Logarithms

1.1.1 Definition

"The Logarithm of a given number to a given base is the index of the power to which the base must be raised in order to equal the given number."

If $a > 0$ and $\neq 1$, then logarithm of a positive number N is defined as the index x of that power of 'a' which equals N *i.e.*, $\log_a N = x$ **iff** $a^x = N \Rightarrow a^{\log_a N} = N, a > 0, a \neq 1$ **and** $N > 0$

It is also known as fundamental logarithmic identity.

The function defined by $f(x) = \log_a x,\ a > 0, a \neq 1$ is called logarithmic function.

Its domain is $(0, \infty)$ and range is R. a is called the base of the logarithmic function.

When base is 'e' then the logarithmic function is called natural or Napierian logarithmic function and when base is 10, then it is called common logarithmic function.

Note : ❑ The logarithm of a number is unique *i.e.* No number can have two different log to a given base.

❑ $\log_e a = \log_e 10 . \log_{10} a$ or $\log_{10} a = \dfrac{\log_e a}{\log_e 10} = 0.434 \log_e a$

1.1.2 Characteristic and Mantissa

(1) The integral part of a logarithm is called the characteristic and the fractional part is called mantissa.

$$\log_{10} N = \underbrace{\text{integer}}_{\text{Characters tics}} + \underbrace{\text{fraction }(+ve)}_{\text{Mantissa}}$$

(2) The mantissa part of log of a number is always kept positive.

(3) If the characteristics of $\log_{10} N$ be n, then the number of digits in N is $(n+1)$

(4) If the characteristics of $\log_{10} N$ be $(-n)$ then there exists $(n-1)$ number of zeros after decimal part of N.

Example: 1 For $y = \log_a x$ to be defined 'a' must be

(a) Any positive real number (b) Any number

(c) $\geq e$ (d) Any positive real number $\neq 1$

Solution: (d) It is obvious (Definition).

Example: 2 Logarithm of $32\sqrt[5]{4}$ to the base $2\sqrt{2}$ is

(a) 3.6 (b) 5 (c) 5.6 (d) None of these

Solution: (a) Let x be the required logarithm , then by definition $(2\sqrt{2})^x = 32\sqrt[5]{4}$

$(2.2^{1/2})^x = 2^5.2^{2/5}$; $\therefore\ 2^{\frac{3x}{2}} = 2^{5+\frac{2}{5}}$

Here, by equating the indices, $\frac{3}{2}x = \frac{27}{5}$, $\therefore x = \frac{18}{5} = 3.6$

1.1.3 Properties of Logarithms

Let m and n be arbitrary positive numbers such that $a>0,\ a\neq 1,\ b>0,\ b\neq 1$ then

(1) $\log_a a = 1,\ \log_a 1 = 0$

(2) $\log_a b.\log_b a = 1 = \log_a a = \log_b b \Rightarrow \log_a b = \dfrac{1}{\log_b a}$

(3) $\log_c a = \log_b a.\ \log_c b$ or $\log_c a = \dfrac{\log_b a}{\log_b c}$

(4) $\log_a(mn) = \log_a m + \log_a n$

(5) $\log_a\left(\dfrac{m}{n}\right) = \log_a m - \log_a n$

(6) $\log_a m^n = n\log_a m$

(7) $a^{\log_a m} = m$

(8) $\log_a\left(\dfrac{1}{n}\right) = -\log_a n$

(9) $\log_{a^\beta} n = \dfrac{1}{\beta}\log_a n$

(10) $\log_{a^\beta} n^\alpha = \dfrac{\alpha}{\beta}\log_a n\ ,\ (\beta\neq 0)$

(11) $a^{\log_c b} = b^{\log_c a}\quad ,\quad (a,b,c>0$ and $c\neq 1)$

Example: 3 The number $\log_2 7$ is

(a) An integer (b) A rational number (c) An irrational number (d) A prime number

Solution: (c) Suppose, if possible, $\log_2 7$ is rational, say p/q where p and q are integers, prime to each other.

Then, $\dfrac{p}{q} = \log_2 7 \quad \Rightarrow 7 = 2^{p/q} \quad \Rightarrow 2^p = 7^q$,

Which is false since L.H.S is even and R.H.S is odd. Obviously $\log_2 7$ is not an integer and hence not a prime number

Example: 4 If $\log_7 2 = m$, then $\log_{49} 28$ is equal to

(a) $2(1+2m)$ (b) $\dfrac{1+2m}{2}$ (c) $\dfrac{2}{1+2m}$ (d) $1+m$

Solution: (b) $\log_{49} 28 = \dfrac{\log 28}{\log 49} = \dfrac{\log 7 + \log 4}{2\log 7} = \dfrac{\log 7}{2\log 7} + \dfrac{\log 4}{2\log 7} = \dfrac{1}{2} + \dfrac{1}{2}\log_7 4$

$= \dfrac{1}{2} + \dfrac{1}{2}.2\log_7 2 = \dfrac{1}{2} + \log_7 2 = \dfrac{1}{2} + m = \dfrac{1+2m}{2}$

Example: 5 If $\log_e\left(\dfrac{a+b}{2}\right) = \dfrac{1}{2}(\log_e a + \log_e b)$, then relation between a and b will be

(a) $a=b$ (b) $a=\dfrac{b}{2}$ (c) $2a=b$ (d) $a=\dfrac{b}{3}$

Solution: (a) $\log_e\left(\frac{a+b}{2}\right)=\frac{1}{2}(\log_e a+\log_e b)=\frac{1}{2}\log_e(ab)=\log_e\sqrt{ab}$

$\Rightarrow \frac{a+b}{2}=\sqrt{ab} \Rightarrow a+b=2\sqrt{ab} \Rightarrow \left(\sqrt{a}-\sqrt{b}\right)^2=0 \Rightarrow \sqrt{a}-\sqrt{b}=0 \Rightarrow \quad a=b$

Example: 6 If $\log_{10} 3=0.477$, the number of digits in 3^{40} is

(a) 18 (b) 19 (c) 20 (d) 21

Solution: (c) Let $y=3^{40}$ is

Taking log both the sides, $\log y=\log 3^{40} \Rightarrow \log y=40\log 3 \Rightarrow \log y=19.08$

$\therefore$ Number of digits in $y=19+1=20$

Example: 7 Which is the correct order for a given number α in increasing order

(a) $\log_2\alpha, \log_3\alpha, \log_e\alpha, \log_{10}\alpha$ (b) $\log_{10}\alpha, \log_3\alpha, \log_e\alpha, \log_2\alpha$

(c) $\log_{10}\alpha, \log_e\alpha, \log_2\alpha, \log_3\alpha$ (d) $\log_3\alpha, \log_e\alpha, \log_2\alpha, \log_{10}\alpha$

Solution: (b) Since 10, 3, e, 2 are in decreasing order

Obviously, $\log_{10}\alpha, \log_3\alpha, \log_e\alpha, \log_2\alpha$ are in increasing order.

