REAL ANALYSIS-I

[MTH1C03]

STUDY MATERIAL

I SEMESTER

CORE COURSE

M.Sc. Mathematics

(2019 Admission onwards)

UNIVERSITY OF CALICUT

SCHOOL OF DISTANCE EDUCATION

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CORE COURSE:

MTH1C03-REAL ANALYSIS-I

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Chapter 1

Basic Topology

1.1 Metric Spaces

The notion of metric is a generalization of distance. The usual concept of distance in our space, which is $\mathbb{R}^3$, the real line $\mathbb{R}$ and in general the Euclidean distance in $\mathbb{R}^n$ are so familiar. The whole properties of this concept of distance can be described in three simple properties-non negativity, symmetry and triangle inequality.

Here onwards, a set is nonepty unless otherwise specified.

Definition 1.1. *Let X be a set. A* **metric** *on X is a function $d : X \times X \to \mathbb{R}$ such that for any three elements p, q and r in X,*

(a) $d(p, q) \geq 0$ *and* $d(p, q) = 0$ *if and only if* $p = q$

(b) **Symmetry***:* $d(p, q) = d(q, p)$

(c) **Triangle inequality***:* $d(p, q) \leq d(p, r) + d(r, q)$

A set together with a metric defined on it is called a **metric space***.*

Example 1.1. *(a)* **The metric space** $\mathbb{R}$. *The usual distance in the set of real numbers $d(p, q) = |p - q|$ is a metric and this metric space is denoted by $\mathbb{R}$. Using the properties of 'modulus' one can easily verify this.*

(b) **The discrete space.** *For any set X the function $d : X \times X \to \mathbb{R}$, defined by*

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y. \end{cases}$$

is a metric on X, called the **discrete metric** *on X. And this space is called* **discrete metric space**. *Note that discrete metric can be defined on any set.*

(c) **The Euclidean space** $\mathbb{R}^k$. *For $\boldsymbol{x} = (x_1, x_2, ..., x_k)$, and $\boldsymbol{y} = (y_1, y_2, ..., y_k)$, where x_i and y_i are real numbers for $1 \leq i, \leq k$; the Euclidean metric is defined by*

$$d(\boldsymbol{x},\ \boldsymbol{y}) = |\boldsymbol{x} - \boldsymbol{y}| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_k - y_k)^2}.$$

The triangle inequality can be proved using the result of dot product, namely $|\boldsymbol{x}|^2 = \boldsymbol{x} \cdot \boldsymbol{x}$ which implies that

$$\begin{aligned} |\boldsymbol{x}+\boldsymbol{y}|^2 &= (\boldsymbol{x}+\boldsymbol{y}) \cdot (\boldsymbol{x}+\boldsymbol{y}) \\ &= \boldsymbol{x} \cdot \boldsymbol{x} + 2\boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{y} \cdot \boldsymbol{y} \\ &\leq |\boldsymbol{x}|^2 + 2|\boldsymbol{x}||\boldsymbol{x}| + |\boldsymbol{y}|^2 \\ &= (|\boldsymbol{x}| + |\boldsymbol{y}|)^2 \end{aligned}$$

So that, $|\boldsymbol{x}+\boldsymbol{y}| \leq |\boldsymbol{x}| + |\boldsymbol{y}|$.

This metric space is denoted by $\mathbb{R}^k$.

(d) **Subspace.** *If (X, d) is a metric space and Y is a subset of X, then Y itself is a metric space, called a subspace of X, in the induced metric obtained by restricting the metric d to the set $Y \times Y$.*

Let $a,\ b \in \mathbb{R}$. If $a < b$, by the **segment** (a, b) we mean the set of all real numbers x such that $a < x < b$. For $a \leq b$, by the **interval** $[a, b]$ we mean the set of all real numbers x such that $a \leq x \leq b$. The **half-open intervals** $[a, b)$ and $(a, b]$, for $a < b$ are respectively the sets $\{x \in \mathbb{R} : a \leq x < b\}$ and $\{x \in \mathbb{R} : a < x \leq b\}$.

Definition 1.2. *Let (X, d) be a metric space.*

(a) *A* **neighbourhood** *of a point $p \in X$ with radius $r > 0$ is the set*

$$N_r(p) = \{q \in X : d(p, q) < r\}.$$

(b) *For a subset A of X, a point $p \in A$ is said to be an* **interior point** *of A if there exists a neighbourhood of p, $N_r(p)$ such that $N_r(p) \subset A$.*

(c) *A subset A of X is said to be* **open** *in X if every point of A is an interior point of A.*

(d) *For a subset A of X, $p \in X$ is said to be a* **limit point** *of A if every neighbourhood of p contains a point $q \neq p$ such that $q \in A$. If $p \in A$ is not a limit point of A, then p is called an* **isolated point** *of A.*

(e) *A subset A of X is said to be* **closed** *if every limit point of A is a point of A.*

(f) *A subset A of X is said to be* **perfect** *if it is closed and every point is a limit point.*

(g) *A subset A of X is said to be* **dense** *in X if every point of X is either a point of A or a limit point of A.*

(h) *If $E \subset X$, and if E' denotes the set of all limit points of E in X, then the* **closure** *of E is the set $\bar{E} = E \bigcup E'$.*

Example 1.2. *(a) In the metric space $\mathbb{R}$ with usual metric, $N_r(x)$ is the open interval $(x - r, x + r)$.*

Since every point of the set $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ is an interior point, it is an open subset of $\mathbb{R}$. In general any segment (a, b) is open in $\mathbb{R}$ and every point is a limit point. It is not closed(why?) and hence not perfect also.

The interval $[a, b]$ is not open(why?), it is closed and all points are limit points, and hence it is perfect.

The set $E = \{1/n : \ n \in \mathbb{N}\}$ has only one limit point in $\mathbb{R}$, namely $x = 0$, which is not a point of E, so that $E' = \{0\}$, hence $\bar{E} = \{1/n : \ n = 1, 2, ...\} \bigcup \{0\}$. Also, no point of E is an interior point of E. So that E is neither open nor closed in $\mathbb{R}$.

(b) *In a discrete metric space (X, d), $N_r(p) = \{q \in X : d(p, q) < r\}$ contains only the element p if $r \leq 1$ and contains every element of X if $r > 1$. Therefore the neighbourhoods are*

$$N_r(p) = \begin{cases} \{p\}, & \text{if } \ r \leq 1 \\ X, & \text{if } \ r > 1. \end{cases}$$

Infact, in a discrete space X, since every singleton set is a neighbourhood of its point, every subset of $\mathbb{X}$ is open. Also no point of a proper subset E of X is a limit point of E, and hence the set of limit points of E is empty and hence every subset is closed as well.

In particular, the interval $[a, b]$ and segments (a, b), both are open as well as closed with respect to discrete metric on the real line.

(c) In the plane $\mathbb{R}^2 = \mathbb{C}$, with the Euclidean metric, the neighbourhood of a point is an open circular disc centred at the point.

(i) The set $A = \{\boldsymbol{x} : |\boldsymbol{x}| < 1\}$ is an open subset. Every point in A as well as every point on the boundary of this disc, that is $\boldsymbol{x} \in \mathbb{R}^2$ with $|\boldsymbol{x}| = 1$ is a limit point of A. So that A is not closed, and not perfect.

(ii) If B is a finite set, then B has no limit point, so is closed; Since every point is an isolated point, B is not perfect, and B is not open also(why?).

(iii) The segment (a, b) is not open in $\mathbb{R}^2$. (why?)

Definition 1.3. *The neighbourhoods in the Euclidean space $\mathbb{R}^k$ are called* **open balls** *and their closures are called* **closed balls***.*

Theorem 1.1. *Every neighbourhood is an open set.*

Proof. Let N be a neighbourhood of a point $x \in X$ with radius r. We want to prove every point of N is an interior point of N.

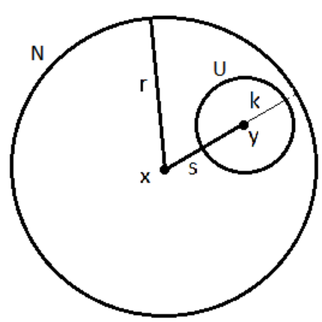


Figure 1

If $y \in N$, then $d(x,y) < r$. Let $d(x,y) = s$, then the neighbourhood U of y of radius k, $0 < k < r - s$ is a proper subset of N. Because for $z \in N_k(y)$, $d(z,y) < k$.

So that

$$d(x,z) \leq d(x,y) + d(y,z) < s + k < s + r - s = r,$$

which implies that $z \in N$ (see Figure 1).

Since z is an arbitrary point of $N_k(y)$, We have $N_k(y) \subset N$. Thus every point is an interior point and the set N is open.

□

Theorem 1.2. *For a subset A of X, $p \in X$ is a limit point of A if and only if every neighbourhood of p contains infinitely many points of A.*

Proof. If every neighbourhood of p contains infinitely many points of A, then p is a limit point of A.

Conversely, if a neighbourhood N of p contains only a finite number of points of A namely, $q_1,\ q_2,\ ...,\ q_n$, let $r = min\{d(p,q_1),\ d(p,q_2),\ ...,\ d(p,q_n)\}$. Then the neighbourhood $N_r(p)$ of p does not contain any point of A and p cannot be a limit point of A, which proves the converse. □

Corollary 1.1. *A finite set has no limit point.*

Definition 1.4. *For a subset A of X the* **complement** *of A, denoted by A^c is the set consisting of all points in X which are not in A.*

Recall the following results from elementary set theory.

Theorem 1.3. *Let X be a set.*

(a) For $E \subset X$, $(E^c)^c = E$.

(b) For any collection $\{E_\alpha\}$ of subsets of X,

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha (E_\alpha^c).$$

(c) For any collection $\{E_\alpha\}$ of subsets of X,

$$\left(\bigcap_\alpha E_\alpha\right)^c = \bigcup_\alpha (E_\alpha^c).$$

Proof. Part (a) follows directly from the definition of complement, which is left as an exercise.

For (b), consider an element

$$x \in \left(\bigcup_{\alpha} E_\alpha \right)^c.$$

Then

$$x \notin \bigcup_{\alpha} E_\alpha.$$

Hence $x \notin E_\alpha$ for any α, which implies that $x \in (E_\alpha)^c$ for all α, and

$$x \in \bigcap_{\alpha} (E_\alpha^c).$$

Proof of (c), similar to the proof of (b), is left as an exercise. □

Theorem 1.4. *A subset E is open in X if and only if E^c is closed in X.*

Proof. Suppose that E is an open subset of X. We have to prove that E^c is closed. Let x be a limit point of E^c. Then every neighbourhood of x contains points of E^c other than x. In other words there is no neighbourhood of x properly contained in E. Which shows that x is not an interior point of E, and since every point of E is an interior point of E, we have $x \in E^c$. Thus every limit point of E^c is a point of E^c, and hence E^c is closed.

Conversely suppose that E^c is closed. In order to get E is open, we have to prove that every point in E is an interior point of E. Let $y \in E$, so that $y \notin E^c$. Being closed E^c contains all its limit points. So that y is not a limit point of E^c. That means y has a neighbourhood N which does not intersect with E^c. Hence $N \subset E$, and y is an interior point of E. □

Equivalently, we have the following result.

Theorem 1.5. *A subset F is closed in X if and only if F^c is open in X.*

Proof. It follows from Theorem 4 using the result $(F^c)^c = F$. □

Theorem 1.6. *In a metric space, we have:*

(a) For any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open.

(b) For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.

(c) For any finite collection $\{G_1, ..., G_n\}$ *of open sets,* $\bigcap_{i=1}^n G_i$ *is open*

(d) For any finite collection $\{F_1, ..., F_n\}$ *of closed sets,* $\bigcup_{i=1}^n F_i$ *is closed.*

Proof. (a) Let $\{G_\alpha\}$ be a collection of open sets in X. If $x \in \bigcup_\alpha G_\alpha$, then $x \in G_\alpha$, for some α. Since G_α is open, x is an interior point of G_α and x has a neighbourhood N such that $N \subset G_\alpha$. Then $N \subset \bigcup_\alpha G_\alpha$, and hence x is an interior point of $\bigcup_\alpha G_\alpha$. Thus $\bigcup_\alpha G_\alpha$ is open.

(b) If F_α is closed for all α, then every $(F_\alpha)^c$ is open, so that by (a), $\bigcup_\alpha (F_\alpha^c) = (\bigcap_\alpha F_\alpha)^c$ is open, hence $\bigcap_\alpha F_\alpha$ is closed.

(c) If $y \in \bigcap_{i=1}^n G_i$, then $y \in G_i$, for $i = 1, 2, ..., n$. Since $G_1, ..., G_n$ are open sets, y is an interior point of each G_i. So that y has neighbourhoods $N_1, ..., N_n$ of radius $r_1, ..., r_n$ respectively such that $y \in N_i \subset G_i$. Put $r = min\{r_1, ..., r_n\}$ and let N be the neghbourhood of y of radius r. Then $N \subset G_i$ for $i = 1, ..., n$, so that $N \subset \bigcap_{i=1}^n G_i$ and hence $\bigcap_{i=1}^n G_i$ is open.

Proof of (d) is left as an exercise. □

Note that arbitrary intersection of open sets need not be open and arbitrary union of closed sets need not be closed.

Example 1.3. *Consider*

$$G_n = (-1/n, 1/n), \; n = 1, 2, 3, ...$$

and

$$F_n = [1/n, 2], \; n = 1, 2, 3, ...$$

Then G_n *is open and* F_n *is closed in* $\mathbb{R}$ *for each* n, *but*

$$\bigcap_{n=1}^{\infty} G_n = \{0\}$$

is not open(it is closed), and

$$\bigcup_{n=1}^{\infty} F_n = (0, 2]$$

is not closed in $\mathbb{R}$.

If $F_n = [1/n, 2 - 1/n],\ n = 1, 2, 3, ...,$ *then* $\bigcup_{n=1}^{\infty} F_n = (0, 2),$ *which is open.*

Thus is arbitrary union of closed sets may be open and arbitrary intersection of open sets may be closed.

Theorem 1.7. *If* X *is a metric space and* $E \subset X$*, then*

(a) $\bar{E}$ *is closed,*

(b) $E = \bar{E}$ *if and only if* E *is closed, and*

(c) $\bar{E} \subset F$ *for every closed set* F *in* X *containing* E*.*

By (a) *and* (c)*, we have* $\bar{E}$ *is the smallest closed subset of* X *that contains* E*.*

Proof. For proving (a), it is enough to prove that $(\bar{E})^c$ is open. If $x \in (\bar{E})^c$, then $x \notin E \bigcup E'$, that is x is neither a point of E nor a limit point of E. Thus x has a neighbourhood N such that $N \subset (\bar{E})^c$, which implies that $(\bar{E})^c$ is open.

Now, by definition $E = \bar{E}$ if and only if $E' \subset E$, that is if and only if E is closed, by the definition of closed sets.

Part (c) follows from the fact that for $E \subset F$, we have $E' \subset F'$, and since F is closed $F' \subset F$. Hence $\bar{E} = E \bigcup E' \subset F$. □

Recall that a subset E of real numbers is said to be **bounded above** if there is a real number M such that $x < M$ for all $x \in E$; **bounded below** if there is a real number K such that $x > K$ for all $x \in E$; and **bounded** if it is both bounded above and bounded below. A real number M such that $x \leq M$ for all $x \in E$ is called an **upper bound** of E and a real number K such that $x \geq K$ for all $x \in E$ is called a **lower bound** of E.

If E is bounded above, the **supremum** or **least upper bound** of E, denoted by *sup* E, is the unique point $a \in \mathbb{R}$ such that

1. for all $x \in E,\ x \leq a$, that is a is an upper bound for E and

2. if $b \in \mathbb{R}$ such that $x \leq b$, for all $x \in E$, then $a \leq b$.

The **infemum or greatest lower bound** for sets bounded below is defined analogously.

Now, we can say that, a set of real numbers is bounded if it is entirely contained in some neighbourhood $(-M, M)$ of zero. In an arbitrary metric space, the boundedness can be defined as:

Definition 1.5. *A subset E of a metric space (X, d) is said to be* **bounded** *if there is a real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in E$.*

Theorem 1.8. *Let E be a nonempty set of real numbers which is bounded above. Let $y = sup\ E$. Then $y \in \bar{E}$. Hence $y \in E$ if E is closed.*

Proof. If $y \in E$, then $y \in \bar{E}$. Assume $y \notin E$. Then $x < y$ for all $x \in E$. Also, for any $h > 0$, $y - h$ is not an upperbound for E, so that there exists a point $x \in E$ such that

$$y - h < x < y,$$

that is

$$x \in (y - h, y + h).$$

This shows that for every $h > 0$ the neighbourhood of y of radius h contains a point of E which implies that y is a limit point of E, and hence $y \in \bar{E}$. □

We have the analogous result for $inf\ E$ of a nonempty set of real numbers bounded below(Exercise).

We have seen that if (X, d) is a metric space and if $Y \subset X$, then Y also is a metric space with respect to the induced metric. Then for $E \subset Y \subset X$, E can be regarded as a subset of Y or as a subset of X. The concepts like neighbourhood of a point in E and openness of E depend on the space in which E is considered as a subset.

Example 1.4. *The real line $\mathbb{R}$ can be regarded as a subset of the plane $\mathbb{R}^2$, by identifying the point $r \in \mathbb{R}$ as the point $(r, 0) \in \mathbb{R}^2$. The segment $(0, 1) \subset \mathbb{R} \subset \mathbb{R}^2$ and $(0, 1)$ is open in $\mathbb{R}$, while it is not open in $\mathbb{R}^2$.*

The open sets of a metric space and the open sets of a subspace are related in the following way.

Theorem 1.9. *Suppose $Y \subset X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X.*

Proof. Suppose E is open relative to Y. That is for every point $x \in E$, there exists an $r_x > 0$ such that the r_x-neighbouhood of x in Y,

$$N(x) = \{y \in Y : \; d(x, y) < r_x\}$$

is a subset of E. So that

$$\bigcup_{x \in E} N(x) \subset E.$$

Now,

$$E \subset \bigcup_{x \in E} N(x),$$

hence

$$E = \bigcup_{x \in E} N(x).$$

If

$$U(x) = \{y \in X : \; d(x, y) < r_x\},$$

then being a neighbourhood, each $U(x)$ and hence $G = \bigcup_{x \in E} U(x)$ is open in X. Also

$$N(x) = U(x) \bigcap Y,$$

so that

$$E = G \bigcap Y.$$

Conversely suppose that G is open in X and $E = Y \bigcap G$. We want to prove E is open in Y. If $y \in E$, then $y \in G$ also. Now G is open in X, so that there exists an $s > 0$ such that the s-neighbourhood of y in X, namely,

$$U(y) = \{p \in X : d(p, y) < s\}$$

is a subset of G. Hence, the s-neighbourhood of y in Y,

$$N(y) = \{p \in Y : d(p, y) < s\} \subset Y \bigcap G = E,$$

which shows that y is an interior point of E and that E is open in Y. □

1.2 Compact Sets

Definition 1.6. *By an* **open cover** *of a set E in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets of X such that*

$$E \subset \bigcup_{\alpha} G_\alpha. \tag{1.1}$$

Any sub collection of $\{G_\alpha\}$ covering E(in the sense that the union of its members contains E) is called a **subcover***, and a subcover having only a finite number of members is called a* **finite subcover***.*

Example 1.5. *The collection*

$$\{(0, \frac{1}{n}) : n \in \mathbb{N}\}$$

is an open cover of the segment $(0, 1)$, and it has finite subcovers(Find one finite subcover).

Now, the collection

$$\{(0, 1 - \frac{1}{n+1}) : n \in \mathbb{N}\}$$

is also an open cover of the segment $(0, 1)$, but it has no finite subcover. (Explain! Hint: Consider the real line geometrically and find what are the sets in the collction.)

Definition 1.7. *A subset K of a metric space X is said to be* **compact** *if every open cover of K contains a finite subcover.*

Example 1.5 shows that the segment $(0, 1)$ is not compact. For an example of compact set, we need some more theoretical tools, because it may not be practical to check whether all open covers of certain set contain finite subcovers.

We first show that unlike openness, compactness of a set K in X will not be affected when the space X is replaced by a subspace Y containing K.

Theorem 1.10. *Suppose $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y.*

Proof. Given $K \subset Y \subset X$. Suppose that K is compact relative to X, which means that every open cover of K, with open sets in X, has a finite subcover. To prove K is compact relative to Y, we have to prove that every open cover of K, with open sets in Y, has finite subcover.

Let $\{E_\alpha : \alpha \in I\}$ be an open cover of K, where E_α is open relative to Y. By Theorem 1.9, we have for each α, $E_\alpha = G_\alpha \bigcap Y$, where G_α is open relative to X. Now

$$K \subset \bigcup_\alpha \{E_\alpha\} \subset \bigcup_\alpha \{G_\alpha\},$$

which shows that $\{G_\alpha\}$ is an open cover of K with open sets in X. Since K is compact relative to X, we have a finite subcollection of G_α namely, $\{G_1, ..., G_n\}$ covering K. That is,

$$K \subset G_1 \bigcup ... \bigcup G_n.$$

Also $K \subset Y$, so that,

$$K \subset \left(\bigcup_{i=1}^{n} G_i\right) \bigcap Y = \bigcup_{i=1}^{n} \left(G_i \bigcap Y\right) = \bigcup_{i=1}^{n} E_i,$$

where $\{E_1, ..., E_n\}$ is a finite subcover of $\{E_\alpha\}$, which completes the necessary part.

Conversely assume that K is compact relative to Y. Let $\{U_\alpha : \alpha \in J\}$ be an open cover of K, where U_α is open relative to X. Again by Theorem 1.9, we have $V_\alpha = U_\alpha \bigcap Y$ is open relative to Y for each α. Now, $K \subset \bigcup_\alpha U_\alpha$ and $K \subset Y$ imply that

$$K \subset \bigcup_{\alpha} V_\alpha.$$

Since K is compact relative to Y, $\{V_\alpha\}$ has a finite subcollection covering K, namely $\{V_1, ...V_m\}$. Thus

$$K \subset \left(\bigcup_{j=1}^{m} V_j\right) = \left(\bigcup_{j=1}^{m} (U_j \bigcap Y)\right) = \left(\bigcup_{j=1}^{m} U_j\right) \bigcap Y,$$

and hence

$$K \subset \left(\bigcup_{j=1}^{m} U_j\right),$$

which shows that $\{U_\alpha\}$ has a finite subcover and hence K is compact relative to X. $\square$

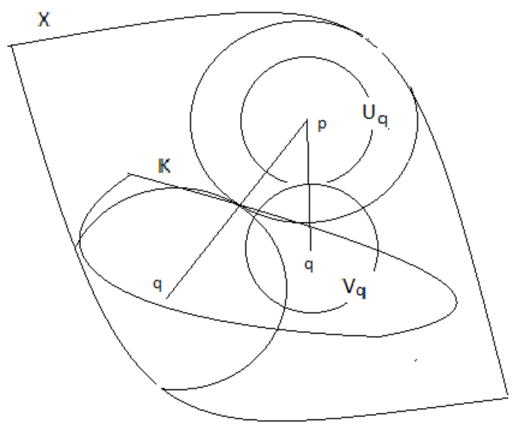
Theorem 1.11. *Compact subsets of metric spaces are closed.*

Proof. Let K be a compact subset of X. To prove K is closed, it is enough to prove that K^c is open. Let $p \in K^c$. To prove p is an interior point of K^c, the only fact we can make use of is the compactness of K.

For each q in K, let $r_q = d(p, q)$. Consider the two neighbourhoods U_q and V_q respectively of the points p and q, each of radius $\frac{r_q}{2}$. Now the collection of neighbourhoods $\{V_q : q \in K\}$ forms an open cover of K. Since K is compact, we have a finite subcollection of $\{V_q : q \in K\}$ covering K. Let

$$K \subset \bigcup_{i=1}^{n} V_i,$$

where $V_i = V_{q_i}$.



Now, every U_q is a neighbourhood of the point p and $U_q \bigcap V_q = \phi$, for each q. So that if $U_i = U_{q_i},\ i = 1, 2, ..., n$, then

$$U = \bigcap_{i=1}^{n} U_i$$

is a neighbourhood of p which does not intersect with $\bigcup_{i=1}^{n} V_i$. Hence

$$U \bigcap K = \phi,$$

which means $U \subset K^c$, and hence K^c is open. □

Theorem 1.12. *Closed subsets of compact sets are compact.*

Proof. Let E be a closed subset of the compact space X. Let $\{G_\alpha\}$ be an open cover of E. We have to find a finite sub collection of $\{G_\alpha\}$ covering E.

Since E being closed, its complement E^c is open. Also,

$$E \subset \bigcup_{\alpha} \{G_\alpha\}.$$

Now $E\bigcup E^c = X$ implies that

$$X \subset \left(\bigcup_{\alpha}\{G_\alpha\}\right)\bigcup\{E^c\}.$$

That is $\{G_\alpha\}\bigcup\{E^c\}$ is an open cover of X.

Now X is compact, so that this open cover has a finite subcover. If this finite subcover does not contain E^c, then it is a finite subcover of $\{G_\alpha\}$ covering E. Otherwise, removal of E^c from the subcover yields a finite subcover of $\{G_\alpha\}$ covering E. □

Corollary 1.2. *If F is closed and K is compact, then $F \cap K$ is compact.*

Theorem 1.13. *If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\cap K_\alpha$ is nonempty.*

Proof. If possible let

$$\bigcap K_\alpha = \phi.$$

Then for any fixed member K_0 of $\{K_\alpha\}$, no point of K_0 belongs to every K_α because, otherwise $\bigcap K_\alpha \neq \phi$. In other words, for each element $x \in K_0$, there exists an α such that $x \in K_\alpha^c$. So that if $G_\alpha = K_\alpha^c$, then

$$K_0 \subset \bigcup G_\alpha.$$

Being compact, K_α is closed for each α. So that, G_α is open and $\{G_\alpha\}$ is an open cover of K_0. Again since K_0 is compact, there is a finite subcover say $\{G_1, ..., G_n\}$ of $\{G_\alpha\}$ such that

$$K_0 \subset \bigcup_{i=1}^{n} G_i = \bigcup_{i=1}^{n}(K_i^c) = \left(\bigcap_{i=1}^{n} K_i\right)^c.$$

Therefore,

$$K_0\bigcap K_1\bigcap\cdots\bigcap K_n = \phi,$$

a contradiction to the hypothesis, which proves the theorem. □

Corollary 1.3. *If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1},\ n = 1, 2, 3, ...,$ then $\bigcap_1^\alpha K_n$ is not empty.*

Proof It follows from the Theorem because intersection of every finite subcollection of $\{K_n\}$ is not empty.(Why?)

Theorem 1.14. *If E is an infinite subset of a compact set K, then E has a limit point in K.*

Proof. Let E be an infinite subset of K. If no point of K is a limit point of E, then every point q of K has a neighbourhood V_q such that $V_q \bigcap E$ is either empty or $\{p\}$. So that no finite subcollection of $\{V_q : q \in K\}$ can cover E, because E is infinite. Hence no finite subcollection of $\{V_q : q \in K\}$ can cover K also. But $\{V_q : q \in K\}$ is an open cover of K and K is compact, we arrive at a contradiction. Hence E has a limit point in K. □

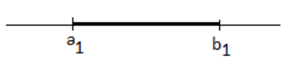
Theorem 1.15. *If $\{I_n\}$ is a sequence of intervals in $\mathbb{R}$, such that $I_n \supset I_{n+1},\ n = 1, 2, 3, ...,$ then $\bigcap_1^\infty I_n$ is not empty.*

Proof. If $I_n = [a_n, b_n],\ n = 1, 2, 3, ...,$ then since $I_n \supset I_{n+1}$, we have

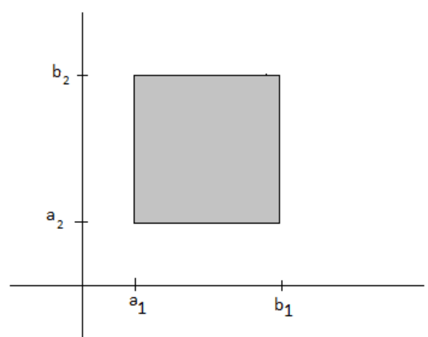
$$a_1 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots \leq b_{n+1} \leq b_n \leq \cdots \leq b_1.$$

Then the set $E = \{a_n : n = 1, 2, 3, ...\}$. is bounded above and if x is the supremum of E, we have $a_n \leq x \leq b_n$, for all n. Thus $x \in I_n$, for $n = 1, 2, 3, ...$ and hence $x \in \bigcap I_n$. □

Definition 1.8. *A subset I of $\mathbb{R}^k$ is called a $k-$**cell** if the coordinates x_i of every point $\mathbf{x} = (x_1, x_2, ..., x_k) \in I$ satisfy the inequality $a_i \leq x_i \leq b_i$, where $a_i < b_i$, for $i = 1, 2, ..., k$.*



1-cell



2-cell

In other words, a $k-$ cell is the cartesian product of k intervals

$$[a_1, b_1] \times [a_2, b_2] \times ... \times [a_k, b_k],$$

where $a_i < b_i$, for $1 \leq i \leq k$. A 1- cell is an interval, a $2-$cell is a rectangle, and so on. Note that a $k-$cell is a bounded perfect subset of $\mathbb{R}^k$.

Theorem 1.16. *Let k be a positive integer. If $\{I_n\}$ is a sequence of k-cells such that $I_n \supset I_{n+1}$ (n=1,2,3,....), then $\bigcap_1^\infty I_n$ is not empty.*

Proof. Let

$$I_n = \{\mathbf{x} = (x_1, ..., x_k) \in \mathbb{R}^k : a_{n,j} \leq x_j \leq b_{n,j},\ 1 \leq j \leq k\},\ n = 1, 2, 3, ...$$

That is

$$I_n = [a_{n,1}, b_{n,1}] \times [a_{n,2}, b_{n,2}] \times \cdots \times [a_{n,k}, b_{n,k}].$$

Then, for each $j,\ 1 \leq j \leq k$ we have a sequence of intervals, say

$$I_{n,j} = [a_{n,j}, b_{n,j}],\ n = 1, 2, 3, ...$$

Given $I_n \supset I_{n+1}$, so that, For each j, the sequence of intervals $\{I_{n,j}\}$ satisfies the hypothesis of Theorem 1.15 namely, $I_{n,j} \supset I_{n+1,j},\ n = 1, 2, 3, ...$ Hence by theorem, for each j, $\bigcap I_{n,j} \neq \phi$. If $x_j^* \in \bigcap I_{n,j}$, then the point

$$\mathbf{x}^* = (x_1^*, ..., x_k^*) \in \bigcap_{n=1}^{\infty} I_n.$$

□

Theorem 1.17. *Every k-cell is compact.*

Proof. Let the $k-$cell be

$$I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k]. \tag{1.2}$$

For example consider a $2-$cell

$$[a_1, b_1] \times [a_2, b_2].$$

The maximum distance between two points in I is

$$d = \{(b_1 - a_1)^2 + (b_2 - a_2)^2\}^{1/2}.$$

That is

$$|\mathbf{x} - \mathbf{y}| \leq d,$$

for any two points $\mathbf{x},\ \mathbf{y} \in I$.

In the general case, for the $k-$cell (1.2), we have

$$d = \{(b_1 - a_1)^2 + (b_2 - a_2)^2 + \cdots + (b_n - a_n)^2\}^{1/2} \tag{1.3}$$

and

$$|\mathbf{x} - \mathbf{y}| \leq d,$$

for any two points $\mathbf{x},\ \mathbf{y} \in I$.

Suppose that I is not compact. Then I has an open cover G_α containing no finite subcover of I. Consider the mid point of a_j and b_j for each j, $c_j = \frac{a_j+b_j}{2}$. Now, for $1 \leq j \leq k$, the intervals $[a_j, c_j]$ and $[c_j, b_j]$ divide I into 2^k $k-$cells.

If all these 2^k subcells are covered by finitely many members of $\{G_\alpha\}$, then I would be covered by a finite subcollection of $\{G_\alpha\}$. So, by our choice of $\{G_\alpha\}$, at least one of these 2^k $k-$cells is not covered by any finite collection of $\{G_\alpha\}$. Let I_1 be such a subcell. Then, the maximum distance d_1 between two points in I_1 is given by $d_1 = (1/2)d$. That is

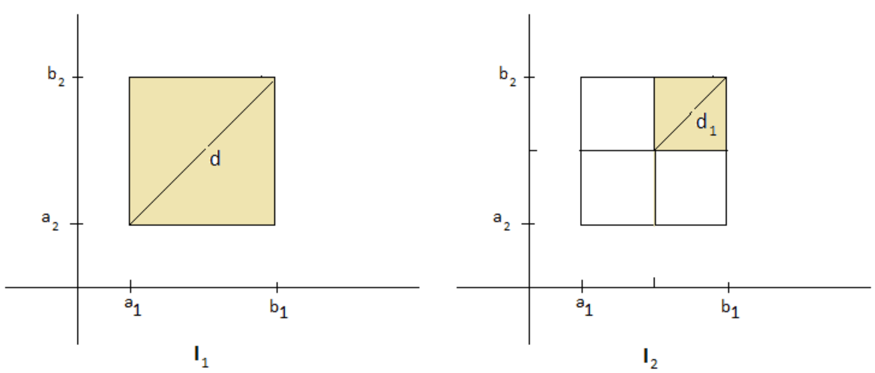
$$|\mathbf{x} - \mathbf{y}| \leq d/2,$$

for any two points $\mathbf{x},\ \mathbf{y} \in I_1$.

Thus we have a k-cell I_1 such that

1. $I \supset I_1$,
2. $|\mathbf{x} - \mathbf{y}| \leq d/2$, for any two points $\mathbf{x},\ \mathbf{y} \in I_1$ and
3. I_1 is not covered by any finite subcollection of $\{G_\alpha\}$,

where d is given by the equation (1.3).



