Module 3

Transportation Problems

Transportation problem is a particular class of LPP. The objective is to transport various amounts of a single homogenous commodity that are stored at several origins, to a number of different destinations in such a way that the demands at the destination are satisfied within the capacity of distribution origins and that the total transportation cost is a minimum.

Mathematical formulation of $m \times n$ transportation problem.

Let the origins be denoted as $O_1, O_2, \dots, O_m$ and the destinations by $D_1, D_2, \dots, D_n$. Let the quantity of the commodity produced at the origins be respectively $a_1, a_2, \dots, a_m$. Let the requirements in the various destinations be $b_1, b_2, \dots, b_n$ respectively. The total quantity produced and the total quantity required must be equal.

ie, $a_1 + a_2 + \dots + a_m = b_1 + b_2 + \dots + b_n$.

$$\sum a_i = \sum b_j$$

Let c_{ij} be the cost of transportation of one unit from the i^{th} origin to the j^{th} destination.

	D_1	D_2	$\cdots$	D_j	$\cdots$	D_n	Available
O_1	C_{11}	C_{12}	$\cdots$	C_{1j}	$\cdots$	C_{1n}	a_1
O_2	C_{21}	C_{22}	$\cdots$	C_{2j}	$\cdots$	C_{2n}	a_2
$\vdots$	$\cdot$	$\cdot$		$\cdot$		$\cdot$	$\vdots$
O_i	C_{i1}	C_{i2}	$\cdots$	C_{ij}	$\cdots$	C_{in}	a_i
$\vdots$	$\cdot$	$\cdot$		$\cdot$		$\cdot$	$\vdots$
O_m	C_{m1}	C_{m2}	$\cdots$	C_{mj}	$\cdots$	C_{mn}	a_m
Required	b_1	b_2	$\cdots$	b_j	$\cdots$	b_n	

The problem is to determine the quantity x_{ij} to be transported from the i^{th} origin to the j^{th} destination such that the total cost $\sum_i \sum_j x_{ij} c_{ij}$ is minimum.

Transportation problem in the form of a LPP.

Let x_{ij} be the number of units transported from the i^{th} origin to the j^{th} destination. c_{ij} be the cost for transporting one unit of the commodity from the i^{th} origin to the j^{th} destination. Let a_i be the units available at the i^{th} origin and b_j be the units required in the j^{th} destination.

Then the problem is

$$\text{Minimise } \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} x_{ij}$$

$$\text{subject to } \sum_{j=1}^{n} x_{ij} = a_i \quad , \quad i = 1, 2, \cdots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad , \quad j = 1, 2, \cdots, n$$

$$x_{ij} \geq 0 \text{ for } i, j$$

<u>Feasible Solution</u> : A feasible solution to a transportation problem is a set of non negative individual allocations which satisfy the row and column sum restrictions.

<u>Basic Feasible Solution</u> : A feasible solution of $m \times n$ transportation problem is said to be a basic feasible solution if the total number of allocations is exactly equal to $m+n-1$.

<u>Optimal Solution</u> : A feasible solution is said to be optimal if it minimises the transportation cost.

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Methods for finding an initial basic feasible solution.

1. North-West Corner Method.

Step 1 : Allocate to the cell (1,1) maximum possible amount, which is the minimum of the row total and the column total. Then either the row total or column total gets exhausted. Hence cross off that row or column.

Step 2 : Consider the reduced matrix. In this matrix allocate to the cell (1,1) maximum possible amount.

Step 3 : Repeat the above steps until all the available quantities are exhausted.

Qn. Find the intitial basic feasible solution to the transportation problem given below by North-West Corner method.

Destinations

Origins	D_1	D_2	D_3	Supply
O_1	2	7	4	5
O_2	3	3	1	8
O_3	5	4	7	7
O_4	1	6	2	14
Demand	7	9	18	

Allocate to cell (1,1) minimum of 5 and 7. ie, 5. Thus O_1 row total is exchausted since the supply of O_1 is completely met So cross off row O_1.

	D_1	D_2	D_3	
O_1	$\boxed{5}$ 2	7	4	~~5~~
O_2	3	3	1	8
O_3	5	4	7	7
O_4	1	6	2	14
	~~7~~ 2	9	18	

Now consider the reduced matrix after deleting O_1 row. Allocate to the cell (1,1) minimum of 8 and 2. ie, 2 Thus column D_1 is exchausted and it is crossed off.

	D_1	D_2	D_3	
O_2	$\boxed{2}$ 3	3	1	~~8~~ 6
O_3	5	4	7	7
O_4	1	6	2	14
	~~2~~	9	18	

Consider the reduced matrix. Allocate to the cell (1,1) minimum of 6 and 9. ie, 6. Thus the O_2 row is crossed off.

	D_2	D_3	
O_2	$\boxed{6}$ 3	1	~~6~~
O_3	4	7	7
O_4	6	2	14
	~~9~~ 3	18	

Allocate to cell (1,1) min of 7 and 3. ie, 3. Thus the D_2 column is crossed off.

