Differentiation

Introduction

The rate of change of one quantity with respect to some another quantity has a great importance. For example, the rate of change of displacement of a particle with respect to time is called its velocity and the rate of change of velocity is called its acceleration.

The rate of change of a quantity 'y' with respect to another quantity 'x' is called the derivative or differential coefficient of y with respect to x.

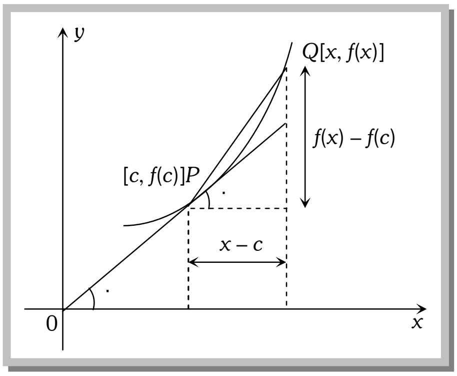
1.1 Derivative at a Point

The derivative of a function at a point $x = a$ is defined by $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ (provided the limit exists and is finite)

The above definition of derivative is also called derivative by first principle.

(1) **Geometrical meaning of derivatives at a point:** Consider the curve $y = f(x)$. Let $f(x)$ be differentiable at $x = c$. Let $P(c, f(c))$ be a point on the curve and $Q(x, f(x))$ be a neighbouring point on the curve. Then,

Slope of the chord $PQ = \frac{f(x) - f(c)}{x - c}$. Taking limit as $Q \to P$, i.e., $x \to c$,



we get $\lim_{Q \to P}$ (slope of the chord PQ) $= \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ ......(i)

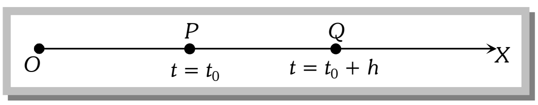
As $Q \to P$, chord PQ becomes tangent at P.

Therefore from (i), we have

Slope of the tangent at $P = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \left(\frac{df(x)}{dx}\right)_{x=c}$.

Note : ❑ Thus, the derivatives of a function at a point $x = c$ is the slope of the tangent to curve, $y = f(x)$ at point $(c, f(c))$.

(2) **Physical interpretation at a point :** Let a particle moves in a straight line OX starting from O towards X. Clearly, the position of the particle at any instant would depend upon the time elapsed. In other words, the distance of the particle from O will be some function f of time t.



Let at any time $t = t_0$, the particle be at P and after a further time h, it is at Q so that $OP = f(t_0)$ and $OQ = f(t_0 + h)$. Hence, the average speed of the particle during the journey from P to Q is $\frac{PQ}{h}$, *i.e.*, $\frac{f(t_0+h) - f(t_0)}{h} = f(t_0, h)$. Taking the limit of $f(t_0, h)$ as $h \to 0$, we get its instantaneous speed to be $\lim_{h\to 0} \frac{f(t_0+h) - f(t_0)}{h}$, which is simply $f'(t_0)$. Thus, if $f(t)$ gives the distance of a moving particle at time t, then the derivative of f at $t = t_0$ represents the instantaneous speed of the particle at the point P, *i.e.*, at time $t = t_0$.

Important Tips

- ☞ *$\frac{dy}{dx}$ is $\frac{d}{dx}(y)$ in which $\frac{d}{dx}$ is simply a symbol of operation and not 'd' divided by dx.*
- ☞ *If $f'(x_0) = \infty$, the function is said to have an infinite derivative at the point x_0. In this case the line tangent to the curve of $y = f(x)$ at the point x_0 is perpendicular to the x-axis*

Example: 1 If $f(2) = 4$, $f'(2) = 1$, then $\lim_{x\to 2} \frac{xf(2) - 2f(x)}{x-2} =$

(a) 1 (b) 2 (c) 3 (d) – 2

Solution: (b) Given $f(2) = 4, f'(2) = 1$

$$\therefore \lim_{x\to 2} \frac{xf(2) - 2f(x)}{x-2} = \lim_{x\to 2} \frac{xf(2) - 2f(2) + 2f(2) - 2f(x)}{x-2} = \lim_{x\to 2} \frac{(x-2)f(2)}{x-2} - \lim_{x\to 2} \frac{2f(x) - 2f(2)}{x-2}$$

$$= f(2) - 2\lim_{x\to 2} \frac{f(x) - f(2)}{x-2} = f(2) - 2f'(2) = 4 - 2(1) = 4 - 2 = 2$$

Trick : Applying L-Hospital rule, we get $\lim_{x\to 2} \frac{f(2) - 2f'(2)}{1} = 2$.