Now subdivide the cell I_1 into 2^k subcells through the midpoints of the component intervals and choose a subcell I_2 of I_1 which is not covered by any finite collection of $\{G_\alpha\}$. Continuing this process, we get a sequence of $k-$cells $I = I_0,\ I_1,\ I_2,\ I_3, ...$ satisfying the following three conditions.

(a) $I_n \supset I_{n+1}$, for $n = 0, 1, 2, ...,$

(b) $|\mathbf{x} - \mathbf{y}| \leq d/(2^n)$, for any two points $\mathbf{x},\ \mathbf{y} \in I_n$ and

(c) I_n is not covered by any finite subcollection of $\{G_\alpha\}$, for each n.

Therefore by Theorem 1.16,

$$\bigcap_{n=0}^{\infty} I_n \neq \phi.$$

Let

$$\mathbf{x_0} \in \bigcap_{n=0}^{\infty} I_n.$$

Since $\{G_\alpha\}$ covers I, that is

$$I \subset \bigcup_{\alpha} G_\alpha,$$

we have

$$\mathbf{x_0} \in G_\alpha,$$

for some α. Now, G_α is open, so that it contains a neghbourhood of $\mathbf{x_0}$. Let

$$N_r(\mathbf{x_0}) \subset G_\alpha. \tag{1.4}$$

Note that from (b), the diameter of I_n is $d/(2^n)$, which decreases as n increases. Choose n large enough so that

$$d/(2^n) < r.$$

Then, for this value of n, say $n = m$, we have

$$|\mathbf{x} - \mathbf{y}| \leq d/(2^m) < r,$$

for any two points $\mathbf{x},\ \mathbf{y} \in I_m$.

Now $\mathbf{x_0} \in I_n$, for all n, so that $\mathbf{x_0} \in I_m$ and

$$|\mathbf{x} - \mathbf{x_0}| \leq d/(2^m) < r, \text{ for all } \mathbf{x} \in I_m.$$

Thus

$$\mathbf{x} \in N_r(\mathbf{x_0}) \text{ for all } \mathbf{x} \in I_m.$$

That is

$$I_m \subset N_r(\mathbf{x_0}) \subset G_\alpha, \text{ using (1.4).}$$

This shows that I_m is covered by a single member of $\{G_\alpha\}$, which contradicts (c). Hence we can conclude that every open cover of I has a finite subcover and that I is compact.

□

Theorem 1.18. *If a set E in $\mathbb{R}^k$ has one of the following three properties, then it has the other two:*

(a) E is closed and bounded.

(b) E is compact.

(c) Every infinite subset of E has a limit point in E.

Recall that we have shown in Theorem 1.14 that every infinite subset of a compact set K, in any metric space, has a limit point in K. Its converse is also true(verify). That is (b) and (c) of Theorem 1.18 are equivalent in any metric space. But (a) does not imply (b) and (c) in general.

Example 1.6. *consider the set $\mathbb{Q}$ of rational numbers with the usual metric $d(p, q) = |p - q|$.*
Let

$$E = \{p \in \mathbb{Q}^+ : 2 < p^2 < 5\}.$$

That is $E = \mathbb{Q} \bigcap (\sqrt{2}, \sqrt{5})$, which is clearly bounded. Also, E is closed in $\mathbb{Q}$.(Exercise 10)

Now, E is not compact. For, consider the open cover $\{G_n\}$ of E in $\mathbb{Q}$, where,

$$G_n = \mathbb{Q} \bigcap (\sqrt{2} + \frac{1}{n+1}, \sqrt{5}).$$

No finite subcollection of $\{G_n\}$ can cover E, because the union of any finite subcollection of $\{G_n\}$ will be $\mathbb{Q} \bigcap (\sqrt{2} + 1/k, \sqrt{5})$, for some natural number k, which cannot cover $\mathbb{Q} \bigcap (\sqrt{2}, \sqrt{5})$.

Proof. $(a) \Rightarrow (b)$**:** Given E is a closed and bounded subset of $\mathbb{R}^k$. Being bounded, E is contained in some $k-$cell I in $\mathbb{R}^k$. Now

$$E \subset I \subset \mathbb{R}^k,$$

then E is closed $\mathbb{R}^k$ implies that E is closed in I also. Hence, being a closed subset of a compact set E is compact.

$(b) \Rightarrow (c)$**:** See Theorem 1.14.

$(c) \Rightarrow (a)$**:** Assume that every infinite subset of $E \subset \mathbb{R}^k$ has a limit point in E.

If E is not bounded, we can find points $\mathbf{x}_n$ in E with

$$|\mathbf{x}_n| > n, \ n = 1, 2, 3, ...$$

Then the set

$$\{\mathbf{x}_n : n = 1, 2, 3, ...\}$$

is an infinite subset of E having no limit point in $\mathbb{R}^k$, a contradiction.

If E is not closed, then E has a limit point $\mathbf{x}_0 \in \mathbb{R}^k$, which is not in E. Then every neighbourhood of $\mathbf{x}_0$ contains points of E.

For $n = 1, 2, 3, ...$, choose a point $\mathbf{x}_n \in E$, which belongs to the $1/n-$neighbourhood of $\mathbf{x}_0$. That is

$$\mathbf{x}_n \in E \text{ and } |\mathbf{x}_n - \mathbf{x}_0| < \frac{1}{n}, \quad n = 1, 2, 3, ...$$

Now, it easy to see that $\mathbf{x}_0$ is a limit point of the set

$$S = \{\mathbf{x}_n : n = 1, 2, 3, ...\}$$

which is an infinite subset of E. We prove that S has no other limit point. For, consider $\mathbf{p} \in \mathbb{R}^k$, we have if $\mathbf{p} \neq \mathbf{x}_0$, then

$$0 < |\mathbf{p} - \mathbf{x}_0| \leq |\mathbf{p} - \mathbf{x}_n| + |\mathbf{x}_n - \mathbf{x}_0| < |\mathbf{p} - \mathbf{x}_n| + 1/n, \quad \text{for all } n.$$

And thus

$$|\mathbf{p} - \mathbf{x}_n| > |\mathbf{p} - \mathbf{x}_0| - 1/n, \quad \text{for all } n.$$

Now, we can find n_0 such that

$$|\mathbf{p} - \mathbf{x}_0| - \frac{1}{n_0} \geq \frac{1}{2}(|\mathbf{p} - \mathbf{x}_0|),$$

and hence

$$|\mathbf{p} - \mathbf{x}_n| > \frac{1}{2}(|\mathbf{p} - \mathbf{x}_0|), \text{ for all } n \geq n_0,$$

which shows that p cannot be a limit point of S. Thus S has no limit point in E, a contradiction. Hence E must be closed. □

The equivalence of (a) and (b) in the above theorem is known as the **Heine-Borel Theorem**.

Theorem 1.19. *Every bounded infinite subset of* $\mathbb{R}^k$ *has a limit point in* $\mathbb{R}^k$.

Proof. Since a bounded subset of $\mathbb{R}^k$ is contained in a $k-$cell in $\mathbb{R}^k$, and $k-$cells are compact, we have an infinite bounded subset of $\mathbb{R}^k$ is an infinite subset of a compact set and hence has a limit point. □

1.3 Perfect Sets

Definition 1.9. *A set A is said to be* **finite** *if either A is empty or there is a bijection between A and the set $\{1, 2, ..., n\}$ for some natural number n.*

A set A is **countable** if there is a bijection between A and the set of natural numbers $\mathbb{N}$.

A set is **uncountable** if it is neither finite nor countable.

Theorem 1.20. *Let P be a nonempty perfect set in $\mathbb{R}^k$. Then P is uncountable.*

Proof. Being perfect, P is closed and every point is a limit point. Now, since P has limit points, by Theorem 1.2, P is infinite.

If possible let P be countable. Then the elements of P can be listed as

$$P = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, ...\}.$$

Now, our aim is to construct a sequence of neighbourhoods $\{V_n\}$ and then a sequence of compact sets $\{K_n\}$ such that $K_n \supset K_{n+1}$, and will arrive at a contradiction.

Choose any neighbourhood V_1 of $\mathbf{x}_1$. Note that

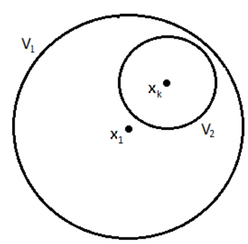
$$V_1 = \{\mathbf{y} \in \mathbb{R}^k : |\mathbf{x}_1 - \mathbf{y}| < r\}$$

and its closure

$$\overline{V_1} = \{\mathbf{y} \in \mathbb{R}^k : |\mathbf{x}_1 - \mathbf{y}| \leq r\}.$$

Now, the neighbourhood V_2 is selected as follows: As x_1 is a limit point of P, V_1 contains many points from P. Choose an $x_k \in V_1 \bigcap P$ and a neighbourhood V_2 of x_k such that

$$\overline{V_2} \subset V_1 \text{ and } x_1 \notin \overline{V_2}.$$



In general, once we selected V_n, choose V_{n+1} in such a way that

$$\overline{V_{n+1}} \subset V_n \text{ and } x_n \notin \overline{V_{n+1}}.$$

Continuing this process, we get a sequence of neighbourhoods

$$\{V_n : n = 1, 2, 3, ...\}$$

such that

$$\overline{V_{n+1}} \subset V_n \text{ and } x_n \notin \overline{V_{n+1}},\ n = 1, 2, 3, ...$$

Now, we get the sequence of compact sets $\{K_n\}$ by taking

$$K_n = \overline{V_n} \bigcap P,\ n = 1, 2, 3, ...$$

For each n, the set $\overline{V_n}$ is closed and bounded subset of $\mathbb{R}^k$, and hence compact, by Heine-Borel Theorem.

Now, both $\overline{V_n}$ and P are closed, so that $\overline{V_n} \bigcap P$ is closed and $K_n = \overline{V_n} \bigcap P$ implies that K_n is a closed subset of the compact set $\overline{V_n}$.

Hence K_n is compact for $n = 1, 2, 3, ...$

By our choice,

$$\overline{V_{n+1}} \subset V_n \subset \overline{V_n}.$$

So that

$$K_{n+1} = \overline{V_{n+1}} \bigcap P \subset \overline{V_n} \bigcap P = K_n, \ n = 1, 2, 3, ...$$

Thus we have a sequence of compact sets $\{K_n\}$ satisfying $K_n \supset K_{n+1}$, and thus by Corollary 1.3,

$$\bigcap_{n=1}^{\infty} K_n \neq \phi.$$

But as $x_n \notin \overline{V_{n+1}}, \ n = 1, 2, 3, ...$; we have $\bigcap K_n$ does not contain any point of P, a contradiction because $K_n \subset P$. Hence we conclude that P is uncountable. □

The Cantor Set

- An uncountable subset of $\mathbb{R}$ containing no segment
- A perfect set
- A compact set
- An uncountable set of measure(length) 0

Construction:

STEP 1. Consider the interval $E_0 = [0, 1]$.
STEP 2. Divide E_0 into three equal parts and remove the middle one third segment, giving $E_1 = [0, 1/3] \bigcup [2/3, 1]$.
STEP 3. Divide each interval in E_1 into three equal parts and remove the middle one third from each, giving $E_2 = [0, 1/9] \bigcup [2/9, 1/3] \bigcup [2/3, 7/9] \bigcup [8/9, 1]$.
Continue the process of subdivision in this way, we get a sequence of intervals:

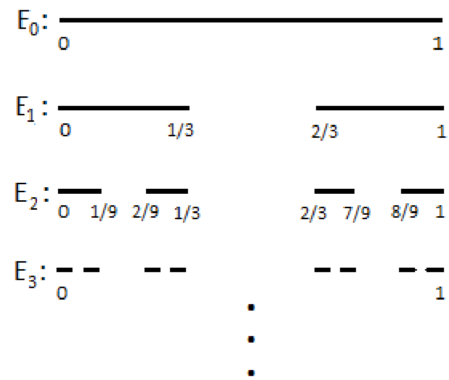
$$E_0 = [0, 1]$$

$$E_1 = [0, 1/3] \bigcup [2/3, 1]$$

$$E_2 = [0, 1/9] \bigcup [2/9, 1/3] \bigcup [2/3, 7/9] \bigcup [8/9, 1]$$

$$\vdots$$

Thus E_n is a union of 2^n intervals, each of length $\dfrac{1}{3^n}$.



Definition 1.10. *The* **Canter set** *is defined as*

$$P = \bigcap_{n=0}^{\infty} E_n.$$

Note that P is nonempty. It contains all end points of each subinterval considering at any step, that is $0, 1, 1/3, 2/3, 1/9, 2/9, ...$

Theorem 1.21. *Cantor set P has the following Properties.*

(a) P is compact.

(b) P contains no segment.

(c) P is perfect.

(d) P is uncountable.

(e) P has measure(length) zero.

Proof. (a) Each E_n and hence P is closed, and $P \subset [0, 1]$ implies that it is bounded. Then by Heine-Borel Theorem, P is compact.

(b) Note that we have removed all segments of the form

$$\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m} \right), \; k = 0, 1, 2, ...; \; m = 1, 2, 3, ...,$$

so that P does not contain a segment of this form.

To prove P contains no segment, we prove that any segment (α, β) in $[0,1]$ contains a segment of the above form.

Consider an arbitrary segment $(\alpha, \beta) \subset [0,1],\ \alpha < \beta$. Choose m in such a way that

$$\frac{1}{3^m} < \frac{\beta - \alpha}{6}.$$

Now,

$$\frac{3k+1}{3^m} = \frac{k}{3^{m-1}} + \frac{1}{3^m}$$

and

$$\frac{3k+2}{3^m} = \frac{k}{3^{m-1}} + \frac{2}{3^m}.$$

Considering the numbers $\frac{1}{3^{m-1}}, \frac{2}{3^{m-1}}, \ldots$, see that we can choose k such that $\frac{k-1}{3^{m-1}} < \alpha$. but $\frac{k}{3^{m-1}} > \alpha$. That is k is the least positive integer satisfying

$$\frac{k}{3^{m-1}} \in (\alpha, \beta).$$

Let $\alpha < a_1 < a_2 < a_3 < a_4 < a_5 < \beta$ be the points dividing the segment (α, β) into six equal parts. Since

$$\mid \frac{k}{3^{m-1}} - \frac{k-1}{3^{m-1}} \mid = \frac{1}{3^{m-1}} = 3\frac{1}{3^m},$$

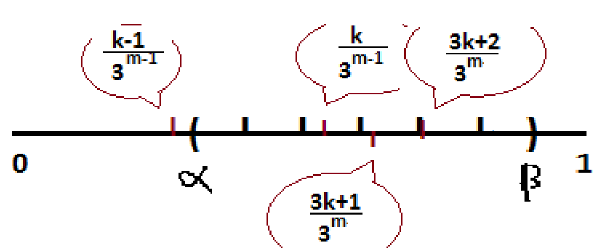
and $\frac{1}{3^m} < \frac{\beta-\alpha}{6}$, we have

$$\frac{k}{3^{m-1}} \in (\alpha, a_3),$$

$$\frac{3k+1}{3^m} = \frac{k}{3^{m-1}} + \frac{1}{3^m} \in (\alpha, a_4)$$

and

$$\frac{3k+2}{3^m} = \frac{k}{3^{m-1}} + \frac{2}{3^m} \in (\alpha, a_5).$$



Hence we have

$$\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \subset (\alpha, \beta).$$

So that P cannot contain a segment.

(c) We have seen in (a) that P is closed. Now to prove every point of P is a limit point of P.

Let $x \in P$ and let N be a neighbourhood of x in $\mathbb{R}$. Since

$$P = \bigcap_{n=0}^{\infty} E_n,$$

we have $x \in E_n,\ n = 0, 1, 2, ...$

Let I_n denote the subinterval of E_n containing x, for $n = 0, 1, 2, ...$ Also note that as n increases, the length of I_n decreases. We can choose n large enough so that $I_n \subset N$.

Now the two end points of this interval I_n belong to both P and N, which shows N contains points in P other than x. Thus x is a limit point and P is perfect.

(d) It follows from (c) and Theorem 1.20.

(e) If $l(A)$ denote the length of A, then

$$l(I_0) = 1$$

$$l(I_1) = 1 - 1/3$$

$$l(I_2) = 1 - (1/3 + 2/9),$$

$$l(I_3) = 1 - (1/3 + 2/9 + 4/27)$$

$$\vdots$$

$$l(I_n) = 1 - \left(\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \cdots + \frac{2^n}{3^{n+1}}\right)$$

$$\vdots$$

Hence

$$\begin{aligned} l(P) &= 1 - \left(\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \cdots + \frac{2^n}{3^{n+1}} + \cdots\right) \\ &= 1 - \frac{1}{3}\left(1 + \frac{2}{3} + \frac{2^2}{3^2} + \cdots + \frac{2^n}{3^n} + \cdots\right) \\ &= 1 - \frac{1}{3}\left(\frac{1}{1-2/3}\right) \\ &= 1 - \frac{1}{3}(3) = 0. \end{aligned}$$

□

1.4 Connected Sets

Definition 1.11. *Two subsets A and B of a metric space X are said to be* **separated** *if both $A \bigcap \overline{B}$ and $B \bigcap \overline{A}$ are empty. That is no point of A lies in the closure of B and no points of B lies in the closure of A.*

Example 1.7. *The segments $(0,1)$ and $(1,2)$ are separated. For, $\overline{(a,b)} = [a,b]$, so that $(0,1) \bigcap \overline{(1,2)} = \phi$ and $\overline{(0,1)} \bigcap (1,2) = \phi$.*

But $(0,1)$ and $[1,2]$ are not separated because, $\overline{(0,1)} \bigcap [1,2] = \{1\}$.

Definition 1.12. *A subset E of X is said to be* **connected** *if E is not a union of two nonempty separated sets. That is E is connected if there do not exist two nonempty separated sets whose union is E.*

The following theorem shows that connected subsets of $\mathbb{R}$ are the intervals(open, closed, semi-open, bounded or unbounded).

Theorem 1.22. *A subset E of $\mathbb{R}$ is connected if and only if it has the following property:*

$$\text{If } x \in E, y \in E \text{ and } x < z < y, \text{ then } z \in E. \tag{1.5}$$

(Remember that in order to prove $p \Rightarrow q$ it is enough if we prove $(-q) \Rightarrow (-p)$.)

Proof. Suppose that E is a subset of $\mathbb{R}$ which does not satisfy the property (1.5). That is there exist $x,\ y \in E$ and some $z \in \mathbb{R}$ such that $x < z < y$ but

$z \notin E$. We prove that E is not connected.

Define $A = E \bigcap(-\infty,\ z)$ and $B = E \bigcap(z,\ \infty)$, then $x \in A$, $y \in B$ and

$$E = A \bigcup B.$$

Thus A and B are nonempty and,

$$A \subset (-\infty,\ z) \text{ and } B \subset (z,\ \infty)$$

imply that

$$\overline{A} \subset (-\infty,\ z] \text{ and } \overline{B} \subset [z,\ \infty),$$

and hence A and B are separated, so that E is not connected.

To prove the converse, suppose that E is not connected. Then there are nonempty separated sets A and B such that $E = A \bigcup B$. Choose $x \in A$ and $y \in B$. Without loss of generality assume that $x < y$ (otherwise interchange x and y in the following arguments).

If $z = sup(A \bigcap [x, y])$, then by Theorem 1.8,

$$z \in \overline{A}.$$

Since $\overline{A} \bigcap B = \phi$, we have

$$z \notin B,$$

and hence $z \neq y$.

Case (i): If $z \notin A$, then we have, $z \notin E$ and $x < z < y$.

Case (ii): If $z \in A$, then $z \notin \overline{B}$, so that $z \notin B$. Hence $z \neq y$, and there exists z_1 such that $z < z_1 < y$ and $z_1 \notin B$.

Since $z = sup(A \bigcap [x, y])$, we have $z_1 \notin A$ also. Thus $z_1 \notin E$ and $x < z_1 < y$. Thus in either case it fails to hold (1.5), hence the proof.

□

Exercises

1. For $x \in \mathbb{R}$ and $y \in \mathbb{R}$, define

$$\begin{aligned} d_1(x,y) &= (x-y)^2 \\ d_2(x,y) &= \sqrt{|x-y|} \\ d_3(x,y) &= |x^2 - y^2| \\ d_4(x,y) &= |x - 2y| \\ d_5(x,y) &= \frac{|x-y|}{1+|x-y|} \end{aligned}$$

   Determine, for each of these, whether it is a metric or not.

2. Prove that in the metric space (X, d), the set X is both open and closed.

3. What are the open and closed balls in $\mathbb{R}^3$?

4. If we regard $\mathbb{R}$ as a subspace of $\mathbb{R}^2$, as in Example 4, is every closed subset of $\mathbb{R}$ a closed subset of $\mathbb{R}^2$?

5. Prove that the closure of (a, b) in $\mathbb{R}$ is $[a, b]$.

6. What are the compact subsets of a discrete space?

7. Prove or disprove:
   (a) Every point in an open subset E of $\mathbb{R}$ is a limit point of E.
   (b) Every point in a closed subset E of $\mathbb{R}$ is a limit point of E.

   Do the same for subsets of $\mathbb{R}^2$.

8. If E° denotes the set of all interior points of E, prove that
   (a) E° is always open.
   (b) E is open if and only if $E^\circ = E$.
   (c) for every open subset G of E, $G \subset E^\circ$.
   (d) the complement of E°, is the closure of the complement of E.

9. Explain how the following pair of sets are related.

(a) E° and $(\overline{E})^\circ$.

(b) E is open if and only if $E^\circ = E$.

(c) $\overline{E}$ and $\overline{E^\circ}$.

(d) $(\overline{E})^\circ$ and $\overline{E^\circ}$

(e) $(\overline{A \bigcup B})$ and $\overline{A} \bigcup \overline{B}$

(f) $(A \bigcap B)^\circ$ and $A^\circ \bigcap B^\circ$

10. Consider the set $\mathbb{Q}$ of rational numbers with the usual metric $d(p, q) = |p - q|$. Prove that the set $E = \{p \in \mathbb{Q}^+ : 2 < p^2 < 5\}$ is a closed subset of $\mathbb{Q}$. Is E open in $\mathbb{Q}$?

11. Verify whether the following are compact subsets of $\mathbb{R}$.

    (a) $[0, \infty)$

    (b) $\{1/n : \ n = 1, 2, 3, ...\} \bigcup \{0\}$

    (c) $\{n + 1/n : \ n = 1, 2, 3, ...\}$

12. Construct a compact set of real numbers with

    (a) exactly one limit point.

    (b) exactly two limit points.

    (c) a countable number of limit points.

13. Is there a nonempty perfect set in $\mathbb{R}$ which contains no rational number?

14. Is there a nonempty perfect set in $\mathbb{R}$ which contains no irrational number?

15. Identify the connected subsets of a discrete space.

    (i) Find a countable dense subset of $\mathbb{R}$.

    (ii) Find a countable dense subset of $\mathbb{R}^k$.

    (iii) Use (i) to prove that every open subset of $\mathbb{R}$ is the union of an at most countable collection of disjoint segments. (A metric space having a countable dense subset is called **separable**)

Chapter 2

Continuity

2.1 Limits of Functions

Definition 2.1. *Let (X, d_X) and (Y, d_Y) be metric spaces, $E \subset X$ and $f : E \to Y$ be a function(also called mapping). For a limit point p of E, we say that*

$$\lim_{x \to p} f(x) = q$$

or

$$f(x) \to q \text{ as } x \to p$$

if there is a point $q \in Y$ satisfying the property: For every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d_Y(f(x), q) < \varepsilon$$

for all points $x \in E$ with

$$0 < d_X(x, p) < \delta.$$

Note:

• If $X = \mathbb{R}$ and $Y = \mathbb{R}$, then $f(x) \to q$ as $x \to p$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - q| < \varepsilon$$

for all points $x \in E$ with

$$|x - p| < \delta.$$

• The point p need not be a point of E. And even if p is a point of E, we may not have

$$\lim_{x \to p} f(x) = f(p).$$

Example 2.1. *Consider the function defined on* $\mathbb{R}$ *by*

$$f(x) = \begin{cases} -1, & \textit{if} \;\; x < 0 \\ 0, & \textit{if} \;\; x = 0 \\ 1, & \textit{if} \;\; x > 1. \end{cases}$$

Note that every point in $\mathbb{R}$ *is a limit point of* $\mathbb{R}$*. And,*

$$\lim_{x \to p} f(x) = \begin{cases} -1, & \textit{if} \;\; p < 0 \\ 1, & \textit{if} \;\; p > 1 \end{cases}$$

but

$$\lim_{x \to 0} f(x)$$

does not exist. This function is called the **sign function** *or* **signum function** *as it explores the sign of a real number, and is often denoted by* **sgn**.

Example 2.2. *Consider the function* $f : [0, \infty) \to \mathbb{R}$ *defined by*

$$f(x) = \begin{cases} 0, & \textit{if} \;\; x = 0 \\ 1/x, & \textit{if} \;\; x > 0. \end{cases}$$

The set of limit points of $[0, \infty)$ *is* $[0, \infty)$*. And,*

$$\lim_{x \to p} f(x) = 1/p, \;\; \textit{if}\; p \in (0, \infty)$$

but

$$\lim_{x \to 0} f(x)$$

does not exist, while $f(0) = 0$*.*

Theorem 2.1. *Let* X *and* Y *be metric spaces,* $E \subset X$*,* $f : E \to Y$ *be a function and* p *be a limit point of* E*. Then*

$$\lim_{x \to p} f(x) = q$$

if and only if

$$\lim_{n \to \infty} f(p_n) = q$$

for every sequence $\{p_n\}$ *in* E *such that*

$$p_n \neq p, \;\; \lim_{n \to \infty} p_n = p.$$

Proof. Suppose that

$$\lim_{x \to p} f(x) = q. \tag{2.1}$$

We have to show that for every sequence $\{p_n\}$ in E converging to p, the sequence $\{f(p_n)\}$ in Y must converge to q.

So, choose a sequence $\{p_n\}$ in E such that

$$p_n \neq p, \ \lim_{n \to \infty} p_n = p.$$

Let $\varepsilon > 0$, (2.1) implies that there exists a $\delta > 0$ such that

$$d_Y(f(x), q) < \varepsilon \text{ for all points } x \in E \text{ with } 0 < d_X(x, p) < \delta. \tag{2.2}$$

Also, for this positive number δ,

$$\lim_{n \to \infty} p_n = p$$

implies that there exists a positive integer N such that

$$d_X(p_n, p) < \delta \text{ for all } n \geq N.$$

Now, $p_n \neq p$, so that $d_X(p_n, p) > 0$. Then by (2.2), we have

$$d_Y(f(p_n), q) < \varepsilon \text{ for } n \geq N,$$

which shows

$$\lim_{n \to \infty} f(p_n) = q.$$

Conversely if (2.1) does not hold, then there exists some $\varepsilon > 0$ such that for every $\delta > 0$, there exists a point $x \in E$, depending on δ, for which

$$d_Y(f(x), q) \geq \varepsilon \text{ but } 0 < d_X(x, p) < \delta. \tag{2.3}$$

Let x_n denotes the point in E satisfying (2.3) for $\delta = 1/n$. Thus we have a sequence $\{x_n\}$ in E such that

$$d_Y(f(x_n), q) \geq \varepsilon \tag{2.4}$$

but

$$0 < d_X(x_n, p) < 1/n. \tag{2.5}$$

Now, as $1/n \to 0$, as $n \to \infty$, (2.5) shows that $x_n \to p$, but (2.4) shows $f(x_n) \nrightarrow q$, Which completes the proof. □

The following corollary is an immediate consequence of Theorem 2.1 and the result that a sequence in a metric space can have at most one limit.

Corollary 2.1. *If f has a limit at p, this limit is unique.*

Definition 2.2. *Let X be a metric space, $E \subset X$ and f and g be two complex valued functions defined on E. Then $f+g$, $f-g$, fg and f/g denote respectively the functions*

$$(f+g)(x) = f(x) + g(x)$$

$$(fg)(x) = f(x)g(x)$$

and

$$(f/g)(x) = \frac{f(x)}{g(x)}, \text{ for } g(x) \neq 0.$$

If $\lambda \in \mathbb{C}$, λf defined by

$$(\lambda f)(x) = \lambda(f(x))$$

also gives a complex function.

Similarly if $\boldsymbol{f}$ and $\boldsymbol{g}$ map E into $\mathbb{R}^k$, and $\lambda \in \mathbb{R}$, we can define the functions

$$(\boldsymbol{f}+\boldsymbol{g})(x) = \boldsymbol{f}(x) + \boldsymbol{g}(x), \quad (\boldsymbol{f}\cdot\boldsymbol{g})(x) = \boldsymbol{f}(x)\cdot\boldsymbol{g}(x) \quad \text{and} \quad (\lambda\boldsymbol{f})(x) = \lambda(\boldsymbol{f}(x),$$

where $\boldsymbol{f}(x)\cdot\boldsymbol{g}(x)$ denotes the dot product of $\boldsymbol{f}(x)$ and $\boldsymbol{g}(x)$.

Note that $\boldsymbol{f}+\boldsymbol{g} : X \to \mathbb{R}^k$ and $\lambda\boldsymbol{f} : X \to \mathbb{R}^k$, but $\boldsymbol{f}\cdot\boldsymbol{g} : X \to \mathbb{R}$.

Theorem 2.2. *Suppose X is a metric space, $E \subset X$, p is a limit point of E, f and g are complex functions on E, and*

$$\lim_{x\to p} f(x) = A, \quad \lim_{x\to p} g(x) = B.$$

Then

$$\lim_{x\to p}(f+g)(x) = A + B,$$

$$\lim_{x\to p}(fg)(x) = AB$$

and

$$\lim_{x\to p}\left(\frac{f}{g}\right)(x) = \frac{A}{B}, \text{ if } B \neq 0.$$

Proof. Using Theorem 2.1, we have

$$\lim_{x\to p} f(x) = A \quad \Rightarrow \quad \lim_{n\to\infty} f(p_n) = A$$

and

$$\lim_{x\to p} g(x) = B \quad \Rightarrow \quad \lim_{n\to\infty} g(p_n) = B$$

for every sequence $\{p_n\}$ in E such that

$$\lim_{n\to\infty} (p_n) = p.$$

From these, we have

$$\lim_{n\to\infty} (f+g)(p_n) = \lim_{n\to\infty} f(p_n) + \lim_{n\to\infty} g(p_n) = A + B,$$

for every sequence $\{p_n\}$ converging to p.
And, again by Theorem 2.1, we have

$$\lim_{x\to p} (f+g)(x) = A + B$$

thus proved the first identity. The proof of second and third are left as an exercise. □

Theorem 2.3. *Suppose X is a metric space, $E \subset X$, p is a limit point of E, $\boldsymbol{f}$ and $\boldsymbol{g}$ map E into $\mathbb{R}^k$, and*

$$\lim_{x\to p} \boldsymbol{f}(x) = \boldsymbol{A}, \quad \lim_{x\to p} \boldsymbol{g}(x) = \boldsymbol{B}.$$

Then

$$\lim_{x\to p} (\boldsymbol{f}+\boldsymbol{g})(x) = \boldsymbol{A} + \boldsymbol{B} \quad \textit{and} \quad \lim_{x\to p} (\boldsymbol{f}\cdot\boldsymbol{g})(x) = \boldsymbol{A}\cdot\boldsymbol{B}.$$

Proof. Exercise □

2.2 Continuous Functions

Definition 2.3. *Suppose $(X,\ d_X)$ and $(Y,\ d_Y)$ are metric spaces, $E \subset X$ and f maps E into Y. Then f is said to be* **continuous at** $p \in E$ *if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$d_Y(f(x),\ f(p)) < \varepsilon$$

for all points $x \in E$ with $d_X(x,\ p) < \delta$.

If f is continuous at every point on E, then f is said to be **continuous on** E.

Example 2.3. *Here we illustrate some particular examples, the reader can verify some simple examples like constant function, identity function, etc. to get a clear picture of the concept of continuity and the* ε, δ *definition.*

(a) The mapping $f : (0, \infty) \to \mathbb{R}$, *defined by* $f(x) = 1/x$, *is continuous on* $(0, \infty)$.