1.1.4 Logarithmic Inequalities

(1) If $a>1, p>1 \Rightarrow \log_a p>0$

(2) If $0<a<1, p>1 \Rightarrow \log_a p<0$

(3) If $a>1,\ 0<p<1 \Rightarrow \log_a p<0$

(4) If $p>a>1 \Rightarrow \log_a p>1$

(5) If $a>p>1 \Rightarrow 0<\log_a p<1$

(6) If $0<a<p<1 \Rightarrow 0<\log_a p<1$

(7) If $0<p<a<1 \Rightarrow \log_a p>1$

(8) If $\log_m a>b \Rightarrow \begin{cases} a>m^b, & \text{if } m>1 \\ a<m^b, & \text{if } 0<m<1 \end{cases}$

(9) $\log_m a<b \Rightarrow \begin{cases} a<m^b, & \text{if } m>1 \\ a>m^b, & \text{if } 0<m<1 \end{cases}$

(10) $\log_p a>\log_p b \Rightarrow a\geq b$ if base p is positive and >1 or $a\leq b$ if base p is positive and < 1 *i.e.,* $0<p<1$

In other words, if base is greater than 1 then inequality remains same and if base is positive but less than 1 then the sign of inequality is reversed.

Example: 8 If $x=\log_3 5,\ y=\log_{17} 25$, which one of the following is correct

(a) $x<y$ (b) $x=y$ (c) $x>y$ (d) None of these

Solution: (c) $y=\log_{17} 25=2\log_{17} 5$

$\therefore \frac{1}{y}=\frac{1}{2}\log_5 17$

$\frac{1}{x} = \log_5 3 = \frac{1}{2}\log_5 9$

Clearly $\frac{1}{y} > \frac{1}{x}$, $\therefore x > y$

Example: 9 If $\log_{0.3}(x-1) < \log_{0.09}(x-1)$, then x lies in the interval

(a) $(2, \infty)$ (b) $(-2, -1)$ (c) $(1, 2)$ (d) None of these

Solution: (a) $\log_{0.3}(x-1) < \log_{(0.3)^2}(x-1) = \frac{1}{2}\log_{0.3}(x-1)$

$\therefore \quad \frac{1}{2}\log_{0.3}(x-1) < 0$

or $\log_{0.3}(x-1) < 0 = \log 1$ or $(x-1) > 1$ or $x > 2$

As base is less than 1, therefore the inequality is reversed, now $x>2 \Rightarrow x$ lies in $(2, \infty)$.

Indices and Surds

1.2.1 Definition of Indices

If a is any non zero real or imaginary number and m is the positive integer, then $a^m = a.a.a.a.........a$ (m times). Here a is called the base and m the index, power or exponent.

1.2.2 Laws of Indices

(1) $a^0 = 1$, $(a \neq 0)$

(2) $a^{-m} = \frac{1}{a^m}$, $(a \neq 0)$

(3) $a^{m+n} = a^m . a^n$, where m and n are rational numbers

(4) $a^{m-n} = \frac{a^m}{a^n}$, where m and n are rational numbers, $a \neq 0$

(5) $(a^m)^n = a^{mn}$

(6) $a^{p/q} = \sqrt[q]{a^p}$

(7) If $x = y$, then $a^x = a^y$, but the converse may not be true.

For example: $(1)^6 = (1)^8$, but $6 \neq 8$

(i) If $a \neq \pm 1$, or 0, then $x = y$

(ii) If $a = 1$, then x, y may be any real number

(iii) If $a=-1$, then x, y may be both even or both odd (iv) If $a=0$, then x, y may be any non-zero real number.

But if we have to solve the equations like $[f(x)]^{\phi(x)}=[f(x)]^{\Psi(x)}$ then we have to solve :

(a) $f(x)=1$ (b) $f(x)=-1$ (c) $f(x)=0$ (d) $\phi(x)=\Psi(x)$

Verification should be done in (b) and (c) cases

(8) $a^m.b^m=(ab)^m$ is not always true

In real domain, $\sqrt{a}\sqrt{b}=\sqrt{(ab)}$, only when $a\geq 0, b\geq 0$

In complex domain, $\sqrt{a}.\sqrt{b}=\sqrt{(ab)}$, if at least one of a and b is positive.

(9) If $a^x=b^x$ then consider the following cases :

(i) If $a\neq \pm b$, then $x=0$

(ii) If $a=b\neq 0$, then x may have any real value

(iii) If $a=-b$, then x is even.

If we have to solve the equation of the form $[f(x)]^{\phi(x)}=[g(x)]^{\phi(x)}$ *i.e.,* same index, different bases, then we have to solve

(a) $f(x)=g(x)$, (b) $f(x)=-g(x)$, (c) $\phi(x)=0$

Verification should be done in (b) and (c) cases.

Example: 1 For $x\neq 0, \left(\frac{x^l}{x^m}\right)^{(l^2+lm+m^2)}\left(\frac{x^m}{x^n}\right)^{(m^2+nm+n^2)}\left(\frac{x^n}{x^l}\right)^{(n^2+nl+l^2)}=$

(a) 1 (b) x (c) Does not exist (d) None of these

Solution: (a) $\left(\frac{x^l}{x^m}\right)^{l^2+lm+m^2}\left(\frac{x^m}{x^n}\right)^{m^2+nm+n^2}\left(\frac{x^n}{x^l}\right)^{n^2+nl+l^2}$

$=(x^{l-m})^{(l^2+lm+m^2)}(x^{m-n})^{m^2+nm+n^2}(x^{n-l})^{n^2+nl+l^2}=x^{l^3-m^3}.x^{m^3-n^3}.x^{n^3-l^3}=x^{l^3-m^3+m^3-n^3+n^3-l^3}=x^0=1$

Example: 2 If $2^x=4^y=8^z$ and $xyz=288$, then $\frac{1}{2x}+\frac{1}{4y}+\frac{1}{8z}=$

(a) 11/48 (b) 11/24 (c) 11/8 (d) 11/96

Solution: (d) $2^x=2^{2y}=2^{3z}$ *i.e.,* $x=2y=3z=k$ (say). Then $xyz=\frac{k^3}{6}=288$, So $k=12$

$\therefore x=12, y=6, z=4$. Therefore, $\frac{1}{2x}+\frac{1}{4y}+\frac{1}{8z}=\frac{11}{96}$

