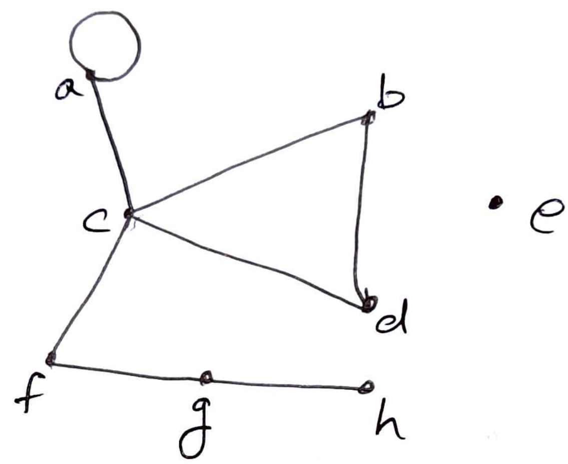
Vertex Degree

Let G be an undirected graph or multigraph. For each vertex v of G, the degree of v written $deg(v)$, is the number of edges in G that are incident with v. Here a loop at a vertex v is considered as two incident edges for v.



$deg(a) = 3$

$deg(b) = deg(d) = deg(f) = deg(g) = 2,$

$deg(c) = 4$

$deg(e) = 0$

$deg(h) = 1$, h is called a pendant vertex.

Theorem

If $G = (V, E)$ is an undirected graph or multigraph.

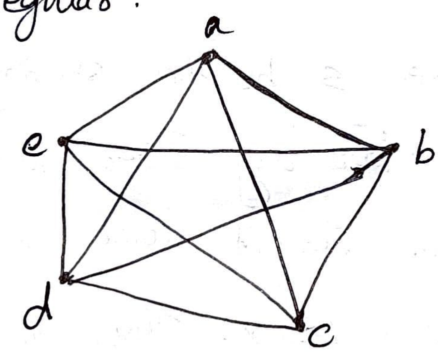
then $\sum_{u \in V} deg(v) = 2|E|$. (sum of the degree of vertices is equal to the twice the number of edges)

Solution

For each edge $\{a,b\}$ in a graph G, it contributes a count of 1 to each of $deg(a)$, $deg(b)$ and consequently a count of 2 to sum of the degree of vertices. Thus $\sum_{v \in V} deg(v) = 2|E|$.

Regular graph

An undirected graph where each vertex has the same degree is called a regular graph. If $deg(v) = k$ for all vertices v, then the graph is called k-regular.



Euler Circuit and Euler Trail

Let $G = (V, E)$ be an undirected graph or multigraph with no isolated vertices. Then G is said to have an Euler circuit if there is a circuit in G that traverses every edge of the graph exactly once. If there is an open trail from a to b in G and this trail traverses each edge in G exactly once, the trail is called an Euler trail.

Theorem

Let $G = (V, E)$ be an undirected graph or multigraph with no isolated vertices. Then G has an Euler circuit if and only if G is connected and every vertex in G has even degree.

Proof

If G has an Euler circuit, then for all $a, b \in V$ there is a trail from a to b. Therefore G is connected.

Let s be the starting vertex of the Euler circuit. For any other vertex v of G, each time the circuit comes to v it then departs from the vertex. Thus the circuit has traversed either two edges that are incident with v or a loop at v. In either case a count of 2 is contributed to $deg(v)$. Since v is not the starting point and each edge incident to v is traversed only once, a count of 2 is obtained each time the circuit passes through v, so $deg(v)$ is even. As for the starting vertex s, the first edge of the circuit must be distinct from the last edge, and because any other visits to s results in a count of 2 for deg(s), we have deg(s) even.

Conversely, let G be connected with every vertex of even degree. The result is true if the number of edges in G is 1 or 2. We proceed now by induction and assume that the result is true for fewer than n edges. If G has n edges, select a vertex s in G as a starting point to build an euler circuit. The graph G is connected and each vertex has even degree, so we can atleast construct a circuit C containing s. If the circuit contains every edge of G, then it it is an euler circuit. If not, remove the edges of the circuit from G, making sure to remove any vertex that would become isolated. The remaining ~~the~~ subgraph k has all vertices of even degree, but it may not be connected. However each component of k is connected and will have an euler circuit. In addition, each of these euler circuits has vertex that is on C. Consequently, starting at s we travel on C until we arrive at a vertex s_1 that is on the euler circuit of a component C_1 of K. Then we traverse this euler circuit and, returning to s_1, continue on C until we reach at a vertex s_2 that is on the euler circuit of component C_2 of K. Since G is finite, as we continue this process we construct an euler circuit for G.

In degree and Out degree

Let $G = (V, E)$ be a directed graph or multigraph. For each $v \in V$,

(a) The incoming, or in degree of v is the number of edges in G that are incident into v, and is denoted by $id(v)$

(b) The outgoing, or out degree of v is the number of edges in G that are incident from v, and is denoted by $od(v)$.

Also for each loop at a vertex v contributes a count of 1 to each $id(v)$ and $od(v)$.

Theorem

Let $G = (V, E)$ be a directed graph or multigraph with no isolated vertices. The graph G has a directed euler circuit if and only if G is connected and $id(v) = od(v)$ for all $v \in V$.

Fleury's Algorithm

Fleury's algorithm produces an euler circuit for a connected graph with no vertices of odd degree. Let G be a connected graph with each vertex of even degree.

