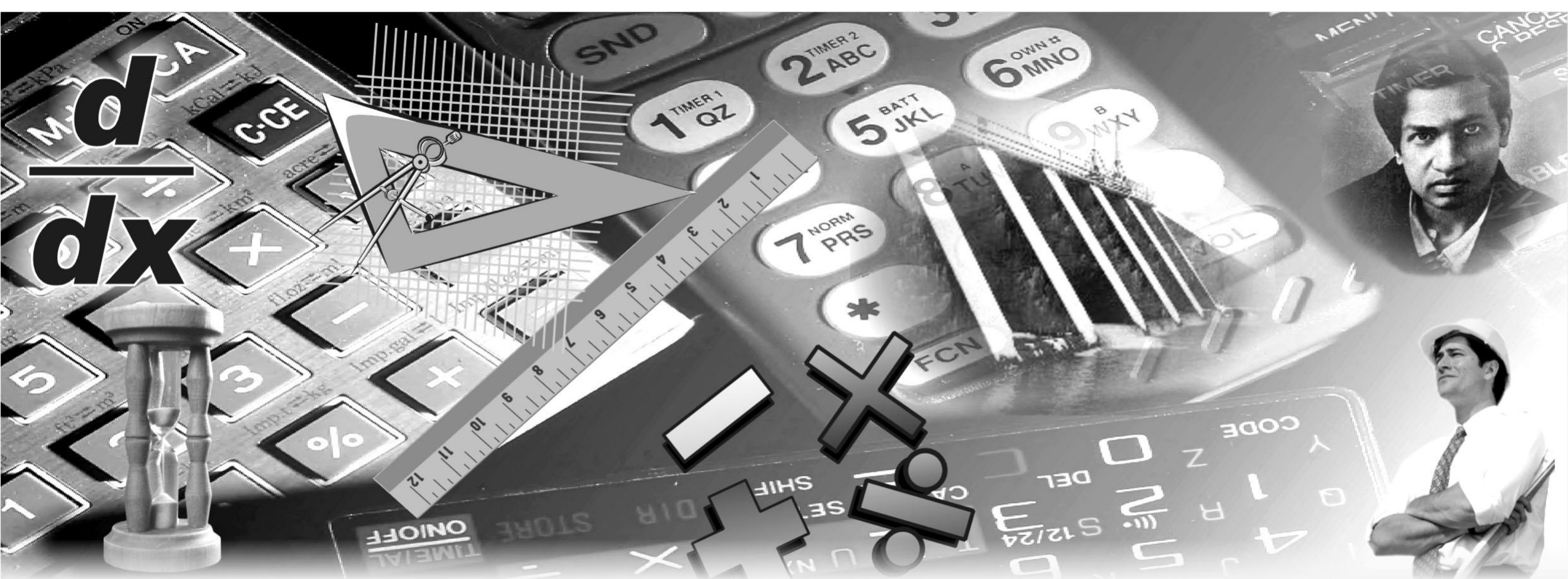

Chapter
9

Determinants and Matrices

Determinants

Definition

Let us consider three homogeneous linear equations

$$a_1x+b_1y+c_1z=0\,,\; a_2x+b_2y+c_2z=0$$

and $a_3x+b_3y+c_3z=0$

Eliminated *x, y, z* from above three equations we obtain

$$a_1(b_2c_3-b_3c_2)-b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2)=0 \quad .....(i)$$

The L.H.S. of (i) is represented by $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Its contains three rows and three columns, it is called a determinant of third order.

The number of elements in a second order is $2^2=4$ and the number of elements in a third order determinant is $3^2=9$.

Rows and columns of a determinant : In a determinant horizontal lines counting from top 1st, 2nd, 3rd,..... respectively known as rows and denoted by $R_1, R_2, R_3, ...$ and vertical lines counting left to right, 1st, 2nd, 3rd,..... respectively known as columns and denoted by $C_1, C_2, C_3, ...$

Properties of determinants

P-1 : The value of determinant remains unchanged, if the rows and the columns are interchanged.

Since the determinant remains unchanged when rows and columns are interchanged, it is obvious that any theorem which is true for 'rows' must also be true for 'columns'.

P-2 : If any two rows (or columns) of a determinant be interchanged, the determinant is unaltered in numerical value but is changed in sign only.

P-3 : If a determinant has two rows (or columns) identical, then its value is zero.

P-4 : If all the elements of any row (or column) be multiplied by the same number, then the value of determinant is multiplied by that number.

P-5 : If each element of any row (or column) can be expressed as a sum of two terms, then the determinant can be expressed as the sum of the determinants.

e.g., $$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

P-6 : The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column)

e.g., $$D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and $$D'=\begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3-na_1 & b_3-nb_1 & c_3-nc_1 \end{vmatrix}.$$

Then $D'=D$.

P-7 : If all elements below leading diagonal or above leading diagonal or except leading diagonal elements are zero then the value of the determinant equal to multiplied of all leading diagonal elements.

P-8 : If a determinant D becomes zero on putting $x=\alpha$, then we say that $(x-\alpha)$ is factor of determinant.

P-9 : It should be noted that while applying operations on determinants then at least one row (or column) must remain unchanged or, Maximum number of operations = order of determinant –1.

P-10 : It should be noted that if the row (or column) which is changed by multiplied a non-zero number, then the determinant will be divided by that number.

Minors and Cofactors

(1) **Minor of an element :** If we take the element of the determinant and delete (remove) the row and column containing that element, the determinant left is called the minor of that element. It is denoted by M_{ij}.

