectors $P_1, P_2, \ldots.$, P_m can be found that are linearly independent such that

$$\xi_1 P_1 + \xi_2 P_2 + \ldots + \xi_m P_m = B, \qquad (15)$$

and $\xi_1 \geq 0$, j=1,2,…,m,

then $X_\xi = |\xi_1 \xi_2 \ldots \xi_m 0\ 0 \ldots 0]'$

which is a b.f.s. is an extreme point (or vertex) of S_F.

Proof: That the point X_ξ belongs to S_F is obvious. Suppose it is not an extreme point. Then two points X_1 and X_2 different from X_ξ exist in S_F such that

$$X_\xi = \lambda X_1 + (1-\lambda)X_2,\ 0 < \lambda < 1,$$

that is, $\xi_1 = \lambda x_{j2} + (1-\lambda)x_{j2},\ j=1,2,\ldots,m,$

and $0=\lambda x_{j1} + (1-\lambda)x_{j2},\ j=m+1,\ldots,n.$

since $X_1, X_2 \in S_F,\ x_{j1}, x_{j2} \geq 0$. Also $0<\lambda<1$.

Hence $x_{j1} = x_{j2}=0,\ j = m + 1,\ldots,n.$

Therefore $X_1 = [x_{11}\ x_{21}\ \ldots\ x_{m1}\ 0\ 0\ \ldots\ 0]^1,$

$$X_2 = [x_{12}\ x_{22}\ \ldots\ x_{m2}\ 0\ 0\ \ldots\ 0]^1.$$

Since X_1, X_2 are solutions of "AX=B,

$$x_{11}P_1 + x_{21}P_2 + \ldots + x_{m1}P_m = B, \tag{16}$$

$$x_{12}P_1 + x_{22}P_2 + \ldots + x_{m2}P_m = B. \tag{17}$$

From (15) and (16),

$$(\xi_1-x_{11})\, P_1 + (\xi_2-x_{21})\, P_2 + \ldots + (\xi_m-x_{m1})P_m=0.$$

But $P_1, P_2,\ldots,P_m$ are, by hypothesis, linearly independent. Therefore

$$\xi_1=x_{11},\ \xi_2 = x_{21},\ldots,\xi_m-x_{m1},$$

or $$X_\xi = X_1,$$

which contradicts the assumption. Hence X_ξ is an extreme point. Proved.

Theorem 3. A vertex of S_F is a basic feasible solution.

(This is the converse of theorem 2.)

Proof: Let $X_\xi=[\xi_1, \xi_2, \ldots,\xi_n]^1$ be a vertex of S_F. Then since $X_\xi \in S_F$, $X_\xi \geq 0$. Let r of the ξ_j' s, j=1,2,…,n, be nonzero, where $r \leq n$. since $m < n$, either $r \leq m$ or $r > m$. If $r \leq m$, X_ξ is obviously a b.f.s. and so the theorem holds.

If $r > m$, then we may put X_ξ as

$$X_\xi = [\xi_1\ \xi_2\ \ldots\ \xi_m,\ 0\ 0\ \ldots 0]^1.$$

where $\xi_j > 0$ for j=1,2,….,r. Since X_ξ is a solution of AX=B, we have

$$\xi_1P_1 + \xi_2P_2 + \ldots + \xi_mP_r = B. \tag{18}$$

As $r > m$, the vectors $P_1, P_2, \ldots, P$, are not linearly independent.

Hence there exist scalars $\propto_1, \propto_2, \ldots, \propto_r$ not all zero such that

$\propto_1 P_1 + \propto_2 P_2 + \ldots + \propto_r P_r = 0$.

Multiplying by $c>0$, we get $c\propto_1 P_1 + c\propto_2 P_2 + \ldots + c\propto_r P_r = 0$. (19)

From (18) and (19),

$(\xi_1 + c\propto_1) P_1 + (\xi_2 + c\propto_2) P_2 + \ldots + (\xi_r + c\propto_r) P_r = B$. (20)

and $(\xi_1 - c\propto_1) P_1 + (\xi_2 - c\propto_2) P_2 + \ldots + (\xi_r - c\propto_r) P_r = B$. (21)

Choose $c > 0$ sufficiently small to make $\xi_j \pm c\propto_j > 0$ for $j=1,2,\ldots,r$.

Then we conclude from (20) and (21) that

$$X_1 = [\xi_1 + c\propto_1 \; \xi_2 + c\propto_2 \ldots \xi_r + c\propto_r \; 0 \; 0 \ldots 0]^1,$$

and
$$X_2 = [\xi_1 - c\propto_1 \; \xi_2 - c\propto_2 \ldots \xi_r - c\propto_r \; 0 \; 0 \ldots 0]^1$$

are feasible solutions. We have now three feasible solutions, X_ξ, X_1 and X_2 which are related through.

$$X_\xi = \frac{1}{2} X_1 + \frac{1}{2} X_2$$

Hence X_ξ is a convex linear combination of X_1 and X_2 which are both different from X_ξ. This means that X_ξ is not a vertex which contradicts our initial assumption.

Hence $r > m$, which means that X_ξ is a basic feasible solution. Proved.

Corollary. Associated with every extreme point of S_F is a set of m linearly independent vectors $P_1, P_2, \ldots, P_m$ of A.

III.7. Optional solutions

Theorem 4. If S_F is nonempty, the objective function f(X) has either an unbounded minimum or it is minimum at a vertex of S_F.

By unbounded minimum we man that there is always an X in S_F such that $f(X) < -N$ where N is as large a positive number as we please. In other words f(X) can be made as small as we please without violating the constraints.

Proof: Two cases arise. Either S_F is bounded or unbounded.

Case (i). S_F is bounded. Then S_F has vertices and every point in S_F is a convex linear combination of its vertices. Let $X_1, X_2, \ldots, X_P$ be the vertices of S_F.

Since S_F is closed and bounded, f(X) is finite for all X in S_F, and so there is a point X_0 in S_F where $f(X_0)$ is minimum. X_0 can be expressed as a convex linear combination of X_r, r=1,2,…,p, and so $X_0 = \sum_{r=1}^{P} \alpha_r X_r,\ \sum_{r=1}^{P} \alpha_r = 1, \alpha_r \geq 0.$

Since f(X) is linear, $f(X_0) = f\left(\sum_{r=1}^{P} \alpha_r X_r\right) = \sum_{r=1}^{P} \alpha_r f(X_r)$

$$\geq \sum_{r=1}^{P} \alpha_r f(X_k) = f(X_k),$$

Where $f(X_k)$ s the least of the values $f(X_r)$ r = 1, 2, … p. But by hypothesis $f(X_0) \leq f(X_k)$.

Therefore $f(X_0) = f(X_k)$

which means that f(X) is minimum at X_k which is a vertex of S_F.

Case (ii), S_F is unbounded. Since it is bounded from below (see theorem 1), S_F has a vertex. Let X_1 be its vertex.

Consider the cone S_c with vertex X_1 and produced by the hyperplanes intersecting at X_1 (see Fig. 2). S_F is a subset of S_c. The edges of S_c are also wholly or partly edges of S_F in the following sense. If Y is a fixed point other than X_1 on an edge of S_c, then any point on the edge is $X=(1-\lambda) X_1+\lambda Y$, $\lambda \geq 0$. If "X is in S_F for all $\lambda \geq 0$, then the edge of S_c is also wholly an edge of S_F (as X_1A in figure). If X is in S_F for $\lambda \leq \lambda_0$ and not in S_F for $\lambda > \lambda_0$ then, the edge of S_c is partly an edge of S_F (as X_1B in figure), and the point $X_2=(1-\lambda_0) x_1+\lambda_0 Y$ is the other extremity of the edge of S_F, the first extremity being the point X_1. X_2 is also a vertex of S_F. Thus moving along any edge of the cone S_c from the vertex X_1, either we shall arrive at another vertex X_2 of S_F or not. In the latter case S_F has an unbounded edge.

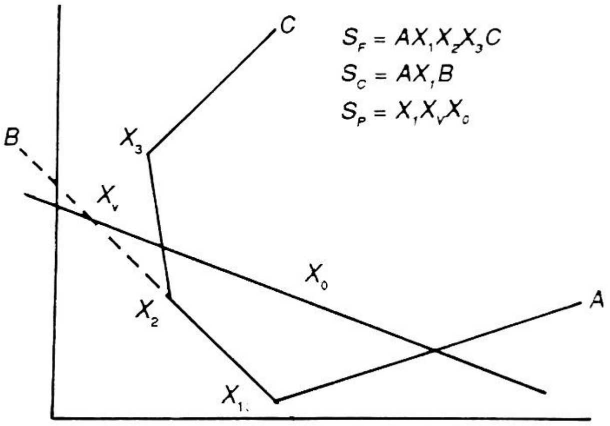


Fig. 2

Now consider $f(X_1)$. One of the two following situations can arise.

(a) For every X on any of the edges of S_c, $f(X_1) \leq f(X)$. In other words f(X) does not decrease as λ increases where $X=(1-\lambda)X_1 + \lambda Y$, $\lambda>0$, and Y is any fixed point on any of the edges.

If possible, let X_0 in S_c be a point different from X_1 such that $f(X_0)$ is minimum for all X in S_c. We can find a hyperplane containing X_0 intersecting all the edges of S_c and thereby producing a bounded polytope $S_P \subseteq S_c$. The vertices of S_P are contained in the edges of S_c. From case (i) f(X) has a minimum in the bounded polytop S_P at one of the vertices of S_p. Let X_v be such a vertex. Then since $X_0 \in S_P \subseteq S_c$, $f(X_0) \leq f(X_v)$. Also $f(X_v) \leq f(X_0)$, for $f(X_v)$ is minimum in S_P. Hence $f(X_0) = f(X_v)$. But $f(X_1) \leq f(X_v)$ because X_v I on an edge of S_c. Also $f(X_v) \leq f(X_1)$ because $f(X_v)$ is minimum in S_P. Hence $f(X_0) = f(X_0)$ which means f(X) is minimum at X_1 in S_c. Since $S_F \subseteq S_c$, in S_F also f(X) is minimum at X_1, a vertex of S_F.

(b) Along some edge of S_c f(X) decreases as λ increases. If this edge is wholly the edge of S_F, then f(X) decreases without limit along this edge, and so f(X) has an unbounded minimum in S_F.

If the edge is partly in S_F, then for $\lambda=\lambda_0$ we arrive at another vertex X_2 of S_F with $f(X_2) < f(X_1)$. We can now apply the same reasoning to X_2 which we have been applying to X_1, namely that either $f(X_2)$ is minimum or $f(X)$ has an unbounded minimum or there is another vertex X_3 of S_F such that $f(X_3) < f(X_2)$. Since the number of vertices of S_F is finite, proceeding like this, if $f(X)$ has not an unbounded minimum, we shall arrive at some vertex of S_F for which $f(X)$ is minimum.

Proved.

Theorem 5. If $f(X)$ is minimum at more than one of the vertices of S_F, then it is minimum at all those points which are the convex linear combinations of these vertices.

Proof: Let $X_1, X_2, \ldots, X_k$ be the vertices of S_F where $f(X)$ is minimum. Then $f(X_1) = f(X_2) = \ldots = f(X_k)$.

Let Y be any convex linear combination of these vertices. Then

$$Y=\sum_{r=1}^{k}\beta_r X_r,\ \sum_{r=1}^{k}\beta_r=1,\ \beta_r\geq 0,$$

and since $f(X)$ is linear $f(Y) = f\left(\sum_{r=1}^{k}\beta_r X_r\right)=\sum_{r=1}^{k}\beta_r f(X_r)$

$$= \sum_{r=1}^{k}\beta_r f(X_1)=f(X_1)$$

which means $f(X)$ is minimum at Y also. Proved.

Definition 3. A solution of (7) and (8) which optimizes the objective function (6) is called an optional solution of the LP problem.

III.8. Summary

If the set S_F of feasible solutions is empty, the problem has no solution. If S_F is nonempty, it is a convex set (polytope) with vertices corresponding to the basic feasible solutions. These are finite in number as they are a subset of basic solutions which are at most ${}^nC_m$ in number.

The convex set S_F may be bounded or unbounded. If bounded, it is a convex polyhedron, and the problem has a solution with f(X) attaining its minimum value at a vertex.

If S_F is unbounded, f(X) may have a finite minimum at a vertex. Or else f(X) may tend to $-\infty$ in which case the solution is unbounded.

III.9. Simplex method

Numerical methods which enable us to compute the solution for numerical values of a_{ij}, b_i and c_j for finite number of variables and constraints have been discovered. The most general and widely used of these methods is called the simplex method.

The simplex method provides an algorithm which consists in moving from one vertex of S_F (one b.f.s) to another in a prescribed manner such that the value of the objective function f(X) at the succeeding vertex is less than at the preceding vertex. The procedure of jumping from vertex to vertex is repeated. If we can reduce f(X) at each jump, then no basis can ever repeat and we can never go back to a vertex already covered. Since the number of vertices is finite, the process must lead to the optimal vertex in a finite number of steps.

III. 10. Canonical form of equations.

Let $x_1, x_2, \ldots, x_m$ be the basic variables corresponding to a certain basis of the equations

$$AX = B \tag{7}$$

These can then be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 - a_{1,m+1}x_{m+1} - \cdots - a_{1n}x_n \\ b_2 - a_{2,m+1}x_{m+1} - \cdots - a_{2n}x_n \\ \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \\ b_m - a_{m,m+1}x_{m+1} - \cdots - a_{mn}x_n \end{bmatrix}$$

The m x m matrix on the left side is nonsingular because the basic vectors which are the columns of this matrix are linearly independent. Premultiplying both sides by its inverse, we get.

$$\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_m \end{bmatrix} = \begin{bmatrix} \bar{b}_1 - \bar{a}_{1,m+1}x_{m+1} - \dots - \bar{a}_{1n}x_n \\ \bar{b}_2 - \bar{a}_{2,m+1}x_{m+1} - \dots - \bar{a}_{2n}x_n \\ \dots \quad \dots \quad \dots \quad \dots \\ \bar{b}_m - \bar{a}_{m,m+1}x_{m+1} - \dots - \bar{a}_{mn}x_n \end{bmatrix}$$

or

$$\begin{aligned} x_1 + \bar{a}_{1,m+1}x_{m+1} + \dots + \bar{a}_{1n}x_n &= \bar{b}_1 \\ x_2 + \bar{a}_{2,m+1}x_{m+1} + \dots + \bar{a}_{2n}x_n &= \bar{b}_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_m + \bar{a}_{m,m+1}x_{m+1} + \dots + \bar{a}_{mn}x_n &= \bar{b}_m \end{aligned}$$

Equations (22) which are equivalent to (7) are called the canonical form of the equations provided $\bar{b}_i \geq 0, i = 1,2,...,m.$ Corresponding to each feasible basis we can get a canonical form, and vice versa. The advantage of putting the equations in a canonical form is that the basis and the corresponding b.f.s. can be immediately known. Since the b.f.s. should have zero values of nonbasic variables, putting $x_{m+1} = x_{m+2} = \ldots = x_n = 0$ in (22), we get the b.f.s. as $(\bar{b}_1 \, \bar{b}_2 \, \bar{b}_3, ... \bar{b}_m \; 0 \; 0 ... 0)$. Thus the right side of (22) gives the values of the basic variables.

Using (22) we can eliminate the basic variables from the objective function (6) and get

$$f(X) = \sum_{i=1}^{m} \bar{b}_i \, c_i + \sum_{j=m+1}^{n} \bar{c}_j x_j \qquad (23)$$

where $$\overline{C_j} = C_j - \sum_{i=1}^{m} c_i \, \bar{a}_{ij}, \; j = m+1, ..., n.$$

It may be noted that the above formula for $\bar{c}_j$, holds even for j=1,2,…,m, it can be seen to give zero values for $\bar{c}$ which is right because x_j, j=1,2,….,m, have been eliminated from (6), and therefore their coefficients in (23) are zero. The advantage of this form is that the value of f(X) for the present b.f.s. is immediately obtained as $\sum_{i=1}^{m} \bar{b}_i c_i$. The coefficients $\bar{c}_j,\ j = m+1, \ldots, n,$ are called the relative cost coefficients.

III. 11. Simplex method (numerical example)

We explain the simplex method through the example of section 2 with the modification that we delete constraint (v).

Introducing slack variables and converting the problem of maximizing f(X) to minimizing –f(X) = Ψ(X), we put the problem in the following standard form.

Minimize $\Psi(X) = -4x_1 - 5x_2$; (24)

subject to

$$\begin{aligned} x_1 - 2x_2 + x_3 \qquad\qquad &= 2, \\ 2x_1 + x_2 \quad + x_4 \qquad &= 6, \\ x_1 + 2x_2 \qquad + x_5 \quad &= 5, \\ -x_1 + x_2 \qquad\qquad + x_6 &= 2; \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0. \end{aligned}$$

Equations (25) are four equations in six variables. Two of the variables, arbitrarily chosen, can be given zero values to obtain a basic solution. To obtain a b.f.s. (or a vertex of the convex set of feasible solutions) zero variables have to be so chosen that the other variables are non-negative.

1 First canonical form. Equations (25) are in canonical form which gives a b.f.s as

$$x_1=0,\ x_2 = 0,\ x_3 = 2,\ x_4 = 6,\ x_5 = 5,\ x_6 = 2;$$

and the corresponding value of Ψ as

$$\Psi = -4x_1 - 5x_2 = 0. \qquad (27)$$

Both x_1 and x_2, the nonbasic variables in terms of which Ψ is expressed, are zero. If either of them is made positive, Ψ will decrease because the coefficients of x_1 and x_2 in (27) are negative. So Ψ at this stage is not minimum. It can be decreased by changing the basis so as to include x_1 or x_2 in place of some other variable which is in the present basis. Let us decide to bring x_2 into the basis. It would be equally reasonable to decide in favour of x_1. But which variable to drop (or be given zero value)? Examining (25) the following are the alternatives (keeping $x_1=0$).

i. Put $x_3 = 0$; then $x_2 = -1$, $x_4=7$, $x_5=7$, $x_6=3$. This is not a feasible solution.

ii. Put $x_4=0$; then $x_2 = 6$, $x_3=14$, $x_5=-7$, $x_6=-4$. This is also not feasible.

iii. Put $x_5=0$; then $x_2 = 5/2$, $x_3=7$, $x_4=7/2$, $x_6=-1/2$. This is also not feasible.

iv. Put $x_6=0$; then $x_2 = 2$, $x_3=6$, $x_4=4$, $x_5=1$. This is a feasible solution.

So, the greatest value that can be given to x_2 without making the solution non-feasible is 2. Putting $x_2 = 2$ would mean putting $x_6=0$ which means x_6 goes out of he basis.

It is easy to discover a simple rule for deciding which variable to drop. Consider the ratios $2/(-2)$, $6/1$, $5/2$, $2/1$ of the right-hand side constant of each of the equations (25) to the coefficient of x_2 in that equation. Of the positive ones the least is $2/1$ corresponding to the last equation which determines the maximum value which can be given to x_2 bringing it into the basis without forcing any other variable to become negative. This also indicates that x_6 should be dropped from the basis. The negative ratio need not be considered because in the corresponding equation x_2 can be made as large as we please without forcing any other variable in that equation to become negative.

II Second canonical form. The new basic variables should therefore be x_3, x_4, x_5, x_2. The canonical form for this basis can be obtained by eliminating x_2 from the first three of the equations (25) with the help of the last, and writing the equations such that the coefficient of each basic variable in its respective equation is 1. The required form is

$$
\begin{aligned}
-x_1 + 2x_6 + x_3 \qquad\qquad &= 6,\\
3x_1 - x_6 \quad + x_4 \qquad &= 4,\\
3x_1 - 2x_6 \qquad + x_5 \quad &= 1,\\
-x_1 + x_6 \qquad\qquad + x_2 &= 2.
\end{aligned}
$$

From this canonical form we get the second b.f.s. as

$x_3 = 6$, $x_4=4$, $x_5 = 1$, $x_2=2$, $x_1 = 0$, $x_6 = 0$.

Also we eliminate x_2 from (24) and express Ψ in terms of the nonbasic variables x_1 and x_6 as

$$\Psi+10=-9x_1+5x_6. \tag{29}$$

This gives the value of Ψ for the present b.f.s. as -10. Notice that since Ψ is expressed in terms of the nonbasic variables which are zero, the constant term occurring in (29) directly gives the value of Ψ. It is an improvement on the first value. Can we reduce it further? Yes, by bringing x1 into the basis because the coefficient of x1 in (29) is negative. But which basic variable to drop? The relevant ratios to be examined in (28) are $6/(-1)$, $4/3$, $1/3$, $2/(-1)$. Keeping the negative ratios out of consideration for reasons already explained, the least ratio is $1/3$ which corresponds to the third of equations (28). So to bring x_1 into the basis x_5 should be dropped.

III Third canonical form. The new basic variables are x_1, x_2, x_3, x_4 and the corresponding canonical form is

$$
\begin{aligned}
\frac{1}{3}x_5 + \frac{4}{3}x_6 + x_3 \qquad\qquad &= \frac{19}{3},\\
-x_5 + x_6 \quad + x_4 \qquad\qquad &= 3,\\
\frac{1}{3}x_5 - \frac{2}{3}x_6 \qquad\qquad + x_1 \quad &= \frac{1}{3},\\
\frac{1}{3}x_5 + \frac{1}{3}x_6 \qquad\qquad\qquad + x_2 &= \frac{7}{3};
\end{aligned}
$$

and Ψ expressed in terms of the nonbasic variables is

$$\Psi+13=3x_2 - x_6. \qquad (31)$$

The b.f.s. is $x_1=1/3$, $x_2=7/3$, $x_3=19/3$, $x_4 = 3$, $x_5=0$, $x_6=0$.

The coefficient of x_6 in (31) is negative and so Ψ can be futher decreased by bringing x_6 back into the basis. The ratios to be observed now are 19/4, 3/1, 7/1; the fourth one being negative is out of consideration. Out of these 3/1 is the least. So x_6 should replace x_4.

IV Fourth canonical form. The new basic variables are x_1, x_2, x_3, x_6 and the corresponding canonical form is

$$\begin{aligned}
-\frac{4}{3}x_4+\frac{5}{3}x_5+x_3 \qquad\qquad &= \frac{7}{3}, \\
x_4 \quad -x_5 \qquad +x_6 \qquad\quad &= 3, \\
\frac{2}{3}x_4-\frac{1}{3}x_5 \qquad\qquad +x_1 \quad &= \frac{7}{3}, \\
-\frac{1}{3}x_4+\frac{2}{3}x_5 \qquad\qquad\qquad +x_2 &= \frac{4}{3};
\end{aligned}$$

and Ψ expressed in terms of the nonbasic variables x_4, x_5 is

$$\Psi+16=x_4 + 2x_5. \qquad (33)$$

The value of Ψ at this stage is -16, and it cannot be further reduced by any change of basis because the coefficients of x_4 and x_5 are positive.

We have come to the end of our search. The minimum value of Ψ is -16 and so the maximum value of f(X) is 16. The optimal solution is

$$x_1 = 7/3,\ x_2 = 4/3,\ x_3 = 7/3,\ x_4=0,\ x_5=0,\ x_6=3.$$

It is instructive to compare results with the graphical solution of the problem. The set of feasible solutions of (2) and (3) [excluding (v)] is the convex polygon OBCDEFO (see Fig. 1). f(X) becomes maximum at the point D. It can be verified that the first, second, third and fourth basic feasible solutions correspond to the verties O, F, E and D respectively.

III. 12. Simplex tableau

The numerical work explained in the last section can be economically organized in a form known as the simplex tableau. The following is the simplex tableau for the proceeding example.

I	Basis	B	P_1	P_2	P_3	P_4	P_5	P_6
	X_3	2	1	-2	1			
	X_4	6	2	1		1		
1	X_5	5	1	2			1	
	X_6	2	-1	1				1
	Ψ	0	-4	-5				
	X_3	6	-1		1			2
	X_4	4	3			1		-1
2	X_5	1	3				1	-2
	X_2	2	-1	1				1
	Ψ	10	-9					5
	X_3	19/3			1		1/3	4/3
	X_4	3				1	-1	1
3	X_1	1/3	1				1/3	-2/3
	X_2	7/3		1			1/3	1/3
	Ψ	13					3	-1
	X_3	7/3			1	-4/3	5/3	
	X_6	3				1	-1	1
4	X_1	7/3	1			2/3	1/3	
	X_2	4/3		1		-1/3	2/3	
	Ψ	16				1	2	

The first column shows the iteration number I. The second and the third columns give the variables in the basis and their values (vector B). the succeeding columns give the vectors P_1, P_2, … of canonical form, which means that row-wise the entries in these columns are the coefficients of x_1, x_2, … in the equations. (For this reason sometimes it is more convenient to write x_1, x_2, … in place of P_1, P_2, … at the top of these columns). For example, the first numerical row in the above table records the first equation in the first canonical form (I=1) which is the equation with 1 as the coefficient of x_1, –2 as the coefficient of x_2, 1 as the coefficient of x_3, and 2 as the right side constant. The equation for the objective function Ψ is also written as a row, putting Ψ in the column for basic variables, its value with sign changed in the next column, and the relative cost coefficients of the nonbasic variables in the respective columns. The equation for Ψ is read, say for I=2, as $\Psi+10 = -9x_1 + 5x_6$, [compare (29)].

The following sequence of steps constitutes one iteration leading from one b.f.s. to another. (I=1 is taken as an example).

i. Examine the relative cost coefficients. If all are non-negative, the current solution is optimal.
ii. If not, pick out the numerically largest negative coefficient (–5). The vector corresponding to it (P_2) is to be brought into the basis. The corresponding basic variable is x_2.
iii. Divide each element of vector B by the corresponding elements of the chosen column vector (P_2). Out of the positive ratios choose the least (2/1). The corresponding basic variable (x_6) has to go out of the basis.
iv. If all the ratios are negative, it means that the value of the incoming variable (whatever it is), can be made as large as we please without violating the feasibility condition. It follows that the problem has an unbounded solution. Iteration stops.

v. Replace x_6 by x_2 in the basic variables column in the table for the next iteration I=2 and rewrite the equation against it so that the coefficient of x_2 is 1. Eliminate x_2 from the rest of the equations in such a way that the coefficients of the basic variables x_3, x_4, x_5 remain 1.
vi. Eliminate x_2 from the equation for Ψ also so that it is expressed in terms of the new nonbasic variables x_1, x_6 only. The entry in the third column of the Ψ equation gives the value of $-\Psi$ at this stage.
vii. Thus the table for I=2 is complete. Go to (i).

III.14. Finding the first b.f.s.; artificial variables.

In the example of section 11 the introduction of slack variables gave a canonical form which immediately led to a b.f.s. providing a starting point for the iterative procedure. This happened because all the constraints were of the type 'less than' and all the constants on the right-had sides of the inequalities were non-negative. But if there is a 'greater than' constraint with non-negative right-hand side or 'less than' constraint with negative right-hand side, then a b.f.s. cannot be obtained right away.

To overcome this difficulty we first put the constraints so that the right-hand side constants are all non-negative. Then we introduce the necessary slack variables. To get a b.f.s. of this system we formulate an auxiliary LP problem whose one b.f.s. can be obtained straightway as in the last example. This auxiliary problem has the property that its optimal solution may immediately give a b.f.s. of the original problem. The auxiliary problem is also solved by the simplex method.

We explain the procedure through the following example.

Minimize $\quad f(X) = 4x_1 + 5x_2;$

subject to $\quad 2x_1 + x_2 \leq 6,$

$\quad x_1 + 2x \leq 5,$

$$x_1+x_2 \geq 1,$$
$$x_1 + 4x_2 \geq 2,$$
$$x_1, x_2 \geq 0.$$

Introducing the slack variables,

$$\begin{aligned}
2x_1 + x_2 + x_3 \qquad\qquad &= 6, \\
x_1 + 2x_2 \quad + x_4 \qquad &= 5, \\
x_1 + x_2 \qquad - x_5 \quad &= 1, \\
x_1 + 4x_2 \qquad\qquad - x_6 &= 2;
\end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

The solution $x_1=0$, $x_2=0$, $x_3=6$, $x_4=5$, $x_5=-1$, $x_6=-2$ is not a basic feasible solution. To get the first b.f.s. to serve as a starting point of iteration for this problem, let us formulate the following auxiliary problem.

Minimize $\qquad g(X) = x_7 + x_8;$

$$\begin{aligned}
2x_1 + x_2 + x_3 \qquad\qquad &= 6, \\
x_1 + 2x_2 \quad + x_4 \qquad &= 5, \\
x_1 + x_2 \qquad - x_5 + x_7 \quad &= 1, \\
x_1 + 4x_2 \qquad\qquad - x_6 + x_8 &= 2;
\end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \geq 0.$$

We have introduced two more variables, x_7 and x_8 called artificial variables, both non-negative and with positive algebraic signs, one in each of the equations which arose from ‘greater than’ constraints. Also the objective function in this new problem is taken as the sum of the artificial variables.

The solution of this problem may be $g(X) = 0$ with $x_7=x_8=0$ and the values of the other variables non-negative with at least two of them zero, because the optimal solution should be a b.f.s. with at most four variables having nonzero values. Then the values of the variables other than the artificial ones should constitute a b.f.s. of the original problem which can become the starting point of iterations for that problem.

Table 3 is the simplex tableau for this example. In phase I we solve the auxiliary problem. Its optimal solution gives the starting b.f.s. for the original problem. At the beginning of phase II we drop the columns for the artificial variables and the row for the function g(X), and carry on the iterations for minimizing f(X). It is convenient to carry the equation for f(X) through phase I also, so that when we start on phase II the expression for f(X) in terms of the nonbasic variables at that stage is readily available.

Phase I starts with the basic variables x_3, x_4, x_7, x_8 and so g(X) should be expressed in terms of the non basic variables as $g(X)–3=–2x_1–5x_2+x_5+x_6$.

Phase	Basis	B	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
	X_3	6	2	1	1		0	0		
	X_4	5	1	2		1	0	0		
	X_7	1	1	1			-1	0	1	
	X_8	2	1	4			0	-1		1
	g	-3	-2	-5			1	1		
	f	0	4	5			0	0		
	X_3	11/2	7/4		1		0	¼		-1/4
	X_4	4	1/2			1	0	½		-1/2
	X_7	1/2	3/4				-1	¼	1	-1/4
	X_2	1/2	¼	1			0	-1/4		1/4
	g	-1/2	-3/4				0	5/4		-5/4
	f	-5/2	11/4				0	5/4		-5/4
	X_3	5	1		1		1		-1	0
	X_4	3	-1			1	2		-2	0
	X_6	2	3				-4	1	4	-1
	X_2	1	1	1			-1		1	0
	g	0	0				0		1	1

f	-5	-1				5		-5	0
X_1	2/3	1				-4/3	1/3		
X_2	1/3		1			1/3	-1/3		
X_3	13/3			1		7/3	-1/3		
X_4	11/3				1	2/3	1/3		
f	-13/3					11/3	1/3		

In general we define the auxiliary (or phase I) problem for the LP problem (6)–(8) as

Minimize $\quad g(X) = \sum_{i=1}^{m} x_{n+i}$

subject to $\quad \sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i$, i=1,2,…,m,

$$X \geq 0,$$

where $X=[x_1 x_2,\ldots x_n x_{m+1}\ldots x_{n+m}]^1$, x_{n+i} being called the artificial variables.

In this problem min g(X) = 0 if and only if $x_{n+i}=0$ for all i. Hence if we solve this problem by the simplex method we get its solution as g(X)=0 only if in its optimal b.f.s. the artificial variables are zero. The optimal values of the rest of the variables, being non-negative, will then satisfy the constraints of the original problem. Moreover, not more than m of these being nonzero, they will constitute a basic feasible solution of the original problem providing a starting point for its solution by the simplex method.

If min g(X) > 0, the conclusion is that there is no feasible solution of auxiliary problem with the values of the artificial variables as zero, and consequently no feasible solution of the original LP problem.

As an alternative to solving the problem in two phases, it is also possible to solve it in one phase after the artificial variables have been introduced. We describe one such method, popularly called the big M method. In this method the original objective function f is replaced by $F=f+M\sum_{i=1}^{m} x_{n+i}$

where x_{n+i} are the artificial variables and M is an arbitrary large number as compared to the coefficients in f. This modified objective function F is minimized subject to the constraints of the original problem. It can be shown that if in the optimal solution of the modified problem all the artificial variables are zero, then that is also the optimal solution of the original problem. If, however, in the optimal solution of the modified problem all the artificial variables are not zero, the conclusion is that the original problem is not feasible. If the modified problem is found to have an unbounded minimum, then the original too, if feasible, is unbounded.

To solve the numerical problem of this section by the big M method, we may take the objective function as $F=4x_1 + 5x_2 + 100\,(x_7+x_8)$.

The iterations would be the same as in table 3 except that the successive rows for F (instead of for f or g) will be as follows:

	B	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
F	-300	-196	-495			100	100		
F	-105/2	-289/4				100	-95/4		495/4
F	-5	-1				5		95	100
F	-13/3					11/3	1/3	289/3	299/3

The starting form of F in terms of the non basic variables is

$$F=300-196x_1-495x_2+100x_5+100x_6.$$

The complete simplex tableau can be written out by inserting the rows for F above in place of the rows for g and f in table 3.

III. 14. Degeneracy

The least of a set of non-negative ratios decides which variable is to be dropped from the basis at a particular stage. It may happen that two or more ratios are equal and the least. In that case a tie occurs as to which variable to drop. One can arbitrarily decide in favour of one, but then it turns out that the variables which tied with it and continue to remain in the basis also become zero. In other words, one or more of the basic variables too have zero value. Such a case is called degenerate.

The difficulty appears in the next iteration when we find that the variable to be brought into the basis and the variable to be dropped both are already zero. The basis, theoretically, is changed, but the value of the objective function remains the same. Geometrically it may be interpreted as the case of two coincident vertices. We change from one to the other but substantially remain where we were. In most cases we go ahead with our iterations and find that following the procedure we eventually change to a substantially different basis which gives an improved value of the objective function, and we finally get the optimal solution.

It may, however, happen that the successive iterations only make us go through a number of (one or more) degenerate bases to arrive back at the degenerate basis from which we started.

We get into a cycle with no apparent way to get out of it. This situation presents some difficulty, but procedures have been discovered to overcome it. Such a situation is very rare. It is claimed that in thousands of linear programming models solved by the simplex method, some of them very large, there is not a case when degeneracy has proved a hurdle. The procedure recommended to deal with such a hurdle, if it occurs, is therefore of theoretical interest, and so we shall omit its discussion.

III. 15. Simplex multipliers

In the simplex method it is necessary at every iteration to express the objective function of f(X) in terms of the non basic variables as given in a general form by (23). We get this equation for each successive canonical form from the preceding canonical form. It is possible to get it for any basis directly from the original equations (6) and (7). This we proceed to explain.

Suppose $(x_1, x_2, \ldots x_m, 0, 0 \ldots 0)$ is a b.f.s. To express f(X) in terms of the non basic variables $x_{m+1}, x_{m+2}, \ldots, x_n$, we may eliminate the basic variables $x_1, x_2, \ldots x_m$ from (6) with the help of (7). With this object in view let us multiply each of the equations (7) by constants $\pi_1, \pi_2, \ldots \pi_m$ respectively and add them to (6). We get $\left(C_1+\sum_{i=1}^{m} a_{i1}\pi_i\right)x_1+\left(c_2+\sum_{i=1}^{m} a_{i2}\pi_i\right)x_2+\ldots+\left(c_n+\sum_{i=1}^{m} a_{in}\pi_i\right)x_n = f+\sum_{i=1}^{m} b_i\pi_i$. (34)

Choose $\pi_1, \pi_2, \ldots \pi_m$ such that the coefficients of $x_1, x_2, \ldots x_m$ vanish, that is, let

$$\sum_{i=1}^{m} a_{ij}\pi_i = -c_j,\ j=1, 2, \ldots, m. \tag{35}$$

Then (34) reduces to

$$F = \sum_{j=m+1}^{n} \bar{c}_j x_j - \sum_{i=1}^{m} b_i \pi_i \tag{36}$$

Where $\bar{c}_j = c_j + \sum_{i=1}^{m} a_{ij}\pi_i,\ j=m+1,\ldots,n.$ (37)

(35) are *m* equations in m unknowns π_i, and on solution give the required values of π_i. In matrix notation we may put (35) as

$$A^1{}_0\Pi = -C'_0$$

where

$$A_0 = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mm} \end{bmatrix},\ \Pi = [\pi_1, \pi_2 \ldots \pi_m]^1,\ C_0 = [c_1\ c_2 \ldots c_m].$$

Hence

$\Pi = -[A^1{}_0]^{-1} C^1{}_0 = -[A_0^{-1}]^1 C^1{}_0$,

or $\Pi^1 = -C_0 A_0^{-1}$. (38)

Here we have defined Π as a column vector and C_0 as a row vector. Π^1, C_0^1, A_0^1 are the transpose of the respective vectors and matrix. The vector Π is called the multiplier vector and its components the simplex multipliers. To calculate Π at any stage the inverse of A_0, the matrix of basic vectors at that stage, is needed. Having calculated π_i for any basis, the value of the objective function for that basis is given by

$$f = -\sum_{i=1}^{m} b_i \, \pi_i \, , \qquad (39)$$

because the terms in (36) involving the nonbasic variables x_{m+1}, x_{m+2}, …, x_n are zero for the simple reason that the variables themselves are zero.

The above discussion is of theoretical interest in a general case when A_0 is any m x m matrix. For, to get A_0^{-1} or to solve equations (35) for π_i, i=1,2,…m, means essentially the same thing, and may not be easy. However, where A_0 arises from constraints converted into equations through slack variables, it becomes easy to read A_0^{-1} and Π from the tables of the original equations and equations in the canonical form with respect to the basis under consideration.

As an example, consider the problem of section 13 and its solution obtained in table 3. The table below shows the initial form of the problem after introducing slack variables, and the final canonical form which gives the optimal solution. We shall show how A_0^{-1} and the simplex multipliers for the optimal solution can be read off from the table 4.

Basis	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_3	6	2	1	1			
x_4	5	1	2		1		
x_5	1	1	1			-1	
x_6	2	1	4				-1
f	0	4	5				
x_1	2/3	1				-4/3	1/3
x_2	1/3		1			1/3	-1/3
x_3	13/3			1		7/3	-1/3
x_4	11/3				1	2/3	1/3
f	-13/3					11/3	1/3

x_1, x_2, x_3, x_4 being the basic variables in the optimal solution, the problem is to find A_0^{-1} where A_0 is the matrix of the coefficient of these variables in the initial form, namely.

$$A_0 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \end{bmatrix}$$

A_0^{-1} operating on the initial matrix of coefficients produces the final matrix of coefficients, that is

$$A_0^{-1} \begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 4 & 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -4/3 & 1/3 \\ 0 & 1 & 0 & 0 & 1/3 & -1/3 \\ 0 & 0 & 1 & 0 & 7/3 & -1/3 \\ 0 & 0 & 0 & 1 & 2/3 & 1/3 \end{bmatrix}$$

or, taking only the submatrix of the last four columns on either side,

$$A_0^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -4/3 & 1/3 \\ 0 & 0 & 1/3 & -1/3 \\ 1 & 0 & 7/3 & -1/3 \\ 0 & 1 & 2/3 & 1/3 \end{bmatrix}$$

Since the inverse of a diagonal matrix with diagonal entries 1 or −1 is the matrix itself, we get

$$A_0^{-1} = \begin{bmatrix} 0 & 0 & -4/3 & 1/3 \\ 0 & 0 & 1/3 & -1/3 \\ 1 & 0 & 7/3 & -1/3 \\ 0 & 1 & 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1/3 \end{bmatrix}$$

or
$$A_0^{-1} = \begin{bmatrix} 0 & 0 & 4/3 & -1/3 \\ 0 & 0 & -1/3 & 1/3 \\ 1 & 0 & -7/3 & 1/3 \\ 0 & 1 & -2/3 & -1/3 \end{bmatrix}$$

The rule for determining A_0^{-1} boils down to this. Observe the columns in the original form which constitute a diagonal matrix with 1 or −1 on the diagonal.

The matrix of columns corresponding to them in the final form multiplied by this diagonal matrix gives the require inverse.

