MTH2C08-TOPOLOGY

STUDY MATERIAL

II SEMESTER
CORE COURSE

M.Sc Mathematics

(2019 Admission ONWARDS)

UNIVERSITY OF CALICUT
SCHOOL OF DISTANCE EDUCATION
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Module 1

Topological spaces

(Reference text: 'INTRODUCTION TO GENERAL TOPOLOGY' by K. D. Joshi)

1.1 Introduction

A metric space is a set endowed with an additional structure, namely, the metric or the distance function. It is clear, therefore, that any concept in the theory of metric spaces, unless it is a purely set-theoretic concept, will have to be defined in terms of the metric. Throughout, (X, d) will be a metric space.

Definition 1. *Let $x_0 \in X$ and r be a positive real number. Then the open ball with centre x_0 and radius r is defined to be the set $\{x \in X : d(x, x_0) < r\}$. It is denoted either by $B_r(x_0)$ or by $B(x_0, r)$. It is also called the open r-ball around x_0. When we want to stress the metric d, we denote it by $B_d(x_0, r)$.*

It is obvious that the open ball $B_d(x_0, r)$ depends not only on x_0 and r, but on the metric d as well. Let d be the discrete metric on a set X. Then for any $x_0 \in X$, $B(x_0, r)$ consists of X or $\{x_0\}$ depending upon whether $r > 1$ or $r \leq 1$. A subset A of X is said to be bounded if the function d is bounded over $A \times A$. If A is a non-empty bounded set, its diameter, denoted by $\delta(A)$ is the number $sup\{d(x, y) : x \in A, y \in A\}$. Note that every open ball is bounded but its diameter may be less than twice its radius.

Proposition 1.1.1. *Let $\{x_n\}$ be a sequence in a metric space (X, d). Then $\{x_n\}$ converges to y in X iff for every open set U containing y, there exists a positive integer N such that for every integer $n \geq N$, $x_n \in U$.*

Proposition 1.1.2. *Let $f : X \to Y$ be a function where X, Y are metric spaces and let $x_0 \in X$. Then f is continuous at x_0 iff for every open set V in Y containing $f(x_0)$, there exists an open set U in Y containing x_0 such that $f(U) \subset V$.*

Theorem 1.1.3. *Let (X, d) be a metric space. Then,*
(i) the empty set ϕ and the entire set X are open,
(ii) the union of any family of open sets is open,
(iii) the intersection of any finite number of open sets is open,
(iv) given distinct points $x, y \in X$ there exist open sets U, V such that $x \in U, y \in V$ and $U \cap V = \phi$.

Proof. (i) and (ii) are trivial consequences of the definition of open sets. For (iii) first consider the case of the intersection of two open sets say A_1 and A_2. Let $x \in A_1 \cap A_2$. Then $x \in A_1$ and $x \in A_2$. Since A_1 is open, there exists $r_1 > 0$ such that $B(x, r_1) \subset A_1$. Similarly since A_2 is open there exists $r_2 > 0$ such, that $B(x, r_2) \subset A_2$. Now let $r = min\{r_1, r_2\}$. Then clearly $B(x, r) \subset B(x, r_1) \cap B(x, r_2) \subset A_1 \cap A_2$. Thus $A_2 \cap A_2$ is open. One can either generalise this argument or use induction to settle the general case.The exceptional case of the intersection of an empty family of open sets is already covered under (i). For (iv) let $x, y \in X$ and $x \neq y$. Then $d(x, y) > 0$. Choose r so that $0 < r < \frac{d(x,y)}{2}$ and let $U = B(x, r), V = B(y, r)$. Then clearly U, V are open sets containing x, y respectively. Also they are mutually disjoint. □

Theorem 1.1.4. *Let $\mathcal{J}$ be the collection of all open sets in a metric space X. Then $\mathcal{J}$ has the following properties:*
(i) ϕ and X belongs to $\mathcal{J}$,
(ii) $\mathcal{J}$ is closed under arbitrary unions,
(iii) $\mathcal{J}$ is closed under finite intersections,
(iv) Given distinct points $x, y \in X$, there exist $U, V \in \mathcal{J}$ such that $x \in \mathcal{J}$, $y \in V$ and $U \cup V = \phi$.

So, in order to define a topological space we take a set X, a certain family $\mathcal{J}$ of its subsets and require that $\mathcal{J}$ satisfy some of the properties listed in the above theorem. We will give the definition of a topological space.

Definition 2. *A topological space is a pair $(X, \mathcal{J})$ where X is a set and $\mathcal{J}$ is a family of subsets of X satisfying:*
(i) $\phi \in \mathcal{J}$ and $X \in \mathcal{J}$,
(ii) $\mathcal{J}$ is closed under arbitrary unions,
(iii) $\mathcal{J}$ is closed under finite intersections.

The family $\mathcal{J}$ is said to be a topology on the set X. Members of $\mathcal{J}$ are said to be open in X or open subsets of X. A sequence $\{x_n\}$ in a topological space $(X, \mathcal{J})$ is said to converge to a point y of X if for every open set U containing y, there exists a positive integer N such that for every integer $n > N$, $x_n \in U$. A topological space is said to be metrisable if its topology can be obtained from a suitable metric on the underlying set. It may happen that two distinct metrics on a set yield the same topology. A trivial case of this occurs when the two metrics are scalar multiples of each by a constant factor. Of course not all topological spaces are metrisable.

Exercise :

1. Prove that the open balls in a metric space are open sets.
2. Prove that a subset A of a metric space X is open iff no sequence in $X \setminus A$ converges to a point of A.

Rough space :

1.2 Examples

1. It may happen that the topology $\mathcal{J}$ on the set X consists only of ϕ and X. It is called the indiscrete topology on X.

2. The other extreme is the so-called discrete topology on X. Here every set is open; in other words the topology coincides with the power set $P(X)$.

3. Let X be any set. A subset A of X is said to be cofinite, if its complement, $X \setminus A$, is finite. Let $\mathcal{J}$ consist of all cofinite subsets of X and the empty set. Then $\mathcal{J}$ is a topology on X and this topology is called cofinite topology.

4. The usual topology on $\mathbf{R}$ is defined as the topology induced by the euclidean metric. Note that with this metric the open balls are just bounded open intervals. So all bounded open intervals are indeed open sets in the usual topology on R. Since unbounded open intervals can be expressed as unions of bounded open intervals, they are also open in the usual topology. Thus all open intervals are open sets of R in the usual topology.

5. There is another, and a stronger topology on $\mathbf{R}$, called the semi-open interval topology. A subset U is said to be open in this topology if for every $x \in \mathbf{R}$, there exists $r > 0$ such that the semi-open interval $[x, x + r)$ is contained in U. The verification that this indeed defines a topology on $\mathbf{R}$ and that the topology so defined is stronger than the usual topology is left to the reader.

6. Let a set X be linearly ordered by $\leq$. Declare a subset A of X to be open if for each $x \in A$ there exist $a, b \in X$ such that $a < x < b$ and the interval (a, b) (i.e. the set $\{y \in X : a < y < b\}$) is contained in A. Assume for the moment that X has no smallest and no largest element. It is easy to show that the collection of open sets is indeed a topology. It is called the order topology induced by the order $\leq$. For the real line $\mathbf{R}$, the topology induced by the usual ordering coincides with the usual topology. However by selecting the set and the ordering $\leq$ suitably one can construct many strange spaces. As an example, compare the usual topology on the plane $\mathbf{R}^2$ with the topology induced by the lexicographic ordering on it.

7. Let X be a set. A subset A of X is said to be cocountable if its complement $X \setminus A$ is countable. Let $\mathcal{J}$ consist of all cocuntable subsets of X and the empty set. Then $\mathcal{J}$ is a topology on X. This topology is called cocountable topology on X.

8. Let $\mathbf{N}$ be set of all positive integers. Consider the cartesian product $\mathbf{N} \times \mathbf{N}$. Let ∞ be any point not in $\mathbf{N} \times \mathbf{N}$ and let $X = (\mathbf{N} \times \mathbf{N}) \cup \{\infty\}$. We will define a topology on X. Let $\mathcal{I}_1$ be the power set $P(\mathbf{N} \times \mathbf{N})$. Clearly $\mathcal{I}_1 \subset P(X)$. Let $\mathcal{I}_2$ be the collection

of those subsets A of X such that $\infty \in A$ and A contains almost all points in almost all rows (the term almost means exception of finitely few). Now, let $\mathbf{I} = \mathcal{I}_1 \cup \mathcal{I}_2$. Then $\mathcal{I}$ is a topology on X.

Definition 3. *The topology $\mathcal{J}_1$ is said to be weaker (or coarser) than the topology $\mathcal{J}_2$ (on the same set) if $\mathcal{J}_1 \subset \mathcal{J}_2$ as subsets of the power set. In this case we also say that $\mathcal{J}_2$ is stronger (or finer) than $\mathcal{J}_1$.*

Remark : The indiscrete topology is the smallest or weakest or coarsest of all while the discrete topology is the largest or the finest or the strongest of all topologies on the same set.

Theorem 1.2.1. *Let X be a set and $\{\mathcal{J}_i | i \in I\}$ be a indexed family of topologies on X. Then $\mathcal{J} = \cap_i \mathcal{J}_i$ is a topology on X. Also $\mathcal{J}$ is weaker than each $\mathcal{J}_i$, $i \in I$.*

Proof. Let us first verify that $\mathcal{J}$ is a topology on X. Clearly the empty set ϕ belongs to each $\mathcal{J}$ since each $\mathcal{J}_i$ is topology on X and so $\phi \in \cap_i \mathcal{J}_i$, i.e. $\phi \in \mathcal{J}$. Similarly $X \in \mathcal{J}$. Next we show that $\mathcal{J}$ is closed under finite intersections. For this let $A_1, A_2, ..., A_n \in \mathcal{J}$ and suppose $A = \cap_i A_i$. To show $A \in \mathcal{J}$. Now, since $\mathcal{J} = \cap_i \mathcal{J}_i$, each $A_i \in \mathcal{J}_i$. But $\mathcal{J}_i$ being a topology on X, is closed under finite intersections. So, $A \in \mathcal{J}_i$ for each i. Hence $A \in \mathcal{J}$. The proof that $\mathcal{J}$ is closed under arbitrary unions is similar and left to the reader. The rest of the theorem is just a general property of intersections. □

Corollary 1.2.1.1. *Let X be a set and $\mathcal{D}$ a family of subsets of X. Then there exists a unique topology $\mathcal{J}$ on X, such that it is the smallest topology on X containing $\mathcal{D}$.*

Proof. Consider the collection of all topologies on X which contain $\mathcal{D}$. This family is non-empty, for the discrete topology surely contains $\mathcal{D}$. Now let $\mathcal{J}$ be the intersection of the members of this collection. Applying the theorem above, $\mathcal{J}$ is a topology on X, it contains $\mathcal{D}$ and clearly it is the smallest topology containing $\mathcal{D}$, for any such topology will be a member of the collection of topologies just considered and hence stronger than its intersection. □

Remark : The topology $\mathcal{J}$ so obtained is said to be generated by the family $\mathcal{D}$.

Exercise :

1. Prove that in the co-countable topology, the only convergent sequences are those which are eventually constant.
2. Prove that the usual topology on the euclidean plane $\mathbf{R}^2$ is strictly weaker than the topology induced on it by the lexicographic ordering.
3. What difficulty would arise in defining the order topology if either had a smallest or a

largest element ?

Rough space :

1.3 Bases and Sub-bases

Definition 4. *Let $(X, \mathcal{J})$ be a topological space. A subfamily $\mathcal{B}$ of $\mathcal{J}$ is said to be a base for $\mathcal{J}$ if every member of $\mathcal{J}$ can be expressed as the union of some members of $\mathcal{B}$.*

In a metric space every open set can be expressed as a union of open balls and consequently the family of all open balls is a base for the topology induced by the metric. It is not necessary to take all open ball, rather the family of all open balls of rational radii is also a base. Indeed it suffices to take balls of radius $1/n, n \in \mathbf{N}$.

Proposition 1.3.1. *Let $(X, \mathcal{J})$ be a topological space and $\mathcal{B} \subset \mathcal{J}$. Then $\mathcal{B}$ is a base for $\mathcal{J}$ iff for any $x \in X$ and any open set G containing x, there exists $B \in \mathcal{B}$ such that $x \in B$ and $B \subset G$.*

Proof. First suppose $\mathcal{B}$ is a base for $\mathcal{J}$. Let $x \in X$ and let an open set G containing x be given. Then G can be written as the union of some members of $\mathcal{B}$, say, $G = \cup_{i \in I} B_i$, where I is an index set and $B_i \in \mathcal{B}$ for all $i \in I$. Since $x \in G$, there exists $j \in I$ such that $x \in B_j$. We take this B_j as the set B required in the assertion. Conversely suppose the given condition holds. Let H be an open set in X, i.e. $H \in \mathcal{J}$. For each $x \in H$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x$ and $B_x \subset H$. Clearly $H = \cup_{x \in H} B_x$. Thus every member of $\mathcal{J}$ can be expressed as the union of some members of $\mathcal{B}$. So $\mathcal{B}$ is a base for $\mathcal{J}$. $\square$

Definition 5. *A space is said to satisfy the second axiom of countability or is said to be second countable if its topology has countable base.*

Definition 6. *A family $\mathcal{U}$ of sets is said to be a cover(or covering) of a set A if A is contained in the union of members of $\mathcal{U}$. A subcover of $\mathcal{U}$ is subfamily $\mathcal{V}$ of $\mathcal{U}$ which itself is a cover of A. If we are in a topological space then a cover is said to be open if all its members are open.*

Theorem 1.3.2. *If a space is second countable then every open cover of it has a countable subcover.*

Proof. Let $(X, \mathcal{J})$ be a space with a countable base $\mathcal{B}$ and let $\mathcal{U}$ be a given open cover of X. First enumerate $\mathcal{B}$ as $\{B_1, B_2, B_3, ...\}$. Now let

$$S = \{n \in \mathbf{N} : B_n \; is \; contained \; in \; some \; member \; of \; \mathcal{U}\}.$$

For each $n \in S$, fix $U_n \in \mathcal{U}$ such that $B_n \subset U_n$. Now let $\mathcal{C} = \{B_n : n \in S\}$ and $\mathcal{V} = \{U_n : n \in S\}$. Clearly $\mathcal{V}$ a countable sub-family of $\mathcal{J}$ and covers X if $\mathcal{V}$ does. So the theorem will be proved if we show $\mathcal{C}$ is a cover of X. For this let $x \in X$. Then $x \in U$ for some $U \in \mathcal{U}$. By proposition above, there is some $k \in \mathbf{N}$ such that $x \in B_k$ and $B_k \subset U$. Clearly then, $k \in S$ and so $B_k \in \mathcal{C}$. So $\mathcal{C}$ and consequently $\mathcal{V}$ is a cover of X. $\square$

Proposition 1.3.3. *Let $\mathcal{J}_1, \mathcal{J}_2$ be two topologies for a set having bases $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. Then $\mathcal{J}_1$ is weaker than $\mathcal{J}_2$ iff every member of $\mathcal{B}_1$ can be expressed as a union of some members of $\mathcal{B}_2$.*

If $\mathcal{B}$ is a base for a topology $\mathcal{J}$ on a set X, then $\mathcal{B}$ generates $\mathcal{J}$, i.e. $\mathcal{J}$ is the smallest topology containing $\mathcal{B}$. It is natural to inquire whether we can start with an arbitrary family $\mathcal{B}$ of subsets of a set X and find a topology $\mathcal{J}$ on X for which $\mathcal{B}$ will be a base. Of course in case such a topology exists, it must be unique. But in general no such topology exists. The following proposition tells precisely which families can be bases for topologies.

Proposition 1.3.4. *Let X be a set and $\mathcal{B}$ a family of its subsets covering X. Then the following statements are equivalent:*
1. There exists a topology on X with $\mathcal{B}$ as a base.
2. For any $B_1, B_2 \in \mathcal{B}$, $B_1 \cap B_2$ can be expressed as the union of some members of $\mathcal{B}$.
3. For any $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.

Proof. (1) $\implies$ (2). Suppose there exists a topology $\mathcal{J}$ on X for which $\mathcal{B}$ is a base. Let $B_1, B_2 \in \mathcal{B}$. Then $B_1, B_2 \in \mathcal{J}$ and so $B_1 \cap B_2 \in \mathcal{J}$ since $\mathcal{J}$ is closed under finite intersections. So, by definition of a base, $B_1 \cap B_2$ can be expressed as the union of some members of $\mathcal{B}$.
The proof of the equivalence of (2) and (3) resembles that of 1.3.1 and is left as an exercise.

It only remains to show (3) $\implies$ (1). Assume that the condition in (3) holds and define $\mathcal{I} = \{G \subset X : \text{ for all } x \in G, \text{ there exists } B \in \mathcal{B} \text{ such that } x \in B \text{ and } B \subset G\}$. We assert that $\mathcal{J}$ is a topology on X. Clearly $\phi \in \mathcal{J}$ while $X \in \mathcal{J}$ since $\mathcal{B}$ is given to be a cover of X. That $\mathcal{J}$ is closed under arbitrary unions is self-evident. It only remains to verify that whenever $G, H \in \mathcal{J}$, $G \cap H \in \mathcal{J}$. Let $x \in G \cap H$. Then $x \in G$ and $x \in H$. So there exist $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1, B_1 \subset G, x \in B_2$ and $B_2 \subset H$. Then $x \in B_1 \cap B_2$. So by (3), there exists $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$. But $B_1 \cap B_2 \subset G \cap H$. So $G \cap H \in \mathcal{I}$. Thus $\mathcal{I}$ is a topology on X and it follows that $\mathcal{B}$ is a base for $\mathcal{J}$. □

Corollary 1.3.4.1. *If $\mathcal{B}$ is a cover of X and $\mathcal{B}$ is closed under finite intersections then $\mathcal{B}$ is a base for a (unique) topology $\mathcal{J}$ on X. Moreover, $\mathcal{J}$ consists precisely of those subsets of X which can be expressed as unions of subfamilies of $\mathcal{B}$.*

Definition 7. *A family $\mathcal{S}$ of subsets of X is said to be a sub-base for a topology $\mathcal{J}$ on X if the family of all finite intersections of members of $\mathcal{S}$ is a base for $\mathcal{J}$.*

Any base for a topology is also a sub-base for the same. In general, however, a sub-base can be chosen to be much smaller than a base. For example, for the usual topology on $\mathbf{R}$, the family of all open intervals of the form (a, ∞) or $(-\infty, b)$ for $a, b \in \mathbf{R}$ (or $\mathbf{Q}$) is a sub-base.

