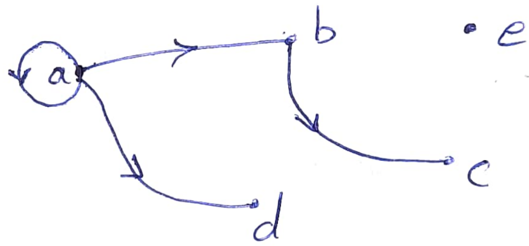
Module - 3

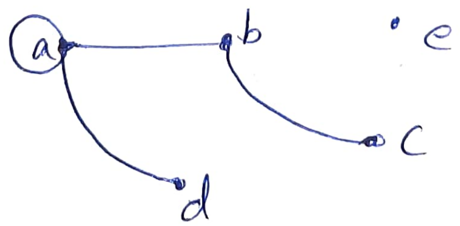
Graph Theory

Graph

Let V be a finite non empty set, and let $E \subseteq V \times V$. The pair (V, E) is then called a directed graph or digraph on V, where V is the set of vertices or nodes and E is its set of edges or arcs. We write $G = (V, E)$ to denote such a graph.



If E is a set of unordered pair of elements taken from V, then $G = (V, E)$ is called an undirected graph.



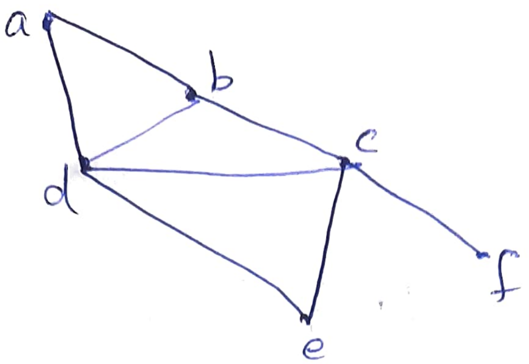
If $G = (V, E)$ is directed or undirected, V is the vertex set of G and E is the edge set of G.

The vertex set of the above graph is $V = \{a, b, c, d, e\}$ and edge set $E = \{(a,a), (a,b), (a,d), (b,c)\}$.

The edge (a,a) is called a loop and vertex e that has no incident edges is called an isolated vertex. For any edge, such as (b,c) the edge is incident with the vertices b and c, b is said to be adjacent to c. The vertex b is called the origin or source of the edge (b,c) and vertex c is the terminus or terminating vertex.

Walk

Let x, y be vertices in an undirected graph $G = (V, E)$. An x-y walk in G is a loop free finite alternating sequence $x = x_0, e_1, x_1, e_2, x_2, e_3, \ldots, e_{n-1}, x_{n-1}, e_n, x_n = y$ of vertices and edges from G, starting at vertex x and ending at vertex y and involving the n edges $e_i = \{x_{i-1}, x_i\}$ where $1 \le i \le n$. The length of this walk is n, the number of edges in the walk. Any x-y walk where $x = y$ is called a closed walk. Otherwise the walk is called open.



(i) $b \to c \to d \to e \to c \to f$ is a b-f walk has length 5 and the vertex c is repeated, but no edge appears more than one.

(2) $\{a,b\}, \{b,d\}, \{d,c\}, \{c,e\}, \{e,d\}, \{d,b\}$.

This is an a-b walk of length 6 in which the vertices d and b are repeated as well as the edge $\{b,d\}$.

Trial

If no edge in the walk is repeated, then the walk is called a trial. The walk $\{d,b\}$ $\{b,c\}$, $\{c,e\}$ is a trial.

Circuit

A Closed trial is called a circuit.

$\{d,b\}$, $\{b,c\}$, $\{c,e\}$ $\{e,d\}$ is a circuit.