Consider the determinant $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$,

then determinant of minors $M = \begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix}$

where $M_{11} =$ minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$M_{12} =$ minor of $a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

$M_{13} =$ minor of $a_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Similarly, we can find the minors of other elements . Using this concept the value of determinant can be

$\Delta = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$

or, $\Delta = -a_{21}M_{21} + a_{22}M_{22} - a_{23}M_{23}$

or, $\Delta = a_{31}M_{31} - a_{32}M_{32} + a_{33}M_{33}$.

(2) **Cofactor of an element :** The cofactor of an element a_{ij} (*i.e.* the element in the i^{th} row and j^{th} column) is defined as $(-1)^{i+j}$ times the minor of that element. It is denoted by C_{ij} or A_{ij} or F_{ij}. $C_{ij} = (-1)^{i+j} M_{ij}$

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then determinant of cofactors is

$$C = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix}$$

where $C_{11} = (-1)^{1+1} M_{11} = +M_{11}$, $C_{12} = (-1)^{1+2} M_{12} = -M_{12}$

and $C_{13} = (-1)^{1+3} M_{13} = +M_{13}$

Similarly, we can find the cofactors of other elements.

Product of two determinants

Let the two determinants of third order be,

$$D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}.$$

Let D be their product.

Then $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

We can also multiply rows by columns or columns by rows or columns by columns.

Differentiation and integration of determinants

(1) **Differentiation of a determinant**

(i) Let $\Delta(x)$ be a determinant of order two. If we write $\Delta(x) = |C_1 \quad C_2|$, where C_1 and C_2 denote the 1st and 2nd columns, then

$\Delta'(x) = |C'_1 \quad C_2| + |C_1 \quad C'_2|$

where C'_i denotes the column which contains the derivative of all the functions in the i^{th} column C_i.

In a similar fashion, if we write $\Delta(x) = \begin{vmatrix} R_1 \\ R_2 \end{vmatrix}$, then

$\Delta'(x) = \begin{vmatrix} R'_1 \\ R_2 \end{vmatrix} + \begin{vmatrix} R_1 \\ R'_2 \end{vmatrix}$

(ii) Let $\Delta(x)$ be a determinant of order three. If we write $\Delta(x) = |C_1 \quad C_2 \quad C_3|$, then

$\Delta'(x) = |C'_1 \quad C_2 \quad C_3| + |C_1 \quad C'_2 \quad C_3| + |C_1 \quad C_2 \quad C'_3|$

and similarly if we consider $\Delta(x) = \begin{vmatrix} R_1 \\ R_2 \\ R_3 \end{vmatrix}$

Then $\Delta'(x) = \begin{vmatrix} R'_1 \\ R_2 \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R'_2 \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2 \\ R'_3 \end{vmatrix}$

(iii) If only one row (or column) consists functions of x and other rows (or columns) are constant, *viz.*

Let $\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$,

Then $\Delta'(x) = \begin{vmatrix} f'_1(x) & f'_2(x) & f'_3(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

And in general $\Delta^n(x) = \begin{vmatrix} f_1^n(x) & f_2^n(x) & f_3^n(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

where n is any positive integer and $f^n(x)$ denotes the n^{th} derivative of $f(x)$.

(2) **Integration of a determinant**

Let $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ l & m & n \end{vmatrix}$, where *a, b, c, l, m* and *n* are constants.

$$\Rightarrow \int_a^b \Delta(x)dx = \begin{vmatrix} \int_a^b f(x)dx & \int_a^b g(x)dx & \int_a^b h(x)dx \\ a & b & c \\ l & m & n \end{vmatrix}$$

If the elements of more than one column or rows are functions of x then the integration can be done only after evaluation/expansion of the determinant.

Application of determinants in solving a system of linear equations

(1) **Solution of system of linear equations in three variables by Cramer's rule :** The solution of the system of linear equations $a_1x + b_1y + c_1z = d_1$ .....(i)

$a_2x + b_2y + c_2z = d_2$ .....(ii)

$a_3x + b_3y + c_3z = d_3$ .....(iii)

Is given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$,

where, $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, and $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Provided that $D \neq 0$

(2) **Conditions for consistency :** For a system of 3 simultaneous linear equations in three unknown variable.

(i) If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$

(ii) If $D = 0$ and $D_1 = D_2 = D_3 = 0$, then the given system of equations is consistent with infinitely many solutions.

(iii) If $D = 0$ and at least one of the determinants D_1, D_2, D_3 is non-zero, then given of equations is inconsistent.

Some special determinants

(1) **Symmetric determinant**

A determinant is called symmetric determinant if for its every element $a_{ij} = a_{ji} \forall i, j$ *e.g.*, $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

(2) **Skew-symmetric determinant :** A determinant is called skew symmetric determinant if for its every element $a_{ij} = -a_{ji} \forall i, j$ *e.g.*, $\begin{vmatrix} 0 & 3 & -1 \\ -3 & 0 & 5 \\ 1 & -5 & 0 \end{vmatrix}$

Every diagonal element of a skew symmetric determinant is always zero.

The value of a skew symmetric determinant of even order is always a perfect square and that of odd order is always zero.

(3) **Cyclic order :** If elements of the rows (or columns) are in cyclic order. *i.e.,* (i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(ii) $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$

$= (a-b)(b-c)(c-a)(ab+bc+ca)$.

(iii) $\begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(v) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc)$

Matrices

Definition

A rectangular arrangement of numbers (which may be real or complex numbers) in rows and columns, is called a matrix. This arrangement is enclosed by small ( ) or big [ ] brackets. The numbers are called the elements of the matrix or entries in the matrix.