Theorem 1.3.5. *Let X be a set, $\mathcal{J}$ a topology on X and $\mathcal{S}$ a family of subsets of X. Then $\mathcal{S}$ is a sub-base for $\mathcal{J}$ iff $\mathcal{S}$ generates $\mathcal{J}$.*

Proof. Let $\mathcal{B}$ be the family of finite intersections of members of $\mathcal{S}$. Suppose first that $\mathcal{S}$ is a sub-base for $\mathcal{J}$. We want to show that $\mathcal{J}$ is the smallest topology on X containing $\mathcal{S}$. Now since $\mathcal{S} \subset \mathcal{B}$ and $\mathcal{B} \subset \mathcal{J}$ we at least have that $\mathcal{J}$ contains $\mathcal{S}$. Suppose $\mathcal{U}$ is some other topology on X such that $\mathcal{S} \subset \mathcal{U}$. We have to show that $\mathcal{J} \subset \mathcal{U}$. Now since $\mathcal{U}$ is closed under finite intersections and $\mathcal{S} \subset \mathcal{U}, \mathcal{U}$ contains all finite intersections of members of $\mathcal{S}$ i.e. $\mathcal{B} \subset \mathcal{U}$. But again since $\mathcal{U}$ is closed under arbitrary unions and each member of $\mathcal{J}$ can be written as union of some members of $\mathcal{B}$ (by definition of a base), it follows that $\mathcal{J} \subset \mathcal{U}$. Conversely suppose $\mathcal{J}$ is the smallest topology containing $\mathcal{S}$. We have to show that $\mathcal{S}$ is a sub-base for $\mathcal{J}$, i.e. that $\mathcal{B}$ is a base for $\mathcal{J}$. Clearly $\mathcal{B} \subset \mathcal{J}$ is closed under finite intersections and $\mathcal{S} \subset \mathcal{J}$. Since $\mathcal{B}$ is closed under finite intersections we know by corollary 1.3.4.1 above that there is a topology $\mathcal{U}$ on X such that $\mathcal{B}$ is a base for $\mathcal{U}$. Every member of $\mathcal{U}$ can be expressed as a union of a sub-family of $\mathcal{B}$ and so is in $\mathcal{J}$ since $\mathcal{B} \subset \mathcal{J}$. This means $\mathcal{U} \subset \mathcal{J}$ and consequently $\mathcal{U} = \mathcal{J}$ since $\mathcal{J}$ is the smallest topology containing $\mathcal{S}$. Thus $\mathcal{B}$ is a base for $\mathcal{J}$ and $\mathcal{S}$ is a sub-base for $\mathcal{J}$. □

Theorem 1.3.6. *Given any family $\mathcal{S}$ of subsets of X, there is a unique topology $\mathcal{J}$ on X having $\mathcal{S}$ as a sub-base. Further, every member of $\mathcal{J}$ can be expressed as the union of sets each of which can be expressed as the intersection of finitely many members of $\mathcal{S}$.*

Proof. The first assertion follows from the last theorem. To prove that every member of $\mathcal{J}$ has the desired form, let $\mathcal{U}$ consist of all subsets of X which can be expressed as unions of members of $\mathcal{B}$ where $\mathcal{B}$ is the family of finite intersections of members of $\mathcal{S}$. By corollary 1.3.4.1 above, $\mathcal{U}$ is the unique topology having $\mathcal{B}$ as a base. Hence $\mathcal{J} = \mathcal{U}$. □

Let n be a positive integer, $\{(X, \mathcal{J}_i) : i = 1, 2, 3, ..., n\}$ be a family of topological spaces and X be the cartesian product $X_1 \times X_2 \times ... \times X_n$. We shall define a certain topology, called the product topology on X. By an open box in X we mean a set of the form $V_1 \times V_2 \times ... \times V_n$ where $V_i \in \mathcal{J}_i$ for $1 = 1, 2, ..., n$. The entire set X is clearly an open box. It is also evident that the intersection of two open boxes is again an open box, although their union need not be. By corollary 1.3.4.1, the family of open boxes is a base for a unique topology $\mathcal{J}$ on X. This topology is called the product topology on X, the space $(X, \mathcal{J})$ is called the topological product of the spaces $(X_1, \mathcal{J}_1), (X_2, \mathcal{J}_2), ..., (X_n, \mathcal{J}_n,)$. The space $(X_i, \mathcal{J}_i)$ is called the ith co-ordinate space or the ith factor of $(X, \mathcal{J})$.
Let $\mathcal{J}_1$ denote the product topology on $\mathbf{R}^n$ while let $\mathcal{J}_2$ denote the usual topology on it (induced by the euclidean metric on $\mathbf{R}^n$). Then we have $\mathcal{J}_1 = \mathcal{J}_2$. The idea behind the proof is to show that $\mathcal{J}_1 \subset \mathcal{J}_2$ and $\mathcal{J}_2 \subset \mathcal{J}_1$, which will imply that $\mathcal{J}_1 = \mathcal{J}_2$. Students are encouraged to go through the details of the proof, which is given in the textbook.

Exercise :

1. Prove that if a space $(X, \mathcal{J})$ has a base B of cardinality a then the cardinality of $\mathcal{J}$ cannot exceed 2^a.
2. Prove that a space is second countable if and only if it has a countable sub-base.
3. Let X_1 and X_2 be two topologies. Prove that the metric topology in $X_1 \times X_2$ coincides with the product topology on it.
4. Prove that if each space $(X_i, \mathcal{J}_i)$ is second countable, for $i = 1, 2, ..., n$ then so is their topological product.

Rough space :

1.4 Closed sets and closure

Definition 8. *Let $(X, \mathcal{J})$ be a topological space. Then a subset A of X is said to be closed in X if its complement $X \setminus A$ is open in X.*

Note : The empty set and the whole set are always open as well as closed in every space. On the other hand, the set of rationals is neither open nor closed in the usual topology on the real line. A set which is both open and closed is sometimes called a clopen set.

Theorem 1.4.1. *Let $\mathcal{C}$ be the family of all closed sets in a topological space $(X, \mathcal{J})$. Then $\mathcal{C}$ has the following properties:*
i. $\phi \in \mathcal{C}, X \in \mathcal{C}$.
ii. $\mathcal{C}$ is closed under arbitrary intersections.
iii. $\mathcal{C}$ is closed under finite unions.
Conversely, given any set X and a family $\mathcal{C}$ of its subsets which satisfies these three properties, there exists a unique topology $\mathcal{J}$ on X such that $\mathcal{C}$ coincides with the family of closed subsets of $(X, \mathcal{J})$.

Proof. The first part follows trivially from the definition of a topology and De Morgan's laws. The converse part is equally trivial once it is clearly understood what it says. Here we are given a set X and some collection $\mathcal{C}$ of its subsets. We are given that properties (i) to (iii) hold for $\mathcal{C}$. The theorem says that given such a family $\mathcal{C} \subset P(X)$ we can define a suitable topology $\mathcal{J}$ on X such that members of $\mathcal{C}$ are precisely the closed subsets of X (w.r.t. the topology $\mathcal{J}$), and that such a topology is unique. Having understood what the theorem says, the proof itself is trivial as we have no choice but to let $\mathcal{J}$ consist of complements (in X) of members of $\mathcal{C}$, i.e. $\mathcal{J} = \{B \subset X : X \setminus B \in \mathcal{C}\}$. That $\mathcal{J}$ is a topology on X follows by applying De Morgan's laws. The open subsets of X are precisely the complements of members of $\mathcal{C}$, and hence the closed subsets of X are precisely the members of $\mathcal{C}$ as asserted. Also this condition determines $\mathcal{J}$ uniquely. $\square$

Definition 9. *The closure of a subset of a topological space is defined as the intersection of all closed subsets containing it. In symbols, if A is a subset of a space $(X, \mathcal{J})$, then its closure is the set $\cap\{C \subset X : C \text{ closed in } X, A \subset C\}$. It is denoted by $\overline{A}$.*

Proposition 1.4.2. *Let A, B be subsets of a topological space $(X, \mathcal{J})$.*
i. $\overline{A}$ is a closed subset of X. Moreover it is the smallest closed subset of X containing A i.e. if C is closed in X and $A \subset C$ then $\overline{A} \subset C$.
ii. $\overline{\phi} = \phi$.
iii. A is closed in X iff $A = \overline{A}$.
iv.$\overline{\overline{A}} = \overline{A}$.
v. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

It is instructive to reformulate some of the properties in the last proposition using the terminology of operators. An operator is just another name for a function, except that the term is generally reserved for those functions whose domains are sets of sets or of functions. For example, if $(X, \mathcal{J})$ is a topological space, then the closure operator associated with it is defined as the function $c : P(X) \to P(X)$ such that $C(A) = \overline{A}$ for each $A \in P(X)$. In terms of the closure operator, (ii) to (v) assume the following forms respectively:
ii'. ϕ is a fixed point of C.
iii'. The fixed points of C are precisely the closed subsets of X.
iv'. C is idempotent, i.e. $C \circ C = C$.
v'. C commutes with finite unions.

Now comes an important question. Suppose we have an abstract operator $\theta : P(X) \to P(X)$. Under what conditions can we find a topology on X whose closure operator will coincide with the given operator θ ?

Theorem 1.4.3. *Let X be a set, $\theta : P(X) \to P(X)$ a function such that*
1. for every $A \in P(X)$, $A \subset \theta(A)$ (this condition is sometimes expressed by saying that θ is an expansive operator),
2. ϕ is a fixed point of θ,
3. θ is idempotent, and
4. θ commutes with finite unions.
Then there exists a unique topology $\mathcal{J}$ on X such that $\mathcal{J}$ coincides with the closure operator associated with $\mathcal{J}$. Conversely, any closure operator satisfies these properties.

Proof. The converse part is already established. For the direct implication, suppose $\theta : P(X) \to P(X)$ satisfies (1) to (4). We want to find a topology $\mathcal{J}$ on X such that for every $A \subset X, \theta(A) = \overline{A}$. If at all such a topology exists then its closed subsets must be precisely the fixed points of θ. This gives us a clue to the construction of $\mathcal{J}$. We let $\mathcal{C} = \{A \subset X : \theta(A) = A\}$ and contend that $\mathcal{C}$ has properties (i) to (iii) of Theorem 1.4.1. Condition (2) shows that $\phi \in \mathcal{C}$ while condition (4) implies that $\mathcal{C}$ is closed under finite unions. To prove that $X \in \mathcal{C}$, we merely note that by (1), $X \subset \theta(X)$ and hence $X = \theta(X)$ since $\theta(X) \subset X$ anyway. It only remains to verify that $\mathcal{C}$ is closed under arbitrary intersections. For this we first note that θ is monotonic, i.e., whenever $A \subset B, \theta(A) \subset \theta(B)$, which follows by writing B as $A \cup (B \setminus A)$ and applying (4). Now let $A = \cap_{i \in I} A_i$ where I is an index set and $A_i \in \mathcal{C}$ for each $i \in I$. We want to show that $A \in \mathcal{C}$, i.e. $\theta(A) = A$. By (1) we already know $A \subset \theta(A)$. Also $\theta(A) \subset \theta(A_i)$ for each $i \in I$ since θ is monotonic, and so $\theta(A) \subset \cap_{i \in I}(A_i)$. But $\theta(A_i) = A_i$, since $A_i \in \mathcal{C}$ for all $i \in I$. Consequently, $\theta(A) \subset A$ and hence $\theta(A) = A$ as desired. So by Theorem 1.4.1, the family $\mathcal{J}$ of complements of members of $\mathcal{C}$ is a topology on X.
It remains to be verified that the closure operator associated with $\mathcal{J}$ coincides with θ. Let

$A \subset X$. Then $\overline{A}$ w.r.t. $\mathcal{J}$ is the intersection of all closed subsets of X containing A. But by construction, closed subsets of X are precisely the fixed points of θ. Hence $\overline{A} = \cap\{B \subset X : A \subset B; \theta(B) = B\}$. Now, whenever $A \subset B$, $\theta(A) \subset \theta(B)$ by monotonicity of θ. So if $A \subset B$ and $\theta(B) = B$ then $\theta(A) \subset B$. But A is the intersection of such B's and so $\theta(A) \subset A$. For the other way inclusion we note that by condition (3), $\theta(A) \in \mathcal{C}$ while by (1) $A \subset \theta(A)$, hence $A \subset \theta(A)$, A being the smallest member of $\mathcal{C}$ containing A. Hence for all $A \subset X$, $\theta(A) = A$ completing the proof. □

Definition 10. *Let A be a subset of a space X. The A is said to be dense in X if $\overline{A} = X$. Trivially, the entire set X is always dense in itself.*

Proposition 1.4.4. *A subset A of a space X is dense in X iff for every nonempty open subset B of X, $A \cap B \neq \phi$.*

Proof. Suppose A is dense in X and B is a non-empty open set in X. If $A \cap B = \phi$ then $A \subset X \setminus B$. Hence $\overline{A} \subset X \setminus B$ since $X \setminus B$ is closed. But then $X \setminus B \subset X$ contradicting that $\overline{A} = X$. So $A \cap B \neq \phi$. Conversely assume that A meets every non-empty open subset of X. This clearly means that the only closed set containing A is X and consequently $\overline{A} = X$. □

Remark : In the topological space of real line with the usual topology the set $\mathbf{Q}$ of all rational numbers, as well as its complement $\mathbf{R} \setminus \mathbf{Q}$ are both dense.

Exercise :

1. Let $(X, \mathcal{J})$ be a topological space and $\mathcal{C}$ be the family of all closed subsets of X. Prove or disprove that $\mathcal{C}$ is the complement of $\mathcal{J}$ in $P(X)$.
2. Let $\mathcal{J}_1, \mathcal{J}_2$ be topologies on a set X and $A \subset X$. If $\mathcal{J}_1$ is weaker than $\mathcal{J}_1$ how are $\overline{A}^{\mathcal{J}_1}$ and $\overline{A}^{\mathcal{J}_2}$ related to each other?
3. If X is a space, $Y \subset X$ and $A \subset Y$, prove that $\overline{A}^{Y} = \overline{A}^{X} \cap Y$.
4. Prove that a second countable space always contains a countable dense subset.
5. Find out the dense subsets of discrete, indiscrete and cofinite spaces.

Rough space :

1.5 Neighbourhoods, Interior and Accumulation Points

Definition 11. *Let $(X, \mathcal{J})$ be a topological space, $x_0 \in X$ and $N \subset X$. Then N is said to be a neighbourhood of x_0 or x_0 is said to be an interior point of N (each w.r.t. $\mathcal{J}$) if there is an open set V such that $x_0 \in V$ and $V \subset N$. The word 'neighbourhood' is sometimes abbreviated to 'nbd'.*

Proposition 1.5.1. *A subset of a topological space is open iff it is a neighbourhood of each of its points.*

Proof. Let X be a topological space and $G \subset X$. First suppose G is open. Then evidently G is a nbd of each of its points. Conversely suppose G is a nbd of each of its points. Then for each $x \in G$, there is an open set V_x such that $x \in V_x$ and $V_x \subset G$. Clearly then, $G = \cup_{x \in G} V_x$, since each V_x is open so is G. $\square$

Note : Trivially if N is a nbd of a point x then so is any superset of N. It is also easy to show that the intersection of any two (and hence finitely many) neighbourhoods of a point is again a neighbourhood of that point.

Definition 12. *Let $(X, \mathcal{J})$ be a space and $A \subset X$. Then the interior (or more precisely the $\mathcal{J}$-interior) of A is defined to be the set of all interior points of A, i.e. the set $\{x \in A : A\ is\ a\ nbd\ of\ x\}$. It is denoted by A^0 or $int(A)$, or $int_{\mathcal{J}}(A)$ when we want to emphasise its dependence upon $\mathcal{J}$.*

Proposition 1.5.2. *Let X be a space and $A \subset X$. Then $int(A)$ is the union of all open sets contained in A. It is also the largest open subset of X contained in A.*

Proof. Let $\mathcal{U}$ be the family of all open sets contained in A ($\mathcal{U}$ is nonempty since $\phi \in \mathcal{U}$). Let $V = \cup_{G \in \mathcal{U}} G$. We have to show $V = int(A)$. Now if $x \in V$ then $x \in G$, for some $G \in \mathcal{U}$. This means A is nbd of x and so $x \in int(A)$. Conversely, let $x \in int(A)$. Then there is an open set H such that $x \in H$ and $H \subset A$. But then, $H \in \mathcal{U}$ and so $H \subset V$. So $x \in V$. This proves the first assertion of the proposition, and also shows that $int(A)$ is an open set contained in A. To see it is the largest such set, supposed G is an open set contained in A. Then $G \in \mathcal{U}$ and so $G \subset int(A)$ by the first assertion. $\square$

Definition 13. *The exterior of a set is defined as the interior of its complement. It is not hard to show that it always coincides with the complement of the closure of the original set. Consequently, it is not an independent concept and does not appear often in topology.*

Definition 14. *Let X be a space and $x \in X$. Let $\mathcal{N}_x$ be the set of all neighbourhoods of x in X (w.r.t. the given topology on X). The family $\mathcal{N}_x$ is called the neighbourhood system at x.*

Proposition 1.5.3. *Let X be a space and for $x \in X$, let $\mathcal{N}_x$ be the neighbourhood system at x. Then,*
i. If $U \in \mathcal{N}_x$, then $x \in U$.
ii. For any $U, V \in \mathcal{N}_x$, then $U \cap V \in \mathcal{N}_x$.
iii. If $V \in \mathcal{N}_x$ and $V \subset U$, then $U \in \mathcal{N}_x$.
iv. A set G is open in X iff $G \in \mathcal{N}_x$ for all $x \in G$.
v. If $U \in \mathcal{N}_x$ then there exists $V \in \mathcal{N}_x$ such that $U \subset V$ and $V \in \mathcal{N}_y$ for all $y \in V$.

Theorem 1.5.4. *Let X be a set and suppose for each $x \in X$, a non-empty family $\mathcal{N}_x$ of subsets of X is given satisfying (i), (ii), (iii) and (v) in the proposition above. Then there is a unique topology $\mathcal{J}$ on X such that for each $x \in X$, $\mathcal{N}_x$ coincides with the family of all neighbourhoods of x w.r.t. $\mathcal{J}$.*

Proof. If at all such a topology exists, then property (iv) gives us a clue for its construction. We let $\mathcal{J} = \{U \subset X : U \in \mathcal{N}_x \; for \; all \; x \in U\}$ and claim $\mathcal{J}$ is a topology on X. Clearly $\phi \in \mathcal{J}$. To show that $X \in \mathcal{J}$ note that for any $x \in X, X \in \mathcal{N}_x$ by (iii) as X is a superset of any member of $\mathcal{N}_x$. Property (ii) shows that $\mathcal{J}$ is closed under finite intersections while using (iii) it follows easily that $\mathcal{J}$ is closed under arbitrary unions. So $\mathcal{J}$ is a topology for X. Note that so far we did not use (i) and (v). With our definition of $\mathcal{J}$, (v) means that for any $x \in X$ and $U \in \mathcal{N}_x$, there exists an open set V (i.e. a member of $\mathcal{J}$) such that $V \in \mathcal{J}$ and $V \subset U$. From (i) we now get that $x \in V$. Thus U contains a member of $\mathcal{J}$ containing x and is therefore a $\mathcal{J}$-neighbourhood of the point x. Hence every member of $\mathcal{N}_x$ is neighbourhood of x w.r.t. $\mathcal{J}$. Conversely let U be a neighbourhood of x w.r.t. $\mathcal{J}$. Then U contains a member V of $\mathcal{J}$ such that $x \in V$ and $V \subset U$. But $V \in \mathcal{N}_x$ by the definition of $\mathcal{J}$ and so by (iii) $U \in \mathcal{N}_x$. Thus neighbourhoods of x w.r.t. $\mathcal{J}$ are precisely the members of $\mathcal{N}_x$ for each $x \in X$ and this completes the proof. □

Definition 15. *Let A be a subset of a topological space X and $y \in X$. Then y is said to be an accumulation point of A if every open set containing y contains at least one point of A other than y.*

In a discrete space no point is an accumulation point of any set while at the other extreme, in an indiscrete space, a point y is an accumulation point of any set A provided only that A contains atleast one point besides y. In the usual topology on the real line, every real number is an accumulation point of the set of rational numbers while the set of integers has no point of accumulation.

