module - 4 **Linear Algebra**

Matrix representation of linear System

Consider the following system of m linear equation in n unknowns $x_1, x_2, \cdots, x_n$:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots \cdots \cdots$$
$$\cdots \cdots \cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

changing to matrix notation, the system can be written as $AX = B$ where the coefficient matrix $A = [a_{jk}]$ is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \text{ and } X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} ; \; B = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{bmatrix}$$

Here X and B are column vectors. We assume that the coefficients a_{jk} are not all zero, so that A is not a zero matrix. Note that X has n components where as B has m components.

The matrix $AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & : & b_2 \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & \vdots & b_m \end{bmatrix}$

Is called the <u>augmented matrix</u>. The augmented matrix AB determines the system completely because it consist all the numbers appearing in.

If all the b_j's are zero, then $AX=0$ is called <u>homogenous system</u>.

Row-echelon form and rank of a matrix

Consider the elementary row operations on matrix

1, Interchange of two rows.
2, Addition of a constant multiple of one row to another row.
3, Multiplication of a row by a non-zero constant

The correspond to the following:

1, Interchange of two equations
2, Addition of a constant multiple of one equation to another equations.
3, Multiplcation of a non-zero constant

Clearly these operations do not alter the solutions set.
A linear system S_1 is row-equivalent to linear system S_2 if S_1 can ~~of~~ be obtained from S_2 by finitely many elementary row operations.
No column operations on the augmented matrix are permitted because they would generally alter the solution set.
In the definition that follows, a non-zero row in a matrix means a row that contains atleast one non-zero entry, a <u>leading entry or pivot</u> of a row refers to the leftmost non-zero entry in a non-zero row.

<u>Echelon Matrix</u> : An $m \times n$ matrix is said to be an ~~ele~~ echelon matrix if
1. all non-zero rows are above any rows of all zeros.
2. the number of zeros proceeding this entry is more than the corresponding number in the previous row.
That is, all entries in a column below a leading entry are zeros.

eg:- $\begin{bmatrix} 2 & -3 & 1 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

Rank of a matrix

Rank of any matrix A is the number of non-zero rows in any echelon matrix equivalent of A.

Q. Reduce to echelon form and hence find the rank of $\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 24 & 54 \\ 0 & -21 & -14 & -15 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 7R_1 \end{matrix}$$

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + \frac{1}{2}R_2$$

Rank $= 2$

Q. Find the rank of $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

~~$R_2 \leftrightarrow R_1 - R_2$~~ $R_1 \leftrightarrow R_2$

~~$R_3 \to R_3 - 3$~~

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & -1 & 2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - 3R_1$;

$R_4 \to R_4 - 6R_1$

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 33 & 22 \end{bmatrix}$$

$R_3 \to 5R_3 - 4R_2$;

$R_4 \to 5R_4 - 9R_2$

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = 3.

Q. find the rank of the matrix : $\begin{bmatrix} -1 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} -1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 4 & 0 \end{bmatrix} \quad R_3 \to R_3 - 4R_2$$

$$\begin{bmatrix} -1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ rank of the matrix is 2.

Q. If the matrix $\begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ 6 & -2 & a & b \end{bmatrix}$ is of rank 2

find the value of a and b.

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ 6 & -2 & a & b \end{bmatrix} \quad R_2 \to R_2 - 2R_1$$

$1 - 2 \times -2$

$-1 - 2 \times 3$

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 5 & -7 & 0 \\ 6 & -2 & a & b \end{bmatrix} \quad R_3 \to R_3 - 6R_1$$

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 5 & -7 & 0 \\ 0 & 10 & a-18 & b-6 \end{bmatrix} \qquad R_3 \rightarrow R_3 - 2R_2$$

$-2 - 6 \times -2$

$= 2 + 12$

$a -$ $\quad 10 - 2 \times 5$

$b -$

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 5 & -7 & 0 \\ 0 & 0 & a-32 & b-6 \end{bmatrix}$$

$= a - 18 - (2 \times 7)$

$= a - 18 - 14$

$= a - 32$

$R_3 \rightarrow R_3$

$\begin{array}{r} 18 \\ 14 \\ \hline 32 \end{array}$

$b - 6 - a2$ $\qquad a - 32 +$

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 5 & -7 & 0 \\ 0 & 0 & a-4 & b-6 \end{bmatrix}$$

$a - 18 + 14$

$a - 4$

$b - 6$

here rank $= 2$

ie, $a - 4 = 0 \qquad b - 6 = 0$

$a = 4 \qquad b = 6$

Linear Independence of vectors

Let $v_1, v_2, \dots, v_n$ be n vectors, then the expression $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$ where $\alpha_1, \alpha_2, \dots, \alpha_n$ are scalars is called linear combination of these vectors.