Order of a matrix

A matrix having m rows and n columns is called a matrix of order $m \times n$ or simply $m \times n$ matrix (read as an m by n matrix). A matrix A of order $m \times n$ is usually written in the following manner

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots a_{1j} & \dots a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots a_{2j} & \dots a_{2n} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ a_{i1} & a_{i2} & a_{i3} & \dots a_{ij} & \dots a_{in} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ a_{m1} & a_{m2} & a_{m3} & \dots a_{mj} & \dots a_{mn} \end{bmatrix} \text{ or } A = [a_{ij}]_{m \times n},$$

where $\begin{matrix} i = 1, 2, \dots. m \\ j = 1, 2, \dots. n \end{matrix}$

Here a_{ij} denotes the element of i^{th} row and j^{th} column.

Example **:** order of matrix $\begin{bmatrix} 3 & -1 & 5 \\ 6 & 2 & -7 \end{bmatrix}$ is 2 × 3.

A matrix of order $m \times n$ contains mn elements. Every row of such a matrix contains n elements and every column contains m elements.

Equality of matrices

Two matrix A and B are said to be equal matrix if they are of same order and their corresponding elements are equal.

Types of matrices

(1) **Row matrix :** A matrix is said to be a row matrix or row vector if it has only one row and any number of columns. *Example* : [5 0 3] is a row matrix of order 1× 3 and [2] is a row matrix of order 1×1.

(2) **Column matrix :** A matrix is said to be a column matrix or column vector if it has only one column and any number of rows.

Example : $\begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$ is a column matrix of order 3 × 1 and [2] is a column matrix of order 1 × 1. Observe that [2] is both a row matrix as well as a column matrix.

(3) **Singleton matrix :** If in a matrix there is only one element then it is called singleton matrix.

Thus, $A=[a_{ij}]_{m\times n}$ is a singleton matrix, if $m=n=1$

Example : [2], [3], [a], [–3] are singleton matrices.

(4) **Null or zero matrix :** If in a matrix all the elements are zero then it is called a zero matrix and it is generally denoted by *O.* Thus $A=[a_{ij}]_{m\times n}$ is a zero matrix if $a_{ij}=0$ for all i and j.

Example : $[0], \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, [0\ 0]$ are all zero matrices, but of different orders.

(5) **Square matrix:** If number of rows and number of columns in a matrix are equal, then it is called a square matrix.

Thus $A=[a_{ij}]_{m\times n}$ is a square matrix if $m=n$.

Example : $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3×3.

(i) If $m \neq n$ then matrix is called a rectangular matrix.

(ii) The elements of a square matrix A for which $i=j$, *i.e.* $a_{11}, a_{22}, a_{33}, \ldots a_{nn}$ are called diagonal elements and the line joining these elements is called the principal diagonal or leading diagonal of matrix *A.*

(6) **Diagonal matrix :** If all elements except the principal diagonal in a square matrix are zero, it is called a diagonal matrix. Thus a square matrix $A=[a_{ij}]$ is a diagonal matrix if $a_{ij}=0$, when $i \neq j$.

Example : $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a diagonal matrix of order 3× 3, which can be denoted by diag [2, 3, 4].

(7) **Identity matrix** : A square matrix in which elements in the main diagonal are all '1' and rest are all zero is called an identity matrix or unit matrix. Thus, the square matrix $A=[a_{ij}]$ is an identity matrix, if

$$a_{ij}=\begin{cases} 1, \text{ if } i=j \\ 0, \text{ if } i\neq j \end{cases}$$

We denote the identity matrix of order n by I_n.

Example : $[1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of order 1, 2 and 3 respectively.

(8) **Scalar matrix** : A square matrix whose all non diagonal elements are zero and diagonal elements are equal is called a scalar matrix. Thus, if $A=[a_{ij}]$ is a square matrix and $a_{ij}=\begin{cases} \alpha, \text{ if } i=j \\ 0, \text{ if } i\neq j \end{cases}$, then A is a scalar matrix.

Unit matrix and null square matrices are also scalar matrices.

(9) **Triangular matrix** : A square matrix $[a_{ij}]$ is said to be triangular matrix if each element above or below the principal diagonal is zero. It is of two types

(i) **Upper triangular matrix :** A square matrix $[a_{ij}]$ is called the upper triangular matrix, if $a_{ij}=0$ when $i>j$.

Example : $\begin{bmatrix} 3 & 1 & 2 \\ 0 & 4 & 3 \\ 0 & 0 & 6 \end{bmatrix}$ is an upper triangular matrix of order 3 × 3.

(ii) **Lower triangular matrix :** A square matrix $[a_{ij}]$ is called the lower triangular matrix, if $a_{ij}=0$ when $i<j$.

Example : $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 2 \end{bmatrix}$ is a lower triangular matrix of order 3 × 3.

Trace of a matrix

The sum of diagonal elements of a square matrix. A is called the trace of matrix A , which is denoted by $tr\,A$.

$$tr\,A=\sum_{i=1}^{n} a_{ii}=a_{11}+a_{22}+\ldots a_{nn}$$

Properties of trace of a matrix

Let $A=[a_{ij}]_{n\times n}$ and $B=[b_{ij}]_{n\times n}$ and λ be a scalar

(i) $tr(\lambda A)=\lambda\, tr(A)$ (ii) $tr(A-B)=tr(A)-tr(B)$

(iii) $tr(AB)=tr(BA)$ (iv) $tr(A)=tr(A')$ or $tr(A^T)$

(v) $tr(I_n)=n$ (vi) $tr\,(0)=0$

(vii) $tr(AB)\neq tr A.\, tr B$

Addition and subtraction of matrices

If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ are two matrices of the same order then their sum $A+B$ is a matrix whose each element is the sum of corresponding elements *i.e.,* $A+B=[a_{ij}+b_{ij}]_{m\times n}$.