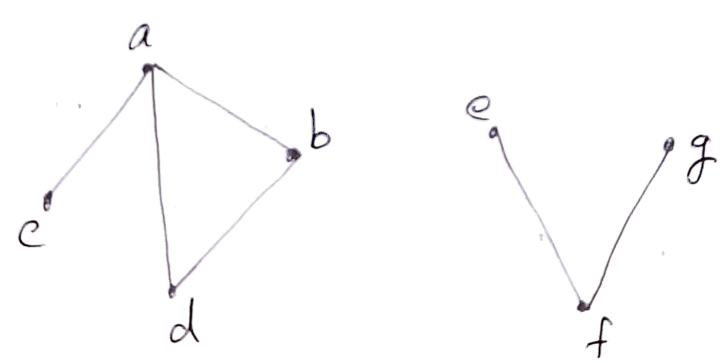
Path If no vertex in the walk occurs more than once, then the walk is called a path. $\{a,b\}, \{b,c\}, \{c,f\}$ is a path.

Cycle A closed path is called a cycle.

$\{a,b\}, \{b,c\}, \{c,d\}\ \{d,a\}$ is a cycle.

Connected graph

Let $G=(V,E)$ be an undirected graph. G is connected if there is a path between two distinct vertices of G. A graph that is not connected is called disconnected.



Consider the undirected graph on $V=\{a,b,c,d,e,f,g\}$. Thus graph is not connected for example, there is no path from a to e.

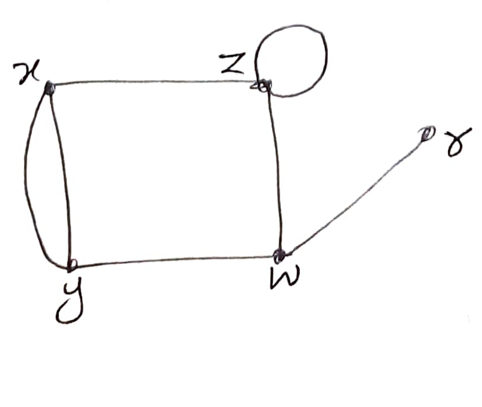
The graph is composed of pieces with vertex sets $V_1 = \{a,b,c,d\}$, $V_2 = \{e,f,g\}$

and edge sets $E_1 = \{ (a,b)\ (a,c)\ (a,d)\ (b,d) \}$

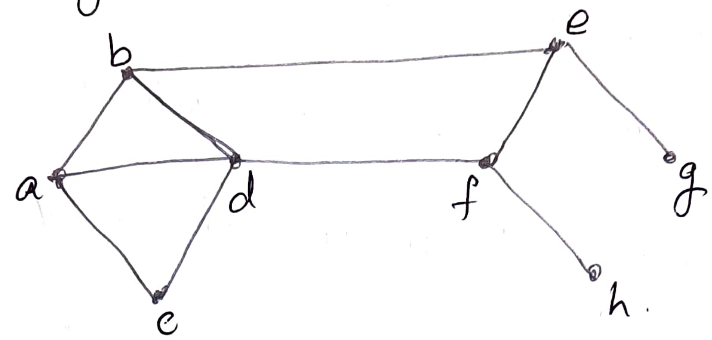
$E_2 = \{ (e,f)\ (f,g) \}$ that are themselves connected, and these pieces are called the components of the graph. Hence an undirected graph G is disconnected if and only if V can be partitioned into atleast two subsets V_1 & V_2. Such that there is no edge in E of the form $\{x, y\}$ where $x \in V_1$ and $y \in V_2$. A graph is connected if and only if it has only one component. For any graph, the number of components of G is denoted by $k(G)$.

Multigraph

Let $G = (V, E)$ be a graph with vertex set V and edge set E. G is called a multigraph if for some $x, y \in V$ there are two or more edges in E.



① Determine which of the following sequences in the graph are walk, closed walk, closed trial path and cycle.



ⓐ b, e, f, g

none.

ⓑ a, b, e, f, d, a, c, d, b.

