Answers 6.5

(1) $x_1 = 7/5,\ x_2 = 3/5$ and $\max z = 59/5$

(2) $x_1 = 4,\ x_2 = 1$ and $\max z = 10$

(3) unbounded solution.

(4) $x_1 = 0,\ x_2 = 11/12,\ x_3 = 5/4$ and $\min z = 67/12$

(5) $x_1 = 1/3,\ x_2 = 1/3$ and $\max z = 4/3$

(6) $x_1 = 7,\ x_2 = 0$ and $\max z = 21$

(7) $x_1 = 45,\ x_2 = 15,\ x_3 = 0$ and $\max z = 675$

(8) $x_1 = 4,\ x_2 = 2$ and $\max z = 10$

(9) $x_1 = 6/5,\ x_2 = 8/5$ and $\max z = 22/5$

(10) $\min z = -11$ where $x_1 = 4,\ x_2 = 5, x_3 = 0$

6.6 ASSIGNMENT PROBLEM

6.6.1 Introduction

There are special cases of LPP whose solution can be obtained by special techniques. They are easier to apply. One of such problems is the assignment problem. It finds many applications in allocation and scheduling.

Suppose we have n jobs (or tasks) that can be performed by n machines (or persons). Assume that each machine can perform each job with varying degree of efficiency (or time or cost). The assignment problem is to find an assignment of these jobs to the machines in such a way that a job is assigned to one and only one machine and a machine is assigned only one job so as to minimize the cost (or time) or maximize the return (or efficiency). Generally, an assignment problem is considered to be a minimization problem since a maximization problem can be converted to a minimization problem.

The solution to this problem was developed by an American Mathematician Harold Kuhn, based on a theorem, first stated by the German Mathematician Dines Koenig and first proved by the Hungarian Mathematician E. Egervary. Thus the method is known as Hungarian Technique. The Koenig's theorem states that in any matrix consisting of elements of two types, the maximum number of independent elements of any one type is equal to the minimum number of lines drawn through rows and columns which cover all elements of that type. The Algorithm stated in this section was developed by the American Mathematician James Munkers as modification of Kuhn's "Hungarian Method" in late 1950. The Koenig's theorem gives that the maximum number of independent zeros in any matrix is equal to the minimum number of lines through rows and columns which cover all zeros in the matrix.

Mathematical Formation of Assignment Problem

Let *Cij* denote the cost if the i[th] job is assigned to the j[th] machine. The $n \times n$ matrix $[c_{i,j}]$ is called the cost matrix.

$$\text{Let } x_{ij} = \begin{cases} 1 & \text{if the ith job is assigned to jth machine} \\ 0 & \text{otherwise} \end{cases}$$

Mathematically, the assignment problem can be expressed as,

Minimize $z = \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij}$

Subject to (i) $\sum_{i=1}^{n} x_{ij} = 1,\ j = 1,2,\dots, n$

(One machine is assigned only one job.)

(ii) $\sum_{j=1}^{n} x_{ij} = 1,\ i = 1, 2, \dots, n$

(One job is assigned to only one machine.)

with x_{ij} is either 0 or 1.

The Assignment Algorithm [Hungarian Technique-Minimization case]

Step 1

Check whether the number of rows and columns of the cost matrix are equal (i.e. the problem is balanced). If not balance it by adding suitable number of dummy rows or columns with zero cost.

Step 2

Subtract from every row its smallest element. Subtract from every column its smallest element.

Step 3

Draw the least number of horizontal and vertical lines to cover all the zeros. If the number of lines is equal to the number of rows or columns, then go to step 5.

Step 4

Identify the smallest uncovered element (not covered by the lines). Subtract it from all the uncovered elements and add the same to the element at the intersection of lines. We get a reduced matrix. Go to step 3.

Step 5

Examine the rows successively to find one with exactly one unmarked zero. Make an assignment there by encircling the zero by O- Cancel all zeros in that column- to show that they cannot be considered for future assignment. Continue until all the rows have been examined. Repeat the above procedure for columns.

Step 6

Repeat Step 5 successively until

(i) no unmarked zero is left or

(ii) There lie more than one unmarked zero in one row or column.

In case (i), the algorithm stops. In case (ii), encircle any one zero arbitrarily and cancel all zeros in that row and column. Go to Step 5.

Step 7

we now have exactly one encircled zero in each row and column. The assignment schedule corresponding to these zeros is the optimal assignment.

Note

In case (ii) of Step 6, the assignment problem will have alternate optimum solution depending upon the choice of the zero (tie break)

In the case of maximization assignment problem, it can be converted into minimization problem by using any one of the following

(a) Multiply every element of the matrix by –1

(b) Subtract every element of the matrix from the maximum element.

Example 30

Solve the following.

		Machines				
		I	II	III	IV	V
	1	11	17	8	16	20
	2	9	7	12	6	15
Jobs	3	13	16	15	12	16
	4	21	24	17	28	26
	5	14	10	12	11	15

(K.U. Nov/Dec. 2002)

Solution

Subtract the smallest element of each row from all its elements (subtract 8 from row 1, 6 from row 2, 12 from row 3, 17 from row 4, 10 from row 5)

3	9	0	8	12
3	1	6	0	9
1	4	3	0	4
4	7	0	11	9
4	0	2	1	5

Subtract the smallest element of each column from all its elements (Subtract 1 from column 1, zero from column 2, 0 from column 3, 0 from column 4 and 4 from column 5)

Draw minimum number of lines to cover the zeros. (One can use the idea of Step 5 and 6 for drawing the lines i.e. identify a row (or column) with exactly one unmarked zero is found. Draw a line in that column (or row). If all rows and columns have more than one zero, draw a line which covers most number of zeros. Repeat the above procedure till all zeros are covered)

Number lines (= 4) $\neq$ number of rows (= 5). The minimum among the uncovered elements is 1. Subtract it from all uncovered elements and add it where the lines cross.

2	9	0	8	8	
2	1	6	0	5	
0	4	3	0	0	L_4
3	7	0	11	5	
3	0	2	1	1	
	L_3	L_1	L_2		

All other elements written as in the above matrix. Draw the minimum number of lines.

1	9	0	8	7	
1	1	6	0	4	
0	5	4	1	0	L_3
2	7	0	11	4	
2	0	2	1	0	L_4
		L_1	L_2		

Here also, number of lines = 4 $\neq$ 5. The minimum among the uncovered elements is 1. Repeat step 4.

0	8	0	8	6	
0	0	6	0	3	L_2
0	5	5	2	0	L_5
1	6	0	11	3	
2	0	3	2	0	L_4
L_3		L_1			

Number of lines - Number of rows =5. Thus we can make assignments. For this write a matrix with only the zeros in their positions

0		⊗		
⊗	⊗		0	
⊗				0
		0		
	0			⊗

The row 4 contains only one zero. Mark it and cancel all zeros in that column. Now row 1 contains only one unmarked zero. Mark it and cancel all zeros in that column. Row 3 contains only one zero. Mark it and cancel all zeros in that column. Row 5 contains only one zero. Mark it and cancel all zeros in that column. Now we have row 2 with exactly one zero. Mark it. The assignment and the corresponding costs are as follows.

Assignment	Cost
(1,I)	11
(2,IV)	6
(3,V)	16
(4,III)	17
(5,II)	10
	Total cost= 60

Example 31

Solve the following assignment problem

	1	2	3	4
A	10	12	19	1
B	5	10	7	8
C	12	14	13	11
D	8	15	11	9

(K.U. Oct/Nov. 2000)

Solution:

9	11	18	0
0	5	2	3
1	3	2	0
0	7	3	1

→

9	8	16	0	
0	2	0	3	L_4
1	0	0	0	L_3
0	4	1	1	
L_2			L_1	

Number lines = 4 = Number of rows.