Similarly, their subtraction $A-B$ is defined as

$$A-B=[a_{ij}-b_{ij}]_{m\times n}$$

Matrix addition and subtraction can be possible only when matrices are of the same order.

Properties of matrix addition : If *A, B* and *C* are matrices of same order, then

(i) $A+B=B+A$ (Commutative law)

(ii) $(A+B)+C=A+(B+C)$ (Associative law)

(iii) $A+O=O+A=A$, where O is zero matrix which is additive identity of the matrix.

(iv) $A+(-A)=0=(-A)+A$, where $(-A)$ is obtained by changing the sign of every element of *A,* which is additive inverse of the matrix.

(v) $\left.\begin{matrix}A+B=A+C\\ B+A=C+A\end{matrix}\right\}\Rightarrow B=C$ (Cancellation law)

Scalar multiplication of matrices

Let $A=[a_{ij}]_{m\times n}$ be a matrix and *k* be a number, then the matrix which is obtained by multiplying every element of *A* by *k* is called scalar multiplication of *A* by *k* and it is denoted by *kA*.

Thus, if $A=[a_{ij}]_{m\times n}$, then $kA=Ak=[ka_{ij}]_{m\times n}$.

Properties of scalar multiplication

If *A*, *B* are matrices of the same order and λ,μ are any two scalars then

(i) $\lambda(A+B)=\lambda A+\lambda B$ (ii) $(\lambda+\mu)A=\lambda A+\mu A$

(iii) $\lambda(\mu A)=(\lambda\mu A)=\mu(\lambda A)$ (iv) $(-\lambda A)=-(\lambda A)=\lambda(-A)$

All the laws of ordinary algebra hold for the addition or subtraction of matrices and their multiplication by scalars.

Multiplication of matrices

Two matrices *A* and *B* are conformable for the product *AB* if the number of columns in *A* (pre-multiplier) is same as the number of rows in *B* (post multiplier). Thus, if $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times p}$ are two matrices of order *m* × *n* and $n\times p$ respectively, then their product *AB* is of order $m\times p$ and is defined as

$$(AB)_{ij}=\sum_{r=1}^{n}a_{ir}b_{rj}=[a_{i1}a_{i2}...a_{in}]\begin{bmatrix}b_{1j}\\ b_{2j}\\ \vdots\\ b_{nj}\end{bmatrix}$$ = (i^{th} row of A)(j^{th} column of *B*) .....(i)

where *i* =1, 2, ..., *m* and *j* =1, 2, ...*p*

Now we define the product of a row matrix and a column matrix.

Let $A=[a_1a_2....a_n]$ be a row matrix and $B=\begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_n\end{bmatrix}$ be a column matrix.

Then $AB=[a_1b_1+a_2b_2+....+a_nb_n]$ .....(ii)

Thus, from (i), $(AB)_{ij}$ = Sum of the product of elements of i^{th} row of *A* with the corresponding elements of j^{th} column of *B.*

Properties of matrix multiplication

If *A*,*B* and *C* are three matrices such that their product is defined, then

(i) $AB\neq BA$, (Generally not commutative)

(ii) $(AB)C=A(BC)$, (Associative Law)

(iii) $IA=A=AI$, where *I* is identity matrix for matrix multiplication.

(iv) $A(B+C)=AB+AC$, (Distributive law)

(v) If $AB=AC\nRightarrow B=C$,(Cancellation law is not applicable)

(vi) If *AB* = 0, it does not mean that *A*= 0 or *B* = 0, again product of two non zero matrix may be a zero matrix.

Positive integral powers of a matrix

The positive integral powers of a matrix *A* are defined only when *A* is a square matrix.

Also then $A^2=A.A$, $A^3=A.A.A=A^2A$.

Also for any positive integers *m* and *n,*

(i) $A^mA^n=A^{m+n}$

(ii) $(A^m)^n=A^{mn}=(A^n)^m$

(iii) $I^n=I, I^m=I$

(iv) $A^0=I_n$, where *A* is a square matrix of order *n*.

Transpose of a matrix

The matrix obtained from a given matrix *A* by changing its rows into columns or columns into rows is called transpose of matrix *A* and is denoted by A^T or A'.

From the definition it is obvious that if order of *A* is *m* × *n*, then order of A^T is *n* × *m.*

***Example*:**

Transpose of matrix $\begin{bmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{bmatrix}_{2\times 3}$ is $\begin{bmatrix}a_1 & b_1\\ a_2 & b_2\\ a_3 & b_3\end{bmatrix}_{3\times 2}$

Properties of transpose : Let *A* and *B* be two matrices then, (i) $(A^T)^T=A$

(ii) $(A+B)^T=A^T+B^T$, *A* and *B* being of the same order

(iii) $(kA)^T=kA^T$, *k* be any scalar (real or complex)

(iv) $(AB)^T=B^TA^T$, *A* and *B* being conformable for the product *AB*

(v) $(A_1A_2A_3....A_{n-1}A_n)^T=A_n^TA_{n-1}^T......A_3^TA_2^TA_1^T$

(vi) $I^T=I$

Special types of matrices

(1) **Symmetric matrix :** A square matrix $A=[a_{ij}]$ is called symmetric matrix if $a_{ij}=a_{ji}$ for all *i, j* or $A^T=A$.