To get the simplex multipliers, by (38)

$$[\pi_1\ \pi_2\ \pi_3\ \pi_4] = -\,[4\ 5\ 0\ 0]\begin{bmatrix} 0 & 0 & 4/3 & -1/3 \\ 0 & 0 & -1/3 & 1/3 \\ 1 & 0 & -7/3 & 1/3 \\ 0 & 1 & -2/3 & -1/3 \end{bmatrix} = [0\ 0 - 11/3 - 1/3]$$

Even otherwise, we can directly observe that the row for f in the final form can only be obtained from the initial form by multiplying the initial rows of the coefficients by 0, 0, –11/3, –1/3 respectively and adding to the initial row for f.

III. 16. Duality in LP Problems

To every LP problem there corresponds another LP problem called its dual. The original problem is called the primal. There exists an important theoretical relationship between the primal and its dual which is of practical use also. Before defining the dual, we shall restate the LP problem in a standard form different from the form defined in equations (6), (7) and (8). This alternative statement is also quite general and is better suited to proving the duality theorems.

We state the general LP problem as

Minimize $\qquad f(X) = CX, \qquad (43)$

Subject to $\qquad AX \geq B, \qquad (44)$

$\qquad X \geq 0, \qquad (45)$

where A is an *m* x *n* matrix, X is a column and C a row *n*-vector and B is a column *m*-vector. The above problem may also be written as

Minimize $f(X) = \sum_{j=1}^{n} c_j x_j$, $\qquad (43)$

subject to $\sum_{j=1}^{n} a_{ij} x_j \geq b_i$, i=1,2,….,m, $\qquad (44)$

$x_j \geq 0$, j=1,2,….,n. $\qquad (45)$

that the above form is quite general follows from the fact that constraints in other form can always be put in form (44). For, an inequality of the type $\sum_j a_{ij} x_j \le b_i$ can be put as $\sum_j \left(-a_{ij} x_j\right) \ge b_i$, and an equation of the type $\sum_j a_{ij} x_j = b_i$ can be put as two inequalities $\sum_j a_{ij} x_j \ge -b_i$, and $-\sum_j a_{ij} x_j \ge -b_i$.

For example, the example of section 3 can be written in the present standard form as

$$
\begin{aligned}
f &= -2x_1 - x_{21} + x_{22} + x_3, \\
-2x_1 + 5x_{21} - 5x_{22} - 3x_3 &\ge -4, \\
3x_1 + 6x_{21} - 6x_{22} - x_3 &\ge 2, \\
x_1 + x_{21} - x_{22} + x_3 &\ge 4, \\
-x_1 - x_{21} + x_{22} - x_3 &\ge -4, \\
x_1, x_{21}, x_{22}, x_3 &\ge 0.
\end{aligned}
$$

Definition 4. If the primal LP problem is in the form (43), (44), (45), then its dual is defined as the following LP problem.

Maximize	$\phi(Y) = B'Y$	(46)
subject to	$A'Y \le C'$,	(47)
	$Y \ge 0.$	(48)

where Y is a column *m*-vector.

It may also be written as:

Maximize	$\phi(Y) = \sum_{i=1}^{m} b_1 y_i$,	(46)
subject to	$\sum_{i=1}^{m} a_{ij} y_i \le c_j,\ j = 1,2,\ldots,n,$	(47)
	$y_i \ge 0$, i=1, 2, …., m.	(48)

The primal dual pair of problems can be defined in other forms also (for example, see problem 28). The equivalence of various definitions can be easily established.

The above definition implies the following correspondence between the primal in the standard form (43), (44), (45), and its dual.

	Primal	Dual
n	n variables	n constraints
m	m constraints	m variables
c_j, j=1,2,…n	cost coefficients	constraint constants
b_i, i=1,2,…m	constraint constants	cost coefficients
variables	$x_j \geq 0$, j==1,2,…n	$y_i \geq 0$, i=1,2,…m
constraints	$\sum_{j=1}^{n} a_{ij} x_j \geq b_i$	$\sum_{j=1}^{m} a_{ij} y_j \leq c_j$
objective function	minimize $\sum_{j=1}^{n} c_j x_j$	maximize $\sum_{j=1}^{n} b_i y_i$

As an example, to write the dual of the example of section 3 we first put it in the standard form (43), (44), (45), as has been done above. We then write the dual as

Maximise $\phi = -4y_1 + 2y_2 + 4y_3 - 4y_4,$

subject to

$$-2y_1 + 3y_2 + y_3 - y_4 \leq -2,$$

$$5y_1 + 6y_2 + y_3 - y_4 \leq -1,$$

$$-5y_1 - 6y_2 - y_3 + y_4 \leq 1,$$

$$-3y_1 - y_2 + y_3 - y_4 \leq 1,$$

$$y_1, y_2, y_3, y_4 \geq 0.$$

We can simplify the above statement of the problem by noticing that y_3, and y_4 occur throughout as $y_3 - y_4$, and so $y_3 - y_4$ can be regarded as a single variable. Let it be denoted by a single symbol, say y_3, (It does not mean that we are making the statement $y_3 - y_4 = y_3$. It is convenient to call the new variable y_3 because the other two variables are y_1 and y_2. We could use any other symbol).

Since the original y_3 and y_4 are both non-negative, $y_3–y_4$, is unrestricted, and so the new variable y_3 is unrestricted. The problem can now be written as

Maximise $\qquad \phi = –4y_1 + 2y_2 + 4y_3,$

subject to

$$–2y_1 + 3y_2 + y_3 \leq – 2,$$
$$5y_1 + 6y_2 + y_3 \leq – 1,$$
$$–5y_1 – 6y_2 – y_3 \leq 1,$$
$$–3y_1 – y_2 + y_3 \leq 1,$$
$$y_1, y_2 \geq 0, y_3 \text{ unrestricted.}$$

We further notice that the second and third constraints are equivalent to one equation. Therefore we may write the above problem as

Maximise $\qquad \phi = –4y_1 + 2y_2 + 4y_3,$

subject to

$$–2y_1 + 3y_2 + y_3 \leq – 2,$$
$$5y_1 + 6y_2 + y_3 \leq = – 1,$$
$$–3y_1 – y_2 + y_3 \leq 1,$$
$$y_1, y_2 \geq 0, y_3 \text{ unrestricted.}$$

This is the dual of the primal problem of section 3 which, for better comparison, we write below.

Maximise $\qquad f = –2x_1 – x_2 + x_3,$

subject to

$$–2x_1 + 5x_2 – 3x_3 \geq – 4,$$
$$3x_1 + 6x_2 – x_3 \geq 2,$$
$$x_1 + x_2 + x_3 = 4,$$
$$x_1, x_3 \geq 0, x_2 \text{ unrestricted.}$$

The interesting point to note in the above primal and dual problems is that the third constraint in the primal is an equation while the third variable in the dual is unrestricted, and the second variable in the primal is unrestricted while the second constraint in the dual is an equation. (for general statement of this property see problems 27, 28).

If we generalize the statement of the standard LP problem to admit that some of the constraints (44) may be equations and the rest inequalities of $\geq$ type, then we get the following rule regarding constraints and variables in the primal and the dual.

Primal	Dual
ith constraint $\geq$ type	$y_i \geq 0$
ith constraint = type	y_i unrestricted
$x_j \geq 0$	jth constraint $\leq$ type
x_j unrestricted	jth constraint = type

MODULE II

DUALITY (Continued)

III. 18 Duality Theorems.

Theorem 6. The dual of the dual is the primal.

Proof. The dual (46), (47), (48) written in the primal form is:

Minimize $\quad -\phi(Y) = -B^1Y,$

subject to $\quad -A^1Y \geq -C^1,\ Y \geq 0.$

Its dual, according to definition, is:

Maximize $\quad \Psi(X) = -CX,$

subject to $\quad -AX \leq -B,\ X \geq 0;$

which may also be written as:

Minimize $\quad f(X) = -\Psi(X) = CX,$

subject to $\quad AX \geq B,\ X \geq 0.$

This is the primal (43), (44), (45). Proved.

Theorem 7. The value of the objective function f(X) for any feasible solution of the primal is not less than the value of the objective function $\phi(Y)$ for any feasible solution of the dual.

Proof: Let us introduce the necessary slack variables in the primal (43), (44), (45) and the dual (46), (47), (48). We obtain

Primal: Minimize $$f(\mathbf{X}) = c_1x_1 + c_2x_2 + \dots + c_nx_n,$$

subject to

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 \dots + a_{1n}x_n - x_{n+1} &= b_1, \\
a_{21}x_1 + a_{22}x_2 \dots + a_{2n}x_n \quad - x_{n+2} &= b_2, \\
\dots\dots\dots\dots\dots\dots & \\
a_{m1}x_1 + a_{m2}x_2 \dots + a_{mn}x_n \quad - x_{n+m} &= b_m, \\
x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_{n+m} &\geq 0.
\end{aligned}$$

Dual: Maximize $$\phi(\mathbf{Y}) = b_1y_1 + b_2y_2 + \dots + b_my_m,$$

subject to

$$\begin{aligned}
a_{11}y_1 + a_{21}y_2 + \dots + a_{m1}y_m + y_{m+1} &= c_1, \\
a_{12}y_1 + a_{22}y_2 + \dots + a_{m2}y_m \quad + y_{m+2} &= c_2, \\
\dots\dots\dots\dots\dots\dots & \\
a_{1n}y_1 + a_{2n}y_2 + \dots + a_{mn}y_m \quad + y_{m+n} &= c_n, \\
y_1, y_2, \dots, y_m, y_{m+1} \dots, y_{m+n} &\geq 0.
\end{aligned}$$

Let $x_1, x_2 \dots x_{n+m}$, and $y_1, y_2, \dots, y_{m+n}$ be any feasible solutions of the primal and the dual respectively. Multiply the primal constraints by $y_1, y_2, \dots y_m$ respectively and add, also multiply the dual constraints by $x_1, x_2, .., x_n$ respectively and add. Thus we obtain two equations. Subtracting one from the other we get.

$f-\phi = x_1y_{m+1} + x_2y_{m+2} + \dots + x_ny_{m+n} + y_1x_{n+1} + y_2x_{n+2} + \dots + y_mx_{n+m}$.

Since all the variables on the right-hand side are non-negative (they are components of feasible solutions),

$$f-\phi \geq 0. \qquad \text{Proved.}$$

Corollary. It immediately follows from above that

$$\min f(X) \geq \max \phi(Y).$$

Theorem 8. The optimum value of f(X) of the primal, if it exists, is equal to the optimum value of $\phi(Y)$ of the dual.

Proof. After introducing slack variables in (44) we get.

$$\sum_{j=1}^{n} a_{ij}\, x_j - x_{n+i} = b_i, i = 1, 2, \dots, m.$$

Let the primal have the optimal solution $(x_1, x_2, \ldots. x_n, x_{n+1}, \ldots., x_{n+m})$. Since it has to be a basic feasible solution, at least *n* of these numbers are zero. Let π_1, $\pi_2, \ldots., \pi_m$ be the simplex multipliers for this solution. Then, as in (34), f(X) for this b.f.s. is given by

$$f(X) + \sum_{i=1}^{m} \pi_i b_i = \sum_{j=1}^{n} \left(c_j + \sum_{i=1}^{m} a_{ij} \pi_i \right) x_j - \sum_{i=1}^{m} \pi_i x_{n+i}.$$

Since f(X) is optimum, from (39),

min f(X) = $-\sum_{i=1}^{m} b_i \pi_i$, and all the relative cost coefficients are non-negative, that is,

$$c_j + \sum_{i=1}^{m} a_{ij} \pi_i \geq 0, j=1, 2, \ldots, n; -\pi_i \geq 0, i=1,2,\ldots,m;$$

$$\text{or } -\sum_{i=1}^{m} a_{ij} \pi_i \leq c_j, -\pi_i \geq 0.$$

The last two inequalities mean that $(-\pi_1, -\pi_2, \ldots., -\pi_m)$ is a solution of (47), (48), that is, a feasible solution of the dual. Corresponding to this solution, from (46).

$$\phi(Y) = -\sum_{i=1}^{m} b_i \pi_i = \min f(X).$$

Thus we have found a feasible solution of the dual such that min f(X) = ϕ(Y), which, by corollary of theorem 7, is possible only when min f(x) = max ϕ(y).
Hence this solution of the dual is optimal. Proved.

Theorem 9. The negative of the simplex multipliers for the optimal solution of the primal are the values of the variables for the optional solution of the dual; and the simplex multipliers for the optimal solution of the dual are the values of the variables for the optimal solution of the primal.

The proof of the first part is implied in the proof of theorem 8, and the second part can be proved likewise.

Theorem 10. If the primal problem is feasible, then it has an unbounded optimum if and only if the dual has no feasible solution, and vice versa.

Proof: Let the primal have an unbounded optimum. It means f(X) has no lower bound, or in other words, there is no number which is less than all possible values of f(X).

If possible, let the dual have a feasible solution. Then ϕ is a definite number corresponding to that solution, and by theorem 7 $\phi \leq f(X)$. This contradicts the conclusion in the last paragraph. So the dual has no feasible solution.

Conversely, let the primal be feasible and the dual infeasible. Let f(X) have a minimum (not unbounded) for feasible X. By theorem 8, min f(X) = max ϕ (Y) over feasible values of Y. Thus a feasible Y exists which contradicts the assumption that the dual is infeasible. Therefore f(X) has an unbounded minimum.

Since the dual of the dual is the primal, the theorem is true if the words dual and primal are interchanged in its enunciation. Proved.

Theorem 11. If, in the optimal solutions of the primal and the dual, (i) a primal variable x_j is positive, then the corresponding dual slack variable y_{m+j} is zero; and (ii) if a primal slack variable x_{n+i} is positive, then the corresponding dual variable y_i is zero; and vice versa.

Proof: It follows from theorem 7 and 8 that for the optimal solutions x_j, j =1, 2, …., n, n+1, …. n+m, of the primal, and y_i, i=1,2,… m, m+1, … m+n, of the dual.

$$x_1y_{m+1} + x_2y_{m+2} + \dots + x_ny_{m+n} + y_1x_{n+1} + y_2x_{n+2} + \dots + y_mx_{n+m} = 0$$

Since an optima solution is feasible, all $x_j \geq 0$, all $y_i \geq 0$. Hence all the terms in the expression on the left side above are non-negative, and since their sum is zero, each term separately should be zero. It follows that in a term like $x_j\, y_{m+j}$, if $x_j>0$ then $y_{m+j}=0$, and if $y_{m+j}>0$ then $x_i=0$. Also in a term like y_ix_{n+i}, if $x_{n+i}>0$ then $y_i=0$, and if $y_i>0$ then $x_{n+i}=0$. Proved.

These conditions are called the complementary slackness conditions. In words they can be stated as follows.

In the optimal solutions of the primal and the dual,

(i) If the jth primal variable $x_j>0$, then the corresponding dual constraint is satisfied as an equation, or, in other words, the constraint is 'tight' (since its slack variable y_{m+j} is zero), and vice versa; and

(ii) If the ith primal constraint is satisfied as a strict inequality, or, in other words, the constraint is 'slack' (since its slack variable x_{n+i} is positive), then the corresponding dual variable y_i is zero, and vice versa.

This theorem is sometimes helpful in determining the optimal solution of the primal from the optimal solution of the dual, or vice versa.

As an example, consider the problem

Maximize	$f=3x_1 + 2x_2 + x_3 + 4x_4,$
Subject to	$2x_1 + 2x_2 + x_3 + 3x_4 \leq 20,$
	$3x_1 + x_2 + 2x_3 + 2x_4 \leq 20,$
	$x_1, x_2, x_3, x_4 \geq 0.$

Its dual is

Minimize	$\phi = 20y_1 + 20y_2,$
Subject to	$2y_1 + 3y_2 \geq 3,$
	$2y_1 + y_2 \geq 2,$
	$y_1 + 2y_2 \geq 1,$
	$3y_1 + 2y_2 \geq 4,$
	$y_1, y_2 \geq 0.$

This is a two-variable problem whose solution can be obtained geometrically as

$$y_1 = 1.2,\ y_2 = 0.2,\ \phi=28.$$

After introducing the slack variables, the primal and dual constraints are

$$2x_1 + 2x_2 + x_3 + 3x_4 + x_5 = 20,$$

$$3x_1 + x_2 + 2x_3\ 2x_4 + x_6 = 20,$$

$$2x_1 + 2x_2 + x_3 + 3x_4 + x_5 = 20,$$

$$3x_1 + x_2 + 2x_3 + 2x_4 + x_6 = 20,$$

$$\begin{aligned}
2y_1 + 3y_2 - y_3 \qquad\qquad &= 3,\\
2y_1 + y_2 \quad - y_4 \qquad &= 2,\\
y_1 + 2y_2 \qquad - y_5 \quad &= 1,\\
3y_1 + 2y_2 \qquad\quad - y_6 &= 4,
\end{aligned}$$

$$x_1, x_2, \ldots x_6, y_1, y_2, \ldots y_6 \geq 0.$$

Substituting the optimal values of y_1 (=1.2) and y_2 (=0.2) in the dual constraints, it follows that the slack variables

$$y_3 = y_6 = 0,\ y_4 > 0,\ y_5 > 0.$$

Thus the second and the third constraints are satisfied as strict inequalities, and so the corresponding primal variables should be zero, that is, $x_2 = 0$, $x_3=3$. Also since the dual variables $y_1>0$, $y_2>0$, it follows that the corresponding primal constraints should be tight, that is, x5 = x6 = 0. The primal constraints thus reduce to

$$2x_1 + 3x_4 = 20,$$

$$3x_1 + 2x_4 = 20,$$

which give $x_1 = 4$, $x_4 = 4$. The optimal solution of the primal is therefore
$x_1=x_4=4$, $x_2 = x_3 = 0$, f=28.

III. 19. Applications of duality

It follows from the duality theorems that, given an LP problem, one may obtain its solution either by solving it or solving its dual. Sometimes the solution of the dual may involve less work. Usually in an LP problem numerical work increases more with the number of constraints than with the number of variables. Since the two get interchanged in the dual problem, if the constraints in the primal far outnumber the variables, then it is generally economical to solve the dual.

It is also possible under certain conditions to avoid the introduction of artiial variables to obtain an initial b.f.s. and thus avoid Phase I part in the simplex procedure. If the introduction of slack variables in the primal leads to a non-feasible basic solution of the primal, but the introduction of slack variables in the dual provides a basic feasible solution of the dual, then also it may be economical to solve the dual. What is more interesting is that in such a case it is also possible to start on the simplex tableau of the primal with a nonfeasible basic solution, and proceed with the iterations with a modified algorithm which finally leads to the optimal solution, provided the cost coefficients satisfy a certain condition. The procedure which we explain in the next section is particularly useful when additional constraints are introduced in a problem after the optimal solution has been obtained under the original set of constraints and the objective is to find the optimal solution to the modified problem without starting work from the very beginning.

III. 20. Dual Simplex Method

Consider the primal and the dual problems in the forms (43), (44), (45) and (46), (47), (48) respectively.

Suppose all $c_j \geq 0$ and all $b_i \leq 0$. Then the basis consisting of the basic variables $x_{n+1}, x_{n+2} \ldots, x_{n+m}$ (which are the slack variables) is feasible and also optimal. Similarly the corresponding basis of the dual is feasible and also optimal. If, however, some of all $b_i > 0$ and all $c_j \geq 0$, then the basis $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ is not feasible for the primal, but the basis $y_{m+1}, y_{m+2}, .., y_{m+n}$ is feasible for the dual. We call this situation primal infeasible and dual feasible. Suppose we start with the simplex algorithm on the dual. We shall be moving through a succession of basic feasible solutions of the dual (which means all $\bar{c}_j \geq 0$) till the final relative cost coefficients $\bar{b}_i$ of the dual are all non-positive. We would have then arrived at the optimal solution. To get the optima basis of the primal from the optimal basis of the dual we shall have to use theorem 9.

It is possible to abridge this procedure by applying a slightly modified algorithm to the primal tableau wherein we start with a non-feasible basic solution but with non-negative cost coefficients. This modified procedure is called the dual simplex method.

Let us write the simplex tableau (table 5) for the primal problem (43)–(45) with a basis consisting of slack variables. We assume that some $b_i>0$ (that is, the values of some basic variables are negative) and all $c_j \geq 0$. The dual simplex method consists in changing a negative basic variable in such a way that the value of the new basic variable in its place would be positive, and the relative cost coefficients for the changed basis till remain non-negative.

Basis	Value	x_1	x_2	..	x_p	.	x_n	x_{n+1}	..	x_{n+m}
x_{n+1}	$-b_1$	$-a_{11}$	$-a_{12}$	..	$-a_{1p}$	..	$-a_{1n}$	1	..	0
x_{n+2}	$-b_2$	$-a_{21}$	$-a_{22}$	..	$-a_{2p}$	..	$-a_{2n}$	0	..	0
.	.	.	.		.		.	.		.
x_{n+r}	$-b_r$	$-a_{r1}$	$-a_{r2}$	..	$-a_{rp}$	..	$-a_{rn}$	0	..	0
.	.	.	.		.		.	.		.
x_{n+m}	$-b_m$	$-a_{m1}$	$-a_{m2}$	..	$-a_{mp}$	..	$-a_{mn}$	0	..	1
f	0	c_1	c_2	..	c_p	..	c_n	0	..	0

For example, let $b_r>0$ so that the corresponding basic variable x_{n+r} is negative. Also let some coefficients $-a_{rj}$ be negative. Let, in particular, $-a_{rp} < 0$. We may replace x_{n+r} by x_p in the basis by dividing the rth equation by $-a_{rp}$ and eliminating x_p from all other equations and also from the last row giving the expression for *f* in terms of the nonbasic variables and relative cost coefficients. This change should be such that no relative cost coefficient becomes negative. This will be so when

$$c_j - \frac{a_{rj}}{a_{rp}} c_p \geq 0,\ j=1,2,\ldots,n+m,$$

or $$\frac{c_r}{a_{rj}} \geq \frac{c_r}{a_{rp}}$$ over all those j for which $-a_{rj} < 0$.

or $$\min_j \frac{c_j}{a_{rj}} = \frac{c_p}{a_{rp}}, \ -a_{rj}<0.$$

This leads to the determination of p. The value of the new basic variable x_p would be $(-b_r)/(-a_{rp})$ which is positive. If for $-b_r < 0$, there is no $-a_{rj}<0$, the problem is infeasible.

We may change the basis in this way step by step, one basic variable in each iteration, till all the basic variables come to have non-negative values. Thus we shall arrive at a basic feasible solution which is optimal.

Notice that in this method we move through a set of points which are not primal feasible taking care all the time that the relative cost coefficients remain non-negative so that the moment we arrive at a feasible basis, we find ourselves at the optimal.

Example : Minimise $f=3x_1 + 5x_2 + 2x_3$

Subject to

$$-x_1 + 2x_2 + 2x_3 \geq 3,$$
$$x_1 + 2x_2 + x_3 \geq 2,$$
$$-2x_1 - x_2 + 2x_3 \geq -4$$
$$x_1, x_2, x_3 \geq 0.$$

Table 7 gives the iterations by the dual simplex algorithm leading to the optimal solution $x_1 = x_2 = 0$, $x_3 = 2$ and the optimum value of f=4.

Basis	Value	x_1	x_2	..	x_p	.	x_n	x_{n+1}	..	x_{n+m}
x_{n+1}	$-b_1$	$-a_{11}$	$-a_{12}$	..	$-a_{1p}$	..	$-a_{1n}$	1	..	0
x_{n+2}	$-b_2$	$-a_{21}$	$-a_{22}$	..	$-a_{2p}$	..	$-a_{2n}$	0	..	0
.	.	.	.		.		.	.		.
x_{n+r}	$-b_r$	$-a_{r1}$	$-a_{r2}$	..	$-a_{rp}$	..	$-a_{rn}$	0	..	0
.	.	.	.		.		.	.		.
x_{n+m}	$-b_m$	$-a_{m1}$	$-a_{m2}$	..	$-a_{mp}$	..	$-a_{mn}$	0	..	1
f	0	c_1	c_2	..	c_p	..	c_n	0	..	0

III. 21. Applications of LP

Example 1: A small manufacturing company produces one-band pocket and two-band table radios. Each two-band model requires twice as much time as one

one-band model. If the company were to produce only two-band models, it could manufacture 150 units per week. The company is licensed to produce in all not more than 250 units per week. The market survey has shown that no more than 100 pieces of two-band model per week could be sold. The company is also committed to supply at least 50 pieces of one-band model per week. If the net profit on the sale of one-band model is Rs. 10 per piece, and on the two-band model Rs. 15 per piece, how should the company plan its production to maximize profit?

The problem is to determine the number of one-band and two-band model radios which the company should produce per week to eam maximum profit. Let these be x_1, x_2 respectively.

Since the profit from the sale per piece of one-band model is Rs. 10, and of two-band is Rs. 15, the total profit per week on x_1 and x_2 pieces of the two models respectively will be $10x_1 + 15x_2$, which it is sought to maximize. This therefore is the objective function to be maximized.

As for the constraints:

i) The production capacity of the company is such that if only two-band radios were to be produced, it will be able to produce only 150 units per week. It takes twice as much time to produce a two-band model as to produce a one-band model.

   Therefore x_1 pieces of one-band model use the same manufacturing capacity as $\frac{1}{2}x_1$ pieces of two-band model. The total capacity used is therefore $\frac{1}{2}x_1 + x_2$ which cannot exceed 150. Hence

$$\frac{1}{2}x_1+x_2\leq 150$$

ii) Since the company is licensed to produce in all not more than 250 pieces per week.

$$x_1+x_2\leq 250$$

iii) Since the demand per week of two-band radios is not more than 100, the company should not produce more than 100 of this type. Hence

$$x_2 \leq 100$$

iv) Also since the manufacturer has a commitment to supply at least 50 one-band models per week.

$$x_2 \leq 50$$

v) Finally since a solution in which either x_1 or x_2 has negative values has no practical significance (making negative number of articles is senseless), x_1, x_2 should be non-negative.

$$x_1 \geq 0, \; x_2 \geq 0$$

Summing up, the mathematical model of the problem which should be solved to provide answer to the company's problem is

Maximize $\quad f = 10x_1 + 15x_2$

Subject to $\quad \frac{1}{2}x_1 + x_2 \leq 150$

$$x_1 + x_2 \leq 250$$

$$x_2 \leq 100$$

$$x_1 \geq 50$$

$$x_1 \geq 0, \; x_2 \geq 0$$

This is a two-variable LP problem easily solvable graphically or by the simplex method. Its solution can be found to be $x_1 = 200$, $x_2 = 50$ with the maximum value of f as 2750. Interpreting in the words of the original problem, the company should manufacture 200 one-band models and 50 two-band models per week to earn the maximum possible profit of Rs. 2750.

Example 2:

A company manufactures three products, A, B and C. Each product has to undergo operations on three types of machines, M_1, M_2, M_3 before they are ready for sale. The time that each product requires on each machine, and the total time per day available on each machine are given in the following table. The table also shows the net profit per unit on the sale of the three products. Formulate the mathematical model for this problem to maximize the total net profit of the company per day, and obtain its solution.

Machine \ Product	Time per unit in minutes A	B	C	Total time available per day in minutes
M_1	1	2	1	480
M_2	2	1	0	540
M_3	1	0	3	510
Profit per unit in Rs	4	3	5	

The problem here is to determine the number of items of the products A, B, C which must be manufactured per day to maximize profits. Let these be x_1, x_2, x_3 respectively. Since there is a profit Rs. 4 per unit on A, Rs. 3 on B and Rs. 5 on C, the total profit on x_1 units of A, x_2 of B and x_3 of C is

$$f=4x_1 + 3x_2 + 5x_3$$

The objective is to maximize this function.

As for the constraints, on machine M_1 time required for processing one unit of A and C each separately is 1 minute, and for one unit of B it is 2 minutes. The total time in minutes required on machine M_1 is therefore $x_1 + 2x_2 + x_3$ which should not exceed 480 minutes. Hence

$$x_1 + 2x_2 + x_3 \leq 480$$

Similarly considering the limitations of time on machines M_2 and M_3 we should ensure that

$$2x_1 + x_2 \leq 540$$

$$x_1 + 3x_3 \leq 510$$

Also since negative values of the variables will be meaningless, we should have

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$$

Combining all these, the mathematical model of the given problem is

Maximize $f=4x_1 + 3x_2 + 5x_3$

Subject to $x_1 + 2x_2 + x_3 \leq 480$

$2x_1 + x_2 \leq 540$

$x_1 + 3x_3 \leq 510$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

(The solution of this problem by the simplex method is left to the student).

The solution is $x_1 = 231$, $x_2 = 78$, $x_3 = 93$, with f=1623. Thus the company should manufacture 231 units of A, 78 units of B and 93 units of C per day. With this production schedule it will earn the maximum possible profit of Rs. 1623 per day.

Example 3: The manager of an agricultural farm of 80 hectares learns that for effective protection against insects, he should spray at least 15 units of chemical A and 20 units of chemical B per hectare. Three brands of insecticides are available in the market which contain these chemicals. One brand contains 4 units of A and 8 units of B per kg and costs Rs. 5 per kg, the second brand contains 12 and 8 units respectively and costs Rs. 8kg, and the third contains 8 and 4 units respectively and costs Rs. 6 per kg. It is also learnt that more than 2.5 kg per hectare of insecticides will be harmful to the crops. Determine the quantity of each insecticide he should buy to minimize the total cost for the whole farm.

Let the quantity of each of the three insecticides used be x_1, x_2, x_3 kg per hectare. Since the cost of these three is Rs. 5, 8 and 6 per kg respectively, the total cost per hectare would be $5x_1 + 8x_2 + 6x_3$. This is the objective function to be minimized.

The first brand of insecticide contains 4 units of chemical A per kg, the second 12 units and the third 8 units. Hence the total content of chemical A is $4x_1 + 12x_2 + 8x_3$ units which should not be less than 15. Hence

$$4x_1 + 12x_2 + 8x_3 \geq 15$$

Similarly the constraint provided by the content of chemical B is

$$8x_1 + 8x_2 + 4x_3 \geq 20$$

Further, not more than a total of 2.5 kg per hectare of insecticides should be sprayed. Hence

$$x_1 + x_2 + x_3 \leq 2.5$$

Also, from the physical nature of the variables

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$$

Summing up, the mathematical model of this problem is

Minimize $\quad f = 5x_1 + 8x_2 + 6x_3$

Subject to $\quad 4x_1 + 12x_2 + 8x_3 \geq 15$

$$8x_1 + 8x_2 + 4x_3 \geq 20$$

$$x_1 + x_2 + x_3 \leq 2.5$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$$

This LP problem can be solved using the big-M or two phase simplex method. Also, since it is a minimization problem in which all the cost coefficients are positive and two of the constraints are $\geq$ type, on introducing the slack variables an optimal but infeasible basis can be found. Hence the problem can also be solved by the dual simplex method. The following table gives the solution by this method.

Basis	Values	x_1	x_2	x_3	x_4	x_5	x_6
x_4	-15	-4	-12	-8	1		
x_5	-20	-8	-8	-4		1	
x_6	5/2	1	1	1			1
f	0	5	8	6			
x_4	-5		-8	-6	1	$-1/2$	
x_1	5/2	1	1	1/2		$-1/2$	
x_6	0		0	1/2		1/8	1
f	$-25/2$		3	7/2		5/8	
x_2	5/8		1	3/4	$-1/8$	1/16	
x_1	15/8	1		$-1/4$	1/8	$-3/16$	
x_6	0			1/2	0	1/8	1
f	$-115/8$			5/4	3/8	7/16	

The optimal solution is $x_1 = 15/8$, $x_2=5/8$, $x_3=0$, with f=115/8. Hence the manager must buy for each hectare 15/8 kg of the first brand of the insecticide and 5/8 kg of the second brand and none of the third. The cost will be Rs. 115/8 per hectare, or Rs. 1150 for the whole farm.

CHAPTER – IV

TRANSPORTATION

IV. 1. Introduction

We shall consider in this chapter some linear programming problems which have special mathematical structure. The general method of solving an LP problem, namely, the simplex method, can be applied to them. But their special features have led to the discovery of simpler algorithms for their solution. Because of the occurrence of fairly large number of physical situations whose mathematical formulation conform or can be made to conform to these special structures, these problems have assumed considerable importance and have been given special names-the transportation problem and the assignment problem.

IV.2. Transportation Problem

The transportation model most apparently arises when we want to determine the minimum cost at which goods can be transported from given origins to specified destinations. Suppose there are m sources (or origins or supply centres) O_i, i=1, 2, ….., m, of a certain commodity and n sinks (or destinations or demand centres) D_j, j=1, 2, ….., n, where it is required. The quantity produced at source O_i is a_i (>0), and the quantity required at sink D_j is b_j (>0). Let us for the present assume that the total supply equals the total demand, that is

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \qquad (1)$$

If this condition is satisfied, the transportation problem is said to be balanced. The cost of unit flow (or transportation of the unit quantity) from each source to each sink is known. The problem is: How to meet the demand that the cost of transportation is minimum? Or in more general terms, what is the flow with minimum cost?

Let x_{ij} be the flow from O_i to D_j. Then the total outflow at O_i and the total inflow at D_j are respectively.

$$\sum_{j=1}^{n} x_{ij} \text{ and } \sum_{i=1}^{m} x_{ij} ,$$

and therefore

$$\sum_{j=1}^{n} x_{ij} = a_i,\ i=1,2,\ldots,m, \tag{2}$$

and

$$\sum_{i=1}^{m} x_{ij} = b_j,\ j=1,2,\ldots,n. \tag{3}$$

Also since the flow, in order to be meaningful, should be either zero or positive, we further impose the condition.

$$x_{ij} \geq 0 \text{ for all i and j.} \tag{4}$$

The cost of flow is

$$f=\sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij}\, x_{ij} \tag{5}$$

where c_{ij} is the cost of unit flow from O_i to D_j. The problem is to find x_{ij} subject to (1), (2), (3) and (4) which minimize the objective function (5). Equation (2) and (3) and the objective function are linear in x_{ij}. Therefore it is an LP problem. It can be solved by the simplex method. Because of the special structure of the problem, a simpler algorithm has been discovered by which a basic feasible solution can be obtained, its optimality or non-optimality can be tested, and, if non-optimal, a change can be made to another basic feasible solution which is nearer the optimal.

IV. 3. Transportation Array

Some special features of the constraint equations in the transportation problem are very well revealed when the equations are visualized in the array form (table 1). Also, as we shall see later, the simplex method of solution when applied to the transportation problem reduces to very simple rules of computation if the equations are written in the array form. Therefore for theoretical as well practical reasons the transportation array is useful.

Visualized in the array form, equations (2) may be called the row equations and equations (3) the column equations. There are m rows and n columns in the array, providing mn number of cells, one for each of the variables. The cell in the *i*th row and the *j*th column, which we cal the (i,j) cell, is the position of the variable x_{ij}. The constants a_i and b_j are placed respectively in an additional column on the right and an additional row below.

Table 1

	D_1	D_2	...	D_j	...	D_n	
O_1	x_{11}	x_{12}	...	x_{1j}	...	x_{1n}	a_1
O_2	x_{21}	x_{22}	...	x_{2j}	...	x_{2n}	a_2
.	.	.	...	.	...	.	..
O_i	x_{i1}	x_{i2}	...	x_{ij}	...	x_{in}	a_i
.	.	.	...	.	...	.	..
O_m	x_{m1}	x_{m2}	...	x_{mj}	...	x_{mn}	a_m
	b_1	b_2	...	b_j	...	b_n	$\Sigma a_i = \Sigma b_j$

In some discussions it may not be necessary to refer to a_i and b_j, and in such cases we shall consider the array to consist of only the mn number of (i,j) cells. It is also not necessary to explicity write x_y in the (i,j) cell. For example, the transportation array for m=3, n=4 would be as in table 2.

Table 2

				a_1
				a_2
				a_3
b_1	b_2	b_3	b_4	

IV. 4. Transportation matrix

Though the array is the most useful form of representation of the transportation equations, their matrix representation is also useful to bring out some of their important features. To put them in their matrix form, it is convenient to multiply one set of constraint equations, say equations (3), by-1, and put them as

$$\left.\begin{aligned} \sum_{j=1}^{n} x_{ij} &= a_i,\ i = 1, 2, \ldots, m, \\ -\sum_{i=1}^{m} x_{ij} &= -b_j,\ j = 1, 2, \ldots, n. \end{aligned}\right\} \tag{6}$$

The advantage of this form will become evident as we proceed. The above equations, in the matrix form, are

$$TX=B. \tag{7}$$

where X is the column vector of elements x_{ij} which are mn in number. The column vector B has m+n elements, m of the type a_i and n of the type $-b_j$. The matrix T is of order (m+n) x (mn).

To get a clear idea of the form of T, let us write equations (6) more explicity for the particular case m=3, n=4, as follows.

$$\begin{array}{cccccccccccccc}
x_{11} & +x_{12} & +x_{13} & +x_{14} & & & & & & & & & = & a_1 \\
& & & & x_{21} & +x_{22} & +x_{23} & +x_{24} & & & & & = & a_2 \\
& & & & & & & & x_{31} & +x_{32} & +x_{33} & +x_{34} & = & a_3 \\
-x_{11} & & & & -x_{21} & & & & -x_{31} & & & & = & -b_1 \\
& -x_{12} & & & & -x_{22} & & & & -x_{32} & & & = & -b_2 \\
& & -x_{13} & & & & -x_{23} & & & & -x_{33} & & = & -b_3 \\
& & & -x_{14} & & & & -x_{24} & & & & -x_{34} & = & -b_4
\end{array} \tag{8}$$

$$\begin{array}{c}
\begin{array}{cccccccccccc} \mathbf{P}_{11} & \mathbf{P}_{12} & \mathbf{P}_{13} & \mathbf{P}_{14} & \mathbf{P}_{21} & \mathbf{P}_{22} & \mathbf{P}_{23} & \mathbf{P}_{24} & \mathbf{P}_{31} & \mathbf{P}_{32} & \mathbf{P}_{33} & \mathbf{P}_{34} \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
-1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1
\end{bmatrix}
\end{array} \tag{9}$$

P_{11}, P_{12}, … have been written above the matrix to denote the column vectors of T corresponding to x_{11}, x_{12},…. for future reference. It is obvious that each column of T has one entry +1, another −1, and all others zero. The general form of T can be written as

$$T = \begin{bmatrix} \mathbf{I}_1 & \mathbf{I}_2 & . & . & \mathbf{I}_m \\ -\mathbf{U}_1 & -\mathbf{U}_2 & . & . & -\mathbf{U}_m \end{bmatrix} \tag{10}$$

Where I_i, i=1,2,…., m, is an m x n matrix in which all entries in the *i*th row are +1 and all other entries are zero, and U_i is an nxn unit matrix. It is important to note that each column of T contains two and only two nonzero entries, one +1 and the other −1.

The correspondence between a variable x_{ij} and a column in T should also be borne in mind. For every x_{ij} there is a column in T which gives the coefficients of that variable in the set of equations (6).

IV. 5. Triangular basis

The constraint equations (2, 3) or (6) are not linearly independent if (1) is satisfied. If (1) does not hold, the equations become inconsistent. This can be easily seen by adding all the equations (6). Unless otherwise mentioned we shall assume that the transportation problem is balanced, that is, (1) holds. Then the m + n equations (6) are linearly dependent.

The same conclusion can be easily drawn from matrix (10). Since each column contains two and only two nonzero entries, +1 and −1, the sum of all rows of T is a zero row, which means that equations (6) are linearly dependent. It is also obvious that deletion of one, any one, row would leave the remaining set of rows linearly independent. Hence the rank of the matrix T is m+n−1, or, in other words, any m+n−1 of the equations (6) are linearly independent. The number of basic variables (chapter 1, section 9) in these equations is therefore m+n−1. A basic solution will consist of at most $m+n-1$ of the variables having nonzero values.

We now enunciate and prove a theorem about the transportation problem which provides a simple method of obtaining a basic solution.

Theorem 1. The transportation problem has a triangular basis.

By triangular basis we mean that the system of equations, when put in terms of the basic variables only, the non basic variables having been put as zero, is triangular that is, the matrix of coefficients of such equations, if necessary, after a permutation of its rows and columns, is triangular. In other words, there is an equation in which only one basic variable occurs; in another equation there is one more basic variable with the total number of basic variables being not more than two; in a third equation another basic variable occurs with the total now being not more than three, and so on.

The theorem can be proved equally easily by referring to the array or referring to the matrix. We prove it first by considering the array.

