Limits

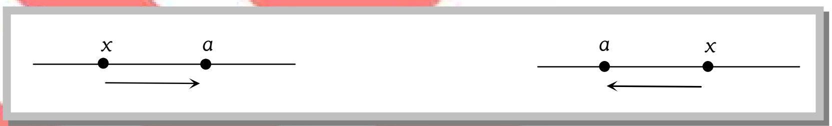
1.1 Limit of a Function

Let $y = f(x)$ be a function of x. If at $x = a, f(x)$ takes indeterminate form, then we consider the values of the function which are very near to 'a'. If these values tend to a definite unique number as x tends to 'a', then the unique number so obtained is called the limit of $f(x)$ at $x = a$ and we write it as $\lim_{x \to a} f(x)$.

(1) **Meaning of 'x ì a':** Let x be a variable and a be the constant. If x assumes values nearer and nearer to 'a' then we say 'x tends to a' and we write '$x \to a$'. It should be noted that as $x \to a$, we have $x \neq a$. By 'x tends to a' we mean that

(i) $x \neq a$ (ii) x assumes values nearer and nearer to 'a' and

(iii) We are not specifying any manner in which x should approach to 'a'. x may approach to a from left or right as shown in figure.



(2) **Left hand and right hand limit :** Consider the values of the functions at the points which are very near to a on the left of a. If these values tend to a definite unique number as x tends to a, then the unique number so obtained is called left-hand limit of $f(x)$ at $x = a$ and symbolically we write it as $f(a-0) = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$

Similarly we can define right-hand limit of $f(x)$ at $x = a$ which is expressed as $f(a+0) = \lim_{x \to a^+} f(x)$ $= \lim_{h \to 0} f(a+h)$.

(3) **Method for finding L.H.L. and R.H.L.**

(i) For finding right hand limit (R.H.L.) of the function, we write $x + h$ in place of x, while for left hand limit (L.H.L.) we write $x - h$ in place of x.

(ii) Then we replace x by 'a' in the function so obtained.

(iii) Lastly we find limit $h \to 0$.

(4) **Existence of limit :** $\lim_{x \to a} f(x)$ exists when,

(i) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist *i.e.* L.H.L. and R.H.L. both exists.

(ii) $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ *i.e.* L.H.L. = R.H.L.

Note : ❒ If a function $f(x)$ takes the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x = a$, then we say that $f(x)$ is indeterminate or meaningless at $x = a$. Other indeterminate forms are $\infty - \infty, \infty \times \infty, 0 \times \infty, 1^\infty, 0^0, \infty^0$

❒ In short, we write L.H.L. for left hand limit and R.H.L. for right hand limit.

- It is not necessary that if the value of a function at some point exists then its limit at that point must exist.

(5) **Sandwich theorem :** If $f(x)$, $g(x)$ and $h(x)$ are any three functions such that, $f(x) \le g(x) \le h(x)$ $\forall x \in$ neighborhood of $x = a$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = l$ (say), then $\lim_{x\to a} g(x) = l$. This theorem is normally applied when the $\lim_{x\to a} g(x)$ can't be obtained by using conventional methods as function $f(x)$ and $h(x)$ can be easily found.

Example: 1 If $f(x) = \begin{cases} x, & \text{when } x > 1 \\ x^2, & \text{when } x < 1 \end{cases}$, then $\lim_{x\to 1} f(x) =$

(a) x^2 (b) x (c) -1 (d) 1

Solution: (d) To find L.H.L. at $x = 1$. *i.e.,*

$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0}(1-h)^2 = \lim_{h\to 0}(1+h^2-2h) = 1$ i.e., $\lim_{x\to 1^-} f(x) = 1$ ....(i)

Now find R.H.L. at x = 1 i.e., $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = 1$ i.e., $\lim_{x\to 1^+} f(x) = 1$ .....(ii)

From (i) and (ii), L.H.L. = R.H.L. $\Rightarrow$ $\lim_{x\to 1} f(x) = 1$.

Example: 2 $\lim_{x\to 2} \frac{|x-2|}{x-2} =$

(a) 1 (b) –1 (c) Does not exist (d) None of these

Solution: (c) L.H.L. $= \lim_{x\to 2^-} \frac{|x-2|}{x-2} = \lim_{h\to 0} \frac{|2-h-2|}{2-h-2} = \lim_{h\to 0} \frac{h}{-h} = -1$ .....(i)

and, R.H.L. $= \lim_{x\to 2^+} \frac{|x-2|}{x-2} = \lim_{h\to 0} \frac{|2+h-2|}{2+h-2} = \lim_{h\to 0} \frac{h}{h} = 1$ .....(ii)

From (i) and (ii) L.H.L. $\ne$ R.H.L. *i.e.* $\lim_{x\to 2} \frac{|x-2|}{x-2}$ does not exist.

Example: 3 If $f(x) = \begin{cases} \frac{2}{5-x}, & \text{when } x < 3 \\ 5-x, & \text{when } x > 3 \end{cases}$, then

(a) $\lim_{x\to 3^+} f(x) = 0$ (b) $\lim_{x\to 3^-} f(x) = 0$ (c) $\lim_{x\to 3^+} f(x) \ne \lim_{x\to 3^-} f(x)$ (d) None of these

Solution: (c) $\lim_{x\to 3+} f(x) = 5-3 = 2$ and $\lim_{x\to 3-} f(x) = \frac{2}{5-3} = 1$

Example: 4 Let the function *f* be defined by the equation $f(x) = \begin{cases} 3x, & \text{if } 0 \le x \le 1 \\ 5-3x, & \text{if } 1 < x \le 2 \end{cases}$, then

(a) $\lim_{x\to 1} f(x) = f(1)$ (b) $\lim_{x\to 1} f(x) = 3$ (c) $\lim_{x\to 1} f(x) = 2$ (d) $\lim_{x\to 1} f(x)$ does not exist

Solution: (d) L.H.L. $= \lim_{x\to 1-0} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} 3(1-h) = \lim_{h\to 0}(3-3h) = 3-3.0 = 3$

R.H.L. $= \lim_{x\to 1+0} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0}[5-3(1+h)] = \lim_{h\to 0}(2-3h) = 2-3.0 = 2$

Hence $\lim_{x\to 1} f(x)$ does not exists.

Example: 5 $\lim_{x\to 0} \frac{|x|}{x} =$

(a) 1 (b) –1 (c) 0 (d) Does not exist

Solution: (d) $\because \lim_{x\to 0-} \frac{|x|}{x} = -1$ and $\lim_{x\to 0+} \frac{|x|}{x} = 1$, hence limit does not exists.

1.2 Fundamental Theorems on Limits

The following theorems are very useful for evaluation of limits if $\lim_{x\to 0} f(x) = l$ and $\lim_{x\to 0} g(x) = m$ (l and m are real numbers) then

(1) $\lim_{x\to a}(f(x)+g(x)) = l+m$ (Sum rule)

(2) $\lim_{x\to a}(f(x)-g(x)) = l-m$ (Difference rule)

(3) $\lim_{x\to a}(f(x).g(x)) = l.m$ (Product rule)

(4) $\lim_{x\to a} k\ f(x) = k.l$ (Constant multiple rule)

(5) $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{l}{m}, m \neq 0$ (Quotient rule)

(6) If $\lim_{x\to a} f(x) = +\infty$ or $-\infty$, then $\lim_{x\to a}\frac{1}{f(x)} = 0$

(7) $\lim_{x\to a}\log\{f(x)\} = \log\{\lim_{x\to a} f(x)\}$

(8) If $f(x) \le g(x)$ for all x, then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$

(9) $\lim_{x\to a}[f(x)]^{g(x)} = \{\lim_{x\to a} f(x)\}^{\lim_{x\to a} g(x)}$

(10) If p and q are integers, then $\lim_{x\to a}(f(x))^{p/q} = l^{p/q}$, provided $(l)^{p/q}$ is a real number.

(11) If $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(m)$ provided 'f' is continuous at $g(x) = m$. *e.g.* $\lim_{x\to a}\ln[f(x)] = \ln(l)$, only if $l > 0$.

1.3 Some Important Expansions

In finding limits, use of expansions of following functions are useful :

(1) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ...$

(2) $a^x = 1 + x\log a + \frac{(x\log a)^2}{2!} + ...$

(3) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

(4) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...,|x|<1$

(5) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - ....,$ where $|x|<1$

(6) $(1+x)^{\frac{1}{x}} = e^{\frac{1}{x}\log(1+x)} = e^{1-\frac{x}{2}+\frac{x^2}{3}.......} = e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 - ....\right)$

(7) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ....$

(8) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$

(9) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + ...$

(10) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$

(11) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...$

(12) $\tanh x = x - \frac{x^3}{3} + 2x^5 - .....$

(13) $\sin^{-1} x = x + 1^2.\frac{x^3}{3!} + 3^2.1^2.\frac{x^5}{5!} + ...$

(14) $\cos^{-1} x = \left(\frac{®}{2}\right) - \sin^{-1} x$

(15) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$

1.4 Methods of Evaluation of Limits

We shall divide the problems of evaluation of limits in five categories.

(1) **Algebraic limits :** Let $f(x)$ be an algebraic function and 'a' be a real number. Then $\lim_{x\to a} f(x)$ is known as an algebraic limit.

(i) **Direct substitution method :** If by direct substitution of the point in the given expression we get a finite number, then the number obtained is the limit of the given expression.

(ii) **Factorisation method :** In this method, numerator and denominator are factorised. The common factors are cancelled and the rest outputs the results.

(iii) **Rationalisation method :** Rationalisation is followed when we have fractional powers (like $\frac{1}{2}, \frac{1}{3}$ etc.) on expressions in numerator or denominator or in both. After rationalisation the terms are factorised which on cancellation gives the result.

(iv) **Based on the form when x ì ã :** In this case expression should be expressed as a function $1/x$ and then after removing indeterminate form, (if it is there) replace $\frac{1}{x}$ by 0.

Step I : Write down the expression in the form of rational function, *i.e.*, $\frac{f(x)}{g(x)}$, if it is not so.

Step II : If k is the highest power of x in numerator and denominator both, then divide each term of numerator and denominator by x^k.

Step III : Use the result $\lim_{x\to\infty} \frac{1}{x^n} = 0$, where $n > 0$.

Note : ❒ **An important result :** If m, n are positive integers and $a_0, b_0 \neq 0$ are non-zero real numbers, then

$$\lim_{x\to\infty} \frac{a_0x^m + a_1x^{m-1} + ... + a_{m-1}x + a_m}{b_0x^n + b_1x^{n-1} + ... + b_{n-1}x + b_n} = \begin{cases} \frac{a_0}{b_0}, \text{ if } m = n \\ 0, \text{ if } m < n \\ \infty, \text{ if } m > n \end{cases}$$

Example: 6 $\lim_{x\to 1} (3x^2 + 4x + 5) =$

(a) 12 (b) –1 (c) Does not exist (d) None of these

Solution: (a) $\lim_{x\to 1} (3x^2 + 4x + 5) = 3(1)^2 + 4(1) + 5 = 12$.

Example: 7 The value of $\lim_{x\to 2} \frac{3^{x/2} - 3}{3^x - 9}$ is

Solution: (c) $\lim_{x\to 2} \frac{3^{x/2} - 3}{(3^{x/2})^2 - (3)^2} = \lim_{x\to 2} \frac{(3^{x/2} - 3)}{(3^{x/2} - 3)(3^{x/2} + 3)} = \frac{1}{6}$.

Example: 8 The value of $\lim_{x\to a} \frac{x^n - a^n}{x - a}$ is

(a) 0 (b) na^{n-1} (c) na^n (d) 1

Solution: (b) $\lim_{x\to a} \frac{x^n - a^n}{x - a} = \lim_{x\to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + .. + a^{n-1})}{(x-a)} = \lim_{x\to a} (x^{n-1} + x^{n-2}a + ... + a^{n-1}) = n.a^{n-1}$.

Example: 9 $\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{x+h}-\frac{1}{x}\right]$ equals

(a) $\frac{1}{2x}$ (b) $-\frac{1}{2x}$ (c) $\frac{1}{x^2}$ (d) $-\frac{1}{x^2}$

Solution: (d) $\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{x+h}-\frac{1}{x}\right]=\lim_{h\to 0}\frac{1}{h}\left[\frac{x-(x+h)}{(x+h)x}\right]=\lim_{h\to 0}\frac{1}{h}\left[\frac{-h}{(x+h)x}\right]=-\frac{1}{x^2}$.

Example: 10 The value of $\lim_{x\to 0}\frac{\sqrt{1-x^2}-\sqrt{1+x^2}}{x^2}$ is

(a) 1 (b) –1 (c) – 2 (d) 0

Solution: (b) $\lim_{x\to 0}\frac{\left(\sqrt{1-x^2}-\sqrt{1+x^2}\right)}{x^2}\frac{\left(\sqrt{1-x^2}+\sqrt{1+x^2}\right)}{\left(\sqrt{1-x^2}+\sqrt{1+x^2}\right)}=\lim_{x\to 0}\frac{(1-x^2)-(1+x^2)}{x^2\left(\sqrt{1-x^2}+\sqrt{1+x^2}\right)}=\frac{-2}{2}=-1$.