Definition 16. *Let A be a subset of a space X. Then the derived set of A, denoted by A', is the set of all accumulation points of A in X. Obviously A' depends not only on A but also on the topology under consideration.*

Theorem 1.5.5. *For a subset A of a space X, $\overline{A} = A \cup A'$.*

Proof. First we claim that $A \cup A'$ is closed or that $X \setminus (A \cup A')$ is open. We do so by showing that $X \setminus (A \cap A')$ is a nbd of each of its points. Let $y \in X \setminus (A \cup A')$. Then since y is not a point of accumulation of A, there exists an open set V containing y such that V contains no point of A except possibly y. But $y \notin A$, so we have $A \cap V = \phi$. We claim $A' \cap V$ is also empty. For, let $z \in A' \cap V$. Then V is an open set containing z which is an accumulation point of A. So $V \cap A$ is nonempty, a contradiction. So $A \cap V = \phi$ and hence $V \subset X \setminus (A \cup A')$. This proves that $A \cap A'$ is closed and since it obviously contains A, it also contains $\overline{A}$ i.e. $\overline{A} \subset A \cup A'$.
For the other way inclusion, it suffices to show that $A' \subset \overline{A}$ since we already have $A \subset \overline{A}$. So let $y \in A'$. If $y \notin \overline{A}$ then $y \in X \setminus \overline{A}$ which is an open set since A is always a closed set. But y is an accumulation point of A. So $(X \setminus A) \cap A \neq \phi$ which is a contradiction since $(X \setminus \overline{A}) \subset (X \setminus A)$ .So $y \in A'$. This completes the proof. □

Theorem 1.5.6. *For a subset A of a space X, $A = \{y \in X : every\, nbd\, of\, y\, meets\, A\, non-vacuously\}$.*

Proof. Let $B = \{y \in X : U \in \mathcal{N}_y \implies U \cap A \neq \phi\}$. We have to show that $B = \overline{A}$. By the theorem above, this amounts to showing that $A \cup A' = B$. First let $y \in A \cup A'$. If $y \in A$ then certainly every nbd of y meets A at least at the point y and so $y \in B$. If $y \in A'$ then too, by the definition of an accumulation point, every nbd of y contains a point of A and so $y \in B$. Thus $A \cup A' \subset B$. Conversely let $y \in B$. If $y \notin A \cup A'$, then $y \notin \overline{A}$ and so $X \setminus \overline{A}$ is a nbd of y which does not meet A, contradicting that $y \in B$. So $B \subset A \cup A'$. Thus $B = A$. □

Definition 17. *Let A be a subset of a space X. Then its boundary or frontier is the set $\overline{A} \cap \overline{(X \setminus A)}$. It is immediate from the definition that the boundary is always a closed set and that the boundary of a set is the same as the boundary of its complement. The boundary of a set A is generally denoted by ∂A or $F(A)$.*

Exercise :

1. Prove that the interior of a set is the same as the complement of the closure of the complement of the set, i.e. for a subset A of a space X, $int(A) = X \setminus \overline{(X \setminus A)}$.
2. Prove that in a metric space X, a point y is in the closure of a set A iff there exists a sequence $\{x_n\}$ such that $x_n \in A$ for all n and $\{x_n\}$ converges to y in X.
3. For a set A in a space X, prove that $\overline{A}$ is the disjoint union of $int(A)$ with the boundary of A.
4. Prove that a set is closed iff it contains its boundary and that it is open iff it is disjoint from its boundary.
5. Let X_1, X_2 be topological spaces and X their topological product. Let $A_1 \subset X_1$, $A_2 \subset X_2$. Prove that

$$\partial(A_1 \times A_2) = (\overline{A_1} \times \partial A_2) \cup (\partial A_1 \times \overline{A_2})$$

6. Prove or disprove that in a metric space, a closed ball is the closure of the open ball with the same centre and radius.

Rough space :

1.6 Continuity and Related Concepts

Definition 18. *Let $f : X \to Y$ be a function and $x_0 \in X$ and $\mathcal{J}, \mathcal{U}$ be topologies on X, Y respectively. Then f is said to be continuous (or more precisely $\mathcal{J} - \mathcal{U}$ continuous) at x_0 if for every $V \in \mathcal{U}$ such that $f(x_0) \in V$,there exists $U \in \mathcal{J}$ such that $x_0 \in U$ and $f(U) \subset V$.*

Proposition 1.6.1. *With the notation above, the following statements are equivalent.*
1. f is continuous at x_0.
2. The inverse image (under f) of every neighbourhood of $f(x_0)$ in Y is a neighbourhood of x_0 in X.
3. For every subset $A \subset X$, $x_0 \in \overline{A}$ implies $f(x_0) \in \overline{f(A)}$.

Definition 19. *Let $f : X \to Y$ be a function and $\mathcal{J}, \mathcal{U}$ be topologies on X, Y respectively. Then f is said to be continuous (or $\mathcal{J} - \mathcal{U}$ continuous) if it is continuous at each point of X.*

Theorem 1.6.2. *Let $(X, \mathcal{J}), (Y, \mathcal{U})$ be spaces and $f : X \to Y$ a function. Then the following statements are equivalent:*
1. f is continuous (i.e. $\mathcal{J} - \mathcal{U}$ continuous).
2. For all $V \in \mathcal{U}$, $f^{-1}(V) \in \mathcal{J}$.
3. There exists a sub-base $\mathcal{S}$ for $\mathcal{U}$ such that $f^{-1}(V) \in \mathcal{J}$ for all $V \in \mathcal{S}$.
4. For any closed subset A of Y, $f^{-1}(A)$ is closed in X.
5. For all $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.

Proof. (1) $\iff$ (2): Assume f is continuous at each point of X and V is an open subset of Y. If $x_0 \in f^{-1}(V)$ then $f(x_0) \in V$. But V is then a neighbourhood of $f(x_0)$ and so, by continuity at x_0, $f^{-1}(V)$ is a neighbourhood of x_0. Thus $f^{-1}(V)$ is a nbd of each of its points and so is an open set in X. Conversely suppose (2) holds and let $x_0 \in X$. Given any open set V containing $f(x_0)$. $f^{-1}(V)$ is an open set containing x_0 and moreover $f(f^{-1}(V)) \subset V$. This shows that f is continuous at x_0.
(2) $\iff$ (3): (2) clearly implies (3). For the converse, we note that the inverse image preserves intersections and unions. Let $V \in \mathcal{U}$. Then V can be written as $\cup_{i \in I} V_i$ where I is an index set and for each $i \in I$, V_i can be written as the intersection of finitely many members of $\mathcal{S}$ say $S_1^i \cap S_2^i \cap .... \cap S_{r_i}^i$, where $r_i \in \mathbf{N}$ and $S_j^i \in \mathcal{S}$ for $1 \leq j \leq r_i$. Then $f^{-1}(S_j^i) \in \mathcal{J}$. But $\mathcal{J}$ is closed under finite intersections and so $f^{-1}(V_i) \in \mathcal{J}$. Further $f^{-1}(V) \in \mathcal{J}$ since $f^{-1}(V) = \cup_{i \in I} f^{-1}(V_i)$ and $\mathcal{J}$ is closed under arbitrary unions. Thus (2) holds.
The equivalence of (2) with (4) follows from the fact that the inverse image preserves complements, i.e. that for any $B \subset Y$, $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$. □

Proposition 1.6.3. *Compositions of continuous functions are continuous. More specifically, if $f : X \to Y, g : Y \to Z$ are functions, with f continuous at x_0 and g continuous at $f(x_0)$*

then $g \circ f$ is continuous at x_0. The identity function on any space is continuous. More generally, if $\mathcal{J}$,$\mathcal{U}$ are topologies on a set X then the identity function $id_x : X \to X$ is $\mathcal{J} - \mathcal{U}$ continuous iff the topology $\mathcal{J}$ is stronger than $\mathcal{U}$. Any function from a discrete space and any function into an indiscrete space is continuous. Finally if $(X, \mathcal{J})$ is a space and $(Y, \mathcal{J}/Y)$ a subspace then the inclusion function $i : Y \to X$ is $\mathcal{J} - \mathcal{J}/\mathcal{Y}$ continuous. A restriction of a continuous function is continuous.

Definition 20. *Let $X_1, X_2, ..., X_n$ be sets and let $X = X_1 \times X_2 \times ... \times X_n$. For each $i = 1, 2, ..., n$, define $\pi_i : X \to X_i$ by $\pi_i(x_1, x_2, ..., x_n) = x_i$. Then π_i is called the projection on X_i, or the i-th projection. It is a surjective function except in the case where some other X_j and hence X is empty. If $x \in X$ then $\pi_i(x)$ is called the i-th coordinate of x.*

Theorem 1.6.4. *Let $\{(X_i, \mathcal{J}_i) : i = 1, 2, ..., n\}$ be a collection of topological spaces and $(X, \mathcal{J})$ their topological product. Then each projection π_i is continuous. Moreover, if Z is any space then a function $f : Z \to X$ is continuous if and only if $\pi_i \circ f : Z \to X_i$, is continuous for all $i = 1, 2, , .., n$.*

Proof. In order to prove that π_i is continuous, let $V \in \mathcal{J}_i$. We have to show that $\pi_i^{-1}(V) \in \mathcal{J}$. But it is easy to check that $\pi_i^{-1}(V) = X_1 \times X_2 \times ... \times X_{i-1} \times V \times X_{i+i} \times ... \times X_n$ because to say that $\pi_i(x) \in V$, for $x \in X$ puts no restriction on other coordinates of x, only the i-th coordinate is required to lie in V. Recalling the definition of product topology $\pi_i^{-1}(V)$ is a member of the defining base and so is open in X. Hence each projection is a continuous function.
For the second part, suppose $f : Z \to X$ is continuous. Then for each i, $\pi_i \circ f$ is continuous as compositions of continuous functions are continuous. Conversely, suppose each of $\pi_1 \circ f : Z \to X_i$ is continuous. Denote $\pi_i \circ f$ by f_i. To prove that f is continuous, it suffices to prove that the inverse image of any member of a base for $(X, \mathcal{J})$ is an open subset of Z. Now, by definition of the product topology, a base for $\mathcal{J}$ consists of all sets V of the form $V_1 \times V_2 \times ... \times V_n$ where $V_i \in \mathcal{J}_i$ for $i = 1, 2, ..., n$. It is then immediate that for $z \in Z$, $f(z) \in V$ iff $\pi_i(f(z)) \in V_i$, for all $i = 1, 2, ..., n$. In other words, $f^{-1}(V) = \cap_{i=1}^{n} f_i^{-1}(V_i)$. But each $f_i^{-1}(V_i)$ is open since f_i is assumed to be continuous. Hence $f^{-1}(V)$, being the intersection of finitely many open sets, is open in Z. This proves that f is continuous and completes the proof.

□

The preceding theorem is of immense use in checking the continuity of functions into euclidean spaces. As a euclidean space is the topological product of a finite number of copies of the real line (with usual topology), the problem reduces to checking the continuity of real-valued functions.

Definition 21. *Let X, Y by spaces. A function $f : X \to Y$ is said to be open (respectively closed) if whenever A is an open (resp. a closed) subset of X, $f(A)$ is open (resp. a closed) subset of Y.*

Definition 22. *A homeomorphsim from a space X to space Y is a bijection $f : X \to Y$ such that both f and f^{-1} are continuous. When such a homeomorphism exists, X is said to be homeomorphic to Y.*

Proposition 1.6.5. *Let X, Y be topological spaces and $f : X \to Y$ a function. Then the following statements are equivalent:*
1. f is a homeomorphism,
2. f is a continuous bijection and f is open,
3. f is a bijection and f^{-1}, f are both open map,
4. there exists a function $g : Y \to X$ such that f, g are continuous, $g \circ f = id_x$ and $f \circ g = id_y$.

Definition 23. *Let $(X, \mathcal{J}), (Y, \mathcal{U})$ be topological spaces. An embedding (or imbedding) of X into Y is a function $e : X \to Y$ which is a homeomorphism when regarded as a function from $(X, \mathcal{J})$ onto $(e(X), \mathcal{U}/e(X))$.*

Proposition 1.6.6. *A function $e : X \to Y$ is an embedding iff it is continuous and one-to-one and for every open set $V \in X$, there exists an open subset W of Y such that $e(V) = W \cap e(X)$.*

Inclusion maps are the most immediate examples of embeddings and as the definition implies, these are the only examples upto homeomorphisms. An important problem in topology is to decide when a space X can be embedded in another space Y i.e. when there exists an embedding from X into Y. This is called the embedding problem. Theorems asserting the embeddability of a space into some other space which is more manageable than the original space are known as embedding theorems.

Exercise :

1. Let $(X, \mathcal{J}), (Y, \mathcal{U})$ be topological spaces such that every function from X to y is $\mathcal{J} - \mathcal{U}$ continuous. Prove that either $\mathcal{J}$ is the discrete topology or $\mathcal{U}$ is the indiscrete topology.
2. Let $(X, \mathcal{J}_1), (X_2, \mathcal{J}_2)$ be topological spaces. Prove that $X_1 \times X_2$ is homeomorphic to $X_2 \times X_1$ each being given the product topology.
3. Let $f : X \to Y$ be a function. The graph of f is defined to be the set $G = \{(x, f(x)) : x \in X\}$. Prove that if f is continuous then G is homeomorphic to X.
4. Let S^1 be the unit circle in the plane, with the relative topology on it. Define $p : R \to S^1$ by $p(x) = (cos(x), sin(x))$. Prove that p is continuous.
5. Prove that the set of all homeomorphisms of a space $(X, \mathcal{J})$ onto itself is a subgroup of

the permutation group of X.

Rough space :

Module 2

Spaces with special properties

2.1 Making Functions Continuous, Quotient Spaces

Problem 1 : Let $\{(Y_i, \mathcal{J}_i) : i \in I\}$ be an indexed family of topological spaces, X any set and $\{f_i : i \in I\}$ an indexed collection of functions such that for each $i \in I$, f_i is a function from X to Y_i. What topology $\mathcal{J}$ on X will make each f_i $\mathcal{J} - \mathcal{J}_i$ continuous?

Problem 2 : Let $\{(Y_i, \mathcal{J}_i) : i \in I\}$ be an index family of topological spaces, X any set and $\{f_i : i \in I\}$ an indexed family of functions from Y, into X. What topology $\mathcal{U}$ on X will make each f_i $\mathcal{J}_i - \mathcal{U}$ continuous?

Either problem can be obtained from the other by reversing the directions in which the functions go. For this reason, the two problems are said to be dual to each other. Their duality will be reflected through their solutions. A trivial solution to the first problem would be to let $\mathcal{J}$ be the discrete topology on X. This is hardly satisfactory. We therefore look for a better solution. Note that if a topology $\mathcal{J}$ on X renders each f_i continuous, then so will any topology stronger than $\mathcal{J}$. The idea then is to find as small a topology on X as possible which will make each f_i continuous. Dually, a trivial solution to the second problem would be to let $\mathcal{U}$ be the indiscrete topology on X. If a topology $\mathcal{U}$ on X renders each f_i continuous then so will any topology weaker than $\mathcal{U}$. The idea then is to find as large a topology on X as possible which will make each function f_i continuous.

Theorem 2.1.1. *With the notation of Problem 1, there exists a unique smallest topology $\mathcal{J}$ on X which makes each f_i continuous.*

Proof. First note that if $\mathcal{D}$ is any topology on X for which $f_i : X \to Y_i$ is $\mathcal{D} - \mathcal{J}_i$ continuous, then $\mathcal{D}$ must contain all sets of the form $f_i^{-1}(V_i)$ as V_i ranges over all open subsets of Y_i. This gives us a clue to the construction of the smallest such topology. We let $\mathcal{S} = \{f_i^{-1}(V_i) : V_i \in \mathcal{J}_i, i \in I\}$. Then $\mathcal{S}$ is a collection of subsets of X. $\mathcal{S}$ itself need not be a topology for X. However we let $\mathcal{J}$ be the topology generated by $\mathcal{S}$, that is, the topology having $\mathcal{S}$ as a sub-base. Then clearly each f_i is $\mathcal{J} - \mathcal{J}_i$ continuous. Further if $\mathcal{D}$ is any other topology on X which makes each f_i continuous, then as we just observed $S \subset \mathcal{U}$ and so $\mathcal{J} \subset \mathcal{U}$. Thus $\mathcal{J}$ is the smallest topology on X which makes each f_i continuous. $\square$

Definition 24. *With the notations above, the weakest topology on X making each f_i continuous is called the weak topology determined by the family of functions $\{f_i : i \in I\}$.*

Theorem 2.1.2. *With the notation of Problem 2, there exists a unique largest topology $\mathcal{U}$ on X which makes each f_i continuous.*

Proof. Let $\mathcal{D}$ be a topology on X which makes each f_i continuous. Then for any $D \in \mathcal{D}$, and $i \in I$, $f_i^{-1}(D)$ is open in Y_i. This puts a restriction on how large $\mathcal{D}$ can be and gives us a clue to the construction of the best possible solution. We let $\mathcal{U} = \{A \subset X : f_i^{-1}(A) \in \mathcal{J}_i \ for \ all \ i \in I\}$. Since the inverse images commute with intersections and unions, it is easy to show that $\mathcal{U}$ is a topology on X and by what we said earlier, it is the strongest topology on Y making each f_i continuous. $\square$

Definition 25. *With the notations above, the strongest topology on X making each f_i continuous is called the strong topology determined by the family of functions $\{f_i : i \in I\}$.*

We shall now discuss few classic situations in which the problem of finding the weak topology is important.

Theorem 2.1.3. *The product topology is the weak topology determined by the projection functions.*

Proof. First, let $\{(X, \mathcal{J}_i) : i = 1, 2, ..., n\}$ be a collection of topological spaces. Let $X = X_1 \times X_2 \times ..... \times X_n$, and let $\pi_i : X \to X_i$ be the i-th projection function for $i = 1, 2, ..., n$. We claim that $\mathcal{J}$ is in fact the smallest topology which makes each π_i continuous. For suppose $\mathcal{U}$ is any other topology on X such that each π_i is $\mathcal{U} - \mathcal{J}_i$ continuous. Let V be an open box in X, say, $V = V_1 \times V_2 \times .... \times V_n$ where $V_i \in \mathcal{J}_i$ for $i = 1, 2, ..., n$. Then clearly $V = \cap_{i=1}^{n} \pi_i^{-1}(V_i)$. But each $\pi_i^{-1}(V_i) \in \mathcal{U}$ since π_i is $\mathcal{U} - \mathcal{J}_i$ continuous. So $V \in \mathcal{U}$ since $\mathcal{U}$ is closed under finite intersections. Now the family of all such open boxes is, by definition, a base for the product topology $\mathcal{J}$ on X. Hence $\mathcal{J} \subset \mathcal{U}$. We have thus proved the following theorem in the case of finite products. $\square$

Theorem 2.1.4. *Let X have the weak topology determined by a family $\{f_i : X \to Y_i : i \in I\}$ of functions where each Y_i is a topological space, I being an index set. Then for any space Z, a function $g : Z \to X$ is continuous iff for each $i \in I$, the composite $f_i \circ g : Z \to Y_i$ is continuous.*

Proof. If g is continuous then so is each $f_i \circ g$, as the compositions of continuous functions are continuous. Conversely, recall that the family $\mathcal{S}$ of all subsets of the form $f_i^{-1}(V_i)$, where V_i is open in Y_i, $i \in I$ is a sub-base for the weak topology on X. So, in order to show that $g : Z \to X$ is continuous, it suffices to show that the inverse image under g of every member of $\mathcal{S}$ is open in Z. Now if $i \in I$ and V_i is open in Y_i, then $g^{-1}(f_i^{-1}(V_i)) = (f_i \circ g)^{-1}(V_i)$ which is open by continuity of $f_i \circ g$. This shows that g is continuous. $\square$

Theorem 2.1.5. *Let X have the strong topology determined by a family $\{f_i : Y_i \to X : i \in I\}$ of functions where each Y_i is a topological space, I being an index set. Then for any space Z, a function $g : X \to Z$ is continuous iff for each $i \in I$ the composite $g \circ f_i : Y_i \to Z$ is continuous.*

Proof. The proof is dual to the previous one. $\square$

Definition 26. *Let $(X, \mathcal{J}), (Y, \mathcal{U})$ be topological spaces and $f : X \to Y$ be an onto function. Then f is said to be a quotient map or Y is said to have the quotient topology w.r.t. X and f if $\mathcal{U}$ is the strong topology generated by the singleton family $\{f\}$. In such a case we also say that Y is a quotient of X.*

Each factor space is indeed a quotient space of the topological product of spaces. Let $\{X_i, \mathcal{J}_i : i = 1, 2, ..., n\}$ be a family of non-empty topological spaces. Let $(X, \mathcal{J})$ be their topological product. Then for any i, the projection $\pi_i : X \to X_i$ is onto. We claim $\mathcal{J}_i$ is the quotient topology w.r.t. X and π_i. Let $V \subset X_i$. If V is open in X_i, then $\pi_i^{-1}(V)$ is open in X by continuity of π_i. On the other hand, if $\pi_i^{-1}(V)$ is open in X then $V = \pi_i(\pi_i^{-1}(V))$ is open in X_i because the projection functions are open. Thus V is open in X_i iff $\pi_i^{-1}(V)$ is open in X and so $\mathcal{J}$ the quotient topology.