Example: 2 If $f(x+y) = f(x).f(y)$ for all x and y and $f(5) = 2$, $f'(0) = 3$, then $f'(5)$ will be

(a) 2 (b) 4 (c) 6 (d) 8

Solution: (c) Let $x = 5, y = 0 \Rightarrow f(5+0) = f(5).f(0)$

$$\Rightarrow f(5) = f(5)f(0) \Rightarrow f(0) = 1$$

Therefore, $f'(5) = \lim_{h\to 0} \frac{f(5+h) - f(5)}{h} = \lim_{h\to 0} \frac{f(5)f(h) - f(5)}{h} = \lim_{h\to 0} 2\left[\frac{f(h) - 1}{h}\right]$ $\{\because f(5) = 2\}$

$$= 2\lim_{h\to 0}\left[\frac{f(h) - f(0)}{h}\right] = 2 \times f'(0) = 2 \times 3 = 6.$$

Example: 3 If $f(a) = 3, f'(a) = -2, g(a) = -1, g'(a) = 4$, then $\lim_{x\to a} \frac{g(x)f(a) - g(a)f(x)}{x-a} =$

Solution: (b) $\lim_{x\to a} \frac{g(x)f(a) - g(a)f(x)}{x-a}$. We add and subtract $g(a)f(a)$ in numerator

$$= \lim_{x\to a} \frac{g(x)f(a) - g(a)f(a) + g(a)f(a) - g(a)f(x)}{x-a} = \lim_{x\to a} f(a)\left[\frac{g(x) - g(a)}{x-a}\right] - \lim_{x\to a} g(a)\left[\frac{f(x) - f(a)}{x-a}\right]$$

$$= f(a)\lim_{x\to a}\left[\frac{g(x) - g(a)}{x-a}\right] - g(a)\lim_{x\to a}\left[\frac{f(x) - f(a)}{x-a}\right] = f(a)g'(a) - g(a)f'(a)$$ [by using first principle formula]

$= 3.4 - (-1)(-2) \quad = 12 - 2 = 10$

Trick : $\lim\limits_{x \to a} \dfrac{g(x)f(a) - g(a)f(x)}{x - a}$

Using L–Hospital's rule, Limit = $\lim\limits_{x \to a} \dfrac{g'(x)f(a) - g(a)f'(x)}{1}$;

Limit $= g'(a)\ f(a) - g(a)f'(a) = (4)(3) - (-1)(-2) = 12 - 2 = 10.$

Example: 4 If $5f(x) + 3f\left(\dfrac{1}{x}\right) = x + 2$ and $y = xf(x)$ then $\left(\dfrac{dy}{dx}\right)_{x=1}$ is equal to

(a) 14 (b) $\dfrac{7}{8}$ (c) 1 (d) None of these

Solution: (b) $\because 5f(x) + 3f\left(\dfrac{1}{x}\right) = x + 2$ ......(i)

Replacing x by $\dfrac{1}{x}$ in (i), $5f\left(\dfrac{1}{x}\right) + 3f(x) = \dfrac{1}{x} + 2$ ......(ii)

On solving equation (i) and (ii), we get, $16f(x) = 5x - \dfrac{3}{x} + 4$, $\therefore\ 16f'(x) = 5 + \dfrac{3}{x^2}$

$\because\ y = xf(x) \Rightarrow \dfrac{dy}{dx} = f(x) + xf'(x) = \dfrac{1}{16}(5x - \dfrac{3}{x} + 4) + x.\dfrac{1}{16}(5 + \dfrac{3}{x^2})$

at $x = 1$, $\dfrac{dy}{dx} = \dfrac{1}{16}(5 - 3 + 4) + \dfrac{1}{16}(5 + 3) = \dfrac{7}{8}$.