Example: 11 $\lim_{x\to 3}\frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$ equals

(a) 1 (b) $\frac{3}{2}$ (c) $\frac{1}{4}$ (d) None of these

Solution: (d) $\lim_{x\to 3}\frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}=\lim_{x\to 3}\frac{(x-3)\left(\sqrt{x-2}+\sqrt{4-x}\right)}{\left(\sqrt{x-2}\right)^2-\left(\sqrt{4-x}\right)^2}$

$=\lim_{x\to 3}\frac{(x-3)\left(\sqrt{x-2}+\sqrt{4-x}\right)}{(2x-6)}=\lim_{x\to 3}\frac{\sqrt{x-2}+\sqrt{4-x}}{2}=\frac{1+1}{2}=1$.

Example: 12 $\lim_{x\to\infty}\frac{ax^2+bx+c}{dx^2+ex+f}=$

(a) $\frac{b}{e}$ (b) $\frac{c}{f}$ (c) $\frac{a}{d}$ (d) $\frac{d}{a}$

Solution: (c) Here the expression assumes the form $\frac{\infty}{\infty}$. We note that the highest power of x in both the numerator and denominator is 2. So we divide each terms in both the numerator and denominator by x^2.

$\lim_{x\to\infty}\frac{ax^2+bx+c}{dx^2+ex+f}=\lim_{x\to\infty}\frac{a+\frac{b}{x}+\frac{c}{x^2}}{d+\frac{e}{x}+\frac{f}{x^2}}=\frac{a+0+0}{d+0+0}=\frac{a}{d}$.

Example: 13 $\lim_{x\to\infty}\left[\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right]$ is equal to

(a) 0 (b) $\frac{1}{2}$ (c) log 2 (d) e^4

Solution: (b) $\lim_{x\to\infty}\left[\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right]=\lim_{x\to\infty}\frac{x+\sqrt{x+\sqrt{x}}-x}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}=\lim_{x\to\infty}\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}=\lim_{x\to\infty}\frac{\sqrt{1+x^{-1/2}}}{\sqrt{1+\sqrt{x^{-1}+x^{-3/2}}}+1}=\frac{1}{2}$.

Example: 14 The values of constants a and b so that $\lim_{x\to\infty}\left(\frac{x^2+1}{x+1}-ax-b\right)=0$ is

(a) $a=0, b=0$ (b) $a=1, b=-1$ (c) $a=-1, b=1$ (d) $a=2, b=-1$

Solution: (b) We have $\lim_{x\to\infty}\left(\frac{x^2+1}{x+1}-ax-b\right)=0 \Rightarrow \lim_{x\to\infty}\frac{x^2(1-a)-x(a+b)+1-b}{x+1}=0$

Since the limit of the given expression is zero, therefore degree of the polynomial in numerator must be less than that of denominator. As the denominator is a first degree polynomial. So, numerator must be a constant *i.e.*, a zero degree polynomial. $\therefore\ 1-a=0$ and $a+b=0 \Rightarrow a=1$ and b = –1. Hence, a = 1 and b = – 1.

Example: 15 $\lim_{x\to 1} x^x =$

(a) 1 (b) ∞ (c) Not defined (d) None of these

Solution: (a) $\lim_{x\to 1} x^x = \left(\lim_{x\to 1} x\right)^{\lim_{x\to 1} x} = 1^1 = 1$

Example: 16 $\lim_{x\to 1} (1+x)^{1/x} =$

(a) 2 (b) e (c) Not defined (d) None of these

Solution: (a) $\lim_{x\to 1} (1+x)^{1/x} = \left(\lim_{x\to 1} (1+x)\right)^{\lim_{x\to 1}\left(\frac{1}{x}\right)} = 2$

Example: 17 The value of the limit of $\frac{x^3-x^2-18}{x-3}$ as x tends to 3 is

(a) 3 (b) 9 (c) 18 (d) 21

Solution: (d) Let $y = \lim_{x\to 3}\frac{x^3-x^2-18}{x-3} = \lim_{x\to 3}(x^2+2x+6) = 9+6+6 = 21$

Example: 18 The value of the limit of $\frac{x^3-8}{(x^2-4)}$ as x tends to 2 is

(a) 3 (b) $\frac{3}{2}$ (c) 1 (d) 0

Solution: (a) $\lim_{x\to 2}\frac{x^3-8}{x^2-4} = \lim_{x\to 2}\frac{(x^2+2x+4)(x-2)}{(x+2)(x-2)} = \lim_{x\to 2}\frac{x^2+2x+4}{x+2} = \frac{4+4+4}{2+2} = 3$.

Example: 19 $\lim_{x\to 0}\frac{x}{\sqrt{1+x}-\sqrt{1-x}}$ is equal to

(a) $\frac{1}{2}$ (b) 2 (c) 1 (d) 0

Solution: (c) $\lim_{x\to 0}\left(\frac{x}{\sqrt{1+x}-\sqrt{1-x}}\right) = \lim_{x\to 0}\left(\frac{x}{\sqrt{1+x}-\sqrt{1-x}}\times\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

$$= \lim_{x\to 0}\left(\frac{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}{1+x-1+x}\right) = \lim_{x\to 0}\left(\frac{\left(\sqrt{1+x}+\sqrt{1-x}\right)}{2}\right) = \frac{2}{2} = 1$$

Example: 20 $\lim_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$ equals

(a) $\frac{2a}{3\sqrt{3}}$ (b) $\frac{2}{3\sqrt{3}}$ (c) 0 (d) None of these

Solution: (b) $\lim_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}} = \lim_{x\to a}\left(\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\right)\times\left(\frac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{a+2x}+\sqrt{3x}}\right)\times\left(\frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{3a+x}+2\sqrt{x}}\right)$

$$= \lim_{x\to a}\left\{\frac{\sqrt{3a+x}+2\sqrt{x}}{3(\sqrt{a+2x}+\sqrt{3x})}\right\} = \frac{2}{3\sqrt{3}}.$$

Example: 21 $\lim_{n\to\infty}\frac{1^{99}+2^{99}+3^{99}+....+n^{99}}{n^{100}} =$

(a) $\frac{99}{100}$ (b) $\frac{1}{100}$ (c) $\frac{1}{99}$ (d) $\frac{1}{101}$

Solution: (b) $\lim\limits_{n\to\infty}\dfrac{1^{99}+2^{99}+3^{99}+....+n^{99}}{n^{100}} = \lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\left(\dfrac{r^{99}}{n^{100}}\right) = \lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{r=1}^{n}\left(\dfrac{r}{n}\right)^{99} = \int_0^1 x^{99}\,dx = \left[\dfrac{x^{100}}{100}\right]_0^1 = \dfrac{1}{100}$.

Example: 22 The values of constants 'a' and 'b' so that $\lim\limits_{x\to\infty}\left(\dfrac{x^2-1}{x+1}-ax-b\right)=2$ is

(a) $a=0, b=0$ (b) $a=1, b=-1$ (c) $a=1, b=-3$ (d) $a=2, b=-1$

Solution: (c) $\lim\limits_{x\to\infty}\left(\dfrac{x^2-1}{x+1}-ax-b\right)=2 \Rightarrow \lim\limits_{x\to\infty} x-1-ax-b=2 \Rightarrow \lim\limits_{x\to\infty} x(1-a)-(1+b)=2$.

Comparing the coefficient of both sides, $1-a=0$ and $1+b=-2 \Rightarrow a=1, b=-3$

Example: 23 $\lim\limits_{n\to\infty}\left[\dfrac{\Sigma n^2}{n^3}\right]=$

(a) $-\dfrac{1}{6}$ (b) $\dfrac{1}{6}$ (c) $\dfrac{1}{3}$ (d) $\dfrac{-1}{3}$

Solution: (c) $\lim\limits_{n\to\infty}\left[\dfrac{n(n+1)(2n+1)}{6n^3}\right] = \lim\limits_{n\to\infty}\dfrac{\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right)}{6} = \dfrac{1}{3}$

Note : ❒ Students should remember that,

$\lim\limits_{n\to\infty}\dfrac{\Sigma n}{n^2}=\dfrac{1}{2}$ and $\lim\limits_{n\to\infty}\dfrac{\Sigma n^2}{n^3}=\dfrac{1}{3}$.

Example: 24 $\lim\limits_{n\to\infty}\left[\dfrac{1}{1-n^2}+\dfrac{2}{1-n^2}+......+\dfrac{n}{1-n^2}\right]$ is equal to

Solution: (b) $\lim\limits_{n\to\infty}\left[\dfrac{1}{1-n^2}+\dfrac{2}{1-n^2}+......+\dfrac{n}{1-n^2}\right] = \lim\limits_{n\to\infty}\dfrac{\Sigma n}{1-n^2} = \dfrac{1}{2}\lim\limits_{n\to\infty}\dfrac{n^2+n}{1-n^2} = -\dfrac{1}{2}$.

Example: 25 If $f(x)=\dfrac{2}{x-3}, g(x)=\dfrac{x-3}{x+4}$ and $h(x)=-\dfrac{2(2x+1)}{x^2+x-12}$ then $\lim\limits_{x\to 3}[f(x)+g(x)+h(x)]$ is

(a) -2 (b) -1 (c) $-\dfrac{2}{7}$ (d) 0

Solution: (c) We have $f(x)+g(x)+h(x)=\dfrac{x^2-4x+17-4x-2}{x^2+x-12} = \dfrac{x^2-8x+15}{x^2+x-12} = \dfrac{(x-3)(x-5)}{(x-3)(x+4)}$

$\therefore \lim\limits_{x\to 3}[f(x)+g(x)+h(x)] = \lim\limits_{x\to 3}\dfrac{(x-3)(x-5)}{(x-3)(x+4)} = -\dfrac{2}{7}$.

Example: 26 If $\lim\limits_{n\to\infty}\left[\dfrac{n!}{n^n}\right]^{1/n}$ equal

(a) e (b) $\dfrac{1}{e}$ (c) $\dfrac{®}{4}$ (d) $\dfrac{4}{®}$

Solution: (b) Let $P=\lim\limits_{n\to\infty}\left(\dfrac{n!}{n^n}\right)^{1/n} \Rightarrow P=\lim\limits_{n\to\infty}\left(\dfrac{1}{n}.\dfrac{2}{n}.\dfrac{3}{n}.\dfrac{4}{n}.........\dfrac{n}{n}\right)^{1/n}$

$\therefore \log P=\dfrac{1}{n}\lim\limits_{n\to\infty}\left(\log\dfrac{1}{n}+\log\dfrac{2}{n}+.........+\log\dfrac{n}{n}\right) \Rightarrow \log P=\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\dfrac{1}{n}\log\dfrac{r}{n}$

$\log P=\int_0^1 \log x\,dx=[x\log x-x]_0^1=(-1)\Rightarrow P=\dfrac{1}{e}$.

Example: 27 If $\lim_{x\to\infty}\left[\frac{x^3+1}{x^2+1}-(ax+b)\right]=2$, then

(a) $a=1$ and $b=1$ (b) $a=1$ and $b=-1$ (c) $a=1$ and $b=-2$ (d) $a=1$ and $b=2$

Solution: (c) $\lim_{x\to\infty}\left(\frac{x^3+1}{x^2+1}-(ax+b)\right)=2 \Rightarrow \lim_{x\to\infty}\left(\frac{x^3(1-a)-bx^2-ax+(1-b)}{x^2+1}\right)=2 \Rightarrow \lim_{x\to\infty}[x^3(1-a)-bx^2-ax+(1-b)]=2(x^2+1)$.

Comparing the coefficients of both sides, $1-a=0$ and $-b=2$ or $a=1, b=-2$.

Example: 28 $\lim_{x\to\infty}\frac{(x+1)^{10}+(x+2)^{10}+.....+(x+100)^{10}}{x^{10}+10^{10}}$ is equal to

(a) 0 (b) 1 (c) 10 (d) 100

Solution: (d) $\lim_{x\to\infty}\frac{(x+1)^{10}+(x+2)^{10}+......+(x+100)^{10}}{x^{10}+10^{10}}=\lim_{x\to\infty}\frac{x^{10}\left[\left(1+\frac{1}{x}\right)^{10}+\left(1+\frac{2}{x}\right)^{10}+...+\left(1+\frac{100}{x}\right)^{10}\right]}{x^{10}\left[1+\frac{10^{10}}{x^{10}}\right]}=100$.

Example: 29 Let $f(x)=4$ and $f'(x)=4$, then $\lim_{x\to2}\frac{xf(2)-2f(x)}{x-2}$ equals

(a) 2 (b) -2 (c) -4 (d) 3

Solution: (c) $y=\lim_{x\to2}\frac{xf(2)-2f(x)}{x-2}$ $\Rightarrow y=\lim_{x\to2}\frac{xf(2)-2f(2)+2f(2)-2f(x)}{x-2}$

$\Rightarrow y=\lim_{x\to2}\frac{-2f(x)+2f(2)+xf(2)-2f(2)}{(x-2)}$ $\Rightarrow y=\lim_{x\to2}-2\frac{[f(x)-f(2)]}{x-2}+\lim_{x\to2}\frac{f(2).(x-2)}{(x-2)}$

$\Rightarrow y=-2\lim_{x\to2}\frac{f(x)-f(2)}{x-2}+f(2)$ $\Rightarrow y=-2\lim_{x\to2}f'(x)+f(2)=-8+4=-4$.