Walk and trial

Vertices (a, b, d) are repeated while no edge is repeated.

(c) d, f, d.

walk, closed walk

ⓓ h.

walk, trial, path.

(e) a, b, e, f, d, c, a

walk, closed walk, trial, closed trial, cycle.

no vertex is repeated, no edge is repeated and sequence starts at a and closes at a so it is a closed walk., closed trial and cycle.

(f) a, c, d, f, e, b, d, a.

walk, Closed walk, trial, closed trial

It is not a cycle because vertices a, d are repeated.

(g) a, b, d, f, e, b, d, c.

walk

(2) From the following figure, determine

(a) a walk from b to d that is not a trial

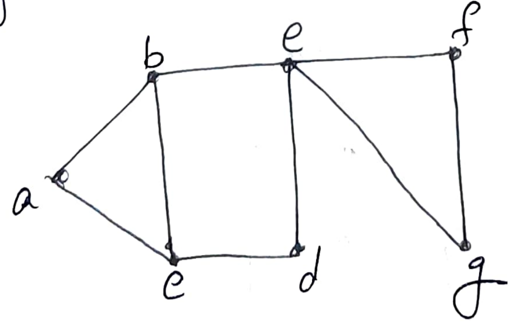
(b) a b-d trial that is not a path

(c) a path from b to d.

(d) a closed walk from b to b that is not a circuit.

(e) a circuit from b to b that is not a cycle.

(f) a cycle from b to b.



(a) (b, e) (e, f) (f, g) (g, e) (e, b) (b, c) (c, d) is a walk but not a trial because the edge (b, e) is repeated.

(b) (b, e) (e, f) (f, g) (g, e) (e, d) is a trial but not a path since vertex e is repeated.

(c) (b, c) (c, d) is a path since no vertex and no edge is repeated

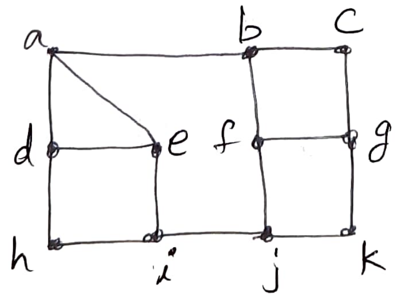
(d) (b, e) (e, f) (f, g) (g, e) (e, b) is a closed walk but is not a circuit because the edge (b, e) is repeated.

(e) (b, e) (e, f) (f, g) (g, e) (e, d) (d, c) (c, b) is a circuit but not a cycle because the vertex e is repeated.

(f) (b, e) (e, d) (d, c) (c, b) is a cycle where no vertex and no edge is repeated.

3) The length of the shortest path from a to b is the distance between two distinct vertices a, b in a connected undirected graph.

Find the distance from d to the other vertices.



a: 1,

h: 1

e: 1, i: 2, b: 2, c: 3, f: 3,

g: 4, j: 3, k: 4

4 ~~3~~. In an undirected graph $G=(V,E)$ with $|V|=v$ and $|E|=e$ and no loops, show that $2e \leq v^2-v$, what is the corresponding result when G is directed.

In a loop free undirected graph the maximum number of edges are. ${}^{v}C_2$.