Example **:** $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

(2) **Skew-symmetric matrix :** A square matrix $A=[a_{ij}]$ is called skew- symmetric matrix if $a_{ij}=-a_{ji}$ for all *i, j* or $A^T=-A$.

Example **:** $\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$

All principal diagonal elements of a skew-symmetric matrix are always zero because for any diagonal element.

$$a_{ij}=-a_{ij}\Rightarrow a_{ij}=0$$

Properties of symmetric and skew-symmetric matrices

(i) If *A* is a square matrix, then $A+A^T, AA^T, A^TA$ are symmetric matrices, while $A-A^T$ is skew- symmetric matrix.

(ii) If *A* is a symmetric matrix, then $-A, KA, A^T, A^n, A^{-1}, B^TAB$ are also symmetric matrices, where $n\in N$, $K\in R$ and *B* is a square matrix of order that of *A*.

(iii) If A is a skew-symmetric matrix, then

(a) A^{2n} is a symmetric matrix for $n\in N$.

(b) A^{2n+1} is a skew-symmetric matrix for $n\in N$.

(c) *kA* is also skew-symmetric matrix, where $k\in R$.

(d) B^TAB is also skew- symmetric matrix where *B* is a square matrix of order that of *A.*

(iv) If *A*, *B* are two symmetric matrices, then

(a) $A\pm B$, $AB+BA$ are also symmetric matrices,

(b) $AB-BA$ is a skew- symmetric matrix,

(c) AB is a symmetric matrix, when $AB=BA$.

(v) If *A, B* are two skew-symmetric matrices, then

(a) $A\pm B$, $AB-BA$ are skew-symmetric matrices,

(b) $AB+BA$ is a symmetric matrix.

(vi) If *A* a skew-symmetric matrix and *C* is a column matrix, then C^TAC is a zero matrix.

(vii) Every square matrix *A* can unequally be expressed as sum of a symmetric and skew-symmetric matrix

i.e., $A=\left[\frac{1}{2}(A+A^T)\right]+\left[\frac{1}{2}(A-A^T)\right]$.

(3) **Singular and Non-singular matrix :** Any square matrix *A* is said to be non-singular if $|A|\neq 0$, and a square matrix *A* is said to be singular if $|A|=0$. Here $|A|$ (or det(*A*) or simply det |*A*| means corresponding determinant of square matrix *A*.

Example **:** $A=\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

then $|A|=\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}=10-12=-2\Rightarrow A$ is a non-singular matrix.

(4) **Hermitian and Skew-hermitian matrix :** A square matrix $A=[a_{ij}]$ is said to be hermitian matrix if

$a_{ij}=\bar{a}_{ji};\ \forall i,j\ i.e., A=A^{\theta}$.

Example **:** $\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$, $\begin{bmatrix} 3 & 3-4i & 5+2i \\ 3+4i & 5 & -2+i \\ 5-2i & -2-i & 2 \end{bmatrix}$

are Hermitian matrices. If *A* is a Hermitian matrix then $a_{ii}=\bar{a}_{ii}\Rightarrow a_{ii}$ is real $\forall i$, thus every diagonal element of a Hermitian matrix must be real.

A square matrix, *A* = $|a_{ij}|$ is said to be a Skew-Hermitian if $a_{ij}=-\bar{a}_{ji}.\forall i,j\ i.e.\ A^{\theta}=-A$. If *A* is a skew-Hermitian matrix, then $a_{ii}=-\bar{a}_{ii}\Rightarrow a_{ii}+\bar{a}_{ii}=0$ *i.e.* a_{ii} must be purely imaginary or zero.

Example **:** $\begin{bmatrix} 0 & -2+i \\ 2-i & 0 \end{bmatrix}$, $\begin{bmatrix} 3i & -3+2i & -1-i \\ 3+2i & -2i & -2-4i \\ 1-i & 2-4i & 0 \end{bmatrix}$

are skew-hermitian matrices.

(5) **Orthogonal matrix :** A square matrix *A* is called orthogonal if $AA^T=I=A^TA$ *i.e.,* if $A^{-1}=A^T$

Example **:** $A=\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

is orthogonal because $A^{-1}=\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}=A^T$

In fact every unit matrix is orthogonal. Determinant of orthonogal matrix is – 1 or 1.

(6) **Idempotent matrix :** A square matrix *A* is called an idempotent matrix if $A^2=A$.

Example **:** $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ is an idempotent matrix, because

$$A^2=\begin{bmatrix} 1/4+1/4 & 1/4+1/4 \\ 1/4+1/4 & 1/4+1/4 \end{bmatrix}=\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}=A.$$

Also, $A=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are idempotent matrices because $A^2=A$ and $B^2=B$.

In fact every unit matrix is indempotent.

(7) **Involutory matrix :** A square matrix A is called an involutory matrix if $A^2=I$ or $A^{-1}=A$

***Example*:**

$A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ is an involutory matrix because

$A^2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$

In fact every unit matrix is involutory.

(8) **Nilpotent matrix :** A square matrix A is called a nilpotent matrix if there exists a $p\in N$ such that $A^p=0$.

Example: $A=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$

is a nilpotent matrix because $A^2=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=0$, (Here $P=2$)

Determinant of every nilpotent matrix is 0.

(9) **Unitary matrix :** A square matrix is said to be unitary, if $\bar{A}'A=I$ since $|\bar{A}'|=|A|$ and $|\bar{A}'A|=|\bar{A}'||A|$ therefore if $\bar{A}'A=I$, we have $|\bar{A}'||A|=1$.