(2) **Trigonometric limits :** To evaluate trigonometric limits the following results are very important.

(i) $\lim_{x\to0}\frac{\sin x}{x}=1=\lim_{x\to0}\frac{x}{\sin x}$

(ii) $\lim_{x\to0}\frac{\tan x}{x}=1=\lim_{x\to0}\frac{x}{\tan x}$

(iii) $\lim_{x\to0}\frac{\sin^{-1}x}{x}=1=\lim_{x\to0}\frac{x}{\sin^{-1}x}$

(iv) $\lim_{x\to0}\frac{\tan^{-1}x}{x}=1=\lim_{x\to0}\frac{x}{\tan^{-1}x}$

(v) $\lim_{x\to0}\frac{\sin x^0}{x}=\frac{\text{®}}{180}$

(vi) $\lim_{x\to0}\cos x=1$

(vii) $\lim_{x\to a}\frac{\sin(x-a)}{x-a}=1$

(viii) $\lim_{x\to a}\frac{\tan(x-a)}{x-a}=1$

(ix) $\lim_{x\to a}\sin^{-1}x=\sin^{-1}a, |a|\le1$

(x) $\lim_{x\to a}\cos^{-1}x=\cos^{-1}a; |a|\le1$

(xi) $\lim_{x\to a}\tan^{-1}x=\tan^{-1}a; -\infty<a<\infty$

(xii) $\lim_{x\to\infty}\frac{\sin x}{x}=\lim_{x\to\infty}\frac{\cos x}{x}=0$

(xiii) $\lim_{x\to\infty}\frac{\sin(1/x)}{(1/x)}=1$

Example: 30 $\lim_{x\to1}(1-x)\tan\left(\frac{\text{®}x}{2}\right)=$

(a) $\frac{\text{®}}{2}$ (b) ® (c) $\frac{2}{\text{®}}$ (d) 0

Solution: (c) $\lim_{x\to1}(1-x)\tan\left(\frac{\text{®}x}{2}\right)$, Put $1-x=y \Rightarrow$ as $x\to1, y\to0$

Thus $\lim\limits_{y\to 0} y \ \tan \dfrac{\circledR(1-y)}{2} = \lim\limits_{y\to 0}\dfrac{2}{\circledR}\cdot\dfrac{\left(\dfrac{\circledR y}{2}\right)}{\tan\left(\dfrac{\circledR y}{2}\right)} = \dfrac{2}{\circledR}\times 1 = \dfrac{2}{\circledR}$.

Example: 31 $\lim\limits_{x\to 1}\dfrac{\sqrt{1-\cos 2(x-1)}}{x-1}$

(a) Exists and it equal $\sqrt{2}$

(b) Exists and it equals $-\sqrt{2}$

(c) Does not exist because $x-1\to 0$

(d) Does not exist because left hand limit is not equal to right hand limit

Solution: (d) $f(1+) = \lim\limits_{h\to 0} f(1+h) = = \lim\limits_{h\to 0}\dfrac{\sqrt{1-\cos 2h}}{h} = \lim\limits_{h\to 0}\sqrt{2}\,\dfrac{\sinh}{h} = \sqrt{2}$

$f(1-) = \lim\limits_{h\to 0} f(1-h) = \lim\limits_{h\to 0}\dfrac{\sqrt{1-\cos(-2h)}}{-h} = \lim\limits_{h\to 0}\sqrt{2}\,\dfrac{\sinh}{-h} = -\sqrt{2}$.

$\therefore$ limit does not exist because left hand limit is not equal to right hand limit.

Example: 32 $\lim\limits_{x\to 0}\dfrac{(1-\cos 2x)\ \sin 5x}{x^2\sin 3x} =$

(a) $\dfrac{10}{3}$ (b) $\dfrac{3}{10}$ (c) $\dfrac{6}{5}$ (d) $\dfrac{5}{6}$

Solution: (a) $\lim\limits_{x\to 0}\dfrac{2\sin^2 x}{x^2}\ \dfrac{\sin 5x}{\sin 3x}\ \dfrac{3x}{3x}\ \dfrac{5x}{5x} = \lim\limits_{x\to 0}\dfrac{2\sin^2 x}{x^2}\cdot\dfrac{3x}{\sin 3x}\cdot\dfrac{\sin 5x}{5x}\cdot\dfrac{5x}{3x} = 2.\dfrac{5}{3} = \dfrac{10}{3}$.

Example: 33 $\lim\limits_{x\to 0}\dfrac{x^3}{\sin x^2} =$

(a) 0 (b) $\dfrac{1}{3}$ (c) 3 (d) $\dfrac{1}{2}$

Solution: (a) $\lim\limits_{x\to 0}\dfrac{x^3}{\sin x^2} = \lim\limits_{x\to 0}\dfrac{x^2}{\sin x^2}\ .\ x = \left(\lim\limits_{x\to 0}\dfrac{x^2}{\sin x^2}\right)\left(\lim\limits_{x\to 0} x\right) = 1.0 = 0.$

Example: 34 $\lim\limits_{x\to 0}\dfrac{\sin 3x+\sin x}{x} =$

(a) $\dfrac{1}{3}$ (b) 3 (c) 4 (d) $\dfrac{1}{4}$

Solution: (c) $\lim\limits_{x\to 0}\dfrac{\sin 3x+\sin x}{x} = \lim\limits_{x\to 0}\dfrac{\sin 3x}{x} + \lim\limits_{x\to 0}\dfrac{\sin x}{x} = \lim\limits_{x\to 0}\dfrac{\sin 3x}{3x}.3 + \lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1.3 + 1 = 4.$

Example: 35 If $f(x) = \begin{cases} x\sin\dfrac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$, then $\lim\limits_{x\to 0} f(x) =$

(a) 1 (b) 0 (c) –1 (d) None of these

Solution: (b) $\lim\limits_{x\to 0} x\sin\left(\dfrac{1}{x}\right) = \left(\lim\limits_{x\to 0} x\right)\left(\lim\limits_{x\to 0}\sin\dfrac{1}{x}\right)$ = 0 × (A number oscillating between – 1 and 1) = 0.

Example: 36 If $f(x) = \begin{cases} \dfrac{\sin[x]}{[x]}, & [x]\neq 0 \\ 0, & [x]=0 \end{cases}$, then $\lim\limits_{x\to 0} f(x)$ equals

(a) 1 (b) 0 (c) –1 (d) Does not exist

Solution: (d) In closed interval of $x=0$ at right hand side [x] =0 and at left hand side $[x]=-1$. Also [0] =0.

Therefore function is defined as $f(x) = \begin{cases} \dfrac{\sin[x]}{[x]}, & (-1 \le x < 0) \\ 0, & (0 \le x < 1) \end{cases}$

$\therefore$ Left hand limit $= \lim\limits_{x \to 0-} f(x) = \lim\limits_{x \to 0-} \dfrac{\sin[x]}{[x]} = \dfrac{\sin(-1)}{-1} = \sin 1^c$

Right hand limit $= 0$, Hence, limit doesn't exist.

Example: 37 $\lim\limits_{x \to 0} \dfrac{\tan x - \sin x}{x^3}$

(a) $\dfrac{1}{2}$ (b) $-\dfrac{1}{2}$ (c) $\dfrac{2}{3}$ (d) None of these

Solution: (a) $\lim\limits_{x \to 0} \dfrac{\tan x - \sin x}{x^3} = \lim\limits_{x \to 0} \dfrac{\sin x - \sin x \cos x}{x^3 \cos x} = \lim\limits_{x \to 0} \dfrac{\sin x \left(2\sin^2 \dfrac{x}{2}\right)}{x^3 \cos x} = \lim\limits_{x \to 0} \left[\dfrac{\sin x}{x} \cdot \dfrac{2}{\cos x} \cdot \dfrac{\sin^2 \dfrac{x}{2}}{\left(\dfrac{x}{2}\right)^2} \cdot \dfrac{1}{4}\right] = \dfrac{1}{2}$

Example: 38 If $f(x) = \dfrac{\sin(e^{x-2} - 1)}{\log(x-1)}$, then $\lim\limits_{x \to 2} f(x)$ is given by

(a) -2 (b) -1 (c) 0 (d) 1

Solution: (d) $\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2} \dfrac{\sin(e^{x-2} - 1)}{\log(t+1)} = \lim\limits_{t \to 0} \dfrac{\sin(e^t - 1)}{\log(t+1)}$. (Putting $x = 2 + t$)

$= \lim\limits_{x \to \infty} \dfrac{\sin(e^t - 1)}{e^t - 1} \cdot \dfrac{e^t - 1}{t} \cdot \dfrac{t}{\log(1+t)} = \lim\limits_{t \to 0} \dfrac{\sin(e^t - 1)}{e^t - 1} \left(\dfrac{1}{1!} + \dfrac{t}{2!} + ...\right) \left[\dfrac{1}{\left(1 - \dfrac{1}{2}t + \dfrac{1}{3}t^2 - .....\right)}\right]$

$= 1.1.1 = 1$ $\left[\because \text{As } t \to 0, e^t - 1 \to 0, \therefore \dfrac{\sin(e^t - 1)}{(e^t - 1)} = 1\right]$

Example: 39 $\lim\limits_{x \to ® / 2} \dfrac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x} =$

(a) $\log a$ (b) $\log 2$ (c) a (d) $\log x$

Solution: (a) $\lim\limits_{x \to ® / 2} \left(\dfrac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x}\right) = \lim\limits_{x \to ® / 2} a^{\cos x} \left(\dfrac{a^{\cot x - \cos x} - 1}{\cot x - \cos x}\right)$

$= a^{\cos(® / 2)} \lim\limits_{x \to ® / 2} \left(\dfrac{a^{\cot x - \cos x} - 1}{\cot x - \cos x}\right) = 1 \log a = \log a$.

Example: 40 If $f(x) = \begin{vmatrix} \sin x & \cos x & \tan x \\ x^3 & x^2 & x \\ 2x & 1 & 1 \end{vmatrix}$, then $\lim\limits_{x \to 0} \dfrac{f(x)}{x^2}$ is

(a) 3 (b) -1 (c) 0 (d) 1

Solution: (d) $f(x) = x(x-1)\sin x - (x^3 - 2x^2)\cos x - x^3 \tan x$

$= x^2 \sin x - x^3 \cos x - x^3 \tan x + 2x^2 \cos x - x \sin x$

Hence, $\lim\limits_{x \to 0} \dfrac{f(x)}{x^2} = \lim\limits_{x \to 0} \left(\sin x - x\cos x - x\tan x + 2\cos x - \dfrac{\sin x}{x}\right) = 0 - 0 - 0 + 2 - 1 = 1$.

Example: 41 If $f(x) = \cot^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right)$ and $g(x) = \cos^{-1}\left(\dfrac{1 - x^2}{1 + x^2}\right)$, then $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{g(x) - g(a)}$, $0 < a < \dfrac{1}{2}$ is

(a) $\dfrac{3}{2(1 + a^2)}$ (b) $\dfrac{3}{2(1 + x^2)}$ (c) $\dfrac{3}{2}$ (d) $-\dfrac{3}{2}$

Solution: (d) $f(x) = \cot^{-1}\left\{\dfrac{3x - x^3}{1 - 3x^2}\right\}$ and $g(x) = \cos^{-1}\left\{\dfrac{1 - x^2}{1 + x^2}\right\}$

Put $x = \tan\theta$ in both equation

$$f(\theta) = \cot^{-1}\left\{\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right\} = \cot^{-1}\{\tan 3\theta\}$$

$$f(\theta) = \cot^{-1}\cot\left(\frac{\pi}{2} - 3\theta\right) = \frac{\pi}{2} - 3\theta \Rightarrow f'(\theta) = -3 \qquad \text{......(i)}$$

$$\text{and } g(\theta) = \cos^{-1}\left\{\frac{1 - \tan^2\theta}{1 + \tan^2\theta}\right\} = \cos^{-1}(\cos 2\theta) = 2\theta \Rightarrow g'(\theta) = 2 \qquad \text{..... (ii)}$$

$$\text{Now } \lim_{x\to a}\left(\frac{f(x) - f(a)}{g(x) - g(a)}\right) = \lim_{x\to a}\left(\frac{f(x) - f(a)}{x - a}\right)\frac{1}{\lim_{x\to a}\left(\dfrac{g(x) - g(a)}{x - a}\right)} = f'(x).\frac{1}{g'(x)} = -3 \times \frac{1}{2} = -\frac{3}{2}.$$

Example: 42 $\displaystyle\lim_{x\to\frac{\pi}{2}}\frac{\left[1 - \tan\left(\frac{x}{2}\right)\right][1 - \sin x]}{\left[1 + \tan\left(\frac{x}{2}\right)\right][\pi - 2x]^3}$ is

Solution: (c) $\displaystyle\lim_{x\to\frac{\pi}{2}}\frac{\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)(1 - \sin x)}{(\pi - 2x)^3}$

Let $x = \dfrac{\pi}{2} + y$, then $y \to 0$ $\Rightarrow \displaystyle\lim_{y\to 0}\frac{\tan\left(\frac{-y}{2}\right)(1 - \cos y)}{(-2y)^3} = \lim_{y\to 0}\frac{-\tan\frac{y}{2}.2\sin^2\frac{y}{2}}{(-8)y^3} = \lim_{y\to 0}\frac{1}{32}\frac{\tan\frac{y}{2}}{\left(\frac{y}{2}\right)}.\left[\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right]^2 = \frac{1}{32}.$

Example: 43 If $\displaystyle\lim_{x\to 0}\frac{[(a - n)nx - \tan x]\sin nx}{x^2} = 0$, where n is non-zero real number, then a is equal to

(a) 0 (b) $\dfrac{n+1}{n}$ (c) n (d) $n + \dfrac{1}{n}$

Solution: (d) $\displaystyle\lim_{x\to 0} n\frac{\sin nx}{nx}.\lim_{x\to 0}\left((a - n)n - \frac{\tan x}{x}\right) = 0 \Rightarrow n[(a - n)n - 1] = 0 \Rightarrow (a - n)n = 1 \Rightarrow a = n + \frac{1}{n}.$

(3) **Logarithmic limits :** To evaluate the logarithmic limits we use following formulae

(i) $\log(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - ....\text{to}\ \infty$ where $-1 < x \le 1$ and expansion is true only if base is e.