			0
⊗		0	
	0	⊗	⊗
0			

Assignment	Cost
(A,4)	1
(B,3)	7
(C,2)	14
(D,I)	8
	Total cost= 30

Example 32

Solve

	J_1	J_2	J_3	J_4
P_1	20	13	7	5
P_2	25	18	13	10
P_3	31	23	18	15
P_4	45	40	23	21

(K.U. April/May. 2001)

Solution

15	8	2	0
15	8	3	0
16	8	3	0
24	19	2	0

→

0	0	0	0	L_1
0	0	1	0	L_2
1	0	1	0	L_3
9	11	0	0	L_4

Number lines = 4 = Number of rows.

All rows and columns have more than one zero. Hence there are alternate optimum solutions. First choose one zero and cancel all zeros in that row and column to break the tie. say (1,1).

0	⊗	⊗	⊗
⊗	0		⊗
	⊗		0
		0	⊗

Next (4,3) is chosen. Since in column 3, there is only one unmarked zero. Again there is a tie. Break the tie arbitrarily, say (2,2). Cancel all zeros in its column and row. The only unmarked zero left is (3,4). Mark it. Thus the optimum solution is (P_1, J_1), (P_2, J_2), (P_3, J_4) and (P_4, J_3). Minimum cost = 20 + 18 + 15 + 23 = 76.

Alternate solutions of the above problem

1. Instead of choosing (1,1) for breaking tie, let us choose (2,1). Again there is a tie. Break it by choosing (1,2). The Solution is

⊗	0	⊗	⊗
0	⊗		⊗
	⊗		0
		0	⊗

(P_1,J_2), (P_2, J_1), (P_3,J_4), (P_4, J_3) and Corresponding Minimum cost = 20 + 18 + 15 + 23 = 76.

2. Another Solution

⊗	⊗	0	⊗
0	⊗		⊗
	0		⊗
		⊗	0

Assignment	Cost
(P_1,J_3)	7
(P_2,J_1)	25
(P_3,J_2)	23
(P_4,J_4)	21
Total cost=	76

3. Another Solution

⊗	⊗	⊗	0
0	⊗		⊗
	0		⊗
		0	⊗

Assignment	Cost
(P_1,J_4)	5
(P_2,J_1)	25
(P_3,J_2)	23
(P_4,J_3)	23
Total cost=	76

Example 33

Solve the following Assignment problem (K.U. April/May. 2003)

	I	II	III	IV
A	2	3	4	5
B	4	5	6	7
C	7	8	9	8
D	3	5	8	4

Solution

Row reduction

0	1	2	3
0	1	2	3
0	1	2	1
0	2	5	1

→

0	0	0	2	L_2
0	0	0	2	L_3
0	0	0	0	L_1
0	1	3	0	L_4

Number of lines = 4

= Number of rows

0	⊗	⊗	
⊗	0	⊗	
⊗	⊗	0	⊗
⊗			0

Assignment	Cost
(A,I)	2
(B,II)	5
(C,III)	9
(D,IV)	4
Total cost= 20	

Example 34 Assign the jobs such that the profit is maximum.

	A	B	C	D
J_1	62	78	50	101
J_2	71	84	61	73
J_3	87	92	111	71
J_4	48	64	87	77

Solution

This is a maximization problem. But we can convert it into a minimization problem by subtracting every element of the given cost matrix from the maximum element 111. So the initial matrix is

49	33	61	10
40	27	50	38
24	19	0	40
63	47	24	34

→

Row Reduction

39	23	51	0
13	0	23	11
24	19	0	40
39	23	0	10

Column Reduction

26	23	51	0	
0	0	23	11	L_3
11	19	0	40	
26	23	0	10	
		L_2	L_1	

15	12	51	0	
0	0	34	22	L_3
0	8	9	40	L_4
15	12	0	10	
		L_2	L_1	

			0
⊗	0		
0		⊗	
		0	

Optimal Assignment is (J_1, D), (J_2, B), (J_3, A) and (J_4, C)

Maximum profit $= 101 + 84 + 87 + 87 = 359$

Example 35 Assign the jobs such that the profit is maximum

		Machine				
		A	B	C	D	E
Job	1	32	38	40	28	40
	2	40	24	28	21	36
	3	41	27	33	30	37
	4	22	38	41	36	36
	5	29	33	40	35	39

Solution

Maximum element = 41. Subtract every element from 41, we get

9	3	1	13	1
1	17	13	20	5
0	14	8	11	4
19	3	0	5	5
12	8	1	6	2

Row Reduction

8	2	0	12	0
0	16	12	19	4
0	14	8	11	4
19	3	0	5	5
11	7	0	5	1

Column Reduction

8	0	0	7	0	L_2
0	14	12	14	4	
0	12	8	6	4	
19	1	0	0	5	L_3
11	5	0	0	1	L_4
L_1					

Number of lines = 4

Number of rows = 5

Modified Matrix

12	0	0	7	0	L_1
0	10	8	10	0	L_2
0	8	4	2	0	L_3
13	4	0	0	5	L_4
15	6	0	0	1	L_5

Number of lines = 5

Number of rows = 5

	0	⊠		⊠
0				⊠
⊠				0
		0	⊠	
		⊠	0	

Assignment	Cost
(1, B)	38
(2, A)	40
(3, E)	37
(4, C)	41
(5, D)	35
Maximum profit =	191

Example 36 Solve the following assignment problem.

	A	*B*	*C*	*D*	*E*
1	2.5	5	1	5	1
2	2	5	1.5	7	3
3	3	6.5	2	8	3
4	3.5	7	2	9	4.5
5	4	7	3	9	6
6	6	9	5	10	6

Solution

The given problem is an unbalanced assignment problem (since number of origins $\neq$ number of destinations). So we add a dummy sixth column 'F' in the cost matrix so as to get the following balanced assignment problem.

	A	5	C	*D*	F	F
1	2.5	5	1	5	1	0
2	2	5	1.5	7	3	0
3	3	6.5	2	8	3	0
4	3.5	7	2	9	4.5	0
5	4	7	3	9	6	0
6	6	9	5	10	6	0

Column Reduction

-0.5	0	0	0	0	0	L_3
0	0	0.5	2	2	0	L_2
1	1.5	1	3	2	0	
1.5	2	1	4	3.5	0	
2	2	2	4	5	0	
4	4	4	5	5	0	
					L_1	

Number of lines = 3 ≠ Number of rows

First modified matrix

-0.5	0	0	0	0	1	L_2
0	0	-0.5	2	2	1	L_3
0	0.5	0	2	1	0	L_4
-0.5	1	0	3	2.5	0	L_5
1	1	1	3	4	0	
3	3	3	4	4	0	
					L_1	

Number of lines = 5
Number of rows = 6

Second modified matrix

-0.5	0	0	0	0	2	L_3
0	0	0.5	2	2	2	
0	0.5	0	2	1	1	
0.5	1	0	3	2.5	1	
0	0	0	2	3	0	
2	2	2	3	3	0	
L_1	L_2	L_1			L_2	

Number of lines = 5 ≠ Number of rows

Third modified matrix

1.5	1	1	0	0	3	L_3
0	0	.5	1	1	2	
0	.5	0	1	0	2	
.5	1	0	2	1.5	1	
0	0	0	1	2	0	
2	2	2	2	2	0	
L_4	L_5	L_1		L_6	L_2	

Number of lines = 6
Number of rows = 6

	A	B	C	D	E	F
1				0	⊗	
2	0	⊗				
3	⊗		⊗		0	
4			0.			
5	⊗	0	⊗			⊗
6						0

Assignment	Cost
(1, D)	5
(2, A)	2
(3, E)	3
(4, C)	2
(5, B)	7
(6, F)	0
Maximum profit =	19

Note that the job 6 is declined since it is assigned to the dummy machine F.