Thus the determinant of unitary matrix is of unit modulus. For a matrix to be unitary it must be non-singular.

Hence $\bar{A}'A=I\Rightarrow A\bar{A}'=I$

(10) **Periodic matrix :** A matrix A will be called a periodic matrix if $A^{k+1}=A$ where k is a positive integer. If, however k is the least positive integer for which $A^{k+1}=A$, then k is said to be the period of A.

(11) **Differentiation of a matrix :** If $A=\begin{bmatrix}f(x) & g(x)\\ h(x) & l(x)\end{bmatrix}$ then $\frac{dA}{dx}=\begin{bmatrix}f'(x) & g'(x)\\ h'(x) & l'(x)\end{bmatrix}$ is a differentiation of matrix A.

Example **:** If $A=\begin{bmatrix}x^2 & \sin x\\ 2x & 2\end{bmatrix}$ then $\frac{dA}{dx}=\begin{bmatrix}2x & \cos x\\ 2 & 0\end{bmatrix}$

(12) **Conjugate of a matrix :** The matrix obtained from any given matrix A containing complex number as its elements, on replacing its elements by the corresponding conjugate complex numbers is called conjugate of A and is denoted by $\bar{A}$.

Example: $A=\begin{bmatrix}1+2i & 2-3i & 3+4i\\ 4-5i & 5+6i & 6-7i\\ 8 & 7+8i & 7\end{bmatrix}$

then $\bar{A}=\begin{bmatrix}1-2i & 2+3i & 3-4i\\ 4+5i & 5-6i & 6+7i\\ 8 & 7-8i & 7\end{bmatrix}$

Properties of conjugates

(i) $\overline{(\bar{A})}=A$ (ii) $\overline{(A+B)}=\bar{A}+\bar{B}$

(iii) $\overline{(\alpha A)}=\bar{\alpha}\bar{A}$, α being any number

(iv) $\overline{(AB)}=\bar{A}\,\bar{B}$, A and B being conformable for multiplication

(13) **Transpose conjugate of a matrix :** The transpose of the conjugate of a matrix A is called transposed conjugate of A and is denoted by A^θ. The conjugate of the transpose of A is the same as the transpose of the conjugate of A *i.e.* $\overline{(A')}=(\bar{A})'=A^\theta$.

If $A=[a_{ij}]_{m\times n}$ then $A^\theta=[b_{ji}]_{n\times m}$ where $b_{ji}=\bar{a}_{ij}$

i.e., the $(j,i)^{th}$ element of $A^\theta=$ the conjugate of $(i,j)^{th}$ element of A.

Example: If $A=\begin{bmatrix}1+2i & 2-3i & 3+4i\\ 4-5i & 5+6i & 6-7i\\ 8 & 7+8i & 7\end{bmatrix}$,

then $A^\theta=\begin{bmatrix}1-2i & 4+5i & 8\\ 2+3i & 5-6i & 7-8i\\ 3-4i & 6+7i & 7\end{bmatrix}$

Properties of transpose conjugate

(i) $(A^\theta)^\theta=A$

(ii) $(A+B)^\theta=A^\theta+B^\theta$

(iii) $(kA)^\theta=\bar{K}A^\theta$, K being any number

(iv) $(AB)^\theta=B^\theta A^\theta$

Adjoint of a square matrix

Let $A=[a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A. Then the transpose of the matrix of cofactors of elements of A is called the adjoint of A and is denoted by *adj A*

Thus, $adj\,A=[C_{ij}]^T\Rightarrow(adj A)_{ij}=C_{ji}=$ cofactor of a_{ji} in A.

If $A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$,

then $adj A=\begin{bmatrix}C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix}^T=\begin{bmatrix}C_{11} & C_{21} & C_{31}\\ C_{12} & C_{22} & C_{32}\\ C_{13} & C_{23} & C_{33}\end{bmatrix}$;

where C_{ij} denotes the cofactor of a_{ij} in A.

Example **:** $A=\begin{bmatrix}p & q\\ r & s\end{bmatrix}$, $C_{11}=s, C_{12}=-r, C_{21}=-q, C_{22}=p$

$\therefore adj A=\begin{bmatrix}s & -r\\ -q & p\end{bmatrix}^T=\begin{bmatrix}s & -q\\ -r & p\end{bmatrix}$

Properties of adjoint matrix : If A, B are square matrices of order n and I_n is corresponding unit matrix, then

(i) $A(adj A)=|A|I_n=(adj A)A$

(Thus A (*adj A*) is always a scalar matrix)

(ii) $|adj A|=|A|^{n-1}$ (iii) $adj(adj A)=|A|^{n-2}A$

(iv) $|adj(adj A)|=|A|^{(n-1)^2}$ (v) $adj(A^T)=(adj A)^T$

(vi) $adj(AB)=(adj B)(adj A)$

(vii) $adj(A^m)=(adj A)^m, m\in N$

(viii) $adj(kA)=k^{n-1}(adj A), k\in R$

(ix) $adj(I_n) = I_n$ (x) $adj(O) = O$

(xi) A is symmetric $adj\ A$ is also symmetric.

(xii) A is diagonal $adj\ A$ is also diagonal.

(xiii) A is triangular $adj\ A$ is also triangular.

(xiv) A is singular $|adj\ A| = 0$

Inverse of a matrix

A non-singular square matrix of order n is invertible if there exists a square matrix B of the same order such that $AB = I_n = BA$.