Suppose $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$ happens only when $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$, then the vectors $v_1, v_2, \dots, v_n$ are linearly independent. If any $\alpha_i \neq 0$, then $v_1, v_2, \dots, v_n$ are linearly dependent.

Determining linear Independence using matrices

Let $v_1, v_2, \dots, v_n$ be the n vertices.

- Form a matrix with these vectors ~~~~ as row vectors.
- Find the rank of this matrix
- If this rank = n, then the vectors are linearly independent.
- If the rank is less than n, then vectors are linearly dependent.

Rank and Linear Independence

1. The rank of a matrix A is the maximum number of linearly independent row vectors of A

2. The rank of a matrix A is the maximum number of linearly independent column vectors of A.

Hence A and A^T have the same rank.

3. Consider p vectors each having n components. If $n < p$, then the vectors are linearly dependent.

Q. Check whether the vectors $[1, 2, 1]$, $[2, 1, 4]$, $[4, 5, 6]$ and $[1, 8, -3]$ are linearly independent in $\mathbb{R}^3$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 4 \\ 4 & 5 & 6 \\ 1 & 8 & -3 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & -3 & 2 \\ 0 & 6 & -4 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 - R_2 \\ R_4 \to R_4 + 2R_2 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here rank $= 2$ which is less than the

number of vectors ∴ hence linearly dependent.

Gauss Elimination Method And Back Substitution

Gauss elimination is a standard method for solving linear systems. It is an exact and systematic elimination process. This method provides an algorithm that ~~perf~~ performs elementary transformations to bring a system of linear equations to the row-echelon form. Since a linear system is completely determined by its augmented matrix, the elimination process can be done by merely considering the matrices.

1) Working principle to check the consistency of Linear system of equations

Consider the system of equations $AX = B$. Reduce the augmented matrix AB to echelon form by Gauss elimination method. Then the following case arise:

- If the rank $[AB] \neq$ rank $[A]$; then the System in inconsistent
- If the rank $[AB]$ = rank $[A]$ = no: of unknowns; then the system is consistent with a unique
- If the rank $[AB]$ = rank $[A]$ < no: of unknowns; then the system is consistent with infinite no: of Solution.

Q. Solve the following linear system by Gauss elimination method

$$x_1 - x_2 + x_3 = 0$$
$$-x_1 + x_2 - x_3 = 0$$
$$10x_2 + 25x_3 = 90$$
$$20x_1 + 10x_2 = 80$$

Soln:

The augmented matrix $AB = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{bmatrix}$

$R_2 \to R_2 + R_1$

$R_4 \to R_4 - 20R_1$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$3R_3 - R_2$

$R_3 \to R_3 + 10R_1$

$25+90$ $10+10$ R_3+10R_1

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 25 & 90 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

hence rank $[AB] = 3$.

$$A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 30 & -20 \\ 0 & 0 & 25 \\ 0 & 0 & 0 \end{bmatrix}$$

rank $[A] = 3$

rank $[A]$ = rank $[AB]$ = 3 = no. of unknowns.

$$x_1 - x_2 + x_3 = 0$$
$$30x_2 - 20x_3 = 80$$
$$25x_3 = 90$$

Q. Test the following system for consistency:

$$3x_1 + 2x_2 + x_3 = 3$$
$$2x_1 + x_2 + x_3 = 0$$
$$6x_1 + 2x_2 + 4x_3 = 6$$

Soln:

$$AB = \begin{bmatrix} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{bmatrix}$$

$$R_2 \to 3R_2 - 2R_1$$
$$R_3 \to R_3 - 2R_1$$

$$\sim \begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & -1 & 1 & -6 \\ 0 & -2 & 2 & 0 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & -1 & 1 & -6 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

Here rank $[AB] \neq$ rank $[A]$. Since

rank $[AB] = 3$ and rank $[A] = 2$. Hence the system is inconsistent.

Q. Solve the following system of equations by Gauss elimination method:

$$7x - 4y - 2z = -6$$
$$16x + 2y + z = 3$$

Soln:

$$AB = \begin{bmatrix} 7 & -4 & -2 & -6 \\ 16 & 2 & 1 & 3 \end{bmatrix}$$

$R_2 \to 7R_2 - 16R_1$

$$\sim \begin{bmatrix} 7 & -4 & -2 & -6 \\ 0 & 78 & 39 & 117 \end{bmatrix}$$

rank $[AB]$ = rank $[A]$ = 2 < the no. of unknowns

hence consistent with infinite number of solutions.

$$7x - 4y - 2z = -6$$

$$78y + 39\cancel{y} = 117$$

$\simeq$

$$7x - 4y - 2z = -6$$

$$2y + z = 3$$

let $y = a$. Then $z = 3 - 2a$ and $x = 0$

hence required solution is $x = 0, y = a$

$z = 3 - 2a$ (Since a is arbitrary, we have infinitely many solutions).