(ii) $\displaystyle\lim_{x\to 0}\frac{\log(1 + x)}{x} = 1$ (iii) $\displaystyle\lim_{x\to e}\log_e x = 1$

(iv) $\displaystyle\lim_{x\to 0}\frac{\log(1 - x)}{x} = -1$ (v) $\displaystyle\lim_{x\to 0}\frac{\log_a(1 + x)}{x} = \log_a e, a > 0, \ne 1$

Example: 44 $\displaystyle\lim_{h\to 0}\frac{\log_e(1 + 2h) - 2\log_e(1 + h)}{h^2}$

(a) –1 (b) 1 (c) 2 (d) –2

Solution: (a) $\displaystyle\lim_{h\to 0}\frac{\log_e(1 + 2h) - 2\log_e(1 + h)}{h^2} = \lim_{x\to a}\frac{\left((2h) - \frac{(2h)^2}{2} + \frac{(2h)^3}{3} - .....\infty\right) - 2\left(h - \frac{h^2}{2} + \frac{h^3}{3} - ......\right)}{h^2}$

$= \lim_{h\to 0} \frac{-h^2 + 2h^3 - ....}{h^2} = \lim_{h\to 0} \frac{h^2\{-1+2h-....\}}{h^2} = \lim_{h\to 0}\{-1+2h+....\} = -1$.

Example: 45 $\lim_{x\to a} \frac{\log\{1+(x-a)\}}{(x-a)} =$

(a) –1 (b) 2 (c) 1 (d) –2

Solution: (c) Let $x-a = y$, when $x \to a,\ y \to 0$,

$\therefore$ The given limit $= \lim_{y\to 0} \frac{\log\{1+y\}}{y} = 1$.

Example: 46 $\lim_{h\to 0} \frac{\log_{10}(1+h)}{h} =$

(a) 1 (b) $\log_{10} e$ (c) $\log_e 10$ (d) None of these

Solution: (b) $\lim_{h\to 0} \frac{\log_e(1+h)}{h} . \frac{1}{\log_e 10} = \log_{10} e$.

Example: 47 If $\lim_{x\to 0} \frac{\log(3+x)-\log(3-x)}{x} = k$, then the value of k is

(a) 0 (b) $-\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

Solution: (c) $\lim_{x\to 0} \frac{\log(3+x)-\log(3-x)}{x} = \lim_{x\to 0} \frac{\log\left(\frac{3+x}{3-x}\right)}{x} = \lim_{x\to 0} \frac{\log\left(\frac{1+(x/3)}{1-(x/3)}\right)}{x}$

$= \lim_{x\to 0} \frac{\log(1+(x/3))}{x} - \lim_{x\to 0} \frac{\log(1-(x/3))}{x} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$.

(4) **Exponential limits :**

(i) **Based on series expansion :** We use $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\infty$

To evaluate the exponential limits we use the following results –

(a) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$ (b) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$ (c) $\lim_{x\to 0} \frac{e^{\text{ª}x} - 1}{x} = \text{ª}$ $(\text{ª} \neq 0)$

(ii) **Based on the form $1^{\tilde{a}}$:** To evaluate the exponential form 1^∞ we use the following results.

(a) If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} \{1 + f(x)\}^{1/g(x)} = e^{\lim_{x\to a} \frac{f(x)}{g(x)}}$, or

when $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$. Then $\lim_{x\to a} \{f(x)\}^{g(x)} = \lim_{x\to a} [1 + f(x) - 1]^{g(x)} = e^{\lim_{x\to a} (f(x)-1)g(x)}$

(b) $\lim_{x\to 0} (1+x)^{1/x} = e$ (c) $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$ (d) $\lim_{x\to 0} (1 + \text{ª}x)^{1/x} = e^{\text{ª}}$ (e) $\lim_{x\to\infty} \left(1 + \frac{\text{ª}}{x}\right)^x = e^{\text{ª}}$

Note : ❒ $\lim_{x\to\infty} a^x = \begin{cases} \infty, \text{if } a > 1 \\ 0, \text{if } a < 1 \end{cases}$ i.e., $a^\infty = \infty$, if $a > 1$ and $a^\infty = 0$ if $a < 1$.

Example: 48 $\lim_{x\to 0} \frac{e^{\sim x} - e^{\ x}}{x} =$

Solution: (d) $\lim_{x\to 0} \frac{e^{\sim x} - e^{\ x}}{x} = \lim_{x\to 0} \frac{(e^{\sim x} - 1) - (e^{\ x} - 1)}{x} = \lim_{x\to 0} \frac{e^{\sim x} - 1}{x} - \lim_{x\to 0} \frac{e^{\ x} - 1}{x} = \sim -$.

Example: 49 The value of $\lim_{x\to 0}\frac{e^x-(1+x)}{x^2}$ is

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) $\frac{1}{4}$

Solution: (b) $\lim_{x\to 0}\frac{e^x-(1+x)}{x^2}=\lim_{x\to 0}\frac{(1+x+\frac{x^2}{2!}+.....)-(1+x)}{x^2}=\lim_{x\to 0}\frac{x^2\left(\frac{1}{2!}+\frac{x}{3!}+\frac{x^2}{4!}+.....\right)}{x^2}=\frac{1}{2!}=\frac{1}{2}$.

Example: 50 $\lim_{x\to 0}\frac{a^x-1}{\sqrt{1+x}-1}$ is equal to

(a) $2\log_e a$ (b) $\frac{1}{2}\log_e a$ (c) $a\log_e 2$ (d) None of these

Solution: (a) $\lim_{x\to 0}\frac{a^x-1}{\sqrt{1+x}-1}=\lim_{x\to 0}\frac{a^x-1}{\sqrt{1+x}-1}.\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}=\lim_{x\to 0}\frac{(a^x-1)(\sqrt{1+x}+1)}{1+x-1}=\lim_{x\to 0}\left(\frac{a^x-1}{x}\right).(\sqrt{1+x}+1)$

$=\left(\lim_{x\to 0}\frac{a^x-1}{x}\right).\left(\lim_{x\to 0}(\sqrt{1+x}+1)\right)=(\log_e a).(2)=2\log_e a$.

Example: 51 The value of $\lim_{x\to\infty}\left(\frac{x+3}{x+1}\right)^{x+2}$ is

(a) e^4 (b) 0 (c) 1 (d) e^2

Solution: (d) $\lim_{x\to\infty}\left(\frac{x+3}{x+1}\right)^{x+2}=\lim_{x\to\infty}\left(1+\frac{2}{x+1}\right)^{\frac{x+1}{2}.(x+2).\frac{2}{(x+1)}}=\lim_{x\to\infty}\left(\left(1+\frac{2}{x+1}\right)^{\frac{x+1}{2}}\right)^{2.\left(\frac{1+\frac{2}{x}}{1+\frac{1}{x}}\right)}=e^{2\lim_{x\to\infty}\left[\left(1+\frac{2}{x}\right)/\left(1+\frac{1}{x}\right)\right]}=e^2$.

Alternative method : $\lim_{x\to\infty}\left(\frac{x+3}{x+1}\right)^{x+2}=\lim_{x\to\infty}\left(1+\frac{2}{x+1}\right)^{x+2}=e^{\lim_{x\to\infty}\frac{2}{x+1}(x+2)}=e^{\lim_{x\to\infty}2\left(\frac{1+\frac{2}{x}}{1+\frac{1}{x}}\right)}=e^2$

Example: 52 If a, b, c, d are positive, then $\lim_{x\to\infty}\left(1+\frac{1}{a+bx}\right)^{c+dx}$

(a) $e^{d/b}$ (b) $e^{c/a}$ (c) $e^{(c+d)/(a+b)}$ (d) e

Solution: (a) $\lim_{x\to\infty}\left(1+\frac{1}{a+bx}\right)^{c+dx}=\lim_{x\to\infty}\left\{\left(1+\frac{1}{a+bx}\right)^{a+bx}\right\}^{\frac{c+dx}{a+bx}}=e^{d/b}$ $\left\{\because \lim_{x\to\infty}\left(1+\frac{1}{a+bx}\right)^{a+bx}=e \text{ and } \lim_{x\to\infty}\frac{c+dx}{a+bx}=\frac{d}{b}\right\}$

Alternative method : $e^{\lim_{x\to\infty}\left(\frac{1}{a+bx}\right)\left(\frac{c+dx}{1}\right)}=e^{d/b}$.

Example: 53 $\lim_{x\to 0}x^x=$

(a) 0 (b) 1 (c) e (d) None of these

Solution: (b) Let $y=x^x\Rightarrow\log y=x\log x$; $\therefore\lim_{y\to 0}\log y=\lim_{x\to 0}x\log x=0=\log 1\Rightarrow\lim_{x\to 0}x^x=1$

Example: 54 The value of $\lim_{x\to 0}\frac{(1+x)^{1/x}-e+\frac{1}{2}ex}{x^2}$ is

(a) $\frac{11e}{24}$ (b) $\frac{-11e}{24}$ (c) $\frac{e}{24}$ (d) None of these

Solution: (a) $(1+x)^{1/x}=e^{\frac{1}{x}\log(1+x)}=e^{\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-....\right)}=e^{1-\frac{x}{2}+\frac{x^2}{3}-.......}=e\,e^{-\frac{x}{2}+\frac{x^2}{3}-......}$

$$= e\left[1+\left(-\frac{x}{2}+\frac{x^2}{3}-.....\right)+\frac{1}{2!}\left(-\frac{x}{2}+\frac{x^2}{3}-.....\right)^2+.....\right] = e\left[1-\frac{x}{2}+\frac{11}{24}x^2-...........\right]$$

$$\therefore \lim_{x\to 0}\frac{(1+x)^{1/x}-e+\frac{ex}{2}}{x^2}=\frac{11e}{24}$$

Example: 55 $\lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}$ equals

(a) ®$/2$ (b) 0 (c) $2/e$ (d) $-e/2$

Solution: (d) $(1+x)^{\frac{1}{x}}=e^{\frac{1}{x}[\log(1+x)]}=e^{\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+....\right)}=e^{\left(1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+....\right)}=e.e^{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+....\right)}$

$$= e\left[1+\frac{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+.....\right)}{1!}+\frac{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+...\right)^2}{2!}+...\right]=\left[e-\frac{ex}{2}+\frac{11e}{24}x^2-..........\right]$$

$$\therefore \lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}=\lim_{x\to 0}\left[\frac{e-\frac{ex}{2}+\frac{11e}{24}x^2.........-e}{x}\right]\Rightarrow \lim_{x\to 0}\left(-\frac{e}{2}-\frac{11e}{24}x+...\right)=-\frac{e}{2}.$$

Example: 56 $\lim_{m\to\infty}\left(\cos\frac{x}{m}\right)^m=$

(a) 0 (b) e (c) $1/e$ (d) 1

Solution: (d) $\lim_{m\to\infty}\left(\cos\frac{x}{m}\right)^m=\lim_{m\to\infty}\left[1+\left(\cos\frac{x}{m}-1\right)\right]^m=\lim_{m\to\infty}\left[1-\left(-\cos\frac{x}{m}+1\right)\right]^m$

$$=\lim_{m\to\infty}\left[1-2\sin^2\frac{x}{2m}\right]^m=e^{\lim_{m\to\infty}-\left(2\sin^2\frac{x}{2m}\right)m}=e^{\lim_{m\to\infty}-2\left(\frac{\sin\frac{x}{2m}}{\frac{x}{2m}}\right)^2\left(\frac{x^2}{4m^2}\right)m}=e^{-2\lim_{m\to\infty}\frac{x^2}{4m}}=e^0=1.$$

Example: 57 $\lim_{n\to\infty}\left(\frac{n^2-n+1}{n^2-n-1}\right)^{n(n-1)}=$

Solution: (b) $\lim_{n\to\infty}\left(\frac{n^2-n+1}{n^2-n-1}\right)^{n(n-1)}=\lim_{n\to\infty}\left(\frac{n(n-1)+1}{n(n-1)-1}\right)^{n(n-1)}=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n(n-1)}\right)^{n(n-1)}}{\left(1-\frac{1}{n(n-1)}\right)^{n(n-1)}}=\frac{e}{e^{-1}}=e^2.$

Alternative Method: $\lim_{n\to\infty}\left(1+\frac{2}{n^2-n-1}\right)^{n(n-1)}=e^{\lim_{n\to\infty}\frac{2n(n-1)}{n^2-n-1}}=e^2.$

(5) L' Hospital's rule : If $f(x)$ and $g(x)$ be two functions of x such that

(i) $\lim_{x\to a}f(x)=\lim_{x\to a}g(x)=0$

(ii) Both are continuous at $x=a$

(iii) Both are differentiable at $x=a$.