Hence $|E| = e \leq {}^{v}C_2$

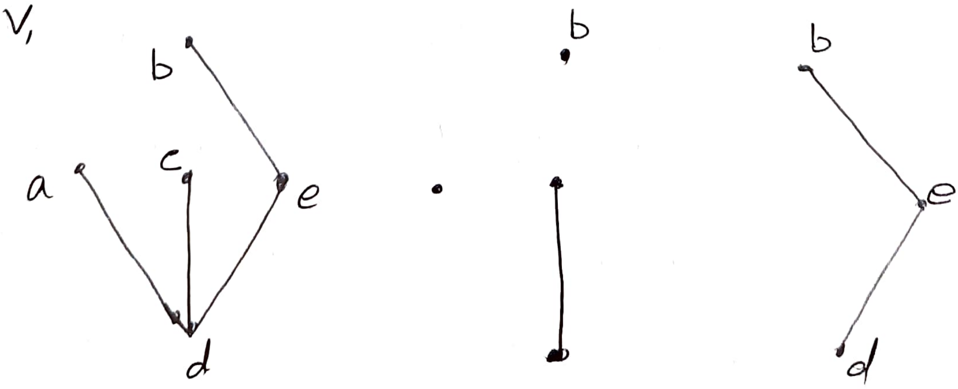
$$e \leq \frac{v(v-1)}{2}$$

$$2e \leq v^2 - v.$$

In a loop free directed graph, $e \leq v^2 - v$

Subgraph:

If $G=(V,E)$ is a graph (directed or undirected) then $G_1 = (V_1, E_1)$ is called a subgraph of G if $\phi \neq V_1 \subseteq V$ and $E_1 \subseteq E$, where each edge in E_1 is incident with vertices in V_1

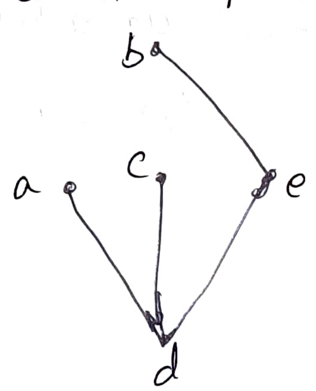


(G) (G_1) (G_2)

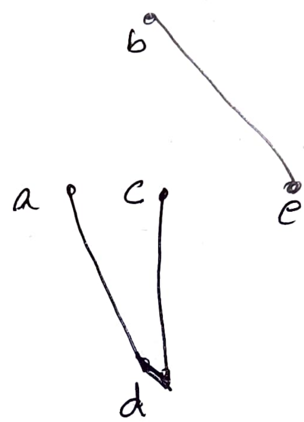
G_1 and G_2 are subgraphs of G.

Spanning subgraph

Let $G = (V, E)$ be a graph, $G_1 = (V_1, E_1)$ is called a spanning subgraph of G if $V_1 = V$ and $E_1 \subseteq E$.



(G)



(G_3)

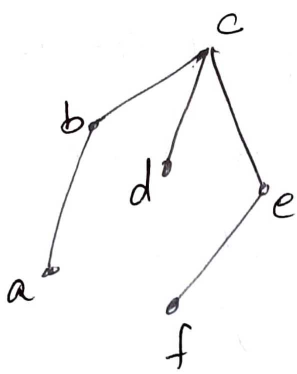
G_3 is a spanning subgraph of G.

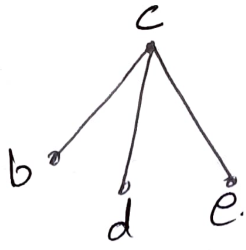
Induced subgraph

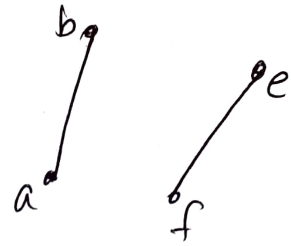
Let $G = (V, E)$ be a graph.

If $\phi \neq U \subseteq V$, the subgraph of G induced by U is the subgraph whose vertex set is U and which contains all edges of the form (x, y), for $x, y \in U$. We denote this graph as $\langle U \rangle$.