Proposition 2.1.6. *Every open, surjective map is a quotient map.*

Proof. Use the same argument as above. $\square$

Proposition 2.1.7. *Every closed, surjective map is a quotient map.*

Proof. Let $f : (X, \mathcal{J}) \to (Y, \mathcal{U})$ be closed, continuous and onto. Let $V \subset Y$. If V is open in Y, then $f^{-1}(V)$ is open in X by continuity of f. On the other hand suppose $f^{-1}(V)$ is open in X. Then $X \setminus f^{-1}(V)$ is closed in $\mathcal{J}$ and $f(X \setminus f^{-1}(V))$ is closed in Y because f is a closed function. Since f is onto, $X \setminus f^{-1}(V))$ equals $Y \setminus V$. Thus V is open in Y. This shows that $\mathcal{U}$ is the strong topology determined by $\mathcal{J}$ and f. Therefore f is a quotient map. $\square$

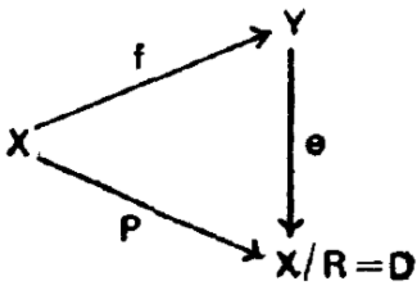
It is not true that every quotient map is either open or closed. The class of quotient maps is quite large inasmuch as any surjective function on any topological space becomes a quotient map if we put the quotient topology on its codomain.

Definition 27. *A topological property is said to be divisible if whenever a space has it, so does every quotient space of it.*

Proposition 2.1.8. *The property of being a discrete space is divisible. In other words, every quotient space of a discrete space is discrete.*

Proof. Let $f : X \to Y$ be a quotient map where X is a discrete space. Let $\mathcal{U}$ be the quotient topology on Y. We have to show that $(Y,\mathcal{U})$ is discrete. Recall that $\mathcal{U}$ is the strongest topology on Y which makes f continuous. But since the domain X is discrete, any topology on Y renders f continuous. So $\mathcal{U}$ is the strongest topology on Y, that is the discrete topology. $\square$

Quotient spaces provide an important tool for the construction of new topological spaces from old ones. For example, let $f : X \to Y$ be a surjective function. Then f determines an equivalence relation R on X defined by xRy iff $f(x) = f(y)$. The equivalence classes of R are precisely the inverse images of singleton subsets of Y. Now let $\mathcal{D}$ be the collection of all equivalence classes under R. $\mathcal{D}$ is called the quotient set of X by R and is also denoted by X/R. There is a canonical function $p : X- \to X/R$, called the projection which assigns to each $x \in X$ its equivalence class under R. The function $\theta : Y \to X/R$ defined by $\theta(y) = p(x)$ for any $x \in f^{-1}(\{y\})$ is obviously a well-defined bijection and the following diagram is commutative.



So far, X was merely a set and f a set-theoretic surjection. Suppose now that X, Y are topological spaces and that f is a quotient map. On X/R, we put the strong topology generated by the projection function p. The function θ then becomes continuous as its composite with f viz., $\theta \circ f$ is continuous. Similarly θ^{-1} is continuous. Thus θ is now not merely a bijection but a homeomorphism. Thus, upto a topological equivalence, we may identify the quotient space Y with the quotient space X/R and the quotient map f with the projection p.

It follows that given a space X, any quotient space of X can be obtained starting from an equivalence relation R on X and putting the strong topology on the set of equivalence classes X/R. This gives us an important tool to generate new topological spaces because there is absolutely no restriction on the relation R on X other than that it be an equivalence relation.

Exercise :

1. Prove that for a metric space, the weak topology determined by the family of all continuous real-valued functions on it coincides with the metric topology.
2. Let X_1 be the set of rational numbers and X_2 the set of all irrational numbers, each with the usual topology. Let i_1, i_2 be the inclusion functions of X_1, X_2 respectively into $\mathbf{R}$. Find the strong topology on $\mathbf{R}$ which makes i_1, i_2 continuous.
3. Let X be a space and $\mathcal{U}$ be an open cover of it. Prove that a function f from X into some space Y is continuous iff for each $U \in \mathcal{U}$, the restriction f_U is continuous.
4. Prove that the composite of two quotient maps is a quotient map.
5. Define a relation R on $\mathbf{R}$ by xRy iff $x - y \in \mathbf{Q}$. Prove that R is an equivalence relation on $\mathcal{R}$ and that the quotient space $\mathcal{R}/R$ is indiscrete.
6. Obtain the torus surface as a quotient space of the unit square.

Rough space :

2.2 Spaces with Special Properties

2.2.1 Smallness Conditions on a Space

Definition 28. *A subset A of a space X is said to be a compact (Lindeloff) subset of X if every cover of A by open subsets of X has a finite (respectively countable) subcover. A space X is said to be compact (Lindeloff) if X is a compact (resp. Lindeloff) subset of itself.*

Proposition 2.2.1. *Let $(X, \mathcal{J})$ be a topological space and $A \subset X$. Then A is a compact (Lindeloff) subset of X if and only if the subspace $(A, \mathcal{J}/A)$ is compact (resp. Lindeloff).*

Proof. We give the argument for compactness. Replacing 'finite' by 'countable' in it, we could use it for Lindeloff subsets. Suppose first that A is a compact subset of X. Let $\mathcal{G}$ be an open cover of the space $(A, \mathcal{J}/A)$. Each member G of $\mathcal{G}$ is of the form $H \cap A$ for some $H \in \mathcal{J}$. For each $G \in \mathcal{G}$, fix $D(G) \in \mathcal{J}$ such that $G = D(G) \cap A$. Then the family $\{D(G) : G \in \mathcal{G}\}$ is a cover of A by open subsets of X. Since A is a compact subset of X, this cover has a finite subcover, say, $\{D(G_i) : i = 1, 2, ..., n\}$ where $G_i \in \mathcal{G}$ for all $i = 1, 2, ..., n$. Clearly then, $\{G_1, G_2, ..., G_n\}$ is a finite sub-cover of $\mathcal{G}$. This shows that the subspace $(A, \mathcal{J}/A)$ is compact.
Conversely suppose the subspace $(A, \mathcal{J}/A)$ is compact. Let $\mathcal{G}$ be a cover of A by open subsets of X. Then $\{G \cap A : G \in \mathcal{G}\}$ is an open cover of the space $(A, \mathcal{J}/A)$. By compactness of the space $(A, \mathcal{J}/A)$, this cover has a finite subcover, say $\{G_i \cap A : i = 1, 2, ..., n\}$ where $G_i \in \mathcal{G}$ for $i = 1, 2, ..., n$. Clearly then, $\{G_1, G_2, ..., G_n\}$ is a finite subfamily of $\mathcal{G}$ covering the set A. Thus A is a compact subset of X. $\square$

The above proposition leads to an interesting concept. Suppose we have a set X, a subset A and two topologies $\mathcal{J}_1, \mathcal{J}_2$ on X such that $\mathcal{J}_1/A = \mathcal{J}_2/A$. If we know that A is a compact subset of $(X, \mathcal{J}_1)$, nn view of the last proposition, it follow that A is also a compact subset of $(X, \mathcal{J}_2)$. The proposition above shows that as long as $\mathcal{J}_1$ and $\mathcal{J}_2$ relativise to the same topology on A, A is a compact subset of $(X, \mathcal{J}_1)$ iff it is a compact subset of $(X, \mathcal{J}_2)$. In other words, compactness of a subset depends only on the topology induced on it, the topology on the entire space is not directly relevant. To stress this point further we note that a similar statement no longer holds for denseness of a subset. It may very well happen that two distinct topologies $\mathcal{J}_1, \mathcal{J}_2$ on a set X induce the same topology on A and still A is a dense subset of X w.r.t. $\mathcal{J}_1$ but not w.r.t. $\mathcal{J}_2$. A trivial example of this occurs when $\mathcal{J}_1$ is the indiscrete topology, $\mathcal{J}_2$ is the discrete topology and A is a singleton subset of X.

Definition 29. *A property of a subset of a topological space is said to be an absolute property if it depends only on the relativised topology on that set, otherwise it is called a relative property. Thus compactness and the property of being a Lindeloff subset are absolute properties while denseness is a relative property. Similarly being an open set is a relative property.*

Definition 30. *A space is said to be separable if it contains a countable dense subset.*

Theorem 2.2.2. *Every second countable space is Lindeloff.*

Proof. Straightforward. □

Theorem 2.2.3. *Every second countable space is separable.*

Proof. We already proved this as part of an exercise. □

Note :

- The converses of both these theorems hold for metric spaces. In general, however, the converses fails.
- Every compact space is Lindeloff. The converse is false. The real line with the usual topology is not compact because the cover by open intervals of the form $(-n, n)$ where n varies overall positive integers, has no finite subcover. However, it is second countable and hence Lindeloff.
- Every cofinite space is compact. For, let X be such a space and let $\mathcal{U}$ be an open cover of X. Take any non-empty member U of $\mathcal{U}$. Then $X \setminus U$ has only finitely many points say $x_1, x_2, ..., x_n$. For each such x_i, choose $U_i \in \mathcal{U}$ such that $x_i \in U_i$. Then $\{U, U_1, U_2, ..., U_n)$ is a finite subcover of $\mathcal{U}$.

Theorem 2.2.4. *Every continuous real-valued function on a compact space is bounded and attains its extrema.*

Proof. Let X be a compact space and suppose $f : X \to \mathbf{R}$ is continuous. First we show f is bounded. For each $x \in X$, let J_x be the open interval $(f(x) - 1, f(x) + 1)$ and let $V_x = f^{-1}(J_x)$. By continuity of f, V_x is an open set containing x. Note that f is bounded on each V_x. Now the family $\{V_x : x \in X\}$ is an open cover of X and by compactness, admits a finite sub-cover say $\{V_{x_1}, V_{x_2}, ..., V_{x_n}\}$. Let $M = max\{(f(x_1), f(x_2), ..., f(x_n)\} + 1$ and let $m = min\{f(x_1), f(x_2), ..., f(x_n)\} - 1$. Now for any $x \in X$ there is some i such that $x \in V_{x_i}$. Then $f(x_i) - 1 < f(x) < f(x_i) + 1$ and so $m < f(x) < M$ showing that f is bounded. It remains to show that f attains its bounds. Let L, λ be respectively the supremum and infimum of f over X. If there is no point x in X for which $f(x) = L$, then we define a new function $g : X \to \mathbf{R}$ by $g(x) = \frac{1}{L - f(x)}$ for all $x \in X$. Then g is continuous. However g is unbounded, for given any $r > 0$, there exists x such that $f(x) > L - \frac{1}{r}$ and hence $g(x) > r$. This contradicts the earlier part of the theorem and shows that f attains the value L. Similarly f attains the infimum A. □

Theorem 2.2.5. *(Lebesgue covering lemma) Let (X, d) be a compact metric space and let $\mathcal{U}$ be an open cover of X. Then there exists a positive real number r such that for any $x \in X$ there exists $V \in \mathcal{U}$ such that $B(x, r) \subset V$.*

Proof. By compactness of the space X, we may suppose that the cover $\mathcal{U}$ is finite say $\mathcal{U} = \{U_1, U_2, ..., U_n\}$. We may also assume that each U_i is nonempty and proper (if $U_i = X$ for some i, then any r will work). Let $A_i = X \setminus U_i$. Then each A_i is a non-empty closed set. Define $f_i : A_i \to \mathbf{R}$ by $f_i(x) = d(x, A_i) = inf\{d(x, y) : y \in A_i\}$. Then f_i is continuous and is positive on U_i for each $i = 1, 2, ..., n$. Define $f : X \to \mathbf{R}$ by $f(x) = max\{f_i(x) : i = 1, 2, ..., n\}$ and f is continuous. Note that f is positive everywhere since for any $x \in X$ there is some i such that $f_i(x) > 0$. So by the last theorem, f has a minimum on X and this minimum is positive. Let r be the minimum. Now let $x \in X$. Then for some $i, f(x) = f_i x$. We contend $B(x, r) \subset U_i$. For suppose $d(x, y) < r$. If $y \notin U_i$ then $y \in A_i$ whence, $r > d(x, y) \geq d(x, A_i) = f_i(x) = f(x)$ contradicting that r is the minimum of f. So $B(x, r) \subset U_i$ and we can let U_i be the desired set $V \in \mathcal{U}$. □

A number r which satisfies the conclusion of the theorem is called a Lebesgue number of the cover $\mathcal{U}$.

Theorem 2.2.6. *Let X be a compact space and suppose $f : X \to Y$ is continuous and onto. Then Y is compact. In other words, every continuous image of a compact space is compact.*

Proof. Let $\mathcal{V}$ be any open cover of Y, Let $\mathcal{U} = \{f^{-1}(V) : V \in \mathcal{V}\}$. Then $\mathcal{V}$ is a cover of X and since f is continuous, it is an open cover of X. Since X is compact some finitely many members of $\mathcal{U}$, say $\{f^{-1}(V_1), f^{-1}(V_2), ..., f^{-1}(V_n)\}$ where $V_1, V_2, ..., V_n \in \mathcal{V}$ cover X. But then $V_1, V_2, ..., V_n$ cover Y since f is onto. So Y is compact. □

Definition 31. *A topological property is said to be preserved under continuous functions if whenever a space has it so does every continuous image of it, that is if $f : X \to Y$ is continuous and onto and X has the given property, then so does Y.*

Properties such as compactness, Lindeloff and separability are preserved under continuous function. However, second countability is not preserved under continuous functions.

Definition 32. *A topological property is hereditary if whenever a space has it, so does every subspace of it. Second countability is a hereditary property. On the other hand separability is not hereditary. The properties of compactness and of being a Lindeloff space are not hereditary.*

Proposition 2.2.7. *If X is a compact (Lindeloff) space and $A \subset X$ is closed in X then A in its relative topology, is also compact*

Proof. Suppose X is compact and $A \subset X$ is closed. Let $\mathcal{U}$ be an open cover of A in the relative topology on A. For each $U \in \mathcal{U}$, fix an open set $V(U)$ in X such that $A \cap V(U) = U$. Then the family $\mathcal{V} = \{V(U) : U \in \mathcal{U}\} \cup \{X \setminus A\}$ is an open cover of X and hence admits a finite subcover consisting of, say, $V(U_1), V(U_2), ..., V(U_n)$ and possibly $X \setminus A$.

But then $\{U_1, U_2, ..., U_n\}$ covers A and is a finite subcover of $\mathcal{U}$. This shows that A in its relative topology, is compact. The proof that A is Lindeloff in case X is, is similar. $\square$

Definition 33. *A topological property is said to be weakly hereditary if whenever a space has it, so does every closed subspace of it. With this terminology, the last proposition shows that compactness and the Lindeloff property are weakly hereditary.*

Definition 34. *Let X be a space and $x \in X$. Then a local base at x is a collection $\mathcal{L}$ of neighbourhoods of x such that given any neighbourhood N of x there exists $L \in \mathcal{L}$ satisfying $L \subset N$.*

The collection of all open neighbourhoods of a point is a local base at that point. If X is a metric space, then the collection of all open balls (closed balls will do as well) centred at a point constitutes a local base at that point. Note that it suffices to take open balls of rational radii, thereby giving a local base which is countable.

Definition 35. *A space is said to be first countable at a point if there exists a countable local base at that point and a space is said to be first countable or to satisfy the first axiom of countability if it is first countable at each point.*

Theorem 2.2.8. *Every second countable space is first countable.*

Proof. Let X be a second countable space and $x \in X$. Suppose $\mathcal{B}$ is a countable base for X. Let $\mathcal{L} = \{B \in \mathcal{B} : x \in B\}$. Clearly $\mathcal{L}$ is countable and the proof will be complete if we show that it is a local base at x. For this, let V be a neighbourhood of x. Then there exists an open set V such that $x \in V \subset N$. Since $\mathcal{B}$ is base, there exists B such that $x \in B \subset V$. But then $B \in \mathcal{L}$, showing that X is a local base at x. $\square$

Remark : The converse of the proposition is false. The real line with the semi-open interval topology is not second countable. However it is first countable.

Theorem 2.2.9. *Let X, Y be spaces, $x \in X$ and $f : X \to Y$ a function. Suppose X is first countable at x. Then f is continuous at x iff for every sequence $\{x_n\}$ which converges to x in X, the sequence $\{f(x_n)\}$ converges to $f(x)$ in Y.*

Proof. The direct implication is easy and does not require that X be first countable at x. For the other way implication suppose $\{V_1, V_2, ..., V_n ...\}$ is a countable local base at x. We let $W_1 = V_1$, $W_2 = V_1 \cap V_2$, $W_3 = V_1 \cap V_2 \cap V_3$,...., $W_n = V_1 \cap V_2 \cap ... \cap V_n$,... etc. Then the collection $\{W_n : n = 1, 2, 3, ...\}$ is also a local base at x. Its advantage over the given base is that it is a nested local base, that is, $W_m \subset W_n$ for $m \geq n$. The reason for switching to W_i's from V_i's will be clear in the course of the proof.
Suppose f is not continuous at x. Then there exists a subset A, of X such that $x \in \overline{A}$ but $f(x) \notin \overline{f(A)}$. Now for each n, W_n is a neighbourhood of x and since $x \cap A$, $W_n \cap A \neq \phi$.

Choose $x_n \in W_n \cap A$, for $n \in \mathbf{N}$. Note that for $m \geq n$, $x_m \in W_n$ since $W_m \subset W_n$. We claim that the sequence $\{x_n\}$ converges to x in X. For, let G be an open set containing x. Then, since $\{W_1, W_2, ..., W_n, ...\}$ is a local base at x, there is some n such that $W_n \subset G$. But then for all $m > n$, $x_m \in W_m$ and so $x_m \in G$. So $\{x_n\}$ converges to x in X. However, for each n, $f(x_n) \in f(A) \subset \overline{f(A)}$. We are assuming that $f(x) \notin \overline{f(A)}$. Then $Y \setminus \overline{f(A)}$ is an open set which contains $f(x)$ but contains no term of the sequence $\{f(x_n)\}$. Consequently $\{f(x_n)\}$ does not converge to $f(x)$ in Y. This contradicts the hypothesis and shows that f is continuous at X.

□

Consider now the space $X = \mathbf{N} \times \mathbf{N} \cup \{\infty\}$ discussed previously. This space is first countable at each $(x, y) \in \mathbf{N} \times \mathbf{N}$. Indeed at any such point the singleton family $\{\{(x, y)\}\}$ is a local base. However, X is not first countable at the point ∞. It is not hard to show this directly, but an indirect argument using the last theorem can be given as follows. Define $f : X \to \mathbf{R}$ by $f(x, y) = 0$ for $(x, y) \in \mathbf{N} \times \mathbf{N}$ and $f(\infty) = 1$. Then f is not continuous at ∞ although the condition in the last theorem is satisfied because all the sequences in X which converge to ∞ are eventually constant. Hence X cannot be first countable at ∞, for otherwise f would be continuous at ∞ by the last theorem. We thus have an example of a countable space (i.e. a space whose underlying set is countable) which is not first countable.