For, choose $\varepsilon > 0$ *and a point* $p \in (0, \infty)$. *To prove* f *is continuous at* p, *we have to find* $\delta > 0$ *such that*

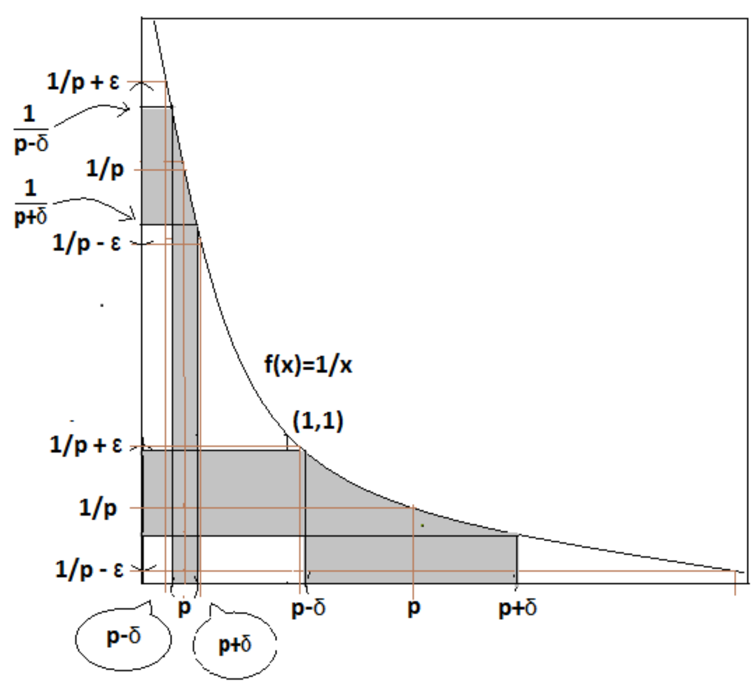
$$|f(x) - f(p)| < \varepsilon \text{ for all } x \in (0, \infty) \text{ satisfying } |x - p| < \delta.$$

Note that

$$|f(x) - f(p)| = |1/x - 1/p|.$$

and that

$$|1/x - 1/p| < \varepsilon \text{ whenever } \left|\frac{x - p}{xp}\right| = \frac{|x - p|}{xp} < \varepsilon. \tag{2.6}$$



Referring the Figure, note that we have to find δ such that

$$\left|\frac{1}{p} - \frac{1}{p-\delta}\right| < \varepsilon \text{ and } \left|\frac{1}{p} - \frac{1}{p+\delta}\right| < \varepsilon.$$

That is to choose δ in such a way that

$$\frac{\delta}{p(p-\delta)} < \varepsilon \text{ and } \frac{\delta}{p(p+\delta)} < \varepsilon$$

or,

$$\delta(1+\varepsilon p) < \varepsilon p^2 \text{ and } \delta(1-\varepsilon p) < \varepsilon p^2.$$

Note that if $\varepsilon p \geq 1$, then any δ will satisfy the second inequality, and if $\varepsilon p < 1$, then

$$\frac{\varepsilon p^2}{1+\varepsilon p} = \min\{\frac{\varepsilon p^2}{1+\varepsilon p}, \frac{\varepsilon p^2}{1-\varepsilon p}\}$$

since both ε and p are positive. Hence if we choose

$$\delta < \frac{\varepsilon p^2}{1+\varepsilon p},$$

then for

$$|x-p| < \delta,$$

we have

$$x > p - \delta$$

and

$$\frac{|x-p|}{xp} < \frac{\delta}{(p-\delta)p} < \frac{\varepsilon p^2/(1+\varepsilon p)}{\left(p - \frac{\varepsilon p^2}{1+\varepsilon p}\right)p} = \varepsilon.$$

Hence from (2.6) we have the result.

(b) *Exercise 2 asks to prove the function $f(x) = sin\ x,\ x \in \mathbb{R}$ is continuous on $\mathbb{R}$.*

Theorem 2.4. *Suppose $(X,\ d_X)$ and $(Y,\ d_Y)$ are metric spaces, $E \subset X$ and f maps E into Y. Also assume that $p \in E$ is a limit point of E. Then f is continuous at p if and only if*

$$\lim_{x \to p} f(x) = f(p).$$

Proof. It follows from the Definitions 2.1 and 2.3. □

Theorem 2.5. *Suppose X, Y and Z are metric spaces and $E \subset X$. Let $f : E \to Y$ and $g : f(E) \to Z$, where $f(E)$ is the range of f. Also let $h : E \to Z$ be the mapping defined by*

$$h(x) = g(f(x)), \quad x \in E.$$

If f is continuous at $p \in E$ and if g is continuous at $f(p)$, then h is continuous at p.

The function h is the **composition** or **composite** of f and g and is usually denoted by $g \circ f$.

Proof. Given f is continuous at p and g is continuous at $f(p)$, so that we have:
(i) For any $\zeta > 0$, there exists $\xi > 0$ such that

$$d_Y(f(x), f(p)) < \zeta \text{ for all } x \in E \text{ with } d_X(x, p) < \xi.$$

(ii) For any $\eta > 0$, there exists $\mu > 0$ such that

$$d_z(g(y), g(f(p))) < \eta \text{ for all } y \in f(E) \text{ with } d_Y(y, f(p)) < \mu.$$

Note that an element $y \in f(E)$ is of the form $f(x)$ for some $x \in E$ and that $\zeta,\ \xi,\ \eta,\ \mu$ are just notations.

In order to get h is continuous at p, we prove:
(iii) For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_Z(h(x), h(p)) < \varepsilon \text{ for all } x \in E \text{ with } d_X(x, p) < \delta.$$

That is

$$d_Z(g(f(x)), g(f(p))) < \varepsilon \text{ for all } x \in E \text{ with } d_X(x, p) < \delta.$$

So, choose $\varepsilon > 0$.

Analysing (i) and (ii) we can see that with $\eta = \varepsilon$, (ii) ensures that there exists $\mu > 0$ such that

$$d_z(g(y), g(f(p))) < \varepsilon \text{ for all } y \in f(E) \text{ with } d_Y(y, f(p)) < \mu.$$

That is

$$d_Z(g(f(x)), g(f(p))) < \varepsilon \text{ for all } x \in E \text{ with } d_Y(f(x), f(p)) < \mu.$$

With $\zeta = \mu$, (i) ensures that there exists $\xi > 0$, say $\xi = \delta$ such that

$$d_Y(f(x), f(p)) < \mu \text{ for all } x \in E \text{ with } d_X(x, p) < \delta.$$

Thus (iii) is proved and hence h is continuous at p. $\square$

Example 2.4. *From Example 2.3(b) and Theorem 2.5, we can conclude that the mapping $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = sin\ \frac{1}{x}$ is continuous on $(0, \infty)$.*

Now, the definition of continuity can be restated as: f is continuous at p if and only if corresponding to every $\varepsilon-$ neighbourhood N of $f(p)$, there is some $\delta-$ neighbourhood U of p such that $f(U) \subset N$.

The next Theorem, in view of this fact, gives a very useful characterization of continuity in terms of open sets. It will reduce the effort of verifying the continuity of a function, from computing δ corresponding to an arbitrary ε, to proving the inverse image of an open set is open or equivalently proving the inverse image of a closed set is closed.

Definition 2.4. *For any mapping $f : X \to Y$, $y \in Y$ and $F \subset Y$,*

$$f^{-1}(y) = \{x \in X : f(x) = y\}$$

and

$$f^{-1}(F) = \{x \in X : f(x) \in F\}.$$

Note that this definition is valid even if the function is not invertible.

Theorem 2.6. *Let X and Y be metric spaces. A mapping $f : X \to Y$ is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.*

Proof. Suppose that f is continuous on X. Let V be an open set in Y. To show that $f^{-1}(V)$ is open in X, we must argue that every point in $f^{-1}(V)$ is an interior point of $f^{-1}(V)$.

Choose $p \in f^{-1}(V)$, so that $f(p) \in V$. Now V is open, $f(p)$ has a neighbourhood $N = N_r(f(p))$ such that $N \subset V$. Since f is continuous at p, for this positive number r, there exists $\delta > 0$ such that

$$d_Y(f(x), f(p)) < r \text{ for all } x \in X \text{ with } d_X(x, p) < \delta. \tag{2.7}$$

In other words (2.7) says

$$f(x) \in N \text{ for all } x \in N_\delta(p) = U.$$

That is

$$f(U) \subset N$$

Since $N \subset V$, we have

$$f(U) \subset V, \text{ and hence } U = N_\delta(p) \subset f^{-1}(V),$$

as required.

Conversely assume that $f^{-1}(V)$ is open in X for all open set V in Y. We have to prove f is continuous on X.

Choose $p \in X$ and $\varepsilon > 0$. Consider the $\varepsilon-$neighbourhood of $f(p)$ in Y, namely,

$$G = N_\varepsilon(f(p)) = \{y \in Y : d_Y(y, f(p)) < \varepsilon.$$

Since every neighbourhood is open, G is open in Y.

By the hypothesis we have $f^{-1}(G)$ is open in X. Now, $f(p) \in G$ so that $p \in f^{-1}(G)$, and hence p is an interior point of $f^{-1}(G)$. Thus, p has a neighbourhood, say

$$K = N_\delta(p) \subset f^{-1}(G).$$

In other words

$$x \in K = N_\delta(p) \Rightarrow x \in f^{-1}(G).$$

That is,

$$\begin{aligned} d_X(x,p) < \delta \Rightarrow & x \in f^{-1}(G) \\ \Rightarrow & f(x) \in G \\ \Rightarrow & d_Y(f(x), f(p)) < \varepsilon, \end{aligned}$$

which shows the continuity of f at p. Since p is an arbitrary point of X, we have f is continuous on X.

$\square$

Corollary 2.2. *Let X and Y be metric spaces. A mapping $f : X \to Y$ is continuous on X if and only if $f^{-1}(C)$ is open in X for every closed set C in Y.*

Proof This result follows from Theorem 1.4, which says, C is closed in Y if and only if its complement C^c is open in Y and the relation $f^{-1}(C^c) = [f^{-1}(C)]^c$.

For complex valued functions, where algebraic operations of functions can be defined, we have the following result.

Theorem 2.7. *Let f and g be complex functions defined on a metric space X. If f and g are continuous on X, then $f+g$ and fg are continuous on X. More over, if $g(x) \neq 0$ for all $x \in X$, then f/g is continuous.*

Proof. The proof is left as an exercise. □

Definition 2.5. *If $\boldsymbol{f}$ maps X into $\mathbb{R}^k$ and*

$$\boldsymbol{f}(x) = (x_1, x_2, ..., x_k),$$

then for each i, $1 \leq i \leq k$,

$$f_i(x) = x_i$$

defines a mapping of X into $\mathbb{R}$.

These real valued functions $f_1, f_2, ..., f_k$ are called the **components** *of $\boldsymbol{f}$.*

Continuity of vector-valued functions is equivalent to the continuity of all its component functions.

Theorem 2.8.

(a) Let $f_1, f_2, ..., f_k$ be real continuous functions on a metric space X and let $\boldsymbol{f}$ be the mapping of X into $\mathbb{R}^k$ defined by

$$\boldsymbol{f}(x) = (f_1(x), f_2(x), ..., f_k(x)), \quad x \in X.$$

Then $\boldsymbol{f}$ is continuous if and only if each of the functions $f_1, f_2, ..., f_k$ is continuous.

(b) If $\boldsymbol{f}$ and $\boldsymbol{g}$ are continuous mappings of X into $\mathbb{R}^k$, then $\boldsymbol{f}+\boldsymbol{g}$ and $\boldsymbol{f}\cdot\boldsymbol{g}$ are continuous on X.

Proof. The metric in $\mathbb{R}^k$ is the Euclidean metric, and

$$|\mathbf{f}(x) - \mathbf{f}(p)| = \left[\sum_{i=1}^{k} |f_i(x) - f_i(p)|^2\right]^{1/2}. \tag{2.8}$$

Since each term of the above sum is non-negative and $|f_j(x) - f_j(p)|$ is one of the terms, for each j, we have

$$|f_j(x) - f_j(p)| \leq \left[\sum_{i=1}^{k} |f_i(x) - f_i(p)|^2\right]^{1/2}.$$

That is

$$|f_j(x) - f_j(p)| \leq |\mathbf{f}(x) - \mathbf{f}(p)|. \tag{2.9}$$

(a) Let $\mathbf{f}$ be continuous. Choose $\varepsilon > 0$ and a point $p \in X$. If $\mathbf{f}$ is continuous at p, we can find $\delta > 0$ such that

$$|\mathbf{f}(x) - \mathbf{f}(p)| < \varepsilon \text{ for all } x \in X \text{ with } d_X(x, p) < \varepsilon.$$

Then by (2.9), we have, for $1 \leq j \leq k$,

$$|f_j(x) - f_j(y)| < \varepsilon \text{ for all } x \in X \text{ with } d_X(x, p) < \delta.$$

Hence f_j is continuous for each j.

Conversely if f_j is continuous for each j, then we can find $\delta_j > 0$ such that

$$|f_j(x) - f_j(p)| < \varepsilon \text{ for all } x \in X \text{ with } d_X(x, p) < \delta_j,\ 1 \leq j \leq k. \tag{2.10}$$

If we choose

$$\delta = \min\{\delta_1, \delta_2, ..., \delta_k\},$$

then every f_j will satisfy

$$|f_j(x) - f_j(p)| < \varepsilon \text{ for all } x \in X \text{ with } d_X(x, p) < \delta.$$

Substituting in (2.8), we get

$$|\mathbf{f}(x) - \mathbf{f}(p)| < \sqrt{k}\varepsilon, \text{ for all } x \in X \text{ with } d_X(x, p) < \delta.$$

Hence $\mathbf{f}$ is continuous.

(If you want to get $|\mathbf{f}(x) - \mathbf{f}(p)| < \varepsilon$, replace ε in (2.10) by $\varepsilon/(\sqrt{k})$, that is obtain δ_j corresponding to $\varepsilon/(\sqrt{k})$.)

(b) The components of $\mathbf{f}+\mathbf{g}$ are $f_1 + g_1, f_2 + g_2, ..., f_k + g_k$; and for all x,

$$\begin{aligned}(\mathbf{f}\cdot\mathbf{g})(x) &= f_1(x)g_1(x)+\cdots+f_k(x)g_k(x)\\ &= (f_1g_1+\cdots+f_kg_k)(x),\end{aligned}$$

so that $\mathbf{f}\cdot\mathbf{g} = f_1g_1+\cdots+f_kg_k$.

If $\mathbf{f}$ and $\mathbf{g}$ are continuous, then by part (a) we have $f_1, f_2, ...f_k, g_1, g_2, ...g_k$ all are continuous and using Theorem 2.7, we get $\mathbf{f}+\mathbf{g}$ and $\mathbf{f}\cdot\mathbf{g}$ are continuous.

□

Example 2.5.

*(i) If $\boldsymbol{x} = (x_1, x_2, ..., x_k) \in \mathbb{R}^k$, then the **coordinate functions***

$$\phi_i : \mathbb{R}^k \to \mathbb{R},\ 1 \le i \le k$$

defined by,

$$\phi_i(\boldsymbol{x}) = x_i$$

are continuous.
For, if $\boldsymbol{x} = (x_1, x_2, ..., x_k)$, $\boldsymbol{y} = (y_1, y_2, ..., y_k) \in \mathbb{R}^k$,

$$|\boldsymbol{x}-\boldsymbol{y}| = \left[\sum_{i=1}^{k} |x_i - y_i|^2\right]^{1/2},$$

so that

$$|\phi_i(\boldsymbol{x}) - \phi_i(\boldsymbol{y})| = |x_i - y_i| \le |\boldsymbol{x}-\boldsymbol{y}|.$$

Hence, given $\varepsilon > 0$, we may take $\delta = \varepsilon$, to get

$$|\phi_i(\boldsymbol{x}) - \phi_i(\boldsymbol{y})| < \varepsilon \text{ for every } \boldsymbol{x} \text{ and } \boldsymbol{y} \text{ with } |\boldsymbol{x}-\boldsymbol{y}| < \delta.$$

(Note that these coordinate functions are precisely the components of the identity mapping of $\mathbb{R}^k$.)

(ii) A function $f : \mathbb{R}^k \to \mathbb{R}$ of the form

$$f(\boldsymbol{x}) = x_1^{n_1}x_2^{n_2}...x_k^{n_k},\ \boldsymbol{x} = (x_1, x_2, ..., x_k) \in \mathbb{R}^k,$$

*where $n_1, n_2, ...n_k$ are non negative integers is called a **monomial**.*
For example, $f(x_1, x_2, x_3) = x_1^2x_2x_3^5$ is a monomial in three variables.

Note that every monomial is a product of coordinate functions. Hence by Theorem 2.7, every monomial is continuous.

(iii) *Since constant functions are also continuous, repeated application of Theorem 2.7 gives every polynomial, given by*

$$P(\boldsymbol{x}) = \sum c_{n_1 \dots n_k} x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}, \ \ \boldsymbol{x} = (x_1, x_2, \dots, x_k) \in \mathbb{R}^k$$

is continuous on $\mathbb{R}^k$. *In the above expression of* P, $c_{n_1 \dots n_k}$ *may be real or complex numbers,* $n_1, \dots, n_k$ *are non-negative integers and the sum includes only finitely many terms.*

Again by Theorem 2.7, the quotient of two continuous functions is continuous, wherever the denominator is nonzero. Thus we have every rational function in $x_1, x_2, \dots, x_k$ *is continuous wherever the denominator is nonzero.*
(A rational function in $x_1, x_2, \dots, x_k$ *is a quotient of two polynomial functions in* $x_1, x_2, \dots, x_k$*.)*

(iv) *The function* $g : \mathbb{R}^k \to \mathbb{R}$, *given by*

$$g(\boldsymbol{x}) = |\boldsymbol{x}|$$

is continuous.

It follows from the inequality

$$|\ |\boldsymbol{x}| - |\boldsymbol{y}|\ | \leq |\boldsymbol{x} - \boldsymbol{y}|, \ \ \boldsymbol{x}, \ \ \boldsymbol{y} \in \mathbb{R}^k.$$

(v) *If* $\boldsymbol{f} : X \to \mathbb{R}^k$ *is continuous then*

$$\phi(p) = |\ \boldsymbol{f}(p)\ |$$

gives a continuous real function on X. *Because* ϕ *is the composition of the function* $\boldsymbol{f}$ *and the modulus function* g *in* (iv) *both are continuous and the composition of continuous functions is continuous.*

2.3 Continuity and Compactness

Theorem 2.9. *Suppose* f *is a continuous mapping of a compact metric space* X *into a metric space* Y. *Then* $f(X)$ *is compact.*

Proof. Let $\{V_\alpha\}$ be an open cover for $f(X)$. Then each V_α is open. Since f is continuous $f^{-1}(V_\alpha)$ is open in X. Now, $\{f^{-1}(V_\alpha)\}$ is an open cover for X and X is compact. Hence there is a finite subcollection of $\{f^{-1}(V_\alpha)\}$, namely $\{f^{-1}(V_1), f^{-1}(V_2), ..., f^{-1}(V_n)\}$ such that

$$X \subset f^{-1}(V_1) \bigcup f^{-1}(V_2) \bigcup ... \bigcup f^{-1}(V_n).$$

That is

$$X \subset f^{-1}(V_1 \bigcup V_2 \bigcup ... \bigcup V_n).$$

So that

$$f(X) \subset V_1 \bigcup V_2 \bigcup ... \bigcup V_n,$$

which shows that $f(X)$ is compact. □

Definition 2.6. *For $E \subset X$, a mapping $\boldsymbol{f} : E \to \mathbb{R}^k$ is said to be* ***bounded*** *if there is a real number M such that $|\boldsymbol{f}(x)| \leq M$ for all $x \in E$.*

Theorem 2.10. *If $\boldsymbol{f}$ is a continuous mapping of a compact metric space X into $\mathbb{R}^k$, then $\boldsymbol{f}(X)$ is closed and bounded. That is the function $\boldsymbol{f}$ is bounded.*

Proof. Since $\mathbf{f}$ is continuous and X is compact, $\mathbf{f}(X)$ is compact, by Theorem 2.9. Now $\mathbf{f}(X) \subset \mathbb{R}^k$ so that by Heine-Borel Theorem(Theorem 1.18) $\mathbf{f}(X)$ is closed and bounded. Hence $\mathbf{f}$ is bounded. □

Thus a continuous function f on a compact metric space is bounded; more over, f attains its maximum and minimum which we prove in next theorem.

If X is not compact we can find continuous functions on X which are not bounded.

Example 2.6. *The mapping $f : (0, 1] \to \mathbb{R}$, defined by $f(x) = 1/x$ is continuous but not bounded, because in neighbourhoods of* 0, *$f(x)$ increases unbounded.*

Theorem 2.11. *Suppose f is continuous real function on a compact metric space X, and*

$$M = \sup_{p \in X} f(p), \quad m = \inf_{p \in X} f(p).$$

Then there exist points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.

Proof. By Theorem 2.9, $f(X)$ is compact. Being a compact subset of $\mathbb{R}$, $f(X)$ is closed and bounded. Hence by Theorem 1.8, we have both M and m are in $f(X)$, and $M = f(p)$ and $m = f(q)$ for some $p,\ q \in X$. □

Theorem 2.12. *Let X and Y be metric spaces and the mapping $f : X \to Y$ be one one and onto. If f is continuous and X is compact, then the inverse mapping $f^{-1} : Y \to X$ defined by*

$$f^{-1}(y) = x,\ \ y \in Y, \textit{if } f(x) = y$$

is continuous.

Proof. Using Theorem 2.6, it is enough to prove that $f(V)$ is open in Y for every open set V of X.

So, let V be an open subset of X. Then its complement V^c, being a closed subset of the compact set X, is compact by Theorem 1.12.

Since f is continuous, we have $f(V^c)$ is compact subset of Y by Theorem 31. Again since compact subsets are closed, we have $f(V^c)$ is closed in Y.

Now, f is one one and onto, so that $f(V^c) = (f(V))^c$. Hence $f(V)$ is open. □

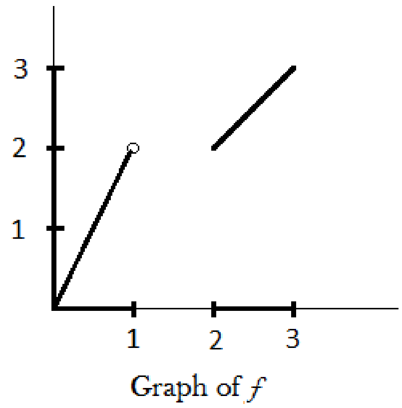
Here again if we skip the compactness of the domain, the result will not hold. In other words, we can find a continuous bijection(one one and onto function) from a metric space X onto a metric space Y such that the inverse function is not continuous.

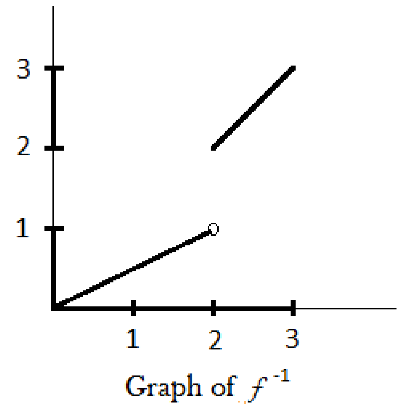
Example 2.7. *Let $X = [0, 1) \bigcup [2, 3]$ and $f : X \to [0, 3]$ be the mapping*

$$f(x) = \begin{cases} 2x, & \textit{if } \ x \in [0, 1) \\ x, & \textit{if } \ x \in [2, 3]. \end{cases}$$

Then f is continuous on X (verify!), one one and onto $[0, 3]$.

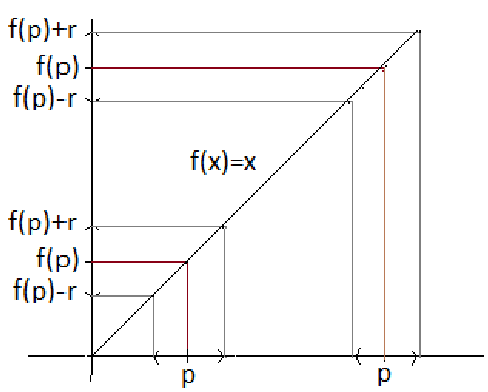
The inverse map $f^{-1} : [0, 3] \to X$ one one and onto but f^{-1} is not continuous at 2(See figure).

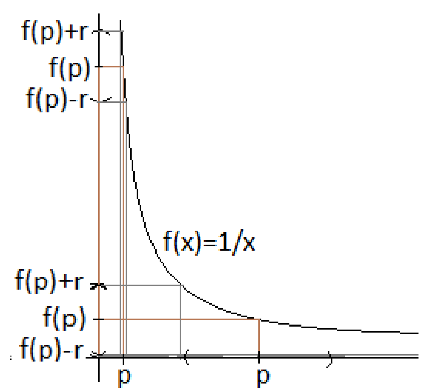




The following figure depicts two continuous functions on $(0, \infty)$;
(i) $f(x) = x$ and
(ii) $f(x) = 1/x$.

Note that, in the first case, for a given $\varepsilon = r$, we can find a common δ for every point p. That is we have a uniformity regardless the choice of the point.





But in the second case as the point p moves closer to 0, δ moves to smaller value, and we cannot find a common value for δ, for all points in the domain, for a fixed ε.(The detailed proof is given in Example 2.8.) This justifies the following definition.

Definition 2.7. *Let X and Y be metric spaces. A mapping $f : X \to Y$ is said to be **uniformly continuous** on X, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$d_Y(f(p), f(q)) < \varepsilon, \text{ for all } p \text{ and } q \text{ with } d_X(p,q) < \varepsilon.$$

Note

(i) Continuity is defined at a point, whereas uniform continuity is defined on a set.
(ii) For continuity at the point p, the value of δ depends on both ε the point p. For uniform continuity on a set E, the value of δ depends only on ε, does not depend any point in E.
(iii) Every uniformly continuous function on X is continuous on X. But the converse fails to hold.

Example 2.8. *For example, we have shown that $f(x) = 1/x,\ x \in (0,\infty)$, is continuous on $(0,\infty)$. We now show that it is not uniformly continuous on $(0,\infty)$.*

Let $\varepsilon = 1/2$. Since the difference between $1/n$ and $1/(n+1)$ decreases as n increases, for any $\delta > 0$, we can find n large enough so that

$$\left|\frac{1}{n} - \frac{1}{n+1}\right| < \delta,$$

but

$$\left|f\left(\frac{1}{n}\right) - f\left(\frac{1}{n+1}\right)\right| = 1 > \varepsilon.$$

The next theorem shows that continuity and uniform continuity are equivalent on compact sets.

Theorem 2.13. *Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X.*

Proof. Let $\varepsilon > 0$ be given. Given f is continuous at every point p in X. Thus, for each $p \in X$, there exists $\delta_p > 0$ such that

$$d_Y(f(p), f(q)) < \varepsilon/2, \text{ for } q \in X \text{ with } d_X(p,q) < \delta_p. \tag{2.11}$$

To find a common δ, we have to use the compactness of X. For, consider the neighbourhoods

$$G_p = \{q \in X : d_X(p,q) < \frac{\delta_p}{2}\}.$$

Then

$$\{G_p : p \in X\}$$

is an open cover of X, and since X is compact, it has a finite subcover, namely

$$\{G_{p_1}, G_{p_2}, ..., G_{p_n}\},$$

thus,

$$X \subset G_{p_1} \bigcup G_{p_2} \bigcup ... \bigcup G_{p_n}. \tag{2.12}$$

Put

$$\delta = \min\{\frac{\delta_{p_1}}{2}, \frac{\delta_{p_2}}{2}, ..., \frac{\delta_{p_n}}{2}\},$$

then $\delta > 0$.

(Note that the infimum of an infinite set of positive real numbers need not be positive(for example $\{1/n : n = 1, 2, ...\}$), but that of a finite set must be positive).

Let $p,\ q \in X$ such that $d_X(p, q) < \delta$. Then by (2.12), there is an integer $m,\ 1 \leq m \leq n$ such that

$$p \in G_{p_m},$$

so that

$$d_X(p, p_m) < \frac{\delta_{p_m}}{2}$$

and (2.11) implies that

$$d_Y(f(p), f(p_m)) < \varepsilon/2.$$

Now, by the property of metric,

$$d_X(q, p_m) \leq d_X(p, q) + d_X(p, p_m) < \delta + \frac{\delta_{p_m}}{2} \leq \delta_{p_m}.$$

Then by (2.11),

$$d_Y(f(p_m), f(q)) < \varepsilon/2,$$

and hence

$$d_Y(f(p), f(q)) \leq d_Y(f(p), f(p_m)) + d_Y(f(p_m), f(q)) < \varepsilon.$$

This completes the proof. □

We have observed that compactness is essential in the hypothesis of the above theorems. Now we give counter examples on an arbitrary non compact subset of real numbers.

Theorem 2.14. *Let E be a noncompact subset in $\mathbb{R}$. Then*

(a) there exists a continuous function on E which is not bounded.

(b) there exists a continuous and bounded function on E which has no maximum.

(c) if E is bounded also, there exists a continuous function on E which is not uniformly continuous.

The construction is similar to that of the function $1/x$ on $(0, \infty)$, where 0 is a limit point of the domain which is not in the domain.

Proof. We establish the results separately in two cases, when E is bounded and when E is unbounded.

Case 1: E is bounded.

Since E is not compact, by Heine-Borel Theorem, we have E is not closed in this case. Then E has a limit point, say x_0 which is not in E. Then the function

$$f(x) = \frac{1}{x - x_0}, \ x \in E$$

is a real continuous on E but not bounded and not uniformly continuous on E.

For, continuity of f follows from Theorem 2.7 and f is unbounded in neighbourhoods of x_0. We prove f is not uniformly continuous on E. Consider $\varepsilon > 0$. Since x_0 is a limit point of E, for each $\delta > 0$, we can choose a point $x \in E$ such that

$$|x - x_0| < \delta.$$

And since f is unbounded near x_0, we can choose y close enough to x_0, such that

$$|y - x| < \delta, \text{ but } |f(y) - f(x)| > \varepsilon.$$

As it is true for every $\delta > 0$, we have f is not uniformly continuous on E.

For a bounded continuous function having no maximum, consider

$$g(x) = \frac{1}{1+(x-x_0)^2},\ x \in E.$$

It is continuous by Theorem 2.7. It is bounded because, $x_0 \notin E$, so that $1+(x-x_0)^2 > 1$ and

$$g(x) < 1 \text{ for all } x \in E.$$

Also as every neighbourhood of x_0 contains points of E, we have

$$\inf_{x\in E}\ (x-x_0) = 0$$

and hence,

$$\sup_{x\in E}\ g(x) = 1.$$

Thus g has no maximum on E because $g(x) < 1$ for all $x \in E$.

Case 2: E is unbounded.

In this case the function $f(x) = x$ establishes (a). For (b), consider

$$h(x) = \frac{x^2}{1+x^2},\ x \in E.$$

Since $x^2 < 1+x^2$ for all x, we have $h(x) < 1$ for all x, and since E is unbounded, x^2 is unbounded, so that

$$\sup_{x\in E}\ h(x) = \sup_{x\in E}\left(1 - \frac{1}{1+x^2}\right) = 1,$$

which is not achieved at any point in E. □

Note that if E is not bounded, then (c) may not be true. For example, consider the set of integers as E. Then every function defined on E is uniformly continuous on E. See Exercise 14.

2.4 Continuity and Connectedness

Theorem 2.15. *If f is a continuous mapping of a metric space X into a metric space Y and if E is a connected subset of X, then $f(E)$ is connected.*

Proof. Given $f : X \to Y$ is continuous and E is a connected subset of X. If $f(E)$ is not connected, then

$$f(E) = A \bigcup B,$$

where A and B are nonempty separated subsets of Y. Consider the sets

$$G = E \bigcap f^{-1}(A) \text{ and } H = E \bigcap f^{-1}(B).$$

We prove that G and H are separated sets and $E = G \bigcup H$.

Since $f(E) = A \bigcup B$, $f(x) \in A \bigcup B$, for all $x \in E$, so that $E \subset f^{-1}(A) \bigcup f^{-1}(B)$ and hence $E = G \bigcup H$.

Also G and H are nonempty, since A and B are so.

Now, $A \subset \overline{A}$ implies that $f^{-1}(A) \subset f^{-1}(\overline{A})$, so that $G \subset f^{-1}(\overline{A})$.

Since $\overline{A}$ is closed and f is continuous, by Corolary 5, we have $f^{-1}(\overline{A})$ is closed. Hence $\overline{G} \subset f^{-1}(\overline{A})$ and $f(\overline{G}) \subset \overline{A}$.

Now, $\overline{A} \bigcap B$ is empty and $f(H) = B$, we have $f(\overline{G}) \bigcap f(H)$ and hence $\overline{G} \bigcap H$ are empty.

In a similar way we can show that $G \bigcap \overline{H}$ is empty. Thus we have E is the union of two nonempty separated sets, a contradiction. Hence $f(E)$ must be connected.

$\square$

The next theorem says that a continuous real function assumes all intermediate values on an interval. This result is known as **Intermediate Value Theorem**.

Theorem 2.16. *Let f be a continuous real function on $[a, b]$. If $f(a) < f(b)$, and if $f(a) < c < f(b)$, then there exists a point $x \in [a, b]$ such that $f(x) = c$.*

Proof. (**Note that it is related to connectedness in $\mathbb{R}$. That will give you the proof idea.**)

Since $[a, b]$ is connected and f is continuous, we have $f([a, b])$ is connected.

Being a connected subset of $\mathbb{R}$, if $x,\ y \in f([a,b])$, and $z \in \mathbb{R}$ such that $x < z < y$, then using Theorem 1.22, we have $z \in f([a,b])$.

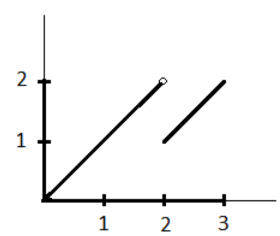
Thus if $f(a) < f(b)$, and if $f(a) < c < f(b)$, then $c \in f([a,b])$, which means $c = f(x)$ for some $x \in [a,b]$.

$\square$

(**Note that a similar result holds if** $f(a) > f(b)$.)

The converse of this is not true in general. For exmple consider the function $f : [0,3] \to [0,2]$ defined by

$$f(x) = \begin{cases} x, & \text{if } \ x \in [0,2) \\ x-1, & \text{if } \ x \in [2,3]. \end{cases}$$



Here we have f attains all values in between 0 and 2, but fails to be continuous at 2.

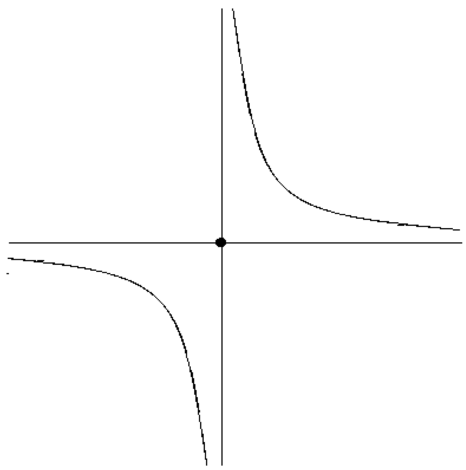
2.5 Discontinuities

If x is a point in the domain of f at which f is not continuous, then f is said to be **discontinuous** at x or f is said to have a discontinuity at x.