In such a case, we say that the inverse of A is B and we write $A^{-1} = B$. The inverse of A is given by $A^{-1} = \frac{1}{|A|}.adjA$.

The necessary and sufficient condition for the existence of the inverse of a square matrix A is that $|A| \neq 0$.

Properties of inverse matrix:

If A and B are invertible matrices of the same order, then

(i) $(A^{-1})^{-1} = A$

(ii) $(A^T)^{-1} = (A^{-1})^T$

(iii) $(AB)^{-1} = B^{-1}A^{-1}$

(iv) $(A^k)^{-1} = (A^{-1})^k, k \in N$ [In particular $(A^2)^{-1} = (A^{-1})^2$]

(v) $adj(A^{-1}) = (adjA)^{-1}$

(vi) $|A^{-1}| = \frac{1}{|A|} = |A|^{-1}$

(vii) A = diag $(a_1 a_2 ... a_n) \Rightarrow A^{-1} = diag(a_1^{-1} a_2^{-1} ... a_n^{-1})$

(viii) A is symmetric A^{-1} is also symmetric.

(ix) A is diagonal, $|A| \neq 0 \Rightarrow A^{-1}$ is also diagonal.

(x) A is a scalar matrix A^{-1} is also a scalar matrix.

(xi) A is triangular, $|A| \neq 0$ A^{-1} is also triangular.

(xii) Every invertible matrix possesses a unique inverse.

(xiii) ***Cancellation law with respect to multiplication***

If A is a non-singular matrix *i.e.*, if $|A| \neq 0$, then A^{-1} exists and $AB = AC \Rightarrow A^{-1}(AB) = A^{-1}(AC)$

$$(A^{-1}A)B = (A^{-1}A)C$$

$$IB = IC \Rightarrow B = C$$

$$AB = AC \Rightarrow B = C \Leftrightarrow |A| \neq 0.$$

Rank of matrix

Definition : Let A be a $m \times n$ matrix. If we retain any r rows and r columns of A we shall have a square sub-matrix of order r. The determinant of the square sub-matrix of order r is called a minor of A order r. Consider any matrix A which is of the order of 3×4 say, $A = \begin{vmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 1 \end{vmatrix}$. It is 3×4 matrix so we can have minors of order 3, 2 or 1. Taking any three rows and three columns minor of order three. Hence minor of order $3 = \begin{vmatrix} 1 & 3 & 4 \\ 1 & 2 & 6 \\ 1 & 5 & 0 \end{vmatrix} = 0$

Making two zeros and expanding above minor is zero. Similarly we can consider any other minor of order 3 and it can be shown to be zero. Minor of order 2 is obtained by taking any two rows and any two columns.

Minor of order $2 = \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = 2 - 3 = -1 \neq 0$. Minor of order 1 is every element of the matrix.

Rank of a matrix: The rank of a given matrix A is said to be r if (1) Every minor of A of order $r+1$ is zero.

(2) There is at least one minor of A of order r which does not vanish. Here we can also say that the rank of a matrix A is said to be r, if (i) Every square submatrix of order $r+1$ is singular.

(ii) There is at least one square submatrix of order r which is non-singular.

The rank r of matrix A is written as $\rho(A) = r$.

Echelon form of a matrix

A matrix A is said to be in Echelon form if either A is the null matrix or A satisfies the following conditions:

(1) Every non- zero row in A precedes every zero row.

(2) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

If can be easily proved that the rank of a matrix in Echelon form is equal to the number of non-zero row of the matrix.

Rank of a matrix in Echelon form : The rank of a matrix in Echelon form is equal to the number of non-zero rows in that matrix.

Homogeneous and non-homogeneous systems of linear equations

A system of equations $AX = B$ is called a homogeneous system if $B = O$. If $B \neq O$, it is called a non-homogeneous system of equations.

e.g., $2x + 5y = 0$

$3x - 2y = 0$

is a homogeneous system of linear equations whereas the system of equations given by

e.g., $2x + 3y = 5$

$x + y = 2$

is a non-homogeneous system of linear equations.

(1) **Solution of Non-homogeneous system of linear equations**

(i) **Matrix method :** If $AX = B$, then $X = A^{-1}B$ gives a unique solution, provided A is non-singular.

But if A is a singular matrix *i.e.,* if $|A| = 0$, then the system of equation $AX = B$ may be consistent with infinitely many solutions or it may be inconsistent.

(ii) Rank method for solution of Non-Homogeneous system $AX = B$

(a) Write down A, B

(b) Write the augmented matrix $[A : B]$

(c) Reduce the augmented matrix to Echelon form by using elementary row operations.

(d) Find the number of non-zero rows in A and $[A : B]$ to find the ranks of A and $[A : B]$ respectively.

(e) If $\rho(A) \neq \rho(A : B)$, then the system is inconsistent.

(f) $\rho(A) = \rho(A : B) =$ the number of unknowns, then the system has a unique solution.

If $\rho(A) = \rho(A : B) <$ number of unknowns, then the system has an infinite number of solutions.

(2) **Solutions of a homogeneous system of linear equations :** Let $AX = O$ be a homogeneous system of 3 linear equations in 3 unknowns.

(a) Write the given system of equations in the form $AX = O$ and write A.

(b) Find $|A|$.

(c) If $|A| \neq 0$, then the system is consistent and $x = y = z = 0$ is the unique solution.

(d) If $|A| = 0$, then the systems of equations has infinitely many solutions. In order to find that put $z = K$ (any real number) and solve any two equations for x and y so obtained with $z = K$ give a solution of the given system of equations.