1.2 Some Standard Differentiation

(1) **Differentiation of algebraic functions**

(i) $\dfrac{d}{dx}x^n = nx^{n-1}, x \in R, n \in R, x > 0$ (ii) $\dfrac{d}{dx}(\sqrt{x}) = \dfrac{1}{2\sqrt{x}}$ (iii) $\dfrac{d}{dx}\left(\dfrac{1}{x^n}\right) = -\dfrac{n}{x^{n+1}}$

(2) **Differentiation of trigonometric functions :** The following formulae can be applied directly while differentiating trigonometric functions

(i) $\dfrac{d}{dx}\sin x = \cos x$ (ii) $\dfrac{d}{dx}\cos x = -\sin x$ (iii) $\dfrac{d}{dx}\tan x = \sec^2 x$

(iv) $\dfrac{d}{dx}\sec x = \sec x \tan x$ (v) $\dfrac{d}{dx}\text{cosec}\ x = -\text{cosec}\ x \cot x$ (vi) $\dfrac{d}{dx}\cot x = -\text{cosec}^2 x$

(3) **Differentiation of logarithmic and exponential functions :** The following formulae can be applied directly when differentiating logarithmic and exponential functions

(i) $\dfrac{d}{dx}\log x = \dfrac{1}{x}$, for $x > 0$ (ii) $\dfrac{d}{dx}e^x = e^x$

(iii) $\dfrac{d}{dx}a^x = a^x \log a$, for $a > 0$ (iv) $\dfrac{d}{dx}\log_a x = \dfrac{1}{x \log a}$, for $x > 0$, $a > 0$, $a \neq 1$

(4) **Differentiation of inverse trigonometrical functions :** The following formulae can be applied directly while differentiating inverse trigonometrical functions

(i) $\dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\sqrt{1-x^2}}$, for $-1 < x < 1$

(ii) $\dfrac{d}{dx}\cos^{-1}x = \dfrac{-1}{\sqrt{1-x^2}}$, for $-1 < x < 1$

(iii) $\dfrac{d}{dx}\sec^{-1}x = \dfrac{1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$

(iv) $\dfrac{d}{dx}\text{cosec}^{-1}x = \dfrac{-1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$

(v) $\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1+x^2}$, for $x \in R$

(vi) $\dfrac{d}{dx}\cot^{-1}x = \dfrac{-1}{1+x^2}$, for $x \in R$

(5) **Differentiation of hyperbolic functions :**

(i) $\dfrac{d}{dx}\sin h\,x = \cos h\,x$

(ii) $\dfrac{d}{dx}\cos h\,x = \sin h\,x$

(iii) $\dfrac{d}{dx}\tan h\,x = \sec h^2 x$

(iv) $\dfrac{d}{dx}\cot h\,x = -\text{cosec}\ h^2 x$

(v) $\dfrac{d}{dx}\sec h\,x = -\sec h\,x \tan h\ x$

(vi) $\dfrac{d}{dx}\text{cosec}\,h\,x = -\text{cosec}\,h\,x\cot h\,x$

(vii) $\dfrac{d}{dx}\sin h^{-1}x = 1/\sqrt{(1+x^2)}$

(viii) $\dfrac{d}{dx}\cos h^{-1}x = 1/\sqrt{(x^2-1)}$

(ix) $\dfrac{d}{dx}\tan h^{-1}x = 1/(x^2-1)$

(x) $\dfrac{d}{dx}\cot h^{-1}x = 1/(1-x^2)$

(xi) $\dfrac{d}{dx}\sec h^{-1}x = -1/x\sqrt{(1-x^2)}$

(xii) $\dfrac{d}{dx}\text{cosec}\,h^{-1}x = -1/x\sqrt{(1+x^2)}$

(6) **Differentiation by inverse trigonometrical substitution:** For trigonometrical substitutions following formulae and substitution should be remembered

(i) $\sin^{-1}x + \cos^{-1}x = \circledR/2$

(ii) $\tan^{-1}x + \cot^{-1}x = \circledR/2$

(iii) $\sec^{-1}x + \text{cosec}^{-1}x = \circledR/2$

(iv) $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right]$

(v) $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left[xy \mp \sqrt{(1-x^2)(1-y^2)}\right]$

(vi) $\tan^{-1}x \pm \tan^{-1}y = \tan^{-1}\left[\dfrac{x \pm y}{1 \mp xy}\right]$