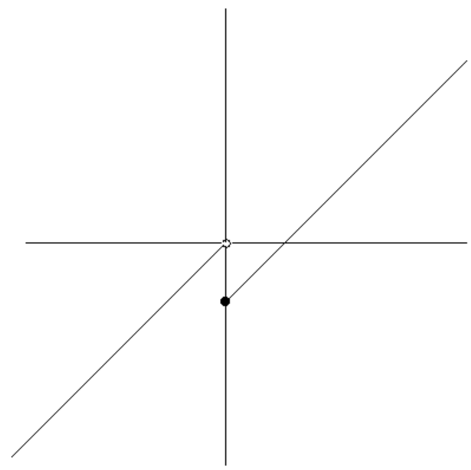
Considering the examples of functions which are not continuous on a domain, we can see that there are two types of discontinuities in general. For example consider the following two functions defined on $\mathbb{R}$:

$$f(x) = \begin{cases} 1/x, & \text{if } \ x \neq 0 \\ 0, & \text{if } \ x = 0. \end{cases} \tag{2.13}$$

$$g(x) = \begin{cases} x, & \text{if } x < 0 \\ x - 1, & \text{if } x \geq 0. \end{cases} \tag{2.14}$$



Graph of f



Graph of g

Both f and g are discontinuous at 0. Considering

$$\lim_{x \to 0} g(x),$$

we see that when we approach 0 through the points on the left of 0, then $g(x)$ approaches 0, while it is -1 when approaching from right. But no such value exist for the limit of $f(x)$ approaching from either side.

Definition 2.8. *Let f be defined on (a, b) and let x be a point such that $a \leq x < b$. The* ***right-hand limit*** *of f at x, denoted by $f(x+)$ is q if $f(t_n) \to q$ as $n \to \infty$ for all sequences $\{t_n\}$ in (x, b) such that $t_n \to x$.*

For a point x such that $a < x \leq b$, the ***left-hand limit*** *of f at x, denoted by $f(x-)$ is q if $f(t_n) \to q$ as $n \to \infty$ for all sequences $\{t_n\}$ in (a, x) such that $t_n \to x$.*

Note that

$$\lim_{t \to x} f(t)$$

exists if and only if

$$f(x+) = f(x-) = \lim_{t \to x} f(t).$$

Definition 2.9. *Let f be defined on (a, b). If f is discontinuous at x, and if $f(x+)$ and $f(x-)$ exist, then f is said to have a* ***discontinuity of the first kind*** *or a* ***simple discontinuity*** *at x. Otherwise the discontinuity is said to be of the* ***second kind****.*

Thus if f has a discontinuity of the first kind at x, then either $f(x+) \neq f(x-)$ or $f(x+) = f(x-) \neq f(x)$.

Example 2.9.

(a) The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

has discontinuity of the second kind at every point, because for every point x any neighbourhood of x contains both rational and irrational points, so that neither $f(x+)$ nor $f(x-)$ exists.

(b) If $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational,} \end{cases}$$

then f is continuous at 0(verify!) and has a discontinuity of the second kind at all other points.

(c) The function $f : (-3, 1) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x + 2, & \text{if } -3 < x < -2 \\ -x - 2, & \text{if } -2 \leq x < 0 \\ x + 2, & \text{if } 0 \leq x < 1 \end{cases}$$

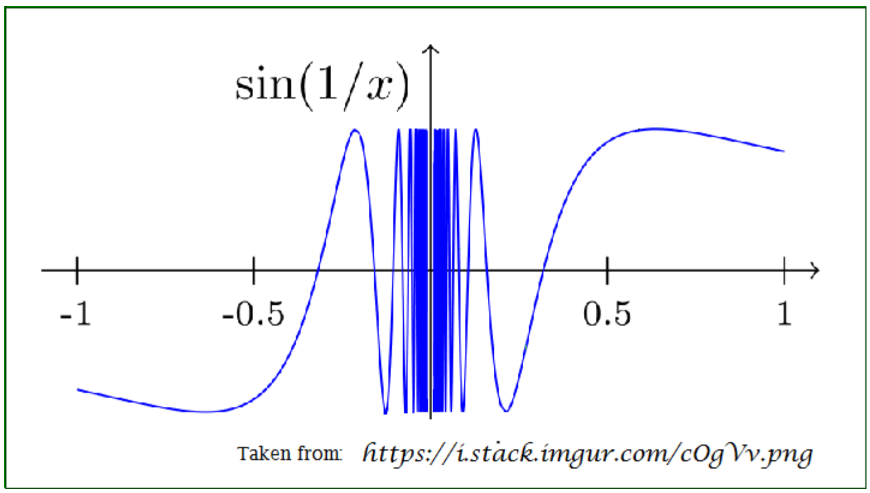
has a simple discontinuity at 0 and is continuous at every other point of $(-3, 1)$.

(d) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} sin\ \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Since $sin\ x = \pm 1$ accordingly for each x of the form $x = \frac{(4n \pm 1)\pi}{2}$, and takes all values from 1 to -1 in between, we have $sin\ \frac{1}{x}$ attains ± 1 at *every point x of the form $\frac{2}{(4n\pm1)\pi}$ and takes all values from 1 to -1 in between.*

Now, the point $\frac{2}{(4n\pm1)\pi}$ tends to 0 as n tends to ∞ and every neighbourhood of 0, however small, contains infinitely many points of the form $\frac{2}{(4n\pm1)\pi}$. So that as x approaches 0, the value of sin $\frac{1}{x}$ oscillates infinitely many times between 1 and -1, and hence neither $f(0+)$ nor $f(0-)$ exists.



Thus f has a discontinuity of the second kind at 0 and is continuous elsewhere, by Example 2.4.

2.6 Monotonic Functions

Monotonic functions are those which never decrease or never increase. We show that for such functions both the left-hand and right-hand limits exist at each point of the domain. In other words if a monotonic function is discontinuous at a point, then the discontinuity must be of the first kind.

Definition 2.10. *Let f be a real function defined on (a, b). Then f is said to be **monotonically increasing** on (a, b) if $a < x < y < b$ implies $f(x) \leq f(y)$. And, f is **monotonically decreasing** on (a, b) if $a < x < y < b$ implies $f(x) \geq f(y)$. By a **monotonic function** we mean a function which is either increasing or decreasing.*

On (a, b), where $a,\ b \in \mathbb{R}$ with $a < b$, the identity function $f(x) = x$, the exponential function $f(x) = exp(x)$ (also denoted by e^x) and the greatest integer function $f(x) = [x]$ are some examples of monotonically increasing functions.

The functions $f(x) = 1 - x,\ x \in \mathbb{R}$; $f(x) = 1/x,\ x \in (0, \infty)$; and $f(x) = |x|,\ x \in (-\infty, 0)$ are examples for monotonically decreasing functions.

Any constant function $f(x) = c$, where c is a fixed real number, is both increasing and decreasing on any segment. The function $f(x) = sin\ x$ is decreasing on $(\frac{\pi}{2}, \frac{3\pi}{2})$ and inceasing on $(\frac{-\pi}{2}, \frac{\pi}{2})$.

Theorem 2.17. *Let f be monotonically increasing on (a, b). Then $f(x+)$ and $f(x-)$ exist at every point x of (a, b). In fact,*

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t).$$

Moreover, if $a < x < y < b$, then

$$f(x+) \leq f(y-).$$

Proof. It is enough to prove that

$$\sup_{a<t<x} f(t)$$

exists and is equal to $f(x-)$, and

$$\inf_{x<t<b} f(t)$$

exists and is equal to $f(x+)$.

Since f is monotonically increasing, the set $\{f(t) : a < t < x\}$, is bounded above with an upper bound $f(x)$. If

$$A = \sup_{a<t<x} f(t),$$

then we have $A \leq f(x)$.

We have to show that $A = f(x-)$.

Since supremum of a set is a limit point of the set, we have,

$$for\ any\ \varepsilon > 0,\ there\ exists\ f(t),\ t \in (a, x)\ such\ that A - \varepsilon < f(t) \leq A. \tag{2.15}$$

Put $t = x - \delta$, (2.15) implies that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $a < x - \delta < x$ and $A - \varepsilon < f(x - \delta) \leq A$.

Then since f is monotonic, we have

$$f(x - \delta) \leq f(t) \leq A, \text{ for } x - \delta < t < x.$$

Thus for $x - \delta < t < x$, we have $A - \varepsilon < f(t) \leq A$, which implies that $f(t)$ approaches A as t approaches x from the left. That is

$$f(x-) = A.$$

A similar proof works for

$$f(x) \leq f(x+) = \inf_{x<t<b} f(t).$$

Now, if $a < x < y < b$, then

$$f(x) \leq f(x+) = \inf_{x<t<b} f(t) = \inf_{x<t<y} f(t),$$

$$f(y-) = \sup_{a<t<y} f(t) = \sup_{x<t<y} f(t)$$

and

$$\inf_{x<t<y} f(t) \leq \sup_{x<t<y} f(t).$$

Thus $f(x+) \leq f(y-)$. □

State and prove the analogous results for monotonically decreasing functions!

Corollary It follows from Theorem 2.17 that a monotonic function does not have a discontinuity of the second kind.

These results lead to an interesting fact that a monotonic function can have at the most a countable number of points of discontinuity.

Theorem 2.18. *Let f be monotonic on (a, b). Then the set of points in (a, b) at which f is discontinuous is at most countable.*

Proof. We associate a rational number to every point in (a, b) at which f is discontinuous. Let f be discontinuous at x. Since $f(x+)$ and $f(x-)$ exist; either

$$f(x-) \neq f(x+)$$

or

$$f(x-) = f(x+) \neq f(x).$$

Since f is monotonic, $f(x-) \leq f(x) \leq f(x+)$, so that the case $f(x-) = f(x+) \neq f(x)$ can not happen. Therefore $f(x-) \neq f(x+)$ and hence

$$f(x-) < f(x+).$$

Then corresponding to every point of discontinuity x of f we can choose a rational number $r(x)$ such that

$$f(x-) < r(x) < f(x+).$$

Now if $x_1 < x_2$, then by Theorem 2.17 we have $f(x_1+) \leq f(x_2-)$, and hence

$$r(x_1) < f(x_1+) \leq f(x_2-) < r(x_2),$$

which means that for $x_1 \neq x_2$, we have $r(x_1) \neq r(x_2)$.

Thus there exists a one to one correspondence between the set of points x in (a, b) at which f is discontinuous and a subset of rational numbers $\{r(x)\}$, which is either finite or countable. Hence the proof.

□

Note that even though a monotonic function has an at most countable number of points of discontinuity, they need not be isolated. In other words, given any countable subset E of (a, b), which may be dense in (a, b), we can construct a monotonic function f on (a, b) which is discontinuous at every point of E, and continuous at all other points of (a, b). Such a function can be constructed as follows.

Construction: Given E is countable. So that we can arrange the points of E in a sequence, say

$$E = \{x_1,\ x_2,\ x_3,\ \ldots\}.$$

Also choose a sequence of positive numbers $\{c_n\}$ such that $\sum c_n$ converges. Then define

$$f(x) = \sum_{x_n < x} c_n, \quad a < x < b,$$

with the convention that the sum is defined to be zero if there are no points x_n smaller than x. Then

(a) f is monotonically increasing on (a, b),

(b) at every point x_n of E,

$$f(x_n+) - f(x_n-) = c_n,$$

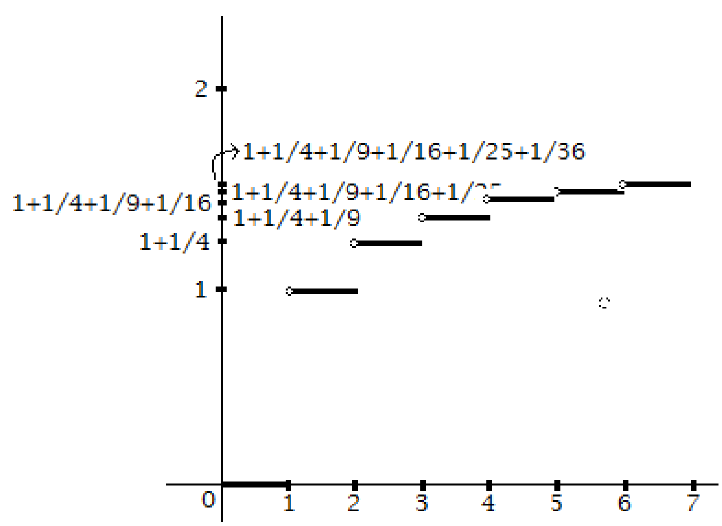
so that f is discontinuous at every point of E,

(c) f is continuous at every other point of (a, b).

Here we have $f(x-) = f(x)$ at all points of (a, b).

For a precise visible example, consider $(a, b) = (0, 10)$ and $E = \{1,\ 2,\ \ldots, 7\}$. Note that here we are not considering a dense subset E, it is only a finite set. Our aim is just to convince the reader, what is the definition of the function f.

Choose the sequence $\{\frac{1}{n^2}\}$ for the sequence $\{c_n\}$ of positive numbers because $\sum \frac{1}{n^2}$ converges. The graph of f is depicted in the following figure.



Construct a monotonically decreasing function satisfying (b) and (c) above!

2.7 Infinite Limits and Limits at Infinity

We have seen that the limit of a function need not always be finite as in the case of $f(x) = 1/x$ as x approaches 0. Also, in many occasions we need to consider the limit of the function at infinity. We extend the definition of limit of functions to the extended real number system, by reformulating it in terms of neighbourhoods.

Definition 2.11. *The* **extended real number system** *consists of the real numbers and two more points $+\infty$ and $-\infty$, with the usual order in $\mathbb{R}$ and*

$$-\infty < x < +\infty$$

for all $x \in \mathbb{R}$.

Thus in the extended real number system, $+\infty$ is an upper bound for any subset and every nonempty subset has a least upper bound. Also if E is a nonempty subset of $\mathbb{R}$ which is not bounded above, then $\sup E = +\infty$ in the extended real number system.

And, $-\infty$ is a lower bound for any subset and every nonempty subset has a greatest lower bound. Also if E is a nonempty subset of $\mathbb{R}$ which is not bounded below, then $\inf E = -\infty$ in the extended real number system.

The extended real number system does not form a field, but we can make the following conventions.

(a) If $x \in \mathbb{R}$, then

$$x + \infty = \infty, \quad x - \infty = -\infty \quad \text{and} \quad \frac{x}{+\infty} = \frac{x}{-\infty} = 0.$$

(b) If $x > 0$, then the products

$$x.(+\infty) = +\infty \quad \text{and} \quad x.(-\infty) = -\infty.$$

(c) If $x < 0$, then the products

$$x.(+\infty) = -\infty \quad \text{and} \quad x.(-\infty) = +\infty.$$

Note that a neighbourhood of a real number x is a segment $(x-\delta,\ x+\delta)$ for some $\delta > 0$.

Definition 2.12. *For any real number* c*, the set* $(c, +\infty) = \{x \in \mathbb{R} : x > c\}$ *is called a neighborhood of* $+\infty$ *and the set* $(-\infty, c) = \{x \in \mathbb{R} : x < c\}$ *is a neighborhood of* $-\infty$*.*

Definition 2.13. *Let* f *be a real function defined on a subset* E *of real numbers. For* A *and* x *in the extended real number system, we say that*

$$f(t) \to A \text{ as } t \to x,$$

if for every neighborhood U *of* A *there is a neighborhood* V *of* x *such that* $V \bigcap E$ *is not empty, and such that* $f(t) \in U$ *for all* $t \in V \bigcap E$*,* $t \neq x$*.*

Notice that when A and x are real numbers, Definition 2.13 coincides with Definition 2.1. The results of Theorem 2.2 in this context can be stated as follows.

Theorem 2.19. *Let* f *and* g *be defined on* E*. Suppose*

$$f(t) \to A \text{ and } g(t) \to B \text{ as } t \to x.$$

Then

(a) $f(t) \to A'$ *implies* $A' = A$

(b) $(f+g)(t) \to A + B$,

(c) $(fg)(t) \to AB$,

(d) $(f/g)(t) \to A/B$

provided the right members of (b),(c) and (d) are defined.

Note that $\infty - \infty,\ 0.\infty,\ \infty/\infty,\ A/0$ are not defined.

Proof. The proof is the same. □

Exercise

1. Prove that constant functions are continuous.

2. Prove that the function $f(x) = sin\ x$ is continuous on $\mathbb{R}$.

3. Suppose f is a real continuous function on a metric space X. If $Z(f) = \{p \in X : f(p) = 0\}$, then prove that $Z(f)$ is closed subset of X.

4. If f is a continuous mapping of a metric space X into a metric space Y, prove that
$$f(\overline{E}) \subset \overline{f(E)}$$
for every subset E of X. Also give an example showing $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.

5. Suppose f is a real function defined on $\mathbb{R}$ such that
$$\lim_{h \to 0}[f(x+h) - f(x-h)] = 0$$
for every $x \in \mathbb{R}$. Does this imply that f is continuous?

6. Let f be a continuous mapping of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$.

7. Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. If $f(p) = g(p)$ for all $p \in E$, prove that $f(x) = g(x)$ for all $x \in X$. (In other words a continuous mapping is determined by its values on a dense subset of its domain.)

8. (i) Let E be a closed subset of $\mathbb{R}$ and let f be a real continuous function defined on E. Prove that there exists a continuous real function g on $\mathbb{R}$ such that $g(x) = f(x)$ for all $x \in E$. (Such a function g is called a continuous extension of f from E to $\mathbb{R}$.)

   (ii) Show that (i) fails to be true if E is not a closed subset.

   (Hint (i): See Exercise 15(iii) in Chapter 1. The complement of E is open. Use straight lines on each of the segments of E^c to define g.)

9. Prove that if f is a real uniformly continuous function defined on a bounded subset E of $\mathbb{R}$, then f is bounded on E.
   Verify whether the result is true or not if we omit the conditions boundedness of E or the uniform continuity of f.

10. If f is a uniformly continuous mapping of a metric space X into a metric space Y, prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for any Cauchy sequence $\{x_n\}$ in X.

11. Let $I = [0, 1]$ be the closed unit interval. Prove that if f is a continuous mapping of I into I, then $f(x) = x$ for some $x \in I$. (In other words every continuous function from I into I has a fixed point.)

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a mapping such that $f(V)$ is open for every open set V. (such mappings are called open mappings.) Prove that if f is continuous, then f is monotonic.

13. For a real number x let $[x]$ denote the greatest integer less than or equal to x, and let $(x) = x - [x]$. Discuss the continuities and discontinuities of $f(x) = [x]$ and $g(x) = (x)$.

14. Prove that every function defined on the set $\mathbb{Z}$ of integers(in usual metric) is uniformly continuous.

15. Discuss the continuity of the function

$$f(x) = \begin{cases} x\ sin\frac{1}{x}, & \text{if } \ x \neq 0 \\ 0, & \text{if } \ x = 0. \end{cases}$$

Chapter 3

Differentiation

In this chapter (except in the last section) we consider real valued functions defined on intervals or segments. And the last section just introduces differentiation of vector valued functions defined on intervals or segments. The differentiation of functions defined on $\mathbb{R}^k$ will be discussed in the second semester course Real Analysis-II.

3.1 The Derivative of a Real Function

Definition 3.1. *Let f be a real valued function defined on $[a,b]$. For any $x \in [a,b]$ define*

$$\phi(t) = \frac{f(t) - f(x)}{t - x}, \qquad a < t < b, \; t \neq x.$$

If

$$f'(x) = \lim_{t \to x} \phi(t)$$

exists, then we say that f is **differentiable at** *x.*

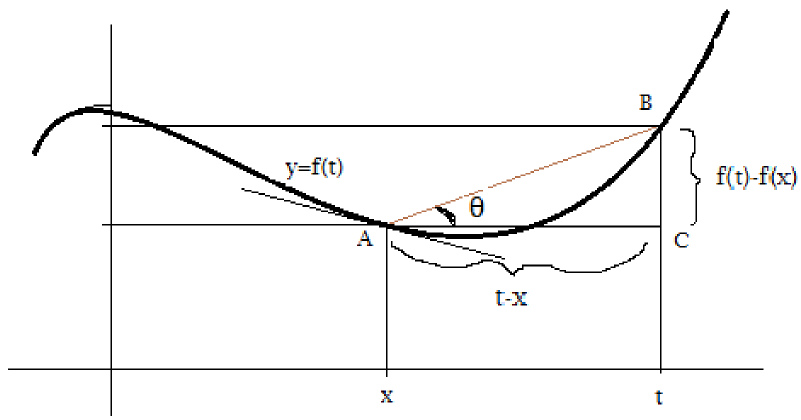
If $f'(x)$ exist for every point x in a subset E of $[a,b]$, then, f is said to be **differentiable on** *E.*

Hence we can associate with f a function f' whose domain is the set of points x at which the above limit exists. The function f' is called the **derivative** *of f.*

Considering the following figure we can see that

$$\frac{f(t) - f(x)}{t - x} = tan\ \theta \ ;$$

and that as t approaches x the line AB turn out to be the tangent to the curve $y = f(t)$ at $(x, f(x))$. This remark is made just to show that the derivative is related to the tangents at the concerned points.



Theorem 3.1. *Let f be defined on $[a, b]$. If f is differentiable at $x \in [a, b]$, then f is continuous at x.*

Proof. In order to prove f is continuous at x, in view of Theorem 2.4, it is enough if we prove

$$\lim_{t \to x} f(t) = f(x).$$

Now,

$$f(t) - f(x) = \frac{f(t) - f(x)}{t - x}(t - x).$$

Given that

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

exists. So that we have

$$\lim_{t \to x}(f(t) - f(x)) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} . \lim_{t \to x}(t - x).$$

That is

$$\lim_{t \to x} f(t) - f(x) = f'(x).0 = 0;$$

and hence

$$\lim_{t \to x} f(t) = f(x),$$

which proves the result. $\square$

The converse of this theorem is not true. In fact, there are functions which are continuous on the whole real line, but nowhere differentiable, we will construct such a function in the last chapter of this Course.

Example 3.1. *As a counter example to the converse of Theorem 3.1, consider the function*

$$f(t) = |t|, \quad t \in \mathbb{R}.$$

We have

$$|t| = \begin{cases} t, & \text{if } t > 0 \\ -t, & \text{if } t < 0. \end{cases}$$

So that, for $x \in \mathbb{R}$,

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \begin{cases} \frac{(t-x)}{t-x} = 1, & \text{if } x > 0 \\ \frac{-(t-x)}{t-x} = -1, & \text{if } x < 0. \end{cases}$$

and at $x = 0$,

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \begin{cases} \frac{(t)}{t} = 1, & \text{if } t > 0 \\ \frac{-(t)}{t} = -1, & \text{if } t < 0. \end{cases}$$

Since any neighbourhood of 0 contain both positive and negative numbers, we conclude that

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

does not exist at $x = 0$.

Hence,

$$f'(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \\ \text{does not exist}, & \text{if } x = 0. \end{cases}$$

Note *that the tangent to the graph of f at the point $(x, f(x))$ is the line $y = -x$, if $x < 0$ and the line $y = x$, if $x > 0$. At the point $x = 0$, the tangent jump from one to the other, when we move from one side to the other side, and at such points the derivative fails to exist.*

At this point, I hope the reader can construct functions continuous on $\mathbb{R}$ but fails to be differentiable at a given finite or countable number of points(Exercise 1).

Theorem 3.2. *Suppose f and g are defined on $[a, b]$ and differentiable at a point $x \in [a, b]$. Then the functions $f + g$, fg and f/g(provided $g(x) \neq 0$) are differentiable at x and*

(a) $(f + g)'(x) = f'(x) + g'(x)$;

(b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ *and*

(c) $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x)-g'(x)f(x)}{g^2(x)}$

Proof. The result (a) follows directly from Theorem 2.4 and Theorem 2.2 (Verify!).

Let $h = fg$. Then for, any $t \in [a, b]$,

$$\begin{aligned}\frac{h(t) - h(x)}{t - x} &= \frac{f(t)g(t) - f(x)g(x)}{t - x}\\ &= \frac{f(t)[g(t) - g(x)] - g(x)[f(x) - f(t)]}{t - x}\\ &= \frac{f(t)[g(t) - g(x)]}{t - x} - \frac{g(x)[f(x) - f(t)]}{t - x}\end{aligned}$$

So that

$$\lim_{t\to x} \frac{h(t) - h(x)}{t - x} = \lim_{t\to x} \frac{f(t)[g(t) - g(x)]}{t - x} + \lim_{t\to x} \frac{g(x)[f(t) - f(x)]}{t - x},$$

which exists since f and g are differentiable at x, and the right hand side is equal to

$$f(x)g'(x) + g(x)f'(x).$$

Thus we proved (b).

Now, if $k = f/g$, then for, any $t \in [a, b]$,

$$\begin{aligned}\frac{k(t) - k(x)}{t - x} &= \frac{[f(t)/g(t)] - [f(x)/g(x)]}{t - x}\\ &= \frac{f(t)g(x) - f(x)g(t)}{g(t)g(x)(t - x)}\\ &= \frac{g(x)[f(t) - f(x)] - f(x)[g(t) - g(x)]}{g(t)g(x)(t - x)}.\end{aligned}$$

So that

$$\lim_{t\to x}\frac{k(t)-k(x)}{t-x}=\lim_{t\to x}\frac{1}{g(t)g(x)}\left(\lim_{t\to x}\frac{g(x)[f(t)-f(x)]}{t-x}-\lim_{t\to x}\frac{f(x)[g(t)-g(x)]}{t-x}\right),$$

which exists and we have

$$(\frac{f}{g})'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}.$$

□

Example 3.2.

1. *If f is the constant function $f(t)=c,\ t\in[a,b]$, where c is any fixed real number, $f(t)-f(x)=c-c=0$, for all $t,\ x\in[a,b]$. Hence $f'(x)=0$ at every point x.*

2. *The identity function $f(t)=t$ has the derivative $f'(x)=1$ at every point x. Verify!*

3. *The derivative of $f(x)=x^2$ can be obtained as $2x$ using Theorem 3.2(b) and 2 above. (Verify!)*

   Repeated application of Theorem 3.2(b) then shows that $f(x)=x^n$ is differentiable, and the derivative is nx^{n-1}, for any integer n, with the restriction $x\neq 0$ if $n<0$.

4. *The examples in 1 to 3 above and Theorem 3.2 then shows that every polynomial is differentiable at every point of the domain and every rational function is differentiable at those points where its denominator is non zero.*

Now we consider the derivative of composition of functions. The following theorem is known as the **Chain rule** for diifferentiation.

Theorem 3.3. *Suppose f is continuous on $[a,b]$, $f'(x)$ exists for some point $x\in[a,b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If $h=f\circ g$ that is*

$$h(t)=f(g(t)),\quad t\in[a,b],$$

then h is differentiable at x and

$$h'(x)=g'(f(x))f'(x).$$

Proof. Given

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

exists. So that we can write

$$\frac{f(t) - f(x)}{t - x} = f'(x) + u(t), \text{ where } u(t) \to 0, \text{ as } t \to x \quad (t \in [a, b]). \tag{3.1}$$

Similarly, since

$$g'(f(x)) = \lim_{t' \to f(x)} \frac{g(t') - g(f(x))}{t' - f(x)}$$

exists, we have

$$\frac{g(t') - g(f(x))}{t' - f(x)} = g'(f(x)) + v(t'),$$

that is

$$g(t') - g(f(x)) = [t' - f(x)][g'(f(x)) + v(t')], \text{ where } v(t') \to 0, \text{ as } t' \to f(x) \ (t' \in I). \tag{3.2}$$

Now,

$$\begin{aligned} \frac{h(t) - h(x)}{t - x} &= \frac{g(f(t)) - g(f(x))}{t - x} \\ &= \frac{[f(t) - f(x)][g'(f(x)) + v(f(t))]}{t - x}, \quad \text{by (3.1)} \\ &= [f'(x) + u(t)][g'(f(x)) + v(f(t))], \quad \text{by (3.2)} \end{aligned}$$

where $u(t) \to 0$, as $t \to x$ and $v(f(t)) \to 0$, as $f(t) \to f(x)$. Since f is continuous on $[a, b]$, we have $f(t) \to f(x)$, as $t \to x$. Hence by using (3.1) and (3.2) we have,

$$h'(x) = \lim_{t \to x} \frac{h(t) - h(x)}{t - x} = [f'(x)][g'(f(x))].$$

□

Example 3.3.

1. *The function sin x is differentiable on* $\mathbb{R}$.

For,

$$\begin{aligned}\lim_{t\to x}\frac{sin\ t - sin\ x}{t-x} &= \lim_{t\to x}\frac{cos\ \frac{t+x}{2} sin\ \frac{t-x}{2}}{t-x}\\ &= \lim_{t\to x} cos\ \frac{t+x}{2}. \lim_{t\to x}\left(\frac{sin\ \frac{t-x}{2}}{\frac{t-x}{2}}\right)\\ &= cos\ x. \lim_{\frac{t-x}{2}\to 0}\left(\frac{sin\ \frac{t-x}{2}}{\frac{t-x}{2}}\right)\\ &= cos\ x.\end{aligned}$$

Hence by using the chain rule and the example 3.2(3), we have the function

$$f(x) = \begin{cases} sin\ \frac{1}{x}, & if\ \ x \neq 0\\ 0, & if\ \ x = 0.\end{cases}$$

is differentiable at every $x \neq 0$ and

$$f'(x) = \left(cos\ \frac{1}{x}\right)\left(\frac{-1}{x^2}\right) = \frac{-cos\ (1/x)}{x^2}.$$

But at $x = 0$, we have seen that f is not continuous and hence not differentiable(Theorem 3.1).

2. *Consider*

$$f(x) = \begin{cases} x\ sin\ \frac{1}{x}, & if\ \ x \neq 0\\ 0, & if\ \ x = 0.\end{cases}$$

It is continuous on $\mathbb{R}$ (Exercise 15, Chapter 2). It follows from Theorem 3.2 and (1) above that f is differentiable at any $x \neq 0$, with derivative

$$f'(x) = sin\ \frac{1}{x} - \frac{cos\ (1/x)}{x}.$$

Now, at $x = 0$, we have for $t \neq 0$,

$$\lim_{t\to 0}\frac{f(t)-f(0)}{t} = \lim_{t\to 0}\frac{t\ sin\ \frac{1}{t} - 0}{t} = \lim_{t\to 0} sin\ \frac{1}{t},$$

which does not exist and f is not differentiable at 0.

3. *If*

$$f(x) = \begin{cases} x^2\ sin\ \frac{1}{x}, & if\ \ x \neq 0\\ 0, & if\ \ x = 0,\end{cases}$$

then f is differentiable on $\mathbb{R}$.

For, at $x \neq 0$, as above we have

$$f'(x) = 2x \ sin \ \frac{1}{x} - cos \ \frac{1}{x};$$

and at $x = 0$ we have

$$\begin{aligned} f'(x) &= \lim_{t \to 0} \frac{f(t) - f(0)}{t} \\ &= \lim_{t \to 0} \frac{t^2 \ sin \ \frac{1}{t} - 0}{t} \\ &= \lim_{t \to 0} t \ sin \ \frac{1}{t} \\ &= 0, \qquad since \ |sin \ \frac{1}{t}| \leq 1. \end{aligned}$$

Thus

$$f'(x) = \begin{cases} 2x \ sin \ \frac{1}{x} - cos \ \frac{1}{x}, & if \ \ x \neq 0 \\ 0, & if \ \ x = 0. \end{cases}$$

3.2 Mean Value Theorems

Definition 3.2. *Let f be a real function defined on a metric space X. Then f is said to have a **local maximum** at a point $p \in X$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \delta$.*

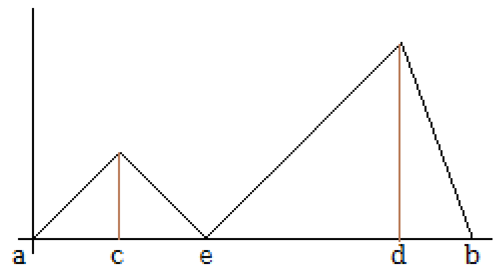
That is if there is a neighbourhood of p in which f attains its maximum at p.

*Let f be a real function defined on a metric space X. Then f is said to have a **local minimum** at a point $p \in X$ if there exists $\delta > 0$ such that $f(p) \leq f(q)$ for all $q \in X$ with $d(p, q) < \delta$.*

That is if there is a neighbourhood of p in which f attains its minimum at p.

For example, $f(x) = sin \ x, \ x \in \mathbb{R}$ has its local maximum at all points of the form $2n\pi + \frac{\pi}{2}, \ n \in \mathbb{Z}$ and local minimum at all points of the form $2n\pi - \frac{\pi}{2}, \ n \in \mathbb{Z}$.

The function whose graph is shown in the following figure has local maximum at two points $x = c$ and $x = d$; and local minimum at three points $x = a,\ x = b$ and $x = e$.



Theorem 3.4. *Let f be defined on $[a,b]$. If f has a local maximum(or local mimnimum) at a point $x \in [a,b]$, and if $f'(x)$ exists, then $f'(x) = 0$.*

Proof. Given f has a local maximum at x. Then by Definition 3.2, there exists a $\delta > 0$ such that

$$f(t) \leq f(x) \text{ for all } t \in X \text{ with } d(x,t) = |x-t| < \delta.$$

Now

$$|x-t| < \delta \text{ means } t \in (x-\delta, x+\delta) \subset [a,b].$$

Then, for $t \neq x$, we have either $x - \delta < t < x$, in which case

$$\frac{f(t)-f(x)}{t-x} \geq 0;$$

or $x < t < x + \delta$, in which case

$$\frac{f(t)-f(x)}{t-x} \leq 0.$$

Thus $f'(x)$ exists if and only if

$$\lim_{t \to x} \frac{f(t)-f(x)}{t-x} = 0.$$

□

Theorem 3.5. ***(The mean value theorem)*** *If f is a real continuous functions on $[a,b]$ which is differentiable in (a,b), then there is a point $x \in (a,b)$ at which*

$$f(b) - f(a) = (b-a)f'(x).$$

It can be regarded as a particular case of the following generalized mean value theorem, with $g(x) = x$. So we are not giving a separate proof for it.