Consistency of a system of linear equation *AX* = *B*, where *A* is a square matrix

In system of linear equations $AX = B, A = (a_{ij})_{n\times n}$ is said to be

(i) Consistent (with unique solution) if $|A| \neq 0$.

i.e., if A is non-singular matrix.

(ii) Inconsistent (It has no solution) if $|A| = 0$ and $(adjA)B$ is a non-null matrix.

(iii) Consistent (with infinitely *m* any solutions) if $|A| = 0$ and $(adjA)B$ is a null matrix.

Cayley-Hamilton theorem

Every matrix satisfies its characteristic equation e.g. let A be a square matrix then $|A - xI| = 0$ is the characteristics equation of A. If $x^3 - 4x^2 - 5x - 7 = 0$ is the characteristic equation for A, then $A^3 - 4A^2 + 5A - 7I = 0$.

Roots of characteristic equation for A are called Eigen values of A or characteristic roots of A or latent roots of A.

If λ is characteristic root of A, then λ^{-1} is characteristic root of A^{-1}.

Geometrical transformations

(1) **Reflexion in the *x*-axis:** If $P'(x', y')$ is the reflexion of the point $P(x, y)$ on the x-axis, then the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ describes the reflexion of a point $P(x, y)$ in the x-axis.

(2) **Reflexion in the *y*-axis**

Here the matrix is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(3) **Reflexion through the origin**

Here the matrix is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(4) **Reflexion in the line *y* = *x***

Here the matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(5) **Reflexion in the line *y* = - *x***

Here the matrix is $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(6) **Reflexion in $y = x\tan\theta$**

Here matrix is $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

(7) **Rotation through an angle**

Here matrix is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Matrices of rotation of axes

We know that if x and y axis are rotated through an angle θ about the origin the new coordinates are given by

$$x = X\cos\theta - Y\sin\theta \text{ and } y = X\sin\theta + Y\cos\theta$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} \Rightarrow \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is the matrix of rotation through an angle θ.

Tips & Tricks

The sum of products of the element of any row with their corresponding cofactor is equal to the value of determinant

$$i.e\ \Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$$

where the capital letters C_{11}, C_{12}, C_{13} *etc.* denote the cofactors of a_{11}, a_{12}, a_{13} *etc.*

In general, it should be noted

$$a_{i1}C_{j1} + a_{i2}C_{j2} + a_{i3}C_{j3} = 0, \text{ if } i \neq j$$

or $a_{1i}C_{1j} + a_{2i}C_{2j} + a_{3i}C_{3j} = 0$, if $i \neq j$

If Δ' is the determinant formed by replacing the elements of a determinant Δ by their corresponding cofactors, then if $\Delta = 0$, then $\Delta^C = 0$, $\Delta' = \Delta^{n-1}$, where n is the order of the determinant.

The system of following homogeneous equations $a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$, $a_3x + b_3y + c_3z = 0$ is always consistent.

If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$, then this system has the unique solution $x = y = z = 0$ known as trivial solution. But if $\Delta = 0$, then this system has an infinite number of solutions. Hence for non-trivial solution $\Delta = 0$.

No element of principal diagonal in a diagonal matrix is zero.

Number of zeros in a diagonal matrix is given by $n^2 - n$ where n is the order of the matrix.

A diagonal matrix of order $n \times n$ having $d_1, d_2, \ldots\ldots, d_n$ as diagonal elements is denoted by $\text{diag}[d_1, d_2, \ldots, d_n]$.

Diagonal matrix is both upper and lower triangular

The multiplication of two diagonal matrices is also a diagonal matrix and

diag $(a_1, a_2, \ldots a_n) \times \text{diag}(b_1, b_2, \ldots b_n) = \text{diag}(a_1b_1, a_2b_2, \ldots a_nb_n)$.

The multiplication of two scalar matrices is also a scalar matrix.

If A and B are two matrices such that AB exists, then BA may or may not exist.

Minimum number of zeros in a triangular matrix is given by $\frac{n(n-1)}{2}$ where n is order of matrix

A triangular matrix $a = [a_{ij}]_{n \times n}$ is called strictly triangular if $a_{ij} = 0$ for $1 \leq i \leq n$.

The multiplication of two triangular matrices is a triangular matrix.

If A is involuntary matrix, then $\frac{1}{2}(I + A)$ and $\frac{1}{2}(I - A)$ are idempotent and $\frac{1}{2}(I + A).(I - A) = 0$.

Trace of a skew symmetric matrix is always 0.

Properties of determinant of a matrix

(i) $|A|$ exists $\Leftrightarrow$ A is square matrix

(ii) $|AB| = |A||B|$

(iii) $|A^T| = |A|$

(iv) $|kA| = k^n |A|$, if A is a square matrix of order n

(v) If A and B are square matrices of same order then

$$|AB| = |BA|$$

(vi) If A is a skew symmetric matrix of odd order then

$$|A| = 0$$

(vii) If $A = \text{diag}(a_1, a_2, \ldots. a_n)$ then $|A| = a_1a_2 \ldots a_n$

(viii) $|A|^n = |A^n|$, $n \in N$.

If a minor of A is zero the corresponding submatrix is singular and if a minor of A is not zero then corresponding submatrix is non-singular.

The adjoint of a square matrix of order 2 can be easily obtained by interchanging the diagonal elements and changing signs of off diagonal elements.

The rank of the null matrix is not defined and the rank of every non-null matrix is greater than or equal to 1.

The rank of a singular square matrix of order n cannot be n.

A Homogeneous system of equations is never inconsistent.