(vii) $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$

(viii) $2\cos^{-1}x = \cos^{-1}(2x^2-1)$

(ix) $2\tan^{-1}x = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

(x) $3\sin^{-1}x = \sin^{-1}(3x-4x^3)$

(xi) $3\cos^{-1}x = \cos^{-1}(4x^3-3x)$

(xii) $3\tan^{-1}x = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$

(xiii) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right)$

(xiv) $\sin^{-1}(-x) = -\sin^{-1}x$

(xv) $\cos^{-1}(-x) = \circledR - \cos^{-1}x$

(xvi) $\tan^{-1}(-x) = -\tan^{-1}x$ or ® $-\tan^{-1}x$ (xvii) $\frac{®}{4} - \tan^{-1}x = \tan^{-1}\left(\frac{1-x}{1+x}\right)$

(7) **Some suitable substitutions**

S. N.	Function	Substitution	S. N.	Function	Substitution
(i)	$\sqrt{a^2-x^2}$	$x = a\sin^{-}$ or $a\cos^{-}$	(ii)	$\sqrt{x^2+a^2}$	$x = a\tan^{-}$ or $a\cot^{-}$
(iii)	$\sqrt{x^2-a^2}$	$x = a\sec^{-}$ or $a\,\mathrm{cosec}^{-}$	(iv)	$\sqrt{\frac{a-x}{a+x}}$	$x = a\cos 2^{-}$
(v)	$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$	$x^2 = a^2\cos 2^{-}$	(vi)	$\sqrt{ax-x^2}$	$x = a\sin^{2-}$
(vii)	$\sqrt{\frac{x}{a+x}}$	$x = a\tan^{2-}$	(viii)	$\sqrt{\frac{x}{a-x}}$	$x = a\sin^{2-}$
(ix)	$\sqrt{(x-a)(x-b)}$	$x = a\sec^{2-} - b\tan^{2-}$	(x)	$\sqrt{(x-a)(b-x)}$	$x = a\cos^{2-} + b\sin^{2-}$

1.3 Theorems for Differentiation

Let $f(x), g(x)$ and $u(x)$ be differentiable functions

(1) If at all points of a certain interval. $f'(x) = 0$, then the function $f(x)$ has a constant value within this interval.

(2) **Chain rule**

(i) **Case I :** If y is a function of u and u is a function of x, then derivative of y with respect to x is $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ or $y = f(u) \Rightarrow \frac{dy}{dx} = f'(u)\frac{du}{dx}$

(ii) **Case II :** If y and x both are expressed in terms of t, y and x both are differentiable with respect to t then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

(3) **Sum and difference rule :** Using linear property $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$

(4) **Product rule :** (i) $\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$ (ii) $\frac{d}{dx}(u.v.w.) = u.v.\frac{dw}{dx} + v.w.\frac{du}{dx} + u.w.\frac{dv}{dx}$

(5) **Scalar multiple rule :** $\frac{d}{dx}(k\,f(x)) = k\frac{d}{dx}f(x)$

(6) **Quotient rule :** $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$, provided $g(x) \neq 0$

Example: 5 The derivative of $f(x) = |x|^3$ at $x = 0$ is

(a) 0 (b) 1 (c) –1 (d) Not defined

Solution: (a) $f(x) = \begin{cases} x^3 & , x \geq 0 \\ -x^3 & , x < 0 \end{cases}$ and $f'(x) = \begin{cases} 3x^2 & , x \geq 0 \\ -3x^2 & , x < 0 \end{cases}$

$f'(0^+) = f'(0^-) = 0$

Example: 6 The first derivative of the function $(\sin 2x\cos 2x\cos 3x + \log_2 2^{x+3})$ with respect to x at $x = ®$ is

(a) 2 (b) -1 (c) $-2+2^{®}\log_e 2$ (d) $-2+\log_e 2$

Solution: (b) $f(x) = \sin 2x.\cos 2x.\cos 3x + \log_2 2^{x+3}$, $f(x) = \frac{1}{2}\sin 4x\cos 3x + (x+3)\log_2 2$, $f(x) = \frac{1}{4}[\sin 7x + \sin x] + x + 3$

Differentiate *w.r.t.* x,

$f'(x) = \frac{1}{4}[7\cos 7x + \cos x] + 1$, $f'(x) = \frac{1}{4}7\cos 7x + \frac{1}{4}\cos x + 1$, $f'(®) = -2+1 = -1$.