Exercise 6.7

(I) Solve the following Assignment problems

(1)

	A	B	C	D	E
I	10	15	13	15	16
II	13	9	18	13	10
III	10	9	12	12	12
IV	15	11	9	9	12
V	11	9	10	14	12

(2)

	A	B	C	D	E
I	13	9	11	21	14
II	16	7	17	24	10
III	15	12	8	17	12
IV	12	6	16	28	11
V	16	15	20	26	15

(3)

	I	II	III	IV	V
A	6	5	8	11	16
B	1	13	16	1	10
C	16	11	8	8	8
D	9	14	12	10	16
E	10	13	11	8	16

(4)

	I	II	III	IV	V
A	10	5	13	15	16
B	3	9	18	13	6
C	10	7	2	2	2
D	5	11	9	7	12
E	7	9	10	4	12

(5)

	I	II	III	IV	V
A	3	5	10	15	8
B	4	7	15	18	8
C	8	12	20	20	12
D	5	5	8	10	6
E	10	10	15	25	10

(6)

	I	II	III	IV
A	5	7	11	6
B	8	5	9	6
C	4	7	10	7
D	10	4	8	3

(7)

	I	II	III	IV	V
1	15	10	25	25	10
2	1	8	10	20	2
3	8	9	17	20	10
4	14	10	25	27	15
5	10	8	25	27	12

(II) Assign the jobs such that the profit is maximum in following problems

(i)

		I	II	III	IV
Jobs	A	16	10	14	11
	B	14	11	15	15
	C	15	15	13	12
	D	13	12	14	15

(ii)

		I	II	III	IV	V
Jobs	A	5	11	10	12	4
	B	2	4	6	3	5
	C	3	12	5	14	6
	D	6	14	4	11	7
	E	7	9	8	12	5

(iii)

		I	II	III	IV
Jobs	A	42	35	28	21
	B	30	25	20	15
	C	30	25	20	15
	D	24	20	16	12

(iv)

		I	II	III	IV	V
Jobs	A	62	78	50	101	82
	B	70	85	60	75	55
	C	88	96	118	85	71
	D	48	64	87	77	80
	E	60	70	98	66	83

(III) Solve the following unbalanced assignment problems

(a)

	I	II	III	IV	V	VI
A	12	10	15	22	18	8
B	10	18	25	15	16	12
C	11	10	3	8	5	9
D	6	14	10	13	13	12
E	8	12	11	7	13	10

(b)

	I	II	III	IV
A	18	24	28	32
B	8	13	17	19
C	10	15	19	22

(c)

	I	II	III	IV	V
A	6	2	5	2	6
B	2	5	8	7	7
C	7	8	6	9	8
D	6	2	3	4	5
E	9	3	8	9	7
F	4	7	4	6	8

(d)

	I	II	III	IV
A	3	6	2	6
B	7	1	4	4
C	3	8	5	8
D	6	4	3	7
E	5	2	4	3
F	5	7	6	2

Answers 6.7

I

(1) (I, A), (II, E), (III, B), (IV, D), (V, C). min. cost =48

(2) (I, C), (II, E), (III, D), (IV, B), (V, A), min. cost =60

(3) (A, II), (B, I), (C, V), (D, III), (E, IV). min. cost =34

(4) (A, II), (B, V), (C, III), (D, I), (E, IV). min. cost =22

(5) (A, III), (B, II), (C, I), (D, IV), (E. V). min. cost =45

(6) (A, 1), (B, II), (C, III), (D, IV). min. cost =23

(7) (1, V), (2, III), (3, IV), (4, II), (5,1). min. cost =60

II

(i) (A, 1), (B, 3), (C, 2), (D, 4) and . Max. profit =61

(ii) (A, 3), (B, 5), (C, 4), (D, 2),(E, 1) and . Max. profit=50

(iii) (A, I), (B, II), (C, III), (D, IV) and . Max. profit =99

(iv) (A, 4), (B, 2), (C, 1), (D, 5),(E, 3) and . Max. profit=452

III

(a) (A, 2), (B, 6), (C, 3), (D, 1),(E, 4) and . Min. cost=38

(b) (A, 1), (B, 2), (C, 3) and . Min. cost=50

(c) (A, 4), (B, 1), (D, 5),(E, 2), (F, 3) and . Min. cost=16

(d) (A, III), (B, II), (C, I), (F, IV) and . Min. cost=83.6

6.7 TRANSPORTATION PROBLEM

6.7.1 Introduction

Many decision problems involve the allocation of supplies from their origins or sources to their destinations or demand centres. The items are transported from the origins to the destinations. Each origin has an available supply of items and each destination has a particular requirement. One must therefore select the plan that most effectively distributes the available supplies to the requesting destinations. The transportation problem (also called the distribution problem) was formulated by Kantorovich (1939) and Hitchcock (1941) and solved by the simplex method of Dantzig. The transportation method has been successfully applied to such diverse problems as production planning, machine scheduling, location analysis, workforce scheduling and media scheduling.

6.7.2 Formation of Transportation Problem.

Let there be m origins O_i, $i = 1,2,...,m$ and n destinations D_j, $j = 1,2,...,n$. Let a_i, $i = 1,2,... m$ denote the amount of items available at the origin O_i and b_j, $j = 1,2 ....$, n denote the amount of items required at the destination D_j, Let c_{ij} denote the unit transportation cost if one unit of item is transported from O_i to D_j. If x_{ij} denote the amount of items transported from the i^{th} origin to the jth destination, then the problem is to find $x_{ij} \geq 0$ so as to

$$\text{Minimize } z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \qquad (1)$$

$$\text{Subject to } \sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, ..., m \qquad (2)$$

$$\text{and } \sum_{i=1}^{m} x_{ij} = b_j, \; j = 1,2,....,n \qquad (3)$$

From (2), we have $\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i$

From (3), we have $\sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j$

Hence for consistency, $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

(i.e. total supply=total demand) and all a_i, $b_j \geq 0$. Thus a balanced transportation problem is to find $x_{ij} \geq 0$ so as to

Minimize $z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$

Subject to $\sum_{j=1}^{n} x_{ij} = a_i$, i= 1,2,..., m, $\sum_{i=1}^{m} x_{ij} = b_{bj}$, j = 1,2... n.

and $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

In applied problems, the constraint (2) may be

$$\sum_{j=1}^{n-1} x_{ij} \leq a_i \text{ with } \sum_{i=1}^{m} a_i \geq \sum_{j=1}^{n-1} b_j .$$

Using the slack variable $x_{in} = a_i - \sum_{j=1}^{n-1} x_{ij}$ for i = 1,2,..., m,

and $b_n = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n-1} b_j$, the problem can be rewritten as a balanced transportation problem. b_n may correspond to the requirement of a dummy origin O_n.One may take $c_{in} = 0$ for i= 1,2......m.

Similarly if (3) is of the form $\sum_{i=1}^{m-1} x_{ij} \leq b_j$ for j = 1,2,..., n with

$\sum_{i=1}^{m-1} a_i \leq \sum_{j=1}^{n} b_j$ using the slack variables $x_{mj} = b_j - \sum_{i=1}^{m-1} x_{ij}$, j = 1,2,... ,n and

$a_m = \sum_{j=1}^{n} b_j - \sum_{i=1}^{m-1} a_i$ the problem can be rewritten as a balanced transportation problem. a_m may correspond to the availability of a dummy destination D_m.

Note

(1) If a problem is unbalanced, we can make it balanced using dummy rows or columns with zero costs.

(2) For a balanced transportation problem, the rank of the matrix of constraints is $m + n - 1$ [From equations (2) and (3) we have $m + n$ equations in mn variables. Since $\sum_i a_i = \sum_j b_j$ one of the equations can be expressed as a linear combination of the others. Hence there are only $m+n$ — 1 linearly independent equations. Thus the rank is $m + n - 1$] Hence as a LPP there will be $m + n -$ 1 basic variables and mn - (m + n –1) = (m – 1)(n –1) non basic variables. Hence every basic solution will be in terms of $m+n-1$ variables and the remaining (m–1) (n–1) variables kept zero.

(3) A basic feasible solution of a transportation problem is called degenerate if the number of non-zero allotments (i.e. the values of x_{ij}) is less than $m + n-1$. i.e. some of the basic variables vanish.

(4) If the a_j's and b_j's are all integers, then the values of x_{ij} in every basic solution are also integers and hence the optimum x_{ij} can be chosen to be integers.

(5) An assignment problem can be considered as a special case of the transportation problem in which the amounts available at each origin and the amounts required at each destination are 1 (one). Similarly a transportation problem is also a special case of the assignment problem in which several of the rows and several of the columns are identical.