Theorem 3.6. ***(Generalized mean value theorem)*** *If f and g are continuous real functions on $[a, b]$ which are differentiable in (a, b), then there is a point $x \in (a, b)$ at which*

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x).$$

Proof. Define

$$h(t) = [f(b) - f(a)]g(t) - [g(b) - g(a)]f(t), \qquad a \le t \le b.$$

Note that $f(b) - f(a)$ and $g(b) - g(a)$ are constants, hence h is continuous on $[a, b]$ and differentiable on (a, b);

$$h(a) = f(b)g(a) - g(b)f(a) = h(b)$$

and

$$h'(t) = [f(b) - f(a)]g'(t) - [g(b) - g(a)]f'(t), \qquad a < t < b.$$

Then to prove the theorem, it is enough to prove that $h'(x) = 0$ for some $x \in (a, b)$. If h is a constant on $[a, b]$, that is if $h(t) = h(a)$ for all $t \in [a, b]$, then we have the result, that is $h'(x) = 0$ for all $x \in (a, b)$.

So let $h(t) > h(a)$ for some $t \in (a, b)$. Then, since $h(a) = h(b)$, and h is continuous on $[a, b]$, there is a point $x \in (a, b)$ at which h attains its maximum (by using Theorem 2.11).

Then Theorem 3.4 shows that $f'(x) = 0$.

If $h(t) < h(a)$ for some $t \in (a, b)$, then using the same arguments, we get a point $x \in (a, b)$ at which h attains its minimum and $f'(x) = 0$. □

Theorem 3.7. *Suppose f is differentiable in (a, b) Then*

(a) if $f'(x) \ge 0$, for all $x \in (a, b)$, then f is monotonically increasing.

(b) if $f'(x) = 0$, for all $x \in (a, b)$, then f is a constant.

(c) if $f'(x) \le 0$, for all $x \in (a, b)$, then f is monotonically decreasing.

Proof. For any pair of numbers x_1 and x_2 in (a, b), applying mean value theorem we have there is a point x between x_1 and x_2 at which

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(x). \tag{3.3}$$

Then if $f'(x) \geq 0$, for all $x \in (a, b)$, then we have $f(x_2) - f(x_1)$ and $(x_2 - x_1)$ have same sign that is either both are non negative or both are non positive for all $x_1,\ x_2$. Which shows f is monotonically increasing.

The proof of (b) and (c) also follows from the equation (3.3). $\square$

3.3 The Continuity of Derivatives

Note that the derivative f' of a function f need not be continuous, as in Example 3.3(3). Even though it assumes all intermediate values, which is shown in the following theorem.

Theorem 3.8. *If f is differentiable in $[a, b]$ such that $f'(a) < \lambda < f'(b)$, for some real λ, then there exists $x \in (a, b)$ such that $f'(x) = \lambda$.*

Proof. Given $f'(a) < \lambda < f'(b)$, that is

$$f'(a) - \lambda < 0 \text{ and } f'(b) - \lambda > 0.$$

So that if

$$g(t) = f(t) - \lambda t, \quad t \in [a, b]$$

then g is differentiable and

$$g'(t) = f'(t) - \lambda.$$

Hence

$$g'(a) < 0 \text{ and } g'(b) > 0.$$

So that there exist $t_1 < t_2$ in (a, b) such that

$$g(t_1) < g(a) \text{ and } g(t_2) < g(b).$$

In other words g decreases in the beginning of (a, b) and increses at the end. Since g is continuous and $[a, b]$ is compact, the above arguments show that g attains its minimum in $[a, b]$ at some point $x \in (a, b)$. Hence $g'(x) = 0$ that is $f'(x) = \lambda$. $\square$

A similar result holds if $f'(a) > f'(b)$.

Corollary 3.1. *If f is differentiable on $[a, b]$ then f' cannot have any simple discontinuity on $[a, b]$.*

3.3.1 L'Hospital's Rule

L'Hospital's Rule is the following theorem which is useful in the evaluation of limits.

Theorem 3.9. *Suppose f and g are real and differentiable in (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$, where $-\infty \leq a < b \leq +\infty$. Suppose*

$$\frac{f'(x)}{g'(x)} \to A \text{ as } x \to a. \tag{3.4}$$

If

$$f(x) \to 0 \text{ and } g(x) \to 0 \text{ as } x \to a, \tag{3.5}$$

or if

$$g(x) \to +\infty \text{ as } x \to a, \tag{3.6}$$

then

$$\frac{f(x)}{g(x)} \to A \text{ as } x \to a. \tag{3.7}$$

Proof. □

3.3.2 Derivatives of Higher Order

If f is differentiable on an interval and the derivative f' is itself differentiable, then the derivative of f' is called the second derivative of f and it is denoted by f''. If f'' is also differentiable, then the derivative is the third derivative of f and is denoted by $f^{(3)}$, and so on.

The n^{th} derivative of f, $f^{(n)}$ (also called the derivative of order n) is thus the derivative of $f^{(n-1)}$.

Theorem 3.10. (Taylor's Theorem) *Suppose f is a real function on $[a, b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a, b]$, and $f^n(t)$ exists for every $t \in (a, b)$. Let α, β be distinct points of $[a, b]$ and define*

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k.$$

Then there exists a point x between α and β such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n.$$

Proof. □

3.3.3 Differentiation of Vector-valued Functions

If f is a complex-valued function defined on $[a, b]$, then the differentiability of f can be defined as in the Definition 3.1 without any change. Also the theorems Theorem 3.1 and Theorem 3.2, and their proofs are valid in this case(verify).

Moreover, if $f(t) = a + ib,\ t \in [a, b]$, then we can define two real valued functions f_1 and f_2 as $f_1(t) = a$ and $f_2(t) = b$. These functions f_1 and f_2 are called respectively the **real** and **imaginary parts** of f.

Then we have

$$f(t) = f_1(t) + if_2(t), \quad t \in [a, b]$$

and f is differentiable at x if and only if both f_1 and f_2 are differentiable at x and

$$f'(x) = f_1'(x) + if_2'(x).$$

Now, for vector-valued functions $\mathbf{f} : [a, b] \to \mathbb{R}^k$, again we may apply Definition 3.1. In this case

$$\phi(t) = \frac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x}, \qquad a < t < b,\ t \neq x$$

is a point in $\mathbb{R}^k$, and the limit

$$\lim_{t \to x} \phi(t)$$

is taken with respect to the norm in $\mathbb{R}^k$.

In other words, $\mathbf{f}\,'(x)$ is the point $\mathbf{a} \in \mathbb{R}^k$ such that

$$\lim_{t \to x} \left| \frac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x} - \mathbf{a} \right| = 0,$$

and $\mathbf{f}\,'$ is also a vector valued function defined on the set of all points in $[a, b]$ at which $\mathbf{f}$ is differentiable.

If $f_1,\ f_2, ..., f_k$ are the components of $\mathbf{f}$, then $\mathbf{f}$ is differentiable at a point x if and only if each of the functions $f_1,\ f_2, ..., f_k$ is differentiable at x and

$$\mathbf{f}\,' = (f_1',\ f_2', ..., f_k').$$

Theorem 3.1 and Theorem 3.2(a) are valid for vector-valued functions also, and Theorem 3.2(b) will be true if the product fg is replaced by the inner product $\mathbf{f.g}$.

But the mean value theorem and L'Hospital's rule fail to be true for complex valued functions itself, as in the following examples.

Example 3.4. *1. Define*

$$f(x) = e^{ix} = cos\ x + i\ sin\ x, \quad x \in [0, 2\pi].$$

Then

$$f'(x) = ie^{ix} = cos\ x + i\ sin\ x, \quad x \in [0, 2\pi],$$

so that

$$|f'(x)| = 1, \quad \textit{for all } x.$$

Now,

$$f(2\pi) - f(0) = 1 - 1 = 0$$

and hence

$$f(2\pi) - f(0) \neq f'(x)(2\pi - 0) \quad \textit{for any } x \in [0, 2\pi].$$

2. Define

$$f(x) = x \quad \textit{and} \quad g(x) = x + x^2 e^{i/x^2}. \quad x \in (0, 1).$$

Then

$$f(x) \to 0 \quad \textit{and} \quad g(x) \to 0 \quad \textit{as} \quad x \to 0$$

and since $|e^{i/x^2}| = 1$*, we have*

$$\frac{f(x)}{g(x)} = \frac{1}{1 + xe^{i/x^2}} \to 1 \quad \textit{as} \quad x \to 0.$$

Now,

$$f'(x) = 1$$

and

$$g'(x) = 1 + 2x\ e^{i/x^2} + x^2 (\frac{-2i}{x^3}) e^{i/x^2},$$

so that

$$|g'(x)| \geq \left| 2x - \frac{2i}{x} \right| - 1 \geq \frac{2}{x} - 1.$$

Hence

$$\left| \frac{f'(x)}{g'(x)} \right| = \frac{1}{|g'(x)|} \leq \frac{x}{2 - x}$$

and so

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = 0.$$

That is L'Hospital's rule fails in this case.

Even though the mean value theorem does not hold as it is for vector valued functions, we have the following consequence of it, which, for applications, is also as useful as mean value theorem.

Theorem 3.11. *Suppose* $\boldsymbol{f}$ *is a continuous mapping of* $[a,b]$ *into* $\mathbb{R}^k$ *and* $\boldsymbol{f}$ *is differentiable in* (a,b)*. Then there exists* $x \in (a,b)$ *such that*

$$|\boldsymbol{f}(b) - \boldsymbol{f}(a)| \leq (b-a)|\boldsymbol{f}'(x)|.$$

Proof. Let

$$\mathbf{f}(b) - \mathbf{f}(a) = \mathbf{z},$$

which is a constant vector in $\mathbb{R}^k$. Define

$$\varphi(t) = \mathbf{z} \,.\, \mathbf{f}(t), \quad t \in [a,b].$$

Then $\varphi : [a,b] \to \mathbb{R}$ is differentiable in (a,b) and by mean value theorem, there exists $x \in (a,b)$ such that

$$(b-a)\varphi'(x) = \varphi(b) - \varphi(a).$$

That is

$$\varphi(b) - \varphi(a) = (b-a)\left(\mathbf{z} \,.\, \mathbf{f}'(x)\right).$$

Now, by the definition of φ, we have

$$\varphi(b) - \varphi(a) = |\mathbf{z}|^2,$$

which implies

$$\begin{aligned} |\mathbf{z}|^2 &= (b-a)|\mathbf{z} \,.\, \mathbf{f}'(x)| \\ &\leq (b-a)|\mathbf{z}|\ |\mathrm{f}'(x)|, \end{aligned}$$

(by using **Schwarz inequality** which states that: If $a_1,\ a_2,\ ...,\ a_n,$ $b_1,\ b_2,\ ...,\ b_n$ are complex numbers, then

$$\left|\sum_{i=1}^{n} a_i\overline{b_i}\right|^2 \leq \sum_{i=1}^{n}|a_i|^2\sum_{i=1}^{n}|b_i|^2.)$$

Hence we have

$$|\mathbf{z}| \leq (b-a)|\mathbf{f}'(x)|,$$

which proves the result since

$$\mathbf{f}(b) - \mathbf{f}(a) = \mathbf{z}.$$

□

3.4 Some Applications

(a) Let f be defined and differentiable for all $x > 0$, and $f'(x) \to 0$ as $x \to \infty$. If $g(x) = f(x+1) - f(x)$, then prove that $g(x) \to 0$ as $x \to \infty$.

Solution: Fix $x > 0$. Since f is differentiable in $(0, \infty)$, we have, in particular f is differentiable in $(x, x+1)$. Then by mean value theorem, there exists $t_x \in (x, x+1)$ such that

$$f'(t_x)[x+1-x] = f(x+1) - f(x) = g(x).$$

That is for each $x > 0$, we have a $t_x \in (x, x+1)$ such that

$$f'(t_x) = g(x).$$

Then as $x \to \infty$, $t_x \to \infty$ and $f'(t_x) \to 0$(given). Hence we get $g(x) \to 0$ as $x \to \infty$.

(b) Suppose (i) f is continuous for $x \geq 0$, (ii) $f(0) = 0$, (iii) $f'(x)$ exists for $x > 0$ and (iv) f' is monotonically increasing. If $g(x) = \frac{f(x)}{x}$, $x > 0$, then g is monotonically increasing.

Solution: Note that $g(x) = \frac{f(x)-f(0)}{x-0}$, $x > 0$.
Using mean value theorem, for every $x > 0$, we have a $t_x \in (0, x)$ such that

$$f'(t_x) = \frac{f(x) - f(0)}{x - 0} = g(x).$$

Now, $t_x \in (0, x)$ and f' is monotonically increasing. So, we get

$$g(x) = \frac{f(x)}{x} = f'(t_x) \leq f'(x),$$

which implies $xf'(x) \geq f(x)$ and hence

$$g'(x) = \frac{xf'(x) - f(x)}{x^2} \geq 0.$$

(c) If $f : [0, 1] \to [0, 1]$ is continuous, then f has a fixed point, that is a point x such that $f(x) = x$.

Solution: We just consider the function

$$g(x) = f(x) - x, \quad x \in [0, 1].$$

Then g is continuous. It is enough if we show $g(x) = 0$ for some $x \in [0, 1]$.

Since the values of f lies in $[0, 1]$, we have in particular

$$f(0) \geq 0, \quad \text{and} \quad f(1) \leq 1.$$

Thus

$$g(0) \geq 0 \quad \text{and} \quad g(1) \leq 0.$$

Then by intermediate value theorem, we must have g attain the value 0 some where in the domain.

Exercise

1. Give an example of a function which is continuous on $\mathbb{R}$ but not differentiable at

   (i) the point 1

   (ii) the points $1, 2, ...10$

   (iii) any integer

2. Discuss the differentiability of the function $f(x) = x|x|, \quad x \in \mathbb{R}$.

3. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and bounded, then prove that, there is a point $x \in \mathbb{R}$ such that $f(x) = x^3$.

4. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and bounded, then prove that, there is a point $x \in \mathbb{R}$ such that $f(x) + x^3 = 0$.

5. Let f be defined on $\mathbb{R}$ such that

$$|f(x) - f(y)| \leq (x - y)^2,$$

   for all real x and y. prove that f is a constant.

6. (i) Suppose f is a real function on $\mathbb{R}$. If f is differentiable and $f'(t) \neq 1$, for any t, prove that f has at most one fixed point.

   (ii) Show that the function f defined by

$$f(t) = t + (1 + e^t)^(- 1)$$

   has no fixed point even though $0 < f'(t) < 1$ for all t.

Chapter 4

The Riemann-Stieltjes Integral

You are familiar with Riemann Integral which you have studied in BSc. and Higher secondary classes. The Riemann-Stieltjes Integration is a generalization of the Riemann integration.

4.1 Definition and Existence of the Integral

Let us define the two concepts one by one using the following terminologies.

Definition 4.1. *Let $[a, b]$ be a given interval. A **partition** P of $[a, b]$ is a finite set of points $x_0,\ x_1,\ \ldots,\ x_n$, where*

$$a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b.$$

Suppose f is a bounded real function defined on $[a, b]$. Corresponding to each partition P of $[a, b]$, we define

$$\Delta x_i = x_i - x_{i-1},$$

$$M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x),$$

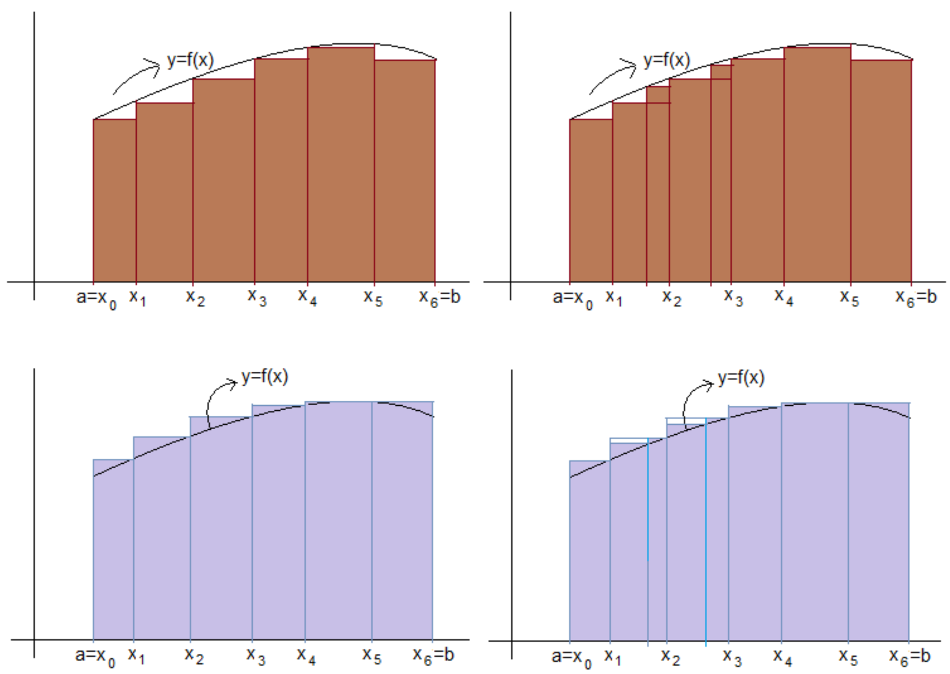
$$m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x),$$

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i, \qquad L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i,$$

and

$$\overline{\int_a^b} f \; dx = \inf_P \; U(P, f), \qquad \underline{\int_a^b} f \; dx = \sup_P \; L(P, f),$$

where the inf *and* sup *are taken over all partitions P of $[a, b]$. The above integrals are called, respectively, the* **upper** *and* **lower Riemann integrals** *of f over $[a, b]$.*



Note that, since f is bounded, there exist real numbers m and M such that

$$m \leq f(x) \leq M, \quad x \in [a, b].$$

So that for every P, we have, $m \leq m_i \leq M_i \leq M$, and hence

$$m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a).$$

Thus both $\inf U(P, f)$ and $\sup L(P, f)$ exist and the upper and lower integrals are defined for a bounded function. Also note that as we include more points in the partition, the upper sum decreases and the lower sum increases(see the above figure) and hence there is a chance that both $\inf U(P, f)$ and $\sup L(P, f)$ may coincide to the area under the curve.

Definition 4.2. *The function f is said to be* **Riemann integrable** *on $[a, b]$ if the upper and lower Riemann integrals are equal; we write $f \in \Re$(the set of Riemannn integrable functions), and the common value of the upper and lower integrals, called the* **Riemann integral** *of f on $[a, b]$, is denoted by*

$$\int_a^b f(x)\ dx \qquad or \qquad \int_a^b f\ dx.$$

Let us work out two simple examples for understanding the definition.

Example 4.1. *1. Let us evaluate*

$$\int_0^2 f(x)\ dx,$$

where $f(x) = 1, \quad x \in [0, 2]$. For every partition $\{x_0,\ x_1,\ ...,\ x_n\}$ of $[0, 2]$, we have $m_i = M_i = 1$, so that

$$U(P, f) = L(P, f) = \sum_{i=1}^{n}(x_i - x_{i-1}) = b - a = 2.$$

Hence

$$\int_a^b f(x)\ dx = 2.$$

2. If

$$f(x) = \begin{cases} 0, & if\ \ x\ is\ rational \\ 1, & if\ \ x\ is\ irrational, \end{cases}$$

then f is not integrable on any interval $[a, b]$, with $a < b$.

For, we have, given any partition $\{x_0,\ x_1,\ ...,\ x_n\}$ of $[a, b]$, $m_i = 0$, and $M_i = 1$. So that,

$$L(P, f) = 0 \ \ and \ \ U(P, f) = b - a > 0.$$

Hence

$$\bar{\int_a^b} f(x)\ dx = b - a, \neq \underline{\int_a^b} f(x)\ dx = 0.$$

Definition 4.3. *Let f be a bounded real function defined on $[a, b]$ and let α be a monotonically increasing function on $[a, b]$ (then α is bounded on $[a, b]$). Corresponding to each partition P of $[a, b]$, we define*

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}).$$

Then $\Delta\alpha_i \geq 0$. *Define*

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i, \qquad L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i,$$

and

$$\overline{\int_a^b} f \ d\alpha = \inf_P \ U(P,f,\alpha), \qquad \underline{\int_a^b} f \ d\alpha = \sup_P \ L(P,f,\alpha),$$

where the inf *and* sup *are taken over all partitions* P *of* $[a,b]$. *The above integrals are called, respectively, the* **upper** *and* **lower Riemann-Stieltjes integrals** *of* f *with respect to* α *over* $[a,b]$.

If the upper and lower Riemann-Stieltjes integrals are equal, we say that f *is* **Riemann-Stieltjes integrable with respect to** α **over** $[a,b]$ *and we write* $f \in \Re(\alpha)$. *The common value of the integrals are called the* **Riemann-Stieltjes integral** *of* f *with respect to* α *over* $[a,b]$ *and is denoted by*

$$\int_a^b f(x) \ d\alpha(x) \qquad or \qquad \int_a^b f \ d\alpha.$$

Note that the Riemann integration is a special case of the Riemann Stieltjes integration, when $\alpha(x) = x$. So that all the properties and theorems are discussed in the general setup- the Riemann Stieltjes integration.

Through out this chapter, when we talk about the integral of f, the function f will be assumed to be bounded and α monotonically increasing on $[a,b]$.

Definition 4.4. *A partition* P^* *of* $[a,b]$ *is a* **refinement** *of* P *if* $P^* \supset P$, *that is every point of* P *is a point of* P^*.

Given two partitions P_1 *and* P_2 *of* $[a,b]$, $P^* = P_1 \cup P_2$ *is a common refinement of* P_1 *and* P_2.

One can easily verify that given a refinement P^* of P, the upper and lower sums satisfy the following relations:

$$L(P,f) \leq L(P^*,f) \leq U(P^*,f) \leq U(P,f).$$

Theorem 4.1. *If* P^* *is a refinement of* P, *then*

$$L(P,f,\alpha) \leq L(P^*,f,\alpha) \qquad and \qquad U(P^*,f,\alpha) \leq U(P,f,\alpha).$$

Proof. Given $P \subset P^*$. Consider the case $P^* = P \cup \{x^*\}$, that is P^* contains just one more point than P. Let

$$x_1 \le x_2 \le \cdots \le x_{i-1} \le x^* \le x_i \le \cdots \le x_n$$

be the points in P^*.

Then

$$L(P, f, \alpha) = \sum_{j=1}^{n} m_j \Delta\alpha_j,$$

where

$$m_j = \inf_{x_{j-1} \le x \le x_j} f(x), \quad \Delta\alpha_j = \alpha(x_j) - \alpha(x_{j-1}), \quad 1 \le j \le n.$$

Now,

$$L(P^*, f, \alpha) = \sum_{j=1}^{i-1} m_j \Delta\alpha_j + w_1[\alpha(x^*) - \alpha(x_{i-1})] + w_2[\alpha(x_i) - \alpha(x^*)] + \sum_{j=i+1}^{n} m_j \Delta\alpha_j$$

where

$$w_1 = \inf_{x_{i-1} \le x \le x^*} f(x) \quad \text{and} \quad w_2 = \inf_{x^* \le x \le x_i} f(x).$$

Notice that

$$w_1 \ge m_i \quad \text{and} \quad w_2 \ge m_i.$$

So that,

$$\begin{aligned} &L(P^*, f, \alpha) - L(P, f, \alpha) \\ &\quad = w_1[\alpha(x^*) - \alpha(x_{i-1})] \ + \ w_2[\alpha(x_i) - \alpha(x^*)] \ - \ m_i \Delta\alpha_i \\ &\quad == (w_1 - m_i)[\alpha(x^*) - \alpha(x_{i-1})] \ + \ (w_2 - m_i)[\alpha(x_i) - \alpha(x^*)] \ge 0. \end{aligned}$$

If P^* contains k points more than P, repeating this argument k times we arrive at the result.

A similar proof holds for $U(P^*, f, \alpha) \le U(P, f, \alpha)$, left as an exercise. □

Theorem 4.2. *The upper and lower integrals are related by the inequality,*

$$\underline{\int_a^b} f \, d\alpha \le \overline{\int_a^b} f \, d\alpha.$$

Proof. Let P_1 and P_2 be any two partitions of $[a, b]$. Consider their common refinement $P^* = P_1 \cup P_2$, we have

$$L(P_1, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P_2, f, \alpha),$$

using Theorem 4.1 and the definition.

That is

$$L(P_1, f, \alpha) \leq U(P_2, f, \alpha),$$

for any two partitions P_1 and P_2 of $[a, b]$.

So that,

$$\sup_P L(P, f, \alpha) \leq U(P_2, f, \alpha)$$

for any partition P_2 of $[a, b]$ (by keeping P_2 fixed and by taking sup over all partitions).

Again, as the above inequality is true for every partition P_2, we have

$$\sup_P L(P, f, \alpha) \leq \inf_P U(P, f, \alpha).$$

That is,

$$\underline{\int_a^b} f \; d\alpha \leq \overline{\int_a^b} f \; d\alpha.$$

□

The following characterization gives a simple criterion for checking the integrability.

Theorem 4.3. *The function* $f \in \Re(\alpha)$ *on* $[a, b]$ *if and only if for every* $\varepsilon > 0$, *there exists a partition* P *such that*

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

Proof. For any partition P, we have

$$L(P, f, \alpha) \leq \underline{\int_a^b} f \; d\alpha \leq \overline{\int_a^b} f \; d\alpha \leq U(P, f, \alpha). \tag{4.1}$$

If for every $\varepsilon > 0$, there exists a partition P such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon,$$

then we have from (4.1),

$$0 \leq \overline{\int_a^b} f \, d\alpha - \underline{\int_a^b} f \, d\alpha < \varepsilon.$$

Since it is true for all $\varepsilon > 0$, we can conclude that

$$\overline{\int_a^b} f \, d\alpha - \underline{\int_a^b} f \, d\alpha = 0,$$

and hence $f \in \Re(\alpha)$.

Conversely let $f \in \Re(\alpha)$. Then we have

$$\sup L(P, f, \alpha) = \underline{\int_a^b} f \, d\alpha = \int_a^b f \, d\alpha = \overline{\int_a^b} f \, d\alpha = \inf U(P, f, \alpha).$$

Given $\varepsilon > 0$, we can find partitions P_1 and P_2 such that

$$U(P_1, f, \alpha) < \int_a^b f \, d\alpha + \frac{\varepsilon}{2}$$

and

$$L(P_2, f, \alpha) > \int_a^b f \, d\alpha - \frac{\varepsilon}{2}$$

Now, for the common refinement $P^* = P_1 \cup P_2$, we have

$$U(P^*, f, \alpha) \leq U(P_1, f, \alpha) < \int_a^b f \, d\alpha + \frac{\varepsilon}{2}$$

and

$$L(P^*, f, \alpha) \geq L(P_2, f, \alpha) > \int_a^b f \, d\alpha - \frac{\varepsilon}{2}$$

That is

$$U(P^*, f, \alpha) - \int_a^b f \, d\alpha < \frac{\varepsilon}{2}$$

and

$$\int_a^b f \, d\alpha - L(P^*, f, \alpha) < \frac{\varepsilon}{2}$$

Adding the above two inequalities, we get

$$U(P^*, f, \alpha) - L(P^*, f, \alpha) < \varepsilon,$$

so that P^* satisfies the requirement. □

Theorem 4.4. *(a) If $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$, for some partition P of $[a,b]$ and for some ε,then*

$$U(P^*,f,\alpha) - L(P^*,f,\alpha) < \varepsilon,$$

for all refinements of P.

(b) If $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$, for $P = \{x_0, x_1, ..., x_n\}$ and if s_i and t_i are any two points in $[x_{i-1}, x_i]$, then

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)|\Delta\alpha_i < \varepsilon.$$

(c) If $f \in \Re(\alpha)$, $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$, for $P = \{x_0, x_1, ..., x_n\}$ and if s_i any point in $[x_{i-1}, x_i]$, then

$$\left| \sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \int_a^b f \ d\alpha \right| < \varepsilon.$$

Proof. Part (a) follows from the inequality

$$L(P,f,\alpha) \le L(P^*,f,\alpha) \le U(P^*,f,\alpha) \le U(P,f,\alpha).$$

Now, we have

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i,$$

and for any two points s_i and t_i in $[x_{i-1}, x_i]$,

$$m_i \le f(s_i), \ f(t_i) \le M_i. \tag{4.2}$$

So that

$$|f(s_i) - f(t_i)| \le M_i - m_i,$$

and hence

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)|\Delta\alpha_i \le U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon,$$

which proves (b).

Again from (4.2), we have

$$L(P,f,\alpha) \le \sum_{i=1}^{n} f(s_i)\Delta\alpha_i \le U(P,f,\alpha)$$

Also since $f \in \Re(\alpha)$, we have

$$L(P, f, \alpha) \leq \int f \; d\alpha_i \leq U(P, f, \alpha).$$

Hence

$$\left| \sum_{i=1}^{n} f(s_i) \Delta\alpha_i - \int_a^b f \; d\alpha \right| < \varepsilon.$$

□

Theorem 4.5. *If f is continuous on $[a, b]$, then $f \in \mathcal{R}(\alpha)$ on $[a, b]$.*

⌈ **Analysis**

Before beginning the proof, let us analyze the situation.

(1) In view of Theorem 4.3, it is enough to prove that for a given $\varepsilon > 0$, there exists a partition P such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

(2) Given f is continuous on $[a, b]$. Since $[a, b]$ is compact, f is uniformly continuous on $[a, b]$.

(3) For $P = \{x_0, x_1, ...x_n\}$,

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i) \Delta(\alpha_i).$$

So that $U(P, f, \alpha) - L(P, f, \alpha)$ can be made less than ε if we get a positive real r such that $M_i - m_i < r$, for all i, which is attained by the uniform continuity of f, because $M_i = f(x)$ and $m_i = f(y)$, or some $x, y \in [x_i, y_i]$.

(4) Once we get $M_i - m_i < r$, for each i, then

$$U(P, f, \alpha) - L(P, f, \alpha) = r(\sum \Delta(\alpha_i)) = r(\alpha(b) - \alpha(a)).$$

Therefore, to ind r such that $r(\alpha(b) - \alpha(a) < \varepsilon$. ⌋

Proof. Choose r such that $r(\alpha(b) - \alpha(a)) < \varepsilon$.
Since f is uniformly continuous on $[a, b]$, for this r, there exists a $\delta > 0$ such that for any two points x, y in $[a, b]$ with $|x - y| < \delta$,

$$|f(x) - f(y)| < r.$$

Now, Consider a partition $P = \{x_0, x_1, ...x_n\}$, of $[a, b]$ such that

$$|x_i - x_{i-1}| < \delta \ \text{ for } \quad 1 \leq i \leq n.$$

Then for all i, if $x, y \in [x_i, x_{i-1}]$, then

$$|x - y| < \delta \quad \text{and} \quad |f(x) - f(y)| < r,$$

in particular,

$$M_i - m_i < r.$$

Therefore,

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta(\alpha_i) < r \sum_{i=1}^{n} \Delta(\alpha_i) = r(\alpha(b) - \alpha(a)) < \varepsilon.$$

□

Theorem 4.6. *If f is monotonic on $[a, b]$ and α is continuous(also monotonic) on $[a, b]$, then $f \in \Re(\alpha)$ on $[a, b]$.*

⌈**Analysis**

(1) α is continuous on $[a, b] \Rightarrow$ α is uniformly continuous on $[a, b]$. So that $|\alpha(x) - \alpha(y)|$ can be made small as we please.

(2) If f is monotonically increasing(similar proof for decreasing f with necessary modifications), then for any partition $P = \{x_0, x_1, ...x_n\}$, $M_i = f(x_i)$ and $m_i = f(x_{i-1})$.

(3) For any $r > 0$, we can make $\Delta(\alpha_i) = \alpha(x_i) - \alpha(x_{i-1}) < r$, for all i, using (1). Then

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta(\alpha_i) < r \left(\sum_{i=1}^{n} (M_i - m_i) \right) = r(f(b) - f(a)),$$

therefore choose r such that $r(f(b) - f(a)) < \varepsilon$. ⌋

Proof. Let $\varepsilon > 0$. Choose $r > 0$ such that $r(f(b) - f(a)) < \varepsilon$.

Since α is uniformly continuous on $[a, b]$, for this r, there exists a δ such that,

$$|\alpha(x) - \alpha(y)| < r, \quad \text{whenever} \quad |x - y| < \delta. \tag{4.3}$$

So that if we choose a partition $P = \{x_0, x_1, ...x_n\}$ of $[a, b]$ such that $|x_i - x_{i-1}| < \delta$, then by (4.3), we have

$$\Delta(\alpha_i) = \alpha(x_i) - \alpha(x_{i-1}) < r.$$

Then

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{i=1}^{n}(M_i - m_i)\Delta(\alpha_i) \\ &< r\left(\sum_{i=1}^{n}(M_i - m_i)\right) \\ &= r(f(b) - f(a)) < \varepsilon, \end{aligned}$$

by our choice of ε. $\square$

Theorem 4.7. *Suppose f is bounded on $[a, b]$, f has only finitely many points of discontiniuity on $[a, b]$, and α is continuous at every point at which f is discontinuous. then $f \in \Re(\alpha)$ on $[a, b]$.*

Proof. Let E be the set of points at which f is discontinuous. Since E is finite, we can find finitely many disjoint intervals $[u_j, v_j] \subset [a, b]$ covering E(covering E means $E \subset \bigcup[u_j, v_j]$).

Let $\varepsilon > 0$.