Example: 7 If $y = |\cos x| + |\sin x|$ then $\frac{dy}{dx}$ at $x = \frac{2®}{3}$ is

(a) $\frac{1-\sqrt{3}}{2}$ (b) 0 (c) $\frac{1}{2}(\sqrt{3}-1)$ (d) None of these

Solution: (c) Around $x = \frac{2®}{3}$, $|\cos x| = -\cos x$ and $|\sin x| = \sin x$

$\therefore\ y = -\cos x + \sin x\ \therefore\ \frac{dy}{dx} = \sin x + \cos x$

At $x = \frac{2®}{3}$, $\frac{dy}{dx} = \sin\frac{2®}{3} + \cos\frac{2®}{3} = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{1}{2}(\sqrt{3}-1)$.

Example: 8 If $f(x) = \log_x(\log x)$, then $f'(x)$ at $x = e$ is

(a) e (b) $1/e$ (c) 1 (d) None of these

Solution: (b) $f(x) = \log_x(\log x) = \frac{\log(\log x)}{\log x} \Rightarrow f'(x) = \frac{\frac{1}{x} - \frac{1}{x}\log(\log x)}{(\log x)^2} \Rightarrow f'(e) = \frac{\frac{1}{e} - 0}{1} = \frac{1}{e}$

Example: 9 If $f(x) = |\log x|$, then for $x \neq 1, f'(x)$ equals

(a) $\frac{1}{x}$ (b) $\frac{1}{|x|}$ (c) $\frac{-1}{x}$ (d) None of these

Solution: (d) $f(x) = |\log x| = \begin{cases} -\log x, & \text{if } 0 < x < 1 \\ \log x, & \text{if } x \geq 1 \end{cases} \Rightarrow f'(x) = \begin{cases} -\frac{1}{x}, & \text{if } 0 < x < 1 \\ \frac{1}{x}, & \text{if } x > 1 \end{cases}$.

Clearly $f'(1^-) = -1$ and $f'(1^+) = 1$, $\therefore$ $f'(x)$ does not exist at $x = 1$

Example: 10 $\frac{d}{dx}\left[\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right\}\right]$ equals to

(a) 1 (b) $\frac{x^2+1}{x^2-4}$ (c) $\frac{x^2-1}{x^2-4}$ (d) $e^x\frac{x^2-1}{x^2-4}$

Solution: (c) Let $y = \left[\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right\}\right] = \log e^x + \log\left(\frac{x-2}{x+2}\right)^{3/4}$

$\Rightarrow y = x + \frac{3}{4}[\log(x-2) - \log(x+2)] \Rightarrow \frac{dy}{dx} = 1 + \frac{3}{4}\left[\frac{1}{x-2} - \frac{1}{x+2}\right] = 1 + \frac{3}{(x^2-4)}$

$\Rightarrow \frac{dy}{dx} = \frac{x^2-1}{x^2-4}$.

Example: 11 If $x = \exp\left\{\tan^{-1}\left(\frac{y - x^2}{x^2}\right)\right\}$ then $\frac{dy}{dx}$ equals

(a) $2x[1 + \tan(\log x)] + x\sec^2(\log x)$ (b) $x[1 + \tan(\log x)] + \sec^2(\log x)$

(c) $2x[1 + \tan(\log x)] + x^2 \sec^2(\log x)$ (d) $2x[1 + \tan(\log x)] + \sec^2(\log x)$

Solution: (a) $x = \exp\left\{\tan^{-1}\left(\frac{y - x^2}{x^2}\right)\right\} \Rightarrow \log x = \tan^{-1}\left(\frac{y - x^2}{x^2}\right)$

$$\Rightarrow \frac{y - x^2}{x^2} = \tan(\log x) \Rightarrow y = x^2 \tan(\log x) + x^2 \Rightarrow \frac{dy}{dx} = 2x.\tan(\log x) + x^2.\frac{\sec^2(\log x)}{x} + 2x$$

$$\Rightarrow \frac{dy}{dx} = 2x\tan(\log x) + x\sec^2(\log x) + 2x \Rightarrow \frac{dy}{dx} = 2x[1 + \tan(\log x)] + x\sec^2(\log x).$$

Example: 12 If $y = \sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$, then $\frac{dy}{dx} =$