(iv) $f'(x)$ and $g'(x)$ are continuous at the point $x=a$, then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$ provided that $g'(a)\neq 0$

Note : ❒ The above rule is also applicable if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$.

❒ If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ assumes the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and $f'(x), g'(x)$ satisfy all the condition embodied in L' Hospital rule, we can repeat the application of this rule on $\frac{f'(x)}{g'(x)}$ to get, $\lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}$. Sometimes it may be necessary to repeat this process a number of times till our goal of evaluating limit is achieved.

Example: 58 $\lim_{x\to 0} \frac{1-\cos mx}{1-\cos nx} =$

(a) m/n (b) n/m (c) $\frac{m^2}{n^2}$ (d) $\frac{n^2}{m^2}$

Solution: (c) $\lim_{x\to 0} \frac{1-\cos mx}{1-\cos nx} = \lim_{x\to 0}\left\{\frac{2\sin^2 \frac{mx}{2}}{2\sin^2\frac{nx}{2}}\right\} = \lim_{x\to 0}\left[\left\{\frac{\sin\frac{mx}{2}}{\frac{mx}{2}}\right\}^2 \frac{m^2x^2}{4} \cdot \frac{1}{\left\{\frac{\sin\frac{nx}{2}}{\frac{nx}{2}}\right\}^2}\cdot\frac{4}{n^2x^2}\right] = \frac{m^2}{n^2}\times 1 = \frac{m^2}{n^2}$

Trick : Apply L-Hospital rule ,

$$\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx} = \lim_{x\to 0}\frac{m\sin mx}{n\sin nx} = \lim_{x\to 0}\frac{m^2\cos mx}{n^2\cos nx} = \frac{m^2}{n^2}.$$

Example: 59 The integer n for which $\lim_{x\to 0}\frac{(\cos x - 1)(\cos x - e^x)}{x^n}$ is a finite non-zero number is

(a) 1 (b) 2 (c) 3 (d) 4

Solution: (c) n cannot be negative integer for then the limit $= 0$

Limit $= \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{2^2(x/2)^2}\frac{e^x-\cos x}{x^{n-2}} = \frac{1}{2}\lim_{x\to 0}\frac{e^x-\cos x}{x^{n-2}}$ ($n \ne 1$ for then the limit $= 0$)

$= \frac{1}{2}\lim_{x\to 0}\frac{e^x+\sin x}{(n-2)x^{n-3}}$. So, if $n=3$, the limit is $\frac{1}{2(n-2)}$ which is finite. If $n=4$, the limit is infinite.

Example: 60 Let $f: R \to R$ be such that $f(1) = 3$ and $f'(1) = 6$. Then $\lim_{x\to 0}\left\{\frac{f(1+x)}{f(1)}\right\}^{\frac{1}{x}}$ equals

(a) 1 (b) $e^{1/2}$ (c) e^2 (d) e^3

Solution: (c) $\lim_{x\to 0}\left\{\frac{f(1+x)}{f(1)}\right\}^{\frac{1}{x}} = e^{\lim_{x\to 0}\frac{1}{x}[\log f(1+x) - \log f(1)]} = e^{\lim_{x\to 0}\frac{f'(1+x)/f(1+x)}{1}} = e^{\frac{f'(1)}{f(1)}} = e^{6/3} = e^2$.

Example: 61 $\lim_{\sim\to ®/4}\frac{\sin\sim - \cos\sim}{\sim - ®/4} =$

(a) $\sqrt{2}$ (b) $1/\sqrt{2}$ (c) 1 (d) None of these

Solution: (a) $\lim\limits_{\sim\to ® /4} \frac{\sin\sim - \cos\sim}{\sim - ® / 4}$ $(\frac{0}{0}$ form$)$ $= \lim\limits_{\sim\to ® /4} \frac{\cos\sim + \sin\sim}{1}$ (By 'L' Hospital rule)

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$.

Example: 62 $\lim\limits_{x\to a} \frac{x^3 - a^3}{x^2 - a^2} =$

(a) 0 (b) Not defined (c) $2a$ (d) $\frac{3a}{2}$

Solution: (d) $\lim\limits_{x\to a} \frac{x^3 - a^3}{x^2 - a^2}$ $\left(\frac{0}{0}\text{form}\right) = \lim\limits_{x\to a} \frac{3x^2}{2x}$ (By 'L' Hospital rule) $= \frac{3a^2}{2a} = \frac{3a}{2}$.

Example: 63 $\lim\limits_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} =$

(a) $1/2\sqrt{x}$ (b) $1/2\sqrt{h}$ (c) Zero (d) None of these

Solution: (a) $\lim\limits_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim\limits_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim\limits_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim\limits_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}}$

Trick : Applying 'L' Hospital's rule, [Differentiating N^r and D^r with respect to h]

We get, $\lim\limits_{h\to 0} \frac{\frac{1}{2\sqrt{x+h}} - 0}{1} = \frac{1}{2\sqrt{x}}$.

Example: 64 $\lim\limits_{\sim\to} \frac{\sin^2\sim - \sin^2}{\sim^2 - \quad 2}$

(a) 0 (b) 1 (c) $\frac{\sin}{}$ (d) $\frac{\sin\ 2}{2}$

Solution: (d) $\lim\limits_{\sim\to} \frac{\sin^2\sim - \sin^2}{\sim^2 - \quad 2} = \lim\limits_{\sim - \to 0} \frac{\sin(\sim + \ )\sin(\sim - \ )}{(\sim + \ )\ (\sim - \ )} = \lim\limits_{\sim - \to 0} \frac{\sin(\sim - \ )}{(\sim - \ )} \lim\limits_{\sim - \to 0} \frac{\sin(\sim + \ )}{(\sim + \ )} = \lim\limits_{\sim\to} \frac{\sin(\sim + \ )}{(\sim + \ )} = \frac{\sin\ 2}{2}$.

Trick : By L' Hospital's rule, $\lim\limits_{\sim\to} \frac{2\sin\sim \cos\sim}{2\sim} = \frac{\sin 2}{2}$.

Example: 65 $\lim\limits_{x\to 0} \frac{\tan 2x - x}{3x - \sin x}$ equals

(a) 2/3 (b) 1/3 (c) 1/2 (d) 0

Solution: (c) $\lim\limits_{x\to 0} \frac{\tan 2x - x}{3x - \sin x} = \lim\limits_{x\to 0} \left\{ \frac{\frac{2\tan 2x}{2x} - 1}{3 - \frac{\sin x}{x}} \right\} = \frac{1}{2}$.

Example: 66 If $G(x) = -\sqrt{25 - x^2}$, then $\lim\limits_{x\to 1} \frac{G(x) - G(1)}{x - 1}$ equals

(a) 1/24 (b) 1/5 (c) $-\sqrt{24}$ (d) None of these

Solution: (d) $\lim\limits_{x\to 1} \frac{G(x) - G(1)}{x - 1} = \lim\limits_{x\to 1} \frac{-\sqrt{25 - x^2} + \sqrt{24}}{x - 1}$ [Multiply both numerator and denominator by $(\sqrt{24} + \sqrt{25 - x^2})$]

$= \lim\limits_{x\to 1} \frac{x + 1}{\sqrt{24} + \sqrt{25 - x^2}} = \frac{1}{\sqrt{24}}$

Alternative method: By L'-Hospital rule, $\lim\limits_{x\to 1} \frac{G'(x)}{1} = \lim\limits_{x\to 1} \frac{-1(-2x)}{2\sqrt{25 - x^2}} = \frac{1}{\sqrt{24}}$

Example: 67 If $f(a)=2, f'(a)=1, g(a)=1, g'(a)=2,$ then $\lim_{x\to a}\frac{g(x)f(a)-g(a)f(x)}{x-a}$ equals

(a) -3 (b) $\frac{1}{3}$ (c) 3 (d) $-\frac{1}{3}$

Solution: (c) Applying L – Hospital's rule, we get, $\lim_{x\to a}\frac{g(x)\,f(a)-g(a)\,f(x)}{x-a}=\lim_{x\to a}\frac{g'(x)\,f(a)-g(a)\,f'(x)}{1}$

$=g'(a)\,f(a)-g(a)\,f'(a)=2\times 2-1\times(1)=3.$

Example: 68 $\lim_{x\to 0}\frac{(1+x)^n-1}{x}=$

(a) n (b) 1 (c) -1 (d) None of these

Solution: (a) $\lim_{x\to 0}\frac{(1+nx+{}^nC_2x^2+......\text{higher powers of } x \text{ to } x^n)-1}{x}=n$

Trick : Apply L- Hospital rule.

Example: 69 $\lim_{x\to 0}\frac{\sin x+\log(1-x)}{x^2}$ is equal to

(a) 0 (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) None of these

Solution: (c) Apply L- Hospital rule, we get, $\lim_{x\to 0}\frac{\cos x-\frac{1}{1-x}}{2x}=\lim_{x\to 0}\frac{-\sin x-\frac{1}{(1-x)^2}}{2}=-\frac{1}{2}$

Alternative method : $\lim_{x\to 0}\frac{\sin x+\log(1-x)}{x^2}=\lim_{x\to 0}\frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-......\right)}{x^2}+\lim_{x\to 0}\frac{\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-......\right)}{x^2}$

$\left(\because \sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-...... \text{ and } \log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}........\right)$

Hence, $\lim_{x\to 0}\frac{\frac{-x^2}{2}-x^3\left(\frac{1}{3!}+\frac{1}{3}\right)-\frac{x^4}{4}....}{x^2}=-\frac{1}{2}.$

Example: 70 $\lim_{x\to 0}\frac{xe^x-\log(1+x)}{x^2}$ equals

Solution: (d) Let $y=\lim_{x\to 0}\frac{xe^x-\log(1+x)}{x^2}$ $\left(\frac{0}{0}\text{form}\right)$

Applying L–Hospital's rule, $y=\lim_{x\to 0}\frac{e^x+xe^x-\frac{1}{1+x}}{2x}$ $\left(\frac{0}{0}\text{form}\right)$

$y=\lim_{x\to 0}\frac{1}{2}\left[e^x+e^x+xe^x+\frac{1}{(1+x)^2}\right]=\lim_{x\to 0}\frac{1}{2}[1+1+0+1]=\frac{3}{2}$

Example: 71 $\lim_{x\to 0}\frac{\sin^{-1}x-\tan^{-1}x}{x^3}$ is equal to

(a) 0 (b) 1 (c) -1 (d) $\frac{1}{2}$

Solution: (d) $\lim_{x\to 0}\frac{\sin^{-1}x-\tan^{-1}x}{x^3}$ $\left(\frac{0}{0}\text{form}\right)$

Applying L-Hospital's rule,

$$= \lim_{x \to 0} \frac{\frac{1}{\sqrt{1-x^2}} - \frac{1}{1+x^2}}{3x^2} \quad \left(\frac{0}{0} \text{form}\right)$$

$$= \lim_{x \to 0} \frac{\frac{-1}{2} \times \frac{-2x}{(1-x^2)^{3/2}} + \frac{2x}{(1+x^2)^2}}{6x} = \lim_{x \to 0} \frac{1}{6}\left[\frac{1}{(1-x^2)^{3/2}} + \frac{2}{(1+x^2)^2}\right] = \frac{1}{2}.$$

Example: 72 $\lim_{x \to 1} \frac{1 + \log x - x}{1 - 2x + x^2} =$

(a) 1 (b) -1 (c) 0 (d) $-\frac{1}{2}$

Solution: (d) Applying L-Hospital's rule, $\lim_{x \to 1} \frac{1 + \log x - x}{1 - 2x + x^2} = \lim_{x \to 1} \frac{\frac{1}{x} - 1}{-2 + 2x} = \lim_{x \to 1} \frac{1 - x}{2x(x-1)}$

Again applying L-Hospital's rule, we get $\lim_{x \to 1} \frac{-1}{4x - 2} = -\frac{1}{2}$

Example: 73 $\lim_{x \to 0} \frac{4^x - 9^x}{x(4^x + 9^x)} =$

(a) $\log\left(\frac{2}{3}\right)$ (b) $\frac{1}{2}\log\left(\frac{3}{2}\right)$ (c) $\frac{1}{2}\log\left(\frac{3}{2}\right)$ (d) $\log\left(\frac{3}{2}\right)$

Solution: (a) $y = \lim_{x \to 0} \frac{4^x - 9^x}{x(4^x + 9^x)} \quad \left(\frac{0}{0} \text{form}\right)$

Using L-Hospital's rule, $y = \lim_{x \to 0} \frac{4^x \log 4 - 9^x \log 9}{(4^x + 9^x) + x(4^x \log 4 + 9^x \log 9)} \Rightarrow y = \frac{\log 4 - \log 9}{2} \Rightarrow y = \frac{\log\left(\frac{2}{3}\right)^2}{2} = \log\frac{2}{3}$.