Example: 3 $\frac{2.3^{n+1}+7.3^{n-1}}{3^{n+2}-2(1/3)^{1-n}}=$

(a) 1 (b) 3 (c) -1 (d) 0

Solution: (a) $\dfrac{2.3^{n+1}+7.3^{n-1}}{3^{n+2}-2\left(\frac{1}{3}\right)^{1-n}}=\dfrac{2.3^{n-1}.3^2+7.3^{n-1}}{3^{n-1}.3^3-2.3^{n-1}}=\dfrac{3^{n-1}[18+7]}{3^{n-1}[27-2]}=1$

Example: 4 If $\left(\frac{2}{3}\right)^{x+2}=\left(\frac{3}{2}\right)^{2-2x}$, then $x=$

(a) 1 (b) 3 (c) 4 (d) 0

Solution: (c) $\left(\frac{2}{3}\right)^{x+2}=\left(\frac{3}{2}\right)^{2-2x}\Rightarrow\left(\frac{2}{3}\right)^{x+2}=\left(\frac{2}{3}\right)^{2x-2}$. Clearly $x+2=2x-2\Rightarrow x=4$

Example: 5 The equation $4^{(x^2+2)}-9.2^{(x^2+2)}+8=0$ has the solution

(a) $x=1$ (b) $x=-1$ (c) $x=\sqrt{2}$ (d) $x=-\sqrt{2}$

Solution: (a, b) $4^{(x^2+2)}-9.2^{(x^2+2)}+8=0\Rightarrow\left(2^{(x^2+2)}\right)^2-9.2^{(x^2+2)}+8=0$

Put $2^{(x^2+2)^2}=y$. Then $y^2-9y+8=0$, which gives $y=8, y=1$

When $y=8\Rightarrow 2^{x^2+2}=8\Rightarrow 2^{x^2+2}=2^3\Rightarrow x^2+2=3\Rightarrow x^2=1\Rightarrow x=1,-1$

When $y=1\Rightarrow 2^{x^2+2}=1\Rightarrow 2^{x^2+2}=2^o\Rightarrow x^2+2=0\Rightarrow x^2=-2$, which is not possible.

1.2.3 Definition of Surds

Any root of a number which can not be exactly found is called a surd.

Let a be a rational number and n is a positive integer. If the n^{th} root of x *i.e.*, $x^{1/n}$ is irrational, then it is called surd of order n.

Order of a surd is indicated by the number denoting the root.

For example $\sqrt{7},\sqrt[3]{9},(11)^{3/5},\sqrt[n]{3}$ are surds of second, third, fifth and n^{th} order respectively.

A second order surd is often called a quadratic surd, a surd of third order is called a cubic surd.

Note : ❑ If a is not rational, $\sqrt[n]{a}$ is not a surd.

For example, $\sqrt{(5+\sqrt{7})}$ is not a surd as $5+\sqrt{7}$ is not a rational number.

1.2.4 Types of Surds

(1) **Simple surd :** A surd consisting of a single term. For example $2\sqrt{3},6\sqrt{5},\sqrt{5}$ etc.

(2) **Pure and mixed surds :** A surd consisting of wholly of an irrational number is called pure surd.

Example : $\sqrt{5},\sqrt[3]{7}$

A surd consisting of the product of a rational number and an irrational number is called a mixed surd.

Example : $5\sqrt{3}$.

(3) **Compound surds :** An expression consisting of the sum or difference of two or more surds.

Example : $\sqrt{5}+\sqrt{2},\ 2-\sqrt{3}+3\sqrt{5}$ etc.

(4) **Similar surds :** If the surds are different multiples of the same surd, they are called similar surds.

Example : $\sqrt{45}, \sqrt{80}$ are similar surds because they are equal to $3\sqrt{5}$ and $4\sqrt{5}$ respectively.

(5) **Binomial surds :** A compound surd consisting of two surds is called a binomial surd.

Example : $\sqrt{5}-\sqrt{2}, 3+\sqrt[3]{2}$ etc.

(6) **Binomial quadratic surds:** Binomial surds consisting of pure (or simple) surds of order two *i.e.,* the surds of the form $a\sqrt{b} \pm c\sqrt{d}$ or $a \pm b\sqrt{c}$ are called binomial quadratic surds.

Two binomial quadratic surds which differ only in the sign which connects their terms are said to be conjugate or complementary to each other. The product of a binomial quadratic surd and its conjugate is always rational.

For example: The conjugate of the surd $2\sqrt{7}+5\sqrt{3}$ is the surd $2\sqrt{7}-5\sqrt{3}$.

1.2.5 Properties of Quadratic Surds

(1) The square root of a rational number cannot be expressed as the sum or difference of a rational number and a quadratic surd.

(2) If two quadratic surds cannot be reduced to others, which have not the same irrational part, their product is irrational.

(3) One quadratic surd cannot be equal to the sum or difference of two others, not having the same irrational part.

(4) If $a+\sqrt{b}=c+\sqrt{d}$, where a and c are rational, and $\sqrt{b}, \sqrt{d}$ are irrational, then $a=c$ and $b=d$.

Example: 6 The greatest number among $\sqrt[3]{9}, \sqrt[4]{11}, \sqrt[6]{17}$ is

(a) $\sqrt[3]{9}$ (b) $\sqrt[4]{11}$ (c) $\sqrt[6]{17}$ (d) None of these

Solution: (a) $\sqrt[3]{9}, \sqrt[4]{11}, \sqrt[6]{17}$

$\because$ *L.C.M* of 3, 4, 6 is 12

$\therefore \sqrt[3]{9}=9^{1/3}=(9^4)^{1/12}=(6561)^{1/12}$, $\sqrt[4]{11}=(11)^{1/4}(11^3)^{1/12}=(1331)^{1/12}$, $\sqrt[6]{17}=(17)^{1/6}=(17^2)^{1/2}=(289)^{1/12}$

Hence $\sqrt[3]{9}$ is the greatest number.