Thus the two problems are identical

(6) If the problem is maximization, it can be converted to minimization case [see example 46]

6.7.3 Solution of a Transportation problem (TP)

Since a TP is identical with an assignment problem, one can use the Hungarian technique to solve the TP. Since the cost matrix will become very large, we do not use this method. Also a TP is a special case of LPP, we can use the same methodology as that of LPP, though we may not employ the simplex algorithm.

The basic steps of solution procedure for a TP are

(i) Find an initial basic feasible solution (IBFS)

(ii) Test for optimality

(iii) If optimal, stop. Otherwise.

(iv) Find an improved basic feasible solution.

(v) Go to step (ii)

We give two methods (i) Northwest corner rule (NWCR) and (ii) Vogel's approximation method (VAM) for finding IBFS. To test the optimality and to find an improved basic feasible solution, we use modified distribution method (MODI). MODI method is also known as u-v method or method of multipliers.

In order to find on IBFS, we first form a matrix table called the transportion table as given below.

Destinations

	D_1	D_2			Availability ↓
	C_{11}	C_{12}	....	C_{1n}	a_1
	C_{21}	C_{22}	....	C_{2n}	
Origins			....		
			....		
			....		
	C_{m1}	C_{m2}	....	C_{mn}	a_m
requirements→	b_1	b_2	....	b_n	

6.7.4 Northwest Corner Rule [For balanced TP]

This is a systematic method. Conventionally, in a map the top side is regarded as north and the left side is west. Thus the north west coner of a transportation table is the cell (1,1). Allocate as many as possible to the cell (1,1). The maximum amount that can be allocated is the smallest of the availability or requirement. The satisfied row or column is deleted and a new table is obtained. If a column and a row are satisfied simultaneously delete both. Adjust the amounts of availability and requirement. Consider the northwest comer of the new table for allotment. Repeat the process till all the rim requirements (i.e. the availability and requirement) are satisfied. This method is quick and easy way to find an IBFS although it may not be near to the optimum solution. This method is illustrated in the following example.

Example 37

Obtain an IBFS to the following TP using NWCR.

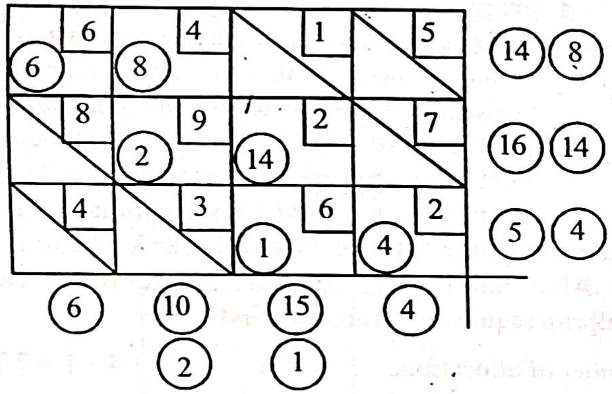
	D_1	D_2	D_3	D_4	a_i
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	4	3	6	2	5
b_j	6	10	15	4	

Solution

Since total supply=total demand=35, the problem is balanced. Rewrite the given table as follows.

Consider the cell (1,1). Min{6,14} = 6. Allocate 6 units there. Since column 1 is exhausted, delete it by crossing out the column. Rewrite the rim values (i.e. a_1 = 14 – 6 = 8 and b_1=6 – 6 = 0). Now the new NW comer is the cell (1,2). Min{8,14} = 8. Allocate 8 units to (1,2). Then row 1 is exhausted. Delete it. The next NW comer is (2,2). min{16, 2} = 2. Allot 2 to (2,2). The column 2 is exhausted. Delete it. The next NW corner is (2,3). min{14,15} = 14. Allot 14 there. The second row is exhausted. Delete it. The NW corner is (3,3). min{5,1} = 1. The 3rd row is exhausted. Delete it. The last NW comer is (3,4) and min{4,4} = 4.

Allot it there. All rim requirements are satisfied.



The total cost = 6×6 + 8×4 + 2×9 + 14×2 + 1×6 + 4×2 = 128.

Here the number of allocations =6. m+n—1=3+4—1=6 The above BFS is non-degenerate.

Example 38

Obtain an IBFS using NWCR

	D_1	D_2	D_3	D_4	
O_1	2	4	3	6	20
O_2	7	3	8	2	10
O_3	2	2	9	11	15
b_j	15	15	8	7	

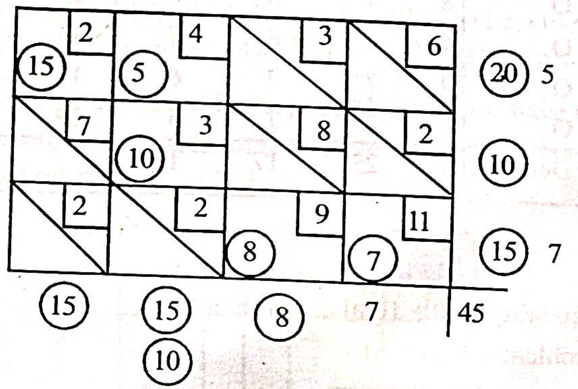
Solution:

Total supply=Total demand = 45 The problem is balanced.

Consider the cell (1,1). Allote min{15,20} = 15 there. Delete column 1. The next NW corner is (1,2). Allot min{15,5} = 5 there and delete row 1. The next NW corner is (2,2). Allot min{10,10} = 10 there. Since both 2nd row and 2nd column rim requirements are satisfied, delete both the 2nd row and 2nd column. The next NW corner is (3,3). Allot min{8,15} = 8 there. Cancel column 3. Allot 7 to (3,4). All rim requirements satisfied.

Hence the total transportation cost = 15×2+5×4+10×3 + 8× 9 + 7× 11 = 229

Here the number of allocations= 5. But m + n — 1 = 3 + 4—1 = 6. Thus the solution is degenerate.



6.7.5 Vogel's Approximation Method (VAM)

In the NWCR we didnot consider the unit transportation cost. Hence this IBFS may be far away from the optimum solution. VAM is a method in which the transportation cost is also considered. Here we calculate the row and column penalties for not using the cell with least cost for allotment. We make an allotment to reduce the maximum penality. Thus the IBFS will be near the optimum solution. The various steps involved in VAM are given below

Step 1

Calculate the row and column penality as the difference between the smallest and the next smallest costs for each row and column

Step 2

Select the row or column with the largest penality. If there is a tie, break it arbitrarily. Choose the cell with least cost in the selected row (or column). Make an allotment there. The maximum allotment is equal to the minimum of the row availability and column requirement. Delete the row or column which is completely satisfied.

Step 3

Repeat Step 1 and 2 till all rim requirements are satisfied.

Example 39

Find an IBFS using YAM

	D_1	D_2	D_3	D_4	Supply
O_1	8	4	6	6	34
O_2	6	6	8	7	15
O_3	9	7	7	6	12
O_4	7	2	7	5	19
Demand	21	25	17	17	

Solution

Total availability = 80 = Total requirements

∴ The problem is balanced.

8 (6)	4 (6)	6 (17)	6 (5)	~~34~~ ~~28~~ 22	(2)	(2) ←	(0)	(0)	(0)
6 (15)	6	8	7	~~15~~		(0)	(0)	(1)	x	x
9	7	7	6 (12)	12		(1)	(1)	(1)	(1)	(1)
7	2 (19)	7	5	~~19~~		(3) ←	x	x	x	x
~~21~~ ~~0~~	~~25~~ ~~0~~	17	17	80						

(1)	(2)	(1)	(1)
(2)↑	(2)	(1)	(0)
(2)↑	x	(1)	(0)
(1)	x	(1)	(0)
x	x	(1)↑	(0)

Calculate the row and column penalities as the difference between the smallest and next smallest costs. The largest penality is 3 for row 4. The least cost in that row is 2 of cell (4,2). Make the allotment min{19,25} = 19 there. The row requirement is satisfied. Delete it and modify the rim values . Again calculate the penalities for the reduced matrix. The largest penality is 2. There are three such values. Choose any one of them. We select the first row. The smallest cost in row 1 is 4 of cell (1,2). Make the allotment min{34,6} = 6 there. Cancel column 2. Calculate the penalities again. The largest is 2 for column 1. The least cost in column 1 is 6 of (2,1). Allot min{15.21} = 15 there. Cancel row 2. Repeat this process till all rim requirements are satisfied

Here number of allocations= 7. $m + n - 1 = 4 + 4 - 1 = 7$ The IBFS is non-degenerate.