Example: 74 If $f(a) = 2$, $f'(a) = 1$, $g(a) = -3$, $g'(a) = -1$, then $\lim_{x \to a} \frac{f(a)\,g(x) - f(x)\,g(a)}{x - a} =$

Solution: (a) $\lim_{x \to a} \frac{f(a)\,g(x) - f(x)\,g(a)}{x - a} \quad \left(\frac{0}{0} \text{form}\right)$

Using L-Hospital's rule, $\lim_{x \to a} \frac{f(a)\,g'(x) - f'(x)\,g(a)}{1 - 0} = f(a) \times g'(a) - f'(a) \times g(a) = 2 \times (-1) - 1 \times (-3) = 1$.

Example: 75 The value of $\lim_{x \to 7} \frac{2 - \sqrt{x-3}}{x^2 - 49}$ is

(a) $\frac{2}{9}$ (b) $-\frac{2}{49}$ (c) $\frac{1}{56}$ (d) $-\frac{1}{56}$

Solution: (d) Applying L-Hospital's rule, $\lim_{x \to 7} \frac{0 - \frac{1}{2\sqrt{x-3}}}{2x} = \lim_{x \to 7} \frac{-1}{4x\sqrt{x-3}} = \frac{-1}{4.7\sqrt{7-3}} = \frac{-1}{56}$.

Example: 76 Let $f(a) = g(a) = k$ and their n^{th} derivatives $f^n(a), g^n(a)$ exist and are not equal for some n. If $\lim_{x \to a} \frac{f(a)\,g(x) - f(a) - g(a)\,f(x) + g(a)}{g(x) - f(x)} = 4$, then the value of k is

(a) 4 (b) 2 (c) 1 (d) 0

Solution: (a) $\lim_{x \to a} \frac{k\,g(x) - k\,f(x)}{g(x) - f(x)} = 4$

By L-Hospital' rule, $\lim_{x\to a} k\left[\frac{g'(x)-f'(x)}{g'(x)-f'(x)}\right]=4$, $\therefore\ k=4$.

Example: 77 The value of $\lim_{x\to 0}\left(\frac{\int_0^{x^2}\sec^2 t\,dt}{x\sin x}\right)$ is

(a) 3 (b) 2 (c) 1 (d) 0

Solution: (c) $\lim_{x\to 0}\frac{\frac{d}{dx}\int_0^{x^2}\sec^2 t\,dt}{\frac{d}{dx}(x\sin x)}=\lim_{x\to 0}\frac{\sec^2 x^2 .2x}{\sin x+x\cos x}$ (By L' –Hospital's rule)

$=\lim_{x\to 0}\frac{2\sec^2 x^2}{\left(\frac{\sin x}{x}+\cos x\right)}=\frac{2\times 1}{1+1}=1$.

Example: 78 $\lim_{x\to ® /6}\left[\frac{3\sin x-\sqrt{3}\cos x}{6x-®}\right]$

(a) $\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $-\sqrt{3}$ (d) $-\frac{1}{\sqrt{3}}$

Solution: (b) Using L–Hospital's rule, $\lim_{x\to ® /6}\frac{3\cos x+\sqrt{3}\sin x}{6}=\frac{3.\frac{\sqrt{3}}{2}+\sqrt{3}.\frac{1}{2}}{6}=\frac{1}{\sqrt{3}}$.

Example: 79 Given that $f'(2)=6$ and $f'(1)=4$, then $\lim_{h\to 0}\frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}=$

(a) Does not exist (b) $-\frac{3}{2}$ (c) $\frac{3}{2}$ (d) 3

Solution: (d) $\lim_{h\to 0}\frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}=\lim_{h\to 0}\frac{f'(2h+2+h^2)(2+2h)}{f'(h-h^2+1)(1-2h)}=\frac{6\times 2}{4\times 1}=3$.

Continuity

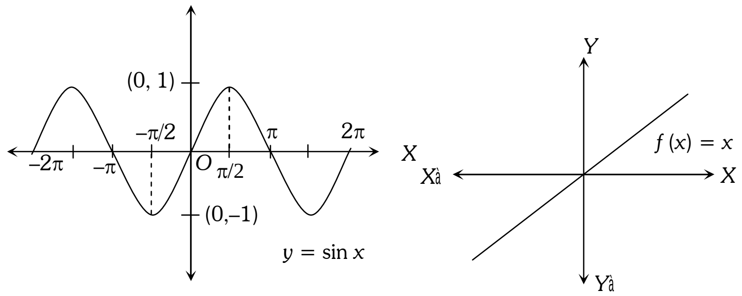
Introduction

The word 'Continuous' means without any break or gap. If the graph of a function has no break, or gap or jump, then it is said to be continuous.

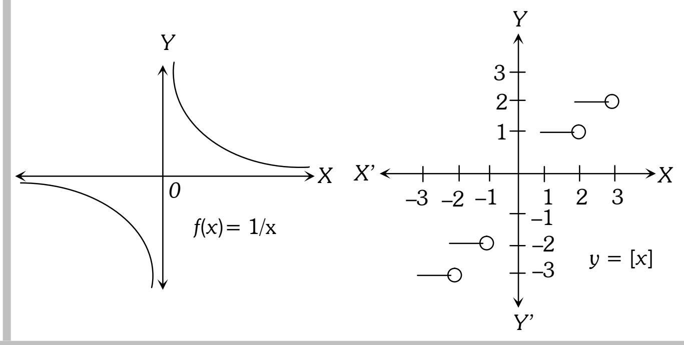
A function which is not continuous is called a discontinuous function.

While studying graphs of functions, we see that graphs of functions $\sin x$, x, $\cos x$, e^x etc. are continuous but greatest integer function $[x]$ has break at every integral point, so it is not continuous. Similarly $\tan x$, $\cot x$, $\sec x$, $\frac{1}{x}$ etc. are also discontinuous function.

Continuous function



Discontinuous function



1.1 Continuity of a Function at a Point

A function $f(x)$ is said to be continuous at a point $x = a$ of its domain iff $\lim_{x\to a} f(x) = f(a)$. *i.e.* a function $f(x)$ is continuous at $x = a$ if and only if it satisfies the following three conditions :

(1) $f(a)$ exists. ('a' lies in the domain of f)

(2) $\lim_{x\to a} f(x)$ exist *i.e.* $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$ or R.H.L. = L.H.L.

(3) $\lim_{x\to a} f(x) = f(a)$ (limit equals the value of function).

Cauchy's definition of continuity : A function f is said to be continuous at a point a of its domain D if for every £ > 0 there exists ¢ > 0 (dependent on £) such that $|x - a| <$ ¢ $\Rightarrow |f(x) - f(a)| <$ £.

Comparing this definition with the definition of limit we find that $f(x)$ is continuous at $x = a$ if $\lim_{x\to a} f(x)$ exists and is equal to $f(a)$ *i.e.*, if $\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a+} f(x)$.

Heine's definition of continuity : A function f is said to be continuous at a point a of its domain D, converging to a, the sequence $< a_n >$ of the points in D converging to a, the sequence $< f(a_n) >$ converges to $f(a)$ *i.e.* $\lim a_n = a \Rightarrow \lim f(a_n) = f(a)$. This definition is mainly used to prove the discontinuity to a function.

Note : ❒ Continuity of a function at a point, we find its limit and value at that point, if these two exist and are equal, then function is continuous at that point.

Formal definition of continuity : The function $f(x)$ is said to be continuous at $x = a$, in its domain if for any arbitrary chosen positive number $\in > 0$, we can find a corresponding number δ depending on $\in$ such that $|f(x) - f(a)| < \in \ \forall x$ for which $0 < |x - a| <$ ¢ .

1.2 Continuity from Left and Right

Function $f(x)$ is said to be

(1) Left continuous at $x = a$ if $\lim_{x\to a-0} f(x) = f(a)$

(2) Right continuous at $x = a$ if $\mathrm{Lim}_{x\to a+0} f(x) = f(a)$.

Thus a function $f(x)$ is continuous at a point $x = a$ if it is left continuous as well as right continuous at $x = a$.

Example: 1 If $f(x) = \begin{cases} x+\lambda, & x<3 \\ 4, & x=3 \\ 3x-5, & x>3 \end{cases}$ is continuous at $x = 3$, then λ =

(a) 4 (b) 3 (c) 2 (d) 1

Solution: (d) L.H.L. at $x = 3$, $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x+\lambda) = \lim_{h\to 0} (3-h+\lambda) = 3+\lambda$ …..(i)

R.H.L. at $x = 3$, $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (3x-5) = \lim_{h\to 0} \{3(3+h)-5\} = 4$ …..(ii)

Value of function $f(3) = 4$ …..(iii)

For continuity at $x = 3$

Limit of function = value of function $3 + \lambda = 4 \Rightarrow \lambda = 1$.

Example: 2 If $f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ k, & x=0 \end{cases}$ is continuous at $x = 0$, then the value of k is

(a) 1 (b) –1 (c) 0 (d) 2

Solution: (c) If function is continuous at $x = 0$, then by the definition of continuity $f(0) = \lim_{x\to 0} f(x)$

since $f(0) = k$. Hence, $f(0) = k = \lim_{x\to 0} (x)\left(\sin\frac{1}{x}\right)$

$\Rightarrow k = 0$ (a finite quantity lies between –1 to 1) $\Rightarrow k = 0$.

Example: 3 If $f(x) = \begin{cases} 2x+1 & \text{when } x<1 \\ k & \text{when } x=1 \\ 5x-2 & \text{when } x>1 \end{cases}$ is continuous at $x = 1$, then the value of k is

(a) 1 (b) 2 (c) 3 (d) 4

Solution: (c) Since $f(x)$ is continuous at $x = 1$,

$\Rightarrow \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$ …..(i)

Now $\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} 2(1-h)+1 = 3$ *i.e.,* $\lim_{x\to 1^-} f(x) = 3$

Similarly, $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} 5(1+h)-2$ *i.e.,* $\lim_{x\to 1^+} f(x) = 3$

So according to equation (i), we have $k = 3$.

Example: 4 The value of k which makes $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ k, & x=0 \end{cases}$ continuous at $x = 0$ is

(a) 8 (b) 1 (c) –1 (d) None of these

Solution: (d) We have $\lim_{x\to 0} f(x) = \lim_{x\to 0} \sin\frac{1}{x}$ = An oscillating number which oscillates between –1 and 1.

Hence, $\lim_{x\to 0} f(x)$ does not exist. Consequently $f(x)$ cannot be continuous at $x = 0$ for any value of k.

Example: 5 The value of m for which the function $f(x) = \begin{cases} mx^2, & x \leq 1 \\ 2x, & x>1 \end{cases}$ is continuous at $x = 1$, is

(a) 0 (b) 1 (c) 2 (d) Does not exist

Solution: (c) LHL = $\lim_{x\to 1^-} f(x) = \lim_{h\to 0} m(1-h)^2 = m$

RHL = $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} 2(1+h) = 2$ and $f(1) = m$

Function is continuous at $x = 1$, $\therefore$ LHL = RHL = $f(1)$

Therefore $m = 2$.

Example: 6 If the function $f(x) = \begin{cases} (\cos x)^{1/x}, & x \neq 0 \\ k & , x = 0 \end{cases}$ is continuous at $x = 0$, then the value of k is

(a) 1 (b) –1 (c) 0 (d) e

Solution: (a) $\lim_{x\to 0} (\cos x)^{1/x} = k \Rightarrow \lim_{x\to 0} \frac{1}{x}\log(\cos x) = \log k \Rightarrow \lim_{x\to 0}\frac{1}{x}\lim_{x\to 0}\log\cos x = \log k \Rightarrow \lim_{x\to 0}\frac{1}{x}\times 0 = \log_e k \Rightarrow k = 1$.

1.3 Continuity of a Function in Open and Closed Interval

Open interval : A function $f(x)$ is said to be continuous in an open interval (a, b) iff it is continuous at every point in that interval.

Note : ❒ This definition implies the non-breakable behavior of the function $f(x)$ in the interval (a, b).

Closed interval : A function $f(x)$ is said to be continuous in a closed interval $[a, b]$ iff,

(1) f is continuous in (a, b)

(2) f is continuous from the right at 'a' i.e. $\lim_{x\to a^+} f(x) = f(a)$

(3) f is continuous from the left at 'b' i.e. $\lim_{x\to b^-} f(x) = f(b)$.