Example: 7 The value of $\dfrac{15}{\sqrt{10}+\sqrt{20}+\sqrt{40}-\sqrt{5}-\sqrt{80}}$ is

(a) $\sqrt{5}(5+\sqrt{2})$ (b) $\sqrt{5}(2+\sqrt{2})$ (c) $\sqrt{5}(1+\sqrt{2})$ (d) $\sqrt{5}(3+\sqrt{2})$

Solution: (c) Given fraction $=\dfrac{15}{\sqrt{10}+\sqrt{20}+\sqrt{40}-\sqrt{5}-\sqrt{80}}=\dfrac{15}{\sqrt{10}+2\sqrt{5}+2\sqrt{10}-\sqrt{5}-4\sqrt{5}}$

$$=\frac{15}{3\sqrt{10}-3\sqrt{5}}=\frac{5}{\sqrt{10}-\sqrt{5}}.\frac{\sqrt{10}+\sqrt{5}}{\sqrt{10}+\sqrt{5}}=\sqrt{10}+\sqrt{5}=\sqrt{5}(\sqrt{2}+1)$$

Example: 8 If $x=\sqrt[3]{(\sqrt{2}+1)}-\sqrt[3]{(\sqrt{2}-1)}$; then $x^3+3x=$

(a) 2 (b) 6 (c) $6x$ (d) None of these

Solution: (a) $x=(\sqrt{2}+1)^{1/3}-(\sqrt{2}-1)^{1/3}$

$x^3=(\sqrt{2}+1)-(\sqrt{2}-1)-3(\sqrt{2}+1)^{1/3}(\sqrt{2}-1)^{1/3}\left[\sqrt[3]{(\sqrt{2}+1)}-\sqrt[3]{\sqrt{2}-1}\right]$

$x^3=2-3(2-1)^{1/3}x \Rightarrow x^3+3x=2$.

1.2.6 Rationalisation Factors

If two surds be such that their product is rational, then each one of them is called rationalising factor of the other.

Thus each of $2\sqrt{3}$ and $\sqrt{3}$ is a rationalising factor of each other. Similarly $\sqrt{3}+\sqrt{2}$ and $\sqrt{3}-\sqrt{2}$ are rationalising factors of each other, as $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})=1$, which is rational.

To find the factor which will rationalize any given binomial surd :

Case I: Suppose the given surd is $\sqrt[p]{a}-\sqrt[q]{b}$

suppose $a^{1/P}=x, b^{1/q}=y$ and let n be the L.C.M. of p and q. Then x^n and y^n are both rational.

Now x^n-y^n is divisible by $x-y$ for all values of n, and

$x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+.....+y^{n-1})$.

Thus the rationalizing factor is $x^{n-1}+x^{n-2}y+x^{n-3}y^2+.....+y^{n-1}$ and the rational product is x^n-y^n.

Case II: Let the given surd be $\sqrt[p]{a}+\sqrt[q]{b}$.

Let x, y, n have the same meaning as in Case I.

(1) If n is even, then x^n-y^n is divisible by $x+y$ & $x^n-y^n=(x+y)(x^{n-1}-x^{n-2}y+x^{n-3}y^3-.....-y^{n-1})$

Thus the rationalizing factor is $x^{n-1}-x^{n-2}y+x^{n-3}y^2-.....-y^{n-1}$ and the rational product is x^n-y^n.

(2) If n is odd, x^n+y^n is divisible by $x+y$, and $x^n+y^n=(x+y)(x^{n-1}-x^{n-2}y+....+y^{n-1})$

Thus the rationalizing factor is $x^{n-1}-x^{n-2}y+.....-xy^{n-2}+y^{n-1}$ and the rational product is x^n+y^n.

Example: 9 The rationalising factor of $a^{1/3}+a^{-1/3}$ is

(a) $a^{1/3}-a^{-1/3}$ (b) $a^{2/3}+a^{-2/3}$ (c) $a^{2/3}-a^{-2/3}$ (d) $a^{2/3}+a^{-2/3}-1$

Solution: (d) Let $x=a^{1/3}, y=a^{-1/3}$ then $a=x^3, a^{-1}=y^3$

$x^3+y^3=(x+y)(x^2-xy+y^2)$

So rationlising factor is (x^2-xy+y^2). Put the value of x and y

Thus the required rationlising factor is $a^{2/3}+a^{-2/3}-1$.

1.2.7 Square Roots of $a+\sqrt{b}$ and $a+\sqrt{b}+\sqrt{c}+\sqrt{d}$ Where $\sqrt{b}$, $\sqrt{c}$, $\sqrt{d}$ are Surds

Let $\sqrt{(a+\sqrt{b})}=\sqrt{x}+\sqrt{y}$, where $x, y>0$ are rational numbers.

Then squaring both sides we have, $a+\sqrt{b}=x+y+2\sqrt{x}\sqrt{y}$

$\Rightarrow a=x+y,\ \sqrt{b}=2\sqrt{xy} \quad \Rightarrow b=4xy$

So, $(x-y)^2=(x+y)^2-4xy=a^2-b$

After solving we can find *x* and *y*.

Similarly square root of $a-\sqrt{b}$ can be found by taking $\sqrt{(a-\sqrt{b})}=\sqrt{x}-\sqrt{y},\ x>y$

To find square root of $a+\sqrt{b}+\sqrt{c}+\sqrt{d}$: Let $\sqrt{(a+\sqrt{b}+\sqrt{c}+\sqrt{d})}=\sqrt{x}+\sqrt{y}+\sqrt{z},(x,y,z>0)$ and take $\sqrt{(a+\sqrt{b}-\sqrt{c}-\sqrt{d})}=\sqrt{x}+\sqrt{y}-\sqrt{z}$. Then by squaring and equating, we get equations in *x, y, z*. On solving these equations, we can find the required square roots.

Note : ❑ If a^2-b is not a perfect square, the square root of $a+\sqrt{b}$ is complicated *i.e.*, we can't find the value of $\sqrt{(a+\sqrt{b})}$ in the form of a compound surd.

❑ If $\sqrt{(a+\sqrt{b})}=\sqrt{x}+\sqrt{y}, x>y$ then $\sqrt{(a-\sqrt{b})}=\sqrt{x}-\sqrt{y}$

❑ $\sqrt{a+\sqrt{(b)}}=\sqrt{\left(\frac{a+\sqrt{a^2-b}}{2}\right)}+\sqrt{\left(\frac{a-\sqrt{a^2-b}}{2}\right)}$

❑ $\sqrt{a-\sqrt{(b)}}=\sqrt{\left(\frac{a+\sqrt{a^2-b}}{2}\right)}-\sqrt{\left(\frac{a-\sqrt{a^2-b}}{2}\right)}$

❑ If *a* is a rational number, $\sqrt{b}, \sqrt{c}, \sqrt{d}$, are surds then

(i) $\sqrt{a+\sqrt{b}+\sqrt{c}+\sqrt{d}}=\sqrt{\frac{bd}{4c}}+\sqrt{\frac{bc}{4d}}+\sqrt{\frac{cd}{4b}}$ (ii) $\sqrt{a-\sqrt{b}-\sqrt{c}+\sqrt{d}}=\sqrt{\frac{bd}{4c}}+\sqrt{\frac{cd}{4b}}+\sqrt{\frac{bc}{4d}}$,