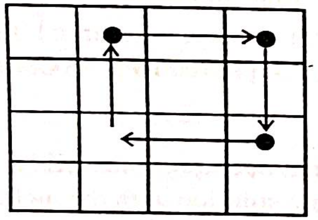
Total cost = 6×8+6×4+17×6+5×6+15×6+12×6+19×2 = 404

MODI Method (The transportation Algorithm)

Consider the IBFS of the problem in Example 3.

8 (6)	4 (6)	6 (17)	6 (5)	34
6 (15)	6	8	7	15
9	7	7	6 (12)	12
7	2 (19)	7	5	19
21	25	17	17	80

The cells (1,1), (1,2), (1,3), (1,4). (2,1), (3,4) and (4,2) containing the basic variables are called basic cells (or occupied cells) and the remaining cells are called non-basic cells. If we want to find another basic solution, we should convert one of the non-basic cells into basic cell and one of the basic cells into non-basic cell. Suppose we want to include the cell (3,2) into the basis, we must make an allotment there. Since the rim requirements (i.e. the availability of each row and requirement of each column) are to be satisfied, inclusion of (3, 2) into the basis will change the allotments of other cells also. If we make an allotment at (3,2), say θ units, then in order to have the column total of column 2 to be 25, θ has to be subtracted from a cell in that column. Subtract θ from (1,2). This affects the row total of row I. (i.e., sum=34) and hence we add θ to the cell (1,4). Again this affects the column total of column 4. Therefore subtract θ from (3,4). Now the rim requirements are satisfied. Thus a change in one cell results in a sequence of changes. The cells which are to be changed are (3,2), (1,2) (1,4) and (3,4). This forms a loop. Occupied cells in the loop are marked by '•'

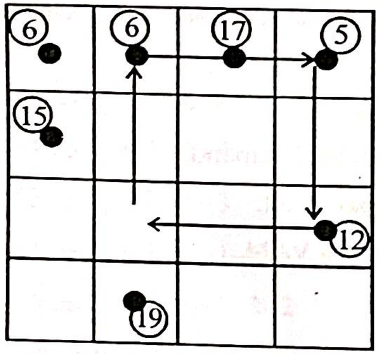


A loop therefore consists of a sequence of cells with the property that

i) two adjacent cells are in a row or column.
ii) three or more adjacent cells are not in a row or column.

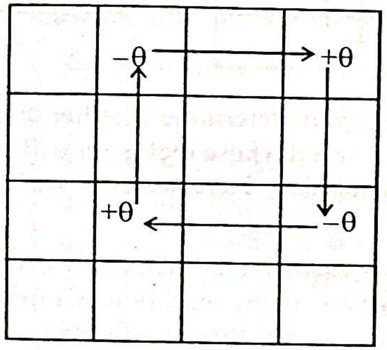
i.e., draw arrows horizontal and vertical alternately starting from an unoccupied cell with the ends of each arrow in an occupied cell and back to the starting cell. There is always a unique loop of this kind

The occupied cells of the given problem are marked in the following table as:-



Note that except at the starting cell, the turning points of the loop are occupied cells. In the part of the loop joining (1,2) and (1,4), there is another occupied cell (1,3). This cell is not a part of the loop since it is not a turning point of the loop. Discard this cell. Thus if we make an allotment of θ units to (3,2), it will result in a $6-\theta$ units in cell (1,2) and $5+\theta$ units in cell (1,4) and $12-\theta$ units in cell (3,4) i.e. change of $+\theta$ in (3,2) results into $-\theta$ in cell (1,2) and $+\theta$ in cell (1,4) and $-\theta$ in cell (3,4)

Since negative allotment is not possible, we must have $6-\theta \geq 0$ and $12-\theta \geq 0$. So maximum values of $\theta = \min\{6,12\} = 6$. Thus we get another basic feasible solution



(6)		(17)	(11)
(15)			
	(6)		(6)
	(19)		

Thus inclusion of different non-basic cells into the basis gives different basic feasible solutions. Note that the loops can take so many shapes. Some of the different loops are shown below:-

Fig. 1

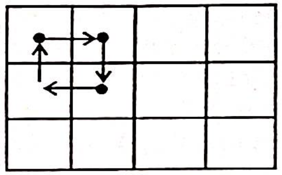


Fig. 2

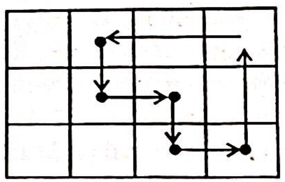
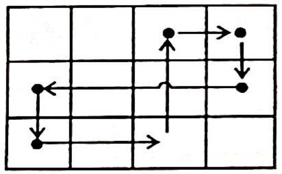


Fig. 3



MODI method helps to determine whether the BFS is optimum or if not, to find a non-basis cell whose inclusion will gives an improved (i.e. reduced total cost) solution. Here we start with an IBFS using either NWCR or VAM.

In this method we associate to each row i a multiplier u_i and to each column j a multiplier v_j with the condition that for each basic cell (i, j), $u_i + v_j = c_{ij}$. Since we have m +n−1 basic cells for a non-degenerate solution, we can find and u_i and v_j, i = 1,2..... m and j = 1,2,..., n by assigning one of these m + n multipliers an arbitrary value. For convenience, let us take $u_1 = 0$. Solving the set of equations $u_i+v_j = c_{ij}$ i = 1,2 ... ,m and j=1,2,... ,n (for occupied cells) we can find all these multipliers. After finding these values, we compute for the non-basic cell, say (p, q), the net evaluation $Z_{pq} - C_{pq} = (u_p + v_q) - C_{pq}$ (note that this is equivalent to $Z_j - C_j$ in the simplex procedure). If $Z_{pq} - C_{pq}$ are all non-positive, the current BFS is optimal. Otherwise we select the cell (p, q) with largest positive net evaluations to enter the basis and obtain an improved BFS. If there is a tie break it arbitrarily.

Thus the Algorithm is :

Step 1 Start with a BFS. Check whether it is non-degenerate. If not make it non-degenerate by alloting a very small quantity $\varepsilon > 0$ to required number of cells so as to make it non-degenerate. (Note that the above cells should be in independent positions. In

otherwords, the basic cells including the new cells do not form a loop).

Step 2 Compute u_i, i = 1,2,.. .m and v_j, j = 1,2,... n with $u_1 = 0$ such that $u_i + v_j = c_{ij}$ for occupied cells. Compute the net evaluations $u_i + v_j - c_{ij}$ for unoccupied cells. If all net evaluations non-positive, the BFS is optimum. If not choose the cell with largest positive net evaluation. This cell is to be included in the basis.

Step 3 Starting from the chosen cell, form a loop through the occupied cells. Put +θ and −θ alternately starting from the chosen cell through the loop. Find minimum of all allotments of the cells containing −θ. Add it where we marked θ and subtract it where we marked −θ. An improved BFS is obtained.

Step 4 Go to Step 1.

Note If the net evaluation of a non-basic cell for an optimum is zero, there is an alternate solution with the inclusion of this cell into the basis.

Example 40 Solve the following TP.

	D_1	D_2	D_3	D_4	Supply
O_1	5	2	4	3	22
O_2	4	8	1	6	15
O_3	4	6	7	5	8
Demand→	7	12	17	9	

Solution

Total supply = 45 = Total demand.

∴ The problem is balanced.

We find an IBFS using VAM.