A subgraph G' of a graph is called an induced subgraph if there exists $\phi \neq U \subseteq V$ where $G' = \langle U \rangle$



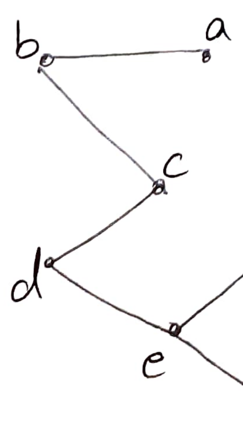




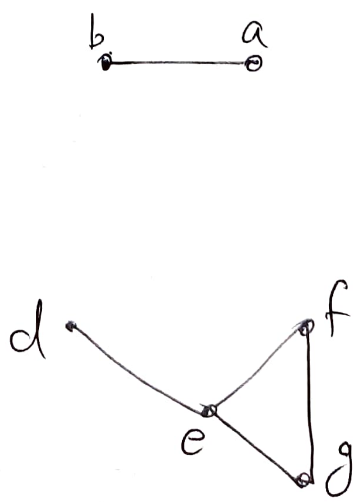
(G) (G_1) (G_2)

$G_1 = \langle U_1 \rangle$ for $U_1 = \{b, c, d, e\}$ and $G_2 = \langle U_2 \rangle$ for $U_2 = \{a, b, e, f\}$ are induced subgraphs of G.

Let v be a vertex in a graph $G = (V, E)$. The subgraph of G denoted by $G - v$ has the vertex set $V_1 = V - \{v\}$ and the edge set $E_1 \subseteq E$, where E_1 contains all the edges in E except for those that are incident with vertex v.



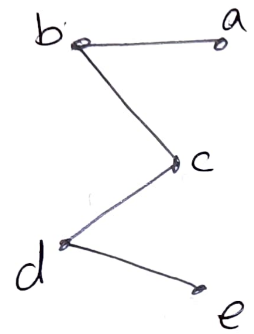
(G)



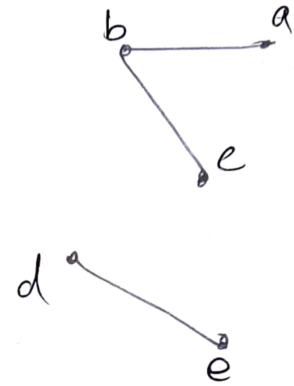
$G_1 = G - c$

If e is an edge of $G=(V,E)$

The subgraph $G-e = (V_1, E_1)$ of G, where the set of edges $E_1 = E - \{e\}$ and the vertex set is unchanged (ie $V_1 = V$).



(G)



$G-e$ for $e = \{c,d\}$.

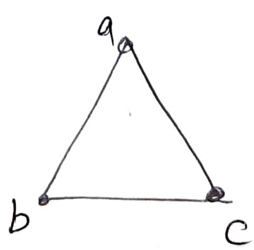
Complete graph.

Let V be a set of n vertices.

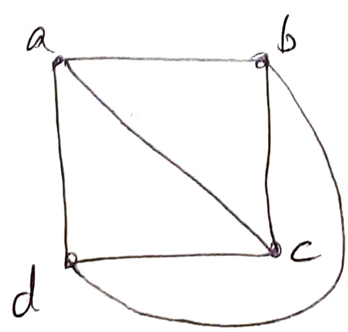
The complete graph on V, denoted by K_n, is a loop free undirected graph, where for all $a,b \in V$, $a \neq b$ there is an edge $\{a,b\}$.

K_1

K_2



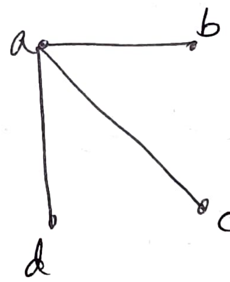
K_3



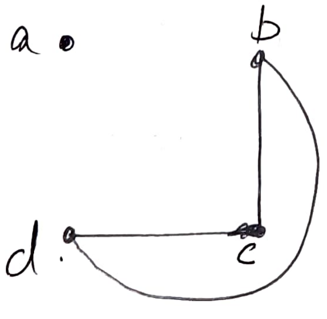
Complement of a graph.

Let G be a loop free undirected graph on n vertices. The complement of G denoted by $\bar{G}$, is the subgraph of K_n consisting of the n vertices in G and all edges that are not in G.