Proof. There cannot be an equation in which there is no basic variable because then the equation cannot be satisfied for $a_i \neq 0$ or $b_j \neq 0$. If possible, let every equation have at least two basic variables. Then there will be at least two basic variables in each now, and so the total number of basic variables will be at least 2m. Also each column equation will have at least two basic variables, and so there will be in all at least 2n basic variables. Thus, if N is the total number of basic variables,

$$N \geq 2m, N \geq 2n.$$

If $m > n$, then $N \geq 2m = m + m > m + n$;

If $m < n$, then $N \geq 2n = n + n > n + m$;

If $m = n$, then $N \geq 2m = m + n$.

So in every case $N \geq m + n$. But, as we have already seen, $N = m + n - 1$. This is a contradiction. Therefore the assumption that there are at least two basic variables in each row and column is wrong. There is therefore at least one equation, row or column, in which there is only one basic variable.

Let the *r*th row equation be such an equation and let x_{rc}, the variable in the *r*th row and the *c*th column, be the only basic variable in it. Then $x_{rc} = a_r$. Eliminate this equation from the system by deleting the *r*th row equation and putting $x_{rc} = a_r$ in the *c*th column equation. The *r*th row then stands cancelled, and b_c is replaced by $b_c^1 = b_c - a_r$.

The resulting system consists of $m-1$ row equations and n column equations, of which $m + n - 2$ are linearly independent. Therefore, the number of basic variables in this system is $m + n - 2$. Repeating the earlier argument we conclude that there is an equation in this reduced system which has only one basic variable. If this equation happens to be the *c*th column equation, in the original system the *c*th colmn equation now contains two basic variables. So we conclude that the original system has an equation which has at most two basic variables. Continuing with this line of reasoning we next prove that there is an equation with at most three basic variables, and so on. We thus prove the theorem.

Alternative proof. Referring to the matrix T, it has $m + n$ rows but is of rank $m + n - 1$. Deleting a row from T we are left with a matrix $\bar{T}$ with $m + n - 1$ rows, and it should be possible to find $m + n - 1$ columns in this matrix which are linearly independent. Let A be the $(m + n - 1) \times (m + n - 1)$ matrix with such linearly independent columns. Each of these columns can at most have two non zero entries, one $+1$ and the other -1. If all the columns have two non zero entries, then the sum of the rows will be a zero row, and so the matrix A will be singular which would mean that its columns are not linearly independent. This will be a contradiction. Hence all the columns cannot have two non zero entries.

The total number of non zero entries in A should therefore be less than 2 (m + n – 1). Since there are only m + n – 1 rows in A and each row must contain at least one non zero entry (otherwise A will not be non singular), there should be at least one row with one non zero entry. This means there should be an equation with only one basic variable. Eliminating this equation from the system, we are left with a nonsingular matrix of order m+n–2, and repeating the argument we must find an equation in this system containing only one basic variable, the original system then having an equation with at most two basic variables. Repeating the argument we prove that the basic variables constitute a triangular system of equations.

We have given two proofs because it is important for the reader to clearly understand the correspondence between the matrix and the array of the transportation problem.

The theorem provides a very simple method of testing whether a given set of m + n – 1 variables is a set of basic variables. For example, for m=3, n = 4, consider the two different sets of six variables shown in tables 3 and 4. We shall test in each case whether they form a triangular set of equations. Considering table 3 first, there is an equation containing only one variable; it is the column equation j=4. Let us

Table 3

x_{11}	x_{12}		
x_{21}		x_{23}	
		x_{33}	x_{34}

Table 4

x_{11}	x_{12}		
x_{21}	x_{22}		
		x_{33}	x_{34}

cross out this column, implying thereby that the variable x_{34} is eliminated from the equations. In the remaining array, the row equation i=3 contains only one variable, namely x_{33}. Crossing out this row, we are left with an array in which column equation j=3 contains only one variable x23. Crossing out the column

$j=3$, in the remaining array the row equation $i=2$ has only one variable x_{21}. Crossing out this row, the column equations of the remaining array contain only one variable each. These variables therefore form a triangular set of equations, and so the variables are basic.

Turning to table 4, we cross out the column $j=4$ which contains only one variable. In the remaining array the column $j=3$ contains only one variable, so we cross it out. Now there is no row or column in the remaining array having only one variable. The variables therefore do not form a triangular system of equations and are not basic.

IV. 6. Finding basic feasible solution

Theorem 1 also provides a practical method of finding a b.f.s. Let us arbitrarily choose x_{rc} as the basic variable which occurs alone in an equation. It can so occur either in the *r*th row or the *c*th column. If we choose the *r*th row equation, then $x_{rc} = a_r$; if we choose the *c*th column equation, then $x_{rc}=b_c$. Suppose $a_r>b_c$. Then if we choose $x_{rc} = a_r$, some other variable in the *c*th column will have to have a negative value in order that the column equation may be satisfied. This will mean going to a non-feasible solution. If on the other hand we choose $x_{rc} = b_c$, the *r*th row equation will need another variable with a positive value for satisfaction which will create no such difficulty. If $a_r < b_c$, the position will be reversed and it will be necessary to choose $x_{rc} = a_r$. The rule is to put $x_{rc} = \min(a_r, b_c)$. If the two are equal, it is immaterial which choice is made. Just now let $a_r < b_c$. Then, following the above rule, put $x_{rc} = a_r$. This satisfies the *r*th row equation. We turn our attention to *c*th column equation. Eliminating x_{rc} from it, we replace b_c by $b_c - a_r$, and then obtain an array with one row (or column) less. With this reduced array we proceed as we did with the first.

The procedure is continued till $m + n - 1$ rows and columns are crossed out and an equal number of variables evaluated. The procedure ensures that the solution so obtained is basic feasible. The last row or column left uncrossed will be automatically satisfied.

It is important to cross out one and not more than one row or column at each stage after choosing a basic variable. In the case when at any stage of the above procedure $a_r = b_c$, we may put $x_{rc} = a_r$ or b_c and may cross out the *r*th row or the *c*th column but not both. If we choose to cross out the *r*th row, then bc is replaced by $b_c - a_r = 0$, and the *c*th column has still to be satisfied by choosing some other variable in the *c*th column to be included in the basis. The value of this variable would be zero. On the other hand if the *c*th column is crossed out first, the *r*th row should be kept open for choosing another basic variable in it whose value would be zero. The resulting basis in either case would be degenerate. We illustrate the method by two numerical examples, the second involving degeneracy.

Example 1: In the numerical problem of table 5

TABLE 5

	D_1	D_2	D_3	D_4	a_i
O_1	10	15			25
O_2		3	20	12	35
O_3				30	30
b_j	10	18	20	42	90

TABLE 6

	D_1	D_2	D_3	D_4	a_i
O_1			20	5	25
O_2				35	35
O_3	10	18		2	30
b_j	10	18	20	42	90

the three origins have capacities 25, 35 and 30, and the four destinations have demands 10, 18, 20 and 42. The total capacity equals the total demand. The number of linearly independent equations and so the number of basic variables is six. Starting with x_{11} as a basic variable, we put $x_{11} = 10$ because $a_1 = 25 > b_1 = 10$. This satisfies the first column equation, and $a_1^1=25-10=15$. Turning to the first row equation, we put $x_{12} = \min(15, 18) = 15$ and $b_2^1 = 18 - 15 = 3$. The first row equation is also satisfied. Next we put $x_{22} = 3$ which satisfies the second column equation. The second row equation should not be satisfied by putting $x_{22} = 32$, because $b_3 = 20 < a_2^1 = 32$. Instead we put $x_{23} = 20$ thus satisfying the third column equation. Proceeding further we put x_{24} = 12 and finally x_{34} = 30, thus satisfying all the equations and getting a b.f.s. as x_{11} = 10, x_{12} = 15, x_{22} = 3, x_{23} = 20, x_{24} = 12, x_{34} = 30, and all other variables zero.

It is not necessary that we should always start with x_{11}. Any variable may be selected to make a start. For example, starting with x_{13} = 20 the b.f.s as given in table 6 is obtained.

Example 2: The example in table 7 illustrates a degenerate case. After setting x_{11} = 10, the first column is crossed and in the first row a_1 = 25 is replaced by 25−10=15 which is equal to b_2. Putting x_{12} = 15 satisfies both the first row and the second column. But we cross the first row only, and put x_{22} = 0 and then cross the second column. Proceeding further we get a b.f.s. as shown in table 7.

TABLE 7

	D_1	D_2	D_3	D_4	a_i
O_1	10	15			25
O_2		0	10		10
O_3			10	40	50
b_j	10	15	20	40	85

IV. 7. Testing for optimality.

To test whether a particular b.f.s. is optimal or not, the objective function should be expressed in terms of the nonbasic variables only by eliminating the basic variables with the help of the constraint equations. The coefficients of the nonbasic variables in the new expression for the objective function are called the relative cost coefficients for the current b.f.s. If all the relative cost coefficients are non-negative, the solution is optimal and the corresponding value of the objective function is minimum. If a relative cost coefficient is negative, the value of the objective function can be further reduced by bringing the corresponding nonbasic variable in the basis in place of some basic variable which is dropped out of the basis.

In the transportation problem the relative cost coefficients can be worked out very easily. Let us for simplicity consider the problem with m=3, n=4 and write equations (2) and (3) and the objective function (5) in the following extended form.

$$
\begin{array}{cccccccccccccl}
x_{11} & +x_{12} & +x_{13} & +x_{14} & & & & & & & & & = a_1 \\
 & & & & x_{21} & +x_{22} & +x_{23} & +x_{24} & & & & & = a_2 \\
 & & & & & & & & x_{31} & +x_{32} & +x_{33} & +x_{34} & = a_3 \\
x_{11} & & & & +x_{21} & & & & +x_{31} & & & & = b_1 \\
 & x_{12} & & & & +x_{22} & & & & +x_{32} & & & = b_2 \\
 & & x_{13} & & & & +x_{23} & & & & +x_{33} & & = b_3 \\
 & & & x_{14} & & & & +x_{24} & & & & +x_{34} & = b_4
\end{array}
\qquad (11)
$$

$c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{14}x_{14} + c_{21}x_{21} + c_{22}x_{22} +$

$+ c_{23}x_{23} + c_{24}x_{24} + c_{31}x_{31} + c_{32}x_{32} + c_{33}x_{33} + c_{34}x_{34} = f.$ (12)

Following table 5, let a feasible basis be x_{11}, x_{12}, x_{22}, x_{23}, x_{24}, x_{34}. To find the relative cost coefficients for this case we have to eliminate these variables from (12). Let π_1, π_2, π_3 be the simplex multipliers for the three row equations and σ_1, σ_2, σ_3, σ_4 for the four column equations respectively. Then in order to eliminate the basic variables we must evaluate these multipliers from the following equations.

$$\pi_1 + \sigma_1 + c_{11} = 0,$$

$$\pi_1 + \sigma_2 + c_{12} = 0,$$

$$\pi_2 + \sigma_2 + c_{22} = 0,$$

$$\pi_2 + \sigma_3 + c_{23} = 0,$$

$$\pi_2 + \sigma_4 + c_{24} = 0,$$

$$\pi_3 + \sigma_4 + c_{34} = 0,$$

These are six equations but seven unknown to be evaluated. In general these are respectively m+n−1 and m + n in number. Obviously there are infinitely many solutions, but anyone would serve our purpose. We may therefore choose any one of the simplex multipliers arbitrarily. The simplest way would be to choose one of these, say π_1 as zero. Putting $\pi_1 = 0$, the other are evaluated quite easily. In tabulated form the rule for evaluation turns out to be very simple. The simplex multipliers for a column and a row should be such that the sum of the two multipliers plus the cost coefficient in the intersecting cell should be zero provided the square is occupied by a basic variable. This rule enables us to evaluate all the simplex multipliers.

Now to evaluate the relative cost coefficients c_{ij}' (the coefficients of the non basic variables x_{ij} in f after the basic variables have been eliminated),

$$C_{ij}^1 = \pi_i + \sigma_j + c_{ij}$$

where the cell (i, j) corresponds to nonbasic variable. This is calculated for all nonbasic variables. If c_{ij}^1 is negative for any (i,j), the present basis is not optimal and the value of f can be improved by bringing the variable x_{ij} in the basis.

Example: Let the cost coefficients c_{ij} in example 1 (table 5) be as shown in the righ-hand bottom corner of each square of table 8. We shall test whether the b.f.s. of table 5 is optimal. Let $\pi_1 = 0$ (written in the second row of the square containing

a_1 (=25)]. Since x_{11} (=10) is a basic variable,

$$\pi_1 + \sigma_1 + c_{11} = 0 \Rightarrow 0 + \sigma_1 + 3 = 0 \Rightarrow \sigma_1 = -3.$$

Also x_{12} (=15) is a basic variable, and so

$$\pi_1 + \sigma_2 + c_{12} = 0 \Rightarrow 0 + \sigma_2 + 2 = 0 \Rightarrow \sigma_2 = -2.$$

Table 8

	D_1	D_2	D_3	D_4	a_i
O_1	10 3	15-u 2	U -3 5	 -3 4	25 0
O_2	 2 4	3+u 1	20-u 7	12 6	35 1
O_3	 6 7	 8 8	 -3 3	30 5	30 2
b_j	10 -3	18 -2	20 -8	42 -7	 90

Similarly, since x_{22} (=3) is a basic variable,

$$\pi_2 + \sigma_2 + c_{22} = 0 \Rightarrow \pi_2 - 2 + 1 = 0 \Rightarrow \pi_2 = 1.$$

In this way all the simplex multipliers are easily evaluated. These are written in the second row in each square for a_i and b_j. Now, since the relative cost coefficient for the nonbasic variable x_{ij} is

$$c_{ij}^1 = \pi_i + \sigma_i + c_{ij},$$

$$c_{13}^1 = 0-8 + 5 = -3,\ c_{14}^1 = 0-7 + 4 = -3,$$

$$c_{21}^1 = 2,\ c_{31}^1 = 6,\ c_{32}^1 = 8,\ c_{33}^1 = -3.$$

We write these in the left bottom comers of the nonbasic (vacant) cells. Since there are negative c_{ij}^1, the present basis is not optimal. The value of the objective function for the present solution is

$$f=10 \times 3 + 15 \times 2 + 3 \times 1 + 20 \times 7 + 12 \times 6 + 30 \times 5 = 425.$$

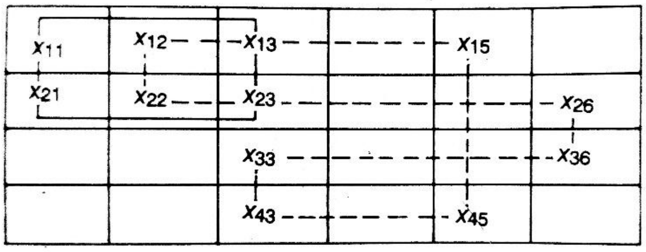
This can be reduced by a change of basis. The candidates to enter the basis are x_{13}, x_{14}, x_{33}.

IV. 8. Loop in transportation array

The procedure for changing the basis is based theoretically on a notion in the transportation array called the loop which we proceed to define and discuss.

Definition: A set of cells L in the transportation array is said to constitute a loop if in every row or column of the array the number of cells belonging to the set is either zero or two.

Suppose (i_1, j_1) is a cell of the loop L. Then there must be cells (i_1, j_2), (i_2, j_2), (i_2, j_3)…, (i_k, j_1) belonging to L. Examples of loops are shown in table 9. One loop consisting of x_{11}, x_{13}, x_{23}, x_{21} is shown by continuous lines, another consisting of x_{13}, x_{15}, x_{45}, x_{43}, x_{33}, x_{36}, x_{26}, x_{23} is shown by dotted lines. The idea is that in a set of



cells forming a loop, starting from any one cell, it is possible to go through all the other cells of the loop and back to the starting cell, visiting each intermediate cell once only and moving alternately along rows and columns of the array.

Theorem 2: The necessary and sufficient condition for a set of column vectors P_{ij} in the matrix $\bar{T}$ (section 5) to be linearly dependent is that the corresponding variables x_{ij} in the transportation array occupy cells a subset of which constitutes a loop.

Proof: To prove that the condition is necessary, let a set of column vectors P_{ij} of the matrix $\bar{T}$ (obtained by deleting a now from T) be linearly dependent. It is more convenient in the present case to denote the column vectors as P(i,j), $i=i_1$, $i_2\ldots$, i_p, …, i_4, $j=j_1$, $j_2\ldots$., j_r …, j_s. Since they are linearly dependent, there exist numbers α(i, j), not all zero, such that

$$\sum_j \sum_i \alpha\,(i,j)\ P\ (i,j) = 0. \qquad (13)$$

Pick up a nonzero multiplier $\alpha(i_p, j_r)$ of the column vector $P(i_p, j_r)$. The corresponding variable $x(i_p, j_r)$ occupies the cell (i_p, j_r) in the array. The row of the column vector $P(i_p, j_r)$ corresponding to the i_pth now of the array contains the entry +1. Therefore in order that (13) is satisfied for this row of the matrix of column vectors, there must be at least one more nonzero entry in that very row of the matrix, that is, there must be another column vector of type $P(i_p, j_r)$ in the given set of column vectors, and its multiplier $\alpha(i_p, j_3)$ should be nonzero. The corresponding variable $x(i_p, j_r)$ occupies the cell (i_p, j_3). Thus the i_pth row of the array contains at least two variables $x(i_p, j_r)$ and $x(i_p, j_3)$ corresponding to the column vectors of the given set.

Also in the column vector $P(i_p, j_r)$ the entry in the row corresponding to the j_rth column of the array is −1. Again, in order that (13) is satisfied for this row of the matrix of the given column vectors, there must be another column of the type $P(i_q, j_z)$ whose multiplier $\alpha(i_q, j_z)$ is nonzero. This means there is a variable $x(i_q, j_z)$ in the cell (i_q, j_z) of the array, making the number of variables in the j_rth column at least two.

We can proceed with this type of argument till we find that for all nonzero multipliers $\alpha(i,j)$ of $P(i,j)$ there must be corresponding variables $x(i, j)$ in the array such that in a row or a column of the array if one of such variables occur, then at least one more of them also occurs. This proves the existence of a loop in the set of variables corresponding to the given set of linearly dependent column vectors.

To prove the converse, let the set of variables $x(i,j)$ corresponding to the column vectors $P(i,j)$, $i=i_1, i_2, \ldots, i_p, \ldots, i_q$, $j=j_1, j_2, \ldots, j_r, \ldots, j_s$, contain a subset forming a loop. Let the subset be $x(i,j_1)$, $x(i_1, j_2)$, $x(i_2, j_2), \ldots, x(i_p, j_r)$, $x(i_p, j_1)$. Let the corresponding column vectors be multiplied by +1 and −1 alternatively and let the remaining column vectors of the given set be multiplied by zero. Then consider the following linear combination of the given set of column vectors.

$$P(i_1, j_1) - P(i_1, j_2) + P(i_2, j_2) - \ldots - P(i_p, j_1)$$

The row of this vector corresponding to the i_1th row of the array becomes zero because only two vectors $P(i_1, j_1)$ and $P(i_1, j_2)$ have nonzero entries in this row and they are each +1 and so their difference becomes zero. Similarly for all other rows of the vector corresponding to other rows and columns of the array. Hence the above linear combination is a zero vector, which means that the given column vectors are linearly dependent.

IV. 9. Changing the basis

Theorem 2 provides a method of changing the basis in a transportation array so as to bring into the basis any desired variable in place of another which is deleted from the basis without making the solution non-feasible.

In section 7 we have seen how to select a variable for entry into the basis. The existing m + n – 1 basic variables along with this new variable become m + n in number. The corresponding m + n column vectors in the matrix $\bar{T}$ are linearly dependent because the matrix is of rank m + n – 1. Hence, by theorem 2, the m + n variables in the transportation array have a loop within themselves. It can be proved that this loop is unique for a particular set of basic variables with a particular additional variable and includes the latter. (We omit the proof). This loop can be easily traced as illustrated in the example of table 8.

Let us, in table 8, decide to bring x_{13} into the basis. This variable x_{13} together with x_{12}, x_{22}, x_{23} which are variables of the existing basis, forms a loop. The values of these basic variables at this stage are

$$x_{12} = 15,\ x_{22} = 3,\ x_{23} = 20$$

If we put $x_{13} = u$ (a constant), and alternately subtract and add u from and to the other variables of the loop so that the equations are still satisfied, we get

$$x_{12} = 15 – u,\ x_{22} = 3 + u=,\ x_{23} = 20 – u.$$

The value of u which would reduce the value of one of these variables to zero without making any of the others negative is $u = 15$, Then $x_{12} = 0$, $x_{22} = 18$, $x_{23} = 5$.

Thus x_{12} goes out of the basis and x_{13} comes in. The new basis is down in table 10. The value of f for this solution is 380 which is an improvement on the previous value (section 7).

TABLE 10

	D_1	D_2	D_3	D_4	a_i
O_1	10		15		25
O_2		18	5	12	35
O_3				30	30
b_j	10	18	20	42	90

TABLE 11

	D_1	D_2	D_3	D_4	a_i
O_1				25	25
O_2	10	18		7	35
O_3			20	10	30
b_j	10	18	20	42	90

The procedure outlined in sections 6, 7 & 9 constitutes the algorithm for the solution of the transportation problem. It is repeated till an optimal solution is obtained. It is left to the reader to proceed with the successive iterations. The optimal solution is shown in table 11 with the minimum value of *f* as 310.

It should be remembered that while the minimum value of the objective function has to be unique, the optimal basis may not be unique. There may be other solutions giving the same value of the objective function. If any $c_{ij}^1 = 0$ at the optimal stage, then an alternative solution exists with the corresponding variable in the basis.

IV. 10. Degeneracy

Degeneracy can occur in transportation problem also. The example in table 7 illustrates a degenerate case. $x_{22} = 0$ is also a basic variable. In practical problems it has seldom proved to be a hurdle. One can proceed to the next b.f.s. according to the prescribed rule and hopefully one would get out of the loop to eventually arrive at the optimal solution. There can, however, be examples in which one is caught in the loop and is unable to get out by the ordinary rule. We shall omit the discussion of the method to overcome this difficulty.

IV. 11. Unbalanced problem

If we remove condition (1) and assume that

$$\sum_{i=1}^{m} a_1 \neq \sum_{j=1}^{n} b_j \ ,$$

The problem (2)-(5) becomes infeasible. In physical sense, if $\Sigma a_i > \Sigma b_j$, there would be surplus left at the sources after all the demands are met, and if $\Sigma a_i < \Sigma b_j$, there would be deficit at the sinks after all the sources have exhausted their capacities. Problems involving surpluses and deficits are common and significant in practical life. They have only to be posed properly to have feasible solutions.

Let us look at the problem of surplus at the sources in the following way. If the total available supply is more than the total demand, the demands at the sinks can be fully met without exhausting the supplies. We may want to know the minimum cost of meeting the demands at all the sinks. The problem then may be

$$\left.\begin{array}{ll}\text{Minimise} & \displaystyle\sum_{j=1}^{n}\sum_{i=1}^{m} c_{ij}x_{ij}, \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} x_{ij} \le a_i,\ i = 1, 2, \ldots, m, \\ & \displaystyle\sum_{i=1}^{m} x_{ij} = b_j,\ j = 1, 2, \ldots, n, \\ & \displaystyle\sum_{i=1}^{m} a_i \ge \sum_{j=1}^{n} b_j, \\ & x_{ij} \ge 0. \end{array}\right\}$$

If equality holds in the third constraint, the total demand can be fully met only with all the supply, and so a feasible solution exists. The problem then reduces to a balanced problem already discussed. If, however, the third constraint is an inequality, the following artifice converts it into a balanced problem.

We create a fictitious sink j=n+1 with demand

$$b_{n+1} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j\,. \qquad (15)$$

The new problem with m sources and n+1 sinks is balanced. Any amount going from a source to the fictitious sink is actually the surplus remaining at that source. Before this balanced problem can be solved, the cost coefficient $c_{i,\,n+1}$ from the *i*th source to the fictitious sink for all i, i = 1, 2, …, m, should be known. It is obviously the cost of surplus lying at the source i. Depending upon the physical nature of the problem, it may be zero or some other number which should be estimated.

The problem of surplus at the sources may be posed in another way also. The demand may be flexible with prescribed minimum at each sink. The supply at each source is fixed and all of it must be transported. The problem is then as follows.

$$\left.\begin{aligned}
&\text{Minimise} && \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij},\\
&\text{subject to} && \sum_{j=1}^{n} x_{ij} = a_i,\ i = 1, 2, \ldots, m,\\
& && \sum_{i=1}^{m} x_{ij} \geq b_j,\ j = 1, 2, \ldots, n,\\
& && \sum_{i=1}^{m} a_i \geq \sum_{j=1}^{n} b_j,\\
& && x_{ij} \geq 0.
\end{aligned}\right\} \tag{16}$$

To solve it we again introduce a fictitious sink j=n+1 with demand b_{n+1} given by (15). But now the cost coefficients for the (n+1)th column in the array should be taken as

$$c_{i,n+1} = c_{i,ri} = \min\,(c_{13}, c_{12}, c_{13}, \ldots, c_{in}),\ i=1,2,\ldots, m.$$

The idea is that with the minimum demands at all the sinks been met, the surplus at a source is transported to that sink for which the cost is minimum. The optimal solution of the balanced problem with m sources and n+1 sinks provides the optimal solution of the original problem after the value in the (i, n+1) cell is added to the value in the (i, r_i) cell for all i, i = 1,2,…., m.

The problem in which the total demand exceeds the total supply may be posed as follows.

$$\left.\begin{array}{ll} \text{Minimise} & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\, x_{ij}, \\ \text{subject to} & \\ & \displaystyle\sum_{j=1}^{n} x_{ij} = a_i,\ i = 1, 2, \ldots, m, \\ & \displaystyle\sum_{i=1}^{m} x_{ij} \geq b_j,\ j = 1, 2, \ldots, n, \\ & \displaystyle\sum_{i=1}^{m} a_i \geq \sum_{j=1}^{n} b_j, \\ & x_{ij} \geq 0. \end{array}\right\} \tag{17}$$

The demand at each sink is not necessarily fully met. The actual supply may fall short of the demand. To solve this problem a fictitious source i=m+1 is introduced with capacity.

$$a_{m+1} = \sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i \,.$$

The cost coefficients in the (m+1)th row should be the cost of deficit at each sink. It again depends upon the physical nature of the problem to estimate how much loss is suffered on account of deficit supply at a sink. The balanced problem of m+1 sources and n sinks so obtained is solved for optimal solution. Suppliers from the fictitious (m+1)th source given by the optimal solution are to be interpreted as deficits in the original problem.

The following is an example of an unbalance problem, also involving degeneracy during its solution.

Example: Table 12 gives the quantity of goods available at four origins 0_i, i=1, 2, 3, 4, and the minimum requirement at three destinations. D_j, j=1,2,3, and the cost of transportation of unit quantity of goods from origins to destinations. The available goods exceed the minimum total requirement, and the excess can be transported to the destinations, but at minimum cost. Find the distribution of goods such that the total cost of transportation is minimum.

TABLE 12

	D_1	D_2	D_3	
O_1	2	1	3	10
O_2	4	5	7	25
O_3	6	0	9	25
O_4	1	3	5	30
	20	20	15	

The total availability is 90, while the minimum requirement is 55. Hence we introduce a fictitious destination with demand 35. Whatever goes to the fictitious destination D_4 from 0_1 should really go to one of the real destinations D_1, D_2 or D_3 such that the cost is minimum. Hence the cost coefficient c_{14} = min (c_{11}, c_{12}, c_{13}) = min (2, 1, 3) = 1. Similarly $c_{24} = 4$, $c_{34} = 0$, $c_{44} = 1$. The balanced transportation array obtained after introducing D_4 and the corresponding cost coefficients is given in table 13(i). It also indicates a basic feasible solution which serves as a starting point for search for the optimal solution. The simplex multipliers and the resulting relative cost coefficients of the non basic variables are also written in the table, as was done in table 8.

Nothing that the relative cost coefficient c^1_{23} is negative and the least (−7), we bring x_{23} into the basis. Putting $x_{23} = u$, we adjust $x_{33} = 5-u$, $x_{32} = 15 + u$, $x_{22} = 5 - u$, and hence set u=5. This makes $x_{33} = x_{22} = 0$. Treating only one of these, x_{33}, as the nonbasic variable and retaining x_{22} as a basic variable, we get the new basic feasible solution as shown in table 13(ii). It is a degenerate solution.

TABLE 13(I)

	D_1	D_2	D_3	D_4	
O_1	9 2	7 1	10 3	7 1	10 6
O_2	20 4	$5-u$ 5	u −7 7	−1 4	25 −5
O_3	7 6	$15+u$ 0	$5-u$ 9	5 0	25 0
O_4	1 1	2 3	−5 5	30 1	30 −1
	20 1	20 0	15 −9	35 0	

TABLE 13(II)

	D_1	D_2	D_3	D_4	
O_1	2 2	0 1	10 3	0 1	10 4
O_2	20 4	$0-u$ 5	5 7	u -1 4	25 0
O_3	7 6	$20+u$ 0	7 9	$5-u$ 0	25 5
O_4	1 1	2 3	2 5	30 1	30 4
	20 -4	20 -5	15 -7	35 -5	

TABLE 13(III)

	D_1	D_2	D_3	D_4	
O_1	2 2	1 1	10 3	1 1	10 4
O_2	20 4	1 5	5 7	0 4	25 0
O_3	6 6	20 0	6 9	5 0	25 4
O_4	0 1	2 3	1 5	30 1	30 3
	20 -4	20 -4	15 -7	35 -4	

For this basic feasible solution c^1_{24} is the only negative relative cost coefficient. Therefore x_{24} is brought into the basis. This necessitates dropping x_{22} out of the basis. The new value of x_{24} is also zero. Table 13 (iii) gives the next basic feasible solution. As all the relative cost coefficients are non-negative in this table, we have obtained the optimal solution. It is $x_{13} = 10$, $x_{21} = 20$, $x_{23} = 5$, $x_{32} = 20$, $x_{34} = 5$, $x_{44} = 30$. The value of x_{34} is to be interpreted as the excess quantity sent from 0_3 to D_2, and similarly the value of x_{44} is the excess quantity from 0_4 to D_1. Thus the optimal distribution in terms of the original problem is: $0_1 \rightarrow D_3$, 10; $0_2 \rightarrow D_1$, 20; $0_2 \rightarrow D_3$, 5; $0_3 \rightarrow D_2$, 25; $O_4 \rightarrow D_1$, 30. The supply to D_1 is 50, to D_2 it is 25, and to D_3 it is 15. The first two destinations receive more than the minimum required, while the third receives just the required minimum. The total cost of transportation is

$10 \times 3 + 20 \times 4 + 5 \times 7 + 20 \times 0 + 5 \times 0 + 30 \times 1 = 175.$

IV. 12. Caterer Problem

What is popularity known as the caterer problem in operations research first arose in connection with number of spare engines required to maintain a fleet of aeroplanes airworthy during a certain period. We shall describe the problem in general terms.

Suppose there is an article which is used once and then sent for repair or servicing before it can be used again. On a job a_1, a_2, …, a_n (positive integers) of these articles are required at times T, 2T, …., nT respectively. The job lasts till nT. The job begins at T with a_1 articles purchased new from the market at a certain price. But at successive times the requirement can be met partly by repaired articles and partly, if necessary, through purchase of new ones. The minimum time of repair is rT and maximum (r++s)T, r and s being positive integers with r + s < n. The quicker the service, the higher the cost of repair, which in any case is less than the price of a new article. The problem is: How to organize purchase and repair of articles so that the job is completed with minimum cost of the articles.

We can look at the problem as follows.

Let x_{ij} be the number of articles received back after repair which were sent for repair at time iT to be returned at time jT, and let c_{ij} be the cost of this repair per article. Then $\sum_{i=1}^{n} x_{ij}$ is the total number of repaired articles available at jT. Of course x_{ij} is meaningless for i$\geq$j and can have nonzero value only if r $\leq$ j – i $\leq$ r + s. The difficulty is easily overcome by putting $c_{ij} = \infty$ for inadmissible values of i, so that the minimum cost expression can never include a nonzero value of x_p for any inadmissible i. Any shortage at time jT will have to be met by urchase of new articles. Let $x_{n+i,\, j}$ be this number. Then

$$\sum_{i=1}^{n} x_{ij} + x_{n+1,\, j} = a_j \text{, j=1, 2, …., n.}$$

The use of the symbol $x_{n+1,\ j}$ for the number of new articles purchased is convenient. It makes up the deficit in the inequality.

$$\sum_{i=1}^{n} x_{ij} \le a_j$$

and is therefore a slack variable. Moreover its introduction helps put the problem in the transportation form.

Also $\sum_{i=1}^{n} x_{ij}$ is the total number of articles sent out for repair at time iT. Again since x_{ij} can be nonzero only for $r \le j - i \le r + s$, we put $c_{ij} = \infty$ for inadmissible values of j. All the articles used at time iT need not be sent for repair, as the job is to last only up to nT and if they cannot be repaired before that time they may as well be left unrepaired. The cost of leaving an article unrepaired may be taken as zero, Let $x_{i,\ n+1}$ be the number of articles used but not sent for repair at time iT. Then

$$\sum_{j=1}^{n} x_{ij} + x_{i,n+1} = a_i,\ i=1,2,\ldots,n.$$

Also we have the non-negativity conditions

$$x_{ij} \ge 0,\ x_{i,n+1} \ge 0,\ x_{n+1,j} \ge 0$$

The objective function to be minimized is $f = \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}\, x_{ij} + c\sum_{j=1}^{n} x_{n+1,\, j}$

where c is the price of the new article.

We may finally put the equations derived above in the standard transportation form.

$$\sum_{i=1}^{n+1} x_{ij} = a_j,\ j = 1, 2, \ldots, n+1;$$

$$\sum_{j=1}^{n+1} x_{ij} = a_i,\ i = 1, 2, \ldots, n+1;$$

$$x_{ij} \ge 0;$$

$$f = \sum_{j=1}^{n+1} \sum_{i=1}^{n+1} c_{ij} x_{ij};$$

provided we can give meanings to $x_{n+1,\ n+1}$, a_{n+1} and $c_{n+1,n+1}$. Let a_{n+1} be a sufficiently large number chosen arbitrarily. A convenient value will be Σa_i which is the total number of articles required on all the days. The variable $x_{n+1,\ n+1}$ can be interpreted as the number of new articles left without being used and so not purchased at all.

The cost of this fictitious transaction may be taken as $c_{n+1,\ n+1} = 0$. This finally puts the problem in the transportation form which can be solved by the standard procedure.

Example: A caterer needs clean table covers every day for six days to meet a contract according to the following schedule.

Days	1	2	3	4	5	6
Number of covers	50	60	80	70	90	100

The cost of a new cover is Rs. 20 while washing charges are Re 1 for return on the fourth day or later, Rs. 2 for return on the third day and Rs 3 for the next day. Find the minimum cost schedule for the purchase and washing of table covers, assuming that after the end of the contract the covers are rejected.

The problem, when put in the transportation form, is as shown in table 15. The table shows an initial b.f.s. and also the optimal solution (bold numbers). The minimum cost is Rs. 2950. The new purchases are 100 required on the first two days. Subsequently the used ones return after washing.

TABLE 15

		Sent for washing → 1	2	3	4	5	6	7 Soiled and rejected	
Received after washing	1	∞	50 **10** 3	**20** 2	**20** 1	1	1	0	50
	2	∞	∞	60 **60** 3	2	1	1	0	60
	3	∞	∞	∞	70 **50** 3	10 **20** 2	**10** 1	0	80
	4	∞	∞	∞	∞	70 **70** 3	2	0	70
	5	∞	∞	∞	∞	∞	90 **90** 3	0	90
	6	∞	∞	∞	∞	∞	∞	100 **100** 0	100
Purchased	7	50 **50** 20	10 **50** 20	20 20	20	10 20	10 20	350 **350** 0	450
		50	60	80	70	90	100	450	

MODULE – III

CHAPTER – V

INTEGER PROGRAMMING

V. 1. Introduction

In many programming problems optimal solution is sought in terms of integral values of the variables, non-integral answers not being meaningful in the context of the situation which gives rise to the problem. For example, if the variables are the numbers of buses on different routes in a town or the numbers of bank branches in different, regions of a country, fractional answers have no meaning. Mathematical programming subject to the constraint that the variables are integers is called integer programming. If some of the variables are restricted to be integers while others are real numbers, the problem is said to be mixed integer programming.

Strictly speaking, if in an LP problem we restrict the variables to integers, the problem becomes non-linear. But it is convenient to call it an integer linear programming problem (ILP) because the constraints and the objective function remain linear if the integral restriction on the variables is ignored. If not all but some of the variables are restricted to be integers, we have a mixed integer linear programming problem (MILP). In general we may have an integer or a mixed integer non-linear programming problem if it is obtained by imposing integer restriction on an otherwise non-linear problem. In this chapter we shall consider only the integer and the mixed integer linear programming problems.

One obvious way of getting an answer to an ILP or MILP is to ignore the integer restrictions on variables and solve it as an ordinary LP problem, and then to round off the optimal solution to nearest integers. When the answers are in the nieghbourhood of large integers, the method gives satisfactory results. For example, if the problem is concerned with human population in a town, a fractional answer giving the number of persons as 3548.68 can be rounded off to 3549 or even to 3550 without any significant error. But if the answer is in the neighbourhood of small integers such rounding off may give a totally wrong answer.

V. 2. ILP in two-dimensional space.

As in the case of an LP problem, it is easy to obtain a graphical solution of an ILP problem if the number of variable is only two. We therefore take a two dimentional ILP problem as an example to bring out the important features of a general ILP problem.

Consider the problem:

Maximise $\phi\,(X) = 3x_1 + 4x_2$; (1)

Subject to $2x_1 + 4x_2 \leq 13$,

$-\ 2x_1 + x_2 \leq 2$

$2x_1 + 2x_2 \geq 1$, (2)

$6x_1 - 4x_2 \leq 15$,

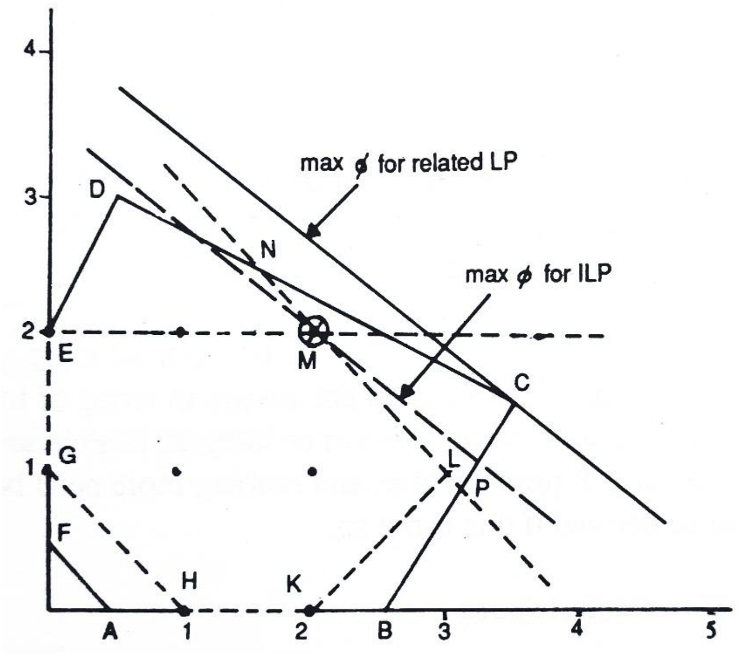
$x_1, x_2 \geq 0$; (3)

x_1, x_2 integers. (4)

This is an ILP problem. If we drop (4) we obtain the related LP problem.

Fig. 1 shows the graphical solution of the related LP problem. The polygon ABCDEF is the convex set of feasible solutions and the point C(x_1 = 7/2, x_2 = 3/2) is the optimal solution with the maximum value of ϕ = 33/2. If we round off (7/2, 3/2) to nearest integers, assuming that ½ may be rounded off to 0 or 1 with equal justification, we get the four points (3, 1), (4, 1), (4, 2), (3, 2). Of these the last three are not feasible. So the only feasible point obtained by rounding off is (3, 1), which makes ϕ=13.