5	2 (12)	4 (2)	3 (8)	~~22~~ ~~10~~ 8	(1)	(1)	(1)	(2)
4	8	1 (15)	6	~~15~~	(3)	(3)	x	x
4 (7)	6	7	5 (1)	~~8~~	(1)	(1)	(1)	(1)
~~7~~	~~12~~	~~17~~ ~~2~~	~~9~~	45				
(0)	↑ (4)	(3)	(2)					
(0)	x	↑ (3)	(2)					
(1)	x	↑ (3)	(2)					
(1)	x	x	↑ (2)					

Total cost = $12\times2+2\times4+8\times3+15\times1+7\times4+1\times5$

$= 104$

Number of allocations = 6

$m + n-1 = 3 + 4-1 = 6$

∴ The IBFS is non-degenerate.

Optimum solution using MODI method

We find u_i and v_j using the result that $u_i + v_j = c_{ij}$ for occupied cells with $u_1 = 0$

$u_1 = 0,\quad u_1+v_2=2 \Rightarrow v_2 = 2-u_1=2$

$u_1+v_3=4 \Rightarrow v_3 = 4-u_1=4$

$u_1+v_4=3 \Rightarrow v_4 = 3-u_1=3$

$u_2+v_3=1 \Rightarrow u_2 = 1-v_3=1-4=-3$

$u_3+v_4=5 \Rightarrow u_3 = 5-v_4=5-3=2$

$u_3+v_1=4 \Rightarrow v_1 = 4-u_3=4-2=2$

5	2 (12)	4 (2)	3 (8)	$u_1 = 0$
4	8	1 (15)	6	$u_2 = -3$
4	6 (7)	7	5 (1)	$u_3 = 2$
$v_1 = 2$	$v_2 = 2$	$v_3 = 4$	$v_4 = 3$	

For the unoccupied cells, find the net evaluations.

Cell	Net evaluation = $u_i + v_j - c_{ij}$
(1,1)	$u_1 + v_1 - c_{11} = 0 + 2-5 = -3$
(2,1)	$u_2 + v_1 - c_{21} = -3 + 2-4 = -5$
(2,2)	$u_2 + v_2 - c_{22} = -3+ 2-8 = -9$
(2,4)	$u_2 + v_4 - c_{24} = -3 + 3-6 = -6$
(3,2)	$u_3 + v_2 - c_{32} = 2 + 2-6 = -2$
(3,3)	$u_3 + v_3 - c_{33} = 2 + 4-7 = -1$

Since all the net evaluations are non-positive, the IBFS is optimum.

So the minimum cost = 104.

Example 41 Solve (K.U. April.2000)

	D_1	D_2	D_3	D_4	$a_i \downarrow$
O_1	19	30	50	10	7
O_2	70	30	40	60	9
O_3	40	8	70	20	18
$b_j \rightarrow$	5	8	7	14	

Solution

Sum of availability = 34 = Sum of requirements.

Hence the problem is balanced.

We find an initial BFS using VAM..

Total cost = 5×19 + 2×10 + 7×40 + 2×60 + 8×8 + 10×20

= 779 Number of allocations = 6

m+n—1 = 3+4—1=6

The IBFS is non-degenerate.

MODI method

We find u_i and v_j using the result that $u_i + v_j = c_{ij}$ for occupied cells with $u_1 = 0$.

19 (5)	30	50	10 (2)	~~7~~ 2	(9)	(9)	(40)	40
70	30	40 (7)	60 (2)	~~9~~	(10)	(20)	(20)	(20)
40	8 (8)	70	20 (10)	~~18~~ ~~10~~	(12)	(20)	(50)	x
~~5~~	~~8~~	~~7~~	~~14~~ 4	34				
(21)	(22)↑	(10)	(10)					
(21)	x	(10)	(10)					
x	x	(10)	(10)					
x	x	(10)	(50)↑					

19 (5)	30	50	10 (2)	$u_1 = 0$
70	30 (7)	40	60 (2)	$u_2 = 50$
40	8 (8)	70	20 (10)	$u_3 = 10$
$v_1 = 19$	$v_2 = -2$	$v_3 = -10$	$v_4 = 10$	

$u_1 = 0$, $u_1+v_1=19 \Rightarrow v_1 = 19-u_1=19-0 = 19$

$u_1+v_4=10 \Rightarrow v_4 = 10-u_1=10$

$u_2+v_4=60 \Rightarrow u_2 = 60-v_4=60-10 = 50$

$u_2+v_3=40 \Rightarrow v_3 = 40-u_2=40-50 = -10$

$u_3+v_4=20 \Rightarrow u_3 = 20-v_4=20-10 = 10$

$u_3+v_2=8 \Rightarrow v_2 = 8-u_3=8-10 = -2$

Calculate the net evaluations for the unoccupied cells $u_i+v_j-c_{ij}$.

Cell	Net evaluation = $u_i + v_j - c_{ij}$
(1,2)	$u_1 + v_2 - c_{12} = 0 + -2-30 = -32$
(1,3)	$u_1 + v_3 - c_{13} = 0 + -10-50 = -60$
(2,1)	$u_2 + v_1 - c_{21} = 50 + 19-70 = -1$
(2,2)	$u_2 + v_2 - c_{22} = 50 - 2-30 = 18$
(3,1)	$u_3 + v_1 - c_{31} = 10 + 19-40 = -11$
(3,3)	$u_3 + v_3 - c_{33} = 10 -10-70 = -70$

We can write the net evaluations in the table at the bottom left comer.

19 (5)	30 −32	50 −60	10 (2)
70 −1	30 +θ 18	40 (7)	60 (2) −θ
40 −11	8 (8) −θ	70 −70	20 (10) +θ

Since not all net evaluations are non-positive, the IBFS is not optimum. The largest positive net evaluation is 18 of cell (2,2). Include that in the basis. Draw a loop starting from (2,2) as in the figure. Mark +θ and −θ alternately starting with (2,2). Min{2,8} = 2 (i.e., minimum of allotments marked −θ). Add 2 to cells (2,2) and (3,4) and subtract 2 from (2,4) and (3,2) to find an improved BFS.

19 (5)	30	50	10 (2)
70	30 (2)	40 (7)	60
40	8 (6)	70	20 (12)

Total cost = 779 − 2×18 = 743.

(Net evaluation 18 means if 1 unit is transformed to this cell a reduction of 18 is there in the total cost. Since we transferred 2 units a reduction of 2×18 = 36 is there in the total cost).

Number of allocations = 6 and m + n − 1 = 3 + 4 − 1 = 6

∴ The improved BFS is non-degenerate.

Again check for optimality.

19 (5)	30 −32	50 −42	10 (2)	$u_1=0$
70 −19	30 (2)	40 (7)	60 −18	$u_2=32$
40 −11	8 (6)	70 −52	20 (12)	$u_3=10$
$v_1=19$	$v_2=-2$	$v_3=8$	$v_4=10$	

(Note that the calculations of u_i, v_j and the net evaluations are marked in a single table above) since all net evaluations are non-positive, the above improved BFS is optimum.

The minimum transportation cost = 743.

Note. We can reduce the number of tables by calculating and marking u_i, v_j, net evaluations and the loops in single matrix as shown in the other examples to follow.

Example 42 Solve the following TP. (K.U. Oct. 2000)

		To			
		1	2	3	Supply
	1	2	7	4	5
From	2	3	3	1	8
	3	5	4	7	7
	4	1	6	2	14
Demand		7	9	18	34

Solution

IBFS using VAM

2 (5)	7	4	~~5~~
3	3	1 (8)	~~8~~
5	4 (7)	7	7
1 (2)	6 (2)	2 (10)	~~14~~ 4
~~7~~ 2	9	~~18~~ ~~10~~	34

Total cost = 5×2 + 8×1 + 7×4 + 2×1 + 2×6 + 10×2 = 80

Number of allocations = 6 = m + n – 1.

∴ The IBFS is non-degenerate.

To find the optimum solution, we apply MODI method.