If $G = K_n$, $\bar{G}$ is a graph consisting of n vertices and no edges. Such a graph is called a null graph. In K_4, the complement of G



is



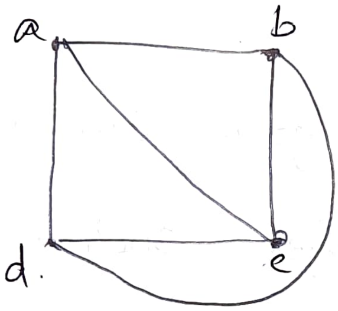
Graph isomorphism

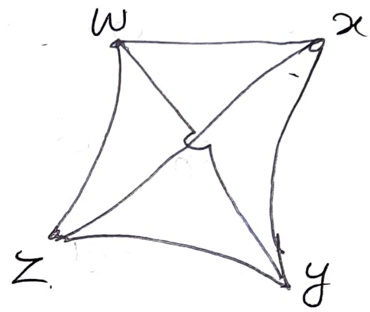
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two undirected graphs. A function $f: V_1 \to V_2$ is called graph isomorphism if.

(a) f is one to one and onto

(b) for all $a, b \in V_1$, $\{a, b\} \in E_1$ if and only if $\{f(a), f(b)\} \in E_2$.

When such a function exists G_1 and G_2 are called isomorphic graphs.

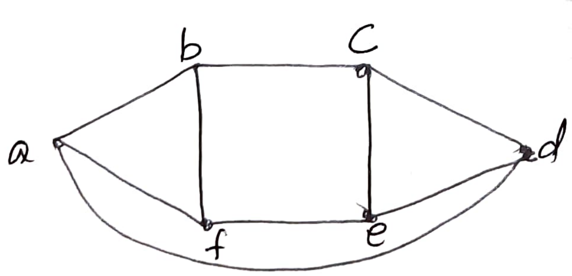


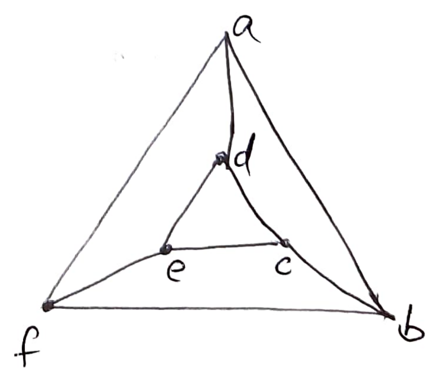


(G_1) (G_2)

G_1 and G_2 are isomorphic graphs with the function f defined by

$f(a) = w$, $f(b) = x$, $f(c) = y$, $f(d) = z$.





The graphs have the same number of 6 vertices and same number of 9 edges.

The two cycles in each graph are.

$\{a, b, f\}$ and $\{c, d, e\}$

Also $\{b, c, e, f\}$, $\{a, b, c, d\}$, $\{a, f, e, d\}$ are cycles of length 4. Thus the graphs are isomorphic.

Problems.

(1) Let G be an undirected graph with n vertices. If G is isomorphic to its own complement $\bar{G}$ (Self Complementary), how many edges must G have?

Solution

Let e_1 be the number of edges in G and e_2 be the number of edges in $\bar{G}$. The number of edges in $K_n = nC_2$.

$$e_1 + e_2 = nC_2.$$

Since G is self complementary $e_1 = e_2$.