Q. $y + z - 2w = 0$

$2x - 3y - 3z + 6w = 2$

$4x + y + z - 2w = 4$

Soln:

$$AB = \left[\begin{array}{cccc|c} 0 & 1 & 1 & -2 & 0 \\ 2 & -3 & -3 & 6 & 2 \\ 4 & 1 & 1 & -2 & 4 \end{array}\right]$$

$\cancel{R_3}\ R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \\ 4 & 1 & 1 & -2 & 4 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_1$

$$\sim \begin{bmatrix} 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 7 & 7 & -14 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 7R_2$

$$\sim \begin{bmatrix} 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank$[AB]$ = rank$[A] = 2 <$ no:of unknowns

hence the system is consistent with infinitely many solution

$$2x - 3y - 8z + 6w = 2$$
$$y + z - 2w = 0$$

$w = a$, $z = b$. hence $y = 2a - b$; $x = 1$.

Q. Solve the following system of 3 equations in 4 unknowns where augmented matrix is

$$\begin{bmatrix} 3 & 2 & 2 & -5 & 8 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{bmatrix}$$

Soln:

Given matrix $\sim \begin{bmatrix} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & 11.1 & 4.4 & -1.1 \end{bmatrix}$

$R_2 \rightarrow R_2 - 0.2R_1$

$R_3 \rightarrow R_3 - 0.4R_1$

$$\sim \begin{bmatrix} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 0.2R_1$; $R_3 \rightarrow R_3 - 0.4R_1$

Back Substitution method

$$3x_1 + 2x_2 + 2x_3 - 5x_4 = 8$$
$$1.1x_2 + 1.1x_3 - 4.4x_4 = 1.1$$

let $x_4 = a$, $x_3 = b$. Then $x_2 = 1 - b + 4a$ and
$x_1 = 2 - a$

If $a = 0$, $b = 1$ then one soln is

$x_1 = 2$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$

Q. Find the values of a and b for which the system of equations $x + y + 2z = 2$, $2x - y + 3z = 10$, $5x - y + az = b$ has
i) no solution ii) unique solution
iii) infinite number of solutions.

Soln: $AB = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & -1 & 3 & 10 \\ 5 & -1 & a & b \end{bmatrix}$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - 5R_1$

$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & 6 \\ 0 & -6 & a-10 & b-10 \end{bmatrix}$

$R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & 6 \\ 0 & 0 & a-8 & b-22 \end{bmatrix}$$

i) if $a = 8$, $b \neq 22$, then rank$[AB] \neq$ rank$[A]$. hence no Solution.

ii) If $a \neq 8$ and b any value, then rank$[AB]$ = rank$[A]$ = no:of unknowns. hence the System has unique Solution.

iii) If $a = 8$, $b = 22$, then rank $[AB]$ = rank$[A]$ < no:of unknowns. Hence the System has infinite no:of Solutions.

Q. Find the values of μ for which the System of equations.
$x + y + z = 1$, $x + 2y + 3z = \mu$, $x + 5y + 9z = \mu^2$ will be consistent. For each value of μ so obtained, find the Solutions of the System.

Soln:

$$AB = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & \mu \\ 1 & 5 & 9 & \mu^2 \end{bmatrix}$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & \mu-1 \\ 0 & 4 & 8 & \mu^2-1 \end{bmatrix}$$

$R_3 \to R_3 - 4R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & \mu-1 \\ 0 & 0 & 0 & (\mu-1)(\mu-3) \end{bmatrix}$$

If $\mu = 1$ or 3 for the system to be consistent with infinite no:of solutions:

when $\mu = 1$, $AB \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solution of this system is $x = 1+a$; $y = -2a$, $z = a$

when $\mu = 3$, $AB \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solution of this system is $x = a-1$, $y = 2-2a$, $z = a$

Homogenous Linear System of Equations

A homogenous linear system of equations:

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n = 0$$

always has the trivial ~~system~~ solution $x_1 = 0, x_2 = 0, \ldots, x_n = 0$. Hence a homogenous linear system is always consistent.