Example: 7 If the function $f(x) = \begin{cases} x + a^2\sqrt{2}\sin x & , \quad 0 \le x < \frac{®}{4} \\ x\cot x + b & , \quad \frac{®}{4} \le x < \frac{®}{2} \\ b\sin 2x - a\cos 2x & , \quad \frac{®}{2} \le x \le ® \end{cases}$, is continuous in the interval $[0, ®]$ then the values of (a, b) are

(a) (–1, –1) (b) (0, 0) (c) (–1, 1) (d) (1, –1)

Solution: (b) Since f is continuous at $x = \frac{®}{4}$; $\therefore f\left(\frac{®}{4}\right) = \underset{h\to 0}{f}\left(\frac{®}{4}+h\right) = \underset{h\to 0}{f}\left(\frac{®}{4}-h\right) \Rightarrow \frac{®}{4}(1) + b = \left(\frac{®}{4}-0\right) + a^2\sqrt{2}\sin\left(\frac{®}{4}-0\right)$

$\Rightarrow \frac{®}{4} + b = \frac{®}{4} + a^2\sqrt{2}\sin\frac{®}{4} \Rightarrow b = a^2\sqrt{2}.\frac{1}{\sqrt{2}} \Rightarrow b = a^2$

Also as f is continuous at $x = \frac{®}{2}$; $\therefore f\left(\frac{®}{2}\right) = \lim_{x\to\frac{®}{2}-0} f(x) = \lim_{h\to 0} f\left(\frac{®}{2}-h\right)$

$\Rightarrow b\sin 2\frac{®}{2} - a\cos 2\frac{®}{2} = \lim_{h\to 0}\left[\left(\frac{®}{2}-h\right)\cot\left(\frac{®}{2}-h\right)+b\right] \Rightarrow b.0 - a(-1) = 0 + b \Rightarrow a = b$.

Hence (0, 0) satisfy the above relations.

Example: 8 If the function $f(x) = \begin{cases} 1+\sin\frac{®x}{2} & \text{for } -\infty < x \le 1 \\ ax+b & \text{for } 1 < x < 3 \\ 6\tan\frac{x®}{12} & \text{for } 3 \le x < 6 \end{cases}$ is continuous in the interval $(-\infty, 6)$ then the values of a and b are respectively

(a) 0, 2 (b) 1, 1 (c) 2, 0 (d) 2, 1

Solution: (c) $\because$ The turning points for $f(x)$ are $x = 1, 3$.

So, $\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0}\left[1+\sin\frac{®}{2}(1-h)\right] = \left[1+\sin\left(\frac{®}{2}-0\right)\right] = 2$

Similarly, $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} a(1+h)+b = a+b$

$\because$ $f(x)$ is continuous at $x = 1$, so $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) = f(1)$

$\Rightarrow \quad 2 = a + b$ ....(i)

Again, $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{h\to 0} f(3-h) = \lim\limits_{h\to 0} a(3-h)+b = 3a+b$ and $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{h\to 0} f(3+h) = \lim\limits_{h\to 0} 6\tan\frac{®}{12}(3+h) = 6$

$f(x)$ is continuous in $(-\infty, 6)$, so it is continuous at $x = 3$ also, so $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x) = f(3)$

$\Rightarrow \quad 3a + b = 6$ ......(ii)

Solving (i) and (ii) $a = 2$, $b = 0$.

Trick : In above type of questions first find out the turning points. For example in above question they are $x = 1,3$. Now find out the values of the function at these points and if they are same then the function is continuous *i.e.*, in above problem.

$$f(x) = \begin{cases} 1+\sin\frac{®}{2}x & ; \quad -\infty < x \le 1, \quad f(1) = 2 \\ ax+b & ; \quad 1 < x < 3 \quad f(1) = a+b,\ f(3) = 3a+b \\ 6\tan\frac{® x}{12} & ; \quad 3 \le x < 6 \quad f(3) = 6 \end{cases}$$

Which gives $2 = a + b$ and $6 = 3a + b$ after solving above linear equations we get $a = 2,\ b = 0$.

Example: 9 If $f(x) = \begin{cases} x\sin x, & \text{when } 0 < x \le \frac{®}{2} \\ \frac{®}{2}\sin(® + x), & \text{when } \frac{®}{2} < x < ® \end{cases}$ then

(a) $f(x)$ is discontinuous at $x = \frac{®}{2}$ (b) $f(x)$ is continuous at $x = \frac{®}{2}$

(c) $f(x)$ is continuous at $x = 0$ (d) None of these

Solution: (a) $\lim\limits_{x\to \frac{®}{2}^-} f(x) = \frac{®}{2}$, $\lim\limits_{x\to\frac{®}{2}^+} f(x) = -\frac{®}{2}$ and $f\left(\frac{®}{2}\right) = \frac{®}{2}$.

Since $\lim\limits_{x+\frac{®}{2}^-} \ne \lim\limits_{x+\frac{®}{2}^+} f(x)$, $\therefore$ Function is discontinuous at $x = \frac{®}{2}$

Example: 10 If $f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2}, & \text{when } x < 0 \\ a, & \text{when } x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{(16+\sqrt{x})}-4}, & \text{when } x > 0 \end{cases}$ is continuous at $x = 0$, then the value of 'a' will be

(a) 8 (b) –8 (c) 4 (d) None of these

Solution: (a) $\lim\limits_{x\to 0^-} f(x) \lim\limits_{x\to 0^-}\left(\dfrac{2\sin^2 2x}{(2x)^2}\right)4 = 8$ and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}[(\sqrt{16+\sqrt{x}})+4] = 8$

Hence $a = 8$.

1.4 Continuous Function

(1) **A list of continuous functions :**

Function *f*(*x*)		Interval in which *f*(*x*) is continuous
(i)	Constant K	$(-\infty, \infty)$
(ii)	x^n, (n is a positive integer)	$(-\infty, \infty)$
(iii)	x^{-n} (n is a positive integer)	$(-\infty, \infty) - \{0\}$
(iv)	$\lvert x - a \rvert$	$(-\infty, \infty)$
(v)	$p(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_n$	$(-\infty, \infty)$
(vi)	$\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial in x	$(-\infty, \infty) - \{x : q(x) = 0\}$
(vii)	$\sin x$	$(-\infty, \infty)$
(viii)	$\cos x$	$(-\infty, \infty)$
(ix)	$\tan x$	$(-\infty, \infty) - \{(2n + 1)$®$/2 : n \in I\}$
(x)	$\cot x$	$(-\infty, \infty) - \{n$® $: n \in I\}$
(xi)	$\sec x$	$(-\infty, \infty) - \{(2n + 1)$®$/2 : n \in I\}$
(xii)	$\text{cosec}\, x$	$(-\infty, \infty) - \{n$® $: n \in I\}$
(xiii)	e^x	$(-\infty, \infty)$
(xiv)	$\log_e x$	$(0, \infty)$

(2) **Properties of continuous functions :** Let $f(x)$ and $g(x)$ be two continuous functions at $x = a$. Then

(i) $cf(x)$ is continuous at $x = a$, where c is any constant

(ii) $f(x) \pm g(x)$ is continuous at $x = a$.

(iii) $f(x) \,.\, g(x)$ is continuous at $x = a$.

(iv) $f(x) / g(x)$ is continuous at $x = a$, provided $g(a) \neq 0$.

Important Tips

☞ *A function $f(x)$ is said to be continuous if it is continuous at each point of its domain.*

☞ *A function $f(x)$ is said to be everywhere continuous if it is continuous on the entire real line R i.e. $(-\infty, \infty)$. eg. polynomial function e^x, $\sin x, \cos x$, constant, x^n, $\lvert x - a \rvert$ etc.*

☞ *Integral function of a continuous function is a continuous function.*

☞ *If g(x) is continuous at $x = a$ and f(x) is continuous at $x = g(a)$ then (fog) (x) is continuous at $x = a$.*

☞ *If f(x) is continuous in a closed interval [a, b] then it is bounded on this interval.*

- If $f(x)$ is a continuous function defined on [a, b] such that $f(a)$ and $f(b)$ are of opposite signs, then there is atleast one value of x for which $f(x)$ vanishes. i.e. if $f(a)>0$, $f(b) < 0$ Ĝ $\exists c \in (a, b)$ such that $f(c) = 0$.
- If $f(x)$ is continuous on [a, b] and maps [a, b] into [a, b] then for some x Č [a, b] we have $f(x) = x$.

(3) **Continuity of composite function :** If the function $u = f(x)$ is continuous at the point $x = a$, and the function $y = g(u)$ is continuous at the point $u = f(a)$, then the composite function $y = (gof)(x) = g(f(x))$ is continuous at the point $x = a$.

1.5 Discontinuous Function

(1) **Discontinuous function :** A function 'f' which is not continuous at a point $x = a$ in its domain is said to be discontinuous there at. The point 'a' is called a point of discontinuity of the function.

The discontinuity may arise due to any of the following situations.

(i) $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ or both may not exist

(ii) $\lim_{x \to a^+} f(x)$ as well as $\lim_{x \to a^-} f(x)$ may exist, but are unequal.

(iii) $\lim_{x \to a^+} f(x)$ as well as $\lim_{x \to a^-} f(x)$ both may exist, but either of the two or both may not be equal to $f(a)$.

Important Tips

- *A function f is said to have removable discontinuity at x = a if* $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$ *but their common value is not equal to f(a).*

  Such a discontinuity can be removed by assigning a suitable value to the function f at x = a.
- *If* $\lim_{x \to a} f(x)$ *does not exist, then we can not remove this discontinuity. So this become a non-removable discontinuity or essential discontinuity.*
- *If f is continuous at x = c and g is discontinuous at x = c, then*

  (a) f + g and f – g are discontinuous (b) f.g may be continuous
- *If f and g are discontinuous at x = c, then f + g, f – g and fg may still be continuous.*
- *Point functions (domain and range consists one value only) is not a continuous function.*

Example: 11 The points of discontinuity of $y = \frac{1}{u^2 + u - 2}$ where $u = \frac{1}{x-1}$ is

(a) $\frac{1}{2}, 1, 2$ (b) $\frac{-1}{2}, 1, -2$ (c) $\frac{1}{2}, -1, 2$ (d) None of these

Solution: (a) The function $u = f(x) = \frac{1}{x-1}$ is discontinuous at the point $x = 1$. The function $y = g(x) = \frac{1}{u^2+u-2} = \frac{1}{(u+2)(u-1)}$ is discontinuous at $u = -2$ and $u = 1$

when $u = -2 \Rightarrow \frac{1}{x-1} = -2 \Rightarrow x = \frac{1}{2}$, when $u = 1 \Rightarrow \frac{1}{x-1} = 1 \Rightarrow x = 2$.

Hence, the composite $y = g(f(x))$ is discontinuous at three points $= \frac{1}{2}, 1, 2$.

Example: 12 The function $f(x) = \frac{\log(1+ax) - \log(1-bx)}{x}$ is not defined at $x = 0$. The value which should be assigned to f at $x = 0$ so that it is continuous at $x = 0$, is

(a) $a - b$ (b) $a + b$ (c) $\log a + \log b$ (d) $\log a - \log b$

Solution: (b) Since limit of a function is $a + b$ as $x \to 0$, therefore to be continuous at $x = 0$, its value must be $a + b$ at $x = 0 \Rightarrow f(0) = a + b$.

Example: 13 If $f(x)=\begin{cases}\dfrac{x^2-4x+3}{x^2-1}, & \text{for } x\neq 1\\ 2 & ,\text{for } x=1\end{cases}$, then

(a) $\lim\limits_{x\to 1^+} f(x)=2$ (b) $\lim\limits_{x\to 1^-} f(x)=3$

(c) $f(x)$ is discontinuous at $x=1$ (d) None of these

Solution: (c) $f(1)=2,\ f(1+)=\lim\limits_{x\to 1+}\dfrac{x^2-4x+3}{x^2-1}=\lim\limits_{x\to 1+}\dfrac{(x-3)}{(x+1)}=-1$

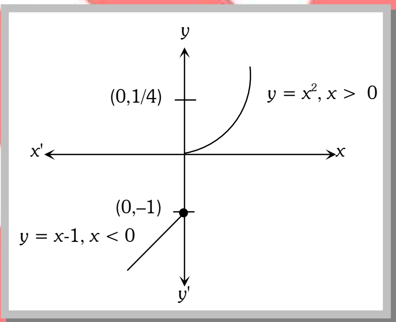
$f(1-)=\lim\limits_{x\to 1-}\dfrac{x^2-4x+3}{x^2-1}=-1\Rightarrow f(1)\neq f(1-)$. Hence the function is discontinuous at $x=1$.

Example: 14 If $f(x)=\begin{cases}x-1, x<0\\ \dfrac{1}{4}, x=0\\ x^2, x>0\end{cases}$, then

(a) $\lim\limits_{x\to 0^+} f(x)=1$ (b) $\lim\limits_{x\to 0^-} f(x)=1$

(c) $f(x)$ is discontinuous at $x=0$ (d) None of these

Solution: (c) Clearly from curve drawn of the given function $f(x)$, it is discontinuous at $x=0$.