	D_2	D_3	
O_3	$\boxed{3}$ 4	7	~~7~~ 4
O_4	6	2	14
	~~3~~	18	

Allocate 4 to cell (1,1) and 14 to cell (2,1)

	D_3	
O_3	$\boxed{4}$ 7	~~4~~
O_4	$\boxed{14}$ 2	14
	~~18~~ 14	

Thus the allocations made to the cells are shown below and the solution is

	D_1	D_2	D_3
O_1	$\boxed{5}$ 2	7	4
O_2	$\boxed{2}$ 3	$\boxed{6}$ 3	1
O_3	5	$\boxed{3}$ 4	$\boxed{4}$ 7
O_4	1	6	$\boxed{14}$ 2

$\therefore$ Total transportation cost $= 5\times 2 + 2\times 3 + 6\times 3 + 3\times 4 + 4\times 7 + 14\times 2$

$= 10+6+18+12+28+2$

$= 102.$

Qn. Determine an initial basic feasible solution to the following transportation problem using the North-West corner rule.

	D_1	D_2	D_3	D_4	Supply
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	4	3	6	2	5
Demand	6	10	15	4	

$\boxed{6}$ 6	$\boxed{8}$ 4	1	5	$\cancel{14}$ $\cancel{8}$
8	$\boxed{2}$ 9	$\boxed{14}$ 2	7	$\cancel{16}$ $\cancel{14}$
4	3	$\boxed{1}$ 6	$\boxed{4}$ 2	$\cancel{5}$ 4
$\cancel{6}$	$\cancel{10}$ $\cancel{2}$	$\cancel{15}$ $\cancel{1}$	4	

Total transportation cost $= 6\times6 + 8\times4 + 2\times9 + 14\times2 + 1\times6 + 4\times2$

$= 36 + 32 + 18 + 28 + 6 + 8$

$= \underline{128}$

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2. <u>Least - Cost Method or Matrix Minima Method.</u>

This method takes into account the minimum unit cost. Choose the cell having the lowest cost in the matrix. Allocate to that cell as much as possible. Thus either a row total or a column total is exhausted. Cross off the corresponding row or column. From the reduced matrix, locate the cell having the lowest cost. Allocate to that cell maximum possible. Continue the process until all the available quantities are exhausted.

Qn 1. Find the initial basic feasible solution to the following transportation problem by lowest cost (entry) method.

F_1	2	7	4	5
F_2	3	3	1	8
F_3	5	4	7	7
F_4	1	6	2	14
	7	9	18	

The lowest cost is 1 in cells (2,3) and (4,1). Select one of these say (2,3). Allocate min of 18 and 8, ie, 8. Thus the total row F_2 is exhausted. Cross of row F_2.

	W_1	W_2	W_3	
F_1	2	7	4	5
F_2	3	3	$\boxed{8}$ 1	8
F_3	5	4	7	7
F_4	1	6	2	14
	7	9	~~18~~ 10	

Allocate to the cell (3,1) min of 7 and 14 ie, 7. and cross off the column W_1.

	W_1	W_2	W_3	
F_1	2	7	4	5
F_3	5	4	7	7
F_4	$\boxed{7}$ 1	6	2	~~14~~ 7
	~~7~~	9	10	

Allocate to the cell (3,2) min of 7 and 10, ie, 7 and cross off row F_4.

	W_2	W_3	
F_1	7	4	5
F_3	4	7	7
F_4	6	$\boxed{7}$ 2	~~7~~
	9	~~10~~ 3	

Allocate 3 to the cell (1,2) and cross off column W_3.

	W_2	W_3	
F_1	7	$\boxed{3}$ 4	~~5~~ 2
F_3	4	7	7
	9	~~3~~	

Allocate 7 to the cell (2,1) and 2 to the cell (1,1)

	W_2	
F_1	$\boxed{2}$ 7	~~2~~
F_3	$\boxed{7}$ 4	~~7~~
	~~9~~ ~~2~~	

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The solution is

	W_1	W_2	W_3	
F_1	2	$\boxed{2}$ 7	$\boxed{3}$ 4	5
F_2	3	3	$\boxed{8}$ 1	8
F_3	5	$\boxed{7}$ 4	7	7
F_4	$\boxed{7}$ 1	6	$\boxed{7}$ 2	14
	7	9	18	

$\therefore$ Total transportation cost $= 2\times 7 + 3\times 4 + 8\times 1 + 7\times 4 + 7\times 1 + 7\times 2$

$= 14 + 12 + 8 + 28 + 7 + 14$

$= 83$

Qn 2. Determine the initial basic feasible solution of the transportation problem by least cost method.