(iii) $\sqrt{a-\sqrt{b}-\sqrt{c}+\sqrt{d}}=\sqrt{\frac{bc}{4d}}-\sqrt{\frac{bd}{4c}}-\sqrt{\frac{cd}{4b}}$

Example: 10 $\sqrt{(3+\sqrt{5})}$ is equal to

(a) $\sqrt{5}+1$ (b) $\sqrt{3}+\sqrt{2}$ (c) $(\sqrt{5}+1)/\sqrt{2}$ (d) $\frac{1}{2}(\sqrt{5}+1)$

Solution: (c) Let $\sqrt{3+\sqrt{5}} = \sqrt{x}+\sqrt{y}$

$3+\sqrt{5} = x+y+2\sqrt{xy}$. Obviously $x+y=3$ and $4xy=5$. So $(x-y)^2 = 9-5=4$ or $(x-y)=2$

After solving $x=\frac{5}{2}, y=\frac{1}{2}$. Hence $\sqrt{3+\sqrt{5}} = \sqrt{\frac{5}{2}}+\sqrt{\frac{1}{2}} = \frac{\sqrt{5}+1}{\sqrt{2}}$

Example: 11 $\sqrt{[10-\sqrt{(24)}-\sqrt{(40)}+\sqrt{(60)}]} =$

(a) $\sqrt{5}+\sqrt{3}+\sqrt{2}$ (b) $\sqrt{5}+\sqrt{3}-\sqrt{2}$ (c) $\sqrt{5}-\sqrt{3}+\sqrt{2}$ (d) $\sqrt{2}+\sqrt{3}-\sqrt{5}$

Solution: (b) Let $10-\sqrt{24}-\sqrt{40}+\sqrt{60} = (\sqrt{a}-\sqrt{b}+\sqrt{c})^2$

$10-\sqrt{24}-\sqrt{40}+\sqrt{60} = a+b+c-2\sqrt{ab}-2\sqrt{bc}+2\sqrt{ca}$, $a,b,c>0$. Then $a+b+c=10$, $ab=6$, $bc=10$, $ca=15$

$a^2b^2c^2 = 900 \Rightarrow abc = 30\ (\neq \pm 30)$. So $a=3,\ \ b=2,\ c=5$

Therefore, $\sqrt{(10-\sqrt{24}-\sqrt{40}+\sqrt{60})} = \pm(\sqrt{3}+\sqrt{5}-\sqrt{2})$

Example: 12 $\sqrt[4]{(17+12\sqrt{2})} =$

(a) $\sqrt{2}+1$ (b) $2^{1/4}(\sqrt{2}+1)$ (c) $2\sqrt{2}+1$ (d) None of these

Solution: (a) $\sqrt{(17+12\sqrt{2})} = \sqrt{[3^2+(2\sqrt{2})^2+2.3.2\sqrt{2}]} = 3+2\sqrt{2}$

$\therefore \sqrt[4]{(17+12\sqrt{2})} = \sqrt{(3+2\sqrt{2})} = \sqrt{2}+1$.

1.2.8 Cube Root of a Binomial Quadratic Surd

If $(a+\sqrt{b})^{1/3} = x+\sqrt{y}$ then $(a-\sqrt{b})^{2/3} = x-\sqrt{y}$, where a is a rational number and b is a surd.

Procedure of finding $(a+\sqrt{b})^{1/3}$ is illustrated with the help of an example :

Taking $(37-30\sqrt{3})^{1/3} = x+\sqrt{y}$ we get on cubing both sides, $37-30\sqrt{3} = x^3+3xy-(3x^2+y)\sqrt{y}$

$\therefore x^3+3xy = 37$

$(3x^2+y)\sqrt{y} = 30\sqrt{3} = 15\sqrt{12}$

As $\sqrt{3}$ can not be reduced, let us assume $y=3$ we get $3x^2+y = 3x^2+3 = 30 \therefore x=3$

Which doesn't satisfy $x^3+3xy=37$

Again taking $y=12$, we get

$3x^2+12=15$, $\therefore x=1$

$x=1, y=12$ satisfy $x^3+3xy=37$

$\therefore \sqrt[3]{37-30\sqrt{3}} = 1-\sqrt{12} = 1-2\sqrt{3}$

Example: 13 $\sqrt[3]{(61-46\sqrt{5})} =$

(a) $1-2\sqrt{5}$ (b) $1-\sqrt{5}$ (c) $2-\sqrt{5}$ (d) None of these

Solution: (a) $\sqrt[3]{61-46\sqrt{5}} = a-\sqrt{b} \Rightarrow 61-46\sqrt{5} = (a-\sqrt{b})^3 = a^3+3ab-(3a^2+b)\sqrt{b}$

$\Rightarrow 61 = a^3+3ab, 46\sqrt{5} = (3a^2+b)\sqrt{b} \Rightarrow 61 = (a^2+3b)a$, $23\sqrt{20} = (3a^2+b)\sqrt{b}$

So $a=1$, $b=20$. Therefore $\sqrt[3]{61-46\sqrt{5}} = 1-\sqrt{20} = 1-2\sqrt{5}$.

1.2.9 Equations Involving Surds

While solving equations involving surds, usually we have to square, on squaring the domain of the equation extends and we may get some extraneous solutions, and so we must verify the solutions and neglect those which do not satisfy the equation.

Note that from $ax=bx$, to conclude $a=b$ is not correct. The correct procedure is $x(a-b)=0$ *i.e.* $x=0$ or $a=b$. Here, necessity of verification is required.

Example: 14 The equation $\sqrt{(x+1)}-\sqrt{(x-1)} = \sqrt{(4x-1)}$, $x \in R$ has

(a) One solution (b) Two solution (c) Four solution (d) No solution

Solution: (d) Given $\sqrt{(x+1)}-\sqrt{(x-1)} = \sqrt{(4x-1)}$ .....(i)

Squaring both sides, we get, $-2\sqrt{(x^2-1)} = 2x-1$

Squaring again, we get, $x=\frac{5}{4}$, which does not satisfy equation (i)

Hence, there is no solution of the given equation.

Partial Fraction

1.3.1 Definition

An expression of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial in x, is called a rational fraction.

(1) **Proper rational functions:** Functions of the form $\frac{f(x)}{g(x)}$,where $f(x)$ and $g(x)$ are polynomials and $g(x) \neq 0$,are called rational functions of x.

If degree of $f(x)$ is less than degree of $g(x)$,then $\frac{f(x)}{g(x)}$ is called a proper rational function.

Example: $\frac{x+2}{x^2+2x+4}$ is a proper rational function.

(2) **Improper rational functions :** If degree of $f(x)$ is greater than or equal to degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called an improper rational function.

For example: $\frac{x^3}{(x-1)(x-2)}$ is an improper rational function.