Non-trivial solution exist if and only if $\text{rank}(A) < n$

Some Important results:

Theorem 1: A homogenous linear system with fewer equations than unknowns has non-trivial solutions

Theorem 2: A homogenous linear system $AX = 0$ where A is a square matrix will have a non-trivial solution iff $|A| = 0$ (singular)

Quadratic Form

Q. Find out what type of conic sections is a quadratic form $Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$ and transform it in principle axis form. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellipse

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ hyperbola

The matrix of quadratic form is

$\downarrow x_1^2$ coff $\quad -\frac{30}{2} = -15$

$$A = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix} \leftarrow x_2^2 \text{ coff.}$$

$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 0$ pair of straight line

take the eigen value by using the characteristic eqn.

$|A - \lambda I| = 0.$

$$\begin{vmatrix} 17-\lambda & -15 \\ 15 & 17-\lambda \end{vmatrix} = 0$$

$(17-\lambda)(17-\lambda) - 15^2 = 0$

$289 - 34\lambda + \lambda^2 - 225 = 0$

$\lambda^2 - 34\lambda - 64 = 0$

$(\lambda - 32)(\lambda - 2) = 0.$

$\lambda = 32$ & $\lambda = 2$

principle axis form is.

$$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2$$

$$Q = 2y_1^2 + 32y_2^2$$

Conic is

$$2y_1^2 + 32y_2^2 = 128$$

$$\frac{2y_1^2}{128} + \frac{32y_2^2}{128} = 1$$

$$\frac{y_1^2}{64} + \frac{y_2^2}{4} = 1$$ Is ellipse

Nature of the quadratic form

A quadratic form is

1) postive ~~depo~~ definite, if all the eign values are postive

2) negative definite, if all the eign values are negative

3) postive semidefinite, if the eignvalues are ~~o and~~ zero and postive.

4) ~~Neg Pos~~
Negative semidefinite, if the eign values are zero and negative.

5, Indefinite, if eign values are both portive and negative.

⇒ The Index of the quadratic form is no:of portive term is the quadratic form

⇒ Signature of the quadratic form is the difference b/w the no:of portive terms and number of negative terms.

Q. Find the nature, Index and Signature of the quadratic form $Q = 3x^2 + 5y^2 + 3z^2 - 2xy + 2xz - 2yz$

here 3 variable ∴ 3×3 matrix

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$$|A - I\lambda| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(5-\lambda)(3-\lambda)-1\right] + 1\left[(-3+\lambda)+1\right] + \left[1-(5-\lambda)\right] = 0$$

$$(3-\lambda)\left[15-5\lambda-3\lambda+\lambda^2-1\right] + \left[\lambda-2\right] + \left[\lambda-4\right] = 0$$

$$\cancel{45-15\lambda+9\lambda+3\lambda^3-3-15\lambda+5\lambda^2+3\lambda^2+\lambda+\lambda-2}$$
$$+\lambda-4=0$$

$$\cancel{45-6\lambda+}$$

$$\cancel{45-3-2-4-15\lambda+9\lambda-15\lambda+\lambda+\lambda+\lambda+3\lambda^2+5\lambda^2+3\lambda^2}$$
$$=0$$

41

$\cancel{36}$

39

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\lambda = 2, 3, 6.$$

~~Principle axis form~~ $\cancel{Q = \lambda_1 x_1^2}$

Is positive definite.

The Index is 3.

Signature is 3.

Diagonaligation

Q. Diagonalized matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

characterstic equation $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

$\lambda = 2, 3, 6$

Eign vector

$\underline{\lambda = 2}$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_2 + x_3 = 0$$

$$-x_1 + 3x_2 - x_3 = 0$$

$$x_1 - x_2 + x_3 = 0$$

$$\frac{x_1}{2} = \frac{-x_2}{0} = \frac{x_3}{-2} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\underline{\lambda = 3}$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x_1 - x_2 + x_3 = 0$$

$$-x_1 + 2x_2 - x_3 = 0$$

$$x_1 - x_2 + 0x_3 = 0$$

$$\frac{x_1}{-1} = \frac{-x_2}{-1} = \frac{x_3}{-1}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$\underline{\lambda = 6}$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x_1 - x_2 + x_3 = 0$$

$$-x_1 - x_2 - x_3 = 0.$$

$$x_1 - x_2 - 3x_3 = 0.$$

$$\frac{-x_1}{2} = \frac{-x_2}{4} = \frac{x_3}{2}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} \operatorname{adj} B$$

$$= \frac{1}{3+2+1} \begin{bmatrix} 3 & +2 & 1 \\ -0 & 2 & -2 \\ -3 & +2 & 1 \end{bmatrix}^T$$

$$= \frac{1}{6} \begin{bmatrix} 3 & 0 & -3 \\ 2 & 2 & 2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$D = B^{-1}AB$$

$$= \frac{1}{6} \begin{bmatrix} 3 & 0 & -3 \\ 2 & 2 & 2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 8 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$