Definition 36. *A space is said to satisfy the countable chain condition if any family of mutually disjoint open sets in it is countable.*

Proposition 2.2.10. *Every separable (and hence every second countable) space satisfies the countable chain condition.*

Proof. Suppose D is a countable dense subset of a space X. Let $\mathcal{Q}$ be a family of open sets in X such that for $G, H \in \mathcal{Q}$, $G \cap H = \phi$ unless $G = H$. We have to show $\mathcal{Q}$ is countable. Suppose first that $\phi \notin \mathcal{Q}$. For each $G \in \mathcal{Q}$, $G \cap D$ is nonempty since D is dense in X. Choose a point $x_G \in G \cap D$ and define the function $f : \mathcal{Q} \to D$ by $f(G) = x_G$. Then f is one-to-one since $x_G = x_H$ would imply $G \cap H \neq \phi$, and hence $G = H$. Since the set D is countable, it follows that $\mathcal{Q}$ is countable. In case the empty set $\phi \in \mathcal{Q}$, we apply the argument to $\mathcal{Q} \setminus \{\phi\}$ and find that it is countable. But then $\mathcal{Q}$ is also countable. □

Exercise :

1. Prove that the co-countable topology on a set (defined analogously to the cofinite topology, by letting the whole set and all countable subsets be closed) makes it into a Lindeloff space.
2. Let x be an uncountable set with the cocountable topology on it. Prove that X is not separable, although it satisfies the countable chain condition.

3. Let X be an uncountable set with the cofinite topology on it. Prove that X is separable but not first countable at any point.
4. Prove that every infinite subset A of a compact space X has at least one accumulation point in X.
5. Using the Lebesgue covering lemma, prove that every continuous function from a compact metric space into another metric space is uniformly continuous.
6. Prove that first countability is a hereditary property.

Rough space :

2.2.2 Connectedness

Definition 37. *A space X is said to be connected if it is impossible to find non-empty subsets A and B of it such that $X = A \cup B$ and $\overline{A} \cap \overline{B} = \phi$. A space which is not connected is called disconnected.*

Proposition 2.2.11. *Let X be a space and A, B subsets of X. Then the following statements are equivalent:*
1. $A \cup B = X$ and $\overline{A} \cap \overline{B} = \phi$.
2. $A \cup B = X$, $A \cap B = \phi$ and A, B are both closed in X.
3. $B = X \setminus A$ and A is clopen (i.e. closed as well as open) in X.
4. $B = X \setminus A$ and ∂A (that is, the boundary of A) is empty.
5. $A \cup B = X$, $A \cap B = \phi$ and A, B are both open in X.

Proposition 2.2.12. *Let X be a space. Then the following are equivalent:*
1. X is connected.
2. X cannot be written as the disjoint union of two nonempty closed subsets.
3. The only clopen subsets of X are ϕ and X.
4. Every nonempty proper subset of X has a nonempty boundary.
5. X cannot be written as the disjoint union of two nonempty open subsets.

Every indiscrete space is connected and that the only connected discrete spaces are those which consist of at most one point. The space of rational numbers is disconnected; given any irrational number a the sets $\{x \in \mathbf{Q} : x < a\}$ and $\{x \in \mathbf{Q} : x > a\}$ are both open in the relative topology on $\mathbf{Q}$ and $\mathbf{Q}$ is clearly their disjoint union. Similarly the set of irrational numbers is disconnected.

Proposition 2.2.13. *Let $f : X \to Y$ be a continuous surjection. Then if X is connected, so is Y.*

Proof. Suppose Y is not connected. Then we can write $Y = A \cup B$ where A, B are disjoint, nonempty and open subsets of Y. But then $X = f^{-1}(A) \cup f^{-1}(B)$. The sets $f^{-1}(A), f^{-1}(B)$ are mutually disjoint, and open since f is continuous. Further each is nonempty since f is onto. This contradicts that X is connected. Hence Y is connected. $\square$

Theorem 2.2.14. *A subset of $\mathbf{R}$ is connected iff it is an interval.*

Proof. First note that a subset $I \subset \mathbf{R}$ is an interval iff it has the property that for any $a, b \in X$, $(a, b) \subset X$. Now if X is not an interval then there exist real numbers a, b, c such that $a < c < b$; $a, b \in X$ and $c \notin X$. Let $A = \{x \in X : x < c\}$ and $B = \{x \in X : x > c\}$. Clearly A, B are disjoint, open subsets of X (in the relative topology) since $A = X \cap (-\infty, c)$ and $B = (c, \infty) \cap X$ and $A \cup B = X$. Further $a \in A$, $b \in B$ and hence A, B are nonempty. This shows that X is not connected.

Conversely suppose X is an interval and that $X = A \cup B$ where $\overline{A} \cap \overline{B} = \phi$, $A \neq \phi$, $B \neq \phi$ where the closure is relative to X. Let $a_0 \in A$, $b_0 \in B$. Without loss of generality we may suppose that $a_0 < b_0$. Now let x be the mid-point of the interval from a_0 to b_0, i.e. $x = \frac{a_0+b_0}{2}$. Then $x \in X$ and so x is exactly in one of the sets A and B. If $x \in A$ we rename it as a_1 and rename b_0 as b_1. If $x \in B$, we rename a_0 as a_1 and x as b_1. In any case $[a_1, b_1]$ is an interval with its left end-point in A and the right end-point in B. We can now take the mid-point of $[a_1, b_1]$ and get an interval $[a_2, b_2]$ of half the length with $a_2 \in A$, $b_2 \in B$. Repeating this process ad infinitum, we get a nested sequence of intervals $\{[a_n, b_n] : n = 0, 1, 2, 3, ...\}$ such that $a_n \in A$ and $b_n \in B$ for all n. Note that $\{a_n\}$ is a bounded monotonically increasing sequence while $\{b_n\}$ is a bounded monotonically decreasing sequence and that $(b_n - a_n) \to 0$ as $n \to \infty$. By the order completeness of $\mathbf{R}$, both sequences converge to a common limit, say c. Note that $c \in X$ since $a_0 \leq c \leq b_0$. Also every neighbourhood of c intersects A as well as B. So $c \in \overline{A} \cap \overline{B}$ a contradiction. Hence X is connected. □

Theorem 2.2.15. *Every closed and bounded interval is compact.*

Proof. Since any closed and bounded interval $[a, b]$ (with $a < b$) is homeomorphic to the unit interval $[0, 1]$, it suffices to prove that $[0, 1]$ is compact. Let $\mathcal{U}$ be an open cover of $[0, 1]$. An element U of $\mathcal{U}$ is open relative to $[0, 1]$ and hence is of the form $V \cap [0, 1]$ where V is an open subset of $\mathbf{R}$. Replacing such U's by the corresponding V's, we get a cover $\mathcal{V}$ of $[0, 1]$ by sets which are open in $\mathbf{R}$ and evidently it suffices to show that $\mathcal{V}$ has a finite subcover. Now let $S = \{t \in [0, 1]$: the interval $[0, t]$ can be covered by finitely many members of $\mathcal{V}\}$. We have to show that $1 \in S$. Evidently $0 \in S$ and so $S \neq \phi$. We claim S is both open and closed in $[0, 1]$. First, let $t \in S$. Then $[0, t]$ can be covered by, say, $V_1, V_2, ..., V_n$ with $t \in V_n$(say). Since V_n is open, there exists $\delta > 0$ such that $(t - \delta, t + \delta) \subset V_n$. Now for any $t' \in (t - \delta, t + \delta) \cap [0, 1]$, the interval $[0, t']$ is also covered by $V_1, V_2, ..., V_n$ showing that $t' \in S$. Hence $(t - \delta, t + \delta) \cap [0, 1] \subset S$ and hence S is open, being a neighbourhood of each of its points.

On the other hand suppose $t \in [0, 1] \setminus S$. Choose $V \in \mathcal{V}$ such that $t \in V$. V being open, there is $\delta > 0$ such that $(t-\delta, t+\delta) \subset V$. We claim that $(t-\delta, t+\delta) \cap [0, 1] \subset [0, 1] \setminus S$. For let $t' \in (t - \delta, t + \delta) \cap [0, 1]$. If $t' \in S$ then $[0, t']$ can be covered by, say, $V_1 V_2, ..., V_n \in \mathcal{V}$. But then $V_1, V_2, ..., V_n$ and V together cover $[0, t]$ contradicting that $t \notin S$. Thus we have shown that $[0, 1] \setminus S$ is also open.

Putting it all together, S is a non-empty clopen subset of $[0, 1]$. But by the theorem above $[0, 1]$ is connected. We are thus forced to conclude that S is the entire interval $[0, 1]$ and hence, in particular, $1 \in S$ as was to be proved. □

Definition 38. *Two subsets A and B of a space X are said to be (mutually) separated if $A \cap \overline{B} = \phi$ and $\overline{A} \cap B = \phi$.*

Proposition 2.2.16. *Let X be a space and C be a connected subset of X (that is, C with the relative topology is a connected space). Suppose $C \subset A \cup B$ where A, B are mutually separated subsets of X. Then either $C \subset A$ or $C \subset B$.*

Proof. Let $G = C \cap A$ and $H = C \cap B$. Then G, H are closed subsets of C since, A, B are closed in $A \cup B$. Also $G \cap H = \phi$. But C is connected. So either $G = \phi$ or $H = \phi$. In the first case $C \subset B$ while in the second, $C \subset A$. □

Theorem 2.2.17. *Let $\mathcal{C}$ be a collection of connected subsets of a space X such that no two members of C are mutually separated. Then $\cup_{C \in \mathcal{C}} C$ is also connected.*

Proof. Let $M = \cup_{C \in \mathcal{C}} C$. If M is not connected then we could write M as $A \cup B$ where A, B are nonempty and mutually separated subsets of X. By the proposition above, for each $C \in \mathcal{C}$ either $C \subset A$ or $C \subset B$. We contend that the same possibility holds for all $C \in \mathcal{C}$, i.e. either $C \subset A$ for all $C \in \mathcal{C}$ or $C \subset B$ for all $C \in \mathcal{C}$. If this is not the case, then there exist $C, D \in \mathcal{C}$ such that $C \subset A$ and $D \subset B$. But, A, B are mutually separated and hence their subsets C, D are also mutually separated contradicting the hypothesis. Thus all members of $\mathcal{C}$ are contained in A or all are contained in B. Accordingly $M = A$ or $M = B$, contradicting that A, B are both non-empty. □

Corollary 2.2.17.1. *Let $\mathcal{C}$ be a collection of connected subsets of a space X and suppose K is a connected subset of X (not necessarily a member of $\mathcal{C}$) such that $C \cap K \neq \phi$ for all $C \in \mathcal{C}$. Then $(\cup_{C \in \mathcal{C}} C) \cup K$ is connected.*

Proof. Let $M = (\cup_{C \in \mathcal{C}} C) \cup K$. Let $\mathcal{D} = \{K \cup C : C \in \mathcal{C}\}$. Clearly $M = \cup_{D \in \mathcal{D}} D$. By the theorem above, each member of $\mathcal{D}$ is connected since it is a union of two connected sets which intersect (and which are therefore not separated). Now any two members of $\mathcal{D}$ have at least points of K in common and so are not mutually separated. So again by the theorem above, M is connected. □

Proposition 2.2.18. *Let X_1, X_2 be connected topological spaces and $X = X_1 \times X_2$ with the product topology. Then X is connected.*

Proof. If either X_1 or X_2 is empty then so is X and the result holds trivially. So assume X_1, X_2 are both non-empty. Fix a point $y_1 \in X_1$. Then the subset $\{y_1\} \times X_2$ is homeomorphic to X_2 and hence is connected. Call it K. For each $x \in X_2$, the set $X_1 \times \{x\}$ is similarly connected and its intersection with K is non-empty. Also note that $X_1 \times X_2 = (\cup_{x \in X_2} X_1 \times \{x\}) \cup K$. So by the corollary above $X_1 \times X_2$ is connected. □

Corollary 2.2.18.1. *The topological product of any finite number of connected spaces is connected.*

Proof. If $X_1, X_2, ..., X_{n-1}, X_n$ are spaces (with $n \geq 2$) then $X_1 \times X_2 \times ... \times X_n$ is homeomorphic to $(X_1 \times ... \times X_{n-1}) \times X_n$.The result follows by induction on n and the last proposition. □

Proposition 2.2.19. *The closure of a connected subset is connected. More generally if C is a connected subset of a space X then any set D such that $C \subset D \subset \overline{C}$ is connected.*

Proof. Suppose C is connected and $C \subset D$. If D is not connected then we can write $D = A \cup B$ where A, B are nonempty, disjoint and closed relative to D. Then $C \cap A$, $C \cap B$ are disjoint closed subsets of C whose union is C. But C is connected. So one of them, say, $C \cap B$ is empty. This means $C \subset A$, and hence $\overline{C}^D \subset A$ where the closure is w.r.t. D. But $\overline{C}^D = \overline{C}^X \cap D = D$ since $D \subset \overline{C}^X$. Hence $A = D$ contradicting that B is nonempty. So D is connected. $\square$

Definition 39. *A component of a space is a maximally connected subset, that is, a connected subset which is not properly contained in any connected subset of that space.*

For example, the space $(-1, 0) \cup (0, 1)$ with the usual topology has two components, $(-1, 0)$ and $(0, 1)$. A space is connected iff it has only one component, namely, the whole space itself. On the other hand in a discrete space the only connected subsets are the empty set and the singleton subsets and hence all components are singleton sets. Such a space is said to be totally disconnected.

Theorem 2.2.20. *a. Components are closed sets,*
b. Any two distinct components are mutually disjoint,
c. Every nonempty connected subset is contained in a unique component,
d. Every space is the disjoint union of its components.

Proof. (a) Let C be a component of a space X. Then C is connected. Then, $\overline{C}$ is also connected. Now $C \subset \overline{C}$. But C is maximal w.r.t. the property of being connected, that is, no proper superset of C can be connected. Hence $C = \overline{C}$ and so C is closed.
(b) Let C, C' be two components. If $C \cap C'$ is nonempty then $C \cup C'$ would be connected. But $C \subset C \cup C'$ and $C' \subset C \cup C'$. So again by maximally of C, C', we get $C \cap C' = \phi$. Thus two distinct components are disjoint.
(c) Let A be a nonempty connected subset of X. Let $\mathcal{C}$ be the collection of all connected subsets of X containing A and let $M = \cup_{C \in \mathcal{C}} C$. Then any two members of C intersect and so M is connected. Clearly $A \subset M$. We claim M is a component. For suppose V is a connected subset of X containing M. Then $N \in \mathcal{C}$ and so $N \subset M$, whence $M = N$. In other words, M is a maximally connected subset of X. Thus every nonempty subset is contained in a component. Such a component is unique since two distinct components are disjoint.
d. This assertion follows from the fact that for any $x \in X$, $\{x\}$ is a connected set and hence there is a unique component C of X such that $x \in C$. $\square$

The number of components of a space is evidently a topological invariant. A direct or an indirect application of this fact is often useful in showing that certain spaces cannot be homeomorphic to each other. As an example, consider [0, 1] and the unit circle S^1 each with the usual topology. Both have one component each. However in $[0, 1]$ there are points x such that $[0, 1] \setminus \{x\}$ is not connected (such points are called cut points). It is clear that if $h : [0, 1] \to S^1$ is a homeomorphism then h would map a cut point of $[0, 1]$ to a cut point of S^1. But it is easy to show that S^1 has no cut points. Hence $[0, 1]$ cannot be homeomorphic to S^1. Clearly the number of cut points is also a topological invariant.

Exercise :

1. Prove that a subset X of $\mathbf{R}$ is an interval iff it has the property that for all $a, b \in X$, $(a, b) \subset X$.
2. Prove that the complement of $\mathbf{Q} \times \mathbf{Q}$ in the plane $\mathbf{R}^2$ is connected.
3. Give rigorous arguments to show that no two spaces in the following list are mutually homemorphic:
i. the closed interval $[0, 1]$
ii. the open interval $(0, 1)$
iii. the semi-open interval $[0, 1)$
iv. a triod, that is a space homeomorphic to the figure Y
v. the unit circle S^1
vi. a space homeomorphic to the figure X
vii. a figure eight curve.
4. Prove or disprove that the interior and the boundary of a connected set are connected.
5. A space X is said to have the fixed point property if every map $f : X \to X$ has a fixed point, i.e. a point $x_0 \in X$ such that $f(x_0) = x_0$.
a. Prove that a space having the fixed point property must be connected.
b. Prove that the unit interval $[0, 1]$ has the fixed point property.
6. Prove that a continuous bijection from $\mathbf{R}$ to $\mathbf{R}$ must be a homeomorphism.
7. Let $A = \{(x, sin1/x) : 0 < x < 1\} \subset \mathbf{R}^2$ and let $X = \overline{A}$ where the closure is w.r.t. the usual topology on $\mathbf{R}^2$. Prove that X is connected.

Rough space :

2.2.3 Local Connectedness and Paths

Definition 40. *A space X is said to be locally connected at a point x in it if for every neighbourhood, N of x (in X) there exists a connected neighbourhood M of x such that $M \subset N$. X is said to be locally connected if it is locally connected at each of its points.*

Proposition 2.2.21. *A space X is locally connected at a point $x \in X$ iff for every neighbourhood N of x, the component of N containing x is a neighbourhood of x.*

Proof. Suppose X is locally connected at x. Let N be a nbd of x and let C be the component of N containing x (caution: C may not be a component of X). We are given that there exists a connected nbd M of x such that $M \subset N$. Then M is contained in a unique component of N and this component must be C since M intersects C. So $M \subset C$ and hence C is a neighbourhood of x. Conversely let N be a nbd of x and let C be the component of N containing x. Then C is a connected neighbourhood of x contained in N and so X is locally connected at x. □

Note : All indiscrete and all discrete spaces are locally connected. Also real line with the usual topology is locally connected since we proved in the last section that all intervals are connected.

Proposition 2.2.22. *For a topological space X the following statements are equivalent:*
i. X is locally connected.
ii. Components of open subsets of X are open (in X).
iii. X has a base consisting of connected subsets.
iv. For every $x \in X$ and every neighbourhood N of x there exists a connected open neighbourhood M of x such that $M \subset N$.

Proof. $(i) \implies (ii)$. Suppose X is locally connected, G is open in X and C is a component of G. We have to show that C is open in X. Let $x \in C$. Then G is a neighbourhood of x and C is the component of G containing x. So by local connectedness at x and the last proposition, C is a neighbourhood of x. Thus C is a neighbourhood of each of its points and therefore C is open in X.
$(ii) \implies (iii)$. Let G be any open subset of X. Write G as the union of its components. All these components are open, connected subsets of X by (ii). This means that every open set can be expressed as the union of some open, connected subsets of X, or in other words that the family of such sets forms a base for X.
$(iii) \implies (iv)$. Let $x \in X$ and N be a neighbourhood of x. Then there exists an open set V such that $x \in V \subset N$. By (iii) and the characterisation of a base, there exists a connected open set M such that $x \in M$ and $M \subset V$. Hence $M \subset N$.
$(iv) \implies (i)$. This is immediate. □

Now we will see an example of a space which is connected but not locally connected. Let X be the set $B \cup A$ where $B = \{(x,0) \in \mathbf{R}^2 : 0 < x \leq 1\}$ and $A = \{(x,y) \in \mathbf{R}^2 : 0 \leq y \leq 1, x = 0 \text{ } or \text{ } x = 1/n$ for some $n \in \mathbf{N}\}$. The set X is pictured below. It consists of infinitely many vertical segments of unit length, including a segment on the y-axis and a horizontal segment along the x-axis. Give X the relative topology induced by the usual topology on the plane. X is then called a 'comb space'. It is easy to show that X is connected, for each of the vertical segments is connected and meets the horizontal segment B which is also connected. However X is not locally connected. For, let V be the open ball (in X, with usual metric) centred at $(0, 1/2)$ and with radius $1/4$. The components of V will be portions of the vertical segments. They will all be open (in X) except the one along the y-axis, namely the component $\{(0,y) \in \mathbf{R}^2 : 1/4 < y < 3/4\}$. Another example is the famous 'topologist's sine curve' pictured below, which is defined as $A = \{(x, sin(1/x)) : 0 < x < 1\} \cap \{0, 0\}$.