Since α is continuous at every point in $x \in E$, and E is compact(being closed subset of $[a, b]$), we can choose the intervals $[u_j, v_j]$ such that

$$\alpha(v_j) - \alpha(u_j) < \varepsilon. \tag{4.4}$$

Also, choose $[u_j, v_j]$ such that every point of E other than a and b is an interior point of $[u_j, v_j]$, in other words,

$$E \cap (a, b) \subset (u_j, v_j).$$

(Our intention is to use the continuity of α on $[u_j, v_j]$ and the continuity of f at the remaining points on $[a, b]$.)

Now, the set $K = [a, b] - \bigcup(u_j, v_j)$ is compact and hence f is uniformly continuous on K.

So that there exists $\delta > 0$ such that

$$\text{for } \ s,\ t \in K \text{ with } |s - t| < \delta,\ |f(s) - f(t) < \varepsilon. \tag{4.5}$$

Now, choose a partition $P = \{x_0, ..., x_n\}$ of $[a, b]$ such that

(a) $u_j \in P$ and $v_j \in P$ for every j;

(b) P contains no point of (u_j, v_j) and

(c) for the subintervals $[x_{i-1}, x_i]$ other than $[u_j, v_j], \Delta\alpha_i < \delta$.

Then

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i. \tag{4.6}$$

Now, split the sum on the right hand side of (4.6) into two- $\sum_1$ and $\sum_2$, that is

$$U(P, f, \alpha) - L(P, f, \alpha) = \textstyle\sum_1 (M_i - m_i)\Delta\alpha_i + \sum_2 (M_i - m_i)\Delta\alpha_i,$$

where $\sum_1$ is extended over all i for which the interval $[x_{i-1}, x_i]$ is of the form $[u_j, v_j]$ and $\sum_2$ is extended over the remaining.

Now, using (4.4), we have

$$\textstyle\sum_1 (M_i - m_i)\Delta\alpha_i < \varepsilon \sum_1 (M_i - m_i) < \varepsilon 2M,$$

because f is bounded, so that $|f(x)| \leq K$, for some K, and$\sum_1 (M_i - m_i) \leq 2Kl = M$, where l is the number of terms in $\sum_1$.

Also,by using (4.5), we have

$$\textstyle\sum_2 (M_i - m_i)\Delta\alpha_i < \varepsilon \sum_2 \Delta\alpha_i < \varepsilon(\alpha(b) - \alpha(a)).$$

Hence we have

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon(\alpha(b) - \alpha(a) + 2M),$$

which completes the proof.

(For getting $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$, the choices of the intervals $[u_j, v_j]$ in (4.4) and δ in (4.5) have to be made for $\frac{\varepsilon}{2M}$ and $\frac{\varepsilon}{\alpha(b)-\alpha(a)}$ respectively, instead of ε.) $\square$

Theorem 4.8. *Suppose $f \in \Re(\alpha)$ on $[a, b]$, $m \leq f \leq M$, ϕ is continuous on $[m, M]$, and $h(x) = \phi(f(x))$ on $[a, b]$. Then $h \in \Re(\alpha)$ on $[a, b]$.*

Proof. Let $\varepsilon > 0$. Our aim is to find a partition $P = \{x_0, ..., x_n\}$ of $[a, b]$ such that

$$U(P, h, \alpha) - L(P, h, \alpha) = \sum_{i=1}^{n}(M_i^* - m_i^*)\Delta\alpha_i < \varepsilon,$$

where

$$M_i^* = \sup_{x \in [x_{i-1}, x_i]} h(x) \quad \text{and} \quad m_i^* = \inf_{x \in [x_{i-1}, x_i]} h(x).$$

(**Note** that $M_i^* - m_i^* = \phi(f(x)) - \phi(f(y))$ for some $x,\ y \in [x_{i-1}, x_i]$, so that we can use the continuity of ϕ.)

Since ϕ is uniformly continuous on $[m, M]$, we can find a $\delta > 0$ such that

$$\delta < \varepsilon,$$

and

$$|\phi(s) - \phi(t) < \varepsilon \quad \text{for all } s,\ t \in [m, M] \quad \text{with} \quad |s - t| < \delta. \tag{4.7}$$

For this δ, there is a partition $P = \{x_0, ..., x_n\}$ of $[a, b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i < \delta^2, \tag{4.8}$$

because $f \in \Re(\alpha)$.

Now, for this partition, consider

$$U(P, h, \alpha) - L(P, h, \alpha) = \sum_{i=1}^{n}(M_i^* - m_i^*)\Delta\alpha_i.$$

We split this sum into two- $\sum_1$ and $\sum_2$, that is

$$U(P, h, \alpha) - L(P, h, \alpha) = \textstyle\sum_1(M_i^* - m_i^*)\Delta\alpha_i + \sum_2(M_i^* - m_i^*)\Delta\alpha_i,$$

where $\sum_1$ is extended over all i for which $M_i - m_i < \delta$ and $\sum_2$ is extended over all i for which $M_i - m_i \geq \delta$.

Now, for $M_i - m_i < \delta$, we have $M_i^* - m_i^* < \varepsilon$, using (4.7). So that

$$\textstyle\sum_1(M_i^* - m_i^*)\Delta\alpha_i < \varepsilon(\alpha(b) - \alpha(a)).$$

And for $M_i - m_i \geq \delta$, we have

$$\delta \sum_2 \Delta\alpha_i \leq \sum_2 (M_i - m_i)\Delta\alpha_i < U(P, f, \alpha) - L(P, f, \alpha) < \delta^2,$$

using (4.8), so that

$$\sum_2 \Delta\alpha_i < \delta < \varepsilon$$

and

$$\sum_2 (M_i^* - m_i^*)\Delta\alpha_i < \sum_2 2K\Delta\alpha_i < 2K\varepsilon,$$

where $K = \sup|\phi(t)|$.

Substituting for $\sum_1$ and $\sum_2$, we get

$$U(P, h, \alpha) - L(P, h, \alpha) < \varepsilon(\alpha(b) - \alpha(a) + 2K),$$

which completes the proof.

$\square$

Example 4.2. *If $f \in \Re(\alpha)$ on $[a, b]$, then $f^2 \in \Re(\alpha)$ on $[a, b]$, where $f^2(x) = (f(x))^2$.*

For, since $f \in \Re(\alpha)$, we have $m \leq f(x) \leq M$ for some m and M. The map $\phi(t) = t^2,\ t \in [m, M]$ is continuous on $[m, M]$. Also,

$$(\phi \circ f)(x) = \phi(f(x)) = (f(x))^2,$$

which implies that $\phi \circ f = f^2$, and hence by Theorem 4.8, we have $f^2 \in \Re(\alpha)$.

But $f^2 \in \Re(\alpha)$ on $[a, b]$, does not imply that $f \in \Re(\alpha)$ on $[a, b]$.
For example let f be the mapping defined on $[a, b]$ by

$$f(x) = \begin{cases} -1, & \text{if } x \text{ is irrational} \\ 1, & \text{if } x \text{ is rational} \end{cases}$$

and $\alpha(x) = x,\ x \in [a, b]$. Then $f \notin \Re(\alpha)$, (Verify!) but $f^2(x) = 1$, so that $f^2 \in \Re(\alpha)$ on $[a, b]$.

4.2 Properties

Theorem 4.9.

(a) If $f_1 \in \Re(\alpha)$ and $f_2 \in \Re(\alpha)$ on $[a,b]$, and c is a constant, then

$$f_1 + f_2 \in \Re(\alpha) \quad \text{and} \quad cf \in \Re(\alpha)$$

on $[a,b]$ and

$$\int_a^b (f_1 + f_2)\ d\alpha = \int_a^b f_1\ d\alpha + \int_a^b f_2\ d\alpha,$$

$$\int_a^b cf\ d\alpha = c\int_a^b f\ d\alpha.$$

(b) If $f_1,\ f_2 \in \Re(\alpha)$ on $[a,b]$, and $f_1(x) \leq f_2(x)$ on $[a,b]$, then

$$\int_a^b f_1\ d\alpha \leq \int_a^b f_2\ d\alpha.$$

(c) If $f \in \Re(\alpha)$ on $[a,b]$, and if $a < c < b$, then $f \in \Re(\alpha)$ on $[a,c]$ and $f \in \Re(\alpha)$ on $[c,b]$, and

$$\int_a^c f\ d\alpha + \int_c^b f\ d\alpha = \int_a^b f\ d\alpha.$$

(d) If $f \in \Re(\alpha)$ on $[a,b]$, and $|f(x)| \leq M$ on $[a,b]$, then

$$\int_a^b f\ d\alpha \leq M(\alpha(b) - \alpha(a)).$$

(e) If $f \in \Re(\alpha_1)$ and $f \in \Re(\alpha_2)$ on $[a,b]$, then $f \in \Re(\alpha_1 + \alpha_2)$ on $[a,b]$ and

$$\int_a^b f\ d(\alpha_1 + \alpha_2) = \int_a^b f\ d\alpha_1 + \int_a^b f\ d\alpha_2.$$

(f) If $f \in \Re(\alpha)$ on $[a,b]$, and c is a positive constant, then $f \in \Re(c\alpha)$ on $[a,b]$ and

$$\int_a^b f\ d(c\alpha) = c\int_a^b f\ d\alpha.$$

Proof. (*a*) Given $f_1 \in \Re(\alpha)$ and $f_2 \in \Re(\alpha)$ on $[a,b]$. So that there are partitions P_1 and P_2 of $[a,b]$ such that

$$U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \varepsilon/2$$

and

$$U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \varepsilon/2.$$

Let $f = f_1 + f_2$. We have (Verify!) if P is the common refinement of P_1 and P_2, then

$$\begin{aligned} L(P_1, f_1, \alpha) + L(P_2, f_2, \alpha) &\le L(P, f_1, \alpha) + L(P, f_2, \alpha) \\ &\le U(P, f_1, \alpha) + U(P, f_2, \alpha) \\ &\le U(P_1, f_1, \alpha) + U(P_2, f_2, \alpha). \end{aligned}$$

Also, we have (Verify!)

$$L(P, f_1, \alpha) + L(P, f_2, \alpha) \le L(P, f, \alpha) \le U(P, f, \alpha) \le U(P, f_1, \alpha) + U(P, f_2, \alpha).$$

Hence

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &\le [U(P, f_1, \alpha) + U(P, f_2, \alpha)] - [L(P, f_1, \alpha) + L(P, f_2, \alpha)] \\ &= [U(P, f_1, \alpha) - L(P, f_1, \alpha)] + [U(P, f_2, \alpha) - L(P, f_2, \alpha)] \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Now, since

$$\int f_j \, d\alpha = \inf_P \; U(P, f_j, \alpha) \quad j = 1, 2;$$

for any $\varepsilon > 0$, we have a partition P for which

$$U(P, f_j, \alpha) < \int f_j \, d\alpha + \varepsilon, \quad j = 1, \; 2.$$

Hence

$$\int f \, d\alpha \le U(P, f, \alpha) \le U(P, f_1, \alpha) + U(P, f_2, \alpha) < \int f_1 \, d\alpha + \int f_2 \, d\alpha + 2\varepsilon.$$

Since it is true for any $\varepsilon > 0$, we have

$$\int f \, d\alpha \le \int f_1 \, d\alpha + \int f_2 \, d\alpha.$$

Similarly considering

$$\int f_j \, d\alpha = \sup_P \; L(P, f_j, \alpha), \quad j = 1, 2;$$

we get
$$\int f\ d\alpha \geq \int f_1\ d\alpha + \int f_2\ d\alpha.$$
Hence we conclude that
$$\int f\ d\alpha = \int f_1\ d\alpha + \int f_2\ d\alpha.$$
The proofs of $(b),\ (c,)\ (d),\ (e)$ and (f) are left as exercise. □

Theorem 4.10. *If $f \in \Re(\alpha)$ and $g \in \Re(\alpha)$ on $[a, b]$, then*

(a) $fg \in \Re(\alpha)$;

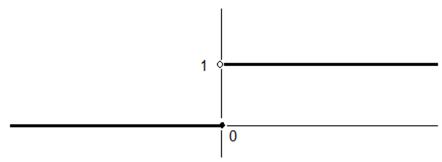
(b) $|f| \in \Re(\alpha)$ *and* $\left|\int_a^b f\ d\alpha\right| \leq \int_a^b |f|\ d\alpha$.

Proof.
(a) By Theorem 4.9(a) and Example 4.2, we have
$$f + g,\ f - g \in \Re(\alpha),$$
and
$$4fg = (f+g)^2 - (f-g)^2 \in \Re(\alpha).$$
Hence
$$fg \in \Re(\alpha).$$
(b) Again by using Theorem 4.8 with the continuous function
$$\phi(t) = |t|,$$
we have
$$|f| \in \Re(\alpha),$$
because
$$(\phi \circ f)(x) = \phi(f(x)) = |f(x)| = |f|(x).$$
In order to prove
$$\left|\int_a^b f\ d\alpha\right| \leq \int_a^b |f|\ d\alpha,$$
it is enough if we prove
$$-\int_a^b |f|\ d\alpha \leq \int_a^b f\ d\alpha \leq \int_a^b |f|\ d\alpha.$$
It follows from Theorem 4.9(b), because
$$-|f|(x) \leq f(x) \leq |f|(x)$$
for all $x \in [a, b]$. □

Definition 4.5. *The function* $I : \mathbb{R} \to \mathbb{R}$ *defined by*

$$I(x) = \begin{cases} 0, & \textit{if} \;\; x \leq 0, \\ 1, & \textit{if} \;\; x > 0 \end{cases}$$

is called the **unit step function**.



Note that I is monotonic. We use the unit step function to express the integral as a series. As a first step, we have the following result.

Theorem 4.11. *If* $a < s < b$, f *is bounded on* $[a, b]$, f *is continuous at* s, *and* $\alpha(x) = I(x - s)$, *then*

$$\int_a^b f \; d\alpha = f(s).$$

Proof. Consider the partition $P = \{x_0, \; x_1, \; x_2, \; x_3\}$, where

$$x_0 = a, \text{ and } x_1 = s < x_2 < x_3 = b.$$

Then

$$U(P, f, \alpha) = M_1(\alpha(s) - \alpha(a)) + M_2(\alpha(x_2) - \alpha(s)) + M_1(\alpha(b) - \alpha(x_2)) = M_2,$$

because

$$\alpha(x) = \begin{cases} 0, & \text{if} \;\; x \leq s, \\ 1, & \text{if} \;\; x > s. \end{cases}$$

Similarly we get

$$L(P, f, \alpha) = m_2.$$

Now,

$$M_2 = \sup_{x \in [s, x_2]} f(x) \text{ and } m_2 = \inf_{x \in [s, x_2]} f(x).$$

This is true for every $x_2 > s$.

Now, given f is continuous at s, so that as x_2 approaches s, we have $f(x_2)$ approaches $f(s)$, and both M_2 and m_2 approaches $f(s)$. Hence we have

$$\int_a^b f \, d\alpha = f(s).$$

$\square$

Theorem 4.12. *Suppose $c_n \geq 0$ for $n = 1, 2, 3, \ldots$, $\sum c_n$ converges, $\{s_n\}$ is a sequence of distinct points in (a, b) and*

$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n).$$

Let f be continuous on $[a, b]$. Then

$$\int_a^b f \, d\alpha = \sum_{n=1}^{\infty} c_n f(s_n).$$

Proof. Note that the series

$$\sum_{n=1}^{\infty} c_n I(x - s_n)$$

converges. For,

$$c_n I(x - s_n) = \begin{cases} 0, & \text{if } \ x \leq s_n, \\ c_n, & \text{if } \ x > s_n. \end{cases}$$

Since $c_n \geq 0$, we have $c_n I(x - s_n) \leq c_n$ for each n. Therefore by comparison test, we get the series

$$\sum_{n=1}^{\infty} c_n I(x - s_n)$$

converges.

Also, $a < s_n < b$ for all n, implies that $I(a - s_n) = 0$ and $I(b - s_n) = 1$, so that

$$\alpha(a) = 0, \quad \alpha(b) = \sum c_n.$$

Let $\varepsilon > 0$. Since $\sum c_n$ converges, we can find a natural number N such that

$$\sum_{n=N+1}^{\infty} c_n < \varepsilon.$$

Write
$$\alpha(x) = \alpha_1(x) + \alpha_2(x),$$
where,
$$\alpha_1(x) = \sum_{n=1}^{N} c_n I(x - s_n) \quad \text{and} \quad \alpha_2(x) = \sum_{n=N+1}^{\infty} c_n I(x - s_n).$$

Then
$$\int_a^b f \ d\alpha_1 = \sum_{n=1}^{N} c_n f(s_n). \tag{4.9}$$
It follows from Theorem 4.11 and Theorem 4.9$(e) - (f)$, because, we can write
$$\alpha_1 = c_1\beta_1 + \cdots + c_N\beta_N,$$
where $\beta_i = I(x - s_i)$, and f is continuous at each s_i.

Now since
$$\alpha_2(a) = 0 \quad \text{and} \quad \alpha_2(b) = \sum_{n=N+1}^{\infty} c_n,$$
we have
$$\alpha_2(b) - \alpha_2(a) < \varepsilon.$$
So that if $M = \sup |f(x)|$, then
$$\left| \int_a^b f \ d\alpha_2 \right| \leq M(\alpha_2(b) - \alpha_2(a)) < M\varepsilon.$$
Now,
$$\int_a^b f \ d\alpha = \int_a^b f \ d\alpha_1 + \int_a^b f \ d\alpha_2.$$
So that
$$\left| \int_a^b f \ d\alpha_2 \right| = \left| \int_a^b f \ d\alpha - \int_a^b f \ d\alpha_1 \right|.$$
Hence
$$\left| \int_a^b f \ d\alpha - \int_a^b f \ d\alpha_1 \right| < M\varepsilon.$$
Note that value of N becomes larger when we choose ε smaller and vice versa. So, if we let $N \to \infty$ in the above inequality, we get
$$\int_a^b f \ d\alpha = \sum_{n=1}^{\infty} c_n f(s_n).$$

□

Theorem 4.13. *Suppose α increases monotonically and $\alpha' \in \Re$ on $[a,b]$. Let f be a bounded real function on $[a,b]$. Then $f \in \Re(\alpha)$ if and only if $f\alpha' \in \Re$. In that case*

$$\int_a^b f \, d\alpha = \int_a^b f(x)\alpha'(x) \, dx.$$

Proof. Given $\varepsilon > 0$, since $\alpha' \in \Re$, there is a partition $P = \{x_0, x_1, ..., x_n\}$ such that

$$U(P,\alpha') - L(P,\alpha') < \varepsilon. \tag{4.10}$$

Applying mean value Theorem for α in the interval $[x_{i-1}, x_i],\ i = 1, 2, ..., n$; we get points $t_i \in [x_{i-1}, x_i],\ i = 1, 2, ..., n$; such that

$$\Delta\alpha_i = \alpha'(t_i)(x_i - x_{i-1}) = \alpha'(t_i)\Delta x_i. \tag{4.11}$$

Now, for any point $s_i \in [x_{i-1}, x_i]$, we have

$$\sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)|\Delta x_i \le U(P,\alpha') - L(P,\alpha') < \varepsilon. \tag{4.12}$$

Since

$$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i = \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i,$$

we have

$$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i = \sum_{i=1}^{n} f(s_i)\left(\alpha'(t_i) - \alpha'(s_i)\right)\Delta x_i.$$

So that,

$$\begin{aligned}\left|\sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i\right| &\le \sum_{i=1}^{n} |f(s_i)||\alpha'(t_i) - \alpha'(s_i)|\Delta x_i \\ &\le M\sum_{i=1}^{n} |\alpha'(t_i) - \alpha'(s_i)|\Delta x_i \\ &< M\varepsilon.\end{aligned}$$

That is,

$$-M\varepsilon < \sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i < M\varepsilon \tag{4.13}$$

The right side inequality of (4.13) gives

$$\begin{aligned}\sum_{i=1}^{n} f(s_i)\Delta\alpha_i &< \sum_{i=1}^{n}(f\alpha')(s_i)\Delta x_i + M\varepsilon \\ &\le U(P, f\alpha') + M\varepsilon.\end{aligned}$$

Since it is true for each $s_i \in [x_{i-1}, x_i]$, we can conclude

$$U(P, f, \alpha) \le U(P, f\alpha') + M\varepsilon.$$

Similarly the left side inequality of (4.13) gives

$$U(P, f\alpha') \le U(P, f, \alpha) + M\varepsilon.$$

Thus

$$|U(P, f\alpha') - U(P, f, \alpha)| \le M\varepsilon. \tag{4.14}$$

Now, (4.10) and hence (4.14) remain true if P is replaced by any refinement. Hence we conclude that

$$\left|\overline{\int_a^b} f\ d\alpha - \overline{\int_a^b} f(x)\alpha'(x)\ dx\right| \le M\varepsilon.$$

Since ε is arbitrary, we have

$$\overline{\int_a^b} f\ d\alpha = \overline{\int_a^b} f(x)\alpha'(x)\ dx.$$

Consider $L(P, f, \alpha)$ and $L(P, f\alpha')$ instead of $U(P, f, \alpha)$ and $U(P, f\alpha')$, the inequalities (4.13) will lead in a same way to the identity

$$\underline{\int_a^b} f\ d\alpha = \underline{\int_a^b} f(x)\alpha'(x)\ dx.$$

Hence under the hypothesis of the theorem,

$$\underline{\int_a^b} f\ d\alpha = \overline{\int_a^b} f\ d\alpha$$

if and only if

$$\underline{\int_a^b} f(x)\alpha'(x)\ dx = \overline{\int_a^b} f(x)\alpha'(x)\ dx.$$

And if so, we have,

$$\int_a^b f\ d\alpha = \int_a^b f(x)\alpha'(x)\ dx.$$

$\square$

Theorem 4.14. *Suppose φ is a strictly increasing continuous function that maps $[A, B]$ onto $[a, b]$. Suppose α increases monotonically on $[a, b]$ and $f \in \Re(\alpha)$ on $[a, b]$. Define β and g on $[A, B]$ by*

$$\beta(y) = \alpha(\varphi(y)) \quad \text{and} \quad g(y) = f(\varphi(y)).$$

Then, $g \in \Re(\beta)$ and

$$\int_A^B g \ d\beta = \int_a^b f \ d\alpha.$$

Proof. Since φ is a strictly increasing, continuous and maps $[A, B]$ onto $[a, b]$, we have, for each partition $P = \{x_0, x_1, ...x_n\}$ of $[a, b]$, the points y_i where $\varphi(y_i) = x_i$ constitute a partition $Q = \{y_0, y_1, ..., y_n\}$ of $[A, B]$.

Also, every partition of $[A, B]$ gives a partition of $[a, b]$.

Now, $g(y) = f(\varphi(y))$ implies that the values taken by f on $[x_{i-1}, x_i]$ is same as the values of g on $[y_{i-1}, i]$. Hence it follows that

$$U(Q, g, \beta) = U(P, f, \alpha), \quad \text{and } L(Q, g, \beta) = L(P, f, \alpha)$$

from which we get the result. (Complete the arguments!) □

If we take $\alpha(x) = x$, in Theorem 4.14, then $\beta = \varphi$. If in addition we assume $\varphi'(= \beta') \in \Re$ on $[A, B]$, Theorem 4.13 gives

$$\int_a^b f(x) \ dx = \int_A^B f(\varphi(y))\varphi'(y) \ dy.$$

4.3 Integration and Differentiation

Theorem 4.15. *Let $f \in \Re$ on $[a, b]$. For $a \leq x \leq b$, define*

$$F(x) = \int_a^x f(t) \ d(t).$$

Then F is continuous on $[a, b]$. If in addition f is continuous at a point x_0 of $[a, b]$, then F is differentiable at x_0 and

$$F'(x_0) = f(x_0).$$

Proof. Given $f \in \Re$ on $[a,b]$. So that f is bounded. Let $|f(x)| < M$ for $x \in [a,b]$. For $a \le x < y \le b$,

$$|F(x) - F(y)| = \left| \int_x^y f(t)\ dt \right| \le M(y-x).$$

So that for any $\varepsilon > 0$, we can make

$$|F(x) - F(y)| < \varepsilon$$

if we have $y - x < \varepsilon/M$, which proves the continuity(uniform) of F on $[a,b]$. Next, suppose f is continuous at x_0. Then for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(t) - f(x_0)| < \varepsilon$$

for all $t \in [a,b]$ with $|t - x_0| < \delta$. If $s,\ t \in [a,b]$ with $x_0 - \delta < s \le x_0 \le t < x_0 + \delta$, then

$$\begin{aligned}
\left| \frac{F(t) - F(s)}{t-s} - f(x_0) \right| &= \left| \frac{1}{t-s} \left(\int_a^t f(x)\ dx - \int_a^s f(x)\ dx \right) - f(x_0) \right| \\
&= \left| \frac{1}{t-s} \left(\int_a^t f(x)\ dx - \int_a^s f(x)\ dx \right) - \frac{1}{t-s} \int_s^t f(x_0)\ dx \right| \\
&= \left| \frac{1}{t-s} \int_s^t [f(x) - f(x_0)]\ dx \right| \\
&< \varepsilon
\end{aligned}$$

by the continuity of f. This is true for all $\varepsilon > 0$, so that we have

$$F'(x_0) = f(x_0).$$

□

Theorem 4.16. (**The fundamental theorem of calculus**) *If $f \in \Re$ on $[a,b]$ and if there is a differentiable function F on $[a,b]$ such that $F' = f$, then*

$$\int_a^b f(x)\ d(x) = F(b) - F(a).$$

Proof. Let $\varepsilon > 0$. There is a partition $P = \{x_0, x_1, ...x_n\}$ of $[a,b]$ such that

$$U(P,f) - L(P,f) < \varepsilon.$$

Applying mean value theorem for F in the subinterval $[x_{i-1}, x_i]$, for $i = 1, ..., n$ we can find points $t_i \in [x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = F'(t_i)(x_i - x_{i-1}) = f(t_i)\Delta x_i,$$

Then

$$\sum_{i=1}^{n} f(t_i)\Delta x_i = F(b) - F(a),$$

and Theorem 4.4(c), implies that

$$\left| F(b) - F(a) - \int_a^b f(x)\ dx \right| < \varepsilon.$$

This proves the result as ε is arbitrary. □

Theorem 4.17. *(***Integration by parts***) Suppose F and G are differentiable functions on $[a,b]$, such that $F' = f \in \Re$ and $G' = g \in \Re$ on $[a,b]$. Then*

$$\int_a^b F(x)g(x)\ d(x) = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx.$$

Proof. Define $H(x) = F(x)G(x)$. Then H is differentiable and $H' \in \Re$. Then applying Theorem 4.16 to H, we get the required result. □

4.4 Integration of Vector-Valued Functions

Definition 4.6. *Let $\boldsymbol{f}$ be a mapping of $[a,b]$ into $\mathbb{R}^k$, with component mappings $f_1,\ f_2,\ ...,\ f_k$. Let α be monotonically increasing on $[a,b]$. We say that $\boldsymbol{f} \in \Re(\alpha)$ on $[a,b]$ if $f_j \in \Re((\alpha))$ on $[a,b]$ for each $j = 1,2,...,k$. And if $\boldsymbol{f} \in \Re((\alpha))$, then*

$$\int_a^b \boldsymbol{f}\, d\alpha = \left(\int_a^b f_1\ d\alpha,\ \int_a^b f_2\ d\alpha,\ .\ .\ .\ ,\ \int_a^b f_k\ d\alpha \right),$$

which is the point in $\mathbb{R}^k$, whose jth coordinate is $\int_a^b f_j\ d\alpha$.

Note that the properties in Theorem 4.9 (a), (b) and (e), and Theorems 4.13, 4.15 and 4.16 are valid for vector-valued integrals also(Verify!). It can be obtained by applying the arguments to the coordinate integrals or component mappings. The analogue of Theorem 4.16 is stated below.

Theorem 4.18. *Suppose $\boldsymbol{f}$ and $\boldsymbol{F}$ map $[a,b]$ into $\mathbb{R}^k$, $\boldsymbol{f} \in \Re$ on $[a,b]$, and $\boldsymbol{F}' = \boldsymbol{f}$. Then*

$$\int_a^b \boldsymbol{f}(t)\ d(t) = \boldsymbol{F}(b) - \boldsymbol{F}(a).$$

Now, part (b) of Theorem 4.10 also is true for vector-valued integrals, its proof requires some more features.

Theorem 4.19. *If $\boldsymbol{f}$ maps $[a, b]$ into $\mathbb{R}^k$, and $\boldsymbol{f} \in \Re(\alpha)$ for some monotonically increasing funtion α on $[a, b]$, then $|\boldsymbol{f}\,| \in \Re(\alpha)$ and*

$$\left| \int_a^b \boldsymbol{f}\, d\alpha \right| \leq \int_a^b |\boldsymbol{f}\,|\ d\alpha.$$

Proof. If $f_1,\ f_2,\ ...,\ f_k$ are the components of $\boldsymbol{f}$, then

$$|\boldsymbol{f}\,| = (f_1^2 + \cdots f_k^2)^{1/2}.$$

Since $\boldsymbol{f} \in \Re(\alpha)$, we have $f_i \in \Re(\alpha)$ for $i = 1, 2, ..., k$.

So that by applying Theorem 4.8 (with $\phi(x) = x^2$) we get $f_i^2 \in \Re(\alpha)$ for $i = 1, 2, ..., k$. Hence

$$f_1^2 + \cdots f_k^2 \in \Re(\alpha).$$

Applying Theorem 4.8 once again with $\phi(x) = \sqrt{x}$, we have

$$|\boldsymbol{f}\,| \in \Re(\alpha).$$

Now in order to prove the inequality relating the integrals we use **Schwarz inequality** which we used to prove Theorem 3.11. Let us use the notations

$$y_i = \int_a^b f_i\ d\alpha \text{ and } \mathbf{y} = \int_a^b \boldsymbol{f}\ d\alpha,$$

, for convenience. Then

$$\mathbf{y} = (y_1,\ y_2,\ ...,\ y_k),$$

$$|\mathbf{y}|^2 = \sum_{i=1}^k y_i^2 = \sum_{i=1}^k (y_i \int_a^b f_i\ d\alpha) = \int_a^b (\sum y_i f_i)\ d\alpha.$$

By Schwarz inequality, we have

$$\sum y_i f_i(t) \leq |\mathbf{y}||\boldsymbol{f}(t)|, \quad a \leq t \leq b.$$

Hence

$$|\mathbf{y}|^2 \leq |\mathbf{y}| \int_a^b |\boldsymbol{f}|\ d\alpha,$$

which shows that

$$\left| \int_a^b \mathbf{f}\ d\alpha \right| = |\mathbf{y}| \leq \int_a^b |\boldsymbol{f}|\ d\alpha, \quad \text{if} \quad |\mathbf{y}| \neq 0.$$

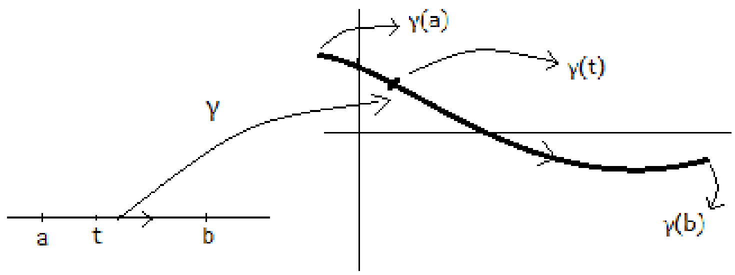
And, if $|\mathbf{y}| = 0$, then the result is obvious. □

4.5 Rectifiable Curves

In this section we consider an application of the theory of integration in geometry.

Definition 4.7. *A* **curve** *in* $\mathbb{R}^k$ *is a continuous mapping* γ *of an interval* $[a,b]$ *into* $\mathbb{R}^k$*. Then we may also say that* γ *is a curve on* $[a,b]$*.*

The curve γ *is called an* **arc** *if it is one-to-one, and if* $\gamma(a)=\gamma(b)$*, then it is a* **closed curve***.*



Note that a curve in $\mathbb{R}^k$ is a subset of $\mathbb{R}^k$, as a collection of some points. But we define it as a mapping $\gamma:[a,b]\to\mathbb{R}^k$, and the corresponding subset of $\mathbb{R}^k$ is then the range of γ.

Also note that different curves may have the same range. For example, consider the mappings

$$\gamma_1:[0,1]\to\mathbb{R}^2 \text{ and } \gamma_2:[0,1/2]\to\mathbb{R}^2$$

defined by

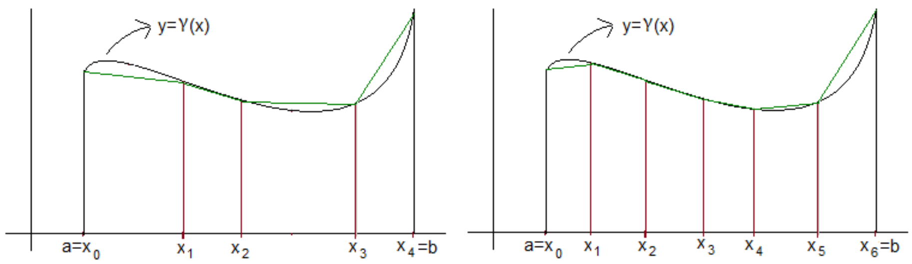
$$\gamma_1(t)=(cos\ t,\ sin\ t) \text{ and } \gamma_2(t)=(cos\ (2t),\ sin(2t)).$$

As t ranges from 0 to 1/2, $2t$ runs from 0 to 1, so that both γ_1 and γ_2 have the same range.

Let $\gamma:[a,b]\to\mathbb{R}^k$ be a curve in $\mathbb{R}^k$. Also let $P=\{x_0,x_1,...,x_n\}$ be a partition of $[a,b]$. Consider the sum

$$\sum_{i=1}^{n}|\gamma(x_i)-\gamma(x_{i-1})|.$$

Note that the i^{th} term of it is the distance betwwen the points $\gamma(x_i)$ and $\gamma(x_{i-1})$. So that this sum gives the length of the polygonal path with vertices at $\gamma(x_i),\ i=0,1,...,n$, which is always less than or equal to length of the curve. Then as the partition becomes finer and finer, this length approach the length of the curve γ.