(3) **Partial fractions :** Any proper rational function can be broken up into a group of different rational fractions, each having a simple factor of the denominator of the original rational function. Each such fraction is called a partial fraction.

If by some process, we can break a given rational function $\frac{f(x)}{g(x)}$ into different fractions, whose denominators are the factors of $g(x)$,then the process of obtaining them is called the resolution or decomposition of $\frac{f(x)}{g(x)}$ into its partial fractions.

1.3.2 Different Cases of Partial Fractions

(1) **When the denominator consists of non-repeated linear factors:** To each linear factor $(x-a)$ occurring once in the denominator of a proper fraction, there corresponds a single partial fraction of the form $\frac{A}{x-a}$, where A is a constant to be determined.

If $g(x) = (x-a_1)(x-a_2)(x-a_3).......(x-a_n)$, then we assume that, $\frac{f(x)}{g(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + ... + \frac{A_n}{x-a_n}$

Where $A_1, A_2, A_3 .... A_n$ are constants, can be determined by equating the numerator of L.H.S. to the numerator of R.H.S. (after L.C.M.) and substituting $x = a_1, a_2 ...... a_n$.

Note : ❑ Remainder of polynomial $f(x)$, when divided by $(x-a)$ is $f(a)$.

e.g., Remainder of $x^2 + 3x - 7$,when divided by $x - 2$ is $(2)^2 + 3(2) - 7 = 3$.

❑ $\frac{px+q}{(x-a)(x-b)} = \frac{pa+q}{(x-a)(a-b)} + \frac{pb+q}{(b-a)(x-b)}$

Example: 1 The remainder obtained when the polynomial $x^{64} + x^{27} + 1$ is divided by $(x+1)$ is

(a) 1 (b) -1 (c) 2 (d) -2

Solution: (a) Remainder of $x^{64} + x^{27} + 1$, when divided by $x+1$ is $(-1)^{64} + (-1)^{27} + 1 = 1 - 1 + 1 = 1$.

Example: 2 If $\frac{2x+3}{(x+1)(x-3)} = \frac{a}{x+1} + \frac{b}{(x-3)}$, then $a+b$

(a) 1 (b) 2 (c) $\frac{9}{4}$ (d) $\frac{-1}{4}$

Solution: (b) $2x+3 = a(x-3)+b(x+1)$

Put $x=-1$; $2(-1)+3 = a(-1-3) \Rightarrow 1 = -4a \Rightarrow a = \frac{-1}{4}$

Now put $x=3$; $2(3)+3 = b(3+1) \Rightarrow 9 = 4b \Rightarrow b = \frac{9}{4}$

Therefore, $a+b = \frac{-1}{4}+\frac{9}{4} = 2$.

Example: 3 If $\frac{3x+a}{x^2-3x+2} = \frac{A}{(x-2)} - \frac{10}{x-1}$, then

(a) $a=7$ (b) $a=-7$ (c) $A=-13$ (d) $A=13$

Solution: (a, d) $\frac{3x+a}{x^2-3x+2} = \frac{A}{(x-2)} - \frac{10}{(x-1)}$

$\Rightarrow (3x+a) = A(x-1)-10(x-2) \Rightarrow 3 = A-10$, $a=-A+20$ (On equating coefficients of x and constant term)

$\Rightarrow A = 13, a = 7$.

(2) **When the denominator consists of linear factors, some repeated:** To each linear factor $(x-a)$ occurring r times in the denominator of a proper rational function, there corresponds a sum of r partial fractions.

Let $g(x) = (x-a)^k (x-a_1)(x-a_2).......(x-a_r)$. Then we assume that

$$\frac{f(x)}{g(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + ... + \frac{A_k}{(x-a)^k} + \frac{B_1}{(x-a_1)} + ... + \frac{B_r}{(x-a_r)}$$

Where $A_1, A_2, ..., A_k$ are constants. To determined the value of constants adopt the procedure as above.

Example: 4 If $\frac{3x+4}{(x+1)^2(x-1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$, then $A =$

(a) $\frac{-1}{2}$ (b) $\frac{15}{4}$ (c) $\frac{7}{4}$ (d) $\frac{-1}{4}$

Solution: (c) We have, $\frac{3x+4}{(x+1)^2(x-1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$

$\Rightarrow 3x+4 = A(x+1)^2 + B(x+1)(x-1) + C(x-1)$

Putting $x=1$, we get $7 = A(2)^2 \Rightarrow A = \frac{7}{4}$.

Example: 5 The partial fraction of $\frac{x^2}{(x-1)^3(x-2)}$ are

(a) $\frac{-1}{(x-1)^3} + \frac{3}{(x-1)^2} - \frac{4}{(x-1)} + \frac{4}{(x-2)}$ (b) $\frac{-1}{(x-1)^3} - \frac{3}{(x-1)^2} + \frac{4}{(x-1)} + \frac{4}{(x-2)}$

(c) $\frac{-1}{(x-1)^3} + \frac{-3}{(x-1)^2} + \frac{-4}{(x-1)} + \frac{4}{(x-2)}$ (d) None of these

Solution: (c) Put the repeated factor $(x-1)=y \Rightarrow x=y+1$

$$\therefore \frac{x^2}{(x-1)^3(x-2)}=\frac{(1+y)^2}{y^3(y-1)}=\frac{1+2y+y^2}{y^3(-1+y)}$$

Dividing the numerator, $1+2y+y^2$ by $-1+y$ till y^3 appears as factor, we get

$$\frac{1+2y+y^2}{-1+y}=(-1-3y-4y^2)+\frac{4y^3}{-1+y}$$

Given expression $=\frac{-1}{y^3}-\frac{3}{y^2}-\frac{4}{y}+\frac{4}{-1+y} = \frac{-1}{(x-1)^3}+\frac{-3}{(x-1)^2}+\frac{-4}{(x-1)}+\frac{4}{(x-2)}$.

(3) **When the denominator consists of non-repeated quadratic factors:** To each irreducible non repeated quadratic factor ax^2+bx+c, there corresponds a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined.