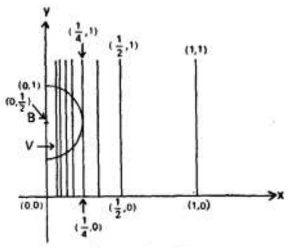


Fig. 1. Comb Space.

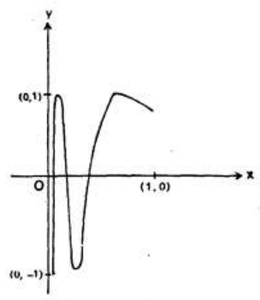


Fig. 2. Topologist's sine curve.

Theorem 2.2.23. *Every open subset of the real line (in the usual topology) can be expressed as the union of mutually disjoint open intervals.*

Proof. Let V be an open subset of $\mathbf{R}$. Write V as the disjoint union of its components. Each such component is a connected subset of $\mathbf{R}$ and hence an interval. But by the last proposition, each component of V is open in $\mathbf{R}$. So V is the disjoint union of open intervals. $\square$

We know that the collection of open intervals is a base for the usual topology on $\mathbf{R}$ and hence that any open subset of $\mathbf{R}$ can be expressed as a union of open intervals. The crux of the preceding theorem is that these intervals could be chosen to be mutually disjoint. There is no analogue of this theorem which holds good for the plane or higher dimensional euclidean spaces.

Proposition 2.2.24. *Every quotient space of a locally connected space is locally connected.*

Proof. Let $f : X \to Y$ be a quotient map and suppose X is locally connected. We have to show that Y is also locally connected. It suffices to prove that components of open subsets of Y are open in Y. So let V be an open subset of Y and let C be a component of V. Then

$f^{-1}(V)$ is an open subset of X and we assert that $f^{-1}(C)$ is a union of some components of $f^{-1}(V)$. This amounts to showing that if x is in $f^{-1}(C)$ and D is the component of $f^{-1}(V)$ containing x, then $D \subset f^{-1}(C)$. Since D is connected and f is continuous, we know that $f(D)$ is connected. Also $f(D) \subset V$ and $f(D) \cap C \neq \phi$ since $x \in f^{-1}(C) \cap D$. So $f(D) \cup C$ is connected. But C is a maximally connected subset of $f^{-1}(V)$. So $f(D) \cup C = C$ or $f(D) \subset C$. Hence $D \subset f^{-1}(C)$ and we have thus shown that $f^{-1}(C)$ is the union of some components of V. But since X is locally connected and $f^{-1}(V)$ is open, each of its components is open in X. So $f^{-1}(C)$is open in X. This shows that C is open in Y as Y has the strong topology generated by $\{f\}$. Thus we have shown that components of open subsets of Y are open and so Y is locally connected. □

Definition 41. *A path in a topological space X is a continuous function α from the unit interval $[0, 1]$ into X. The points $\alpha(0)$ and $\alpha(1)$ are called respectively the initial and the terminal points or the beginning and the end of α. We say α is a path from $\alpha(0)$ to $\alpha(1)$ or that it joins $\alpha(0)$ to $\alpha(1)$.*
We could of course replace the unit interval by any closed, bounded interval $[a, b]$ where $a < b$. However, topologically the generality gained is illusory since $[a, b]$ is homeomorphic to $[0, 1]$. A path is sometimes also called a curve; however, this term is generally reserved for paths in a euclidean space. A path a is said to be simple if the function α is injective and it is said to be closed if $\alpha(0) = \alpha(1)$. A simple closed path is a closed path α which is injective except for $\alpha(0) = \alpha(1)$.

Definition 42. *A space X is said to be path-connected if for every two points $x, y \in X$, there exists a path α in X such that $\alpha(0) = x$ and $\alpha(1) = y$.*

Clearly the real line, all euclidean spaces and any convex subsets of them are all path-connected. The unit sphere S^n is also path-connected for $n > 1$ since given any two distinct points on S^n there is an arc of a great circle joining them and it can be parametrised as a path (in fact as a simple path). It is easy to show that the topological product of path-connected spaces is path-connected.

Proposition 2.2.25. *Every path-connected space is connected.*

Proof. Let X be a path-connected space. If X is empty then certainly it is connected. Suppose X is non-empty. Fix some point $x_0 \in X$. For each $x \in X$, we are given a path α_x in X such that $\alpha_x(0) = x_0$ and $\alpha_x(1) = x$. Let C_x be the range of α_x. Then C_x is connected since it is a continuous image of the unit interval which is connected. Clearly $X = \cup_{x \in X} C_x$. For any two $x, y \in X$, $x_0 \in C_x \cap C_y$ and so C_x, C_y are not mutually separated. So X is connected. □

The converse of Proposition is false. A classic counterexample is the topologists's sine curve.

Proposition 2.2.26. *In any space X, the binary relation defined by letting $x \sim y$ for $x, y \in X$ iff there exists a path in X from x to y, is an equivalence relation.*

Proof. Reflexivity of the relation follows trivially by considering constant paths. For symmetry, suppose α is a path in X from x to y. Define $\beta : [0,1] \to X$ by $\beta(t) = \alpha(1-t)$ for $t \in [0,1]$. Now β is continuous since it is the composition $\alpha \circ f$ where $f : [0,1] \to [0,1]$ is the map $f(t) = (1-t)$, $t \in [0,1]$. Then β is a path from y to x. For transitivity, suppose α is a path from x to y and β is a path from y to z. Define $\gamma : [0,1] \to X$ by

$$\gamma(t) = \begin{cases} \alpha(2t), & if\ 0 \leq t \leq 1/2 \\ \beta(2t-1), & if\ 1/2 \leq t \leq 1 \end{cases}$$

Then γ is well-defined since $\alpha(1) = y = \beta(0)$. Also β is continuous since its restriction to each of the two closed subsets $[0,1/2]$ and $[1/2,1]$ of $[0,1]$ is continuous. Clearly γ is a path in X from x to z and so $x \sim z$. Hence $\sim$ is an equivalence relation on X. □

Definition 43. *The equivalence classes under the equivalence relation defined above are called the path-components of the space X.*

Proposition 2.2.27. *A subset C is a path-component of a space X iff C is a maximal subset of X w.r.t. the property of being path-connected.*

Proof. Suppose C is a path-component of a space X. We claim C is path-connected. Let $x, y \in C$. We certainly know that there is a path a in X from x to y. We assert that the range of a is contained in C. For otherwise, there exists $s \in (0,1)$ such that $\alpha(s) \notin C$. Now define $\beta : [0,1] \to X$ by $\beta(t) = \alpha(st)$. Then β is a path in X from x to $\alpha(s)$. This means that x and $\alpha(s)$ are in the same equivalence class under $\sim$, contradicting that $\alpha(s) \notin C$. Hence α can be considered as a path in C, showing that C is path-connected. Moreover if D is a proper superset of C, then a point of $D \setminus C$ cannot be joined to a point in C by a path in X, and certainly not by a path in D, showing that D is not path connected. Hence C is a maximally path-connected subset of X.
Conversely suppose C is path-connected and is not a proper subset of any path-connected subset of X. Then any two points of C can be joined by a path in C and hence, a fortiori, by a path in X. Hence C is contained in some equivalence class under $\sim$, say $C \subset C'$. But as we just saw, every path-component of X is path-connected. In particular C' is path-connected and so $C = C'$ by maximality of C. Thus C is a path-component of X. □

Exercise :

1. Let $X_1, X_2, ..., X_n$ be topological spaces and X be their topological product. Suppose X_i is locally connected at a point x_1 for $i = 1, 2, ..., n$. Let $x = (x_1, x_2, ..., x_n) \in X$. Prove that X is locally connected at x.

2. Prove that an open subspace of a locally connected space is locally connected.
3. Prove that path-connectedness is preserved under continuous functions.
4. Prove that the topological product of a finite number of non-empty spaces is path-connected iff each coordinate space is so.
5. Prove that a connected, locally path-connected space is path-connected.

Rough space :

Module 3

Separation Axioms

Every concept in topology is defined in terms of open sets, in order to make non-trivial and interesting statements about a space, it is necessary that the space possess a fairly rich collection of open sets. In this section we shall study various degrees of such richness. We shall define a number of related conditions all of which assert the existence of open sets which will contain something but which will also exclude something else. For this reason, the conditions are known as separation axioms.

3.1 Hierarchy of Separation Axioms

The separation axioms are of various degrees of strengths and they are called T_0, T_1, T_2, T_3 and T_4 axioms in ascending order of strength.

Definition 44. *A topological space X is said to satisfy the T_0-axiom, or is said to be a T_0 space if given any two distinct points in X, there exists an open set which contains one of them but not the other.*

T_0 axiom is the weakest separation axiom. For if a space X is not T_0, then there would exist two distinct points x, y in X such that every open set in X either contains both x and y or else contains neither of them. In such a case, x and y may as well be regarded as topologically identical.
Examples of T_0 spaces include all metric spaces. As another example, let $\mathcal{J}$ be the topology on $\mathbf{R}$ whose members are ϕ, $\mathbf{R}$ and all sets of the form (a, ∞) for $a \in \mathbf{R}$. Note that in this space, for $x, y \in \mathbf{R}$ with $x < y$, there exists an open set containing y but not x although there exists no open set which contains x but not y.

Definition 45. *A space X is said to satisfy the T_1-axiom or is said to be a T_1-space if for every two distinct points x and $y \in X$, there exists an open set containing x but not y (and hence also another open set containing y but not x).*

Every T_1 space is also T_0 and the space $(\mathbf{R}, \mathcal{J})$ above shows that the converse is false. Thus the T_1-axiom is strictly stronger than T_0. The essential point is that given two distinct points, the T_0-axiom merely requires that at least one of them can be separated from the other by an open set whereas the T_1-axiom requires that each one of them can be separated from the other.

Proposition 3.1.1. *For a topological space $(X, \mathcal{J})$ the following are equivalent:*
1. The space X is a T_1 space.
2. For any $x \in X$, the singleton set $\{x\}$ is closed.
3. Every finite subset of X is closed.
4. The topology $\mathcal{J}$ is stronger than the cofinite topology on X.

Proof. Only the equivalence of (1) with (2) needs to be established. The rest follow from properties of closed sets and the definition of the cofinite topology. Assume (1) holds and let $x \in X$. We claim that $X \setminus \{x\}$ is a neighbourhood of each of its points. For, let $y \in X \setminus \{x\}$. Then x, y are distinct points of X and so by the T_1-axiom, there exists an open set, say, V such that $y \in V$ and $x \notin V$. But this means $V \subset X \setminus \{x\}$ and hence $X \setminus \{x\}$ is a neighbourhood of y. So $X \setminus \{x\}$ is open and therefore $\{x\}$ is closed in X. Conversely assume (2). Let x, y be distinct points of X. Then $X \setminus \{y\}$ is an open set which contains x but not y. Thus the T_1-axiom holds in X. $\square$

Proposition 3.1.2. *Suppose y is an accumulation point of a subset A of a T_1 space X. Then every neighbourhood of y contains infinitely many points of A.*

Proof. Let N be a neighbourhood of y and let $F = A \cap (N \setminus \{y\})$. We claim that the set F is infinite. For, if not, $X \setminus F$ will be an open set containing y and so $N \cap (X \setminus F)$ will be a neighbourhood of y. Evidently this neighbourhood contains no point of A, except possibly y, contradicting that y is an accumulation point of A. So F is infinite, showing that every neighbourhood of y contains infinitely many points of A. $\square$

Definition 46. *A space X is said to satisfy the T_2 axiom (or the Hausdorff property) or is said to be a T_2 (or Hausdorff) space if for every distinct point $x, y \in X$ there exist disjoint open sets U, V in X such that $x \in U$ and $y \in V$.*

Every T_2 space is T_1 since the condition in the definition implies that U contains x but not y and that V contains y but not x. The converse is false. An infinite set with the cofinite topology is T_1 but not T_2, in fact no two open sets in it are disjoint unless one of them is empty. All metric spaces are T_2. However, there are spaces which are T_2 but which are not metrisable, the real line with the semi-open interval topology is such a space.

Proposition 3.1.3. *In a Hausdorff space, limits of sequences are unique.*

Proof. Let $\{x_n\}$ be a sequence in a Hausdorff space X and suppose $x_n \to x$ and $x_n \to y$ as $n \to \infty$. We have to show that $x = y$. If not, then there exist open sets, U, V in X such that $x \in U$, $y \in V$ and $U \cap V = \phi$. Then there exist $N_1, N_2 \in \mathbf{N}$ such that $x_n \in U$ for all $n \geq N_1$ and $x_n \in V$ for all $n \geq N_2$. Let m be an integer greater than both N_1 and N_2. Then $x_m \in U \cap V$ contradicting that $U \cap V = \phi$. So $x = y$ and thus limits of sequences in X when they exist, are unique. □

Definition 47. *A space X is said to be regular at a point $x \in X$ if for every closed subset C of X not containing x, there exist disjoint open sets U, V such that $x \in U$ and $C \subset V$. X is said to be regular if it is regular at each of its points.*

Definition 48. *A space X is said to be normal if for every two disjoint closed subsets C and D of X there exist two disjoint open sets U and V suchthat $C \subset U$ and $D \subset V$.*

Informally, a space is regular if every point can be separated from every closed subset (not containing it) and it is normal if every two mutually disjoint closed subsets can be separated from each other. These conditions can be satisfied in a vacous manner if the space fails to have very many closed sets. For example, an indiscrete space is regular because if x is a point not in a closed set C then C must be the empty set and hence we can take (actually we have to take) U and V to be respectively the whole space and the empty set. Similarly an indiscrete space is normal because the only way two closed subsets of it can be mutually disjoint is when one of them is empty. This argument also shows that the space $(\mathbf{R}, \mathcal{J})$ considered above is normal. Note however that it is not regular.

Definition 49. *A topological space is said to satisfy the T_3-axiom or is said to be a T_3-space if it is regular and T_1.*

Definition 50. *A topological space is said to satisfy the T_4-axiom or is said to be a T_4-space if it is normal and T_1.*

Theorem 3.1.4. *The axioms T_0, T_1, T_2, T_3 and T_4 form a hierarchy of progressively stronger conditions.*

Proof. We have to show that each axiom from T_1 onwards implies the preceding one. We have already seen that T_1 implies T_0 and T_2 implies T_1. The implication that T_3 implies T_2 follows easily from the definition of regularity by taking the closed set to be a sigleton set. Note that normality does not, by itself, imply regularity as we saw in an example above. However, in presence of the T_1-axiom, we can apply the condition of normality to a singleton set and a given closed set and see that regularity holds. Hence T_4 implies T_3. □

An example of a space which is Hausdorff but not regular is provided by the following topology on the real line. Let $\mathcal{U}$ be the usual topology on $\mathbf{R}$ and let C be the set $\{1/n : n \in \mathbf{N}\}$. Let $\mathcal{S}$ be the smallest topology on $\mathbf{R}$ containing $\mathcal{U} \cup \{\mathbf{R} \setminus C\}$. Then $\mathcal{S}$ makes $\mathbf{R}$ a T_2

space since $\mathcal{S}$ is stronger than $\mathcal{U}$ and $(\mathbf{R}, \mathcal{U})$ is a T_2-space. Note that C is closed in $\mathbf{R}$ w.r.t. $\mathcal{S}$ (although not with respect to $\mathcal{U}$) and that $0 \notin C$. Thus $(\mathbf{R}, \mathcal{S})$ is not a regular space.

Theorem 3.1.5. *All metric spaces are T_4 (and hence T_3 as well).*

Proof. We already know that metric spaces are Hausdorff and hence T_1. It only remains to show that they are normal. Let (X, d) be a metric space and let C, D be disjoint, closed subsets of X. If either C or D is empty then we could separate them by the empty set and the set X. Suppose then that both C, D are non-empty. Define $f : X \to \mathbf{R}$ by $f(x) = d(x, C) - d(x, D)$ for $x \in X$. Then f is continuous, being the difference of two continuous real valued functions. Note that $d(x, C) = 0$ iff $x \in C$ and $d(x, D) = 0$ iff $x \in D$. So f is positive on D and negative on C. Now let $U = f^{-1}(-\infty, 0)$ and $V = f^{-1}(0, \infty)$. Then U, V are open subsets of X since they are the inverse images of open subsets of $\mathbf{R}$. Also $C \subset U$ and $D \subset V$ as noted above. Finally $U \cap V = \phi$ completing the proof that X is normal. □

Definition 51. *A space X is said to be completely regular if for any point $x \in X$ and closed set C not containing x, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \in C$, where the continuity is w.r.t. the usual topology on the unit interval $[0, 1]$. A space is said to be a Tychonoff space if it is completely regular and T_1.*

In a completely regular space, a point can be separated (or distinguished) from a closed set by means of a continuous real-valued function. This condition is significantly different from the earlier separation axioms which assert separation by means of mutually disjoint open sets.

Theorem 3.1.6. *Every completely regular space is regular. Every Tychonoff space is T_3.*

Proof. Suppose X is a completely regular space. Let $x \in X$ and C be a closed subset of X not containing x. We are given a continuous function $f : X \to [0, 1]$ which assumes the value 0 at x and the value 1 at all points of C. Now $[0, 1]$ is a Hausdorff space (in fact it is a metric space). Let G, H be disjoint open sets in $[0, 1]$ containing 0 and 1 respectively. Let $U = f^{-1}(G)$ and $V = f^{-1}(H)$. Then U, V are mutually disjoint, open sets in X and clearly $x \in U$, $C \subset V$. This shows that X is a regular space whenever it is completely regular. If in addition X is also T_1 then X is T_3 whenever it is a Tychonoff space. □

The converse of this proposition is false. Normality does not imply complete regularity, in fact, it does not even imply regularity as we saw above. However, in presence of the T_1 condition, a normal space is a Tychonoff space. This is a non-trivial result which we shall prove later. It shows that T_4 is stronger than Tychonoff while the proposition above shows that Tychonoff itself is stronger than T_3. For this reason, Tychonoff spaces are sometimes called $T_{3\frac{1}{2}}$-spaces.

Proposition 3.1.7. *For a topological space X the following statements are equivalent.*
1. X is regular.
2. For any $x \in X$ and any open set G containing x there exists an open set H containing x such that $H \subset G$.
3. The family of all closed neighbourhoods of any point of X forms a local base at that point.

Proof. $(1) \implies (2)$. Suppose X is regular, $x \in X$ and G is an open set containing x. Then $X \setminus G$ is a closed set not containing x. So by regularity of X there exist open sets U, V such that $x \in U$, $(X \setminus G) \subset V$ and $U \cap V = \phi$. Then $U \subset X \setminus V$ and hence $\overline{U} \subset X \setminus V$ since $X \setminus V$ is a closed set. But $X \setminus V \subset G$ and thus $\overline{U} \subset G$. So we can let $H = U$ and (2) holds.
$(2) \implies (3)$. Let $x \in X$ and N be a neighbourhood of $x \in X$. Let G be the interior of N in X. Then G is an open set containing x and so by (2) there exists an open set H such that $x \in H$ and $\overline{H} \subset G$. Then $\overline{H}$ is a closed neighbourhood of x contained in N. Hence the family of all closed neighbourhoods of x is a local base at x.
$(3) \implies (1)$. Suppose $x \in X$ and C is a closed subset not containing x. Then $X \setminus C$ is a neighbourhood of x. So by (3), there exists a closed neighbourhood M of x such that $M \subset X \setminus C$. Let $U = int(M)$ and $V = X \setminus M$. Then U, V are mutually disjoint open sets and clearly $x \in U$. Also $C \subset V$. This shows that X is a regular space, □

Proposition 3.1.8. *For a topological space X the following are equivalent:*
1. X is normal.
2. For any closed set C and any open set G containing C, there exists an open set H such that $C \subset H$ and $\overline{H} \subset G$.
3. For any closed set C and any open set G containing C, there exists an open set H and a closed set K such that $C \subset H \subset K \subset G$.