Example: 15 Let $f(x)=\begin{cases}(1+|\sin x|)^{\frac{a}{|\sin x|}} & , -\dfrac{\text{®}}{6}<x<0\\ b & , x=0\\ e^{\frac{\tan 2x}{\tan 3x}} & , 0<x<\dfrac{\text{®}}{6}\end{cases}$, then the values of a and b if f is continuous at $x=0$, are respectively

(a) $\dfrac{2}{3},\dfrac{3}{2}$ (b) $\dfrac{2}{3},e^{2/3}$ (c) $\dfrac{3}{2},e^{3/2}$ (d) None of these

Solution: (b) $f(x)=\begin{cases}(1+|\sin x|)^{\frac{a}{|\sin x|}} & ; -\left(\dfrac{\text{®}}{6}\right)<x<0\\ b & ; x=0\\ e^{\frac{\tan 2x}{\tan 3x}} & ; 0<x<\left(\dfrac{\text{®}}{6}\right)\end{cases}$

For $f(x)$ to be continuous at $x=0$

$\Rightarrow \lim\limits_{x\to 0^-} f(x)=f(0)=\lim\limits_{x\to 0^+} f(x)\ \Rightarrow\ \lim\limits_{x\to 0}(1+|\sin x|)^{\frac{a}{|\sin x|}}=e^{\lim\limits_{x\to 0^-}\left(|\sin x|\frac{a}{|\sin x|}\right)}=e^a$

Now, $\lim\limits_{x\to 0^+} e^{\tan 2x/\tan 3x}=\lim\limits_{x\to 0^+} e^{\left(\frac{\tan 2x}{2x}.2x\right)\Big/\left(\frac{\tan 3x}{3x}.3x\right)}=\lim\limits_{x\to 0^+} e^{2/3}=e^{2/3}$.

$\therefore\ e^a = b = e^{2/3} \Rightarrow a = \frac{2}{3}$ and $b = e^{2/3}$.

Example: 16 Let $f(x)$ be defined for all $x > 0$ and be continuous. Let $f(x)$ satisfy $f\left(\frac{x}{y}\right) = f(x) - f(y)$ for all x, y and $f(e) = 1$, then

(a) $f(x) = \text{In } x$ (b) $f(x)$ is bounded (c) $f\left(\frac{1}{x}\right) \to 0$ as $x \to 0$ (d) $xf(x) \to 1$ as $x \to 0$

Solution: (a) Let $f(x) = \text{In }(x), x > 0$ $f(x) = \text{In }(x)$ is a continuous function of x for every positive value of x.

$f\left(\frac{x}{y}\right) = \text{In}\left(\frac{x}{y}\right) = \text{In }(x) - \text{In }(y) = f(x) - f(y)$.

Example: 17 Let $f(x) = [x]\sin\left(\frac{\circledR}{[x+1]}\right)$, where [.] denotes the greatest integer function. The domain of f is ….. and the points of discontinuity of f in the domain are

(a) $\{x \in R \mid x \in [-1, 0)\}, I - \{0\}$ (b) $\{x \in R \mid x \notin [1, 0)\}, I - \{0\}$

(c) $\{x \in R \mid x \notin [-1, 0)\},\ I - \{0\}$ (d) None of these

Solution: (c) Note that $[x+1] = 0$ if $0 \le x + 1 < 1$

i.e. $[x+1] - 0$ if $-1 \le x < 0$.

Thus domain of f is $R - [-1, 0) = \{x \notin [-1, 0)\}$

We have $\sin\left(\frac{\circledR}{[x+1]}\right)$ is continuous at all points of $R - [-1, 0)$ and $[x]$ is continuous on $R - I$, where I denotes the set of integers.

Thus the points where f can possibly be discontinuous are……, $-3, -2, -1, 0\,1, 2,$ ……….. But for $0 \le x < 1, [x] = 0$ and $\sin\left(\frac{\circledR}{[x+1]}\right)$ is defined.

Therefore $f(x) = 0$ for $0 \le x < 1$.

Also $f(x)$ is not defined on $-1 \le x < 0$.

Therefore, continuity of f at 0 means continuity of f from right at 0. Since f is continuous from right at $0, f$ is continuous at 0. Hence set of points of discontinuities of f is $I - \{0\}$.

Example: 18 If the function $f(x) = \frac{2x - \sin^{-1} x}{2x + \tan^{-1} x}, (x \ne 0)$ is continuous at each point of its domain, then the value of $f(0)$ is

(a) 2 (b) 1/3 (c) 2/3 (d) – 1/3

Solution: (b) $f(x) = \lim_{x \to 0}\left(\frac{2x - \sin^{-1} x}{2x + \tan^{-1} x}\right) = f(0)$ $,\left(\frac{0}{0}\text{ form}\right)$

Applying L-Hospital's rule, $f(0) = \lim_{x \to 0} \frac{\left(2 - \frac{1}{\sqrt{1 - x^2}}\right)}{\left(2 + \frac{1}{1 + x^2}\right)} = \frac{2 - 1}{2 + 1} = \frac{1}{3}$

Trick : $f(0) = \lim_{x \to 0} \frac{2x - \sin^{-1} x}{2x + \tan^{-1} x} \Rightarrow \lim_{x \to 0} \frac{2 - \frac{\sin^{-1} x}{x}}{2 + \frac{\tan^{-1} x}{x}} = \frac{2-1}{2+1} = \frac{1}{3}$.

Example: 19 The values of A and B such that the function $f(x) = \begin{cases} -2\sin x, & x \le -\frac{\pi}{2} \\ A\sin x + B, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \cos x, & x \ge \frac{\pi}{2} \end{cases}$, is continuous everywhere are

(a) $A = 0, B = 1$ (b) $A = 1, B = 1$ (c) $A = -1, B = 1$ (d) $A = -1, B = 0$

Solution: (c) For continuity at all $x \in R$, we must have $f\left(-\frac{\pi}{2}\right) = \lim_{x \to (-\pi/2)^-} (-2\sin x) = \lim_{x \to (-\pi/2)^+} (A\sin x + B)$

$\Rightarrow 2 = -A + B$ .....(i)

and $f\left(\frac{\pi}{2}\right) = \lim_{x \to (\pi/2)^-} (A\sin x + B) = \lim_{x \to (\pi/2)^+} (\cos x)$

$\Rightarrow 0 = A + B$ ....(ii)

From (i) and (ii), $A = -1$ and $B = 1$.

Example: 20 If $f(x) = \frac{x^2 - 10x + 25}{x^2 - 7x + 10}$ for $x \ne 5$ and f is continuous at $x = 5$, then $f(5) =$

(a) 0 (b) 5 (c) 10 (d) 25

Solution: (a) $f(5) = \lim_{x \to 5} f(x) = \lim_{x \to 5} \frac{x^2 - 10x + 25}{x^2 - 7x + 10} = \lim_{x \to 5} \frac{(x-5)^2}{(x-2)(x-5)} = \frac{5-5}{5-2} = 0$.

Example: 21 In order that the function $f(x) = (x+1)^{\cot x}$ is continuous at $x = 0$, $f(0)$ must be defined as

(a) $f(0) = \frac{1}{e}$ (b) $f(0) = 0$ (c) $f(0) = e$ (d) None of these

Solution: (c) For continuity at 0, we must have $f(0) = \lim_{x \to 0} f(x)$

$= \lim_{x \to 0} (x+1)^{\cot x} = \lim_{x \to 0} \left\{ (1+x)^{\frac{1}{x}} \right\}^{x \cot x} = \lim_{x \to 0} \left\{ (1+x)^{\frac{1}{x}} \right\}^{\lim_{x \to 0} \left(\frac{x}{\tan x} \right)} = e^1 = e$.

Example: 22 The function $f(x) = \sin |x|$ is

(a) Continuous for all x (b) Continuous only at certain points

(c) Differentiable at all points (d) None of these

Solution: (a) It is obvious.

Example: 23 If $f(x) = \begin{cases} \frac{1 - \sin x}{\pi - 2x}, & x \ne \frac{\pi}{2} \\ \alpha, & x = \frac{\pi}{2} \end{cases}$ be continuous at $x = \frac{\pi}{2}$, then value of α is

(a) -1 (b) 1 (c) 0 (d) 2

Solution: (c) $f(x)$ is continuous at $x = \frac{\pi}{2}$, then $\lim_{x \to \pi/2} f(x) = f(0)$ or $\alpha = \lim_{x \to \pi/2} \frac{1 - \sin x}{\pi - 2x}$, $\left(\frac{0}{0} \text{ form} \right)$

Applying L-Hospital's rule, $\alpha = \lim_{x \to \pi/2} \frac{-\cos x}{-2} \Rightarrow \alpha = \lim_{x \to \pi/2} \frac{\cos x}{2} = 0$.

Example: 24 If $f(x)=\dfrac{2-\sqrt{x+4}}{\sin 2x}$; $(x \neq 0)$, is continuous function at $x=0$, then $f(0)$ equals

(a) $\dfrac{1}{4}$ (b) $-\dfrac{1}{4}$ (c) $\dfrac{1}{8}$ (d) $-\dfrac{1}{8}$

Solution: (d) If $f(x)$ is continuous at $x=0$, then, $f(0)=\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\dfrac{2-\sqrt{x+4}}{\sin 2x}$, $\left(\dfrac{0}{0}\text{ form}\right)$

Using L–Hospital's rule, $f(0)=\lim\limits_{x\to 0}\dfrac{\left(-\dfrac{1}{2\sqrt{x+4}}\right)}{2\cos 2x}=-\dfrac{1}{8}$.

Example: 25 If function $f(x)=\begin{cases} x & \text{if } x \text{ is rational} \\ 1-x & \text{if } x \text{ is irrational}\end{cases}$, then $f(x)$ is continuous at ...... number of points

(a) ∞ (b) 1 (c) 0 (d) None of these

Solution: (c) At no point, function is continuous.

Example: 26 The function defined by $f(x)=\begin{cases}\left(x^2+e^{\frac{1}{2-x}}\right)^{-1} & , \; x\neq 2 \\ k & , \; x=2\end{cases}$, is continuous from right at the point $x=2$, then k is equal to

(a) 0 (b) 1/4 (c) –1/4 (d) None of these

Solution: (b) $f(x)=\left[x^2+e^{\frac{1}{2-x}}\right]^{-1}$ and $f(2)=k$

If $f(x)$ is continuous from right at $x=2$ then $\lim\limits_{x\to 2^+} f(x)=f(2)=k$

$\Rightarrow \lim\limits_{x\to 2^+}\left[x^2+e^{\frac{1}{2-x}}\right]^{-1}=k \Rightarrow k=\lim\limits_{h\to 0} f(2+h) \Rightarrow k=\lim\limits_{h\to 0}\left[(2+h)^2+e^{\frac{1}{2-(2+h)}}\right]^{-1}$

$\Rightarrow k=\lim\limits_{h\to 0}\left[4+h^2+4h+e^{-1/h}\right]^{-1} \Rightarrow k=[4+0+0+e^{-\infty}]^{-1} \Rightarrow k=\dfrac{1}{4}$.

Example: 27 The function $f(x)=\dfrac{1-\sin x+\cos x}{1+\sin x+\cos x}$ is not defined at $x=®$. The value of $f(®)$, so that $f(x)$ is continuous at $x=®$, is

(a) $-\dfrac{1}{2}$ (b) $\dfrac{1}{2}$ (c) -1 (d) 1

Solution: (c) $\lim\limits_{x\to ®} f(x)=\lim\limits_{x\to ®}\dfrac{2\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}=\lim\limits_{x\to ®}\dfrac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}=\lim\limits_{x\to ®}\tan\left(\dfrac{®}{4}-\dfrac{x}{2}\right)$

$\therefore$ At $x=®$, $f(®)=-\tan\dfrac{®}{4}=-1$.

Example: 28 If $f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x} & ,\text{for } -1\leq x<0 \\ 2x^2+3x-2 & ,\text{for } 0\leq x\leq 1\end{cases}$ is continuous at $x=0$, then $k=$

(a) -4 (b) -3 (c) -2 (d) -1

Solution: (c) L.H.L. $= \lim_{x \to 0^-} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x} = k$

R.H.L. $= \lim_{x \to 0^+} (2x^2 + 3x - 2) = -2$

Since it is continuous, hence L.H.L = R.H.L $\Rightarrow k = -2$.

Example: 29 The function $f(x) = |x| + \frac{|x|}{x}$ is

(a) Continuous at the origin

(b) Discontinuous at the origin because $|x|$ is discontinuous there

(c) Discontinuous at the origin because $\frac{|x|}{x}$ is discontinuous there

(d) Discontinuous at the origin because both $|x|$ and $\frac{|x|}{x}$ are discontinuous there

Solution: (c) $|x|$ is continuous at $x = 0$ and $\frac{|x|}{x}$ is discontinuous at $x = 0$

$\therefore f(x) = |x| + \frac{|x|}{x}$ is discontinuous at $x = 0$.

*****