Consider now the given ILP problem. We restrict x_1, x_2 to be integers, and so the set of feasible solution are the points in the polygon ABCDEF whose coordinates are integers. Such points, marked x in fig. 1, are (1, 0), (2, 0), (0, 1), (1, 1), (2, 1), (3, 1), (0, 2), (1, 2), (2, 2). Among these points the objective function ϕ is maximum at (2, 2) with ϕ=14. Thus rounding off gives a wrong answer.



The set of feasible solution of the ILP problems is not convex, because it consists of the isolated nine points given above. If we obtain the convex hull of this non-convex set of feasible solutions, we get the polygon EGHKLM. Every vertex of this convex polygon is a feasible solution of the ILP problem. Let us consider the LP problem:

Maximize $\qquad \phi(X) = 3x_1 + 4x_2 \qquad (1)$

Subject to X $\in$ convex hull of feasible solutions of (2), (3) and (4).

The optimal solution of this problem can be seen to be the point M (2, 2) which is the optimal solution of the given ILP. We may therefore conjecture that the optimal solution of an ILP problem is the same as the optimal solution to an LP problem whose objective function is the same as that of the ILP problem but whose constraints are such that the convex set of feasible solutions turns out to be the convex hull of the set of feasible solutions of the ILP problem.

V. 3. General ILP and MILP problems

A vector $X \in E_n$ shall be called an integer vector if its components x_i for all i, i=1, 2, …., n, are integers; it shall be called a mixed integer vector if x_i is integer for $i \in J$ where $J \subset \{1, 2, 3, …., n\}$.

We enunciate the general ILP or MILP problem as follows:

Minimize $\quad f(X) = CX, \quad$ (5)

Subject to $\quad AX = B, \quad$ (6)

$\quad X \geq 0, \quad$ (7)

X an integer or a mixed integer vector. (8)

If we drop constraint (8) we are left with the related LP problem. A solution of (6), (7), (8) is obviously a solution of (6), (7). Therefore, if T_F denotes the set of feasible solutions of the ILP or the MILP problem, and S_F the set of feasible solutions of the related LP problem, the $T_F \subseteq S_F$. Since S_F, if nonempty, is a convex set and every point of T_F is in S_F, the convex linear combinations of points in T_F are also in S_F. Hence $[T_F]$, the convex hull of T_F, is a subset of S_F. thus

$$T_F \subseteq [T_F] \subseteq S_F. \tag{9}$$

The ILP or MILP problem (5) – (8) may now be stated as follows also:

$$\left.\begin{array}{ll} \text{Minimise} & f(X) = CX, \\ \text{subject to} & X \in T_F \end{array}\right\} \tag{10}$$

The related LP problem is:

$$\left.\begin{array}{ll} Minimise & f(X)=CX, \\ subject\ to & X \in S_F \end{array}\right\} \quad (11)$$

We state another LP problem associated with the above:

$$\left.\begin{array}{ll} Minimise & f(X)=CX, \\ subject\ to & X \in [T_F] \end{array}\right\} \quad (12)$$

We prove three theorem concerning the solutions of these problems.

Theorem 1. If an optimal solution of (11) exists and T_F is nonempty, then optimal solutions of (10) and (12) exist. Also the optimal solution of (11) is a lower bound for the optimal solutions of (10) and (12).

Proof. Let X_0 be an optimal solution of (11). Then for all X in S_F.

$$F(X_0) \leq f(X).$$

Let $Y \in T_F$. Then, from (9), $Y \in S_F$, and so

$F(X_0) \leq f(Y)$.

This means that f(Y), $Y \in T_F$, has a lower bound, and so (10) has an optimal solution. Similarly we prove that (12) has an optimal solution. The second part of the theorem also stands proved. Proved.

Theorem 2. If an optimal solution of (11) is an integer or a mixed integer vector as required by (8), then it is also an optimal solution of (10).

Proof. Let X_0 be an optimal solution of (11) satisfying (8). Then $X_0 \in T_F$ and so T_F is nonempty. Let, if possible, X_0 be not an optimal solution of (10). But, from theorem 1, an optimal solution exists. Let it be Y_0. Then $Y_0 \in T_F$ and

$$f(Y_0) < f(X_0).$$

Since $T_F \subseteq S_F$, $Y_0 \in S_F$. Also $X_0 \in S_F$. The above inequality then implies that X_0 is not an optimal solution of (11) which contradicts our hypothesis. Hence X_0 is an optimal solution of (10). Proved

Theorem 3. An optimal solution of (10) is an optimal solution of (12). Conversely, a basic optimal solution of (12) is an optimal solution of (10).

Proof. Let $X_0 \in T_F$ be an optimal solution of (10). Then for all $X \in T_F$.

$$f(X_0) \leq f(X). \qquad (13)$$

Let Y be any point in (T_F). Then Y is a convex linear combination of some points X_i, i=1,2,…,r, of T_F, that is

$$Y = \sum_{i=1}^{r} \lambda_i X_i, \lambda_i \geq 0, \sum_{i=1}^{r} \lambda_i = 1.$$

Since $X_0 \in T_F$, X_0 is also in $[T_F]$. Let Y be different from X_0, and if possible, let

$$f(Y) < f(X_0) \qquad (14)$$

$$\Rightarrow f\left(\sum_{i=1}^{r} \lambda_i X_i\right) < f(X_0),$$

$$\Rightarrow \sum_{i=1}^{r} \lambda_i \, f(X_i) < f(X_0), \text{ since } f(X) \text{ is linear},$$

$$\Rightarrow f(X_i) \sum_{i=1}^{r} \lambda_i < f(X_0),$$

$$\Rightarrow f(X_k) < f(X_0), \qquad (15)$$

where $f(X_k) = \min f(X_i)$.

But X_k, being one of the X_i's, is in T_F, and so (15) contradicts (13). Therefore (14) is not true, and consequently.

$F(Y) \geq f(X_0)$

Which means X_0 is an optimal solution of (12).

To prove the converse, let X_0 be an optimal solution of (12). Then X_0 is a vertex of $[T_F]$, and so X_0 is in T_F (chapter 1, theorem 15). Let X be any other point in T_F. The it is in $[T_F]$, and so

$$f(X_0) \leq f(X).$$

which means X_0 is an optimal solution of (10). Proved.

The above theorem leads to the conclusion that to solve an ILP or MILP problem one has only to solve the associated LP problem whose set of feasible solutions is the convex hull of the set of feasible solutions of the original problem. It is, however, not easy to find the required convex hull, and therefore the theorem provides only theoretical insight and not any practical method of solution. The practical methods generally recommended fall in two categories, commonly called (i) the cutting plane method and (ii) the branch and bound method. We proceed to discuss these methods, in each case first explaining the underlying idea with the help of numerical example of section 2.

V. 4. Example of section 2 continued.

We go back to the example of section 2 to illustrate the cutting plane method. Suppose we introduce an additional constraint in the problem which has the effect of cutting out the portion NPC from the polygon ABCDEF (Fig. 1). Since the equation of the straight line ML is $x_1 + x_2 = 4$, such a constraint is

$$x_1 + x_2 \leq 4.$$

This cuts out the optimal solution C of the LP problem without cutting out any of its integral feasible solutions. The solution to the LP problem with this additional constrain is the point N (3/2, 5/2) which again is nonintegral. Let us cut this out by introducing the constraint corresponding to the line EM which is

$x_2 \leq 2$.

This again does not cut out any of the integral feasible solutions.

With the two additional constraints, the original LP problem is now modified so that its set of feasible solutions is the polygon ABPMEF which still contains all the integral feasible solutions of the original problem. The optimal solution of this modified problem is the point M (2, 2). Since this is integral, it is the solution of the original ILP problem.

The additional constraints are called cuts. By introducing suitable cuts one by one and solving an LP problem every time, we could hope to arrive at the solution of the ILP problem. The important question now is – How to find suitable cuts.

V.5. Cutting planes

We confine our discussion to the general ILP problem (5)-(8), leaving the MILP problem for later comment. We therefore assume that X is an integer vector. The related LP problem is (5) – (7).

Let a basic optimal solutions of the related LP problem be $[x_1, x_2, \ldots\ldots, x_m, 0, \ldots\ldots 0]$, and let the corresponding canonical from of equations (6).

$x_1 + a^1_{1, m+1}\, x_{m+1} + \ldots. + a^1_{1n}\, x_n = b_1^1,$

$x_2 + a^1_{2, m+1}\, x_{m+1} + \ldots. + a^1_{2n}\, x_n = b_2^1,$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$ (16)

$x_m + a^1_{m, m+1}\, x_{m+1} + \ldots.. + a^1_{mn}\, x_n = b^1_m$

since the solution is necessarily feasible,

$$x_i = b_i^1 \geq 0,\ i=1,2,\ldots..,m.$$

If all the b_i^1 are integers, we have the solution of the ILP problem, and nothing more need be done. In general this may not be so. Let a particular b_i^1 be noninteger. The corresponding equation in (16) is

$$x_1 + a^1_{i, m+1}\, x_{m+1} + \ldots + a^1_{in}\, x_n = b_i^1. \tag{17}$$

$$\text{Let } b'_I = [b_i'] + \beta_I, \tag{18}$$

$$\text{and } a_{ij} = [a_{ij}] + \alpha_{ij},\ j=m+1,\ldots., n, \tag{19}$$

where $[b_i^1]$ is the greatest integer less than b_i^1, and $[a^1_{ij}]$ is the greatest integer less than or equal to a^1_{ij}. Then

$[b_i^1] \geq 0,\ 0<\beta_i < 1,\ 0\leq\alpha_{ij} < 1.$

Equation (17) can now be written as

$$x_i - [b_i^1] + \sum_{j=m+1}^{n} \left[a^1_{ij}\right] x_j = \beta_i - \sum_{j=m+1}^{n} \alpha_{ij}\, x_j\,,$$

This equation, being one of the constraints, must be satisfied by every feasible solution of the ILP as well as the related LP problem. For an integer feasible solution the left side should be an integer and so the right side too should be an integer. Also, since $0 \le \alpha_{ij} < 1$ and x_j, being feasible, is non-negative,

$$\sum_{j=m+1}^{n} \alpha_{ij} x_j \ge 0.$$

Hence $$\beta_i - \sum_{j=m+1}^{n} a_{ij} x_j \text{ is an integer} \le \beta_i.$$

But $0<\beta_I < 1$. Therefore for an integer feasible solution.

$$\beta_i - \sum_{j=m+1}^{n} \alpha_{ij} x_j \ge 0,$$

or $$-\sum_{j=m+1}^{n} \alpha_{ij} x_j \le -\beta_i. \qquad (20)$$

But for the optimal solution of the related problem with which we started, $x_j = 0$, $j=m+1, \ldots. n$, and so

$$\beta_i - \sum_{j=m+1}^{n} \alpha_{ij} x_j = \beta_i > 0.$$

Thus we have discovered a linear constrains (20) which is satisfied by integer solutions of the problem but cuts out the optimal solution of the LP problem provided it is nonintegral. This, therefore, provides a suitable new constraint. (20) with equality sign is the corresponding cutting plane. We note that β_i and α_{ij} in (20) are defined by (18) and (19) respectively.

We add the constraint (20) to the set of constraints (6) and solve the modified problem. If its optimal solution is integral, we stop, otherwise we again obtain a cutting plane to cut out this optimal solution but not any of the integer feasible solutions. We go on doing this till we get an integer optimal solution. It has been proved that the cutting plane method terminates in a finite number of iterations either with the integer optimal solution or with the conclusion that the given problem is not feasible.

The successively modified LP problem obtained after adding each time a constraint of the type (20) are best solved by the dual simplex method. Constraint (20) leads to the constraint equation.

$$-\sum_{j=m+1}^{n}\alpha_{ij}x_j + y = -\beta_i,$$

where y is a slack variable. We add this constraint to the simplex tableau as it stands at the optimal stage of the preceding LP problem. A basic solution of the modified problem consists of the basic solution at the preceding stage along with $y=-\beta_i$. But this solution is not feasible. It is, however, dual feasible because $c_j \geq 0$, j=m+1, …., n, at this stage. Hence we apply the dual simplex algorithm to proceed further to obtain a solution which is both primal and dual feasible and therefore optimal.

V.6. Example

We illustrate the cutting plane method explained in the preceding section through the example of section 2. Introducing the slack variables x_3, x_4, x_5, x_6 and the artificial variable x_7 (which we introduce to obtain a basic feasible solution by first solving the Phase I problem which minimizes w), the problem can be written as

$$\begin{array}{lll}
\text{Minimise } f = & -3x_1 - 4x_2, & \text{(Phase II)} \\
\text{minimise } w = & x_7, & \text{(Phase I)} \\
\text{subject to} & 2x_1 + 4x_2 + x_3 & = 13, \\
 & -2x_1 + x_2 \quad + x_4 & = 2, \\
 & 6x_1 - 4x_2 \quad\quad + x_5 & = 15, \\
 & 2x_1 + 2x_2 \quad\quad\quad - x_6 + x_7 & = 1; \\
 & x_1, x_2, \ldots, x_7 \geq 0 \text{ and integers.} &
\end{array}$$

Table 1 is the simplex tableau for a complete solution. Phase I ends after the iteration I=2 when get a basic feasible solution of the related LP problem. The end of Phase II at iteration I=4 gives the optimal solution of this problem. This is nonintegeral, and so cutting planes in the form of additional constraints are included, one at a time, in subsequent iterations till the integer optimal solution is reached.

In the optimal solution of the related LP problem $x_1 = 7/2$. The corresponding equation of type (17) is

$$x_1 + \frac{2}{16}x_3 + \frac{2}{16}x_5 = \frac{7}{2}$$

or $$x_1 + \left(0 + \frac{1}{8}\right)x_3 + \left(0 + \frac{1}{8}\right)x_5 = 3 + \frac{1}{2}$$

The required constraint, by (20), is therefore

$$\frac{1}{8}x_3 - \frac{1}{8}x_5 \le \frac{1}{2},$$

Or, after introducing the slack variable x_8,

$$\frac{2}{16}x_3 - \frac{2}{16}x_5 + x_8 = -\frac{1}{2}$$

The constraints is added to the problem as it stands at stage I=4 producing the problem at stage I=5. The dual simplex algorithm is used to proceed the further and the optimal solution of the current problem is obtained in I=6. This is also non-integral. Hence from the equation.

$$x_2 + \frac{1}{4}x_3 - \frac{1}{2}x_8 = \frac{7}{4'}$$

or $$x_2 + \left(0 + \frac{1}{4}\right)x_3 + \left(-1 + \frac{1}{2}\right)x_8 = 1 + \frac{3}{4'}$$

a second constraint is obtained as

$$-\frac{1}{4}x_3 - \frac{1}{2}x_8 \le -\frac{3}{4'}$$

Or $$-\frac{1}{4}x_3 - \frac{1}{2}x_8 + x_9 = \frac{3}{4'}$$

Again the optimal solution of the current problem is obtained by the dual simplex method (I=8). The optimal solution is not yet integral. A third cutting plane is added in I=9 which finally gives the integral optimal solution in I=10.

TABLE 1

I	Basis	Value	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
	x_3	13	2	4	1					
	x_4	2	−2	1		1				
1	x_5	15	6	−4			1			
	x_7	1	2	2				−1	1	
	$-f$	0	−3	−4						
	$-w$	−1	−2	−2				1		
	x_3	12		2	1	·		1	−1	
	x_4	3		3		1		−1	1	
2	x_5	12		−10			1	3	−3	Initial b.f.s. of related LP
	x_1	1/2	1	1				−1/2	1/2	
	$-f$	3/2		−1				3/2	3/2	
	$-w$	0		0				0	1	End of Phase I
	x_3	8		16/3	1		−1/3			
	x_4	7		−1/3		1	1/3			
3	x_6	4		−10/3			1/3	1		
	x_1	5/2	1	−2/3			1/6			
	$-f$	15/2		−6			1/2			
	x_2	3/2		1	3/16		−1/16			Optimal solution of related
4	x_4	15/2			1/16	1	5/16			LP
	x_6	9			10/16		2/16	1		
	x_1	7/2	1		2/16		2/16			Eq. giving cutting plane
	$-f$	33/2			9/8		1/8			End of Phase II

I	Basis	Value	x_1	x_2	x_3	x_4	x_5	x_6	x_8	x_9	
	x_2	3/2		1	3/16		−1/16				
	x_4	15/2			1/16	1	5/16				
5	x_6	9			10/16		2/16	1			
	x_1	7/2	1		2/16		2/16				
	x_8	−1/2			−2/16		−2/16		1		First cutting plane
	$-f$	33/2			9/8		1/8				
	x_2	7/4		1	1/4				−1/2		Eq. giving cutting plane
	x_4	25/4			−1/4	1			5/2		
6	x_6	17/2			1/2			1	1		
	x_1	3	1		0				1		
	x_5	4			1		1		−8		
	$-f$	16			1				1		
	x_2	7/4		1	1/4				−1/2		
	x_4	25/4			−1/4	1			5/2		
	x_6	17/2			1/2			1	1		
7	x_1	3	1		0				1		
	x_5	4			1		1		−8		
	x_9	−3/4			−1/4				−1/2	1	Second cutting plane
	$-f$	16			1				1		

V. 7. Remarks on cutting plane methods.

The above method of obtaining cutting planes is only one of the several methods of generating cutting planes which have been proposed by various authors and which can be found in the vast literature on integer programming. It has been proved that the cutting plane method solves the ILP problem in a finite number of steps, either giving an integer optimal solution or indicating that a feasible solution does not exist.

One disadvantage of the method is that the number of steps can be very large sometimes even in problems which apparently look simple. The number of constraints goes on increasing leading to increased volume of numerical work. Some relief can be obtained by dropping out a cutting plane from the simplex tableau once it becomes superfluous due to subsequent addition of other cutting planes. This happen when the slack variable in that cutting plane becomes a basic variable with a positive value in the simplex tableau. For example, in tables 1, let us follow the part played by the first cutting plane introduced at the stage I=5. We find that the slack variable x8 in the constraint added at stage I=5 is nonbasic for the optimal solution at stage I=6. This means that at this stage $x_8=0$ and so the optimal solution lies on the cutting plane.

$$\frac{1}{8}x_3 - \frac{1}{8}x_3 = -\frac{1}{2}$$

Later on after a second cutting plane has been introduced at I=7, x_8 appears as a basic variable with positive value at I=8. This means that the optimal solution at this stage does not lie on the first cutting plane but within the region.

$$\frac{1}{8}x_3 - \frac{1}{8}x_5 < -\frac{1}{2}.$$

It remains so throughout subsequent work. The first cutting plane plays no active part in the remaining stages of the solution. It could therefore as well be erased from the simplex tableau after I=8.

Cutting plane methods can be applied to MILP problems also. There are rules by which cutting planes for mixed integer problems can be obtained. We, however, omit these as, in general, the cutting plane method has been found to be less suitable than the other method, the branch and bound method, which we proceed to discuss.

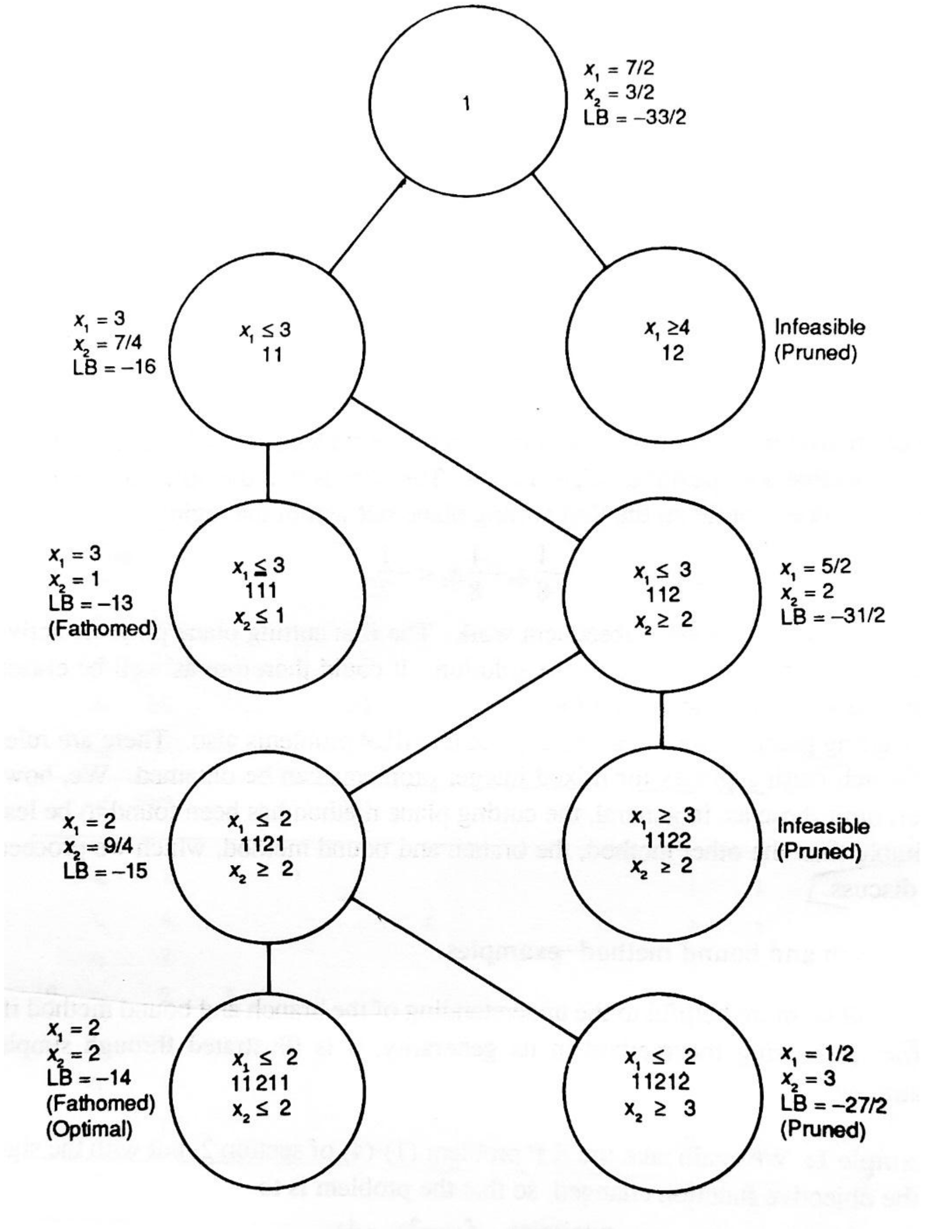
V.8. Branch and bound method – examples

It will be more helpful to the understanding of the branch and bound method if, before discussing the method in its generally, it is illustrated through simple examples.

Example 1: We again take the ILP problem (1)-(4) of section 2, but with the sign of the objective function changed, so that the problem is to

Minimize $f=-3x_1 - 4x_2$.

Dropping the constraint (4) we get the related LP problem. Its solution is f=−33/2 with x_1 = 7/2, x_2 = 3/2 (as can be easily obtained graphically or otherwise). Obviously – 33/2 is a lower bound (LB) for the objective function f of this problem. Let us call the related LP as problem 1, and say that a LB of the objective function f of this problem is – 33/2 with x_1 = 7/2, x_2 = 3/2. (We adopt this phraseology because, as we shall see later, what is essential to the branch and bound method is not the exact minimum value of the objective function but a lower bound to it). In Fig. 2 circle 1 at the top with information regarding the LB and the corresponding values of the variables indicates this situation.



Let us divide problem 1 into two subproblems, problem 11 and problems 12, by imposing the constraints $x_1 \leq 3$ and $x_1 \geq 4$ respectively on problem 1. These consolutions, the interval $3 < x, < 4$ in which $x_1 = 7/2$ lies can be left out, and further probes need be made only in regions $x1 \geq 4$ and $x1 \leq 3$. (We could, with equal justification, impose the constraints $x_2 \leq 1$ and $x_2 \geq 2$ first). This operation of replacing a problem by two subproblems is called branching. Problem 1 which may be called the parent problem has been branched into problems 11 and 12.

Solution to problem 11 can be obtained, again graphically or otherwise, as $x_1 = 3$, $x_2 = 7/7$, with $f = -16$. Following the phraseology explained above, -16 is the LB of the problem with values $x_1 = 3$, $x_2 = 7/4$. Problem 12 is easily found to be infeasible. It is therefore left out of further consideration, or, in standard branch and bound terminology, pruned. Problem 11 is further branched into two problems, 111 and 112, by imposing the additional constraints $x_2 = 7/4$. The LB of problem 111 is found to be – 13 for $x_1 = 3$, $x_2 = 1$. This is an integer solution, and is therefore a possible candidate for the optimal solution of the original ILP problem. Moreover, no other feasible integer solution of problem 111 need the be found out as the one already found gives the lowest value of f. We say that problem 111 has been fathomed. There is no need to branch it further, but its LB integer solution should be kept in view as a possible candidate for the optimal solution of the given ILP problem.

Problem 112 is found to have an LB = - 31/2 with $x_1 = 5/2$, $x_2 = 2$. Since its LB is lower than the LB of problem 111, it may be concealing integer solutions which into problem 1121 and 1122. The latter is found to be infeasible and is therefore pruned. The former gives the LB – 15 for $x_1 = 2$, $x_2 = 9/4$. Since this LB is lower than the LB of the fathomed problems 111, we branch problem 1121 into problems 11211 and 11212. The former has the LB-14 for $x_1 = 2$, $x_2 = 2$. Since this solution is integral, this problem stands fathomed, and its solution gives a possible candidate for the optimal solution of the original ILP problem is not fathomed, but since its LB is higher than he LB of the fathomed problem 11211, it cannot possibly conceal an integer solution which may be a candidate for optimality. Hence this problem is also left out of consideration or pruned.

Now all the subproblems have been fathomed or pruned or branched. The fathomed problem which gives the lowest LB for the objective function gives the optimal solution of the original ILP problem. Thus $x_1 = 2$, $x_2 = 2$, $f=-14$ is the required solution.

Example 2:

Minimise $f = 3x_4 + 4x_5 + 5x_6$,

Subject to $2x_1 + 2x_4 - 4x_5 + 2x_6 = 3$,

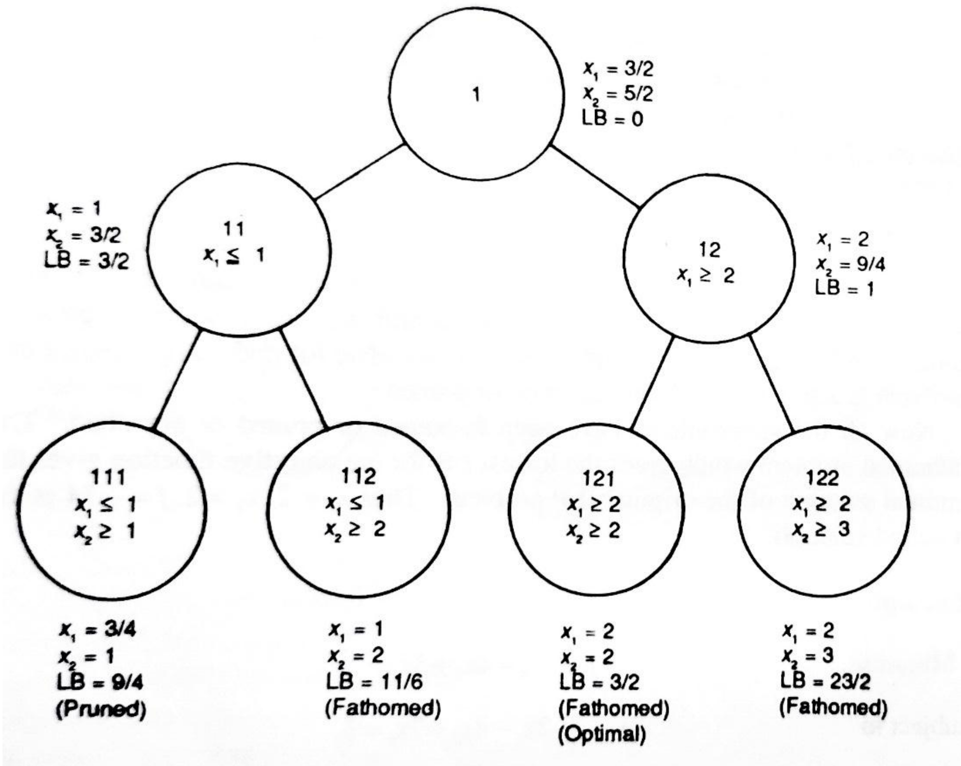
$2x_2 + 4x_4 + 2x_5 - 2x_6 = 5$,

$x_3 - x_4 + x_5 + x_6 = 4$,

$x_1, x_2, \ldots\ldots, x_6 \geq 0$, x_1, x_2 integers.

Since only two of the six variables are constrained to be integers, the problem is of mixed integer programming.

Deleting the integer constraints, we get the related LP problem, whose minimal solution is easily found as $x_1 = 3/2$, $x_2 = 5/2$, $x_3 = 4$ $x_4 = x_5 = x_6 =0$, giving the LB of f as zero. This is problem 1 of Fig. 3. Since x_1 is required to be an integer, we branch problem 1 into problems 11 and 12 by introducing respectively the constraints $x_1, \leq 1$ and $x_1 \geq 2$ indicated by the value x1 = 3/2 which lies between 1 and 2. The solution to these two problems can be found by the dual simplex method, and are shown in the figure. Since problems have optimal solutions in which the variable x_2 is non-integral, none of the problems has been fathomed. Nor any of them has been pruned (that is, not to be considered further). So both problems 11 and 12 are branched into problems 111, 112, 121, 122, with additional constraints respectively as $x_2 \leq 1$, $x_2 \geq 2$, $x_2 \leq 2$, $x_2 \geq 3$, indicated by the value $x_2 = 3/2$ in problem 11 and $x_2 = 9/4$ in problem 12. Again the four problems can be solved by the dual simplex method to give the solutions as written in the figure. Problems 112, 121 and 122 stand fathomed as the optimal solution in each case is integeral in 3/2, the LB of problem 121. Hence it is pruned. Among the fathomed problems the least LB is provided by problem 121. The therefore gives the solution of the original problem.



V.9. Branch and bound method – general description

As the name implies and as is also clear from the above examples, the branch and bound method consists of two strategies, alternately followed, till the desired solution is obtained. One strategy consists in branching a problem into two subproblems, and the other in solving each of the two subproblems to obtain the minimum or a suitable lower bound of the objective function, if the original problem is to minimise the objective. (If the problem is to maximise an objective function, the lower bound is replaced by the upper bound).

Let the problem be of the MILP type in which the variables $X_j, j = 1,2,\ldots..,r$, are integers and $j = r + 1 \quad n$, are real numbers. The problem of

ILP, by the branch and bound method, is only a special case of MILP, with $r = n$, and needs no separate discussion.

We start with the related LP problem, hereafter designated as problem 1, and solve it to obtain a lower bound of its objective function. Let us suppose, for the present, that it is the actual minimum that we are able to determine. We presume that this minimum and the corresponding optimal (minimal) solution can be found without much difficulty. If the optimal solution happens to satisfy the integer constraint also, it is the optimal solution of the given MILP, and nothing more need be done. If not, then the value of at least one of the variables x_j, j =1,2,..., r, in that optimal solution is not integral. Let x_P be one such variable, and at the optimal let $x_P = b$, where b is not an integer. Let [b] be the largest integer less than b. Since b, being feasible, is non-negative, [b] is also non-negative.

Formulate two subproblems, designed as problems 11 and 12, by imposing on problem 1 the additional constraints $x_p \leq [b]$ and $x_p \geq [b] + 1$ respectively. This operation is called *branching.* In effect, the set of feasible solutions of the MILP is partitioned into two subsets, and the optimal solution which we are seeking is in one subset or the other, provided it exists.

Each of the two subproblems 11 and 12 is now treated as an independent problem, and subjected to the same operation as problem 1, namely, obtaining the minimum of the objective function, and then, if necessary, branching. This 'branch and bound' process is continued through resulting subproblems which fan out from problem 1 as a *tree.* Branching terminates when any of the following three conditions arise.

i. The subproblem yields an optimal solution which satisfies the integer constraint on all the variables xj,j= 1,2,.., r, the subproblem is then said to have been *fathomed.*

ii. The optimum (minimum) value of the objective function in the subproblem is not lower than the minimum value of the objective function in a subproblem which has been fathomed.
iii. The subproblem turns out to be infeasible.

The reasons to terminate branching in the above three cases are as follows. In *case* (i) the optimal solution with required integer constraint out of the subset of feasible solutions of that subproblem has been obtained, and no further probe in that subproblem is necessary. In case (ii), since an integer optimal solution which is lower than the optimal solution of the subproblem has been discovered in the set *of* feasible solutions of another subproblem, the former subproblem needs no further probe, as it cannot be concealing a solution which would make the objective function lower than what has been discovered in the latter subproblem. In case (iii) the *subproblem* obviously cannot contain the required solution. Subproblems falling under cases (ii) and (iii) arc said to be *pruned.*

When all the subproblems obtained through branching have been either fathomed or pruned, the branch and bound algorithm terminates. The fathomed subproblem with the lowest minimum gives the answer to the original problem.

The branch and bound method is partially enumerative. The set of feasible *solutions* is successively partitioned into subsets and those subsets which cannot *contain* the optimal solution are deleted from further consideration. The criterion for deletion is provided by the lower bound of the objective function for the feasible values in that subset.

It is sometimes difficult or strenuous to determine the minimum of the objective function in a problem. The reason why in the branch and bound method the stress is on a lower bound and not the minimum of the objective function is that any suitable lower bound and not necessarily the exact minimum is needed to decide whether a subset of feasible solutions should be further probed or deleted. Of course the closer a lower bound is to the minimum the better, but one has to balance the time spent in determining the minimum against the time spent in going ahead with further branching. If a suitable lower bound is more easily determined than the minimum, it is worthwhile saving time here. Branching being easier than finding the minimum, bulk of the total time spent in solving a problem by the branch and bound method is spent in the latter operation, and so whatever time can be saved on it should be saved. It may result in more branching, but the overall effort is less.

There are several strategies recommended for determining a lower bound. We shall briefly mention only two of them. The detailed discussion would be omitted. One method consists in ignoring the constraints which appear to be difficult, and minimising the objective function subject to the remaining constraints. This minimum is certainly not higher than the minimum under all the constraints, and so can serve as the required lower bound. Another method is to construct another objective function which is not greater than the given objective function for any feasible solution of the original problem, and determination of whose minimum is comparatively easier than that of the original function.

CHAPTER VI

SENSITIVITY ANALYSIS

VI. 1. Introduction

In the preceding chapters we primarily discussed methods for solving linear programming problems. However, solving an LP in itself is not the end of the story. In most real life problems we want to find not only an optimal solution but also to know as to what happens to this optimal solution when changes are made in the initial system. It would be preferable to determine the effect of these changes on the optimal solution without having to solve a modified problem from the very beginning. In sensitivity analysis (also called post-optimality analysis) we develop methods to do this. A more general problem is to study the effects on the optimal solution of an *LP* as some parameter of it undergoes continuous change in its value. The procedures developed for doing this are known as parametric programming techniques.

In linear programming our aim so far has been to get as large (or small) a value of the objective function as is possible without violating any of the constraints. It may happen that in doing so other considerations which may also be important are ignored. In many practical problems, instead of maximizing or minimizing the objective function, it may be considered better to be satisfied with setting up a certain value of the objective function as a reasonable goal, and then try to achieve it as closely as possible. This approach is known as goal programming.

There can also be multiobjective linear programming problems in which it is desirable to optimize simultaneously more than one objective function satisfying the same set of constraints. The objectives may be conflicting, and it may not be possible to find a solution that accomplishes their simultaneous optimization. But one may still try to get the *best* solution, defining the best in some satisfactory manner. One can also visualize multiobjective goal programming problems in which different goals are set for different objective functions, it being desired that the objective functions achieve these goals as closely as possible.

VI. 2 Sensitivity Analysis

In an LP the optimal solution is dependent on the values of the cost coefficients c_j, the constants b_i, occurring on the right side of the constraint equations, and the coefficients a_{ij} in the constraints. In real life problems the values of these coefficients are seldom known with certainty because many of them are functions of some uncontrolled parameters. For instance, the future demands, the cost of raw materials, or the cost of eneigy resources cannot be accurately predicted. Hence the problem is not satisfactorily solved with the mere determination of the optimal solution. Each variation in the values of the data coefficients changes the problem which may affect the optimal solution found earlier. However, it is not always necessary to solve the whole problem afresh to determine the new optimal solution. In the following sections we discuss methods of starting out from the optimal solution already obtained to determine the new optimal solution under the following modifications.

i. Changes in the values of b_i;
ii. Changes in the values of c_j;
iii. Changes in the values of a_{ij};
iv. Introduction of new variables;
v. Introduction of new constraints;
vi. Deletion of certain variables;
vii. Deletion of some constraints.

VI. 3. Changes in b_i

For the LP problem:

Minimize f(x) = CX, subject to AX = B, $X \geq 0$, (1)

let the optimal basis be

$$X_0 = [x_1\ x_2\ \ldots,\ x_m]^1,\ X_0 \geq 0,$$

x_{m+1}, x_{m+2}, …., x_n being the nonbasic variables for this solution. The corresponding relative cost coefficients $\overline{c_j}$ given by

$$\overline{c_j} = c_j - \sum_{i=1}^{m} c_i\, \bar{a}_{ij}\ ,\ j=1,2,\ldots,n, \qquad (2)$$

are all nonnegative. Also since the nonbasic variables have zero values in the optimal solution, the constraints in (1) reduce to

$$A_0X_0=B,$$

So that

$$X_0=A_0^{-1}B,$$

Where

$$A_0 = \begin{bmatrix} a_{11} & a_{12} & \ldots. & a_{1m} \\ a_{21} & a_{22} & \ldots. & a_{2m} \\ . & . & . & . \\ . & . & . & . \\ a_{m1} & a_{m2} & \ldots & a_{mm} \end{bmatrix}$$

Let B change to B + $\triangle$ B where $\triangle B = [\triangle b_1, \triangle b_2 \ldots \triangle b_m]'$, everything else in (1) remaining the same. Then the new values of the variables of the earlier optimal basis are given by

$$X_0 + \triangle X_0 = A_0^{-1} (B + \triangle B)$$

Now if $A_0^{-1} (B + \triangle B) \geq 0$ (3)

the original optimal basis continues to be feasible for the new problem. It will also be an optimal basis if all the relative cost coefficients of the modified problem for this basis are nonnegative. The relative cost coefficients, by (2), are independent of B, and so remain unchanged. So they remain nonnegative. Hence, if (3) holds, the original optimal basis is still optimal. The new value of the objective function is given by f $(X_0+\triangle X_0)$.

If, however, $\triangle B$ is such that (3) does not hold, that is, the new values of the variables in the basis Xo are not all nonnegative, then the new solution $X_0+\triangle X_0$ is not feasible. In such a case we may replace the values of the basic variables in the earlier optimal solution by their new values and proceed further by the big M or the two-phase or the dual simplex method to obtain the new optimal solution. If too many components of $X_0+\triangle X_0$ are negative, it may be more economical to solve the new problem *ab initio.*

Example: Consider the problem of section 13, chapter 3. Its optimal solution, as obtained there, is given in table 1.

Basis	B	P_1	P_2	P_3	P_4	P_5	P_6
x_1	2/3	1				-4/3	1/3
x_2	1/3		1			1/3	-1/3
x_3	13/3			1		7/3	-1/3
x_4	11/3				1	2/3	1/3
f	–13/3					11/3	1/3

From this we find that

$X_0 = [x_1\ x_2\ x_3\ x_4]' = [2/3\ 1/3\ 13/3\ 11/3]'$

Also, from the original problem,

$C_0 = [4\ 5\ 0\ 0]$, $B = [6\ 5\ 1\ 2]'$ and,

$$A_0 \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \end{bmatrix}, A_0^{-1} \begin{bmatrix} 0 & 0 & 4/3 & -1/3 \\ 0 & 0 & -1/3 & 1/3 \\ 1 & 0 & -7/3 & 1/3 \\ 0 & 1 & -2/3 & -1/3 \end{bmatrix},$$

We proceed to consider three cases of changes in B.

(i) Let $B + \triangle B = [7\ 4\ 1\ 2]'$. (Notice that only b_1 and b_2 change).