2 (5)	7 0	4 −1	$u_1 = 0$
3 −3	3 2	1 (2)	$u_2 = -2$
5 −6	4 (7)	7 −7	$u_3 = -3$
1 (2)	6 (2) θ	2 (10) +θ	$u_4 = -1$
$v_1 = 2$	$v_2 = 7$	$v_3 = 3$	

θ= min{allocations containing – θ} = min{2,8} = 2. The improved BFS is

2 (5)	7 −2	4 −1	$u_1 = 0$
3 −3	3 (2)	1 (6)	$u_2 = -2$
5 −4	4 (7)	7 −5	$u_3 = -1$
1 (2)	6 −2	2 (12)	$u_4 = -1$
$v_1 = 2$	$v_2 = 5$	$v_3 = 3$	

(Number of allocations = 6 = m + n- 1. The improved BFS is non-degenerate. Test for optimality)

Since all net evaluations non-positive, the improved BFS is optimum. So the optimum cost = 5×2 + 2×3 + 6×1 + 7×4 + 2×1 + 12×2

= 76

Example 43 Solve the following TP.

	D_1	D_2	D_3	D_4	Supply
S_1	11	20	7	8	50
S_2	21	16	10	12	40
S_3	8	12	18	9	70
Demand	30	25	35	40	

Solution

Total supply = 160 and total demand = 130. As the two totals are not equal, the given problem is unbalanced. So we introduce a dummy destination D_5 with requirement 160 – 130 = 30. The cost of transportation from each source to D_5 is zero. Thus we get the following balanced problem.

IBFS using VAM

11	20	7 (25)	8 (25)	0	50		(7)	(1)	(1)	(1)	(1)
21	16	10 (10)	12	0 (30)	~~40~~ 10		(10)	(2)	(2)	(2)	(2)
8 (30)	12 (25)	18	9 (15)	0	~~70~~ ~~45~~ ~~15~~		(8)	(1)	(1)	(11)	x
30	25	35	40 25	30	160						
(3)	(4)	(3)	(1)	(0)							
(3)	(4)	(3)	(1)	x							
(3)	x	(3)	(1)	x							
x	x	(3)	(1)	x							
x	x	(3)	(4)	x							

Number of allocations = 7 and m+n-l=3+5-l=7 .

∴ IBFS is non-degenerate.

Optimum solution using MODI

11 / −4	20 / −9	7 / (25)	8 / (25)	0 / −3	$u_1=0$
21 / −11	16 / −2	10 / (10)	12 / −1	0 / (30)	$u_2=3$
8 / (30)	12 / (25)	18 / −10	9 / (15)	0 / −2	$u_3=1$
$v_1=7$	$v_2=11$	$v_3=7$	$v_4=8$	$v_5=-3$	

Since all net evaluations non-positive, the IBFS is optimum.

Optimum cost = 25×7+25×8+10×10+30×0+30×8+25×12+15×9

= 1150

Example 44 Solve the following TP. (K.U. Oct/Nov. 1997)

	D_1	D_2	D_3	D_4	D_5	a_i
O_1	4	7	3	8	2	4
O_2	1	4	7	3	8	7
O_3	7	2	4	7	7	9
O_4	4	8	2	4	7	2
b_j	8	3	7	2	2	

Solution

$\sum a_i = 22$ and $\sum b_j = 22$.

∴ The problem is balanced.

Using VAM, we get the following IBFS

4 (1)	7	3 (1)	8	2 (2)	4
1 (7)	4	7	3	8	7
7	2 (3)	4 (6)	7	7	9
4	8	2	4 (2)	7	2
8	3	7	2	2	22

Number of allocations = 7 and m + n — 1 = 4 + 5—1 = 8. Since the number of allocations < m + n — 1, the IBFS degenerates. To resolve degeneracy, we require one more basic cell in an independent position. Among the independent unoccupied cells, the cell (4,3) has the least cost. So we allocate ε to that cell. Thus the non-degenerate initial BFS is

4 (1)	7 −6	3 (1)	8 −3	2 (2)	$u_1=0$
1 (7)	4 −6	7 −7	3 −1	8 −9	$u_2=-3$
7 −2	2 (3)	4 (6)	7 −1	7 −4	$u_3=1$
4 −1	8 −8	2 (ε)	4 (2)	7 −6	$u_4=-1$
$v_1=4$	$v_2=1$	$v_3=3$	$v_4=5$	$v_5=2$	

Since all net evaluations non-positive, the IBFS is optimum.

Optimum cost $= 1\times4+1\times3+2\times2+7\times1+3\times2+6\times4+\varepsilon\times2+2\times4$

$= 56$ as $\varepsilon\to 0$

Note In order to convert a degenerate solution to a non-degenerate one we allocate a very small quantity denoted by e to one or more unoccupied cells in independent positions. These cells can be taken as independent cells with least costs so that the number of iterations may be reduced. It is not advisable to allocate ε to a cell in the dummy row or column. If degeneracy occur during the iteration, allocate ε to one or more of recently vacated cells so as to make the number of basic cells m + n−1.

Example 45 Solve

	W_1	W_2	W_3	W_4	W_5	Supply
F_1	4	3	1	2	6	40
F_2	5	2	3	4	5	30
F_3	3	5	6	3	2	20
F_4	2	4	4	5	3	10
Demand :	30	30	15	20	5	100

Solution

We find the IBFS using Nortwest comer rule. The aim of finding an IBFS by NWCR is only to show as to how degeneracy occurs during the solution stages. Otherwise, the optimal solution can be obtained immediately by using VAM.

IBFS using NWCR

4 −θ 30	3 +θ 10	1 3	2 −1	6 7	$U_1=0$
5 −2	2 −θ 20	3 10 +θ	4 −4	5 −7	$U_2=-1$
3 3	5 30	6 −θ 5	3 15 +θ	2 −1	$U_3=2$
2 6 +θ	4 3	4 4	5 5 −θ	3 5	$U_4=4$
$V_1=4$	$V_2=3$	$V_3=4$	$V_4=1$	$V_5=-1$	

Number of allocations = 8 = m + n- 1. The IBFS is non-degenerate. The largest net evaluation is 6 of cell (4,1).

Now $\theta=\min\{\text{allocations containing} -\theta\}$

$=\min\{5, 5, 20, 30\} = 5.$

The improved BFS is

4 (25)	3 (15)	1	2	6
5	2 (15)	3 (15)	4	5
3	5	6	3 (20)	2
2 (5)	4	4	5	3 (5)

Number of allocations = 7 and m+n–1 = 8. Since the number of allocations < m + n– 1, the improved solution degenerates. To resolve degeneracy, we require one more basic cell in an independent position. We allocate ε to the recently vacated cell (4,4) [because it has the lowest cost]

The rest of the procedure will be exactly the same as explained earlier. This way, the optimal solution can be obtained. The optimal solution is in $x_{11} = 5$, $x_{13} = 15$, $x_{14} = 20$, $x_{22} = 30$, $x_{31} = 15$, $x_{35} = 5$, $x_{41} = 10$.

Example 46 A company has three factories F_1, F_2 and F_3 which supply warehouses at Wi_1, W_2 and W_3. Weekly factory capacities are 200, 160 and 90 units respectively and weekly warehouse requirements are 180, 120 and 150 units respectively. Unit costs (in Rupees) are as follows.

Factories	Warehouse W_1	W_2	W_3	Supply
F_1	16	20	12	200
F_2	14	8	18	160
F_3	26	24	16	90
Demand	180	120	150	450

Determine the optimal distribution for this company to maximise shipping costs. (K.U. April. 1997)

Solution

The maximization problem can be converted into a minimization problem by changing the signs of figures of the profit matrix by multiplying each entry by -1. But it is inconvenient to deal with negative figures. So we form a new matrix. Subtract all entries in the given matrix from the highest cost. Then the problem can be solved using minimization algorithm. Thus the matrix reduces to

10	6	14	200
12	18	8	160
0	2	10	90
180	120	150	

(The highest cost is 26. Subtract every cost from 26 to obtain the above matrix) Using VAM we find an IBFS

10 (80)	6 (120)	14	~~200~~ ~~80~~	(4)	(4)	(4)
12 (10)	18	8 (150)	~~160~~ ~~10~~	(4)	(4)	(4)
0 (90)	2	10	~~90~~	(2)	x	x
~~180~~ ~~90~~	~~120~~	150				
(10)↑	(4)	(2)				
(2)	(12)↑	(6)				
(2)	x	↑(6)				

MODI Method

Number of allocations = 5; $m + n - 1 = 3 + 3 - 1 = 5$

$\therefore$ IBFS is non-degenerate

10 (80)	6 (120)	14 −8	$u_1=0$
12 (10)	18 −10	8 (150)	$u_2=2$
0 (90)	2 −6	10 −14	$u_3=-10$
$v_1=10$	$v_2=6$	$v_3=6$	

All net evaluations are non-positive. Hence the BFS is optimum.