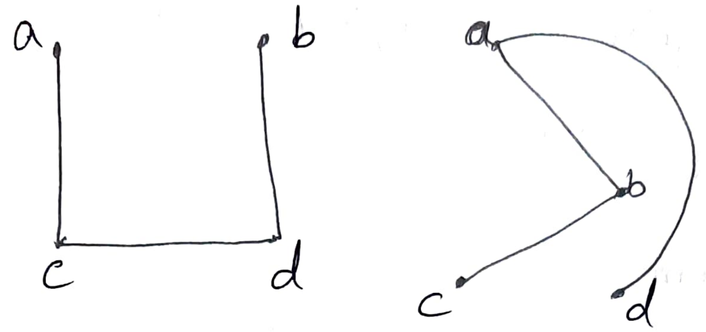
$$2e_1 = nC_2$$

$$e_1 = \frac{nC_2}{2}$$

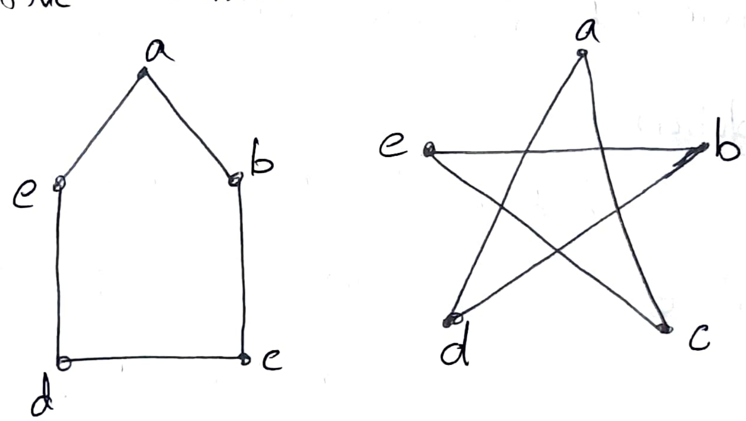
$$e_1 = \frac{1}{2} \frac{n(n-1)}{2}$$

$$e_1 = \frac{n(n-1)}{4}.$$

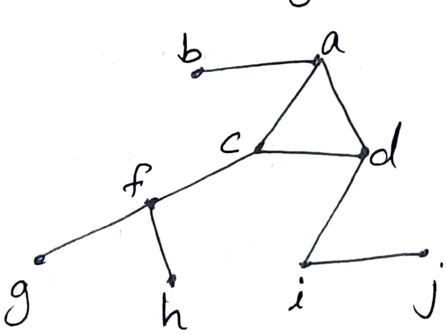
(2) Example of a self complementary graph with four vertices



(3) Example of a self complementary graph with five vertices



(4) Let G be the graph



(a) How many connected subgraphs of G have four vertices and include a cycle.

(b) Draw the subgraph of G induced by the set of vertices $U = \{b, c, d, f, i, j\}$

(c) Let $e = \{c, f\}$. Draw the subgraph

~~$G = V$~~ $G - e$

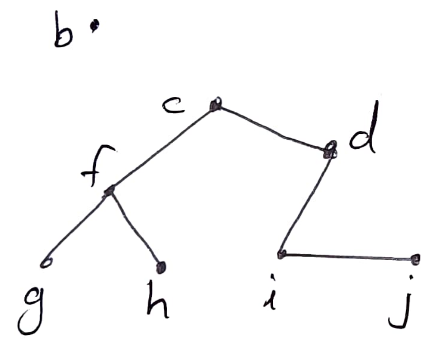
(d) How many spanning subgraphs are there for the graph G.

(e) How many connected spanning subgraphs are there for G.

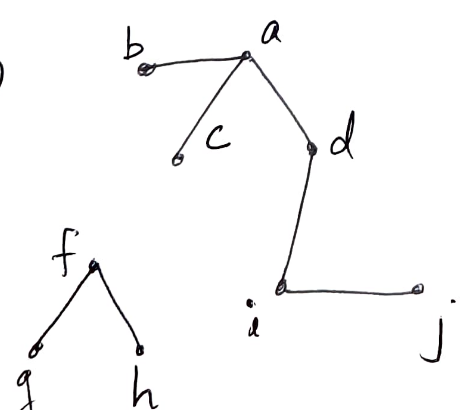
Solution

(a) 3

(b)



(c)



(d) $2^9 = 512$

(e) 3.