Then A_0^{-1} (B + $\triangle B$) =[2/3 1/3 16/3 8/3]'$\geq$0. Thus the original optimal basis remains feasible and hence optimal. The new optimal values of the basic variables are x_1 = 2/3, x_2 = 1/3, x_3 = 16/3, x_4 = 8/3, and the new optimal value of the objective function is f= 13/3 which is the same as before. This is so because the values of only those basic variables have changed which do not occur in the original form of the objective function. In fact, from the form of A_0^{-1} it follows that any change in the value of b_1 or b_2 will not affect the values of x_1 and x_2, and therefore the value of the objective function, as long as the new basis is feasible.

(ii) Let $B + \triangle B = [6\ 5\ 1\ 1]^1$.

In this case $A_0^{-1} (B + \triangle B) = [1\ 0\ 4\ 4]^1$.

Thus the original optimal basis is still feasible with the optimal values of the basic variables as x_i = 1, x_2 = 0, x_3 = 4, x_4 = 4. However, the optimal value of the objective function is now 4 which is different from the earlier value.

(iii) Let $B + \triangle B = [6\ 5\ 2\ 1]^1$.

Now $A_0^{-1} (B + \triangle B) = [7/3\ {-1/3}\ 5/3\ 10/3]^1$.

In this case the original optimal basis X_0 becomes infeasible. Since the relative cost coefficients remain unchanged and so non negative, from this point onwards we may proceed by the dual simplex method to obtain the optimal solution of the modified problem. The new values of the basic variables are x_1 = 7/3, x_2 = –1/3, x_3 = 5/3, x_4 = 10/3, and the new value of f is 23/3. With these values in the simplex table and doing one iteration of the dual simplex method, we get the new optimal solution as x_1 = 2, x_2 = 0, x_3 = 2, x_4 = 3, x_5 = 0, x_6 = 1; f=8.

VI. 4. Changes in c_j

If c_j are changed to c_j^*, everything else in the problem remaining the same, then the changed relative cost coefficients of the non basic variables are given by (2) as

$$\overline{c_j^*} = c_j^* - \sum_{i=1}^{m} c_i^* \overline{a}_{ij},\ j = m+1, \ldots, n.$$

These may not all be nonnegative. If $\overline{c_j^*}$ is negative for some *j*, then this would mean that the basic feasible solution X_0 which was earlier optimal is now not optimal. So from this point onwards.further iterations may be done with c_j in the simplex table replaced by $\overline{c_j^*}$, $j = m+1, \ldots, n$, to obtain the new optimal solution.

If c_j^* are such that all $\overline{c_j^*}$ are nonnegative, then the original optimal basis remains optimal and the optimal values of the basic variables also remain unchanged. Optimum value of the objective function, however, will be different since the cost coefficients have changed. In the particular case when $c_j^* = c_j$ for the basic variables, even the value of the objective function will not change.

Example: Suppose in the example of section 3 $c_1 = 4$, $c_2 = 5$ are changed to $c_1^* = 5$, $c_2^* = 6$. Then, using (2),

$$\overline{c_5^*} = -(-4/3)c_1^* - (1/3)c_2^* = (1/3)(4c_1^* - c_2^*) = 14/3$$

$$\overline{c_6^*} = -(1/3)c_1^* - (-1/3)c_2^* = (1/3)(c_2^* - c_1^*) = 1/3 \tag{4}$$

As $\overline{c_5^*}$ and $\overline{c_5^*}$ are both nonnegative, the original optimal basis is still optimal, and there is no change in the optimal values of the basic variables. The new value of the objective function is $f = c_1^* x_1 + c_2^* x_2 = 16/3$. From (4) it is evident that $\overline{c_5^*}$ and $\overline{c_6^*}$ will be nonnegative as long as $c_1^* \leq c_2^* \leq 4c_j^*$. So the original optimal basis will remain optimal so long as the changed cost coefficients satisfy this condition.

Next suppose that c_1 and c_2 change to $c_1^* = 5$, $c_2^* = 1$. Then $\vec{c}_5^* = 19/3$ and $\vec{c}_6^* = -4/3$. Since $\vec{c}_6^*$ is negative, the original optimal basis cases to be optimal. Replacing the previous value 11/3 of $\vec{c}_5$ by $\vec{c}_5^* = 19/3$, and the value 1/3 of $\vec{c}_6$ by $\vec{c}_6^* = -4/3$, and the original entry 23/3 for the value of f by its new value 11/3 in table 1, we can do one more iteration to obtain the new optimal solution $x_1 = 0$, $x_2 = 1$, $x_3 = 5$, $x_4 = 3$, $x_5 = 0$, $x_6 = 2$; $f = 1$).

VI. 5. Changes in a_{ij}

If the changes arc in a_{ik}, where x_k is a nonbasic variable of the original optimal solution, then the modified value $\vec{c}_k^*$ c_k of $\bar{c}_k$ may be found by using equation.

$$\vec{c}_k = c_k + \sum_{i=1}^{m} a_{ik}^* \pi_i \qquad (5)$$

where a_{ik}^* are new values of a_{ik}. If $\bar{c}_k^* \geq 0$, the original optimal solution is still optimal. If $\bar{c}_k^* < 0$, then further iterations of the simplex method may be done to find the new optimal solution. For this purpose the values of $\bar{a}_{ik}^*$, the modified values of a_{ik}^* in the original optimal table, may be calculated by the formula.

$$[\bar{a}_{1k}^* \bar{a}_{2k}^* \ldots \bar{a}_{mk}^*]' = A_0^{-1}[a_{1k}^* a_{2k}^* \ldots a_{mk}^*]'.$$

If x_k is a basic variable in the original optimal solution, then the procedure may be as follows.

Introduce a new variable x_k^* in the system with coefficients $\bar{a}_{ik}^*$ and $c_k^* = c_k$. In this new problem treat the original variable x_k *as* an artificial variable and use phase I of the two-phase simplex method to eliminate it, and then proceed to phase II to get the new optimal solution.

Example: To illustrate the above procedures, we take the example of the section of linear programing. There the problem has been solved by the dual simplex method, and its optimal solution is given in table 2.

Table 2

Basis	B	P_1	P_2	P_3	P_4	P_5	P_6
x_3	2	1	2	1		–1	
x_4	1	3	2		1	–2	
x_6	8	4	5			–2	1
f	–4	1	1			2	

From the above, the optimal basis is

$X_0 = [x_3\ x_4\ x_6]^1 = [2\ 1\ 8]^1$, with $C_0 = [2\ 0\ 0]$,

and $A_0 = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ $Hence\ A_0^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

and, the simplex multipliers are

$\Pi' = -C_0 A_0^{-1} = [0 \text{-} 2\ 0]$ (7)

We consider three cases of changes in a_{ij}.

i. Let the values of a_{11}, a_{21}, a_{31} change from (–1,1,2) to (1, –1, –2). The changes are in the coefficients of x_1 which is a nonbasic variable. In this case

$$[a_{11}^*\ a_{21}^*\ a_{31}^*] = [1 - 1 - 2],$$

and so $\vec{c_1} = c_k^* + \sum_{i=1}^{3} a_{i1}^* \pi_i$ = 3+1 x 0 + (-1) (-2) + (-2) x 0 = 5 > 0.

Therefore the original optimal solution is still optimal.

ii. Suppose the changed values of a_{11}, a_{21}, a_{31} are 1,2, - 2.

In this case $\left[a^*_{11}\ a^*_{21}\ a^*_{31}\right]=\left[1 \quad 2 \quad -2\right]$, and so $\overline{c^*_1}=-1$. The original optimal solution, therefore, ceases to be optimal. The entries in the column for P_1 in table 2 are modified to

$$[\overline{a}^*_{11} \quad \overline{a}^*_{21} \quad \overline{a}^*_{31}]' = A_0^{-1}[a^*_{11} \quad a^*_{21} \quad a^*_{31}]' = A_0^{-1}[1 \quad 2 \quad -2]' = [2 \quad 3 \quad 2]'$$

The modified table appears as

Basis	B	P_1	P_2	P_3	P_4	P_5	P_6
x_3	2	2	2	1		–1	
x_4	1	3	2		1	–2	
x_6	8	2	5			–2	1
f	–4	–1	1			2	

On performing one iteration of the simplex method we find the new optimal solution to be $x_1 = 1/3$, $x_2 = 0$, $x_3 = 4/3; f=11/3$.

iii. Let a_{13}, a_{23}, a_{33} change from (2,1,-2) to (1, -1,2). Since the change is in the coefficients of x_3 which is a basic variable in the original optimal solution, to study the effect of this change we introduce a new variable x^*_3 with coefficients $a^*_{13}=1, a^*_{23}=-1, a^*_{33}=2$ *and* $c^*_3=c_3=2.$

From these $\overline{c^*_3}=4, \overline{a^*_{13}}=-1, \overline{a^*_{23}}=-3,\ a^*_{33}=0.$ Now the simplex table (given below) for finding the new optimal solution by the two- phase method, treating the original variable x_3 as an artificial variable, is obtained by introducing a column P^*_3 in table 2. The objective function g for phase I is given by $g=x_3$, which in terms of non basic variables is $g=2-x_1-2x_2 + x_5 + x^*_3$.

Basis	B	P_1	P_2	P_3	P_4	P_5	P_6	P_3^*
x_3	2	1	2	1		–1		–1
x_4	1	3	2		1	–2		–3
x_6	8	4	5			–2	1	0
f	–4	1	1			2		4
g	–2	–1	–2			1		1

Proceeding from here onwards, the new optimal solution is found to be $x_1=0$, $x_2=5/4$, x_3 (or x_3) = ½, $x_4 = 0$, $x_5 = 0$, $x_6 = 7/4$; f = 29/4.

VI. 6. Introduction of new variables

Let- the new variables be x_{n+k}, k = 1,2,3,..., and their coefficients be $a_{i,\ n+k,}$ i=1,2,…m, and c_{n+k}. Since the number of constraints remains the same, the number of basic variables also remains the same, and so the original optimal solution gives a basic feasible solution of the new problem. The relative cost coefficients corresponding to the newly introduced cost coefficients c_{n+k} would be given by (5) as

$$\bar{c}_{n+k} = c_{n+k} + \sum_{i=1}^{m} a_{i,n+k}\pi_i, k = 1, 2, 3, \ldots$$

If all these are nonnegative, the original optimal solution remains optimal for the new problem. If not, then from this point onwards iterations may be done to obtain the new optimal solution taking into account the new variables.

Example: Suppose in the example of section 3 we introduce a new variable x_7 such that (i) $c_7 = 2$, $a_{17} = 1$, $a_{27} = -1$, $a_{37} = -3$, $a_{47} = 3$, and (ii) $c_7 = 2$, $a_{17} = 1$, $a_{27} = -1$, $a_{37} = 3$, $a_{47} = 3$, and wish in each case to determine the new optimal solution. For the optimal basis of the original problem (table 1).

$X_0 = [x_1\ x_2\ x_3\ x_4]^1 = [2/3\ \ 1/3\ \ 13/3\ \ 11/3]^1$.

the simplex multipliers, by (7), are

$\Pi' = -C_0 A^{-1}{}_0 = -[4\ 5\ 0\ 0]\ A^{-1}{}_0 = [0\ 0\ \ -11/3 - 1/3].$

Hence, by (5), the value of $\overline{C_7}$ corresponding to c_7 for case (i) is

$$\overline{c_7} = c_7 + \sum_{i=1}^{4} a_{i7}\,\pi_i = 12 > 0.$$

Therefore in case (i) the original optimal basis remains optimal, and the optimum value of the objective function remains unchanged.

In case (ii), proceeding similarly, $\bar{c}_7 = -10$. This being negative, the original optimal basis is now not optimal, and further iterations are necessary to get the new optimal. The starting table for finding the new optimal solution will be the same as of the original optimal solution, (table 1), with an additional column P_7 in which $\left[\bar{a}_{17}\ \bar{a}_{27}\ \bar{a}_{37}\ \bar{a}_{47}\right]^1 = A^{-1}0\ [a_{17}\ a_{27}\ a_{37}\ a_{47}]^1 = [3\ 0\ \ -5\ \ -4]^1$ and $\bar{c}_7 = -10$.

Thus the starting table for further iterations is

Basis	B	P_1	P_2	P_3	P_4	P_5	P_6	P_7
x_1	2/3	1				–4/3	1/3	3
x_2	1/3		1			1/3	-1/3	0
x_3	13/3			1		7/3	-1/3	-5
x_4	11/3				1	2/3	1/3	-4
f	–13/3					11/3	1/3	-10

The new optimal solution after two iterations turns out to be $x_1 = x_2 = 0$, $x_3 = 16/3$, $x_4 = 17/3$, $x_5 = 1$, $x_6 = 0$, $x_7 = 2/3$; $f = 4/3$.

VI. 7. Introduction of new constraints

If K is the set of feasible solutions of the original problem and K' the set of feasible solutions of the modified problem obtained by introducing new constraints, then $K' \subseteq K$. If the original optimal solution X_0 satisfies the new constraints, then X_0 is in K^1, and since f (X_0) is minimum in K, it is also minimum in K^1. In this case, therefore, the original optimal solution continues to be optimal.

If some or all of the new constraints are violated by X_0, then the problem has to be solved further by taking into consideration the new constraints. Each new constraint in the form of an inequality gives rise to a slack variable, and, if necessary, also an artificial variable. For a constraint in the form of an equation, an artificial variable may be introduced. A start is made with the feasible basis consisting of the variables in the original optimal solution and the slack or artificial variables (as the need be) of the new constraints. The problem may now be solved by the two-phase or the big M method. If all the additional constraints are inequalities, the problem may also be solved, without introducing artificial variables, by the dual simplex method.

Example: Let us introduce the additional constraint $3x_1 - 2x_2 \leq 2$ in the example of section 3. The original optimal solution, $x_1 = 2/3$, $x_2 = 1/3$, does not violate this constraint. Hence it continues to be the optimal solution of the modified problem.

However, if the new constraint is $3x_I - 2x_2 \geq 2$, the situation becomes different. The original optimal solution, (table 1) $x_1 = 2/3$, $x_2 = 1/3$, violates this constraint. In order to obtain the new optimal solution, we introduce the slack variable x_7 in the new constraint, and write it as

$$3x_1 - 2x_2 - x_7 = 2.$$

Eliminating the basic variables x_1, x_2 from this equation with the help of the first two equations in the original optimal table, we put this equation as

$$-\frac{14}{3}x_5 + \frac{5}{3}x_6 + x_7 = -\frac{2}{3}$$

and introduce it in table 1 as follows.

Basis	B	P_1	P_2	P_3	P_4	P_5	P_6	P_7
x_1	2/3	1				–4/3	1/3	
x_2	1/3		1			1/3	-1/3	
x_3	13/3			1		2/3	1/3	
x_4	11/3				1	2/3	1/3	
x_7	-2/3					-14/3	5/3	1
f	–13/3					11/3	1/3	

Doing one iteration of the dual simplex method, we obtain the new optimal solution as $x_1 = 6/7$, $x_2 = 2/7$; $f=34/7$.

VI. 8 Deletion of variables

If the deleted variable is a nonbasic variable or a basic variable with a zero value in the optimal basis, then the original optimal solution remains unchanged, because the zero value of the variable in the optimal solution makes the variable nonexistent in effect.

If the variable to be deleted is a basic variable with positive value in the optimal solution, its removal will affect the optimal solution. To obtain the new optimal solution, we should delete from the original optimal table the column corresponding to the deleted variable. Also this variable should be dropped from the basis column. This leaves the equation against the deleted basic variable in a form which is not canonical, and the number of basic variables in the system one short. We may now introduce an artificial variable in this equation, and proceed to obtain the solution by the two-phase or the big *M* method. As an alternative, another approach is also possible which involves the dual simplex method. After dropping the deleted variable from the table, the sign of all the entries of the row corresponding to that variable are changed. This leaves the equation essentially unchanged. Then a new basic variable is introduced in this equation with+1 as its coefficient. This makes the new basis, which includes this variable, infeasible, but the relative cost coefficients remain unchanged as nonnegative. Therefore from here onwards the dual simplex method may be employed to obtain the new optimal solution, with the new variable becoming nonbasic, which can then be dropped without affecting the optimal solution.

Example: In the example of section 5, if we delete the variable x_x which is non-basic in the optimal solution, (table 2), the modified problem has the same optimal solution.

However, if x_3 is deleted, then in table 2 column P_3 disappears and from the basis column x_3 goes. Following the second method suggested above, we change the signs of all entries in the row which corresponded to x_3. Then introduce a new variable x_7 with coefficient +1 in that row, treating it as a basic variable. Thus we get the following table, from which after two iterations of the dual simplex method we get the new optimal solution (ignoring the entries in column P_7) as $x_1 = 0$, $x_2 = 3/2$; f= 15/2.

Basis	B	P_1	P_2	P_4	P_5	P_6	P_7
x_7	–2	–1	–2		1		1
x_4	1	3	2	1	–2		
x_3	8	4	5		–2	1	
f	–4	1	1		2		
x_2	3/2	–1/2	1	–1/2			–1
x_5	1	–2		–1	1		–1
x_6	5/2	5/2		1/2		1	3
f	–15/2	11/2		5/2			3

VI. 9. Deletion of constraints

If the constraint to be deleted is such that its slack variable has a positive value in the optimal solution, then its deletion leaves the optimal solution unchanged.

This is so because the constraint is not being satisfied as an equality by the optimal solution, and therefore it is ineffective in determining the optimal solution. There are other constraints, those which are satisfied as equations, which determine it. Therefore the ineffective constraint may be deleted from the problem without doing any damage to the optimal solution.

If the constraint to be deleted has zero value for its slack variable in the optimal solution, that is, if it is being satisfied as an equality, then the modified problem may have a different optimal solution. Let the constraint to be deleted be of the type

$$\sum_{j=1}^{n} a_{kj} x_j \leq b_k,$$

so that after introducing the slack variable it becomes

$$\sum_{j=1}^{n} a_{kj} x_j + u_k = b_k,\ u_k \geq 0.$$

There are two ways of looking at the process of deletion of this constraint. The obvious one is to delete it from the problem, but in that case we shall have to solve the problem *ab initio.* The other is to say that

$$\text{either } \sum_{j=1}^{n} a_{kj} x_j \leq b_k \text{ or } \sum_{j=1}^{n} a_{kj} x_j \geq b_k.$$

(Notice that it is one constraint *or* the other, not one *and* the other.) This also, in effect, removes the constraint. The two alternative constraints can be combined into a single equation by introducing a slack variable which is not restricted in sign:

$$\sum_{j=1}^{n} a_{kj} x_j + s_k = b_k,\ s_k \text{ unrestricted in sign},$$

which is equivalent to

$$\sum_{j=1}^{n} a_{kj}\, x_j + u_k - v_k = b_k,\ u_k \geq 0,\ v_k \geq 0.$$

u_k can be identified with the slack variable already occurring in the original constraint, and so now we have only to introduce another variable v_k with coefficient -1 in the original constraint to get the modified problem. The problem so modified is equivalent to the problem obtained by deleting the constraint from the original problem. The problem with the additional variable can now be solved by the method of section 6.

Similarly, if the constraint to be deleted is of the type

$$\sum_{j=1}^{n} a_{kj}\, x_j \geq b_k$$

or, on introducing the slack variable,

$$\sum_{j=1}^{n} a_{kj}\, x_j - v_k = b_k,\ v_k \geq 0,$$

we have to introduce a variable u_k with coefficient +1, so that the constraint becomes

$$\sum_{j=1}^{n} a_{kj}\, x_j + u_k - v_k = b_k,\ u_k \geq 0,\ v_k \geq 0,$$

which, in effect, deletes the constraint.

Finally, if the constraint to be deleted is of the type

$$\sum_{j=1}^{n} a_{kj}\, x_j = b_k,$$

we may introduce two new variable, u_k, y_k, with coefficients +1 and -1 respectively, in the constraint to get the desired effect of deleting the constraint.

Example: In the optimal solution of the example of section 5, (table 2), the variables x_4 and x_6, which are respectively the slack variables of the first and the third constraint of the original problem, are positive. Hence the deletion of the first or the third constraint from the problem leaves the optimal solution unchanged. But the slack variable x_5 of the second constraint is zero in the original optimal solution. Therefore if the second constraint is deleted, the optimal solution will change. Since this constraint is of $\geq$ type, to determine the optimal solution of the modified problem, we introduce a new variable x_7 with coefficient +1, in addition to the old slak variable x_5 (with coefficient —1), to get the constraint

$$x_1+2x_2+x_3-x_5+x_7 = 2.$$

The other two constraints remain the same. The coefficients of x_7 in the three constraint equations and the objective function are $a_{17} = 0$, $a_{27} = 1$, $a_{37} = 0$, $c_7=0$. The corresponding values in the original optimal solution, (table 2), can be calculated to be $\bar{a}_{17} = 1, \bar{a}_{27} = 2, \bar{a}_{37} = 2, \bar{c}_7 = -2$, so that the starting simplex table for finding the new optimal solution is

Basis	B	P_1	P_2	P_3	P_4	P_5	P_6	P_7
x_3	2	1	2	1		–1		1
x_4	1	3	2		1	–2		2
x_3	8	4	5			–2	1	2
f	–4	1	1			2		–2

One iteration on the above table gives the new optimal solution as x, = 0, $x_2 = 0$, $x_3 = 3/2$; $f = 3$. It can be verified that this solution satisfies the first and the third constraint but not the second, as it now stands deleted.

VI. 10. Parametric linear programming

So far in this chapter we discussed the effect of changes in the values of the input data of an *LP* problem on its original optimal solution. The changes considered were discrete. We shall now assume that the coefficients in the problem vary continuously as a function of some parameter. The analysis of the effect of this functional dependence, hereafter called parametric variation, on the optimal solution of the problem, is called parametric linear programming. Parametric variation can be linear or nonlinear. The nonlinear case will not be considered here, as the computations in that case become too cumbersome.

In the subsequent sections we shall consider linear parametric variations in (i) the cost coefficients c_j, (ii) the right hand entries b_i of the constraints, (iii) the coefficients a_{ij} and (iv) c_j, b_i and a_{ij} simultaneously.

Parametric linear programming is essentially based on the same concepts as sensitivity analysis. Assuming that the coefficients which are varying are linear functions of a parameter λ, the general strategy adopted is the following. We first compute the optimal solution for $\lambda = 0$. Then using optimality and feasibility conditions, we find the range of values of λ for which this optimal solution remains optimal and feasible. Suppose this range is $(0, \lambda_1)$. This means that any increase in the value of λ beyond λ_1 will make the present optimal solution nonoptimal or infeasible. At $\lambda=\lambda_1$ we determine a new optimal solution and find the range (λ_1, λ_2) of the values of λ for which this new optimal solution remains feasible and optimal. The process is repeated at λ_2 and continued till a value of λ is reached beyond which either the optimal solution does not change or does not exist. A similar strategy is adopted for investigating the effect of variations for the negative values of λ.

CHAPTER VII

FLOW AND POTENTIAL IN NETWORKS

VII. 1. Introduction

Networks are familiar diagrams in electrical theory; they are easily visualized in transportation or communication systems like roads, railways or pipelines, nerves or blood vessels. A large variety of mathematical problems are presented by networks, ranging from puzzles for children to intricate problems challenging mathematicians. Many problems, particularly those which involve sequential operations or different but related states or stages, are conveniently described diagrammatically as networks. Sometimes a problem with no such apparent structure assumes a mathematical form which is best understood and solved by interpreting it as a network.

A network, in its more generalized and abstract sense, is called a graph. In this chapter we shall discuss some linear programming problems of such special forms that the ideas of graph theory help in their solution.

VII. 2. Graphs: Definition and notation.

A *graph $G(V, U)$* or simply G (when there is no ambiguity) is defined as a set V of elements v_j, j=1,2,…..,n, which can be represented as points, and a set U of pairs (v_j, v_k). $v_j, v_k \in V$, which can be represented as arcs joining points of V. The elements of V are called *vertices* and the elements of U *arcs.* We shall denote the elements of U as either u_i, i = 1,2,……,m, or as (v_j, v_k).

If (v_j, v_k) are ordered pairs, we represent them by *directed arcs,* that is, arcs carrying arrow marks on them denoting the direction v_j to v_k A graph with directed arcs is called a *directed graph.* Unless otherwise stated, we shall assume that a graph $G(V, U)$ is directed.

The graph is said to be *finite* when V and *U* are finite sets. We shall restrict our discussion to finite graphs only.

We shall denote a graph diagrammatically as shown in Fig. 1. The vertices are shown as small circles, with vertex v_j denoted as *j*. The examples given in this section to explain the defined terms refer to this figure.

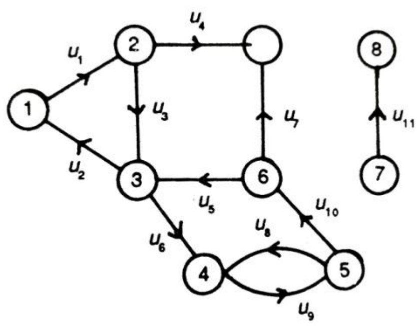


Fig. 1

An arc (directed or undirected) is said to be *incident with* a vertex which it joins to some other vertex. It *connects* the two vertices. The directed arc $u_i = (v_j, v_k)$ is said to be *incident from* or *going from* v_j and *incident to* or *going to* v_k; v_j is called the *initial* vertex and v_k the *terminal* vertex of the arc (v_j, v_k).

A *subgraph* of *G(V, U)* (Fig. 1) is defined as a graph $G_1(V_1, U_1)$ with $V_1 \subseteq V$ and U_1 containing all those arcs of *G* which connect the vertices of G_1. For example, in the figure, if $V_1 = \{v_1, v_2, v_3\}$ and $U_1 = \{u_1, u_2, u_3\}$, then $G_1(V_1, U_1)$ is a subgraph of *G*. A *partial graph* of *G(V, U)* is a graph $G_2(V, U_2)$ which contains all the vertices of *G* and some of its arcs $(U_2 \subseteq U)$. For example, if we erase some arcs, say u_1, u_2, from Fig. 1 we shall be left with a partial graph of the original graph.

Let V_1 and V_2 be two subsets of V such that they have no common vertex, and let $u_i = (v_j, v_k)$ be an arc such that $v_j \in V_1$, $v_k \in V_2$. Then u_i is said to be *incident from* or *going from* V_1 and *incident to* or *going to* V_2. It is *incident with* both V_1 and V_2 and is said to *connect* them. In the figure, if $V_1 = \{v_2, v_3\}$ and $V_2 = \{v_6, v_9\}$, then u_4 connects V_1 and V_2. It goes from V_1 to V_2 and is incident with both. We shall denote by $\Omega(V_k)$ the set of arcs of *G(V, U)* incident with a subset V_k of V, by $\Omega^+(V_k)$ the set of arcs incident to V_k, and by $\Omega^-(V_k)$ the set of arcs incident from V_k. In the figure, if $V_1 = \{v_2, v_3\}$, then

$\Omega+(V_1) = \{u_1,u_5\}$, $\Omega^-(V_1) = \{u_2, u_4,u_6\}$ and $\Omega(V_1) = \{u_1, u_2, u_4, u_5, u_6\}$.

A sequence of arcs $(u_l, u_2, \ldots, u_{k+1}, \ldots\ldots u_q)$ of a graph such that every intermediate arc u_k has one vertex common with the arc u_{k-1} and another common with u_{k+1} is called a *chain. For* example, the sequence (u_2, u_3, u_4, u_7) in the figure is a chain. We may also denote a chain by the vertices which it connects, for example, the above chain may also be written as $(v_l, v_3, v_2, v_9, v_6)$.

A chain becomes a *cycle* if in the sequence of arcs no arc is used twice and the first arc has a vertex common with the last arc, and this vertex is not common with any intermediate arc. For example, the (u_3, u_5, u_7, u_4) in the figure is a cycle.

A path is a chain in which all the arcs are directed in the same sense such that the terminal vertex of the preceding arc is the initial vertex of the succeeding arc. In the figure the sequence of arcs (u_1, u_3, u_6, u_9) is a path. We may also denote the path in terms of the vertices as $(v_1, v_2, v_3, v_4, v_5)$. A path is a chain, but every chain is not a path.

A *circuit* is a cycle in which all the arcs arc directed in the same sense. The cycles (u_1, u_3, u_2) and (u_8, u_9) are circuits.

A graph is said to be *connected* if for every pair of vertices there is a chain connecting the two. The graph in Fig. 1 is not connected because there is no chain connecting, for instance, v_6 to v_7 or v_2 to v_8. If we erase the vertices v_7, v_8, and the arc u_{11} we shall be left with a connected graph. If v_a. is a vertex of a graph, then the set formed by v_a and all other vertices which are connected to v_a by chains, and the set of arcs connecting them, form a *component* of the graph. A connected graph has only one component. If a graph is not connected, it has at least two components. The graph of Fig. 1 has two components, one consisting of vertices v_7, v_8 and the arc u_{11} and the other the remaining portion.

A graph is *strongly connected* if there is a path connecting every pair of vertices in it. Telephones in a town are the vertices of a strongly connected graph. Radio receivers and transmitters form a connected graph but not strongly connected, because there is a path from a transmitter to a receiver but not one from a receiver to a transmitter.

A *tree* is defined as a connected graph with at least two vertices and no cycles (Fig. 2). It can be proved that a tree with n vertices has n -1 arcs, and that every pair of vertices is joined by one and only one chain. If we delete an arc from a tree, the resulting graph is not connected, and if we add an arc, a cycle is formed. As the name indicates, a natural tree is the best example of a graphical tree, the branches forming the arcs and the extremities of the branches forming the vertices.

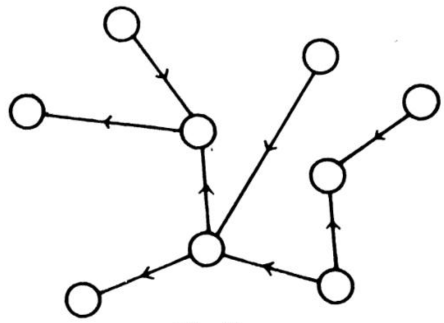


Fig. 2

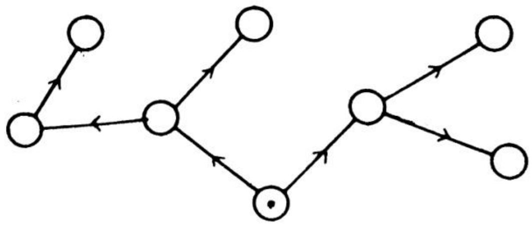


Fig. 3

A vertex which is connected to every other vertex of the graph by a path is called a *centre* of the graph. A graph may or may not have a centre, or may have many centres. Every vertex of a strongly connected graph is a centre. The tree of Fig. 2 has no centre. A tree can at the most have only one centre.

A tree with a centre is called an *arborescence* (Fig. 3). In the figure the centre is marked ⊙. In an arborcscence all the arcs incident with the centre go from it and all the other arcs are directed in the same sensed

VII. 3 Minimum Path Problem

Let a number x_{jk} *be* associated with each arc $(v_j, v_k$ of a graph $G(V, U)$, and let v_a and v_b be two vertices of the graph. There may be a number of paths from v_a to v_b. For each path we define the *length* of the path as Σx_{jk} where the summation is over the sequence of arcs forming the path. The problem is to find the path of the smallest length.

The term *length* is used here in a generalized sense of any real number associated with the arc and should not be regarded as a geometrical distance. A road map connecting towns is a graph and the distance along a road between any two towns is the length of a path within the present definition of the term, but this is only a particular case. The time or the cost involved in going from one town to another is also a *length* under the present definition. There may be more abstract situations in which the length is not even non-negative. In general x_{jk} is a real number, unrestricted in sign.

Many methods and algorithms have been suggested for solving the problem of the minimum path. We shall describe two algorithms here, one applicable only to the case when $x_{jk} \geq 0$ for all arcs and the other for the general case when x_{jk} is unrestricted

I All arc lengths non-negative. Let f_j denote the minimum path from v_a to v_j. We have to find f_b. Obviously $f_a=0$

Let V_p be a subset of V such that v_a is in V_p and v_b is not in V_P Further suppose that f_j for every v_j in V_P has been determined. Now determine $f_j + x_{jk}$ for every v_j in V_P and v_k not in V_P such that (v_j, v_k) is an arc incident from V_p.

Let

$$f_r + x_{rs} = \min (f_j + x_{jk})$$

where $v_r \in V_P$ and $v_s \notin V_P$. Then the minimum path from v_a to v_s is given by

$$f_s = f_r + x_{rs}.$$

This is so because to reach v_s we must leave V_P and $f_r + x_{rs}$ is the least of all paths going out of V_P along single arcs. Any alternative path to v_s can either be along some other single arc going out of V_P to v_s which would be larger, or along some other arc going out of V_p to some other point and then to v_s which would be larger still.

Now form an enlarged subset of V_{P+1} *of V* defined by

$$V_{P+1} = V_p \cup \{v_s\},$$

and repeat the operation. Suppose we start with $p = 0$ with V_0 consisting of a single vertex v_a and $f_a = 0$. Following the procedure described above the sets V_1, V_2,.... V_p, V_{p+1},... are formed. As soon as we arrive at a set in this sequence which includes v_b, we have found f_b,. If no such set can be found, there is no path connecting v_a to v_b.

Example: Find the minimum path from v_0 to v, in the graph of Fig. *4* in which the number along a directed arc denotes its length.

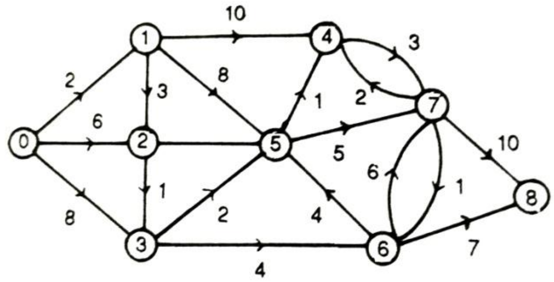


Fig. 4

Table 1 shows the iteration according to the algorithm explained above. In the V_P column are listed the vertices in the subset V_P. Under f are written the least distances to these vertices from v_0. $\Omega^{-}(V_P)$ are the arcs incident from $V_{P'}$ (v_j, v_k) being written as (j, k). Under x are given the lengths of the arcs. F_s, is the minimum of f+x, and v_s is the vertex to which this minimum distance leads and which in the next iteration is included in the enlarged subset V_{P+1}.

TABLE 1

p	V_p	f	$\Omega^{-}(V_p)$	x	$f+x$	f_s	v_s
0	0	0	(0, 1)	2	2	2	1
			(0, 2)	6	6		
			(0, 3)	8	8		
1	0	0	(0, 2)	6	6		
			(0, 3)	8	8		
	1	2	(1, 2)	3	5	5	2
			(1, 4)	10	12		
			(1, 5)	8	10		
2	0	0	(0, 3)	8	8		
	1	2	(1, 4)	10	12		
			(1, 5)	8	10		
	2	5	(2, 3)	1	6	6	3
			(2, 5)	1	6	6	5
3	0	0					
	1	2	(1, 4)	10	12		
	2	5					
	3	6	(3, 6)	4	10		
	5	6	(5, 4)	1	7	7	4
			(5, 7)	5	11		
4	0	0					
	1	2					
	2	5					
	3	6	(3, 6)	4	10	10	6
	4	7	(4, 7)	3	10	10	7
	5	6	(5, 7)	5	11		
5	0	0					
	1	2					
	2	5					
	3	6					
	4	7					
	5	6					
	6	10	(6, 8)	7	17	17	8
	7	10	(7, 8)	10	20		

The minimum path is found to be of length 17 and goes through the vertices (0,1,2,3,6,8).

It should be appreciated that actually drawing the graph is not essential either to the description of the problem or to its solution. The problem is completely enunciated if all the vertices, arcs and arc lengths are specified. In fact in a large problem with many vertices and arcs drawing a figure may be neither practicable nor necessary.

II Arc lengths unrestricted in sign.

Let v_a, v_b be two vertices in the graph *G(V, U)* whose arc lengths are real numbers, positive, negative or zero. We have to find the minimum path from v_a to v_b. We assume that there are no circuits in the graph whose arc lengths add up to a negative number. For, if there is any such circuit, one can go around and round it and decrease the length of the path without limit, getting an unbounded solution.

Construct an arborescence $A_1(V_1, U_1)$, $V_1 \subseteq V$, $U_1 \subseteq U$, with centre v_a and V_1 containing all those vertices of V which can be reached from v_a along a path, and U_1 containing some arcs of *U* which are necessary to construct the arborescence. If V_1 contains v_b, a path connects v_a to v_b. In a particular arborescence this path is unique. There may be many arborescence and therefore many paths. A_1 is any one arborescence. If in any problem only one arborescence is possible, there is only one path from v_a to v_b, and that is the solution. If V_1 does not contain v_b there is no path from v_a to v_b and the problem has no solution.

The method of construction of the arborescence is straightforward. Mark out the arcs going from v_a. From the vertices so reached mark out the arcs (not necessarily all of them) going out to the other vertices. No vertex should be reached by more than one arc, that is, not more than one arc should be incident to any vertex. If there is a vertex to which no arc is incident it cannot be reached from v_a and so is left out. No arc incident to v_a should be drawn.

Let f_j denote the length of the path from v_a to any vertex v_j in the arborescence. The arborescence determines f_j uniquely for each v_j in V_1, but f_j is not necessarily minimum. Let (v_k, *v_j)* be an arc in *G* but not in *A_1*. Consider the length $f_k + x_{kj}$ and compare it with *f_j*. If $f_j \le f_k + x_{kj}$, make no change. If $f_j > f_k+x_{kj}$, delete the arc incident to v_j in A_1 and include instead the arc (v_k, *v_j)*. This modifies the arborescence from A_1 to A_2 and reduces f_j to its new value $f_k + x_{kj}$, the reduction in the value of f_j being $f_j – f_k – x_{kj}$. The lengths of the paths to the vertices going through v_j are also reduced by the same amount These adjustments are made and thus the new values of *f_j* for all v_j in *A_2* are calculated.

Now repeat the operation in *A_2*, that is, select a vertex and see if any alternative arc gives a smaller path to it If yes, modify *A_2* to A_3 and adjust f_j accordingly. Ultimately an arborescence *A_r* is reached which cannot be further changed by the above procedure. A_r marks out the minimum path to each v_j from v_a, and f_b in this arborescence is the minimum path to v_b. The proof is as follows.

Proof. Let (v_a, v_1, v_2,..., v_b) be any path in *G* from v_a to v_b. Its length is $x_{a1} + x_{12} + ... + x_{pb}$. The vertices in this path are in *A_r* also because *A_r* contains all those vertices of *G* which can be reached from v_a. By the property of A_r given in the last paragraph, for every vertex v_j in A_r and for every arc (v_k, *v_j)* in *G_r,*

$$F_j \le f_k + x_{kj},$$

$$\text{or } f_j – f_k \le x_{kj},$$

because otherwise A,could have been further modified. Writing these inequalities for all vertices of the above path,

$$f_1 – f_a \le x_{a1},$$

$$f_2 – f_1 \le x_{12},$$

$$\ldots\ldots\ldots\ldots\ldots..$$

$$f_b – f_p \le x_{pb},$$

Adding, we get,

$$f_b - f_a \le x_{a1} + x_{12} + x_{23} + \ldots + x_{pb},$$

or since $f_a = 0$,

$$f_b \le x_{a1} + x_{12} + x_{23} + \ldots + x_{pb}.$$

Thus we prove that no path from v_a to v_b in *G* can be smaller than f_b. Since the path of length f_b is also in *G,* this path is the minimum. Proved.

The path of maximum length can be found either by changing the signs of the lengths of all arcs and then finding the minimum path, or by reversing the inequality $f_j > f_k + x_{kj}$ to $f_j < f_k + x_{kj}$ as the criterion for changing an arc in the arborescence, so that at every stage a greater path is selected against a smaller one.

Example: Find the minimum path from v_0 to v_7 in the graph *G* of Fig. 5. Notice that it has no circuit whose length is negative.

Draw an arborescence A_1 (Fig. 6) with centre v_0 consisting of all those vertices of the graph which can be reached from v_0, (v_8 is thus excluded), and the necessary number of arcs. Notice that there can be many such arborescences. A_1 is one of them.

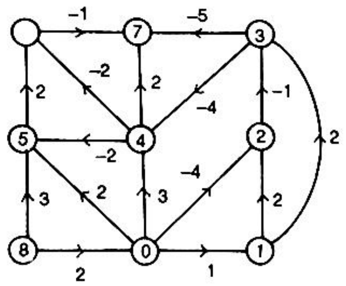


Fig. 5

The lengths f_j of the paths from v_0 to different vertices v_j of A_1 are as follows. $f_0=0$, $f_1 = 1$, $f_2=-4$, $f_3 = 3$, $f_4 = 3$, $f_5 = 2$, $f_6 = 4$, $f_7 = 5$. Consider the vertex v_2. There is an arc (v_1, v_2) in *G* which is not in A_1, such that

$$f_2=-4 < f_1 + x_{12} = 1+2 = 3.$$

So we leave A_1 unchanged.