$\therefore$ Optimum cost $= 80\times16 + 120\times20 + 10\times14 + 150\times18 + 90\times26$

$= 8860$

Note For maximization case, we can also use the following rules.

(i) In VAM we find the penality as the differences between the greatest and next greatest entries. Identify the row or column with maximum penality. Allocate at the cell with maximum cost.

(ii) In MODI method, for optimum solution we should have all net evaluation non-negative. For the identification of entering basic cell, choose the cell with most negative net evaluation.

16 (80)	20 (120)	12	~~200~~ ~~80~~	(4)	(4)	(4)
14 (10)	8	18 (150)	~~160~~ ~~10~~	(4)	(4)	(4)
26 (90)	24	16	~~90~~	(2)	x	x
~~180~~ 90	~~120~~	150				
(10)↑	(4)	(2)				
(2)	(12)↑	(6)				

MODI

16 (80)	20 (120)	12 8	$u_1=0$
14 (10)	8 10	18 (150)	$u_2=-2$
26 (90)	24 6	16 14	$u_3=10$
$v_1=16$	$v_2=20$	$v_3=20$	

All net evaluation non-negative. Hence solution is optimum.

Example 47 (Alternate solution)

Consider the problem

48	60	56	58	140
45	55	53	60	260
50	65	60	62	360
52	64	55	61	220
200	320	250	210	

Solution

Applying VAM, we get IBFS as

48	60 (60)	56 (30)	58 (50)
45	55 (260)	53	60
50 (200)	65	60	62 (160)
52	64	55 (220)	61

No: of allocation = 7

m + n — 1=4 + 4 — 1 = 7 ∴ the IBFS is non-degenerate.

MODI method

48 −2	60 (60)	56 (30) −θ	58 (50) +θ	$U_1=0$
45 −4	55 (200)	53 −2	60 −7	$U_2=-5$
50 (200)	65 −1	60 +θ	62 (160) −θ	$U_3=4$
52 −7	64 −5	55 (220)	61 −4	$U_4=-1$
$V_1=46$	$V_2=60$	$V_2=56$	$V_4=58$	

All net evaluations are non-positive. ∴ IBFS is optimum. Since the net evaluation of cell (3,3) is zero, there is an alternate solution by including this cell in the basis.

Thus we get

48 −2	60 (60)	56 0	58 (80)	$u_1=0$
45 −4	55 (260)	53 −2	60 −7	$u_2=-5$
50 (200)	65 −1	60 (30)	62 (130)	$u_3=4$
52 −7	64 −5	55 (220)	61 −4	$u_4=-1$
$v_1=46$	$v_2=60$	$v_3=56$	$v_4=58$	

All net evaluation non-positive. Hence solution is optimum. Note that the total cost for both these optimum solution is 54500.

Exercise 6.8 Solve the following transportation problems.

(1)

	To			a_i
	6	8	4	14
From	4	9	8	12
	1	2	6	5
b_j	6	10	15	31

(2)

	1	2	3	4	$a_i\downarrow$
1	21	16	25	13	11
2	17	18	14	23	13
3	32	27	18	41	19
$b_j\rightarrow$	6	10	12	15	

(3)

	To				
	15	10	17	18	2
From	16	13	12	13	6
	12	17	20	11	7
	3	3	4	5	

(4)

	To				
	11	12	13	14	10
From	21	22	23	24	6
	31	32	33	34	15
	2	9	3	17	

(5)

To

From	2	1	6	4	5	90
	7	5	4	3	2	50
	4	2	3	1	7	80
	6	5	2	6	4	30
	60	20	50	40	80	

(6)

To

From	4	3	6	5	20
	7	10	5	6	30
	8	9	12	7	50
	15	35	20	30.	

(7)

To

From	7	6	4	5	9	40
	8	5	6	7	8	30
	6	8	9	6	5	20
	5	7	7	8	6	10
	30	30	15	20	5	

(8)

To

From	4	6	8	13	50
	13	11	10	8	70
	14	4	10	13	30
	9	11	13	8	50
	25	35	105	20	

(9)

To

From	8	10	7	6	50
	12	9	4	7	40
	9	11	10	8	30
	25	32	40	23	

(10)

To

From	5	1	7	10
	6	4	6	80
	3	2	5	15
	75	20	50	

(11)

To

From	12	14	16	50
	16	12	14	50
	20	16	12	60
	24	26	28	30
	32	28	24	50
	60	120	40	

(12)

To

From	7	4	5
	6	8	15
	3	9	9
	15	6	

(13)

To

From	1	2	1	4	5	2	30
	3	3	2	1	4	3	50
	4	2	5	9	6	2	75
	3	1	7	3	4	6	20
	20	40	30	10	50	25	

(14)

To

From	2	3	11	7	6
	1	0	6	1	1
	5	8	15	9	10
	7	5	3	2	

(15) Solve the following TP to maximize profits.

(a)

To

From	15	51	42	33	23
	80	42	26	81	44
	90	40	66	60	33
	23	31	16	30	100

(b)

To

From	40	25	22	33	100
	44	35	30	30	30
	38	38	28	30	70
	40	20	60	30	

Answers 6.8

(1) $x_{13}=14, x_{21}=6, x_{22}=5, x_{23}=1, x_{32}=5$, and min cost = 143.

(2) $x_{13}=11, x_{21}=6, x_{22}=3, x_{24}=4, x_{32}=7, x_{33}=12$, and min cost =796.

(3) $x_{12}=2, x_{22}=1; x_{23}=4, x_{24}=1, x_{31}=3, x_{34}=4$ and min cost = 174.

(4) $x_{31}=2, x_{22}=6, x_{32}=3, x_{33}=3, x_{14}=10, x_{34}=7$ and opt cost = 767.

(5) $x_{11}=60, x_{12}=20, x_{15}=10, x_{25}=50, x_{33}=40, x_{34}=40, x_{43}=10, x_{45}=20$. and opt cost = 550.

(6) $x_{12}=20, x_{21}=10, x_{23}=20, x_{31}=5, x_{32}=15, x_{34}=30$, and opt cost=6

(7) $x_{11}=5, x_{13}=15, x_{14}=20, x_{22}=30, x_{31}=15, x_{35}=5, x_{41}=10$, and opt =510.

(8) $x_{11}=25, x_{12}=5, x_{13}=20, x_{23}=70, x_{32}=30, x_{43}=15, x_{44}=20, x_{45}=15$, opt cost =1465.

(9) $x_{11}=25, x_{12}=2, x_{14}=23, x_{23}=40, x_{32}=30$, and opt cost = 748.

(10) $x_{12}=10, x_{21}=60, x_{22}=10, x_{23}=10, x_{31}=15, x_{43}=40$, and opt cost = 515.

(11) $x_{11}=50, x_{22}=50, x_{32}=20, x_{33}=40, x_{41}=10, x_{42}=20, x_{52}=30, x_{54}=20$ and opt cost = 3600.

(12) $x_{12}=5, x_{21}=6, x_{22}=1, x_{23}=8, x_{31}=9$, and opt cost = 91.

(13) $x_{11}=20, x_{13}=10, x_{23}=20, x_{24}=10, x_{25}=20, x_{32}=40, x_{35}=10, x_{36}=25, x_{46}=20$.

(14) $x_{12}=5, x_{13}=1, x_{23}=1, x_{31}=7, x_{33}=1, x_{34}=2$ and opt cost = 100.

(15) (a) $x_{12}=23, x_{21}=6, x_{22}=8, x_{24}=30, x_{31}=17, x_{33}=16$ and max profit = 7005.

(15) (b) $x_{11}=20, x_{14}=30, x_{21}=20, x_{23}=10, x_{32}=20, x_{33}=50$ and max profit = 5130.