Definition 4.8. *Let* $\gamma : [a, b] \to \mathbb{R}^k$ *be a curve in* $\mathbb{R}^k$. *For each partition* $P = \{x_0, x_1, ..., x_n\}$ *of* $[a, b]$, *define*

$$\Lambda(P, \gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|.$$

The **length** *of* γ *is defined by*

$$\Lambda(\gamma) = \sup_P \Lambda(P, \gamma),$$

where the supremum is taken over all partitions of $[a, b]$.

If $\Lambda(\gamma) < \infty$, *we say that* γ *is* **rectifiable.**

These are related to integrals. In certain cases, $\Lambda(\gamma)$ is given by an integral, for example when γ is continuously differentiable, that is γ is differentiable and its derivative $\gamma\,'$ is continous.

Theorem 4.20. *Let* $\gamma : [a, b] \to \mathbb{R}^k$ *be a curve in* $\mathbb{R}^k$. *If* $\gamma\,'$ *is continuous on* $[a, b]$, *then* γ *is rectifiable and*

$$\Lambda(\gamma) = \int_a^b |\gamma\,'(t)|\ dt.$$

Proof. Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a, b]$. We have,

$$\int_{x_{i-1}}^{x_i} \gamma\,'(t)\ dt = \gamma(x_i) - \gamma(x_{i-1}).$$

So that

$$|\gamma(x_i) - \gamma(x_{i-1})| = \left| \int_{x_{i-1}}^{x_i} \gamma\,'(t)\ dt \right| \leq \int_{x_{i-1}}^{x_i} |\gamma\,'(t)|\ dt.$$

So that

$$\Lambda(P, \gamma) \leq \int_a^b |\gamma\,'(t)|\ dt,$$

for every partition P, and hence

$$\Lambda(\gamma) \leq \int_a^b |\gamma\,'(t)|\ dt. \tag{4.15}$$

For the reverse inequality, consider $\varepsilon > 0$. Since $\gamma\,'$ is continuous on $[a, b]$, we have $\gamma\,'$ is uniformly continuous on $[a, b]$. Then there exists $\delta > 0$ such that

$$|\gamma\,'(s) - \gamma\,'(t)| < \varepsilon \ \text{ if } \ |s - t| < \delta.$$

Therefore, if we consider a partition $P = \{x_0, ..., x_n\}$ of $[a, b]$, with $\Delta(x_i) < \delta$, for all i, then we have,

$$|\gamma\,'(x_i) - \gamma\,'(t)| < \varepsilon \ \text{ for all } \ t \in [x_{i-1}, x_i].$$

Hence,

$$|\gamma\,'(t)| \leq |\gamma\,'(x_i)| + \varepsilon,$$

and

$$\begin{aligned}
\int_{x_{i-1}}^{x_i} |\gamma\,'(t)|\ dt &\leq \int_{x_{i-1}}^{x_i} (|\gamma\,'(x_i)| + \varepsilon)\ dt \\
&= |\gamma\,'(x_i)|\ \Delta x_i + \varepsilon\ \Delta x_i \\
&= \left| \int_{x_{i-1}}^{x_i} \gamma\,'(x_i)\ dt \right| + \varepsilon\ \Delta x_i \\
&= \left| \int_{x_{i-1}}^{x_i} [\gamma\,'(t) + \gamma\,'(x_i) - \gamma\,'(t)]\ dt \right| + \varepsilon\ \Delta x_i \\
&\leq \left| \int_{x_{i-1}}^{x_i} \gamma\,'(t)\ dt \right| + \left| \int_{x_{i-1}}^{x_i} [\gamma\,'(x_i) - \gamma\,'(t)]\ dt \right| + \varepsilon\ \Delta x_i \\
&\leq |\gamma(x_i) - \gamma(x_{i-1})| + 2\varepsilon\ \Delta x_i.
\end{aligned}$$

Now,

$$\begin{aligned}
\int_a^b |\gamma\,'(t)|\ dt &= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (|\gamma\,'(t)|\ dt \\
&\leq \sum_{i=1}^{n} [\ |\gamma(x_i) - \gamma(x_{i-1})| + 2\varepsilon\ \Delta x_i\ ] \\
&= \Lambda(P, \gamma) + 2\varepsilon(b - a) \\
&\leq \Lambda(\gamma) + 2\varepsilon(b - a).
\end{aligned}$$

Since ε is arbitrary, we get,

$$\int_a^b |\gamma\,'(t)|\; dt \leq \Lambda(\gamma),$$

which completes the proof.

$\square$

Exercises

1. Suppose α increases on $[a, b]$, $x_0 \in [a, b]$, α is continuous at x_0, $f(x_0) = 1$ and $f(x) = 0$, for $x \neq x_0$. Prove that $f \in \Re(\alpha)$ and $\int f \; d\alpha = 0$.

2. If $f \geq 0$, f is continuous on $[a, b]$, and if $\int_a^b f(x) \; dx = 0$, then prove that $f(x) = 0$ for all $x \in [a, b]$.

3. Compare exercises 1 and 2.

4. Suppose f is a bounded real function on $[a, b]$. Example 4.2 shows that $f^2 \in \Re$ on $[a, b]$ Does not imply that $f \in \Re$ on $[a, b]$? What will happen if we replace f^2 by f^3?

5. Let **P** be the Cantor set(described in Chapter 1). Let f be a bounded real function defined on $[0, 1]$, which is continuous at every point outside **P**. Prove that $f \in \Re$ on [0,1]. (Hint: **P** can be covered by finitely many segments whose total length can be made as small as we please. Proceed as in Theorem 4.7)

6. For $1 < s < \infty$, define
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$
(It is called the Riemann zeta function). Prove that
$$\zeta(s) = s \int_1^{\infty} \frac{[x]}{x^{s+1}} \; dx$$
and that
$$\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{x - [x]}{x^{s+1}} \; dx,$$
where $[x]$ denotes the greatest integer less than or equal to x.

Chapter 5

Sequences and Series of Functions

In this chapter we consider complex-valued(which includes real-valued also) functions.

Definition 5.1. *Suppose $\{f_n\}$, $n = 1, 2, 3, \cdots$ is a sequence of functions defined on a set E. Suppose that $\{f_n(x)\}$ converges for every $x \in E$. We can then define the function f by*

$$f(x) = \lim_{n\to\infty} f_n(x), \quad (x \in E)$$

Under these circumstances we say that $\{f_n\}$ **converges pointwise** on E and that f is the **pointwise limit**, or limit function, of $\{f_n\}$.

Similiarly, if $\sum f_n(x)$ converges for every $x \in E$, and if we define

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \quad (x \in E)$$

then we say the function f is the **pointwise limit of the series** $\sum f_n$

Remark 5.1. *Suppose $\{f_n\}$ converges pointwise to f and suppose each of the functions f_n is continuous, then is it necessary that f is continuous?*

In view of Theorem 2.4, this question can be restated in terms of limit as: for all $a \in E$, whether

$$\lim_{x\to a}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to a} f_n(x).$$

Similiarly given a sequence $\{f_n\}$ of Riemann integrable functions that converges to a function f, the question arises that whether the limit function *is integrable and whether*

$$\lim_{n\to\infty}\int f_n = \int \lim_{n\to\infty} f_n.$$

The following examples give an insight into such situations and we will see that mere convergence is not enough for the limiting function to have a property that the functions under consideration have in common.

Example 5.1.

1. *Consider*

$$f_n(x) = x^n, \quad n = 1,\ 2,\ \ldots; \quad x \in [0,1].$$

Then f_n is continuous for each n; for $x < 1$, we have $x^n \to 0$, as $n \to \infty$; and $f_n(1) = 1$ for all n. Thus the pointwise limit of $\{f_n\}$ is f, where

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x = 1, \end{cases}$$

which is not continuous.

2. *For $m = 1, 2, 3, \cdots$, $n = 1, 2, 3, \cdots$, let*

$$S_{m,n} = \frac{m}{m+n}$$

Then, for every fixed n

$$\lim_{m\to\infty} S_{m,n} = \lim_{m\to\infty}\frac{m}{m+n} = 1$$

So that

$$\lim_{n\to\infty}\lim_{m\to\infty} S_{m,n} = 1$$

On the other hand, for every fixed m,

$$\lim_{n\to\infty} S_{m,n} = 0$$

So that

$$\lim_{m\to\infty}\lim_{n\to\infty} S_{m,n} = 0$$

hence

$$\lim_{n\to\infty}\lim_{m\to\infty} S_{m,n} \neq \lim_{m\to\infty}\lim_{n\to\infty} S_{m,n}.$$

Thus the order of preference is important while applying limit.

3. *Let*

$$f_n(x) = \frac{x^2}{(1+x^2)^n} \quad (x \text{ is real}; n = 0, 1, 2, \cdots)$$

and consider

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$$

Since $f_n(0) = 0$, *we have* $f(0) = 0$.
For $x \neq 0$

$$\begin{aligned} f(x) &= x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots \\ &= \frac{x^2}{1 - \frac{1}{1+x^2}} \\ &= 1 + x^2 \end{aligned}$$

Hence

$$f(x) = \begin{cases} 0 & \text{If } \; x = 0 \\ 1 + x^2 & \text{If } \; x \neq 0 \end{cases}$$

Clearly f *is not continuous at* $x \neq 0$, *but each* f_n *is continuous. So that a convergent series of continuous functions may have discontinuous sum.*

4. *For* $m = 1, 2, 3, \cdots$, *put*

$$f_m(x) = \lim_{n\to\infty} [\cos\,(m!\pi x)]^{2n}$$

When $m!x$ *is an integer*, $\cos\,(m!\pi x)$ *is either* 1 *or* -1. *Therefore*

$$f_m(x) = 1$$

When $m!x$ *is not an integer,*

$$-1 < \cos\,(m!\pi x) < 1$$

Hence

$$f_m(x) = \lim_{n\to\infty} [\cos\,(m!\pi x)]^{2n} = 0$$

Now let

$$f(x) = \lim_{m\to\infty} f_m(x).$$

For irrational x, $m!x$ is never an integer, so that $f_m(x) = 0$ for every m; hence $f(x) = 0$. For rational x, say $x = p/q$, where p and q are integers, we see that $m!x$ is an integer if $m \geq q$, so that $f(x) = 1$. Hence

$$\lim_{m\to\infty} \lim_{n\to\infty} [\cos\ (m!\pi x)]^{2n} = \begin{cases} 0 & x \ \ irrational \\ 1 & x \ \ rational \end{cases}$$

We have thus obtained an everywhere discontinuous limit function, which is not Riemann-integrable.

5. *Let*

$$f_n(x) = \frac{\sin(nx)}{\sqrt{n}} \qquad (x \ real,\ n = 1, 2, 3, \cdots).$$

Since $|sin\ nx| \leq 1$, we get

$$f(x) = \lim_{n\to\infty} f_n(x) = 0,$$

and hence $f'(x) = 0$.

Now, f_n are differentiable,

$$f_n'(x) = \sqrt{n}\cos(nx),$$

and

$$f_n'(0) = \sqrt{n} \longrightarrow +\infty \quad as \quad n \longrightarrow \infty.$$

Therefore $f_n'(0)$ does not converge to $f'(0)$

6. *Let*

$$f_n(x) = n^2 x(1 - x^2)^n \qquad (0 \leq x \leq 1, \quad n = 1, 2, 3, \cdots)$$

For $0 < x \leq 1$, we have $0 \leq 1 - x^2 < 1$, so that

$$\lim_{n\to\infty} f_n(x) = 0.$$

Also, $f_n(0) = 0$.

Thus

$$f(x) = \lim_{n\to\infty} f_n(x) = 0 \qquad (0 \leq x \leq 1).$$

Now

$$\int_0^1 f_n(x)dx = \frac{n^2}{2(n+1)} \longrightarrow +\infty, \quad as \quad n \to \infty.$$

Hence

$$\int_0^1 f_n(x)dx \quad \textit{does not converges to} \int_0^1 f(x)dx$$

Suppose we have $f_n(x) = nx(1-x^2)^n$*, then*

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx = \lim_{n\to\infty}\frac{n}{2n+1} = \frac{1}{2}$$

Whereas

$$\int_0^1 \left[\lim_{n\to\infty} f_n(x)\right] dx = 0$$

Thus the limit of the integral need not be equal to the integral of the limit, even if both finite.

5.1 Uniform Convergence

Definition 5.2. *We say that a sequence of functions* $\{f_n\}, n = 1, 2, 3, \cdots,$ **converges uniformly** *on* E *to a function* f *if for every* $\epsilon > 0$ *there is an integer* N *such that* $n \geq N$ *implies*

$$|f_n(x) - f(x)| \leq \epsilon \tag{5.1}$$

for all $x \in E$

It is clear that every uniformly convergent sequence is pointwise convergent.

Note that $\{f_n\}$ converges pointwise on E if there is a function f such that, for every $\epsilon > 0$, and for every $x \in E$, there exists an integer N, depending on ϵ and on x, such that inequality (5.1) holds for all $n \geq N$; and $\{f_n\}$ converges uniformly on E, if for each $\epsilon > 0$, there is an integer N such that (5.1) holds for all $n \geq N$ independent of the choice of $x \in E$.

Definition 5.3. *We say that the* **series** $\sum f_n(x)$ **converges uniformly** *on* E *if the sequence* $\{s_n(x)\}$ *of partial sums defined by*

$$\sum_{i=1}^{n} f_i(x) = s_n(x)$$

converges uniformly on E*.*

Theorem 5.1. Cauchy criterian for uniform convergence
The sequence of functions $\{f_n\}$, defined on E, converges uniformly on E if and only if for every $\epsilon > 0$ there exist an integer N such that $m \geq N, n \geq N, x \in E$ implies

$$|f_n(x) - f_m(x)| \leq \epsilon.$$

Proof. Suppose $\{f_n\}$ converges uniformly on E and let f be the limit function. Then there is an integer N such that

$$|f_n(x) - f(x)| \leq \frac{\epsilon}{2}, \tag{5.2}$$

for all $n \geq N$ and for all $x \in E$. So that

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \leq \epsilon$$

if $n \geq N, m \geq N, x \in E$.

Conversely, suppose the Cauchy condition holds.

Then for each $x \in E$, the sequence $\{f_n(x)\}$ is Cauchy(as it is a sequence of complex numbers, by our general assumption that all functions considered in this chapter are complex-valued). So that $\{f_n(x)\}$ converges to, say $y \in \mathbb{C}$. Define

$$f(x) = y, \quad x \in E.$$

We prove that the sequence $\{f_n\}$ converges to f and the convergence is uniform.

Let $\epsilon > 0$ and choose N such that (5.2) holds. Fix n, and let $m \longrightarrow \infty$ in (5.2). Since $f_m(x) \longrightarrow f(x)$ as $m \longrightarrow \infty$, this gives

$$|f_n(x) - f(x)| \leq \epsilon$$

for every $n \geq N$ and for all $x \in E$, which completes the proof. □

Next two theorems give some criteria which may help us in some situations, the first one is obvious.

Theorem 5.2. *Suppose*

$$\lim_{n \to \infty} f_n(x) = f(x) \qquad (x \in E).$$

Put

$$M_n = \sup_{x \in E} |f_n(x) - f(x)|.$$

Then $f_n \longrightarrow f$ uniformly on E if and only if $M_n \longrightarrow 0$ as $n \longrightarrow 0$.

Theorem 5.3. *Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose*

$$|f_n(x)| \leq M_n \quad (x \in E,\ n = 1, 2, 3, \cdots).$$

Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

Proof. Note that M_n are non-negative reals. Suppose $\sum M_n$ converges. Then the sequence $\{s_n\}$, where $s_n = \sum_{i=1}^{n} M_i$, is a Cauchy sequence in $\mathbb{R}$, and hence it is convergent. Thus for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $m, n \geq N$

$$|s_n - s_m| < \epsilon$$

that is

$$\sum_{i=m+1}^{n} M_i < \epsilon \quad \text{for all} \quad n > m > N.$$

Now, if

$$s_n(x) = \sum_{i=1}^{n} f_i(x), \qquad x \in E$$

Then

$$|s_n(x) - s_m(x)| = \left| \sum_{i=m+1}^{n} f_i(x) \right| \leq \sum_{i=m+1}^{n} M_i < \epsilon.$$

Thus Theorem 5.1 shows that the sequence of partial sums of the series $\sum f_n$ converges uniformly on E, and hence $\sum f_n$ converges uniformly on E. □

Here the converse is not true.

5.2 Uniform Convergence and Continuity

We see that uniform convergence ensures the continuity of the limit function of sequence of continuous functions.

Theorem 5.4. *Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, and suppose that*

$$\lim_{t \to x} f_n(t) = A_n \qquad (n = 1, 2, ...)$$

Then $\{A_n\}$ converges, and

$$\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n.$$

In other words, the conclusion is that

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\ \lim_{t\to x} f_n(t).$$

Proof. Let $\epsilon > 0$. Given that $f_n \to f$ uniformly. Then by Cauchy criterion there exists a positive integer N such that for every $n \geq N$, $m \geq N$ and $t \in E$,

$$|f_n(t) - f_m(t)| \leq \epsilon.$$

Letting $t \to x$, we obtain

$$|A_n - A_m| \leq \epsilon, \quad \text{for all} \quad n, m \geq N$$

Hence $\{A_n\}$ is a Cauchy sequence of numbers. Therefore $\{A_n\}$ converges. Assume $A_n \to A$.

Now we have to prove that $\lim_{t\to x} f(t) = A$. We can write

$$\begin{aligned} |f(t) - A| &= |f(t) - f_n(t) + f_n(t) - A_n + A_n - A| \\ &\leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A| \end{aligned}$$

Since $f_n \to f$ uniformly, we can choose a large n such that, for every $t \in E$,

$$|f_n(t) - f(t)| \leq \epsilon/3.$$

Also as $A_n \to A$, we have for large values of n,

$$|A_n - A| \leq \epsilon/3.$$

Now choose n large enough such that

$$|f_n(t) - f(t)| \leq \epsilon/3 \quad \text{for all} \;\; t \in E \;\; \text{and} \;\; |A_n - A| \leq \epsilon/3.$$

Again since $\lim_{t\to x} f_n(t) = A_n$, we can choose a neighborhood N of x such that

$$|f_n(t) - A_n| \leq \epsilon/3 \;\; \text{whenever} \;\; t \in N \cap E.$$

Hence,

$$|f(t) - A| \leq \epsilon \;\; \text{whenever} \;\; t \in N \cap E,$$

which shows $\lim_{t\to x} f(t) = A$ and hence the result. □

Theorem 5.5. *If $\{f_n\}$ is a sequence of continuous function on E and if $f_n \to f$ uniformly on E, then f is continuous on E.*

Proof. Let $a \in E$. Given that each function f_n is continuous on E. Therefore

$$\lim_{t \to a} f_n(t) = f_n(a).$$

Now,

$$\begin{aligned}
\lim_{t \to a} f(t) &= \lim_{t \to a} \lim_{n \to \infty} f_n(t) \\
&= \lim_{n \to \infty} \lim_{t \to a} f_n(t), \text{ using Theorem 5.4} \\
&= \lim_{n \to \infty} f_n(a) \\
&= f(a)
\end{aligned}$$

Hence f is continuous. □

Theorem 5.6. *Suppose K is compact and*

(a) $\{f_n\}$ *is a sequence of real continuous functions on K.*

(b) $\{f_n\}$ *converges pointwise to a continuous function f on K.*

(c) $f_n(x) \geq f_{n+1}(x) \quad \forall x \in K, \quad n = 1, 2, ...$

Then $f_n \to f$ uniformly on K.

Proof. Put

$$g_n = f_n - f.$$

Then g_n is continuous on K, which is obtained from the given hypothesis that f and f_n are continuous. Also $g_n \to 0$ pointwise and $g_n(x) \geq g_{n+1}(x)$. Now we have to prove that $g_n \to 0$ uniformly.

Since $f_n(x)$ is decreasing and $f_n(x) \to f(x)$, we have $f_n(x) \geq f(x)$ for all n, so that, $g_n(x) \geq 0$, for all $x \in K$.
Let $\epsilon > 0$. Define

$$K_n = \{x \in K : g_n(x) \geq \epsilon\}$$

Since g_n is continuous, K_n is closed in K. Also since K is compact, K_n is compact which is from the result 'Every closed subset of a compact subset is compact'.

Now,

$$\bigcap_{n=1}^{\infty} K_n = \phi.$$

For, we have for all $x \in K$ $g_n(x) \to 0$ as $n \to \infty$.
Therefore there exist $p \in N$ such that $g_n(x) < \epsilon \quad \forall\ n \geq p$.

In particular, $g_p(x) < \epsilon$, that is, $x \notin K_p$. So that x cannot be in $\bigcap_{n=1}^{\infty} K_n$.

Thus $\{K_{n_1},\ K_{n_2}, \ldots\}$ is a collection of compact sets such that

$$\bigcap_{n=1}^{\infty} K_n = \phi.$$

Then there must exist a finite collection $K_{n_1}, K_{n_2}, ..., K_{n_k}$, such that

$$\bigcap_{n=1}^{k} K_n = \phi.$$

Since $g_n(x) \geq g_{n+1}(x)$, we have $K_n \supset K_{n+1}$. So that if $N = \max\{n_1, n_2...n_k\}$, then $K_N \subset K_{n_i}$, for all i and

$$\bigcap_{i=1}^{k} K_{n_i} = K_N.$$

That is, $K_N = \phi$ for some $N \in \{1, 2, \ldots\}$. Thus we get, $g_N(x) < \epsilon$, for all $x \in K$
But we have $g_n(x) \leq g_N(x)$, for all $x \in K$ and for all $n \geq N$, which implies that

$$g_n(x) < \epsilon, \quad \text{for all} \quad x \in K \text{ and for all} \quad n \geq N.$$

Thus we have

$$0 \leq g_n(x) < \epsilon, \quad \text{for all} \quad x \in K \quad \text{and} \quad \forall n \geq N.$$

That is $g_n \to 0$ uniformly and hence $f_n \to f$ uniformly of K and thereby the result. □

Definition 5.4. *Let X be a metric space, $\mathcal{C}(X)$ denote the set of all complex valued, continuous, bounded functions with domain X. We define the* **supremum norm** *for each $f \in \mathcal{C}(X)$,*

$$\|f\| = \underset{x \in X}{sup}\ |f(x)|.$$

Then $\|f\| < \infty$, since f is bounded.

Note that it defines a norm on $\mathcal{C}(X)$. If $h = f + g$, Then,

$$\begin{aligned} |h(x)| &= |(f+g)(x)| \\ &= |f(x) + g(x)| \\ &\leq |f(x)| + |g(x)| \end{aligned}$$

Thus,

$$|h(x)| \leq \|f\| + \|g\| \quad \forall x \in X.$$

So that,

$$\sup_{x \in X} |h(x)| \leq \|f\| + \|g\|,$$

and hence

$$\|f + g\| \leq \|f\| + \|g\|.$$

Now, for $f, g \in \mathcal{C}(X)$, define $d(f, g) = \|f - g\|$. Then d is a metric on $\mathcal{C}(X)$. or $(\mathcal{C}(X),\ d)$ is a metric space.

Theorem 5.7. *A sequence $\{f_n\}$ converges to f with respect to the metric of $\mathcal{C}(X)$ if and only if $f_n \to f$ uniformly on X.*

Proof. *Necessary condition:* Suppose $f_n \to f$ with respect to the metric d of $\mathcal{C}(X)$.
Therefore, for every $\epsilon > 0$ there exist $n \in \mathbb{N}$ such that

$$d(f_n, f) < \epsilon, \quad \forall\ n \geq N.$$

That is

$$||f_n - f|| < \epsilon, \quad \forall n \geq N,$$

or,

$$\sup_{x \in X} |f_n(x) - f(x)| < \epsilon, \quad \forall n \geq N,$$

which shows

$$|f_n(x) - f(x)| < \epsilon, \quad \forall n \geq N \text{ and } x \in X.$$

Therefore, $f_n \to f$ uniformly on X.

Sufficient condition: Suppose $f_n \to f$ uniformly.

For a given $\epsilon > 0$, there exist a k such that

$$|f_n(x) - f(x)| \leq \epsilon, \quad \forall x \in X \text{ and } n \geq k$$

Therefore,

$$\sup_{x \in X} |f_n(x) - f(x)| \leq \epsilon, \quad \forall n \geq k$$

Hence,

$$||f_n - f|| \quad \leq \quad \epsilon, \quad \forall n \geq k,$$

that is

$$d(f_n, f) \quad \leq \quad \epsilon, \quad \forall n \geq k.$$

Hence, $f_n \to f$ with respect to the metric of $\mathcal{C}(X)$ and follows the desired result.

□

Definition 5.5. *A metric space (X, d) is called complete if every Cauchy sequence in X converges in X under the metric d.*

Theorem 5.8. *$\mathcal{C}(X)$ is a complete metric space under the above metric.*

Proof. Let (f_n) be a Cauchy sequence in $\mathcal{C}(X)$. Then given $\epsilon > 0$, there exists a natural number N such that $d(f_n, f_m) < \epsilon, \quad \forall n, m \geq N$. Hence,

$$||f_n - f_m|| < \epsilon, \quad \forall\ n, m \geq N,$$

that is,

$$\sup_{x \in X} |f_n(x) - f_m(x)| < \epsilon, \quad \forall\ n, m \geq N$$

and hence

$$|f_n(x) - f_m(x)| < \epsilon, \quad \forall\ x \in X \text{ and } \forall\ n, m \geq N.$$

Then by Cauchy criterion, we have $\{f_n\}$ converges uniformly to, say f.

It remains to prove $f \in \mathcal{C}(X)$. Since each f_n is continuous and f_n converges to f uniformly, we get f is continuous.
As $f_n \to f$ uniformly, there exists $p \in N$ such that

$$|f_n(x) - f(x)| \quad \leq \quad 1, \quad \forall\ n \geq p \text{ and } \forall\ x \in X.$$

$$\text{In particular, } |f_p(x) - f(x)| \quad \leq \quad 1, \quad \forall\ x \in X,$$

$$\text{that is } f_p(x) - 1 \leq f(x) \leq f_p(x) + 1.$$

Also since, $f_p \in \mathcal{C}(X)$, then f_p is bounded. Therefore there exist $M \in \mathbb{N}$ such that

$$|f_p(x)| \leq M, \quad \forall\ x \in X.$$

Hence,

$$-M \leq f_p(x) \leq M,$$

so that,

$$-M - 1 \leq f(x) \leq M + 1$$

$$|f(x)| \leq M + 1 \quad \forall x \in X$$

Hence, $f \in \mathcal{C}(X)$.
This implies $\mathcal{C}(X)$ is complete and hence the result. □

5.3 Uniform Convergence and Integration

Theorem 5.9. *Let α be monotonically increasing on $[a,b]$. Suppose $f_n \in \Re(\alpha)$ on $[a,b]$ for $n = 1,2,3,\ldots,$ and suppose $f_n \to f$ uniformly on $[a,b]$. Then $f \in \Re(\alpha)$ on $[a,b]$, and*

$$\int_a^b f\ d\alpha = \lim_{n\to\infty} \int_a^b f_n\ d\alpha$$

Proof. Suppose $f_n \to f$ uniformly and suppose each function is real.
Let,

$$\epsilon_n \quad = \quad \sup_{x\in[a,b]} |f_n(x) - f(x)|$$

Hence we have,

$$|f(x) - f_n(x)| \leq \epsilon_n, \qquad \forall\ x \in [a,b], \quad \forall\ n.$$

That is

$$f_n(x) - \epsilon_n \leq f(x) \leq f_n(x) + \epsilon_n, \quad \forall\ x \in [a,b], \quad \forall\ n.$$

Hence we get,

$$\int_a^{\overline{b}} f\ d\alpha\ \leq \int_a^b (f_n + \epsilon_n)\ d\alpha$$

and

$$\int_a^b (f_n - \epsilon_n)\ d\alpha\ \leq \int_{\underline{a}}^b f\ d\alpha.$$

Combining these results we get,

$$\int_a^b (f_n - \epsilon_n)\ d\alpha\ \leq \int_{\underline{a}}^b f\ d\alpha\ \leq \int_a^{\overline{b}} f\ d\alpha\ \leq \int_a^b (f_n + \epsilon_n)\ d\alpha. \tag{5.3}$$

So that

$$0 \leq \int_a^{\overline{b}} f\ d\alpha - \int_{\underline{a}}^b f\ d\alpha\ \leq 2\epsilon_n[\alpha(b) - \alpha(a)].$$

Therefore, $f \in \mathcal{R}(\alpha)$.
Again from (5.3), we have

$$\int_a^b f_n\ d\alpha - \int_a^b \epsilon_n\ d\alpha \leq \int_a^b f\ d\alpha \leq \int_a^b f_n\ d\alpha + \int_a^b \epsilon_n\ d\alpha.$$

Subtracting $\int_a^b f_n \, d\alpha$ from each, we get

$$-\epsilon_n(\alpha(b) - \alpha(a)) \leq \int_a^b f \, d\alpha - \int_a^b f_n \, d\alpha \leq \epsilon_n(\alpha(b) - \alpha(a))$$

That is,

$$\left| \int_a^b f \, d\alpha - \int_a^b f_n \, d\alpha \right| \leq \epsilon_n(\alpha(b) - \alpha(a))$$

Since $\epsilon_n \to 0$ as $n \to \infty$,

$$\left| \int_a^b f \, d\alpha - \int_a^b f_n \, d\alpha \right| \to 0, \quad \text{as} \quad n \to \infty$$

Therefore,

$$\int_a^b f \, d\alpha = \lim_{n \to \infty} \int_a^b f_n \, d\alpha$$

and hence the result. □

Corollary 5.1. *If $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$ and if the series*

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (a \leq x \leq b),$$

converge uniformly on $[a, b]$, then

$$\int_a^b f \, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n \, d\alpha$$

Proof. Given that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to f. Therefore the sequence of n^{th} partial sums, $S_n = f_1 + f_2 + ... + f_n$ converges uniformly to f.
We know that if $f, g \in \mathcal{R}(\alpha)$ then $f + g \in \mathcal{R}(\alpha)$.
Since $f_1, f_2, ..., f_n \in \mathcal{R}(\alpha)$ then, $S_n \in \mathcal{R}(\alpha)$.

As S_n converges uniformly to f, then by the Theorem 5.9, $f \in \mathcal{R}(\alpha)$ and

$$\begin{aligned}
\int_a^b f\ d\alpha &= \lim_{n\to\infty}\int_a^b S_n\ d\alpha \\
&= \lim_{n\to\infty}\int_a^b \sum_{i=1}^{n} f_i\ d\alpha \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\int_a^b f_i\ d\alpha, \quad \text{by the properties of integration} \\
&= \sum_{i=1}^{\infty}\int_a^b f_i\ d\alpha.
\end{aligned}$$

□

5.4 Uniform Convergence and Differentiation

Example 5.1(5) shows that the function

$$f_n(x) = \frac{sin\ nx}{\sqrt{n}}$$

converges to 0, as $n \to \infty$. This convergence is uniform. But even then the sequence of derivatives

$$f_n'(x) = \sqrt{n}\ cos\ nx$$

does not converge to a point.

Thus the uniform convergence of the sequence $\{f_n\}$ to a function f is not enough for its derivatives f_n' to converge to f'.

Theorem 5.10. *Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ converges uniformly on $[a, b]$, to a function f, and*

$$f'(x) = \lim_{n\to\infty} f_n'(x) \qquad (a \leq x \leq b)$$

Proof. Let $\epsilon > 0$. Given that $\{f_n(x_0)\}$ converges. Therefore, it is a Cauchy sequence. Hence, by definition, there exist $N_1 \in \mathbb{N}$ such that

$$|f_n(x_0) - f_m(x_0)| < \epsilon/2 \quad \forall\ m, n \geq N_1.$$

Also given that, $\{f_n'\}$ converges uniformly. Therefore, it is a Cauchy sequence. Hence, by definition, there exist $N_2 \in \mathbb{N}$ such that

$$|f_n'(x) - f_m'(x)| < \epsilon/2(b-a) \quad \forall\, m, n \geq N_2 \text{ and } \forall\, a \leq x \leq b.$$

Then for $N = \max\{N_1, N_2\}$, we have

$$|f_n(x_0) - f_m(x_0)| < \epsilon/2 \quad \text{and} \quad |f_n'(x) - f_m'(x)| < \epsilon/2(b-a),$$

for all $m, n \geq N$, and for all $x \in [a, b]$.

Now applying Generalised Mean Value theorem to $(f_n - f_m)$ in the interval $[x, t] \subset [a, b]$, there exists $y \in [x, t]$ such that

$$\begin{aligned}
|(f_n - f_m)(t) - (f_n - f_m)(x)| &\leq |t - x|\, |(f_n - f_m)'(y)| \\
&= |t - x|\, |f_n'(y) - f_m'(y)|, \\
&< |t - x|\, \epsilon/2(b-a), \quad \forall n, m \geq N \quad (5.4) \\
&< \epsilon/2, \quad \forall n, m \geq N.
\end{aligned}$$

Hence,

$$|f_n(t) - f_m(t) - f_n(x) - f_m(x)| < \epsilon/2, \quad \forall\, n, m \geq N \text{ and } \forall\, x,\ t \in [a, b].$$

Also,

$$\begin{aligned}
|f_n(x) - f_m(x)| &= |f_n(x) - f_m(x) - f_n(x_0) + f_n(x_0) - f_m(x_0) + f_m(x_0)| \\
&\leq |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)| \\
&< \epsilon/2 + \epsilon/2 = \epsilon, \quad \forall\, n, m \geq N \text{ and } \forall\, x \in [a, b].
\end{aligned}$$

Thus

$$|f_n(x) - f_m(x)| < \epsilon \quad \forall\, n, m \geq N \text{ and } \forall\, x \in [a, b]$$

Therefore by Cauchy criterion, we have f_n converges uniformly on $[a, b]$, say, to f.