Now consider the vertex v_3. There is an arc (v_2, v_3) in G which is not in A_1 such that

$$f_3=3>f_2+x_{23} = -4-1= -5.$$

So we delete the arc (v_1, v_3) which is incident to v_3 in A_1 and instead include the arc (v_2, v_3). This gives us a new arborescence A_2 with $f_3 = -5$. Since no vertex is reached in A_1 through v_3, all other f_j remain unchanged.

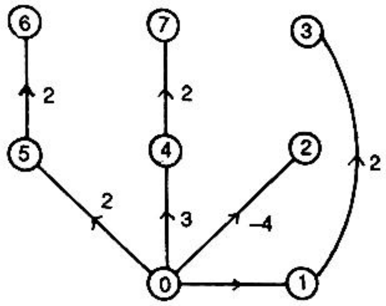


Fig. 6

Coming now to v_4 in A_2 (figure not drawn), arc (v_3, v_4) is in G but not in A_2 such that $f_4 = 3 > f_3 + x_{34} = -5 - 4 = -9$. So we delete the arc (v_0, v_4), include (v_3, v_4), get another arborescence A_3 with $f_4 = -9$ and consequently $f_7 = -7$.

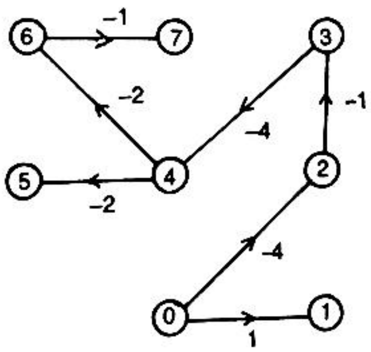


Fig. 7

Continuing like this we finally get the arborescence (Fig. 7) which cannot be further modified. No alternative arc decreases the length of the path from v_0 to any vertex. This is seen by testing for every possible alternative arc. The minimum path from v_0 to v_7 is $(v_0, v_2, v_3, v_4, v_6, v_7)$ with length -12.

VII. 4 Spanning tree of minimum length

Let *G(V, U)* be a connected graph with undirected arcs, and let T(V, *U')* be a tree such that $U' \subseteq U$. The set of vertices of *T* is the same as that of G, while all the arcs of *T* are arcs of G also. Then *T(V, U')* is said to be a *spanning* tree of *G(V, U)* or T is said to *span G.* In Fig. 8 a spanning tree is shown in thick lines and the graph which it spans consists of thick and thin lines. It is obvious that a spanning tree of a graph is not unique. For example in Fig. 8 if we remove u_1 and introduce u_2 in the tree, we still have a spanning tree.

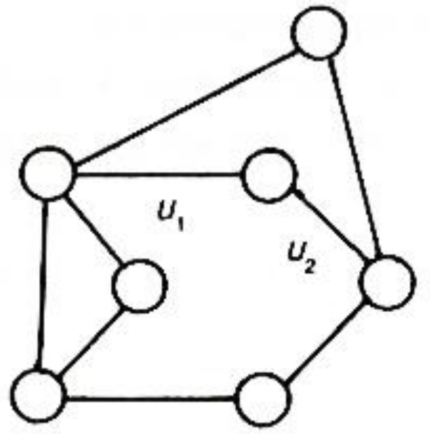


Fig. 8

The problem is to find the spanning tree of minimum length if the length of each arc of G is given as a non-negative number. The problem is of practical significance in laying down pipelines or telephone wires connecting towns.

We denote the arc length between the vertices v_j and v_k by x_{jk}. Since the arcs are undirected, $x_{jk}=x_{kj}$ for all arcs. Also $x_{Jk} \geq 0$ for all arcs.

We describe an algorithm for solving this problem which needs the condition that no two arcs of *G* are of the same length. Therefore we assume that this condition holds. In case some arcs are of-equal length, it is possible to introduce small differences in their lengths in such a way that these changes do not affect the optimal solution. For example, if

$$x_{jk} = x_{pi} = x_{rs},$$

we put $x_{pq} = x_{jk} + e$, $x_{rs} = x_{jk} + 2e$, $e > 0$,

keeping *e* small enough as not to make x_{rs} equal to or greater than any other arc length which is greater than x_{jk}.

The algorithm is as follows. Consider the set of vertices *V* of the graph *G (V, U).* Since *G* is connected, every vertex has one or more arcs incident with it Also since all the arcs in U are of unequal length, for each vertex, among all the arcs incident with it, there is one which is of the smallest length. We shall say that this arc connects the vertex to its *nearest neighbour.*

Consider each vertex of *V* one by one and connect it to its nearest neighbour. In this way we shall get a partial graph $G_1(V, U_1)$ of *G* consisting of all its vertices and some of its arcs. G_1 may, in general, be an unconnected graph with several components, each of which is connected within itself.

Let each of these components be treated as if it is a single vertex. We shall thus have a new set of 'vertices', each ‘vertex’ being a component of the graph G_1. Every one of these 'vertices' have arcs of the graph G connecting it with one or more of the other 'vertices'. For, if it were not so, G will not be connected. The arc of the smallest length between two 'vertices' will be taken as the measure of the 'distance' between them. We again consider each of these 'vertices' one by one and connect each 'vertex' to its nearest neighbour, that is the one which is at the least distance from it. This operation will result in another graph $G_2(V, U_2)$ which again may have components.

Again let each of these components be treated as 'vertices' and let each 'vertex' be connected to its nearest neighbour, yielding a graph $G_3(V, U_3)$. We repeat this procedure till a connected graph $G_p(V, U_p)$ is obtained. This graph is the required spanning tree.

Before giving a proof of this algorithm we work out an example to clarify the procedure described above.

Example: Find the minimum spanning tree of the graph *G(V,U) of* Fig. 9. Notice that it is an undirected graph.

Going through all the vertices v_1 to v_8 and drawing the arc connecting each to its nearest neighbour, we get the graph $G_1(V, U_1)$ of Fig. 10. The nearest neighbour of v_1 is v_3, of v_2 is v_3, of v_3 is v_2, and so on. The graph G_1 is not connected. It has three components A_1, A_2, A_3 We treat them as three 'vertices'. The arcs of G connecting A_1 to A_2 are of lengths 14,18, 8, 16, 11 and so the distance between A_1 and A_2 is 8. Similarly the distance between A_2 and A_3 is 9. Also since there is no arc connecting A_1 and A_3, the distance between them is ∞. The nearest neighbour of A_1 is A_2 and of A_2 is A_1. So we connect the two by arc (v_2, v_5) which measures the distance between the two.

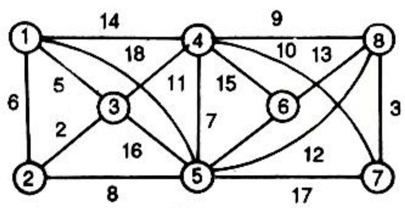


Fig. 9

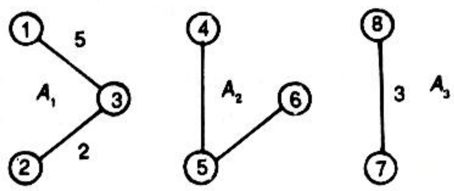


Fig. 10

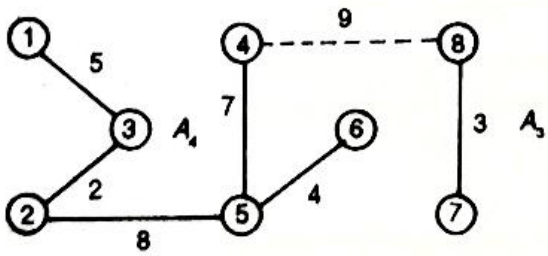


Fig. 11

Thus we get the graph $G_2(V, U_2)$ of Fig. 11 which has two components, A_4 and A_3 Since they are only two, each is the nearest neighbour of the other, and so we connect them with the arc (v_4, v_g) (shown dotted) which measures the distance between them. Thus we get a single connected graph which is the smallest spanning tree. The length of the tree is 38.

We now prove the algorithm.

Proof. Since $G(V, U)$ is connected and no two arcs are of equal length, every vertex of G has a unique nearest neighbour. By connecting every vertex to its nearest neighbour we get the graph $G_1(V, U_1)$ which has components. When these components are connected, each to its nearest neighbour, we get another graph $G_2(V, U_2)$ which has components. By repeating this operation we finally get a graph $G_p(V, U_p)$ which has one single component, and thus the final product is a connected graph.

This graph has no cycles. For, if possible, let it have a cycle $(v_a, v_b, v_c, \ldots v_d, v_a)$. Let us mark arrows on the arcs in this cycle (remember that originally they are undirected) which indicate the nearest neighbour of each vertex. For example, if the nearest neighbour of v_b is v_c we mark the arrow from v_k to v_c. In this way every arc in the cycle will receive an arrow. Because if, for instance. (v_a, v_b) does not get an arrow, then neither v_b is the nearest neighbour of v_a. nor v_a *of* v_b, and so the arc (v_a, v_b) should not have been there at all.

After all the arcs in the cycle have been marked, there will be one of the two following situations.

i) All the arcs are marked in the same sense. Let us suppose they have been marked as $v_a \to v_b$, $v_b \to v_c, \ldots v_d \to v_a$. Then since v_c and not v_a is the nearest neighbour of v_b, $x_{ab} > x_{bc}$. Similarly for other vertices; so we find that

$$x_{ab} > x_{bc} > \ldots > x_{da}$$

But $x_{ab} > x_{da}$ means that the nearest neighbour of v_a is not v_b. This is a contradiction. Therefore all the arcs cannot have arrow marks in the same sense.

(ii) Some arcs are marked in one sense and some in the opposite. Then there must be a vertex, say v_b, such that the arrows to both its neighbours v_a and v_c are directed away from v_b. This would mean v_b is nearest to v, and also to v_c. Since the arcs are of different lengths, this is not possible. So this case also leads to a contradiction.

Thus we conclude that the assumption of the existence of a cycle leads to a contradiction in every case. So the assumption itself is wrong. There are therefore no cycles in the graph $G_P(V, U_P)$.

We thus prove that the graph we get by this algorithm is a connected graph without cycles, that is, it is a tree. Since it includes all the vertices of G, it spans G.

To prove that it is the smallest spanning tree, let us suppose, if possible, that the smallest spanning tree $T(V, U')$ is not the same as G_p (V, U_P). Then there must be an arc in G_p which is not in T, otherwise the two will not be different Let this arc be (v_a, v_b) such that v_b is the nearest neighbour of v_a. Let us introduce this arc in T. A cycle will be formed, because any additional arc in a tree produces a cycle. Let this cycle be (v_a, v_b, v_c …., v_d, v_a). Since v_b is the nearest neighbour of v_a, the length of the arc (v_d, v_a) is greater than that of (v_a, v_b). So if we delete the arc (v_d, v_a) from T and introduce (v_a, v_b), the length of the resulting tree will decrease, which would mean *T* is not of minimum length, which is contrary to hypothesis. So there is not an arc of G_p which is not in *T*. But the total number of arcs in G_p is the same as in T, because both are trees spanning the same graph and should have $n - 1$ arcs where *n* is the. number of vertices in G. Thus G_p is the same as T, the minimum spanning tree of *G)* Proved.

VII. 5. Scheduling of sequential activities.

The problem of minimum path finds an important application in scheduling and coordinating various activities in a project so as to complete it in minimum time at a given cost. Also it is possible to estimate the least rise in cost or the maximum saving possible if certain activities are speeded up or slowed down to finish the project within a prescribed period.

A project involves a number of activities, operations or jobs which we identify as vertices v_a …., v_j ….., v_k ..., v_a of a graph, v, represents the beginning and v_b the end of the project Each job v_j requires some time for its completion. It may not be possible to start on a job unless some specified time has been spent on some other job or jobs. The problem is to find the minimum time in which the project can be finished and the time schedule for each job.

Let c_{jk} be the time required on job v_j before job v_k can start. It is the time interval between the start of the two jobs, v_j preceding v_k. We indicate this information by drawing the arc (v_j, v_k) and associating the length c_{jk} with it. The time required to complete v_j is represented by the arc (v_j, v_b) of length c_{jb}, as it would mean that the time c_{jb} should be spent on v_j before the end v_b can be reached. Also if v_j can start only after some time has passed from the beginning of the project, we may indicate it by c_{aj}. All arcs (v_j, v_k) with lengths c_{jk} will in this way denote a sequential relationship in terms of time among various jobs.

Each sequence of jobs which must be done before work on v_j can begin is represented by a path connecting v_a to v_j. The longest of these paths determines the earliest time v_j can start. In this way the longest path joining v_a to v_b gives the minimum time of completion of the project. The problem thus reduces to finding the maximum path with arc lengths c_{jk} (or the minimum path with arc lengths $-c_{jk}$). This path is called the *critical path.*

Example: A building activity has been analyzed as follows, V_g stands for a job.

(i) v_1 and v_2 can start simultaneously, each one taking 10 days to finish.

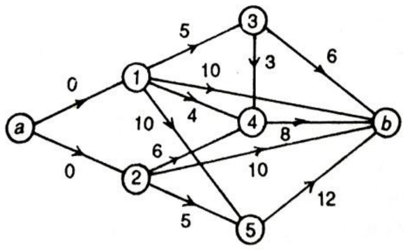
(ii) v_3 can start after 5 days and v_4 after 4 days of starting v_1.

(iii) v_4 can start after 3 days of work on v_3 and 6 days of work on v_2.

(iv) v_5 can start after v_1 is finished and v_2 is half done.

(v) v_3, v_4 and v_5 take respectively 6, 8 and 12 days to finish. Find the critical path and the minimum time for completion.

Fig. 14 is the graph of the activity, vertices $v_0$and v_b representing the start and the finish, and the other vertices the jobs to be done in between. The arc lengths denote the time between the start of two jobs.



The arborescence giving the maximum path is shown in Fig. 15. The critical path is (v_a, v_1, v_5, v_0) of length 22 days which is the minimum time of completion of the work.

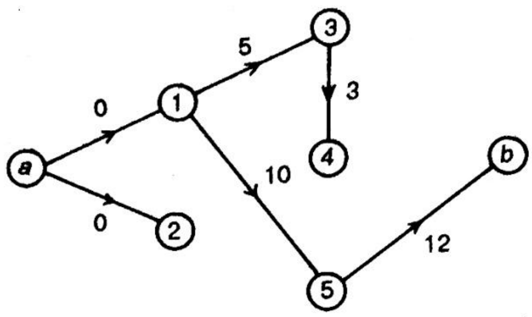


Fig. 15

Having determined the critical path, another type of question can be raised. Suppose it is possible to reduce time on some jobs but at an increased cost. To keep matters simple let us assume that cost of a job increases linearly as time of its completion decreases within certain limits. What is the least increase in cost if the time of completion of the project is decreased by a certain period? If the maximum path is to be reduced, some arc lengths must decrease. We have to determine the decrease which costs the least.

Let α_j be the increase in cost for a unit decrease in time for the completion of the job v_j. Normally α_j would be positive. But there can be situations when slowing down a job results in increased cost and speeding it up leads to some saving. Therefore α_j, in general, are real numbers unrestricted in sign.

We pick up the paths which need reduction in time and examine each vertex v_j in these paths in the order of α_j increasing, beginning with the smallest α_j and reduce the time there as much as is necessary and possible. When the reduction in time in the concerned paths has been made, we stop further reduction and add up the cost. We illustrate by the following example.

Example: In the previous example the work is required to be finished in 16 days. The following table gives the normal values of c_{jb} (same as in the previous example), the minimum possible values of c_{jb}, and the increase α_j in cost at v_j for a unit decrease in time. Find the minimum additional cost at which the work can be completed.

	v_j	1	2	3	4	5
Normal	c_{jb}	10	10	6	8	12
Minimum	c_{jb}	7	8	4	6	8
	α_j	3	1	2	2	2

The paths from v_a to v_1 exceeding the length of 16 days are (v_a, v_1 v_5, v_b) and (v_a, v_2, v_5, v_b). The former is of length 22 days and so needs a reduction of 6 days, while the latter is of length 17 and so needs to be reduced by 1 day. The jobs (or vertices) involved are v_1, v_2, v_5. Of these reduction at v_2 is the cheapest. So we start with $\alpha_2 = 1$. The path through v_2 needs reduction of 1 day only which we get by putting c_{25} = 4 days at a cost of 1. This brings the path (v_a, v_2, v_5, v_b) to 16. In the other path the reduction required is still 6, because the arc (v_2, v_5) is not in it. The vertices to be examined are now v_1 and v_5 of which reduction at v_5 is cheaper. So we should reduce c_{5b} by 6. That would make c_{5b} = 6, but the minimum possible value of c_{5b} is 8. So we reduce it to 8 days at a cost 4 x 2 = 8. The remaining reduction of 2 days can come only at v_1 at a cost 2 x 3 = 6. But reducing c_{56} to 8 days makes reduction in c_{25} unnecessary. So we restore c_{25} to its original value of 5 days. The minimum additional cost of doing the work in 16 days is thus 14.

The method of coordinating and scheduling of activities described in this section is commonly referred to as CPM (critical path method), and is helpful in maintaining progress in construction projects, manufacturing and assembly works, etc. We have given an example of how increase or decrease in cost can be estimated for a specified completion time. There can be variations of this problem. For example, it should be possible to estimate the minimum completion time for a given total cost.

Another similar procedure, called PERT (project evaluation and review technique), goes further and takes into account chance variations in completion times of various jobs to estimate the total expected time of completion. Since we are keeping stochastic and probabilistic considerations completely out of the scope of the present work, we omit further discussion.

VII. 6. Maximum flow problem

Like potential, flow in a network is a familiar concept in electrical theory. Flow of liquid through a network of pipelines, or of traffic through a network of roads, or of production through assembly lines are other examples of network flows. In physical terms the basic condition of flow in a network is that at every vertex the total flow in should be equal to the total flow out, that is, there should be no accumulation of whatever stuff is flowing. To extend the idea to more abstract situations it is necessary to give a precise definition of flow in a graph.

Let x_i, be a real number associated with every arc u_i, i = 1,2,..., m, of a graph *G(V, U)* such that for every vertex v_j,

$$\Sigma_1 x_i = \Sigma_2 x_i, \; j = 1,2,\ldots n,$$

where the left-hand side summation Σ_1 is on all arcs going to v_j and the right-hand side summation is on all arcs going from v_j. Then x, is said to be a *flow in the arc* u_i, and the set $\{x_i\}$, $i = 1,2\ldots,m$, is said to be a *flow in the graph G.*

To state the problem of maximum flow in a network we define a graph as follows.

Let *G(V, U)* be a graph (Fig. 16) with *V* as the set of *n* + 2 vertices v_a, v_1, v_2, v_3, ..., v_n, v_b, and *U* as the set of *m* + 1 arcs u_0, u_1, u_2, ….., u_m. The vertices v_a and v_b and the arc u_o play a special role in this graph, v_a is called the *source* and v_b the *sink,* and the arc u_0 connects v_b to v_a. It is the only arc going from v_b and also the only arc going to v_a. Other arcs incident with v_a are such that they all go from v_a. Similarly all other arcs incident with v_b go to v_b.

With every arc u_i, i=1, 2, …., m, (except u_0), is associated a real number $c_i \geq 0$ called the *capacity* of the arc.

Let $\{x_i\}$ be a flow in the graph *G* such that $0 \leq x_i \leq c_i$, $i = 1, 2,\ldots, m$. Notice that x_0 as the flow in the arc u_0 is defined but the capacity of the arc u_0 is not defined and so there is no constraint on x_0. By the definition of flow and because x_0 is the only flow out at v_b and in at v_a,

$$\begin{aligned} \text{total flow in at } v_b &= \text{total flow out at } v_b \\ &= x_o \\ &= \text{total flow in at } v_a \\ &= \text{total flow out at } v_a. \end{aligned}$$

All that flows out at v_a flows in at v_b. This explains why v_a and v_b have been called the source and the sink respectively. The arc u_0 serves as a mathematical device to bring the flow in the network within the definition of a flow in a graph. We shall call u_0 the *return arc*.

The problem is to determine the maximum flow out at the source (= maximum flow in at the sink). More precisely, the problem is to find the flow $\{x_i\}$ such that

$$x_0 \text{ is maximum}$$

subject to $0 < x_i < c_i$, $i = 1, 2,.., m$.

We first describe an algorithm which solves the problem. The proof will be given later.

(i) Start by assuming a feasible flow. In the absence of any better guess, it is always possible to start with $x_i = 0$ for all i.

(ii) Divide the set V of vertices into two subsets, W_l and W_2, such that each vertex is either in W_1 or in W_2 but not in both. To begin with let $W_1 = \{v_a\}$, all other vertices being in W_2.

(iii) Adopt the following procedure of transferring a vertex from W_2 to W_1. Let $v_j \in W_1$, $v_k \in W_2$.

(a) if (v_j, v_k) is an arc u_i, and $x_i < c_i$, transfer v_k to W_1;

(b) if (v_k, v_j) is an arc u_i, and $x_i > 0$, transfer v_k to W_1;

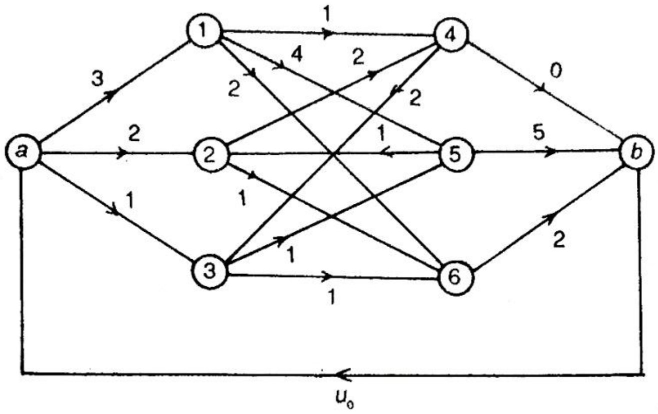
(c) otherwise do not transfer v_k to W_1.

Go on transferring vertices from W_2 to W_1 like this. If v_b is transferred to W_1 by this procedure, the flow is not optimal.

(iv) If the flow is not optimal, increase x_i in arc of category (a) in which $x_i < c_i$ and decrease x_i in arc of category (b) in which $x_t > 0$ so that the flow remains feasible and at least one arc gets capacity flow. Go back to (ii).

Repeating operations (ii), (iii) and (iv) we shall come to a stage when v_b cannot be transferred to W_1 by operation (iii). The flow at that stage is optimal.

Example: In the graph of Fig. 16, numbers along arcs are values of c_i. Find the maximum flow in the graph.



Assuming the initial flow as zero in all arcs, let $W_1 = \{v_a\}$. There is an arc (v_a, v_I) from v_a in W_1, to v_1 in W_2 in which the flow (zero) is less than its capacity 3. Therefore by criterion (iiia) transfer v_I to W_I. Now there is an arc (v_1, v_4) with $v_1 \in W_1$ and $v_4 \in W_2$ such that the flow in it is less than its capacity 1. So we transfer v_4 to W_1. In the arc (v_4, v_b), $v_4 \in W_l$, $v_b \in W_2$, the flow is zero which is equal to its capacity and so we cannot transfer v_b to W_1 for this reason. But there is another arc (v_4, v_3), $v_4 \in W_1$, $v_3 \in W_2$, which is such that v_3 is transferrable to W_1. Further, because the flow in arc (v_3, v_5), $v_3 \in W_1$ $v_5 \in W_2$ is below capacity, v_5 is transferred to W_1, and finally because the arc (v_5, v_b) satisfies the same criterion, v_b is transferred to W_1. Thus it is possible to transfer v_b to W_1, and so the flow is not optimal.

We have gone along the chain $(v_a, v_1, v_4, v_3, v_5, v_b)$. The least capacity in this chain is 1. So in each arc of this chain and also in the return arc (v_b, v_a) increase the flow to 1, keeping the flow as it was in all other arcs. The modified flow is feasible because in each arc it is less than or equal to its capacity, and also at every vertex the flow in equals the flow out.

The above reasoning is repeated with every modified feasible flow until it is not possible to transfer v_b to W_1. The iterations are shown in table 3. In each feasible flow the numbers in ( ) indicate the chain along which it is possible to proceed to bring v_b into W_1. The asterisk indicates that the flow in the corresponding arc is equal to its capacity and cannot be further increased.

TABLE 3

Arcs	Capacity c_i	Feasible flows I	II	III	IV	V	VI
(*a*, 1)	3	(0)	(1)	3*	3*	3*	3*
(*a*, 2)	2	0	0	(0)	(1)	2*	2*
(*a*, 3)	1	0	0	0	0	(0)	1*
(1, 4)	1	(0)	1*	1*	1*	(1*)	0
(1, 5)	4	0	(0)	2	2	(2)	3
(1, 6)	2	0	0	0	0	0	0
(2, 4)	2	0	0	0	(0)	1	1
(2, 6)	1	0	0	(0)	1*	1*	1*
(3, 5)	1	(0)	1*	1*	1*	1*	1*
(3, 6)	1	0	0	0	(0)	1*	1*
(4, 3)	2	(0)	1	1	(1)	(2*)	1
(4, *b*)	0	0*	0*	0*	0*	0*	0*
(5, 2)	1	0	0	0	0	0	0
(5, *b*)	5	(0)	(1)	3	3	(3)	4
(6, *b*)	2	0	0	(0)	(1)	2*	2*
(*b*, *a*)		0	1	3	4	5	6

The change from flow V to flow VI deserves to be followed carefully. The chain in V the flow through which has been modified is (v_a, v_3, v_4, v_1, v_s, v_b). We argue as follows. Starting with $W_1 = \{v_a\}$, v_3 can be transferred to W_1 by criterion (iiia). There is no unsaturated arc going out from v_3, both (v_3, v_6) and (v_3, v_5) carrying capacity flows. But (v_4, v_3) is an arc such that $v_3 \in W_1$, $v_4 \in W_2$, and the flow in it is 2 which is greater than zero. Hence, by criterion (iiib), v_4 is transferred to W_1. Again there is an arc (v_1, v_4) with $v_4 \in W_2$ and $v_1 \in W_2$ and with the flow in it greater than zero. So v_1 is also transferred to W_1 by criterion (iiib). This time there is an arc (v_1, v_5) with $v_1 \in W_1$ $v_5 \in W_2$ with flow 2 in it which is less than its capacity 4. Consequently, by criterion (iiia), v_5 is transferred to W_1 and finally, by the same criterion, v_b is transferred to W_1. So the flow is not optimal. In this chain arcs (v_4, v_3) and (v_1, v_4) occur in reverse directions. We reduce flows in them by 1 and increase flows in other arcs of the chain by 1 thereby saturating the arc (v_a, v_3).

The iterations stop at this stage because no matter how we try we cannot bring v_b into W_1. In fact we cannot even proceed one step from the initial position of W_1 containing only one point v_a. This is so because the arcs going out from v_a are all saturated and so neither v_1, nor v_2 nor v_3 can be brought in W_1. The maximum flow in the graph is 6.

We now proceed to prove the algorithm. We begin with a definition.

Definition 1. If in the graph G(V,U) of the maximum flow problem, W_2 is a subset of V such that $v_b \in W_2$ $v_a \notin W_2$, then the set of arcs $\Omega^+(W_2)$ (arcs incident to W_2) is said to be a cut. The capacity of the cut is the sum of the capacities of the arcs contained in the cut.

Theorem 1. For any feasible flow $\{x_i\}$, i=1,2,….,m, in the graph, the flow x_0 in the return arc is not greater than the capacity of any cut in the graph.

Proof. Let $\Omega^+(W_2)$ be any cut. Consider the flow in the arcs going to and going from W_2. The flow in should be equal to the flow out.

Therefore

$$\Sigma_1 x_i = x_0 + \Sigma_2 x_i,$$

Where Σ_1 and Σ_2 respectively denote summations over the arcs going to and going from W_2 (except u_0). Since $x_i \geq 0$ for all i,

$$\Sigma_i x_i \geq x_0,$$

Also $x_i \leq c_i$ for all I. Therefore

$$\Sigma_1 c_i \geq x_0.$$

Where $\Sigma_1 c_i$ is the capacity of the cut $\Omega^+(W_2)$. Proved.

Theorem 2. The algorithm described earlier in this section solves the problem of the maximum flow.

Proof. Suppose by the application of the algorithm a stage is reached when no vertex of W_2 can be transferred to W_1 by the prescribed procedure and $v_b \in W_2$. The set of arcs $\Omega^+(W_2)$ is a cut. Let $u_i \in \Omega^+(W_2)$. It means u_i is an arc (v_p, v_q) where $v_p \in W_l$, $v_q \in W_2$. The flow in this arc should be saturated, that is $x_i = c_i$, because if it were not so it would have been possible to transfer v_q from W_2 to W_1 by criterion (iiia), which is contrary to hypothesis.

Again, let $u_j \in \Omega^-(W_2)$, $j \neq 0$. It means u_j is an arc (v_r, v_s) where $v_r \in W_2$, $v_s \in W_1$. The flow in this arc should be zero, because if it were not so, it would have been possible to transfer v, from W_2 to W_1 by criterion (iiib), which again is contrary to hypothesis.

We conclude that the flow into W_2 is Σc_i, summation being over all $u_j \in \Omega^+(W_2)$, and the flow out of W_2 is only in the return arc u_0, because it is the only arc going from W_2 carrying a nonzero flow. Let the flow in u_0 be y_0. Then, since the flow in and out of W_2 should balance.

$$\Sigma c_i = y_0,$$

where Σc_i, is the capacity of the cut obtained by the application of the algorithm. But from theorem 1, for any flow x_0 in u_0.

$$x_0 \leq \Sigma c_i,$$

where Σc_i is the capacity of any cut. It follows that

$$y_0 = \mathit{max}\ x_0,$$

which means the algorithm leads to finding out the maximum flow. Proved.

THEOREM 3. The maximum flow in a graph is equal to the minimum of the capacities of all possible cuts in it.

Proof. By theorem I, $x_0 \leq \Sigma c_i$,

Therefore max $x_0 \leq$ min $\Sigma c_{i.}$

But we have seen in the course of the proof of theorem 2 that there is a cut corresponding to which the flow in is equal to cut capacity. Necessarily this flow should be maximum and the corresponding cut capacity should be the least of all cut capacities. Proved.

This theorem is generally known as the max-flow min-cut theorem.

CHAPTER VIII
THEORY OF GAMES

VIII. 1. Introduction

In all the various types of optimization problems considered so far the assumption has been that there is a single decision maker whose interest lies in choosing the variables in such a way as to optimize the objective function, there; being no conflict in deciding what the objective is. There are, however, situations in which there are two or more decision makers, each one making decisions (that is, choosing variables) to optimize *his* objective function which may be in conflict with the objectives of others. Trade and commerce, battles and wars, various types of games, and many other activities present situations in which different parties compete to achieve their own objective and prevent others from achieving theirs. Mathematical models of such situations and their solutions form the subject matter of the theory of games.

Game is defined as an activity between two or more persons involving *moves* by each person according to a set of rules, at the end of which each person receives some benefit or satisfaction or suffers loss (negative benefit).

The set of rules defines the game. Going through the set of rules once by the participants defines a *play.* There can be various types of games. They can be classified on the basis of the following characteristics.

(i) *Chance or strategy:* If in a game the moves are determined by chance, we call it a *game of chance,* if they are determined by skill, it is a *game of strategy.* In general a game may involve partly strategy and partly chance. We shall discuss the simplest models of games of strategy only.

(ii) *Number of persons:* A game is called an n-person game if the number of persons playing it is *n.* (Here 'person' means an individual or group aiming at one objective.)

(iii) *Number of moves:* These may be finite or infinite.

(iv) Number of alternatives (or choices) available to each person per move: *These also may be finite or infinite.*

A *finite game* has a finite number of moves, each involving a finite number of alternatives. Otherwise the game is infinite.

(v) *Information available to players of the past moves of other players:* The two extreme cases are, (a) no information at all, (b) complete information available. There can be cases in between in which information is partly available.

(vi) *Pay off:* It is a quantitative measure of the satisfaction a person gets at the end of the play. It is a real-valued function of the variables in the game.

Let p_i be the pay off to the person P_i i=1,2,..., n, in an n-person game. Then if $\sum_{i=1}^{n} p_i = 0$, the game is said to be a *zero-sum game*.

VIII. 2. Matrix (or rectangular) games

A rnatrix game is a zero-sum two-person game with the following mathematical model. (The name 'rectangular' or 'matrix' has no other significance except that the game can be described in a rectangular matrix form).

The player P_1 has m choices $i, i = 1,2,\ldots,m$, the rows of a matrix, while P_2 has n choices j, $j = 1,2,\ldots,n$, the columns of the same matrix. The mxn matrix A = $\{a_{ij}\}$ gives the *pay off to* P_1 for all possible combinations of the choices. Since it is a zero-sum game, the pay off to P_2 is the matrix -A. Conventionally the pay off matrix to P_1, the player who chooses row-wise, is taken as the matrix of the game (table 1).

TABLE 1

		P_2			
	$i \backslash j$	1	2	...	n
	1	a_{11}	a_{12}	...	a_{1n}
P_1	2	a_{21}	a_{22}	...	a_{2n}
	—	—	—	—	—
	m	a_{m1}	a_{m2}	...	a_{mn}

The game is played as follows. P_1 chooses a value of i and P_2 choose a value of j without each knowing what the other has chosen. Then the choices are disclosed, and P_1 receives a_{ij} (or P_2 pays a_{ij}). Here we shall discuss matrix games.

VIII. 3. Problem of game theory

To solve a mathematical model of a game is to investigate whether there is an optimal way to play it, that is, whether there exists any rational argument in favour of playing it one way or the other. Briefly, the problem is to discover, if any, the optimal strategy.

This is explained further in the following examples.

Example 1: Consider the following matrix game.

		P_2			
	$i \backslash j$	1	2	3	4
	1	4	−2	−4	−1
	2	3	1	−1	2
P_1	3	2	3	−2	−2
	4	−1	−3	−3	1
	5	−3	2	−2	−3

P_1 wishes to obtain the largest possible a_{ij} by choosing some i, i = 1 , 2 , … . 5 , while P_2 is determined to make P_1's gain the minimum possible by his choice of j, j=1, 2,3,4. We shall call P_1 the maximizing player and P_2 the minimizing player. It would be rational for P_1 to argue as follows.

"If I choose i= 1, then it could happen that P_2 chooses j=3 in which case I gain only–4. Similarly for my other choices i = 2,3,4,5, P_2 can force me to get only –1, –2, –3, –3 respectively by his choice of j. Thus the best choice for me is to opt for i = 2, for this assures me at least the gain –1. In general, I should try to maximize my least gain, or find out

$$\max_{i} \min_{j} a_{ij}."$$

P_2 can argue similaiiy to keep P_1's gain the least. By his choosing j = 1,2,3,4, P_1*'s* gain can be respectively as high as 4, 3, –1, 2. So P_2 should settle for $j = 3$

because that would minimize P_1's gain. In general, he should find out

$$\min_{j} \max_{i} a_{ij}$$

It turns out in the present problem that

$$\max_{i} \min_{j} a_{ij} = \min_{j} \max_{i} a_{ij}$$

and so the arguments of P_1 and P_2 lead to the same pay off. It may not always happen, as in the following example.

Example 2: Consider the following game.

		P_2		
	$i \backslash j$	1	2	3
	1	2	−3	7
P_1	2	−7	4	−5
	3	5	−6	6

Arguing as in example I, in this problem

$$\max_{i} \min_{j} a_{ij} = -3,$$

$$\min_{j} \max_{i} a_{ij} = 4.$$

The two are not equal. Notice that

$$\max \min a_u < \min \max a_{ij}$$

If, a matrix $\{a_{ij}\}$ is such that

$$\max_{i} \min_{j} a_{ij} - \min_{j} \max_{i} a_{ij} => a_r s$$

the matrix is said to have a saddle point at (r, *s)*. In a game whose pay off matrix is of this type, the *optimal strategies* of players P_1 and P_2 are said to be $i = r$ and j=s respectively, and a_{rs} is said to be the value of the game. Example 1 is of this type. But, in general, a matrix need not be of this type, as example 2 shows, and a saddle point as defined above may not exist. The above definitions of optimal strategy and value of the game are therefore not adequate to cover all cases and need to be generalized. The definition of a saddle point of a function of several variables and some theorems connected with it form the basis of such a generalization. We therefore first present these theorems.

VIII. 4. Minimax theorem, saddle point

Let $f(X, Y)$ be a real-valued function of two vectors X and Y, $X \in E_n$, $Y \in E_m$. Suppose this function is such that if X is kept fixed at some value and Y is varied, then $f(X_1, Y)$ has a minimum for some value of Y. We denote this value by

$$\phi = \min_{y} f(X,Y).$$

If we give to X some other fixed value, we may find another value of ϕ. Thus for different values of X we can obtain values of ϕ, assuming that ϕ exists in every case. This means that ϕ is a function of X and we may write

$$\phi(X) = \min_{y} f(X, Y).$$

Let us now suppose that $\phi(X)$ has a maximum for some value of X. We may write it as

$$\max_{x} \phi(X) = \max_{x} \min_{y} f(X, Y).$$

Similarly the expression

$$\min_{y}\max_{x} f(X, Y)$$

is interpreted. Here we first find a maximum of f(X, Y) with respect to X keeping Y fixed, and then find the minimum of the function so obtained with respect to Y.

THEOREM 1. Let f(X, Y) *be such that both* $\max_{x}\min_{y}$ f(X, Y) *and* $\min_{y}\max_{x}$ f(X, Y) exist. Then

$$\max_{x}\min_{y} f(X, Y) \le \min_{y}\max_{x} f(X, Y). \qquad (1)$$

Proof. Let X_0 and Y_0 be some arbitrarily chosen points in E_n and E_m respectively. Then

$$\min_{y} f(X_0, Y) \le f(X_0, Y_o)$$

$$\text{and } \max_{x} f(X, Y_0) \ge f(X_0, Y_0).$$

$$\text{Hence } \min_{y} f(X_0,Y) \le \max_{x} f(X, Y_0).$$

But Y_0 is arbitrarily chosen and could have been any point in E_m, and for every one of them the inequality should hold. Even if we had chosen Y_0 to be that point for which max f(X, Y) has the least value, the inequality shall be true. "So

$$x \qquad \text{Min}_{y}\, f(X_0, Y) \le \min_{y}\max_{x} f(X, Y).$$

Also since X_0 is any point in E_n, the inequality will hold even if we choose that X_0 which makes min f(X,Y) maximum. Therefore

$$x \qquad \text{Max}_{x}\min_{y} f(X, Y) \le \min_{y}\max_{x} f(X, Y). \qquad \text{Proved.}$$

Corollary 1. Let $\{a_{ij}\}$ be an $m \times n$ matrix. Then

$$\max_{i} \min_{j} a_{ij} \leq \min_{j} \max_{i} a_{ij}. \tag{2}$$

We have only to regard as a real-valued function $f(i, j) = a_{ij}$ of two variables i and *j* where $i = 1,2, \ldots\ldots, m$, *and* $j = 1,2,..., n$, and (2) follows immediately from (1).

We defined the saddle point of a function in chapter 2, section 7. Because of its importance in the present context we state the definition again.

Definition 1. A point (X_0, Y_0), $X_0 \in E$, $Y_0 \in E_m$, is said to be a saddle point of $f(X, Y)$if

$f(X, Y_0) \leq f(X_0, Y_0) \leq f(X_0, Y)$. (3)

We now prove the following theorem on the existence of a saddle point.

Theorem 2. Let f(X, Y) be such that both $\max_{x} \min_{y} f(X, Y)$ and $\min_{y} \max_{x} f(X, Y)$ exist. Then the necessary and sufficient condition for the existence of a saddle point (X_0, Y_0) of f(X, Y) is that

$$f(X_0, Y_0) = \max_{x} \min_{y} f(X, Y) = \min_{y} \max_{x} f(X, Y). \tag{4}$$

Proof.