Example : $\frac{4x^2+2x+3}{(x^2+4x+9)(x-2)(x+3)}=\frac{Ax+B}{x^2+4x+9}+\frac{C}{x-2}+\frac{D}{x+3}$

Note :

- $\frac{px+q}{x^2(x-a)}=\frac{-q}{ax^2}-\frac{pa+q}{a^2x}+\frac{pa+q}{a^2(x-a)}$
- $\frac{px+q}{x(x-a)^2}=\frac{q}{a^2x}-\frac{q}{a^2(x-a)}+\frac{pa+q}{a(x-a)^2}$
- $\frac{px+q}{x(x^2+a^2)}=\frac{q}{a^2x}+\frac{pa^2-qx}{a^2(x^2+a^2)}$

Example: 6 The partial fractions of $\frac{3x-1}{(1-x+x^2)(2+x)}$ are

(a) $\frac{x}{(x^2-x+1)}+\frac{1}{x+2}$ (b) $\frac{1}{x^2-x+1}+\frac{x}{x+2}$ (c) $\frac{x}{x^2-x+1}-\frac{1}{x+2}$ (d) $\frac{-1}{x^2-x+1}+\frac{x}{x+2}$

Solution: (c) $\frac{3x-1}{(1-x+x^2)(2+x)}=\frac{Ax+B}{x^2-x+1}+\frac{C}{x+2}$

$\Rightarrow (3x-1)=(Ax+B)(x+2)+C(x^2-x+1)$

Comparing the coefficient of like terms, we get $A+C=0$, $2A+B-C=3$, $2B+C=-1$ $\Rightarrow A=1$, $B=0$, $C=-1$

$$\therefore \frac{3x-1}{(1-x+x^2)(2+x)}=\frac{x}{x^2-x+1}-\frac{1}{x+2}.$$

Example: 7 If $\frac{(x+1)^2}{x^3+x}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$, then $\sin^{-1}\left(\frac{A}{C}\right)=$

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

Solution: (a) $\frac{(x+1)^2}{x^3+x}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$

$\Rightarrow (x+1)^2 = A(x^2+1)+(Bx+C)x \Rightarrow A+B=1$, $C=2$, $A=1 \Rightarrow B=0$

Therefore $\sin^{-1}\left(\frac{A}{C}\right)=\sin^{-1}\left(\frac{1}{2}\right)=30^o=\frac{\pi}{6}$.

(4) **When the denominator consists of repeated quadratic factors:** To each irreducible quadratic factor ax^2+bx+c occurring r times in the denominator of a proper rational fraction there corresponds a sum of r partial fractions of the form.

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\text{.........}+\frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

Where, A's and B's are constants to be determined.

Example: 8 If $\frac{x}{(x-1)(x^2+1)^2}=\frac{1}{4}\left[\frac{1}{(x-1)}-\frac{x+1}{x^2+1}\right]+y$ then $y=$

(a) $\frac{(1-x)}{2(x^2+1)^2}$ (b) $\frac{(1-x)}{3(x^2+1)}$ (c) $\frac{1+x}{2(x^2-1)^2}$ (d) None of these

Solution: (a) $\frac{x}{(x-1)(x^2+1)^2}=\frac{1}{4}\left[\frac{1}{(x-1)}-\frac{x+1}{x^2+1}\right]+y$

$\Rightarrow \frac{x}{(x-1)(x^2+1)^2}=\frac{1}{4}\left[\frac{1}{(x-1)}-\frac{x+1}{x^2+1}\right]+\frac{Ax+B}{(x^2+1)^2}$

$\Rightarrow 4x=(x^2+1)^2-(x+1)(x-1)(x^2+1)+4(Ax+B)(x-1)$

$\Rightarrow 4A+2=0$, $4B-4A=4 \Rightarrow A=\frac{-1}{2}$, $B=\frac{1}{2}$

$\therefore y=\frac{Ax+B}{(x^2+1)^2}=\frac{1}{2}\frac{(1-x)}{(x^2+1)^2}$

1.3.3 Partial Fractions of Improper Rational Functions

If degree of $f(x)$ is greater than or equal to degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called an improper rational function and every rational function can be transformed to a proper rational function by dividing the numerator by the denominator.

We divide the numerator by denominator until a remainder is obtained which is of lower degree than the denominator.

i.e., $\dfrac{f(x)}{g(x)} = Q(x) + \dfrac{R(x)}{g(x)}$, where degree of $R(x)$ < degree of $g(x)$.

For example, $\dfrac{x^3}{x^2-5x+6}$ is an improper rational function and can be expressed as $(x+5)+\dfrac{19x-30}{x^2-5x+6}$ which is the sum of a polynomial $(x+5)$ and a proper rational function $\dfrac{19x-30}{x^2-5x+6}$.

Example: 9 If $\dfrac{x^3-6x^2+10x-2}{x^2-5x+6} = f(x) + \dfrac{A}{(x-2)} + \dfrac{B}{(x-3)}$, then $f(x) =$

(a) $x-1$ (b) $x+1$ (c) x (d) None of these

Solution: (a)

$$
\begin{array}{r|l}
x^2-5x+6 & x^3-6x^2+10x-2 \;\big(\; x-1 \\
 & \underline{\underset{-}{x^3}\underset{+}{-5x^2}\underset{-}{+6x}} \\
 & -x^2+4x-2 \\
 & \underline{\underset{+}{-x^2}\underset{-}{+5x}\underset{+}{-6}} \\
 & -x+4
\end{array}
$$

$\therefore\ f(x) = x-1$.

1.3.4. General Method of Finding out the Constants

(1) Express the given fraction into its partial fractions in accordance with the rules written above.

(2) Then multiply both sides by the denominator of the given fraction and you will get an identity which will hold for all values of x.

(3) Equate the coefficients of like powers of x in the resulting identity and solve the equations so obtained simultaneously to find the various constant is short method. Sometimes, we substitute particular values of the variable x in the identity obtained after clearing of fractions to find some or all the constants. For non-repeated linear factors, the values of x used as those for which the denominator of the corresponding partial fractions become zero.

Note: ❑ If the given fraction is improper, then before finding partial fractions, the given fraction must be expressed as sum of a polynomial and a proper fraction by division.

Important Tips

☞ *Some times a suitable substitution transforms the given function to a rational fraction which can be integrated by breaking it into partial fractions.*

Example: 10 The coefficient of x^n in the expression $\dfrac{5x+6}{(2+x)(1-x)}$ when expanded in ascending order is **[MNR 1993]**

(a) $\dfrac{-2}{3}\dfrac{(-1)^n}{2^n}+\dfrac{11}{3}$ (b) $\dfrac{2}{3}+\dfrac{(-1)^n}{2^n}-\dfrac{11}{3}$ (c) $-\dfrac{2}{3}+\dfrac{(-1)^n}{3}-\dfrac{11}{2^n}$ (d) None of these

Solution: (a) $\dfrac{5x+6}{(2+x)(1+x)} = \dfrac{\frac{-4}{3}}{2+x} + \dfrac{\frac{11}{3}}{1-x}$

Rewriting the denominators for expressions, we get

$$= \frac{\frac{-4}{3}}{2\left(1+\frac{x}{2}\right)} + \frac{\frac{11}{3}}{1-x} = \frac{-2}{3}\left(1+\frac{x}{2}\right)^{-1} + \frac{11}{3}(1-x)^{-1}$$

$$= \frac{-2}{3}\left[1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + ... + (-1)^n \frac{x^n}{2^n} + ...\right] + \frac{11}{3}[1 + x + x^2 + ... + x^n + ...]$$

The coefficient of x^n in the given expression is $\frac{-2}{3}(-1)^n \frac{1}{2^n} + \frac{11}{3}$.