Proof. The argument is similar to that of the last proposition and is left to the reader. Note that (3) can be informally stated as, in any closed-open inclusion we can insert an open-closed inclusion. □

Note: Every discrete space satisfies all the separation axioms mentioned so far. Since any space whatsoever can be expressed as the continuous image of a discrete space, it is clear that none of the properties defined in this section is preserved under continuous functions. However, most of them are hereditary, with the exception of normality.

Proposition 3.1.9. *Regularity is a hereditary property.*

Proof. Suppose X is a regular space and Y is a subspace of X. Let $y \in Y$ and D be a closed subset of Y not containing y. Then D is of the form $C \cap Y$ where C is a closed subset of X. Note that $y \notin C$ for otherwise $y \in D$. Hence by regularity of X, there exist open sets U, V (in X) such that $y \in U$, $C \subset V$ and $U \cap V = \phi$. Let $G = U \cap Y$, $H = V \cap Y$. Then G, H are open in the relative topology on Y. Also $y \in G$,$D \subset H$ and $G \cap H = \phi$. Thus the space Y (with the relative topology) is regular. $\square$

Finally, as regards divisibility, it turns out that none of the separation axioms is divisible. If on $\mathbf{R}$ we define an equivalence relation R by letting xRy iff $x - y \in \mathbf{Q}$, then the quotient space X/R is indiscrete. Thus we see that none of the T_0, T_1, T_2, T_3, T_4 and Tychonoff conditions is divisible.

Exercise :

1. Prove that the properties T_0, T_1, T_2 and complete regularity are all hereditary.
2. Prove that the co-countable topology on an uncountable set does not make it a Hausdorff space although limits of sequences in it are unique.
3. For a set Y, the diagonal ΔY is defined to be the set $\{(y, y) \in Y \times Y : y \in Y\}$. Prove that a space Y is T_2 iff the diagonal ΔY is a closed subset of $Y \times Y$ in the product topology.
4. Let Y be a Hausdorff space. Prove that for any space X and any two maps $f, g : X \to Y$ the set $\{x \in X : f(x) = g(x)\}$ is closed in X.
5. Let R be an equivalence relation on a space X. Prove that the quotient space X/R is T_1 if and only if the equivalence classes of R are closed in X.

Rough space :

3.2 Compactness and Separation Axioms

Suppose we have a Hausdorff space X. This means that any two distinct singleton subsets say $\{x\}$, $\{y\}$ of X can be separated from each other by disjoint open sets. Suppose we replace the singleton set $\{y\}$ by a finite subset F. It is then easy to show that $\{x\}$ and F can be separated from each other by disjoint open sets. Indeed, suppose the distinct elements of F are $y_1, y_2, ..., y_n$. For $i = 1, 2, ..., n$ let U_i, V_i be open sets such that $x \in U_i$, $y_i \in V_i$ and $U_i \cap V_i = \phi$. Now let $U = \cap_{i=1}^n U_i$ and $V = \cup_{i=1}^n V_i$. Then, clearly U, V are open sets, $x \in U$, $F \subset V$ and $U \cap V = \phi$. We have used here that the intersection of finitely many open sets is open. The argument will break down precisely at this point in case the set F is infinite. If, however, F is compact, things are not so bad, as we will see in the proposition below.

Proposition 3.2.1. *Let X be a Hausdorff space, $x \in X$ and F a compact subset of X not containing x. Then there exist open sets U, V such that $x \in U$, $F \subset V$ and $U \cap V = \phi$.*

Proof. For each $y \in F$, there exist open sets U_y, V_y such that $x \in U_y$,$y \in V_y$ and $U_y \cap V_y = \phi$. The family $\{V_y : y \in F\}$ is an open cover of F. Since F is compact, there is a finite subcover, say $\{V_{y_1}, V_{y_2}, ..., V_{y_n}\}$. Let $U = \cap_{i=1}^n U_{y_i}$, and $V = \cup_{i=1}^n V_{y_i}$. Then U, V are disjoint open subsets, $x \in U$ and $F \subset V$. □

Corollary 3.2.1.1. *A compact subset in a Hausdorff space is closed.*

Proof. Suppose X is a T_2 space and F is a compact subset of X. Then by the proposition above, for any $x \in X \backslash F$ there exist open sets U, V such that $x \in U$, $F \subset V$ and $U \cap V = \phi$. In particular, $U \cap F = \phi$ and hence $U \subset X \setminus F$. Thus $X \setminus F$ is a neighbourhood of each of its points. So $X \setminus F$ is open and F closed. □

Corollary 3.2.1.2. *Every map from a compact space into a T_2 space is closed. The range of such a map is a quotient space of the domain.*

Proof. Suppose $f : X \to Y$ is continuous where X is compact and Y is Hausdorff. Let C be a closed subset of X. Then C is compact and so $f(C)$ is compact since continuous image of a compact set is compact. But then $f(C)$ is closed in Y by the corollary above. Hence images of closed sets in X are closed in Y, i.e. the map f is closed. Let Z be the range of f. Then f is a map from X onto Z is a quotient map since f is a closed surjective map. Consequently Z is a quotient space of X. □

Corollary 3.2.1.3. *A continuous bijection from a compact space onto a Hausdorff space is a homeomorphism.*

Proof. Let $f : X \to Y$ be a continuous bijection where X is compact and Y is Hausdorff. We claim f is open. Let G be an open subset of X. Then $X \setminus G$ is closed and hence

$f(X \setminus G)$ is closed in Y by the corollary above. But $f(X \setminus G) = Y \setminus f(G)$ because f is a bijection. So $f(G)$ is open in Y. Thus f is a continuous, open bijection and hence a homeomorphism. □

Corollary 3.2.1.4. *Every continuous, one-to-one function from a compact space into a Hausdorff space is an embedding.*

Proof. This is immediate from the last corollary. □

An application of the above corollaries is to prove that there can be no continuous one-to-one map from the unit circle S^1, into the real line. For if f were such a map then $f(S^1)$ would be homeomorphic to S^1 by the corollary above since S^1 is a compact space. But then $f(S^1)$ would be a compact, connected subspace of $\mathbf{R}$, whence $f(S^1)$ must be a closed, bounded interval. But such an interval cannot be homeomorphic to S^1 since it has a cut point whereas S^1 has no cut points.

Another interesting consequence is that if X is any set then a compact topology on X (i.e. topology on X which makes it a compact space) cannot be properly stronger than a Hausdorff topology. For, suppose $\mathcal{J}_1, \mathcal{J}_2$ are,topologies on X such that $(X, \mathcal{J}_1)$ is compact and $(X, \mathcal{J}_2)$ is Hausdorff. If $\mathcal{J}_2 \subset \mathcal{J}_1$ then the identity function $id_x : X \to X$ is $\mathcal{J}_1 - \mathcal{J}_2$ continuous and hence a homeomorphism. So $\mathcal{J}_1 = \mathcal{J}_2$. It follows that in the class of all compact topologies on a set, every Hausdorff topology is maximal and that in the class of all Hausdorff topologies on a set, every compact topology is a minimal one.

Theorem 3.2.2. *Every compact Hausdorff space is a T_3 space.*

Proof. Let X be a compact, Hausdorff space. Then every closed subset of X is compact and so the space X is regular by Proposition (3.2.1). Since X is also T_1 (being T_2), the result follows. □

Proposition 3.2.3. *Let X be a regular space, C a closed subset of X and F compact subset of X, such that $C \cap F = \phi$. Then there exist open sets U, V such that $C \subset U$, $F \subset V$ and $U \cap V = \phi$.*

Proof. Use arguments analogous to those of Proposition (3.2.1) □

Theorem 3.2.4. *Every regular, Lindeloff space is normal.*

Proof. Let X be regular, Lindeloff space and let C, D be disjoint, closed subsets of X. By the characterisation of regularity, we get for each $x \in C$, an open set U_x containing x such that $\overline{U_x} \subset X \setminus D$ and similarly for each $y \in D$ an open set V_y containing y such that $V_y \subset X \setminus C$. The sets C, D are closed subsets of a Lindeloff space and hence they are themselves Lindeloff spaces . So the open covers $\{U_x : x \in C\}$ and $\{V_y : y \in D\}$ of C, D respectively have countable subcovers say $\{U_n : n = 1, 2, ...\}$ and $\{V_n : n = 1, 2, ...\}$. It

is now tempting to let $U = \cup_n U_n$ and $V = \cup_n V_n$. Unfortunately although $\overline{U_n} \subset X \setminus D$ for each n, we cannot deduce from it that $\overline{U} \subset X \setminus D$ because the closure of U may be larger than $\cup_n \overline{U_n}$ (however, this argument would be valid in case X were compact, for then the subcover could be chosen to be finite and the closure operator does commute with finite unions).
For each $n \in \mathbf{N}$, let $G_n = U_n \setminus \cup_{i=1}^n \overline{V_i}$ and $H_n = V_n \setminus \cup_{i=1}^n \overline{U_i}$. Note that G_n, H_n are open sets for all n. Let $G = \cup_{i=1}^\infty G_i$ and $H = \cup_{i=1}^\infty H_i$ We contend $C \subset G$. For, let $x \in C$. Then $x \in U_n$ for some $n \in \mathbf{N}$. Also $x \notin V_m$ for all m since $V_m \subset X \subset C$ for all m. Hence $x \in G_n$ and so $x \in G$. Similarly $D \subset H$. Thus G, H are open sets in X containing C and D respectively and to complete the proof we need only show that $G \cap H = \phi$. If this is not so, then there exist $m, n \in \mathbf{N}$ such that $G_m \cap H_n \neq \phi$. Without loss of generality we may suppose $m \leq n$. Let $x \in G_m \cap H_n$. Then $x \in U_m \subset \overline{U_m}$ which contradicts that $x \in H_n$. So $G \cap H = \phi$ and X is normal. □

Corollary 3.2.4.1. *Every regular, second countable space is normal.*

Proof. This is immediate since every second countable space is a Lindeloff space □

Corollary 3.2.4.2. *Every compact Hausdorff space is T_4.*

Proof. We have already seen that every compact Hausdorff space is regular and hence by previous theorem it is also normal. Moreover, it is a T_1-space since it is T_2-space. Putting together, every compact Hausdorff space is T_4. □

Theorem 3.2.5. *Let A, B be compact subsets of topological spaces X, Y respectively. Let W be an open subset of $X \times Y$ containing the rectangle $A \times B$. Then there exist open sets U, V in X, Y respectively such that $A \subset U$, $B \subset V$ and $U \times V \subset W$.*

Proof. The result is trivial if either A or B is empty. So assume A, B are both nonempty. Fix $b \in B$. For each $a \in A$, W is an open neighbourhood of the point $(a, b) \in X \times Y$. So by the definition of the product topology, there exist open sets G_a, H_a in X, Y respectively such that $a \in G_a$, $b \in H_a$ and $G_a \times H_a \subset W$. The family $\{G_a : a \in A\}$ is an open cover of the compact set A. Let $\{G_{a_1}, G_{a_2}, ..., G_{a_m}\}$ be a finite subcover. Let $G_b = \cup_{i=1}^n G_{a_i}$ and $H_b = \cap_{i=1}^n H_{a_i}$. Then G_b, H_b are open sets in X, Y respectively such that $A \subset G_b$, $b \in H_b$ and $G_b \times H_b \subset W$. These sets depend on the point $b \in B$.
We now let b vary over B. For each $b \in B$ we find open sets G_b, H_b as above. The family $\{H_b : b \in B\}$ is an open cover of the compact set B. Let $\{H_{b_1}, H_{b_2}, ..., H_{b_m}\}$ be a finite subcover. Let $U = \cap_{i=1}^m G_{b_i}$, and $V = \cup\{H_{b_i} : i = 1, ..., m\}$. Then U, V clearly have the desired properties. □

Informally, the says that any neighbourhood of a compact rectangle contains an open rectangular neighbourhood.

Proposition 3.2.6. *Let X be a completely regular space. Suppose F is a compact subset of X, C is a closed subset of X and $F \cap C = \phi$. Then there exists a continuous function from X into the unit interval which takes the value 0 at all points of F and the value 1 at all points of C.*

Proof. If F is the empty set, the function which is identically 1 will work. Similarly if C is empty, the identically zero function will work. Let us assume then that C and F are both non-empty. For each $x \in F$, there exists a map $f_x : X \to [0, 1]$ such that $f_x(x) = 0$ and $f_x(y) = 1$ for all $y \in C$. If the set F were finite we could easily get the result by taking the minimum (or the product) of the finite family of functions $\{f_x : x \in F\}$. Since F is compact (and not necessarily finite), we do the next best thing, namely, to apply a standard compactness argument. For each $x \in F$, let U_x be the set $f_x^{-1}([0, 1/2))$. Then U_x is an open set containing x and so the family $\{U_x : x \in F\}$ is an open cover of the set F. By compactness of F, there exists a finite subcover, say, $\{U_{x_1}, U_{x_2}, ..., U_{x_n}\}$. Now define $f : X \to [0, 1]$ by $f(x) = min\{f_{x_1}(x), f_{x_2}(x), ..., f_{x_n}(x)\}$ for $x \in X$. Then f is continuous. Also f assumes the value 1 at all points of C since each f_{x_i} does so for $i = 1, 2, ..., n$. However, f may not vanish identically on the set F. This difficulty can be corrected as follows. We certainly know that $f(F)$ is a subset of the semi-open interval $[0, 1/2)$ since for $x \in F$, there exists i such that $0 \leq f_{x_i}(x) < 1/2$ and $f(x) \leq f_{x_i}(x)$. Now let g be a continuous function from the unit interval $[0, 1]$ into itself such that $g([1, 1/2)) = \{0\}$ and $g(1) = 1$. The composite $g \circ f$ vanishes identically on F and assumes the value 1 at all points of C. □

When the members of a decomposition of a space X are compact and the projection map p is closed, the quotient space shares many of the nice properties of the original space. We now study a few results regarding this. Let X will be a space and $\mathcal{D}$ be some decomposition of X. We shall regard $\mathcal{D}$ as a quotient space of X and $p : X \to \mathcal{D}$ will denote the projection map. Note that for a subset S of X, $p^{-1}(p(S))$ is in general large than S. If R denotes the equivalence relation on X corresponding to the decomposition $\mathcal{D}$ then $p^{-1}(p(S))$ is the set $\{x \in X : xRy$ for some $y \in S\}$. It is clear that $p^{-1}(p(S)) = S$ if and only if S is the union of some members of $\mathcal{D}$ and that this is the case iff every member of $\mathcal{D}$ is either completely contained in S or completely contained in $X \setminus S$. There is a name for such sets.

Definition 52. *A subset S of X is said to saturated (w.r.t. the decomposition $\mathcal{D}$) if $p^{-1}(p(S)) = S$, equivalently S is saturated if there exists a subfamily $\mathcal{C}$ of $\mathcal{D}$ such that $S = \cup_{C \in \mathcal{C}} C$.*

Note : For any $A \subset X$, the set $p^{-1}(p(A))$ is always saturated. Also the complement of a saturated subset is saturated. If A, B are mutually disjoint and at least one of them is saturated then $p(A)$ and $p(B)$ are mutually disjoint.

Proposition 3.2.7. *With the notation above, the quotient map $p : X \to \mathcal{D}$ is closed if and only if for any $D \in \mathcal{D}$ and any open set G (in X) containing D, there exists a saturated* *open set H such that $D \subset H \subset G$.*

Proof. Assume first that p is closed. Let $D \in \mathcal{D}$ and an open subset G (of X) containing D be given. Let $K = p^{-1}(p(X \setminus G))$ and $H = X \setminus K$. Then K is closed since p is closed and continuous. So H is open in X and clearly it is saturated since K is so. Also $H \subset G$ since $X \setminus G \subset K$. It remains to show that $D \subset H$. For this, let $x \in D$. Since $D \subset G$, it follows that $p(x) = D \neq p(X \setminus G)$. So $x \notin K$ and hence $x \in H$ as desired.
Conversely assume that the given condition holds and suppose C is a closed subset of X. We have to show that $p(C)$ is closed in $\mathcal{D}$. In view of the fact that $\mathcal{D}$ has the quotient topology on it, this amounts to showing that $p^{-1}(p(C))$ is closed in X. So let $V = X \setminus p^{-1}(p(C))$. We claim that V is a neighbourhood of each of its points and hence is open. Note first that V is the union of those members of $\mathcal{D}$ which are disjoint from C. Hence V is saturated and does not intersect C. Now let $x \in V$. Let D be the unique member of $\mathcal{D}$ containing x. Then $D \subset X \setminus C$ which is open. By the given condition there exists a saturated open set H such that $D \subset H \subset X \setminus C$. Then H is the union of some members of $\mathcal{D}$. None of these members intersects C. So $H \subset V$. Thus $x \in D \subset H \subset V$, showing that V is a neighbourhood of x. □

Theorem 3.2.8. *Suppose $\mathcal{D}$ is a decomposition of a space X each of whose members is compact and suppose the projection $p : X \to \mathcal{D}$ is closed. Then the quotient space $\mathcal{D}$ is Hausdorff or regular according as X is Hausdorff or regular.*

Proof. Assume first that X is Hausdorff. Let C, D be distinct elements of $\mathcal{D}$. Then C, D are compact subsets of X and X is T_2. we can get open subsets U, V of X such that $C \subset U$, $D \subset V$ and $U \cap V = \phi$. By the last proposition, there exist saturated open sets G, H such that $C \subset G \subset U$ and $D \subset H \subset V$. Clearly $p(G), p(H)$ are mutually disjoint subsets of $\mathcal{D}$, containing C, D respectively. Also $p^{-1}(p(G)) = G$ is open in X and so $p(G)$ is open in $\mathcal{D}$ by definition of the quotient topology. Similarly $p(H)$ is open. It thus follows that $\mathcal{D}$ is a Hausdorff space.
Next, suppose X is regular. Let $A \in \mathcal{D}$ and suppose C is a closed subset of $\mathcal{D}$ not containing A. Then $p^{-1}(C)$ is a closed subset of X which is disjoint from A. Now, there exist open subsets U, V containing A and $p^{-1}(C)$ respectively such that $U \cap V = \phi$. Note that $p^{-1}(C)$ is the union of some members of $\mathcal{D}$. For each of these we apply the last proposition and get a saturated open subset contained in V. The union of all such open saturated sets gives an open, saturated subset H such that $p^{-1}(C) \subset H \subset V$. Also there exists a saturated open subset G such that $A \subset G \subset U$. The rest of the argument is now similar to that given in the last paragraph. □

Before moving on to the next theorem, we introduce some notation. For a subset V of X let $K(V)$ denote the union of those members of $\mathcal{D}$ which are contained in V. Evidently $K(V) \subset V$ and $X \setminus K(V) = p^{-1}(p(X \setminus V))$. It thus follows that if p is closed and V is open then $K(V)$ is open. Moreover, $p(K(V))$ is open since $p^{-1}(p(K(V))) = K(V)$.