(i) To prove that the condition is necessary, let (X_0, Y_0) be a saddle point such that (3) holds. Since

$$f(X, Y_0) \leq f(X_0, Y_0) \text{ for all } X \in E_n,$$

$$\max_{x} f(X, Y_0) \leq f(X_0, Y_0).$$

$$\text{But } \min_{y} [\max_{x} f(X, Y)] \leq \max_{x} f(X_0, Y_0).$$

$$\text{Therefore } \min_{y} \max_{x} f(X, Y) \leq f(X_0, Y_0).$$

Again, from (3), since

$$f(X_0, Y_0) \leq f(X_0, Y) \text{ for all } Y \in E_m,$$

$$f(X_0, Y_0) \leq \min_{y} f(X_0, Y).$$

$$\text{But } \min_{Y} f(X_0, Y) \leq \max_{X} [\min_{Y} f(X, Y)].$$

Hence $$f(X_0, Y) \leq \max_{X} \min_{Y} f(X, Y).$$

Thus we find that

$$\min_{Y} \max_{X} f(X, Y) \leq f(X_0, Y_0) \leq \max_{X} \min_{Y} f(X, Y).$$

But from theorem 1,

$$\min_{Y} \max_{X} f(X, Y) \geq \max_{X} \min_{Y} f(X, Y).$$

The only conclusion from the above two statements is that

$$\max_{X} \min_{Y} f(X, Y) = \min_{Y} \max_{X} f(X, Y) = f(X_0, Y_0).$$

(ii) To prove that the condition is sufficient, let (4) be true. Let the maximum of $\min_{Y} f(X,Y)$ occur at X_0, and the minimum of $\max_{X} f(X, Y)$ occur at Y_0. Then, from (4)

$$\min_{Y} f(X_0, Y) = \max_{X} f(X, Y_0). \quad (5)$$

But by definition of minimum,

$$\min_{Y} f(X_0, Y) \leq f(X_0, Y_0),$$

and so from (5),

$$\max_{X} f(X, Y_0) \leq f(X_0, Y_0),$$

Which means that

$$f(X, Y_0) \leq f(X_0, Y_0) \text{ for all } X.$$

Also, by definition of maximum,

$$\max_{X} f(X, Y_0) \geq f(X_0, Y_0),$$

which so, again from (5).

$$\min_{Y} f(X_0, Y) \geq f(X_0, Y_0)$$

which means

$$f(X_0, Y) \geq f(X_0, Y_0) \text{ for all } Y.$$

Thus we find that

$$f(X, Y_0) \leq f(X_0, Y_0) \leq f(X_0, Y)$$

Which, by definition, means that (X_0, Y_0) is a saddle point of $f(X, Y)$.

Proved.

Corrollary 2. Let $\{a_{ij}\}$ be an m x n matrix. Then the necessary and sufficient condition that $\{a_{ij}\}$ has a saddle point at i=r, j=s is that

$$a_{rs} = \max_{i} \min_{j} a_{ij} = \min_{j} \max_{i} a_{ij}. \quad (6)$$

As in corollary 1, regarding $\{a_{ij}\}$ as a real-valued function of two variables i and j, (6) follows immediately from (4).

VIII. 5. Strategies and pay off

As was mentioned at the end of section 3, in a matrix game, if the pay off matrix $\{a_{ij}\}$ has a saddle point (r, s), then $i = r, j=s$ are the optimal strategies of the game and the pay off a_{rs} is called the value of the game. If the matrix has no saddle point, the game has no optimal strategies in the above sense. By introducing probability with choice and mathematical expectation with pay off, the concept of optimal strategy can be extended to apply to all matrices. This we proceed to do.

Let P_1 choose a particular i, $i = 1,2,\ldots,m$, with probability x_i. We may also interpret it as the relative frequency with which P_1 chooses i in a large number of plays of the game. The probabilities x_i $i = 1, 2, \ldots\ldots, m$, constitute the strategy of P_1. Similarly if P_2 chooses a particular j with probability y_j, the probabilities y_j, $j = 1, 2, \ldots, n$, are the strategy of P_2.

Definition 2. *The vector* $X = \{x_i\}$ *of nonnegative numbers* x_i, *such that* $\sum_{i=1}^{m} x_i = 1,$ is defined as the mixed strategy of P_1. Similarly the vector $Y = \{y_j\}$ of nonnegative numbers y_j, such that $\sum_{j=1}^{n} y_j = 1,$ defines the mixed strategy of P_2.

For the sake of brevity we define S_m as the set of ordered m-tuples of nonnegative numbers whose sum is unity, and say that $X \in S_m$. Similarly $Y \in S_n$. Unless otherwise mentioned it will be assumed throughout this chapter that $X \in S_m$ and $Y \in S_n$, where X and Y are mixed strategies of P_1 and P_2 respectively.

Definition 3. The mixed strategy $X = \xi_i$ whose ith component is unity and all other components are equal to zero is called a pure strategy of P_1. Similarly $Y=\eta_j$, where all the components of Y except the jth are zero, is called a pure strategy of P_2.

Definition 4. *The mathematical expectation or the payoff function* $E(X, Y)$ *in the game whose payoff matrix is* $A = \{a_{ij}\}$ *is defined as*

$$E(X,Y) = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i a_{ij} y_j = X'AY$$

where X and Y are the mixed strategies of P_1 and P_2.

Following the argument given in section 3, it is reasonable to postulate that P_1 should choose X so as to maximize his least expectation and P_2 should choose Y so as to minimize *P_1's* greatest expectation. Thus P_1 aims at $\max\limits_{x} \min\limits_{y} E(X, Y)$ and P_2 aims at $\min\limits_{y} \max\limits_{x} E(X, Y)$.

D*efinition 5*. If $\max\limits_{x} \min\limits_{y} E(X, Y) = \min\limits_{y} \max\limits_{x} E(X, Y) = E(X_0, Y_0)$, *then* (X_0, Y_0) *is* defined as the strategic saddle point of the game, X_0 and Y_0 are defined as the optimal strategies, and $v = E(X_0, Y_0)$ is the value of the game.

According to a theorem, known as the fundamental theorem of the theory of rectangular games, a strategic saddle point always exists. Before turning to the theoretical aspects let us consider the following example.

Example: Consider the following matrix game.

$$
\begin{array}{cc|cc}
 & & P_2 & \\
 & j & 1 & 2 \\
 & i & & \\
\hline
 & 1 & 5 & 1 \\
P_1 & & & \\
 & 2 & 3 & 4
\end{array}
$$

The above matrix is without a saddle point, as

$$\max_{i} \min_{j} a_{ij} = 3 \neq \min_{j} \max_{i} a_{ij} = 4.$$

Let the mixed strategies of P_1 and P_2 be $X = [x_1\ x_2]$ and $Y = [y_1\ y_2]$. Then

$$E(X, Y) = 5x_1y_1 + 3x_2y_1 + x_1y_2 + 4x_2y_2$$

where $x_1 + x_2 = 1$, $y_1 + y_2 = 1$.

Eliminating x_2, y_2, we get

$$E(X,Y) = 5x_1y_1 - 3x_1 - y_1 + 4$$

$$= 5\left(x_1 - \frac{1}{5}\right)\left(y_1 - \frac{3}{5}\right) + \frac{17}{5}.$$

If P_1 chooses $x_1 = 1/5$, he ensures that his expectation is at least 17/5. He cannot be sure of more than 17/5, because by choosing $y_1 = 3/5$, P_2 can keep $E(X, Y)$ down to 17/5. So P_1 might as well settle for 17/5 and play $X_0 = [1/5, 4/5]$, and P_2

reconcile to - 17/5 and play Y_0 = [3/5, 2/5]. These are the optimal strategies for P_1 and P_2. The value of the game is 17/5, and (X_0, Y_0) is a saddle point of $E(X, Y)$.

VIII. 6. Theorems of matrix games

We begin with a theorem which is required in the proof of the fundamental theorem of games.

Theorem 3. Let A be an $m \times n$ matrix, and let P_j and Q_i, $j=1,2,\ldots,n$, $i=1,2,\ldots,m$, be its column and row vectors respectively. Then either (i) there exists a Y in S_n such that $Q_i Y \leq 0$ for all i, or (ii) there exists an X in S_m such that $X' P_j > 0$ for all j.

Proof. Let $\xi_i \in S_m$ be a vector such that its ith component is unity and all other components are zero. Consider the m + n points

$\xi_1, \xi_2, \ldots, \xi_m, P_1, P_2, \ldots, P_n$

belonging to E_m. Let C be the convex hull of the $m + n$ points. Then the origin 0 of E_m is either in C or not in C. We consider the two case separately.

(i) Let 0 be in C. Then 0 can be expressed as a convex linear combination of the $m + n$ points which span C (chapter 1, section 15). Hence there exist

$$[\lambda_1, \lambda_2, \ldots, \lambda_m, \mu_1, \mu_2, \ldots, \mu_n] \in S_{m+n} \tag{7}$$

Such that

$$\sum_{i=1}^{m} \lambda_i \xi_i + \sum_{j=1}^{n} \mu_j P_j = 0,$$

or

$$\lambda_i + \sum_{j=1}^{n} \mu_j a_{ij} = 0, i = 1,2,\ldots, m. \tag{8}$$

since $\lambda_i \geq 0$,

$$\sum_{i=1}^{n} \mu_j a_{ij} \leq 0, i = 1,2,\ldots, m. \tag{9}$$

Also $$\sum_{j=1}^{n} \mu_j > 0,$$

for, if it is equal to zero, each μ_j should be zero, which from (8) would mean that each λ_i should also be zero. This would contradict hypothesis (7), and therefore is not possible. Dividing (9) $\sum_{j=1}^{n} \mu_j$ by we get

$$\left(\sum_{j=1}^{n} \mu_j \, a_{ij}\right) / \sum_{j=1}^{n} \mu_j \leq 0.$$

Putting $$y_j = \mu_j / \sum_{i=1}^{n} \mu_j$$

We get $$\sum_{j=1}^{n} y_j \, a_{ij} \leq 0,$$

or $$Q_i \, Y \leq 0, \; i = 1, 2, \ldots, m \qquad (10)$$

This proves alternative (i) of the theorem.

(ii) Let $0 \in C$. Then by the theorem on separating hyperplanes there exists a hyperplane containing 0, say $BZ = 0$, such that C is contained in the halfspace $BZ > 0$. In particular, since $\xi_i \in C$,

$$B\xi_i > 0,$$

or $$b_i > 0, \; i = 1,2,\ldots,m,$$

and therefore

$$\sum_{i=1}^{m} b_i > 0,$$

where b_i is the ith component of B. Also $P_j \in C$ and so

$$BP_j > 0, \; j = 1, 2, \ldots, n,$$

or $$\sum_{i=1}^{m} b_i \, a_{ij} > 0.$$

Dividing by $\sum_{i=1}^{m} b_i$ and putting $x_i = b_i / \sum_{i=1}^{m} b_i$, we get

$$\sum_{i=1}^{m} x_i \, a_{ij} > 0 \,,$$

or $$X^1 P_j > 0, \; j=1,2,\ldots,n, \tag{11}$$

which proves alternative (ii) of the theorem.

We now state and prove the fundamental theorem of rectangular games.

THEOREM 4. *For an m* x *n matrix game both* $\max_X \min_Y E(X, Y)$ *and* $\min_Y \max_X E(X, Y)$ *exist and are equal.*

Proof. E(X, Y) is a continuous linear function of X defined over the closed and bounded subset S_m of E_m for each Y in S_n. Therefore $\max_X E(X, Y)$ exists and is a continuous function of Y. Since S_n is also closed and bounded, $\min_Y \max_X E(X,Y)$ exists. Similarly we prove that $\max_X \min_Y E(X, Y)$ also exists.

From theorem 3 either (10) or (11) holds. Let (11) hold. Then multiplying (11) by the component y_j- of Y and summing for all j, we get

$$E(X,Y) = \sum_{j=1}^{n} \sum_{i=1}^{m} x_i \, a_{ij} \, y_j > 0$$

for all Y. Hence $$\min_Y E(X,Y) > 0,$$

and consequently $$\max_X \min_Y E(X, Y) > 0.$$

If, on the other hand, (10) holds, then by a similar argument we conclude that

$$\min_Y \max_X E(X, Y) \leq 0.$$

At least one of the above two inequalities must hold, and so

$$\max_X \min_Y E(X,Y) < 0 < \min_Y \max_X E(X,Y) \text{ is not true.} \tag{12}$$

Let A_k be the matrix $\{a_{ij} - k\}$ formed by subtracting k from each of the elements of A, and let its expectation function be $E_k(X, Y)$. Then

$$E_k(X,Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i \left(a_{ij} - k\right) y_j \,,$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} x_i a_{ij} y_j - k\sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j ,$$

$$= E(X,Y) - k. \qquad (13)$$

Since A is any matrix, what is true for A is true for A_k. Therefore, from (12), $\max_x \min_y E_k(X, Y) < 0 < \min_y \max_x E_k(X, Y)$ is not true,

or using (13), $\max_x \min_y E(X, Y) < k < \min_y \max_x E(X, Y)$ is also not true (14)

for any value of *k*. The only conclusion from (12) and (14) is that $\max_x \min_y E(X, Y) < \min_y \max_x E(X, Y)$ is false.

Therefore

$$\max_x \min_y E(X,Y) \geq \min_y \max_x E(X,Y).$$

But from theorem 1

$$\max_x \min_y E(X,Y) \leq \min_y \max_x E(X,Y).$$

Hence

$$\max_x \min_y E(X,Y) = \min_y \max_x E(X,Y) \qquad (15)$$

Proved.

By theorem 2, (15) is a necessary and sufficient condition for a point $(X_0. Y_0)$, $X_0 \in S_m$ $Y_0 \in S_n$ *to* exist such that

$$E(X_0, Y_0) = \max_x \min_y E(X, Y) = \min_y \max_x E(X, Y),$$

and

$$E(X, Y_0) \leq E(X_0, Y_0) \leq E(X_0, Y), \qquad (16)$$

for all $X \in S_m$, $Y \in S_n$.

By definition 5, (X_0, Y_0) is a strategic saddle point, $E(X_0, Y_0)$ is the value of the game and X_0, Y_0 are the optimal strategies.

We thus conclude that every matrix game has a value and an optimal strategy for each player.

THEOREM 5. *Condition* (16) *is equivalent to*

$$E\,(\xi_i, Y_0) \le E(X_0, Y_0) \le X_0, \eta_j) \qquad (17)$$

where ξ_i, i = 1,2, ..., m and η_j, j =1,2,..., n, *are the pure strategies.*

Proof. To prove the equivalence of (16) and (17) we have to prove that (17) is a necessary and sufficient condition for the existence of (16).

That the condition is necessary is obvious. For, (16) holds for all $X \in S_m$. and $Y \in S_n$, and ξ_i and are in S_m and S_n respectively.

To prove that (17) is sufficient for (16), we notice that $E(i_I, Y) = \sum_{j=1}^{n} a_{ij}\, y_j$,

because the ith component of ξ_i is unity and all the other components are zero.

Similarly

$$E(X, \eta_j) = \sum_{j=1}^{n} x_i\, a_{ij}\ .$$

$$\text{Hence } \sum_{i=1}^{m} E(\xi_i, Y) x_i = E(X,Y),$$

$$\text{and } \sum_{j=1}^{n} E(X, \eta_j) y_j = E(X,Y).$$

Now let (17) be true, that is, let

$$E\,(\xi_i, Y_0) \le E\,(X_0, Y_0) \le E(X_0, \eta_j) y_j$$

$$\Rightarrow \sum_{i=1}^{m} E\,(\xi_i, Y_0)\, x_i \le \sum_{i=1}^{m} E\,(X_0, Y_0) x_i,\ \sum_{j=1}^{n} E\,(X_0, Y_0) y_j \le \sum_{j=1}^{n} E\,(X_0, \eta_j) y_j$$

$$\Rightarrow E(X, Y_0) \le E(X_0, Y_0) \le E(X_0, Y),$$

$$\text{since } \sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1. \text{ Proved.}$$

THEOREM 6. *Condition* (17) *reduces to strict equality for at least one i and one j, that is* $\max\limits_{i} E(\xi_i, Y_0) = E(X_0, Y_0) = \min\limits_{j} E(X_o, \eta j)$ (18)

Proof. If possible let

$$E(X_o,Y_0) < \min E(X_0, \eta_j).$$

Then $E(X_0,Y_0) < E(X_0, \eta j)$ for all j,

$\Rightarrow E(X_0, Y_0)\, y_{j0} < E(X_0, m_j)\, y_{j0}$ for all *j*,

where y_{j0} is a component of Y_0,

$$\Rightarrow \sum_{j=1}^{n} E\,(X_0,Y_0)y_{j0} < \sum_{j=1}^{n} E\,(X_0,\eta_j)y_{j0},$$

$$\Rightarrow E(X_0,Y_0) < E(X_0,Y_0),$$

which is absurd.

Hence $E(X_0, Y_o) = \min\limits_{j} E(X_0, \eta_j)$.

Similarly we can prove that

$$E(X_0, Y_0) = \max_{i} E(\xi_i,Y_0). \qquad \text{Proved.}$$

VIII. 7. Graphical solution

We can make use of the theorems discussed above to obtain solution of a rect-angular game, but the methods are laborious. More direct methods are available. We first give a simple graphical method applicable to 2 x *n* or m x 2 rectangular games.

Consider the following problem.

		P_2			
		1	2	3	4
P_1	1	19	15	17	16
	2	0	20	15	5

It is clear that there is no saddle point in this game. Let the mixed strategy of P_l be X = [x 1— x], Then

$$E(X,\eta_1) = 19x + 0(1-x)= 19x \qquad \text{(i)}$$

$$E(X, \eta_2) = 15x + 20(1-x) = 20-5x \qquad \text{(ii)}$$

$$E(X, \eta_3) = 17x + 15(1-x)= 15 + 2x \qquad \text{(iii)}$$

$$E(X, \eta_4) = 16x + 5(1-x) = 5 + 11x \qquad \text{(iv)}$$

We plot *E* against *x* in the domain $0 \le x \le 1$, and obtain Fig, 1, The four lines represent the four expected payoffs to P_1. The continuous piecewise linear curve *OBCD* gives the least expectation for any value of x,$0 \le x \le 1$. P_1 must choose x so as to maximize his least expectation Obviously *C* is the point where the least expectation is the maximum. It is the intersection of lines (ii) and (iv), giving x = 15/16, So the optimal strategy for P_1 is X_0 = [15/16, 1/16], and the value of the game is v =$E(X_0,\eta_2) = E(X_0, \eta_4)$ = 245/16.

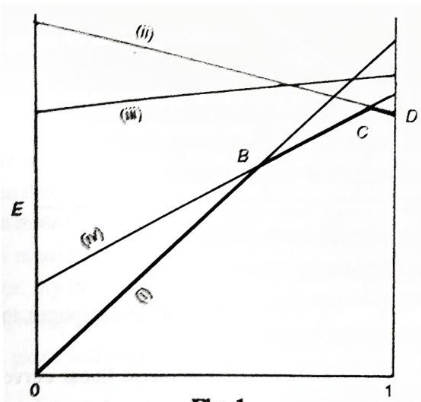


Fig. 1

To find the optimal strategy for P_2; let $Y_0 = [y_1\ y_2\ y_3\ y_4]$, $y_1 + y_2 + y_3 + y_4 = 1$. Then

$$E(X_0, Y_0) = \frac{285}{16}y_1 + \frac{245}{16}y_2 + \frac{270}{16}y_3 + \frac{245}{16}y_4 = \frac{245}{16},$$

$$\Rightarrow \frac{285}{16}y_1 + \frac{270}{16}y_3 + \frac{245}{16}(1 - y_1 - y_3) = \frac{245}{16},$$

$$\Rightarrow \frac{40}{16}y_1 + \frac{25}{16}y_3 = 0,$$

$\Rightarrow y_1 = y_3 = 0$, sine $y_1 \geq 0$, $y_3 \geq 0$.

Hence $\quad Y_0 = [0\ y_2\ 0\ y_4] = [0\ y_2\ 0\ 1-y_2]$.

Now $\quad E(\xi_1, Y_0) = 15y_2 + 16y_4 = 16-y_2 \quad$ (v)

$$E(\xi_2, Y_0) = 20y_2 + 5y_4 = 5 + 15y_2 \quad \text{(vi)}$$

This time we plot E against y_2 in the domain $0 \leq y_2 \leq 1$. The curve *ABC* in Fig.2 represents the maximum payoff to P_1 for any value of y_2. P_2 must choose y_2 so as to minimize this payoff. This happens at the point *B for* which

$$16-y_2 = 5 + 15y_2,$$

or $\quad y_2 = 11/16.$

Consequently

$$y_4 = 1 - y_2 = 5/16.$$

The optimal strategy of P_2 is thus $Y_0 = [0\ \ 11/16\ \ 0\ \ 5/16]$.

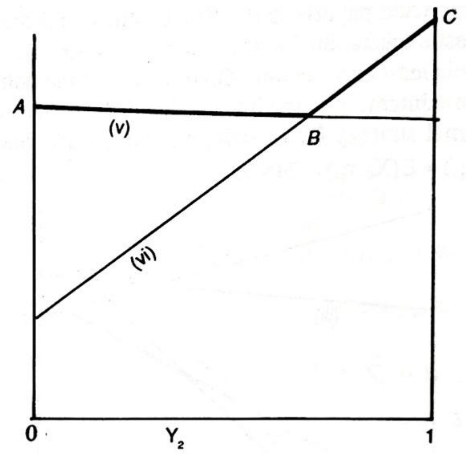


Fig. 2

It can be verified that $E(X_0, Y_0) = 245/16$. This completes the solution of the problem.

To solve an m *x* 2 game graphically, piecewise linear curve representing the maximum expectation for any y_1 should be drawn and the point where it attains its minimum would give the optimal strategy of P_2 and the value of the game. The optimal strategy of P_1 can be found as above.

VIII. 8. Notion of dominance

Sometimes a row or a column in the payoff matrix of a game is obviously ineffective in influencing the optimal strategies and the value of the game. For example, consider the game

		P_2			
	j / i	1	2	3	4
	1	4	−8	7	−2
P_1	2	3	−9	2	−3
	3	−2	6	8	2

Notice the rows 1 and 2. For every j, $a_{ij} > a_{2j}$. Whatever the choice of P_2, P_1 will do better by choosing i = 1 rather than i = 2. The second row therefore should not play any part in the strategy of P_1 or, in other words, the probability associated with it should be zero. The solution of the above game would be the same as that of the game with the payoff matrix

$$\begin{bmatrix} 4 & -8 & 7 & -2 \\ -2 & 6 & 8 & 2 \end{bmatrix}$$

Thp problem is thus simplified. As another example, in the following matrix, the first column does not play any part in deciding the strategy of P_2 and so may be left out of consideration.

$$\begin{bmatrix} 3 & 1 & -1 \\ -1 & -2 & 3 \\ 4 & 3 & -3 \end{bmatrix}$$

VIII. 9. Rectangular game as an LP problem

It can be shown that the problem of solving a rectangular game is equivalent to solving a problem of linear programming. This provides one of the methods of solving a matrix game problem. We shall only explain here how to convert the problem of a matrix game into an equivalent LP problem.

Let the m x n matrix A = {a_{ij}} be the payoff matrix of the game and v its value, v is a real number. By increasing every a_{ij} by a suitable positive number k, we may form a matrix A_k = {$a_{ij}+k$} = {a_{ij}'} where every $a_{ij}^1 > 0$. The expectation function E_k (X, Y) of the game with payoff matrix A_k is given by equation (13) as

$$E_k(X,Y) = E(X,Y) + k.$$

By such a transformation the optimal strategies of the game do not change, but the value of the game is increased by k, and it is ensured that this new value is positive.

Let us assume that, if necessary, after this transformation, the matrix of the game is A = {a_{ij}}, where $a_{ij} > 0$ for all i and j, and the value of the game is v > 0. Let $X_0 = [X_1\ X_2 ...\ x_m$ and $Y_0 = [y_1\ y_2\ ...\ y_n]$ be optimal strategies of P_1 and P_2 respectively. Then, from (17), for all j,

$$E(X_0, \eta_j) \geq E(X_0, Y_0) = v,$$

or $\sum_{i=1}^{m} a_{ij}\, x_i \geq v,\ j = 1, 2, ..., n,$

subject to $\sum_{i=1}^{m} x_i = 1,$

and $x_i \geq 0$, i=1,2,…..,m.

Since v>0, dividing (19) throughout by v, we get

$$\sum_{i=1}^{m} a_{ij}\, x_i' \geq 1,\ j=1,2,\ldots.,n,$$

subject to $$\sum_{i=1}^{m} x_i' = 1/v, x_i' \geq 0.$$

The strategy of P_1 is to maximize v. Therefore he has to choose x_t', such that

Subject to

$$\left.\begin{array}{l} f = \sum_{i=1}^{m} x_i' \text{ is minimum} \\ \sum_{i=1}^{m} a_{ij} x_i' \geq 1, \quad j = 1,2,\ldots,n, \\ x_i' \geq 0, \qquad i = 1,2,\ldots,m. \end{array}\right\} \tag{20}$$

This is an LP problem put in the standard primal form. The value of the game is $v = l/f_{min}$, and the optimal strategy of P_1 is $\{x_i\} = \{x_i'v)$ where $\{x_i^1\}$ is the optimal solution of the LP problem.

If we start from the inequality

$$E(\xi_i, Y_0) \leq E(X_0, Y_0) = v$$

of (17), we shall get the LP problem as

$$\left.\begin{array}{l} \phi = \sum_{j=1}^{n} y_j' \text{ is maximum} \\ \sum_{j=1}^{n} a_{ij} y_j' \leq 1, \quad i = 1,2,\ldots,m, \\ y_j' \geq 0, \qquad j = 1,2,\ldots,n, \end{array}\right\} \tag{21}$$

which is the dual of (20). One may either solve the primal or the dual to get the solution of the game.

PROBLEMS

1. Prove that $f(x) = x^2$, $x \in R$, is a convex function.

2. Prove that $f(X) = ||X||$, $X \in E_n$, is a convex function.

3. Prove that the linear function $f(X) = CX$, $X \in E_n$, is both convex and concave.

4. Prove that $f(X) = 2x_1^2 + 2x_2^2 + 4x_3^2 + 2x_1x_2 + 2x_1x_3 + 4x_2x_3$ is a convex function.

5. Sow that the sum of two convex functions is a convex function.

6. Prove that every positive linear combination of convex functions in K is a convex function in K. (This is a generalization of the above problem).

7. Prove that for f(X) to be convex in K, it is necessary and sufficient that for any positive integer m and for $X_i \in K$, i=1,2,….,m.

   $f(\lambda_1 x_1+\lambda_2 x_2 + \ldots+\lambda_m x_m) \le \lambda_1 f(x_1) + \lambda_2 f(x_2) + \ldots+\lambda_m f(X_m)$

   where $\sum_{i=1}^{m} \lambda_i = 1, \lambda_i \ge 0.$

8. Solve graphically.

   a. Maximize $5x_1 + 3x_2$ subject to $4x_1 + 5x_2 \le 10$, $5x_1 + 2x_2 \le 10$, $3x_1 + 8x_2 \le 12$, $x_1 \ge 0$, $x_2 \ge 0$.

   b. Maximize $4x_1 + 5x_2$ subject to $x_1 - 2x_2 \le 2$, $2x_1 + x_2 \le 6$, $x_1 + 2x_2 \le 5$, $-x_1 + x_2 \le 2$, $x_1 + x_2 \ge 1$, $x_1 \ge 0$, $x_2 \ge 0$.

   c. Minimize $x_1 + 3x_2$ subject to $x_1 + x_2 \ge 3$, $-x_1 + x_2 \le 2$, $x_1 - 2x_2 \le 2$, $x_1 \ge 0$, $x_2 \ge 0$.

   d. Maximize $5x_1 - x_2$ subject to $x_1 + x_2 \ge 2$, $x_1 \ge 2$, $x_1 + 2x_2 \le 2$, $2x_1 + x_2 \le 2$, $x_1 \ge 0$, $x_2 \ge 0$.

   e. Maximize $2x_1 + x_2$ subject to $x_1 - 3x_2 \le 3$, $x_1 \le 8$, $2x_1 + x_2 \le 20$, $x_1 + 3x_2 \le 30$, $-x_1 + x_2 \le 6$, $x_1 \ge 0$, $x_2 \ge 0$.

f. Minimize $-5x_1 - 3x_2$ subject to $x_1 + x_1 \leq 2$, $5x_1 + 2x_2 \leq 10$, $3x_1 + 8x_2 \leq 12$, $x_1 \geq 0$, $x_2 \geq 0$.

g. Maximize $4x_1 + 2x_2$ subject to $x_1 + x_2 \leq 8$, $x_1 = 4$, $x_1 \geq 0$, $x_2 \geq 0$. Does the optimal solution.

Change if the constraint $x_1 = 4$ is changed to (i) $x_1 \leq 4$ (ii) $x_1 \geq 4$? [(i) No (ii) Yes]

9. Show graphically that

a. The problem: Maximize $3x_1 + 4x_2$ subject to $4x_1 + 3x_2 \geq 12$, $x_1 + 2x_2 \leq 2$, $x_1 \geq 0$, $x_2 \geq 0$, has no feasible extreme points. What can be concluded regarding the solution of the problem? [No solution]

b. At the optimal solution of the problem: Maximize $6x_1 - 4x_2$ subject to $x_1 - x_2 \leq 1$, $3x_1 - 2x_2 \leq 6$, $x_1 \geq 0$, $x_2 \geq 0$, the values of x_1 and x_2 can be increased indefinitely while the value of the objective function remains unchanged.

10. Write the following LP problems in the standard form of section 3 (minimize CX subject to AX=B, X$\geq$0).

a. Minimize $x_1 + x_2 - x_3$ subject to $x_1 + x_2 \geq 2$, $x_1 - x_3 \leq 4$, $2x_1 - x_2 + x_3 \geq 1$, $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$.

b. Maximize $2x_1 + 3x_2 + 5x_3$ subject to $x_1 + x_2 - x_3 \geq -5$, $-6x_1 + 7x_2 - 9x_3 \leq 4$, $x_1 + x_2 + 4x_3 = 10$, $x_1 \geq 0$, $x_2 \geq 0$, x_3 unrestricted in sign.

c. Minimize $-3x_1 + 4x_2 - 2x_3 + 5x_4$ subject to $-2x_1 + 3x_2 - x_3 + 2x_4 \geq 2$, $4x_1 - x_2 + 2x_3 - x_4 = -2$, $x_1 + x_2 + 3x_3 - x_4 \leq 14$, $x_1 \geq 0$, $x_2 \geq 0$, x_3 and x_4 unrestricted in sign.

11. For the problem: Maximize $-6x_1 + 2x_2 - 4x_3 + 5x_4$ subject to $4x_1 - x_2 + 2x_3 + 3x_4 \leq 1$, $x_2 + 4x_3 - 2x_4 \leq 2$, $x_1, x_2, x_3, x_4 \geq 0$, determine (i) the maximum number of possible basic solutions, (ii) the basic feasible solutions, (iii) the feasible extreme point solutions, (iv) the optimal solution.

[(i)15; (ii) (0,8,0,3,0,0), (3/4,2,0,0,0,0), (0,0,0,1/3,0,8/3), (1/4,0,0,0,0,2), (0,0,1/2,0,0,0), (0,0,0,0,1,2); (iii) same as in (ii); (iv) (0,8,0,3,0,0); 31]

12. Solve (i) and (ii) of problem 1 by simplex method. In each case follow the solution steps of the simplex method by interpreting them as the shift from one vertex of the feasible region to another vertex in the graphical solutions obtained earlier.

13. Solve the following problems using (a) the big M method, (b) the two-phase simplex method. (i) Problem 8 (iii). (ii) Problem 8 (iv).

    (iii) Minimize $2x_1 – 3x_2 + 6x_3$ subject to $3x_1 – 4x_2 – 6x_3 \leq 2$, $2x_1 + x_2 + 2x_3 \geq 11$, $x_1 + 3x_2 – 2x_3 \leq 5$, $x_1, x_2, x_3 \geq 0$. [9; $x_1 = 0$, $x_2 = 4$, $x_3=7/12$]

14. Solve the following problems using simplex method.

    (i) Maximize $5x_1 – 3x_2 + 4x_3$ subject to $x_1 – x_2 \leq 1$, $– 3x_1 + 2x_2 + 2x_3 \leq 1$, $4x_1 – x_3 = 1$, $x_2 \geq 0$, $x_3 \geq 0$, x_1, unrestricted in sign.

    [43/5; $x_1=3/5$, $x_2 = 0$, $x_3=7/5$]

    (ii) Maximize $5x_1 + 3x_2 + x_3$ subject to $2x_1 + x_2 + x_3 = 3$, $– x_1 + 2x_3 = 4$, $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$.

    [5; $x_1 = 0$, $x_2 = 1$, $x_3 = 2$]

15. Use the simplex method to verify that the following problem has no finite optimal solution.

    Maximize $2x_1 + x_2$ subject to $x_1 – x_2 – x_3 \leq 1$, $x_1 – 2x_2 + x_3 \leq 2$, $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$.

    From the final simplex tale construct a feasible solution with value of the objective function greater than 2000.

    [$x_1 = 1$, $x_2 = 2000$, $x_3 = 0$. There can be many more]

16. Solve the following problem by the simplex method, taking the entering variable at each iteration to be the non basic variable with the most negative relative cost coefficient at that stage.

Minimize $x_1-2x_2-3x_3$ subject to $-2x_1 + 4x_3 \leq 12$, $-4x_1 + 8x_2 + 3x_3 \leq 10$, $3x_1 + 2x_2 - x_3 \leq 7$, $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$.

Now solve this problem again, selecting the entering variable at each iteration as the non basic variable with the least negative relative cost coefficient at that stage. Do you get the same solution in both cases? [Yes]

17. Show that the LP problem: Minimize f(X) subject to AX=B, $X \geq D \geq 0$, can be converted into the form: Minimize f(Y) subject to AY = = K, $Y \geq 0$. Use this approach to solve the problem: Minimize $-3x_1-x_2-2x_3$ subject to $2x_1 + 3x_2 - x_3 \leq 28$, $-2x_1 + 2x_2 + 5x_3 \leq 33$, $x_1 + x_2 - 2x_3 \leq 37$,
$X_1 \geq 2$, $x_2 \geq 1$, $x_3 \geq 3$. [$-175/2$, $x_1 = 39/2$, $x_2 = 1$, $x_3 = 14$]

18. Show that the simplex method can be used to solve a system of linear equations or inequalities, and use this approach to solve the following sets.
    i. $2x_1 + x_2 - x_3 + 2x_4 = 4$, $x_1 - x_2 + x_3 + x_4 = 2$, $x_1 - x_3 + 3x_4 = 3$, $x_1, x_2, x_3, x_4 \geq 0$.
    ii. $2x_1 - 4x_2 = 1$, $2x_1 - 3x_2 - 2x_3 \geq 3$, $x_1 - x_2 - 6x_3 \leq 5$, $x_1, x_2, x_3 \geq 0$.

    [Hint. Introduce an artificial variable in each of the equations (after converting inequalities into equations), and then minimize an objective function which is the sum of the artificial variables. Thus we get a set of basic variables expressed in terms of the other variables which can be given arbitrary values to get any number of solutions.
    i. $x_2 = 3-2x_1$, $x_3 = 3 - 2x_1$, $x_4 = 2 - x_1$
    ii. $x_1=(19/2) + 12x_3 - 2x_5$, $x_2 = (9/2) + 6x_3 - x_5$, $x_4 = (5/2) + 4x_3 - x_5$]

19. For the problem: Maximize $f=3x_1 + 4x_2+x_3+5x_4$ subject to $4x_1 + 3x_2 - 2x_3 + x_4 \leq 10$, $3x_1 + 8x_2 - 3x_3 + 2x_4 \leq 20$, $-5x_1 + 4x_2 - 6x_3 + 4x_4 \geq -20$, $x_1, x_2, x_3, x_4 \geq 0$; one iteration in the simplex table is as follows. (x_5, x_6, x_7 are the slack variables).

Basis	Value	P_1	P_2	P_3	P_4	P_5	P_6	P_7
x_4					1		2	
x_3							1	
x_7							2	
$-f$	-50	-38	33			-17	11	

Without actually performing the simplex iterations, find the following missing entries. (i) P_7 column, (ii) P_2 column, (iii) coefficient of x_4 in the objective function row, (iv) values of the basic variables x_4, x_3, x_7, [Hint. $X=A^{-1}B$]. [(i) (0,0,1), (ii) (7,2,12), (iii) 0, (iv) (10,0,60)].

20. Each of the following tables represents an iteration of the simplex tableau.

Basis	Value	P_1	P_2	P_3	P_4
		Table 1			
x_1	2	1	1	0	2
x_3	5	0	2	1	1
f	-10	0	3	0	0
		Table 2			
x_1	10	1	0	1	2
x_2	2	0	1	2	1
f	-20	0	0	-3	1
		Table 3			
x_2	20	1	1	-2	0
x_4	-10	-3	0	0	1
f	0	3	0	10	0
		Table 4			
x_3	0	-1	0	1	3
x_2	10	3	1	0	2
f	-20	12	0	0	-3

Select one or more of the following conditions that best describe the results indicated in each table.

(i) The solution is not basic feasible.

(ii) Optimal solution has been obtained.

(iii) An optimal solution has been obtained, but it is not unique. Find others.

(iv) Improvement in the value of the objective function is still possible. Which variable should enter the basis, and which should go out?

(v) The solution is degenerate. Which variable is causing degeneracy?

[(i) Table 3, (ii) Table 1, (iii) Table 1, ($x_3 = 4$, $x_4 = 1$, $f = 10$, (iv) Table 2, x_3 should replace x_2 in the basis, (v) Table 4, x_3].

21. Let A_0 be the matrix of a feasible basis of the LP problem as stated by (6), (7), (8), and II be the multiplier vector for this basis, and C_0 the row vector of cost coefficients corresponding to the basic variables. Prove that $f(X) = -B^1\Pi = C_0(A_0^{-1}B)$. (Hint. See section 15).

22. Show that an LP problem (with objective function required to be minimum) has a unique optimal solution if there exists an optimal basis such that the relative cost coefficients of the nonbasic variables are strictly positive. If one

of these relative cost coefficients, say c_k' is zero, then there exists another optimal solution with x_k *in* the basis, provided there is a positive coefficient a'_{ik} of x_k in the canonical form of equations at that stage.

23. Show that the problem:

Minimize $f = -x_4 + 7x_5 + x_6 + 2x_7$

Subject to $x_1 + x_4 + x_5 + x_6 + x_7 = 1,$

$$x_2 + \frac{1}{2}x_4 - \frac{11}{2}x_5 - \frac{5}{2}x_6 + 9x_7 = 0,$$

$$x_3 + \frac{1}{2}x_4 - \frac{3}{2}x_5 - \frac{1}{2}x_6 + x_7 = 0;$$

$$x_1, \ldots, x_7 \geq 0;$$

is degenerate and cycling occurs if the initial basis (x_1, x_2, x_3) is successively modified as follows:

Incoming:	x_4	x_5	x_6	x_7	x_2	x_3
Outgoing:	x_2	x_3	x_4	x_5	x_6	x_7;

but no cycling occurs and the optimal solution is immediately obtained if x_3 is replaced by x_4. (Marshall and Suurballe).

24. In the problem: Maximize $20x_1 + x_2 + 10x_3$ subject to $x_1 + 4x_2 - x_3 \leq 20$. $x_1 + x_2 \leq 10$, $3x_1 + 5x_2 - 3x_3 \leq 50$, $x_1, x_2, x_3 \geq 0$, in which direction is the solution space unbounded? Without any computations what can be said about the optimal solution to the problem?

[Solution space is unbounded in the direction of x_3. Optimal solution is also unbounded].

Duality

25. Write the duals of the following LP problems. In each case verify that the dual of the dual is the primal.

(i) Minimize $6x_1 + 3x_2$ subject to $3x_1 + 4x_2 + x_3 \geq 5$, $6x_1 - 3x_2 + x_3 \geq 2$, $x_1, x_2, x_3 \geq 0$.

(ii) Minimize $x_1 - 3x_2 - 2x_3$ subject to $2x_1 - 4x_2 \geq 12$, $3x_1 - x_2 + 2x_3 \leq 7$, $-4x_1 + 3x_2 + 8x_3 = 10$, $x_1 \geq 0$, $x_2 \geq 0$, x_3, unrestricted.