Step 1: Select a vertex $a \in V$ as the starting vertex of the euler circuit.

Step 2: Suppose $a-b-c-\cdots-d$ is the sequence of vertices considered so far. Then

(a) At d if there is only one edge $\{d,e\}$, extend the sequence to $a-b-c-\cdots-d-e$ and then delete the edge $\{d,e\}$ from E and vertex e from V.

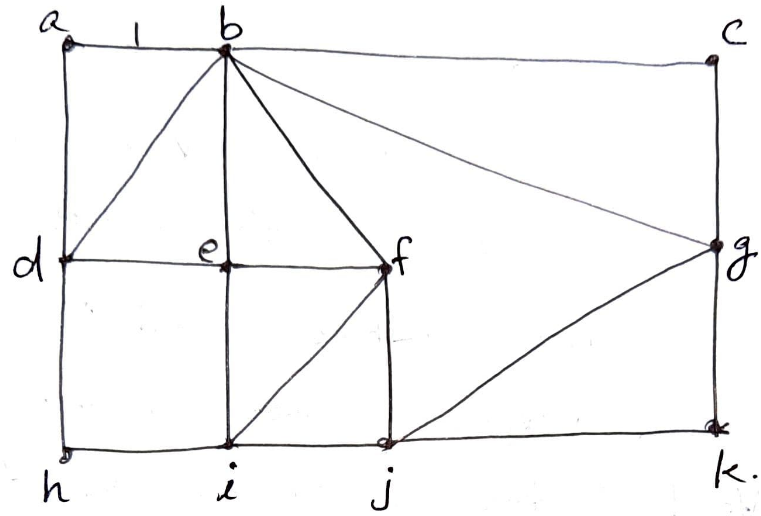
(b) On the other hand, suppose at d, there are several edges, choose an edge that is not a bridge to the remaining graph

[An edge is called a bridge in a connected graph if the deletion of this edge disconnects the graph].

Now extend the sequence to $a-b-c-\cdots-d-e$ and delete the edge $\{d,e\}$ from E.

Step 3: Repeat step 2 until no edges remains in E.

(i) Use Fleury's algorithm to find an euler circuit in the given graph. If the edge $\{d, e\}$ is removed from this graph, find an euler trail for the resulting subgraph.



Choose arbitrarily the vertex 'a' as the starting vertex.

Sl. No.	Current path	Next edge	Reason
1	a	$\{a, b\}$	No edge from a is ~~the~~ a bridge. Choose any one edge $\{a, b\}$ or $\{a, d\}$. Suppose choose the edge $\{a, b\}$. Mark 1 in the graph.
2	a-b	$\{b, c\}$	None of the five edges at b is a bridge. ~~&~~ Choose arbitrarily any edge, say $\{b, c\}$.
3	a-b-c	$\{c, g\}$.	As c there is only one edge $\{c, g\}$. Take $\{c, g\}$.

Sl.No.	Current path	Next edge.	Reason
4.	a-b-c-g	{g,k}	None of the three edges at g is a bridge. choose any one, say edge {g, k}
5.	a-b-c-g-k	{k, j}	Chose {k, j} since that is the only edge at vertex k.
6.	a-b-c-g-k-j	{j, g}	choose any one edge out of the three edges
7.	a-b-c-g-k-j-g	{g, b}.	At g only one edge {g, b}
8.	a-b-c-g-k-j-g-b	{b, f}	None of the three edges at b are bridges. choose any one.
9	a-b-c-g-k-j-g-b-f	{f, j}	Choose any one of the three edges at f.
10.	a-b-c-g-k-j-g-b-f-j	{j, i}	Choose the only remaining edge {j, i}
11.	a-b-c-g-k-j-g-b-f-j-i	{i, f}	choose any of the edge at i, take {i, f}
12.	a-b-c-g-k-j-g-b-f-j-i-~~b~~f	{f, e} ~~{h, d}~~	choose only one edge ~~{h, d}~~ {f, e}
13.	a-b-c-g-k-j-g-~~b-f-j-i~~ b-f-j-i-f-e.	{e, i}.	Choose any one of the three edges.

Sl. No.	Current path	Next edge	Reason
14	a-b-c-g-k-j-g-b-f-j-i-f-e-i	{i,h}	choose only one edge. {i,h} at i
15	a-b-c-g-k-j-g-b-f-j-i-f-e-i-h.	{h,d}	Choose only one edge {h,d} at h.
16.	a-b-c-g-k-j-g-b-f-j-i-f-e-i-h-d	{d,e}	choose out of the three edges at d.
17.	a-b-c-g-k-j-g-b-f-j-i-f-e-i-h-d-e	{e,b}	choose only one balance edge {e,b}
18	a-b-c-g-k-j-g-b-f-j-i-f-e-i-h-d-e-b	{b,d}	only one edge remains at b
19.	a-b-c-g-k-j-g-b-f-j-i-f-e-i-h-d-e-b-d.	{d,a}	Only one edge remains at d.

Thus an Euler circuit is a-b-c-g-k-j-g-b-f-j-i-f-e-i-h-d-e-b-d-a.

(b) If the edge {d,e} is removed, Euler trail is
d-a-b-d-h-i-e-f-i-j-f-b-c-g-k-j-g-b-e.