Theorem 3.2.9. *With the hypothesis of the last theorem, if X is second countable, so is $\mathcal{D}$.*

Proof. Let $\mathcal{B}$ be a countable base for X. Let $\mathcal{U}$ be the family of all finite unions of members of $\mathcal{B}$. Then $\mathcal{U}$ is also countable. Let $\mathcal{L} = \{p(K(U)) : U \in \mathcal{U}\}$. Then $\mathcal{L}$ is a countable family of open sets in $\mathcal{D}$. We contend that X is a base for the quotient topology on $\mathcal{D}$. For this, let $A \in \mathcal{D}$ and G be an open subset of $\mathcal{D}$ containing A. Then $p^{-1}(G)$ is an open subset of X containing A. Since A is compact, we can find a finite number of members of $\mathcal{B}$ covering A whose union is contained in $p^{-1}(G)$. This means, there exists $U \in \mathcal{U}$ such that $A \subset U \subset p^{-1}(G)$. Then $p(K(U))$ is an open subset of $\mathcal{D}$ containing A and contained in G. Since $p(K(U)) \in \mathcal{L}$ by definition, it follows that $\mathcal{L}$ is a base for $\mathcal{D}$. $\square$

Exercise :

1. Prove that the unit circle S^1 is compact.
2. For any map $f : S^1 \to \mathbf{R}$ prove that there exists a point $x_0 \in S^1$ such that $f(x_0) = f(-x_0)$. (Hint: Consider the sets $\{x \in S^1 : f(x) > f(-x)\}$ and $\{x \in S^1 : f(x) < f(-x)\}$ and use connectedness of S^1. Note that this result is stronger than saying that S^1 cannot be embedded in $\mathbf{R}$.)
3. Let A, B be closed subsets of S^1 such that $S^1 = A \cup B$. Prove that at least one of A and B contains a pair of mutually antipodal points.
4. Prove that the closure of a compact subset of a regular space is compact.
5. Prove that the real line with the semi-open interval topology is normal.

Rough space :

3.3 The Urysohn Characterisation of Normality

We have seen that complete regularity implies regularity i.e. separation of a point from a closed set by means of a continuous function implies separation by open sets. This also holds for separation of two closed sets as the following proposition.

Proposition 3.3.1. *Let A, B be subsets of a space X and suppose there exists a continuous function $f : X \to [0, 1]$, such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. Then there exist disjoint open sets U, V such that $A \subset U$ and $B \subset V$.*

Proof. We simply choose any two disjoint open sets G, H in $[0, 1]$ containing 0 and 1 respectively (for example we could let $G = [0, 1/2]$ and $H = (1/2, 1)$ and set $U = f^{-1}(G)$ and $V = f^{-1}(H)$. The assertion follows from the given properties of f. □

Corollary 3.3.1.1. *If a space X has the property that for any two mutually disjoint closed subsets A, B of it, there exists a continuous function $f : X \to [0, 1]$ taking the value 0 at all points of A and the value 1 at all points of B, then X is normal.*

Proof. This follows immediately from the last proposition and the definition of normality. □

The interesting thing is that the converse of the corollary above is true. This is a non-trivial result due to Urysohn and is known as the **Urysohn characterisation of normality**. In this section, our aim is at proving this result.

Theorem 3.3.2. *A topological space X is normal if and only if it has the property that for every two mutually disjoint, closed subsets A, B of X, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.*

Before we move on with the proof of the theorem, we will introduce couple of lemmas which we will use to prove the theorem.

We are given some information about the subsets of X and we want to construct a continuous function $f : X \to [0, 1]$ with some special properties. Let us see how the existence of a continuous function $f : X \to [0, 1]$ implies the existence of a certain family of subsets of X.

Lemma 3.3.3. *Let $f : X \to [0, 1]$ be continuous. For each $t \in \mathbf{R}$ let $F_t = \{x \in X : f(x) < t)\}$. Then the indexed family $\{F_t : t \in \mathbf{R}\}$ has the following properties:*
i. F_t is an open subset of X for each $t \in \mathcal{R}$.
ii. $F_t = \phi$ for $t < 0$ (actually F_0 is also 0 but this is not very important.)
iii. $F_t = X$ for $t > 1$.
iv. For any $s, t \in \mathbf{R}$, $s < t \implies \overline{F_s} \subset F_t$. Moreover, for each $x \in X$, $f(x) = inf\{t \in \mathbf{Q} : x \in F_t\}$.

Proof. Note that F_t is the inverse image (under f) of the set $(-\infty, t)$ which is open in $\mathbf{R}$. So, by continuity of f, each F_t is open, showing (i),(ii) and (iii) follow easily from the fact that f takes values in the unit interval. For (iv) let r be any number between s and t and let $C = \{x \in X : f(x) \leq r\}$. Then $F_s \subset C \subset F_t$. But C is closed in X by continuity of f. So $\overline{F_s} \subset C$ and $\overline{F_S} \subset F_r$.
For the remainder of the lemma, let $x \in X$ and let G_x be the set $\{t \in \mathbf{Q} : x \in F_t\}$. We have to show $f(x) = inf G_x$. G_x is nonempty because $t \in G_x$ for all rational $t > f(x)$. Also $t \geq 0$ for all $t \in G_x$ and so G_x is bounded below. Let $y = inf G_x$. Now, for any $t \in G_x$, $x \in F_t$ and so $f(x) < t$. Hence $f(x) \leq inf G_x$ i.e. $f(x) \leq y$. Suppose $f(x) < y$. Let q be a rational number between $f(x)$ and y. Then $q \notin G_x$ since $q < y$. Hence $x \notin F_q$. So $f(x) \geq q$, which is a contradiction. Hence we get $f(x) = y$. $\square$

The lemma above shows that a continuous function $f : X \to [0, 1]$ induces a certain indexed family of open subsets of X and moreover that we can recover the function f if we knew some of these sets, namely the sets F_t for $t \in \mathbf{Q}$. This is exactly what the next lemma shows us.

Lemma 3.3.4. *Let X be a topological space and suppose $\{F_t : t \in \mathbf{Q}\}$ is a family of sets in X such that*
1. F_t is open in X for each $t \in \mathbf{Q}$.
2. $F_t = \phi$ for $t \in \mathbf{Q}$, $t < 0$.
3. $F_t = X$ for $t \in \mathbf{Q}$, $t > 1$.
4. $\overline{F_s} \subset F_t$ for $s, t \in \mathbf{Q}$, $s < t$.
For $x \in X$, let $f(x) = inf\{t \in \mathbf{Q} : x \in F_t\}$. Then f is a continuous real-valued function on X and it takes values in the unit interval $[0, 1]$.

Proof. For $x \in X$, let $G_x = \{t \in \mathbf{Q}. : x \in F_t\}$. Condition (3) shows that G_x is non-empty and condition (2) shows that it is bounded below (by 0) for all $x \in X$. So the function $f(x) = inf(G_x)$ is certainly a well-defined, real-valued function on X. Also conditions (2) and (3) easily imply that for each $x \in X$, $0 \leq f(x) \leq 1$ and hence f takes values in the unit interval. It only remains to prove that f is continuous. For this, note-that the family of all intervals of the form $(-\infty, a)$ or (b, ∞) for $a, b \in \mathbf{R}$ is a sub-base for the usual topology on $\mathbf{R}$. Hence, continuity of f will be established if we can prove that for any $s \in \mathbf{R}$, the sets $\{x \in X : f(x) < s\}$ and $\{x \in X : f(x) > s\}$ are open in X.
Let $s \in \mathbf{R}$. Let H be the set $\{x \in X : f(x) < s\}$. It is tempting to think that H is precisely the set F_s and hence is trivially open. Unfortunately this need not be so even when s is rational. Nevertheless, we claim $H = \cup\{F_t : t \in \mathbf{Q}, t < s\}$. For, suppose first, $x \in H$. Then $f(x) < s$. Since $f(x) = inf(G_x)$ and $f(x) < s$, there exists $q \in G_x$ such that $q < s$. This means that $x \in F_q$ and hence $x \in \cup\{F_t : t \in \mathbf{Q}; t < s\}$. Conversely, we have to show that if $t \in \mathbf{Q}$ and $t < s$ then $F_t \subset H$. Let $x \in F_t$. Then clearly $f(x) \leq t$ and so $f(x) < s$, showing $x \in H$. Thus we have shown that H is the union of F_t's, for $t \in \mathbf{Q}$, $t < s$. But

each F_t is an open subset of X by (1) and so H is open in X.
Next, let $K = \{x \in X : f(x) > s\}$. We show that K is open in X by showing that its complement $X \setminus K$ is closed. To do this, we claim that $X \setminus K = \cap\{\overline{F_t} : t \in \mathbf{Q}, t > s\}$. For, suppose first that $x \in X \setminus K$. Then $f(x) \leq s$. Suppose $t \in \mathbf{Q}$ and $t > s$. Then $f(x) < t$. Since $f(x) = inf(G_x)$, there exists $q \in G_x$ such that $q < t$. But then $x \in F_q$ and by (4), $\overline{F_q} \subset F_t$. So $x \in \overline{F_t}$ for all $t \in \mathbf{Q}$ for which $t > s$. Conversely suppose $x \in \cap\{\overline{F_t} : t \in \mathbf{Q}, t > s\}$. We must show that $f(x) \leq s$. If not, then $s < f(x)$. Let q, t be rational numbers such that $s < q < t < f(x)$. Then clearly $x \notin \overline{F_q}$ for otherwise $x \in F_t$ by (4), and so $t \in G_x$ violating that $f(x) = inf(G_x)$. Thus $q \in \mathbf{Q}$, $q > s$ and $x \notin \overline{F_q}$, a contradiction. This establishes that $X \setminus K$ is an intersection of closed sets and therefore is closed in X. As noted before, this completes the proof of the continuity of f and of the lemma. □

In view of the preceding lemmas, the problem of finding a continuous function on a space reduces to the problem of constructing a family $\{F_t : t \in \mathbf{Q}\}$ of subsets with certain conditions. Countability of $\mathbf{Q}$ allows us to apply an inductive method in the construction.
We will now complete the proof of the theorem. We are given disjoint closed subsets A and B of a normal space X. We want a continuous real-valued function f on X such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. We define a family of sets $\{F_t : t \in \mathbf{Q}\}$ satisfying the conditions of the last lemma. For $t < 0$, and $t > 1$ we have no choice but to let F_t be respectively the empty set and the set X. Let $F_1 = X \setminus B$. Define F_0 to be any open set containing A such that $\overline{F_0} \subset X \setminus B$ (such a set exists by the characterisation of normality). For rational numbers between 0 and 1 we proceed as follows.

Enumerate the set of rationals in $[0, 1]$ as $\{q_0, q_1, q_2, q_3, ..., q_n, ...\}$ with $q_0 = 0$ and $q_1 = 1$. Now F_{q_0}, F_{q_1} are already defined. Consider q_2. Clearly $q_0 < q_2 < q_1$. Define F_{q_2} to be any open set such that $\overline{F_{q_0}} \subset F_{q_2} \subset \overline{F_{q_2}} \subset F_{q_1}$. Such a set exists by normality of X. Now suppose $n \geq 3$ and that the open sets $F_{q_1}, F_{q_2}, ..., F_{q_{n-1}}$ have already been defined so as to satisfy condition (4) of the lemma. Consider q_n. Let q_i be the largest among those of $q_0, q_1, q_2, ..., q_{n-1}$ which are less than q_n, i.e. $q_1 = max\{q_r : 0 \leq r \leq n-1, q_r < q_n\}$. Similarly let q_j be the smallest among those of $q_0, .., q_{n-1}$ which are greater than q_n. Then $q_i < q_n < q_j$. By the inductive hypothesis $\overline{F_{q_i}} \subset F_{q_j}$. By normality of X, there exists an open set F_{q_n} such that $\overline{F_{q_1}} \subset F_{q_n}$ and $\overline{F_{q_n}} \subset F_{q_j}$. Then condition (4) continues to be satisfied with this F_{q_n} included in the set of the F's defined so far. This completes the inductive step in the definition and also shows that the family of sets $\{F_t : t \in \mathbf{Q}\}$ satisfies all the conditions in the last lemma. So the function defined by $f(x) = inf\{t \in \mathbf{Q} : x \in F_t\}$ is a continuous function from X into $[0, 1]$. Now, if $x \in A$ then $x \in F_0$ and $f(x) = 0$. Similarly if $x \in B$ then $x \notin F_t$ for any $t \in \mathbf{Q}$, $t \leq 1$ and so $f(x) = 1$. Thus the proof is complete.

Corollary 3.3.4.1. *All T_4 spaces are completely regular and hence Tychonoff.*

Proof. Simply apply the theorem to the case where one of the closed sets is a singleton. $\square$

Note : It need not be true that f is 0 only on A and 1 only on B. There may be points outside $A \cup B$ at which f is 0 or 1. In other words we merely claim that $A \subset f^{-1}(\{0\})$ and $B \subset f^{-1}(\{1\})$ and not that $A = f^{-1}(\{0\})$ or $B = f^{-1}(\{1\})$. A function whose existence is asserted by the Urysohn's lemma is called a Urysohn function.

Exercise :

1. Suppose X is a metric space and A, B are non-empty disjoint, closed subsets of X. Prove that there exists a continuous function $f : X \to [0, 1]$ such that $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$.
2. Prove that every continuous real-valued function on an indiscrete space is constant, even though such a space is always normal.
3. Prove that a connected, T_4 space with at least two points must be uncountable (i.e. the underlying set must be uncountable). (Hint: A Urysohn function on such a space must be onto).
4. Prove that there exists no countable, connected, T_3 space.

Rough space :

3.4 Tietze Characterisation of Normality

Suppose X is a topological space, A is a subset of X and $f : A \to \mathbf{R}$ is a continuous function (w.r.t. the subspace topology on A and the usual topology on $\mathbf{R}$), We want to find a continuous extension of f to the space X, that is we want a continuous function $F : X \to \mathbf{R}$ such that for any $x \in A$, $F(x) = f(x)$. Of course such an extension may not always exist. For example, let $X = [0, 1]$, $A = (0, 1]$ and define $f : A \to \mathbf{R}$ by $f(x) = sin(1/x)$. Then f cannot be continuously extended to $[0.1]$ because there exist sequences $\{x_n\}$,$\{y_n\}$ in A which converge to 0 in $[0, 1]$ such that $\{f(x_n)\}$, $\{f(y_n)\}$ converge to distinct limits in $\mathbf{R}$ and this would violate the continuity of any extension of f to $[0, 1]$.

Proposition 3.4.1. *Let A be a subset of a space X and let $f : A \to \mathbf{R}$ be continuous. Then any two extensions of f to X agree on $\overline{A}$. In other words, if at all an extension of f exists its values on $\overline{A}$ are uniquely determined by values of f on A.*

Proof. Suppose $F, G : X \to \mathbf{R}$ are both extensions of f. Let $C = \{x \in X : F(x) = G(x)\}$. Clearly $A \subset C$. But since $\mathbf{R}$ is a Hausdorff space, C is closed in X. Hence $A \subset C$ which implies the result. □

The general problem of extending f from A to X can be broken into two steps: (i) to extend f from A to $\overline{A}$, and (ii) to extend it further from $\overline{A}$ to X. The proposition just proved says that the first part has at most one solution and the example given above shows that there may actually be no solution. In this section we will be concerned with the second part of the extension problem. Let us then suppose that f has already been extended somehow to $\overline{A}$ and inquire whether it can further be extended to X. This means that without loss of generality, A may be regarded as a closed subset of X. Even then, an extension of f may not always exist. In fact the existence of such extensions puts a strong condition on the space X as we see in the following proposition.

Proposition 3.4.2. *Suppose a topological space X has the property that for every closed subset A of X, every continuous real valued function on A has a continuous extension to X. Then X is normal.*

Proof. Let B and C be disjoint closed subsets of X. Let $A = B \cup C$ and define $f : A \to \mathbf{R}$ by $f(x) = 0$ for $x \in B$ and $f(x) = 1$ for $x \in C$. Then A is a closed subset of X. Also the function f is well defined and continuous. By hypothesis, there exists a continuous function $F : X \to \mathbf{R}$ which extends f. Then $F(x) = 0$ for $x \in B$ and $F(x) = 1$ for $x \in C$. From this, as we have seen many times, it follows that there are disjoint open sets containing B and C respectively. Hence the space X is normal. □

The interesting point is that its converse is true. Proposition (3.4.2) along with its converse is called the **Tietze characterisation of normality**. The proof of the converse requires the use of Urysohn's lemma and the notion of uniform convergence of sequences of functions.

Definition 53. *Let X be a topological space and (Y, d) a metric space. Then a sequence of functions $\{f_n\}$ from X to Y is said to converge uniformly on X to a function $f : X \to Y$ if for every $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that for all $n \geq N$, and for all $x \in X$, $d(f_n(x), f(x)) < \epsilon$. The sequence $\{f_n\}$ is said to converge point wise to f if for every $x \in X$ the sequence $\{f_n(x)\}$ converges to $f(x)$ in Y.*

Proposition 3.4.3. *Let $X, (Y, d), \{f_n\}$ and f be as above and suppose $\{f_n\}$ converges to f uniformly. Then if each f_n is continuous, so is f.*

Proof. Let $x_0 \in X$ and let V be an open neighbourhood of $f(x_0)$ in Y. Choose $\epsilon > 0$ so that $B(f(x_0), \epsilon) \subset V$. Let $N \in \mathbf{N}$ be such that for all $n \geq N$ and for all $x \in X$, $d(f_n(x), f(x)) < \epsilon/3$. Since f_N is continuous at x_0 there exists an open neighbourhood W of x_0 such that for all $x \in W$, $d(f_N(x), f_N(x_0)) < \epsilon/3$. Now for any $x \in W$ we haved

$$\begin{aligned} d(f(x), f(x_0)) \leq& d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0)) \\ <& \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

Hence $f(W) \subset B(f(x_0), \epsilon) \subset V$ showing that f is continuous at x_0. Since $x_0 \in X$ was arbitrary, f is continuous. □

Proposition 3.4.4. *Let $\sum_{n=1}^{\infty} M_n$ be a convergent series of non-negative real numbers. Suppose $\{f_n\}$ is a sequence of real valued functions on a space X such that for each $x \in X$ and $n \in \mathbf{N}$, $|f_n(x)| \leq M_n$. Then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly to a real valued function on X.*

Proof. By the comparison test for series, for each $x \in X$, the series $\sum_{n=1}^{\infty} f_n(x)$ is absolutely convergent. Denote its sum by $f(x)$. Then f is a real valued function on X. Let $\{s_n\}$ be the sequence of the partial sums of the series $\sum_{n=1}^{\infty} f_n$. We claim that $\{s_n\}$ converges to f uniformly on X. Note that for each $x \in X$ and $n \in \mathcal{N}$, $|s_n(x) - f(x)| \leq \sum_{k=n+1}^{\infty} M_k$. Since the series $\sum_{n=1}^{\infty} M_k$ is given to be convergent, given $\epsilon > 0$, we can find $N \in \mathbf{N}$ such that for all $n \geq N$, $\sum_{k=n+1}^{\infty} M_k < \epsilon$. But then for any $x \in$ and any $n \geq N$, $|s_n(x) - f(x)| < \epsilon$. Hence $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on X. □

We now have all the machinery to prove the Tietze extension theorem. First we establish it for functions into the interval $[-1, 1]$.

Theorem 3.4.5. *Let A be a closed subset of a normal space X and suppose $f : A \to [-1, 1]$ is a continuous function. Then there exists a continuous function $F : X \to [-1, 1]$ such that $F(x) = f(x)$ for all $x \in A$.*

Proof. Refer the textbook. □

Theorem 3.4.6. *Let A be a closed subset of a normal space X and suppose $f : A \to (-1, 1)$ is continuous. Then there exists a continuous function $F : X \to (-1, 1)$ such that $F(x) = f(x)$ for all $x \in A$.*

Proof. Refer the textbook. $\square$

Since any open interval and the real line are homeomorphic to $(-1, 1)$, any continuous real-valued function on a closed subset of a normal space can be extended continuously to the whole space, thus proving the theorem.

Exercise :

1. Prove that there exists a map $r : \mathbf{R} \to [0, \infty)$ such that $r(x) = x$ for all $x \in [0, \infty)$. (in other words r is a right inverse to the inclusion map of $[0, \infty)$ into $\mathbf{R}$. (Such a map is called a retraction and in the present case we say that $[0, \infty)$ is a retract of $\mathbf{R}$).
2. Prove that if A is a closed subset of a normal space X then any map $f : A \to [0, \infty)$ can be continuously extended to a map from X to $[0, \infty)$.
3. Does the Tietze extension theorem hold for maps into $\mathbf{R}$ if on $\mathbf{R}$ we put the semi-open interval topology?

Rough space :