(iii) Maximize $x_1 + 6x_2 + 4x_3 + 6x_4$ subject to $2x_1 + 3x_2 + 17x_3 + 80x_4 \leq 48$, $8x_1 + 4x_2 + 4x_3 + 4x_4 = 21$, $x_2, x_3 \geq 0$, x_1, x_4 unrestricted in sign.

[(iii) Minimize $48y_1 + 21y_2$ subject to $2y_1 + 8y_2 = 1$, $3y_1 + 4y_2 \geq 6$, $17y_1 + 4y_2 \geq 4$, $80y_1 + 4y_2 = 6$, $y_1 \geq 0$, y_2 unrestricted].

26. For the problem: Minimize $x_1 + x_2$ subject to $2x_1 + x_2 \geq 8$, $3x_1 + 7x_2 \geq 21$. $x_1 \geq 0$, $x_2 \geq 0$; (i) find the dual, (ii) solve the primal and the dual graphically and verify theorem 8, (ii) find the simplex multipliers for the optimal solution in either case and verify theorem 9.

$$\left[For\ primal\left(-\frac{4}{11}, -\frac{1}{11}\right),\ for\ dual\left(\frac{35}{11}, \frac{18}{11}\right) \right]$$

27. Solve graphically to show that the following problem has an unbounded solution. Write its dual and solve it to verify that it has no solution.

Maximize $3x_1 + 4x_2$ subject to $x_2 - x_1 \leq 1$, $x_1 + x_2 \geq 4$, $x_1 - 3x_2 \leq 3$, $x_1 \geq 0$, $x_2 \geq 0$.

28. Solve the following problem by the simplex method. Also solve it by solving its dual graphically. Maximize $y_1 + y_2 + y_3$ subject to $2y_1 + y_2 + 2y_3 \leq 2$, $4y_1 + 2y_2 + y_3 \leq 2$, $y_j \geq 0$ for $j = 1, 2, 3$.

[$y_1 = 0$, $y_2 = 2/3$, $y_3 = 2/3$; 4/3]

29. The optimal solution of the problem: Maximize $z = 5x_1 + 3x_2 + 2x_3$ subject to $x_1 - 6x_2 - 5x_3 \leq 40$, $x_1 + 2x_2 + 5x_3 = 30$, $x_1, x_2, x_3 \geq 0$; is given by the following table.

Basis	Value	x_1	x_2	x_3	x_4
x_4	10	0	−8	−10	1
x_1	30	1	2	5	0
−z	150	0	7	23	0

Write the dual problem, and find its solution from this table.

[Minimize $40y_1 + 30y_2$ subject to $y_1 + y_2 \geq 5$, $-6y_1 + 2y_2 \geq 3$, $-5y_1 + 5y_2 \geq 2$, $y_1 \geq 0$, y_2 unrestricted; $y_1 = 0$, $y_2 = 5$; 150].

30. Show graphically that the problem: Maximize $6x_1 + 8x_2$ subject to $-x_1 + x_2 \geq 2$, $-5x_1 + 5x_2 \leq 3$, $x_1 \geq 0$, $x_2 \geq 0$; has no feasible solution. Write its dual and show that it has no unbounded solution. How do you reconcile this example with the statement of theorem 10?

    [Theorem 10 states that if the primal has no feasible solution, then the dual has an unbounded solution provided it has at all a feasible solution]

31. Write the dual of the LP problem: Maximize $2x_1 + 5x_2 + 3x_3$ subject to $4x_1 + x_3 \leq 420$, $2x_2 + 3x_3 \leq 460$, $2x_1 + x_2 + x_3 \leq 500$, $x_1, x_2, x_3 \geq 0$. Without carrying out the simplex computation on either the primal or the dual problem, estimate a range for the optimal value of the objective function.

    [An obvious feasible solution of the primal is $x_1 = x_3 = 0$, $x_2 = 230$, which gives the value of the objective function as 1150. Similarly an obvious solution of the dual is $y_1 = y_2 = 0$, $y_3 = 5$, which gives the objective value function as 2500. Thus we estimate the range as (1150, 2500)].

32. Prove that if for the optimal solution of the primal the slack variable x_{n+r} is nonzero, then the variable y_r in the optimal solution of the dual is zero, and vice versa. (Hint. Examine the expression for $f-\phi$ in theorem 7. All the variables are non-negative, and for optimality $f = \phi$).

33. Prove the second part of theorem 9.

34. Prove that if the *k*th constraint of the primal is an equality then the dual variable y_k is unrestricted in sign. (Hint Replace the equality by two inequalities numbered k_1 and k_2. Then the corresponding dual variables y_{k1} and y_{k2} occur as $y_{k1} - y_{k2}$ and each is non-negative).

35. Let the primal be written in the form: Minimize $f(X) = CX$ subject to $AX = B$. $X \geq 0$. Show that its dual is: Maximize $\phi(Y) = B'Y$ subject to $A'Y \leq C^1$, (Y unrestricted).

36. Solve the following problems by the dual simplex method.

    i. Minimize $2x_1+3x_2$ subject to $2x_1 + 3x_2 \leq 30$, $x_1 + 2x_2 \geq 10$, $x_1 \geq 0$, $x_2 \geq 0$.

[(0,5); 15]

ii. Minimize $x_1 + 3x_2 + 2x_3$ subject to $4x_1 - 5x_2 + 7x_3 \leq 8$, $2x_1 - 4x_2 + 2x_3 \geq 2$, $x_1 - 3x_2 + 2x_3 \leq 2$, $x_1, x_2, x_3 \geq 0$. [(1,0,0); 1]

iii. Minimize $5x_1 + 7x_2 + 6x_3 + 3x_4$ subject to $4x_1 + 6x_2 + 5x_3 - 3x_4 \geq 12$, $x_1 + 5x_2 + 2x_3 + x_4 \geq 8$, $-6x_1 + x_2 - 5x_4 \geq 8$, $x_1, x_2, x_3, x_4 \geq 0$. [(0, 8,0,0); 56]

37. What type of problems can be solved by the dual simplex method? Which of the problems 8 are of this type? Solve them by the dual simplex method.

[Only 8 (iii). $x_1 = 8/3$, $x_2 = 1/3$; 11/3]

Applications

Set up mathematical models for the following problems and solve them graphically, if possible, otherwise by the simplex method.

38. A person has the option of investing Rs 10,000 in two plans, A and B. Plan A guarantees a return of 50 paisa on each rupee invested after a period of three years, and plan B guarantees that each rupee invested will earn one and a half rupee after six years. How should the person invest his money to maximize his earnings in a period of six years, if he is not willing to invest more than 60% in plan B? [4000, 6000]

39. A company manufactures two products, A and B, each on separate sets of machines. The production capacity is 60 units of A and 75 units of B per day. Each unit of A requires 10 pieces of a component, and each unit of B requires 8 pieces of the same component. The maximum availability of this component is 800 pieces a day. The profit per unit on A is Rs 3 and on B Rs 2. Determine the optimum daily production of each product to maximize profit.

[60, 25]

40. A farmer has to plant trees of two kinds A and B in a land 4400 m^2 in area. Each A tree requires at least 25 m^2 and B tree at least 40 m^2 of land. The annual water requirement of A is 30 units and of B 15 units per tree, while at most 3300 units of water is available. It is also estimated that the ratio of the number

of B trees to the number of A trees should not be less than 6/19 and not more than 17/8. The return per tree from A trees is expected to be one and half times as much as from B trees. What should be the number of trees of each kind so that the expected return is maximum?

[80, 60]

41. A woman worker has two typos of jobs in a handicraft centre (i) spinning thread (11) knitting patterns from the thread so produced. She produces one unit of thread per hour and one unit of pattern per hour, and is paid one rupee per unit of thread and three rupees per unit of pattern. She wants to earn not less than Rs 6 per day and wants to work not more than 6 hours a day. The thread spun should not exceed the thread consumed by more than 2 units. The centre desires that her earnings from knitting should not exceed her earnings from spinning by Rs. 10. Selling profit is Re 1 per unit of thread and Rs 2 per unit of pattern. How many units of thread and pattern should the woman produce everyday to maximize the profit from sales?

[2, 4]

42. A housing corporation proposes to build houses of type A, B, C with cost and return per thousand as given below:

Type	Cost in Rs. 10^4	Return per month
A	2	50
B	5	150
C	8	300

The total investment should not exceed Rs 4 x 10^7. How many houses of each type should be built to maximize return if at least 50% of the houses should be of type A, and not more than 20% houses should be of type C? [487,293, 195]

43. A manufacturer produces three models A, B, C of a certain product. He uses two types of raw material, I and II. The availability of these raw materials and their requirement in each of these three models are given below.

Raw Material	Requirement per unit of			Availability
	A	B	C	
I	3	3	2	4000
II	2	3	5	6000

The labour time for model C is half that of A and the labour time for model B is three-fourth that of A. The production capacity of the factory is equivalent to 1500 units of A per week. The manufacturer is committed to supply 200 units of B and 200 units of C per week. Market survey indicates that there is a profit of Rs 24, Rs 20 and Rs 15 per unit of A, B and C respectively. What number of units of each model must be produced to maximize profit.

[1000, 200, 200]

44. A student, on the eve of examination, has 100 hours to prepare three subjects A, B and C. For every one hour of study he hopes to get 1 mark in A, 2 marks in B and 3 marks in C. He has already secured 60, 70 and 67 marks in course work in each of these subjects respectively. He must get at least 40, 30, and 33 more to pass in each of these subjects. Also he cannot gel more than 100 marks in any written paper. For how many hours should he study each subject to maximize his total marks in the examination. [40,27,33]

45. A contractor generally undertakes construction of two types of buildings, say A and B. The profit per building is Rs 5,000 for type A and Rs 4,000 for type B. Building of type A requires 50,000 bricks, 3 quintals of steel, 2 quintals of roofing iron and 10 quintals of concrete. Building of type B requires 40,000 bricks, 4 quintals of steel, 3 quintals of roofing iron, and 20 quintals of concrete. However, only 3, 20, 000 bricks, 24 quintals of steel, 20 quintals of roofing iron and 160 quintals of concrete are available. How many buildings of the two types should he undertake to maximize his profits? [4, 3]

46. Four products, A, B, C, D, are successively processed on two machines, M_1, M_2. The machine time required on each machine, and the total time available on it is given in the following table.

	Time per unit (hours)				Total time available (hours)
	A	B	C	D	
M_1	2	1	1	2	380
M_2	2	4	3	1	500

The total cost of producing one unit of each product is taken equal to the cost of machine time. The cost per hour on machines M, and M_2 are Rs 15 and Rs 10 respectively. If the sale price per unit of the product A, B, C, D is Rs 65, Rs 70, Rs 55 and Rs 45 respectively, how many units of each product should be produced to maximize profit? What is the maximum profit?

[(170, 40, 0, 0); Rs 3150]

47. A farmer has the choice to plant wheat, com, cereals and soyabeans in his fields consisting of 20 hectares of land. The profits he expects per hectare of these crops are respectively Rs 3000, Rs 2000, Rs 1000 and Rs 4000. These crops respectively require 500, 800, 400 and 500 kg per hectare of fertilizers, and 60, 40, 20 and 30 kg per hectare of insecticides. Also it takes respectively 3, 3, 1 and 2 man-days to cultivate a hectare of these crops. How many hectares of each crop will yield maximum total profit if the farmer's resources limit fertilizers to 15000 kg, insecticides to 1000 kg, and man-days to 50 and he also decides to use not more than 50% of his land for cereals and soyabeans.

[10, 0, 0,10]

48. A fertilizer firm produces four types of fertilizers, say A, B, C, D. The amount of nitrogen, phosphate and potash needed for production of 10 kg of the fertilizer each of these varieties is as follows.

Amount needed per 10 kg

Fertilizer	Nitrogen	Phosphate	Potash
A	5	2	3
B	3	1	4
C	2	3	2
D	3	4	3

The firm has a supply of 500 kg of nitrogen, 400 kg of phosphate and 400 kg of potash per day. If the company makes profit of Rs 5, Rs 4, Rs 5, Rs 2 per kg on fertilizer A, B, C, D respectively, and is committed to supply 100 kg of variety D per day, determine the amount of fertilizer of each type which the firm should produce each day to maximize profit.

[556,112,792,100]

49. A manufacturer produces three products, A, B, C. Each product is to be processed on two machines, M_1, M_2. The following table gives the time required on the machines.

Product	Machine time in hours	
	M_1	M_2
A	0.5	0.2
B	0.7	0.8
C	0.9	1.05

There are 105 working hours available on each of die machines. The operating cost of M_1 is Rs 5 per hour, and of M_2 Rs 4 per hour. The production requirements are at least 40 units of A, 40 units of B and 60 units of C. Use dual simplex method to find how many units of each product should be produced to minimize the production cost. [40,40,60 units, Rs 922]

50. A workshop manufactures two alloys, A and B, and sells them at profit of Rs 4 and Rs 5 per kg respectively. Whereas alloy A requires nickel, chromium, germanium and magnesium in the ratios 3:4:3:2, alloy B requires them in the ratios 2:3**:1:2.** Also 8 kg of nickel, 12 kg of chromium, 7 kg of germanium and 6 kg of magnesium are available to the workshop per day. Set up the

mathematical model for this problem to maximize profit. Write and solve its dual by the dual simplex method. Use this solution to obtain the solution of the original problem. [Dual solution: (2,0,0,1); primal solution: (24, 8)]

51. Solve the transportation problem for minimum cost with the cost coefficients, demands and supplies as given in the following table. Obtain three optimal solutions.

	D_1	D_2	D_3	D_4	
0_1	1	2	-2	3	70
0_2	2	4	0	1	38
0_3	1	2	-2	5	32
	40	28	30	42	

[86]

52. Solve the following transportation problem for minimum cost starting with the degenerate solution $x_{l2} = 30$. $x_{21} = 40$, $x_{32} = 20$, $x_{43} = 60$.

	D_1	D_2	D_3	
0_1	4	5	2	30
0_2	4	1	3	40
0_3	3	6	2	20
0_4	2	3	7	60
	40	50	60	

[300]

53. A farmer has three farms A, *B, C,* which need respectively 100. 300 and 50 units of water (in suitable units) annually. The canal can supply 150 units and tubewell 200 units while the balance is left at the mercy of rain god. The following table shows the costs per unit of water in a dry year when the rains totally fail, the third row giving the costs of failure of rain. Find how the canal and tubewell water should be utilized to minimize the total cost.

	A	B	C	
Canal	3	5	7	150
Tubewell	6	4	10	200
Failure rain	8	10	3	100
	100	300	50	

(Tubewell: 200 to B; Canal: 100 to A, 50 to B]

54. Food bags have to be lifted by three different types of aircraft A_1, A_2, A_3 from an airport and dropped in flood affected villages V_1, V_2, V_3, V_4, V_s. The quantity of food (in suitable units) that can be carried in one trip by aircraft A_i to village V_j is given in the following table. The total number of trips that A_i can make in a day is given in the last column. The number of trips possible each day to village V_i is given in the last row. Find the number of trips each aircraft should make on each village so that the total quantity of food transported in a day is maximum.

	V_1	V_2	V_3	V_4	V_5	
A_1	10	8	6	9	12	50
A_2	5	3	8	4	10	90
A_3	7	9	6	10	4	60
	100	80	70	40	20	

(Hint To balance the problem introduce a fictitious aircraft which carries no food. To convert the problem to a minimum one, change the sign of each c_{ji}.)

[A_1:50 trips to V_1; A_2:70 trips to V_3; 20 to V_5; A_3:20 trips to V_2; 40 to V_4]

55. Power stations P_j, j = 1, 2, 3, 4, run on coal found in mines Q_i, i = 1,2, 3. Quantity of coal produced *at* Q_i *is* a_i, and maximum possible production of power at P_j is b_j. The cost of production of unit quantity of coal at Q is c_i and its cost of transportation to Pj is c_{ij}. A unit quantity of coal produces h_j units of power at P_j. How should coal be distributed so that the total cost of coal at power stations is minimum, and what is the power produced at each station? Solve the problem for the following data.

i	c_{ij}				c_i	a_i
1	4	3	2	1	10	750
2	3	5	6	2	15	350
3	6	4	3	3	20	400
h_j	1/2	1/2	1/3	1/4		
b_j	100	150	300	200		
j	1	2	3	4		

[x_{13}=100, x_{14}=650, x_{21}=200, x_{24}=150, x_{33}=400; Power produced: 100, 0, 500/3, 200]

56. A company has three factories manufacturing the same product and five sole agencies in different parts of the country. Production costs differ from factory to factory, and sale prices from agency to agency. Find the production and distribution schedule most profitable to the company, given the following data.

Factory		1	2	3	Sale Price	Demand
Production cost		20	22	18		
	Agencies					
Transportation cost	1	3	9	4	30	80
	2	1	7	5	32	100
	3	5	8	3	31	75
	4	7	3	2	34	45
	5	4	6	7	29	125
		150	200	125		

57. There are forest areas F_1, F_2, F_3, F_4, and timber depots D_1, D_2, D_3. The following table gives the produce of each forest area, the minimum timber required at each depot to attract buyers, and the cost of transportation per unit of timber from each forest area to each depot. Find the distribution of the entire forest produce for minimum cost of transportation.

	D_1	D_2	D_3	
F_1	3	4	6	100
F_2	7	3	8	80
F_3	6	4	5	0
F_4	7	5	2	120
	110	110	60	

58. A company wishes to manufacture five products, A, B, C, D, E, in its three workshops W_1, W_2, W3, whose production capacities are 90, 40, 60 units respectively. The potential sales of the five products are 60, 30, 40, 70, 40 units respectively. Product D cannot be produced in workshop W_3. What quantity of each product should be produced at each workshop to minimize the cost? The production costs per unit are given in the following table.

	A	B	C	D	E
W_1	14	19	20	16	21
W_2	13	20	15	16	19
W_3	18	15	18	-	20

(Hint, Take the production cost of D in W_3 as infinity).

[$A \to W_1$, 20; $A \to W_2$, 40; $B \to W_3$, 30; $C \to W_3$, 30; $D \to W_1$, 70; no production of E. Total cost: 2910].

Solve problems 59 to 63 by the cutting plane method.

59. Minimise $4x_1 + 5x_2$ subject to $3x_1 + x_2 \geq 2$, $x_1 + 4x_2 \geq 5$, $3x_1 + 2x_2 \geq 7$; x_1, x_2 non-negative integers. [13; (2, 1)]

60. Maximize $x_1 + x_2$ subect to $7x_1 - 6x_2 \leq 5$, $6x_1 + 3x_2 \geq 7$, $-3x_1 + 8x_2 \leq 6$; x_1, x_2 non-negative integers. [2; (1, 1)]

61. Maximise $x_1 + x_2$ subject to $2x_1 \leq 3$, $2x_1 + 2x_2 \geq 5$, $-2x_1 + 2x_2 \leq 1$; x_1, x_2 non-negative integers. [Infeasible]

62. Minimize $3x_1 - x_2$ subject to $-10x_1 + 6x_2 \leq 15$, $14x_1 + 18x_2 \geq 63$; x_1, x_2 non-negative integers. [−1; (1, 4)]

63. Minimize $-2x_1 - 3x_2$ subject to $2x_1 + 2x_2 \leq 7$, $0 \leq x_1 \leq 2$, $0 \leq x_2 \leq 2$; x_1, x_2 integers. [−8; (1, 2)]

Solve problems 64 to 68 by the branch and bound method.

64. Maximise $11x_1 + 21x_2$ subject to $4x_1 + 7x_2 + x_3 = 13$; x_1, x_2, x_3 non-negative integers. [33; (3, 0, 1)]

65. Minimize $9x_1 + 10x_2$ subject to $0 \leq x_1 \leq 10$, $0 \leq x_2 \leq 8$, $3x_1 + 5x_2 \geq 45$; x_2 integer. [95; (5/3, 8)]

66. Maximize $13x_1 + 3x_2 + 3x_3$ subject to $7x_1 + 6x_2 - 3x_3 \leq 8$, $7x_1 - 3x_2 - + 6x_3 \leq 8$; x_1, x_2, x_3 non-negative integers. [13; (1, 0, 0)]

67. Maximize $x_1 + 2x_2$ subject to $5x_1 + 7x_2 \leq 21$, $-x_1 + 3x_2 \leq 8$; x_1, x_2 non-negative integers. [5; (1, 2)]

68. Same as 5.

69. Formulae the following knapsack problem as an ILP.

There are n objects, j=1, 2, …, n, whose weights are w_j and values v_j. They have to be chosen to be packed in a knapsack so that total value of the objects chosen is maximum subject to their total weight not exceeding W.

$$\left[Maximise \sum_j v_j\, x_j \; subject \sum_j w_j\, x_j \leq W, x_j = 0\, or\, 1 \right]$$

70. Solve the knapsack problem (as formulated above) with the following data.

(i)

Object	Weight	Value
j	w_j	v_j
1	2	10
2	2	14
3	3	18
4	6	48
5	8	80

Knapsack capacity W = 12 [Max value 98, with objects 3 and 5]

(ii)

Object	Weight	Value
j	w_j	v_j
1	3	12
2	4	12
3	3	9
4	6	30
5	10	20
6	12	12

Knapsack capacity W = 14 [Max value 54, with objects 1,2,4]

71. Maximise $2x_1 + 5x_2$ subject to $0 \leq x_1 \leq 8$, $0 \leq x_2 \leq 8$, and either $4 - x_1 \geq 0$ or $4 - x_2 \geq 0$. [48; (4, 8)]

(Hint: Introduce two 0−1 variables y_1, y_2 such that $4 - x_1 + 10y_1 \geq 0$, $4 - x_2 + 10y_2 \geq 0$, $y_1 + y_2 = 1$, 10 being a suitably large number].

72. In a network of streets and junctions, the junctions are denoted by j = 1, 2,... and the streets connecting the junctions by (i, j). Fire-hydrants have to be installed at some of the junctions such that every street connected to a junction has access to the fire-hydrant at that junction. The cost of installing the fire-hydrant at junction *j* is c_j. Formulate the problem as an integer program to minimise the cost of installing the fire-hydrants so that each street has access to at least one hydrant. Solve the problem for the following data.

j = 1 2 3 4 5 6

c_j = 4 6 10 8 7 9

(i, j) = (1,2), (1,4), (1,6), (2,4), (3,5), (3,6), (4,5).

(Hint: Minimise $\Sigma c_j x_j$ subject to $x_j = 0$ or l, $x_i + x_j \geq 1$, for every given (i, *j)*.

[Hydrants at j=1, 5 at cost 11]

73. Minimise $3x_1 + 2x_2 + f_1 + f_2$,

subject to $5x_1 + 2x_2 \geq 10$,

$3x_1 + 5x_2 \geq 15$,

$f_1 = 5$ if $x_1 > 0$, $f_1 = 0$ if $x_1 = 0$,

$f_1 = 2$ if $x_2 > 0$, $f_2 = 0$ if $x_2 = 0$. [12; (0, 5)]

74. In a factory 4000 units of a certain product are to be manufactured. There are three machines on which it can be manufactured. The set up cost, the production cost per unit and the maximum production capacity for each machine are tabulated below. The objective is to minimize the total cost of producing the entire lot. Formulate the problem as an integer programme, and solve it.

Machine	Set up	Production cost/unit	Capacity
I	400	10	2400
II	600	4	1600
III	200	20	1200

75. Solve graphically the LP problem: maximize $f=4x_1+8x_2$, subject to $x_1+2x_2\geq 20$, $2x_1 + 2x_2 \leq 100$, $x_1 - 3x_2 \leq 0$, $4x_1-x_2\geq 0$, $x_1 \geq 0$, $x_2 \geq 0$. Also analyse graphically how the optimal solution is modified when the following changes are introduced in the problem, (one at a time);

i. objective function is replaced by $8x_1 + 4x_2$;

ii. right hand side of the second constraint is changed to 50;

iii. the coefficients of x_2 in the constraints are changed from (2, 2, -3, -1) to (2, 1, -2,-1);

iv. fourth constraint is deleted;

v. a new constraint $2x_1+x_2 \geq 10$ is introduced.

[(10. 40), 360; (i) (37.5, 12.5), 350; (ii) (5, 20), 180; (iii) (50/3,200/3), 600; (iv) (0, 50), 400; (v) no change.]

76. Solve the above problem using simplex method, and analyse the effects of the changes using sensitivity analysis methods.

77. Solve by simplex method the problem: maximize $f=-5x_1+l3x_2+5x_3$, subject to $12x_1 + 10x_2 + 4x_3 \leq 90$, $-x_1, + 3x_2+x3 \leq 20$, $x_1, x_2, x_3 \geq 0$. Use the sensitivity analysis approach to investigate the effects on the optimal solution of the following changes introduced one at a time:

i. right side of the second constraint is changed to 30;

ii. coefficient of x_2 in the objective function changes to 8;

iii. coefficient of x_1 in the objective function changes to – 2, and in the constraints from 12, – 1 to 5,10 respectively;

iv. a new variable is introduced with coefficient 10 in the objective function and 5 and 3 respectively in the first and second constraints;

v. variable x_3 is deleted from the problem;

vi. a new constraint $2x_1 + 5x_2 + 3x_3 \leq 50$ is introduced.

Verify your answers by solving the modified problems *ab initio* by the simplex method.

[(0,0,20), 100; (i) (0,9,0), 117; (ii, iii, iv) no change; (v) (0,20/3), 260/3; (vi) (0,5/2,25/2), 95].

78. For the problem: maximize $f=x_1-x_2 + 2x_3$, subject to $x_1-x_2+x_3 \leq 4$, $x_1+ x_2- x_3\leq 3$, $2x_1 - 2x_2 + 3x_3 \leq 15$, $x_1, x_2, x_3 \geq 0$, assuming x_4, x_5, x_6 respectively as the slack variables for the three constraints, the optimal table is the following.

Basis	Value	x_1	x_2	x_3	x_4	x_5	x_6
x_3	21	4		1		2	1
x_4	7	2			1	1	0
x_2	24	5	1			3	1
$-f$	18	2				1	1

Carry out the sensitivity analysis for each of the following changes:

i. coefficient of x_1 in the objective function changes to 2;

ii. coefficients of x_1 in the problem become $c_1 = 4_1$ $a_{11} = 1$, $a_{21} = 2$, $a_{31} = 3$;

iii. coefficients of x_2 and x_3 change to $c_2 = -2$, $a_{12} = 2$, $a_{22} = 3$, $a_{32} = -1$, $c_3 = 1$, $a_{l3} = 3$, $a_{23} = -2$, $a_{33} = 1$;

iv. right hand side vector changes from [4 3 15] to [2 4 20];

v. objective function changes to $3x_1+x_2+5x_3$;

vi. first constraint is deleted;

vii. a new constraint $2x_1 + x_2+2x_3 \leq 60$ is introduced;

viii. third constraint changes to $4x_1 - x_2 + 2x_3 \leq 12$.

[(i), (ii) no change; (iii) (17/5, 0, 1/5), 18/5; (iv) (0, 32, 28), 24; (v) (0, 24, 21), 129; (vi) no change; (vii) (0,150/7, 135/7), 120/7; (viii) (0,18,15), 12; or (0,4, 8), 12],

79. A company manufactures three products. A, B and C, using the same raw material and the same labour force. The mathematical model of the problem formulated to maximize the profits is;

Maximize $f=5x_1 + 3x_2 + x_3$,

subject to $5x_1 + 6x_2 + 3x_3 \leq 45$, (labour)

$5x_1 + 3x_2 + 4x_3 \leq 30$, (material)

$x_1, x_2, x_3 \geq 0$;

where x_1, x_2. x_3 are the amounts of the products A, B, C. The optimal solution to this problem is given in the following table, where x_4, x_5 are the slack variables in the first and second constraints respectively.

Basis	Value	x_1	x_2	x_3	x_4	x_5
x_2	5		1	-1/3	1/3	-1/3
x_1	3	1		1	-1/5	2/5
$-f$	30			3	0	1

Use sensitivity analysis to find the new optimal solution if the following changes, one at a time, arc made in the data:

i. coefficient of x_2 in the expression for f is changed to 2;

ii. available material increases from 30 to 60 units;

iii. per unit requirement of the material for the production of C is reduced from 4 to 2 units;

iv. a constraint $3x_1 + 2x_2 + x_3 \leq 20$, expressing limitation of supervisory staff is added.

[(i) (6, 0, 0). 30; (ii) (9, 0, 0), 45; (iii. iv) no change].

80. In problem 76 determine the effect on the optimal solution if changes (i) and (ii) are introduced in succession, (that is, the solution obtained after introducing change (i) is subjected to change (ii). Also solve the problem when both the changes are introduced simultaneously. Is the optimal solution in the two cases the same?

Repeat this analysis for the successive and simultaneous occurrence of changes (ii) and (iii).

What inference can be drawn regarding the technique to carry out sensitivity analysis when changes of more than one type are introduced simultaneously in the data?

[Changes may be introduced in succession.]

81. Use the approach suggested in problem 80 to determine the effect on the optimal solution of problem 79 of introducing simultaneously (a) changes (i) and (ii), (b) changes (ii) and (iii). [In both cases (9, 0, 0), 45].

82. The following table gives the optimal solution to a LP problem of the type: Maximize f = CX. subject to AX = B, $X \geq 0$.

Basis	Value	x_1	x_2	x_3	x_4	x_5
x_1	1	1		1	3	-1
x_2	2		1	1	-1	2
–f	8			4	3	4

x_4, x_5 are the slack variables respectively in the two constraints with right hand sides b_1, and b_2. The values of the cost coefficients are $c_1 = 2$, $c_2 = 3$, $c_3 = 1$.

i. How much can the coefficient c_1 be increased before the current basis ceases to be optimal? Answer the same question with respect to c_3.

ii. How much can the value of b_1 be varied before the present basis (x_1, x_2) ceases to be feasible? (It is not necessary to know the value of b_1 to answer this question).

iii. Find the optimal solution by the dual simplex method when b_l is increased by 3.

[(i) c_1, c_3 separately can be increased by 4; (ii) $7/15 \leq b_1 \leq 14/5$; (ii) (7, 0, 0),14],

83. Explain why the algorithm for finding the minimum path when all arc lengths are nonnegative, given in section VII-3, is not applicable to the general case when arc lengths can be negative also.

84. Find the minimum path from v, to v_8 in the graph with arcs and arc lengths (i) given below. Solve the problem by both the algorithms given in section VII-3 and compare the numerical work involved.

ii. Find the minimum path from v_1 to v_8 in the same graph with arc lengths (ii). In the following table (i, j) denotes the arc (v_i, v_j).

Arc		(1,2)	(1,3)	(1,4)	(2,3)	(2,6)	(2,5)	(3,5)	(3,4)	(4,7)
Length	(i)	1	4	11	2	8	7	3	7	3
Length	(ii)	-1	4	-11	2	-8	7	-3	7	3
Arc		(5,6)	(5,8)	(6,3)	(6,4)	(6,7)	(6,8)	(7,3)	(7,8)	
Length	(i)	1	12	4	2	6	10	2	2	
Length	(ii)	1	12	4	2	6	-10	-2	2	

[(i) 15; (ii)-22]

85. In each case of problem 84, is there a maximum path from v_1 to v_8? Explain with reasons. Identify circuits with positive lengths.

[Maximum unbounded because of circuits with positive lengths]

86. Find the minimum spanning tree in the following undirected graph. Arc (v_i, v_j) is denoted as (i,j).

Arc	(1,2)	(1,3)	(1,4)	(2,3)	(2,8)	(2,10)	(3,4)	(3,8)	(4,5)	(4,6)
Length	7	4	8	3	9	14	4	10	15	12
Arc	(4,8)	(5,6)	(5,7)	(6,7)	(6,8)	(6,9)	(7,9)	(8,9)	(8,10)	(9,10)
Length	10	4	1	2	20	16	18	3	4	6

[42]

87. Five villages in a hilly region are to be connected by roads. The direct distance (in km) between each pair of villages along a possible road and its cost of construction per km (in 10^4 rupees) are given in the following table (distances are given in the upper triangle and costs in the lower triangle). Find the minimum cost at which all the villages can be connected, and the roads which should be constructed.

Distance

		1	2	3	4	5
	1		18	12	15	10
	2	3		15	8	22
Costs	3	4	3		6	20
	4	5	5	6		7
	5	2	2	5	7	

(Hint Construct the spanning tree of minimum cost).

[Rs 140x10^4; (1,5,2,4,3)]

Project scheduling

88. A project consists of activities *A, B, C,.... M.* In the following data *X–Y= c* means *Y* can start after c days of work on X. *A, B, C* can start simultaneously. *K* and *M* are the last activities and take 14 and 13 days respectively. *A–D = 4, B–F = 6, B–E = 3, C–E = 4. D–H = 5, D–F = 3, E–F=10, F–G = 4,G–I= 12, H–I=3. H–J = 3, J–K=8, I–K=7, I–L* = 7, L–M = 9. Find the least time of completion of the project If activities *K* and *L* both need a crane, and only one crane is available, how should the crane be used so that the project is completed with the least delay? [59; use crane first on *L*, resulting delay 1 day]

89. Tasks A, *B*, *C*,.... H, I constitute a project. The notation $X<Y$ means that the task X must be finished before Y can begin. With this notation

$$A<D,\ A<E,\ B<F,\ D<F,\ C<G,\ C<H,\ F<I,\ G<I.$$

The time (in days) of completion of each task in as follows.

Task	A	B	C	D	E	F	G	H	I
Time	8	10	7	9	16	7	8	14	9

Draw a graph to represent the sequence of tasks and find the minimum time of completion of the project [33 days]

90. The project of problem 89 is required to be completed as early as possible. How soon can it be completed and at what additional minimum cost with the following data?

Task	A	B	C	D	E	F	G	H	I
Increase in cost for each day less	1	2	-	3	4	1	-	6	4
Minimum time	6	7	7	8	13	6	8	11	8

[28 days; cost 16]

Maximum flow

91. Find the maximum flow in the graph with the following arcs and arc capacities, flow in each arc being non-negative. Arc (v_j, v_k) is denoted as (j, k). v_a is the source and v_b the sink.

Arc	(a,1)	(a,2)	(a,3)	(1,4)	(1,5)	(1,6)	(2,4)	(2,5)	(2,6)
Capacity	2	2	2	1	1	1	1	1	1
Arc	(3,4)	(3,5)	(3,6)	(4,b)	(5,b)	(6,b)			
Capacity	1	1	1	2	2	2			

92. Find the maximum non-negative flow in the network described below, arc (v_j, v_k) being denoted as (j, k). v_a is the source and v_b the sink.

Arc	(a,1)	(a,2)	(1,2)	(1,3)	(1,4)	(2,4)	(3,2)	(3,4)	(4,3)	(3,b)	(4,b)
Capacity	8	10	3	4	2	8	3	4	2	10	9

[14]

93.Find the maximum flow in the network with the following data, flow in arcs not necessarily being non-negative. The arc (v_j, v_k) is denoted as *(j, k)* and the flow limit (b, c) means that the constraint on the flow x_i is $b_i \le x_i \le c_i$. v_a is the source and v_b the sink.

Arc	(a,1)	(a,2)	(1,2)	(1,3)	(2,4)	(3,4)	(3,b)	(4,b)
(b_i, c_i)	(0,10)	(0,5)	(-2,3)	(7,10)	(-3,5)	(-1,1)	(0,8)	(0,4)

[12]

94.*Families $a_1, a_2,$ ..... a_m decide to go on a picnic in cars b_1, b_2 .... b_n. The number of persons in* family a_i is c_i and the seating capacity of car b_j *is* k_j. Assuming that the total seating capacity is not less than the total number of persons, it is required to allot persons to cars such that in car b_j the number of persons from the same family should not exceed h_j. Formulate the problem as that of maximum flow, and solve it for the following data.

i	1	2	3	4	5
c_i	2	3	4	4	2

j	1	2	3
k_j	5	5	5
h_j	2	2	2

(Hint Let a_0 be the source, b_0 the sink, (a_0, a_i) an arc with capacity c_i, (b_j, b_0) an arc with capacity k_j, (a_i, b_j) an arc with capacity h_j).

95.Convoys of army vehicles have to go from stations a_i, i = 1,2,3,4, to b_j, j = 1,2,3, at night The maximum number of vehicles leaving a_i or arriving at b_j is different for each station due to limited parking space, and is given in die following table. Each a_i is connected to each b_j by road. For secrecy reasons no convoy should consist of more than 15 vehicles.

Station	a_1	a_2	a_3	a_4	b_1	b_2	b_3
Parking capacity (no. of vehicles)	40	30	25	55	50	30	45

Find how the vehicles should be sent so that the total number of vehicles moved is maximum. Is the optimal solution unique? If not, find two alternatives.

(Hint Let a_0 be a source connected to each a_i and b_0 be a sink connected to each b_j, with capacity of arc (a_0, a_i) or (b_j, b_0) equal to parking capacity of a_i or b_j. Let the capacity of each arc (a_i, b_j) be 15. Find the maximum flow).

[Maximum vehicles 125, distribution not unique. See answer to problem 96]

96. Solve problem 95 if the convoy on each road should consist of not more than 15 and not less than 7 vehicles. Is there a solution to this problem if the least strength of each convoy is 8 vehicles?

[125 vehicles, (i) is the solution to this problem. Both (i) and (ii) are solutions of problem 95.

	(i)	(ii)		(i)	(ii)		(i)	(ii)
(a_1, b_1)	15	15	(a_1, b_2)	7	5	(a_1, b_3)	15	5
(a_2, b_1)	15	15	(a_2, b_2)	7	0	(a_2, b_3)	8	15
(a_3, b_1)	9	5	(a_3, b_2)	9	10	(a_3, b_3)	7	10
(a_4, b_1)	11	15	(a_4, b_2)	7	15	(a_4, b_3)	15	15]

97. Examine the following payoff matrices for saddle points. In case the saddle point exists, find the optimal strategies and value of the game. In every case verify that

$$\max_i \min_j a_{ij} \leq \min_j \max_i a_{ij}$$

(i) $\begin{bmatrix} 1 & 3 \\ -2 & 10 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & -1 & -2 \\ 1 & 0 & 1 \\ -2 & -1 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} -5 & 3 & 1 & 20 \\ 5 & 5 & 4 & 6 \\ -4 & -2 & 0 & -5 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 3 & 6 \\ 2 & 1 & 3 \\ 6 & 2 & 1 \end{bmatrix}$

(vi) $\begin{bmatrix} 0 & 2 & -3 & 0 \\ -2 & 0 & 0 & 3 \\ 3 & 0 & 0 & -4 \\ 0 & -3 & 4 & 0 \end{bmatrix}$ (vii) $\begin{bmatrix} 3 & 2 & 4 & 0 \\ 3 & 4 & 2 & 4 \\ 4 & 2 & 4 & 0 \\ 0 & 4 & 0 & 8 \end{bmatrix}$

98. Solve the games with the following payoff matrices

$$\text{(i)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

99. Solve graphically the games whose payoff matrices are the following.

$$\text{(i)} \begin{bmatrix} 2 & 7 \\ 3 & 5 \\ 11 & 2 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \quad \text{(iii)} \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{bmatrix}$$

100. Use the notion of dominance to simplify the following payoff matrices and then solve the game.

$$\text{(i)} \begin{bmatrix} 0 & 5 & -4 \\ 3 & 9 & -6 \\ 3 & -1 & 2 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 0 & 2 & 1 \\ 4 & 3 & 1 & 3 & 2 \\ 4 & 3 & 4 & -1 & 2 \end{bmatrix}$$

101. Write both the primal and the dual LP problems corresponding to the rectangular games with the following payoff matrices. Solve the game by solving the LP problem by simplex method

$$\text{(i)} \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{bmatrix}$$

102. Show that an alternative formulation of an LP problem equivalent to the problem of strategy of P_1 in a rectangular game with payoff matrix

$$A = \{a_{ij}\}, \; i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n, \text{ is:}$$

Maximize
$$v = \sum_{i=1}^{m} a_{in} x_i - x_{m+n}$$

subject to
$$\sum_{i=1}^{m} (a_{ij} - a_{in}) x_i - x_{m+j} + x_{m+n} = 0, \; j = 1, 2, \ldots, n,$$

$$\sum_{i=1}^{m} x_i = 1, \quad x_i \ge 0.$$

Write the corresponding LP problem for the strategy of P_2 and show that it is the dual of the *above.*

103. *Following* the formulation suggested in the above problem, formulate the primal and dual LP problems equivalent to the matrix games of problems 3 and 5 above, and solve them by the simplex method.

BOOKS FINISHED