Fundamentals

\

1.4.1. Square Root

$\sqrt{1} = \pm 1$ is not correct.

$\sqrt{1} = 1$ is correct.

In fact ' $\sqrt{\ }$ ' is the symbol for "positive square root" and ' $-\sqrt{\ }$ ' is symbol for negative square root.

* $\sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0 \end{cases}$ e.g. $\sqrt{(-1)^2} = |-1| = 1$

1.4.2. Modulus Fundamental

$y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

|x| denotes the magnitude of 'x' which is non negative.

i.e. $|x| \geq 0$ for any real 'x' and $\nless 0$.

One should remember t hat $|x|^2 = x^2$ for all real 'x'.

Similarly if we simplify $|2x - 3|$ it is as below-

$$|12x - 3| \begin{cases} 2x - 3 & \text{if } 2x - 3 \geq 0 \text{ i.e. } x \geq 3/2 \\ -(2x - 3) & \text{if } 2x - 3 < 0 \text{ i.e. } x < 3/2 \end{cases}$$

Ans. ± 2

1.4.3. Generalised Results

If $k > 0$ then

$|f(x)| = k \Rightarrow f(x) = \pm k$

$|f(x)| < k \Rightarrow -k < f(x) < k$

$|f(x)| > k \Rightarrow f(x) < -k$ or $f(x) > k$

If $k < 0$ then

$|f(x)| = k \Rightarrow$ no solution

$|f(x)| < k \Rightarrow$ no solution

$|f(x)| > k \Rightarrow$ all real values of x in the domain of f(x)

Note that writing-

$|f(x)| < k \Rightarrow f(x) < \pm k$ is wrong and

$|f(x)| > k \Rightarrow f(x) > \pm k$ is wrong

Example : $|2x+3| < 5$.

Solution : $-5 < (2x+3) < 5$

$-8 < 2x < 2$

$-4 < x < 1$

Example : $|3x-5| > 4$

Solution : $(3x-5) < -4$ or $(3x-5) > 4$

$x < \frac{1}{3}$ or $x > 3$

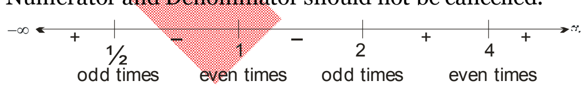
1.4.4. Sign scheme for rational/polynomial function

1. Put numerator and Denominator equal to zero separately. For polynomial function only Numerator = 0 (because Denominator $\neq 0$) Find the roots.
2. Plot number line. Mark these points (roots) on the number line in terms of their increasing order.
3. Thus the whole number line is divided into a number of subintervals.
4. Now if a root is repeated even times the sign of the function will remain the same in the two adjacent sub-intervals of the root.
5. If a root is repeated odd times, the sign of the function will be different in the two adjacent sub-intervals of the roots.

Example : Find the values of 'x' satisfying the inequality $f(x) = \frac{(2x-1)(x-1)^2(x-2)^2}{(x-2)(x-4)^4} > 0$.

Solution :

First, we have to discuss the sign scheme for f(x). It should be noted that the common factor (x – 2) in the Numerator and Denominator should not be cancelled.



Now putting Numerator $= (2x-1)(x-1)^2(x-2)^2 = 0$

$\Rightarrow x = 1/2, 1, 1, 2, 2$

Denominator $= (x-2)(x-4)^4 = 0$

$\Rightarrow x = 2, 4, 4, 4, 4$

Sign scheme for f(x) is as given below.

Illustration : We test the sign of f(x) at any real number (except roots) say x = 5. We should that f(x) comes out to be positive. So f(x)>0 for all 'x' in the sub-interval x>4. Now use the steps mentioned above. Since 4 is repeated even times. f(x) is positive in the sub-interval just left of 4 also. Root '2' is repeated odd times and hence the sign of f(x) will alternate. Clearly f(x) is negative in the sub-interval just left of 2 and so on.

Since we have to find values of 'x' for which f(x) > 0

$\therefore\ -\infty < x < 1/2$ or $2 < x < 4$ or $4 < x < \infty$ (from the sign scheme)

If we have to solve for $f(x) \le 0$

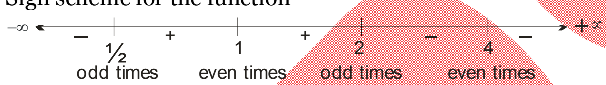
Then $1/2 \le x \le 1$ or $1 \le x < 2$ i.e. $1/2 \le x < 2$

Example : Find the values of 'x' for which $f(x) = (x-2)^2(1-x)(x-3)^3(x-4)^2 \le 0$.

Solution : Since D' is unity, put $N' = 0 \Rightarrow (x-2)^2(1-x)(x-3)^3(x-4)^2 = 0$ roots of the function are

$x = 2, 2, 1, 3, 3, 3, 4, 4.$

Sign scheme for the function-



We have to find those values of 'x' for which $f(x) \le 0$

From the sign scheme, we get $-\infty < x \le 1$ or $3 \le x \le 4$ or $4 \le x \le \infty$ or $x = 2$

Example : $\big|\,|2x+3| + 1\big| = 4$.

Solution : $|2x+3| + 1 = \pm 4 \Rightarrow |2x+3| = 3, -5$

Discarding –ve value

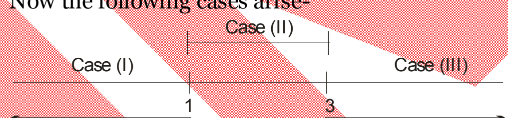
$\Rightarrow |2x+3| \Rightarrow 2x + 3 = \pm 3$

$\Rightarrow 2x = -6, 0 \Rightarrow x = -3, 0$

Example : $|x-1| + |x-3| = 2$.

Solution : First find the roots of each modulus terms which are here 1 and 3.

Now the following cases arise-



If x < 1

Then $x - 1 < 0$ and $x - 3 < 0$

$\therefore\ |x-1| = -(x-1)$ and $|x-3| = -(x-3)$

The equation is- $-(x-1) - (x-3) = 2 \Rightarrow -2x + 4 = 2 \Rightarrow x = 1$

x = 1 doesn't satisfy the condition (A) so it is not solution.

***