Now, fix a point $x \in [a, b]$. Define

$$\phi_n(t) = \frac{f_n(x) - f_n(t)}{x - t} \quad \text{and} \quad \phi(t) = \frac{f(x) - f(t)}{x - t} \qquad \forall\, t \in [a, b],\ t \neq x$$

Therefore,

$$\lim_{t \to x} \phi_n(t) = f_n'(x)$$

Hence,

$$|\phi_n(t) - \phi_m(t)| = \frac{|f_n(x) - f_n(t) - f_m(x) + f_m(t)|}{|x - t|}$$

By equation (5.4), we get,

$$\begin{aligned}
|\phi_n(t) - \phi_m(t)| &\leq \frac{\epsilon|x - t|}{2(b-a)|x-t|}, \quad \forall\, n, m \geq N, \quad \forall\, t \in [a, b] \\
|\phi_n(t) - \phi_m(t)| &< \frac{\epsilon}{2(b-a)}, \quad \forall\, n, m \geq N, \quad \forall\, t \in [a, b]
\end{aligned}$$

Hence by Cauchy criterion, $\{\phi_n\}$ converges uniformly.
Also by the given hypothesis, $\{f_n\} \to f$ we have,

$$\lim_{n\to\infty} \phi_n(t) = \frac{f(x) - f(t)}{x - t} = \phi(t) \qquad a \leq x \leq b, \quad t \neq x$$

We obtained that, $\phi_n \to \phi$ uniformly. Now combining the above results and the Theorem 5.4, we get,

$$\lim_{t\to x}\ \lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty}\ \lim_{t\to x} \phi_n(t).$$

That is

$$\lim_{t\to x} \phi(x) = \lim_{n\to\infty} f'_n(x)$$

Hence,

$$f'(x) = \lim_{n\to\infty} f'_n(x) \qquad (a \leq x \leq b)$$

and thereby the result. □

Theorem 5.11. *There exists a real continuous function on the real line which is nowhere differentiable.*

Proof. Define

$$\phi(x) = |x|, \qquad -1 \leq x \leq 1.$$

Extend the function ϕ to all real x by defining as

$$\phi(x + 2) = \phi(x).$$

Then for all $s,\ t$ we have

$$|\phi(s) - \phi(t)| \leq |x - t|.$$

For if $s,\ t \in [-1, 1]$, then we have

$$\phi(s) = |s| \qquad \text{and} \quad \phi(t) = |t|.$$

Hence,

$$|\phi(s) - \phi(t)| = |\ |s| - |t|\ | \leq |s - t|.$$

From the extension it follows that

$$|\phi(s) - \phi(t)| \leq |s - t| \qquad \forall\ s,\ t \in \mathbb{R},$$

so that, ϕ is continuous on $\mathbb{R}$.
Define,

$$f(x) = \sum_{n=0}^{\infty} (3/4)^n \phi(4^n\ x).$$

We have,

$$(3/4)^n \phi(4^n\ x) \leq (3/4)^n$$

Since $\sum_{n=0}^{\infty} (3/4)^n$ is an infinite series with common ratio $(3/4) < 1$, $\sum_{n=0}^{\infty} (3/4)^n$ converges.

Therefore by Theorem 5.3(Weierstrass test),

$$\sum_{n=0}^{\infty} (3/4)^n \phi(4^n\ x)$$

converges uniformly.

Hence, $f(x)$ is well defined. Now,

$$f_n(x) = (3/4)^n \phi(4^n\ x)$$

is continuous for all n.
Let

$$S_n = f_1 + f_2 + \cdots + f_n.$$

Since the series converges uniformly to f and f_n is continuous, then $S_n \to f$ uniformly.
Also since each S_n is continuous, f is continuous.
Claim: f is not differentiable at any $x \in \mathbb{R}$.
Fix a positive integer 'm' and put

$$\delta_m = \pm(1/2)\ 4^{-m},$$

where the sign is choosen so that no integer lies between $4^m\ x$ and $4^m\ (x+\delta_m)$. This is possible since $|4^m\ \delta_m| = 1/2$.

Define

$$\gamma_n = \frac{\phi(4^n(x+\delta_m)) - \phi(4^n\ x)}{\delta_m}.$$

When $n > m$, we have $4^n\delta_m$ is an even integer, so that $\phi(4^n x + 4^n\delta_m) = \phi(4^n x)$ and hence $\gamma_n = 0$.

Now if $0 \leq n \leq m$,

$$|\gamma_n| = \left|\frac{\phi(4^n(x+\delta_m)) - \phi(4^n\ x)}{\delta_m}\right| \leq \left|\frac{4^n\delta_m}{\delta_m}\right| = 4^n.$$

Also,

$|\gamma_m| = 4^m$, so that

$$\begin{aligned}
\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| &= \left|\frac{\sum_{n=0}^{\infty}(3/4)^n\phi(4^n(x+\delta_m)) - \sum_{n=0}^{\infty}(3/4)^n\phi(4^n x)}{\delta_m}\right| \\
&= \left|\frac{\sum_{n=0}^{\infty}(3/4)^n[\phi(4^n(x+\delta_m)) - \phi(4^n x)]}{\delta_m}\right| \\
&= \left|\sum_{n=0}^{\infty}(3/4)^n\gamma_n\right| \\
&= \left|\sum_{n=0}^{m}(3/4)^n\gamma_n\right| \quad \text{since } \gamma_n = 0, \quad \text{for} \quad n > m \\
&= \left|3^m + \sum_{n=0}^{m-1}(3/4)^n\gamma_n\right| \quad \text{since } \gamma_m = 4^m \\
&\geq \ 3^m - \sum_{n=0}^{m-1} 3^n \\
&= \ 3^m - \frac{(3^m - 1)}{3-1} \\
&= \ \frac{3^m + 1}{2}
\end{aligned}$$

Since, $\frac{3^m+1}{2} \to \infty$ as $m \to \infty$, and $\delta_m \to 0 \iff m \to \infty$, we have,

$$\lim_{\delta_m \to 0} \frac{f(x+\delta_m) - f(x)}{\delta_m}$$

does not exist. That is $f(x)$ is not differentiable. Since x is arbitrary, f is not differentiable and hence the desired result. □

5.5 Equicontinous of Families of Functions

We know the result that every bounded sequence of complex numbers has a convergent subsequence. But it is not true for sequence of functions. Now, regarding boundedness of sequence of functions, two types of boundedness can be defined.

Definition 5.6. *Let $\{f_n\}$ be a sequence of functions defined on a set E. We say that $\{f_n\}$ is* **pointwise bounded** *on E if for each $x \in E$, the sequence $\{f_n(x)\}$ is bounded, that is, for each $x \in E$, there exist a finite number M_x such that*

$$|f_n(x)| < M_x \qquad n = 1, 2, 3, ...$$

Thus if $\{f_n\}$ is pointwise bounded on E, then we can define a finite valued function ϕ on E such that

$$|f_n(x)| < \phi(x) \qquad x \in E, \quad n = 1, 2, 3, ...,$$

where $\phi(x) = M_x$.

We say that $\{f_n\}$ is **uniformly bounded** *on E, if there exist a number M such that*

$$|f_n(x)| \leq M \quad \textit{for all} \quad x \in E, \quad \textit{and} \quad n = 1, 2, 3, ...$$

Clearly uniform boundedness implies pointwise boundedness. But converse need not be true.

If $\{f_n\}$ is pointwise bounded on E, there need not exist a subsequence of $\{f_n\}$ converging pointwise. But if E_1 is a countable subset of E, it is always possible to find a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E_1$

In fact, even if $\{f_n\}$ is a uniformly bounded sequence of continuous functions on a compact set E, there need not exist a subsequence which converges pointwise on E.

Example 5.2. *Let*

$$f_n(x) = \sin nx \qquad (0 \leq x \leq 2\pi, n = 1, 2, ...).$$

Since

$$|\sin nx| \leq 1,$$

$\{f_n\}$ is uniformly bounded. Suppose there exist a subsequence $\{n_k\}$ such that $\{\sin n_k x\}$ converges, for every $x \in [0, 2\pi]$. Therefore

$$\lim_{k \to \infty} (\sin n_k x - \sin n_{k+1} x) = 0 \qquad (0 \leq x \leq 2\pi)$$

Therefore

$$\lim_{k\to\infty} (\sin n_k x - \sin n_{k+1} x)^2 = 0 \qquad (0 \le x \le 2\pi)$$

Hence

$$\int_0^{2\pi} \lim_{k\to\infty} (\sin n_k x - \sin n_{k+1} x)^2 dx = 0$$

By dominated convergence theorem

$$\int_0^{2\pi} \lim_{k\to\infty} (\sin n_k x - \sin n_{k+1} x)^2 dx = \lim_{k\to\infty} \int_0^{2\pi} (\sin n_k x - \sin n_{k+1} x)^2 dx$$

But calculating(Verify!) the integral, we get

$$\lim_{k\to\infty} \int_0^{2\pi} (\sin n_k x - \sin n_{k+1} x)^2 dx = 2\pi.$$

Therefore our assumption was wrong, hence there does not exist a subsequence which converges pointwise on $[0, 2\pi]$.

Also note that, in case of functions, every convergent sequence need not contain a uniformly convergent subsequence.

Example 5.3. *Let*

$$f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2} \qquad (0 \le x \le 1, n = 1, 2, 3, ...).$$

Then $|f_n(x)| \le 1$*, so that* $\{f_n\}$ *is uniformly bounded on* $[0, 1]$*. Also*

$$\lim_{n\to\infty} f_n(x) = 0 \qquad (0 \le x \le 1)$$

but

$$f_n\left(\frac{1}{n}\right) = 1.$$

So that no subsequence can converge uniformly on $[0, 1]$.

Definition 5.7. *A family* $\mathcal{F}$ *of complex functions* f *defined on a set* E *in a metric space* X *is said to be* **equicontinuous** *on* E *if for every* $\epsilon > 0$ *there exist a* $\delta > 0$ *such that*

$$|f(x) - f(y)| < \epsilon$$

whenever $d(x, y) < \delta, x \in E, y \in E$*, and* $f \in \mathcal{F}$*. Here* d *denote the metric of* X.

Note that every member of an equicontinuous family is uniformly continuous.

Compare the concepts of continuity, uniform continuity and equicontinuity!!!

Theorem 5.12. *If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, then$\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E$.*

Proof. Given E is countable. Let

$$E = \{x_1,\ x_2,\ ...\}.$$

Now, by the hypothesis, $\{f_n(x_1)\}$ is a bounded sequence of complex numbers, so that it has a convergent subsequence, say $\{f_{1,k} :\ k = 1, 2, ...\}$. That is $\{f_{1,k}(x_1)\}$ converges as $k \to \infty$.

Now, Since $\{f_{1,k}\}$ is a subsequence of $\{f_n\}$, we have $\{f_{1,k}(x_2)\}$ is bounded, and hence it has a convergent subsequence, say, $\{f_{2,k}$. Thus $\{f_{2,k}(x_2)\}$ converges as $k \to \infty$.

Proceeding like this, we get a sequence of sequences

$$\begin{array}{lccccccccccc}
S_1 : & f_{1,1} & f_{1,2} & f_{1,3} & \cdot & \cdot & \cdot & f_{1,k} & \cdot & \cdot & \cdot \\
S_2 : & f_{2,1} & f_{2,2} & f_{2,3} & \cdot & \cdot & \cdot & f_{2,k} & \cdot & \cdot & \cdot \\
S_3 : & f_{3,1} & f_{3,2} & f_{3,3} & \cdot & \cdot & \cdot & f_{3,k} & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

having the following properties.

(a) S_n is a subsequence of S_{n-1} for $n = 2, 3, ...$
(b) $\{f_{n,k}(x_n\}$ converges as $k \to \infty$.

(c) The order in which the functions appear is same in all sequences S_n.
Now, consider the sequence

$$S = f_{1,1},\ f_{2,2},\ f_{3,3}, \cdots$$

By our construction, the tail of S, excluding the first $n-1$ terms, is a subsequence of S_n, for each n. Hence we have $\{f_{n,n}(x_j)\}$ converges as $n \to \infty$, for every $x_j \in E$. □

Theorem 5.13. *If K is a compact metric space, $f_n \in \mathcal{C}(K)$ for $n = 1, 2, ...,$ and if $\{f_n\}$ converges uniformly on K, then $\{f_n\}$ is equicontinuous on K.*

Proof. Let $\varepsilon > 0$. Since $\{f_n\}$ converges uniformly on K, there is an integer N such that

$$|f_n(x) - f_N(x)| < \varepsilon, \quad \text{for all} \quad n > N,$$

which implies that the supremum norm

$$\|f_n - f_N\| < \varepsilon, \quad \text{for all} \quad n > N.$$

Also since K is compact, each f_n is uniformly continuous on K. So, we can find a $\delta > 0$ such that

$$|f_i(x) - f_i(y)| < \varepsilon \tag{5.5}$$

for $x,\ y \in K$ such that $d(x, y) < \delta$ and $1 \le i \le N$.

Thus, if $n > N$ and $d(x, y) < \delta$,

$$|f_n(x) - f_n(y)| \le |f_n(x) - f_N(y)| + |f_N(y) - f_n(y)| < 3\varepsilon. \tag{5.6}$$

Inequalities (5.5) and (5.6) prove the theorem. □

Theorem 5.14. *If K is a compact metric space, $f_n \in \mathcal{C}(K)$ for $n = 1, 2, ...,$ and if $\{f_n\}$ is pointwise bounded and equicontinuous on K, then*

(a) $\{f_n\}$ is uniformly bounded on K and

(b) $\{f_n\}$ contains a uniformly convergent subsequence.

5.6 The Stone-Weierstrass Theorem

The next theorem due to Weierstrass gives every continuous complex valued function on $[a, b]$ is the uniform limit of a sequence of polynomial functions. We will then consider Stone's generalization of this result.

Theorem 5.15. *(Weierstrass) If f is a continuous complex function on $[a, b]$, there exists a sequence of polynomials P_n such that*

$$\lim_{n\to\infty} P_n(x) = f(x)$$

uniformly on $[a, b]$. If is real then P_n can be chosen real.

Proof. It is enough to prove the theorem in the case $[a, b] = [0, 1]$. Now, f is continuous on $[0, 1]$ implies that f is uniformly continuous on $[0, 1]$.

Also, assume that

$$f(0) = f(1) = 0.$$

For, consider the function

$$g(x) = f(x) - f(0) - x[f(1) - f(0)], \quad 0 \leq x \leq 1.$$

Here $g(0) = g(1) = 0$, and if g is the uniform limit of a sequence of polynomials, then

$$f(x) = g(x) + f(0) + x[f(1) - f(0)], \quad 0 \leq x \leq 1$$

shows f also can be obtained as a uniform limit of a sequence of polynomial functions. So, it is enough if the theorem is proved for f satisfying $f(0) = f(1) = 0$.

Now we extend f to the whole real line without affecting the uniform continuity. It can be done by defining $f(x) = 0$ for $x \notin [0, 1]$. Thus we have f is uniformly continuous on $\mathbb{R}$. Define

$$Q_n(x) = c_n(1 - x^2)^n, \quad n = 1, 2, 3, ..., \tag{5.7}$$

where c_n is chosen so that

$$\int_{-1}^{1} Q_n(x)\ dx = 1, \quad n = 1, 2, 3, ..., \tag{5.8}$$

Then we have

$$c_n < \sqrt{n}.$$

Then for any $\delta > 0$,

$$Q_n(x) \leq \sqrt{n}(1 - \delta^2)^n, \quad \delta \leq |x| \leq 1, \quad n = 1, 2, 3, ..., \tag{5.9}$$

This shows $Q_n \to 0$ uniformly in $\delta \leq |x| \leq 1$. Now define

$$P_n(x) = \int_{-1}^{1} f(x + t)Q_n(t)\ dt, \quad 0 \leq x \leq 1. \ \ n = 1, 2, 3, ... \tag{5.10}$$

By applying the technique of change of variable, we get,

$$P_n(x) = \int_{0}^{1} f(t)Q_n(t - x)\ dt, \quad 0 \leq x \leq 1. \ \ n = 1, 2, 3, ..., \tag{5.11}$$

which shows P_n is a polynomial in x. Thus we have a sequence of polynomials $\{P_n\}$, which are real if f is a real valued function. We prove $\{P_n\}$ converges to f uniformly on $[0,1]$.

Let $\varepsilon > 0$. Since f is uniformly continuous, we can find a $\delta > 0$ such that

$$|y-x| < \delta \quad \text{implies} \quad |f(y)-f(x)| < \frac{\varepsilon}{2}.$$

Let $M = \sup |f(x)|$. For $0 \le x \le 1$, we have

$$\begin{aligned}
|P_n(x) - f(x)| &= \left| \int_{-1}^{1} [f(x+t) - f(x)] Q_n(t)\ dt \right|, \quad \text{by (5.8) and (5.10)} \\
&\le \int_{-1}^{1} |f(x+t) - f(x)| Q_n(t)\ dt, \ \text{since} \quad Q_n(x) \ge 0 \\
&= \int_{-1}^{-\delta} |f(x+t) - f(x)| Q_n(t)\ dt \\
&\quad + \int_{-\delta}^{\delta} |f(x+t) - f(x)| Q_n(t)\ dt \\
&\quad + \int_{\delta}^{1} |f(x+t) - f(x)| Q_n(t)\ dt \\
&\le 2M \int_{-1}^{-\delta} Q_n(t)\ dt + \frac{\varepsilon}{2} \int_{-\delta}^{\delta} Q_n(t)\ dt + 2M \int_{\delta}^{1} Q_n(t)\ dt \\
&\le 4M\sqrt{n}(1-\delta^2)^n + \frac{\varepsilon}{2}, \quad \text{by (5.9)} \\
&< \varepsilon
\end{aligned}$$

for large enough n, which proves the theorem □

Corollary 5.2. *For every interval $[-a,a]$, there is a sequence of real polynomials P_n such that*

$$P_n(0) = 0 \quad \textit{and} \quad \lim_{n\to\infty} P_n(x) = |x|$$

uniformly on $[-a,a]$.

Proof. Since $|x|$ is a continuous function on any interval $[-a,a]$, by Theorem 5.15, there exists a sequence$\{P_n^*\}$ of real polynomials which converges to $|x|$ uniformly on $[-a,a]$.

In particular

$$P_n^* \to 0, \quad \text{as} \quad n \to \infty.$$

Now, for each $n = 1,2,3,...,$

$$P_n(x) = P_n^*(x) - P_n^*(0)$$

is a polynomial and and

$$\lim_{n\to\infty} P_n(x) = |x|$$

uniformly on $[-a, a]$. □

It is because of some particular properties of polynomials which make Weierstrass theorem possible. Let us identify these properties and then discuss Stone's generalization of Weierstrass theorem.

Definition 5.8. *A family $\mathcal{A}$ of complex functions defined on a set E said to be an* **algebra** *if for every $f \in \mathcal{A}$, $g \in \mathcal{A}$ and for every complex constant c, we have*

(i) $f + g \in \mathcal{A}$,

(ii) $fg \in \mathcal{A}$ *and*

(iii) $cf \in \mathcal{A}$.

That is if $\mathcal{A}$ is closed under addition, multiplication and scalar multiplication. For defining algebra of real functions (iii) is only required to hold for real numbers.

For example the set of all polynomials is an algebra(verify!).

Definition 5.9. *$\mathcal{A}$ is said to be* **uniformly closed** *if for every sequence $\{f_n : n = 1, 2, ...\}$ of functions in $\mathcal{A}$, such that $f_n \to f$ uniformly on E, the function $f \in \mathcal{A}$. That is $\mathcal{A}$ is uniformly closed if it contains the uniform limits of all sequences which converge uniformly.*

The **uniform closure** *of $\mathcal{A}$ is the set of all functions which are limits of uniformly convergent sequences in $\mathcal{A}$.*

We will see that Weierstrass Theorem shows that the set of continuous functions on $[a, b]$ is the uniform closure of the algebra of polynomials on $[a, b]$.

Theorem 5.16. *Let $\mathcal{B}$ be the uniform closure of an algebra $\mathcal{A}$ of bounded functions. Then $\mathcal{B}$ is a uniformly closed algebra.*

Proof. Given $\mathcal{B}$ is the uniform closure of $\mathcal{A}$.

To prove $\mathcal{B}$ is an algebra, we have to prove $\mathcal{B}$ is closed under addition, multiplication and scalar multiplication.

For, let $f \in \mathcal{B}$ and $g \in \mathcal{B}$. Then by definition there are uniformly convergent sequences $\{f_n\}$ and $\{g_n\}$ such that $f_n \in \mathcal{A}$, $g_n \in \mathcal{A}$ and $f_n \to f$, $g_n \to g$.

It is enough to prove

$$f_n + g_n \to f + g, \quad f_n g_n \to fg \text{ and } cf \to cf,$$

where each convergence is uniform. (details of the proof is left as an exercise!!!)

Now, to prove $\mathcal{B}$ is uniformly closed, we have to prove that: if $\{f_n\}$ is a sequence in $\mathcal{B}$ such that $f_n \to f$ uniformly, then $f \in \mathcal{B}$.

It is enough if we can find f as a uniform limit of some sequence of functions in $\mathcal{A}$.

Since $f_n \in \mathcal{B}$, for each n, we have a sequence namely $\{f_{jn} : j = 1, 2, 3, ...\}$, of functions such that $f_{jn} \in \mathcal{A}$ and converging uniformly to f_n.

$$\begin{array}{cccccc}
f_{11} & f_{12} & f_{13} & \cdots & f_{1n} & \cdots \\
f_{21} & f_{22} & f_{23} & \cdots & f_{2n} & \cdots \\
f_{31} & f_{32} & f_{33} & \cdots & f_{3n} & \cdots \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdots \\
\downarrow & \downarrow & \downarrow & \cdots & \downarrow & \cdots \\
f_1 & f_2 & f_3 & \cdots & f_n & \cdots
\end{array}$$

Now, if we consider the functions on the diagonal, we see that they converge to f uniformly, since $f_n \to f$ uniformly. Thus, we have the sequence $\{f_{nn}\}$ converges uniformly to f(prove in detail!), and hence $f \in \mathcal{B}$, and $\mathcal{B}$ is uniformly closed. $\square$

Definition 5.10. *Let $\mathcal{A}$ be a family of functions on a set E. Then $\mathcal{A}$ is said to* **separate points** *on E if to each pair of distinct points x_1 and x_2 in E, there corresponds a point $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$.*

We say that $\mathcal{A}$ **vanishes at no point** *of E, if to each $x \in E$, there corresponds a function $f \in \mathcal{A}$ such that $f(x) \neq 0$.*

Example 5.4. *The algebra of polynomials in one variable on $\mathbb{R}$ separate points and vanishes at no points.*

The set of even polynomials on [-1,1] forms an algebra, which does not separate points on [-1,1], since $f(x) = f(-x)$.

The set of polynomials functions on $\mathbb{R}$ with $x-2$ as a factor is an algebra, because if f and g are such functions, then so is $f+g$, fg and cf, for $c \in \mathbb{R}$. But $f(2) = 0$, for all f.

The following theorem shows that an algebra which vanishes at no point and separate points provides some thing more.

Theorem 5.17. *Let $\mathcal{A}$ be an algebra of functions on a set E, $\mathcal{A}$ separate points on E and $\mathcal{A}$ vanishes at no point of E. Suppose x_1 and x_2 are distinct points of E, and $c_1,\ c_2$ are constants(real or complex according as $\mathcal{A}$ is a real or complex algebra). Then $\mathcal{A}$ contains a function f such that*

$$f(x_1) = c_1 \quad \textit{and} \quad f(x_2) = c_2.$$

Proof. Since $\mathcal{A}$ separates points on E, there is $g \in \mathcal{A}$ for which

$$g(x_1) \neq g(x_2)$$

Again since $\mathcal{A}$ vanishes at no points of E, there exist $h,\ k \in \mathcal{A}$ such that

$$h(x_1) \neq 0 \quad \text{and} \quad k(x_2) \neq 0.$$

so that if we can find functions u and v in $\mathcal{A}$ such that

$$u(x_1) = v(x_2) = 0, \quad u(x_2) \neq 0, \quad \text{and} \quad v(x_1) \neq 0,$$

then the function

$$f = \frac{c_1 v}{v(x_1)} + \frac{c_2 u}{u(x_2)}$$

has the desired property. and, there are such functions u and v in $\mathcal{A}$, namely

$$u = gk - g(x_1)k \quad \text{and} \quad v = gh - g(x_2)h,$$

and hence the proof. □

Now, we are in a position to consider Stone's generalization of the Weierstrass Theorem.

Theorem 5.18. *(***Stone-Weierstrass Theorem***) Let $\mathcal{A}$ be an algebra of real continuous functions on a compact set K. If $\mathcal{A}$ separate points on K and if $\mathcal{A}$ vanishes at no point of K, then the uniform closure $\mathcal{B}$ of $\mathcal{A}$ consists of all real continuous functions on K.*

Proof. We prove the theorem in four steps, by proving,
Step 1 If $f \in \mathcal{B}$, then $|f| \in \mathcal{B}$.
Step 2 If $f,\ g \in \mathcal{B}$, then $\max\{f,\ g\} \in \mathcal{B}$ and $\min\{f,\ g\} \in \mathcal{B}$.
Step 3 Given a real function f, continuous on K, a point $x \in K$, and $\varepsilon > 0$, there exists a function $g_x \in \mathcal{B}$ such that

$$g_x(x) = f(x) \text{ and } g_x(t) > f(t) - \varepsilon, \quad t \in K, \quad t \neq x.$$

Step 4 Given a real function f, continuous on K, and $\varepsilon > 0$, there exists a function $h \in \mathcal{B}$ such that

$$|h(x) - f(x)| < \varepsilon, \quad x \neq K.$$

Step 4 shows that any real continuous function on K is in the uniform closure of $\mathcal{B}$. Since $\mathcal{B}$ is uniformly closed, its uniform closure is $\mathcal{B}$ itself, which proves the Theorem.

Step 1: Let $f \in \mathcal{B}$. Since K is compact, f is bounded on K.. Let

$$a = \sup\ |f(x)|, \quad x \in K.$$

Using Corollary 5.2, we can find a sequence of real polynomials P_n such that $P_n(0) = 0$ and

$$\lim_{n\to\infty} P_n(y) = |y|,$$

uniformly on $[-a, a]$. Therefore, given $\varepsilon > 0$, we can find real numbers $c_1, c_2, ..., c_n$ such that

$$\left|\sum_{i=1}^{n} c_i y^i - |y|\right| < \varepsilon, \quad -a \leq y \leq a. \tag{5.12}$$

Consider the function

$$g = \sum_{i=1}^{n} c_i f^i.$$

Since $f \in \mathcal{B}$, and $\mathcal{B}$ is an algebra, we have $g \in \mathcal{B}$. Now,

$$g(x) = \sum_{i=1}^{n} c_i f(x)^i$$

and since $|f(x)| \in [-a, a]$, for all $x \in K$, inequality (5.12) gives

$$|g(x) - |f(x)|| < \varepsilon, \quad x \in K.$$

This shows that $|f| \in \mathcal{B}$, since $\mathcal{B}$ is uniformly closed.

Step 2: It follows from Step 1, because of the identities

$$\max\{f, g\} = \frac{f+g}{2} + \frac{|f-g|}{2},$$

$$\min\{f, g\} = \frac{f+g}{2} - \frac{|f-g|}{2}.$$

By repeated application of the result, we can see that it is true for a finite set of functions. That is if $f_1, ..., f_n \in \mathcal{B}$, then

$$\max\{f_1, ..., f_n\} \in \mathcal{B} \quad \text{and} \quad \min\{f_1, ..., f_n\} \in \mathcal{B}.$$

Step 3: Since $\mathcal{A} \subset \mathcal{B}$, $\mathcal{A}$ separate points on K and $\mathcal{A}$ vanishes at no point of K, using Theorem 5.17, we have for every $y \in K$, there exists a function $h_y \in \mathcal{B}$ such that

$$h_y(x) = f(x), \quad h_y(y) = f(y).$$

Then the continuity of h at y implies that there is an open set J_y containing y such that,

$$f(t) - h_y(t) < \varepsilon, \quad \text{for all} \quad t \in J_y.$$

Now, $\{J_y : y \in K\}$ forms an open cover for K, and since K is compact there are points $y_1, ..., y_n \in K$ such that

$$K \subset J_{y_1} \cup ... \cup J_{y_n}.$$

Then the function

$$g_x = \max\{h_{y_1}, ..., h_{y_n}\} \in \mathcal{B},$$

by Step 2, and g_x satisfies the properties required in Step 3.

Step 4: Consider the function g_x, for each $x \in K$, constructed in Step 3. The continuity of g_x shows that there exists open set V_x containing x such that

$$g_x(t) - f(t) < \varepsilon, \quad \text{for all} \quad t \in V_x.$$

Again using the compactness of K, we have a finite number of points $x_1, ..., x_m$ such that

$$K \subset V_{x_1} \cup ... \cup V_{x_m}.$$

Then the function

$$h = \min\{g_{x_1}, ..., g_{x_m}\} \in \mathcal{B},$$

by Step 2, and h satisfies the properties required in Step 4(Verify!). Hence the proof. □

However, This result fails in complex algebras. But it will hold if an additional condition is imposed on $\mathcal{A}$, namely, $\mathcal{A}$ is **self-adjoint**, that is for every $f \in \mathcal{A}$, its complex conjugate $\overline{f} \in \mathcal{A}$. The conjugate $\overline{f}$ is defined by $\overline{f}(x) = \overline{f(x)}$.

Theorem 5.19. *Suppose $\mathcal{A}$ is a self-adjoint algebra of complex continuous functions on a compact set K, $\mathcal{A}$ separates points on K and $\mathcal{A}$ vanishes at no point of K. Then the uniform closure $\mathcal{B}$ of $\mathcal{A}$ consists of all complex continuous functions on K. That is $\mathcal{A}$ is dense in $\mathcal{C}(K)$.*

Proof. Let $\mathcal{A}_R$ be the set of all real functions on K which belong to $\mathcal{A}$.

Now, for $f \in \mathcal{A}$, write

$$f = u + iv,$$

where u and v are real functions. More over,

$$2u = f + \overline{f} \in \mathcal{A},$$

since $\mathcal{A}$ is self-adjoint. Thus we have

$$u \in \mathcal{A}_R.$$

If $x_1 \neq x_2$, there exists $f \in \mathcal{A}$ such that $f(x_1) = 1, \quad f(x_2) = 0$; so that $u(x_1) \neq u(x_2)$, which shows that $\mathcal{A}_R$ separates points on K.

Again for $x \in K$, there exists $g \in \mathcal{A}$ such that $g(x) \neq 0$. Then we can find a complex number λ such that $\lambda g(x) > 0$; so that if $\lambda g = u + iv$, then $u \in \mathcal{A}_R$ and $u(x) > 0$. That is $\mathcal{A}_R$ vanishes at no point of K.

Applying Theorem 5.18 to $\mathcal{A}_R$, we get every real continuous function on K lies in in the uniform closure of $\mathcal{A}_R$, and hence lies in $\mathcal{B}$.

Now, every complex continuous function f on K, $f = u + iv$, where $u,\ v \in \mathcal{B}$, and hence $f \in \mathcal{B}$.

□

Exercises

1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded

   Solution: Let $\{f_n\}$ be a uniformly convergent sequence of bounded functions on E. Then there exists a natural number N such that

   $$|f_n(x) - f_m(x)| < 1, \quad \text{for all} \;\; n,\ m \geq N, \quad x \in E.$$

In particular,

$$|f_n(x) - f_N(x)| < 1, \quad \text{for all} \quad n \geq N, \quad x \in E.$$

This implies that

$$|\,|f_n(x)| - |f_N(x)|\,| \leq |f_n(x) - f_N(x)| < 1, \quad \text{for all} \quad n \geq N, \quad x \in E.$$

Hence

$$|f_n(x)| < 1 + |f_N(x)|, \quad \text{for all} \quad n \geq N, \quad x \in E.$$

Since f_n are bounded, for each $n = 1, 2, ...$, there exist M_n such that

$$|f_n(x)| \leq M_n, \quad \text{for all} \quad x \in E.$$

Hence the above inequality shows

$$|f_n(x)| < 1 + M_N, \quad \text{for all} \quad n \geq N, \quad x \in E.$$

Thus, if

$$K = \max\{M_1, ..., M_N\},$$

we get

$$|f_n(x)| < 1 + K, \quad \text{for all} \quad x \in E, \quad \text{and for all} \quad n.$$

That is $\{f_n\}$ is uniformly bounded.

2. Prove that on a finite set, pointwise convergence and uniform convergence are the same.

3. Prove that if $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E and $\{f_n g_n\}$ converges on E. If in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, then prove that $\{f_n g_n\}$ converges uniformly on E.

4. Construct sequences $\{f_n\}$ and $\{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E.

   Solution: Consider the functions

$$f_n(x) = x \quad \text{and} \quad g_n(x) = \frac{1}{n}, \quad x \in \mathbb{R}, \quad n = 1, 2, ...$$

   Then

$$f_n g_n(x) = \frac{x}{n}.$$

   Verify that $\{f_n\}$ and $\{g_n\}$ converge uniformly on $\mathbb{R}$ but $\{f_n g_n\}$ does not converge uniformly on $\mathbb{R}$.

5. For $n = 1, 2, ...,$ define

$$f_n(x) = \frac{x}{1 + nx^2}, \quad x \in \mathbb{R}.$$

Show that $\{f_n\}$ converges uniformly to a function f, and that the equation

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

is correct if $x \neq 0$, but not correct if $x = 0$.