Destinations

		A	B	C	D	Supply
Origins	1	1	5	3	3	34
	2	3	3	1	2	15
	3	0	2	2	3	12
	4	2	7	2	4	19
Demands		21	25	17	17	

The solution is

	A	B	C	D	
1	$\boxed{9}$ 1	$\boxed{8}$ 5	3	$\boxed{17}$ 3	~~34~~ ~~25~~ 8
2	3	3	$\boxed{15}$ 1	2	~~15~~
3	$\boxed{12}$ 0	2	2	3	~~12~~
4	2	$\boxed{17}$ 7	$\boxed{2}$ 2	4	~~19~~ 17
	~~21~~ ~~9~~	~~25~~ 17	~~17~~ ~~2~~	~~17~~	

$$\therefore \text{Total cost} = (9\times1) + (8\times5) + (17\times3) + (15\times1) + (12\times0) + (17\times7) + (2\times2)$$

$$= 9 + 40 + 51 + 15 + 0 + 119 + 4$$

$$= 238$$

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3. Vogel's Approximation Method.

Step 1

Write the difference between the smallest and the second smallest costs in each column below the corresponding column, within brackets. Similarly write the differences in each row to the right of the corresponding row. These differences are known as penalty.

Step 2

Select the row or column having the largest penalty and allocate the maximum possible amount to the cell with lowest cost in that row or column. Thus either the row total or column total is completely exhausted. Cross off that row or column.

Step 3

For the reduced matrix apply step 1 and step 2 until all rows and columns totals are exhausted.

Note: The initial basic feasible solution obtained by Vogel's method is more close to the optimal solution that the solution obtained by the other two methods.

Qn 1. Find the initial solution for the transportation problem by Vogel's method.

	W_1	W_2	W_3	Supply
F_1	2	7	4	5
F_2	3	3	1	8
F_3	5	4	7	7
F_4	1	6	2	14
Demand	7	9	18	

Maximum penalty 2 is associated with rows F_1 and F_2. Choose row F_1 and allocate min (5,7) = 5 to the cell (1,1) which has the lowest cost. Cross off row F_1

	W_1	W_2	W_3	
F_1	$\boxed{5}$ 2	7	4	~~5~~ (2)
F_2	3	3	1	8 (2)
F_3	5	4	7	7 (1)
F_4	1	6	2	14 (1)
	2 ~~7~~ (1)	9 (1)	18 (1)	

In the reduced matrix, the maximum penalty 2 is associated with row F_2 and column W_1. Select any one of these, say row F_2. Allocate min (8, 18) = 8 to the cell (1,3). Cross off row F_2

	W_1	W_2	W_3	
F_2	3	3	$\boxed{8}$ 1	~~8~~ (2)
F_3	5	4	7	7 (1)
F_4	1	6	2	14 (1)
	2 (2)	9 (1)	~~18~~ 10 (1)	

Maximum penalty 5 in column W_3. Allocate min (14, 10) = 10 to the cell (2,3). Cross off column W_3.

	W_1	W_2	W_3	
F_3	5	4	7	7 (1)
F_4	1	6	$\boxed{10}$ 2	~~14~~ 4 (1)
	2 (4)	9 (2)	~~10~~ (5)	

Allocate 2 to the cell (2,1)
Cross off column W_1.

	W_1	W_2	
F_3	5	4	7 (1)
F_4	$\boxed{2}$ 1	6	~~4~~ (5) 2
	~~2~~ (4)	9 (2)	

Allocate 7 to cell (1,1) and 2 to cell (2,1)

	W_2	
F_3	$\boxed{7}$ 4	~~7~~
F_4	$\boxed{2}$ 6	~~2~~
	~~9~~ ~~2~~	

The solution is

	W_1	W_2	W_3
F_1	$\boxed{5}$ 2	7	4
F_2	3	3	$\boxed{8}$ 1
F_3	5	$\boxed{7}$ 4	7
F_4	$\boxed{2}$ 1	$\boxed{2}$ 6	$\boxed{10}$ 2

$$\text{Total cost} = (5\times2) + (8\times1) + (7\times4) + (2\times1) + (2\times6) + (10\times2)$$
$$= 10 + 8 + 28 + 2 + 12 + 20$$
$$= 80$$

Qn 2 Find an initial basic feasible solution using Vogel's method to the following transportation problem.

	1	2	3	4	Supply
A	21	16	15	3	11
B	17	18	14	23	13
C	32	27	18	41	19
Demand	6	10	12	15	

The solution is

	1	2	3	4			
A	21	16	15	$\boxed{11}$ 3	~~11~~	(12)	
B	$\boxed{6}$ 17	$\boxed{3}$ 18	14	$\boxed{4}$ 23	9 3 ~~13~~	(3)	(3) (4)
C	32	$\boxed{7}$ 27	$\boxed{12}$ 18	41	~~19~~ 7	(9)	(9) (9)
	~~6~~	~~10~~ ~~7~~	~~12~~	~~15~~ 4			
	(4)	(2)	(1)	(20) ↑			
	(15) ↑	(9)	(4)	(18) ↑			

Total cost $= (11 \times 3) + (6 \times 17) + (3 \times 18) + (4 \times 23) + (7 \times 27) + (12 \times 18)$

$= 686$