(a) 0 (b) $\frac{1}{\sqrt{x}+1}$ (c) 1 (d) None of these

Solution: (a) $y = \sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) = \cos^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) = \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = 0$ $\left\{\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right\}$

Example: 13 $\frac{d}{dx}\tan^{-1}\left[\frac{\cos x - \sin x}{\cos x + \sin x}\right]$

(a) $\frac{1}{2(1+x^2)}$ (b) $\frac{1}{1+x^2}$ (c) 1 (d) -1

Solution: (d) $\frac{d}{dx}\tan^{-1}\left[\frac{\cos x - \sin x}{\cos x + \sin x}\right] = \frac{d}{dx}\tan^{-1}\left[\tan\left(\frac{\pi}{4} - x\right)\right] = -1$.

Example: 14 $\frac{d}{dx}\left[\sin^2\cot^{-1}\left\{\sqrt{\frac{1-x}{1+x}}\right\}\right]$ equals

(a) -1 (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) 1

Solution: (b) Let $y = \sin^2\cot^{-1}\left\{\sqrt{\frac{1-x}{1+x}}\right\}$

Put $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$

$$\Rightarrow y = \sin^2\cot^{-1}\left\{\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right\} = \sin^2\cot^{-1}\left(\tan\frac{\theta}{2}\right) \Rightarrow y = \sin^2\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \cos^2\frac{\theta}{2} = \frac{1}{2}(1+\cos\theta) = \frac{1}{2}(1+x)$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}$$

Example: 15 If $y = \cos^{-1}\left(\frac{5\cos x - 12\sin x}{13}\right)$, $x \in \left(0, \frac{\pi}{2}\right)$, then $\frac{dy}{dx}$ is equal to

(a) 1 (b) -1 (c) 0 (d) None of these

Solution: (a) Let $\cos\alpha = \frac{5}{13}$. Then $\sin\alpha = \frac{12}{13}$. So, $y = \cos^{-1}\{\cos\alpha.\cos x - \sin\alpha.\sin x\}$

$\therefore y = \cos^{-1}\{\cos(x+\alpha)\} = x + \alpha$ ($\because$ $x + \alpha$ is in the first or the second quadrant)

$$\therefore \frac{dy}{dx} = 1.$$

Example: 16 $\frac{d}{dx}\cosh^{-1}(\sec x) =$

(a) $\sec x$ (b) $\sin x$ (c) $\tan x$ (d) $\operatorname{cosec} x$

Solution: (a) We know that $\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$, $\frac{d}{dx}\cosh^{-1}(\sec x) = \frac{1}{\sqrt{\sec^2 x - 1}}\sec x \tan x = \frac{\sec x \tan x}{\tan x} = \sec x$.

Example: 17 $\frac{d}{dx}\left[\left(\frac{\tan^2 2x - \tan^2 x}{1-\tan^2 2x \tan^2 x}\right)\cot 3x\right]$

(a) $\tan 2x \tan x$ (b) $\tan 3x \tan x$ (c) $\sec^2 x$ (d) $\sec x \tan x$

Solution: (c) Let $y = \frac{\tan^2 2x - \tan^2 x}{1-\tan^2 2x \tan^2 x} = \frac{(\tan 2x - \tan x)}{(1+\tan 2x \tan x)}\frac{(\tan 2x + \tan x)}{(1-\tan 2x \tan x)} = \tan(2x-x)\tan(2x+x) = \tan x \tan 3x$.

$\therefore \frac{d}{dx}[y.\cot 3x] = \frac{d}{dx}[\tan x] = \sec^2 x$.

Example: 18 If $f(x) = \cot^{-1}\left(\frac{x^x - x^{-x}}{2}\right)$, then $f'(1)$ is equal to

(a) -1 (b) 1 (c) $\log 2$ (d) $-\log 2$

Solution: (a) $f(x) = \cot^{-1}\left(\frac{x^x - x^{-x}}{2}\right)$

Put $x^x = \tan\theta$, $\therefore y = f(x) = \cot^{-1}\left(\frac{\tan^2\theta - 1}{2\tan\theta}\right) = \cot^{-1}(-\cot 2\theta) = \pi - \cot^{-1}(\cot 2\theta)$

$\Rightarrow y = \pi - 2\theta = \pi - 2\tan^{-1}(x^x) \Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^{2x}}.x^x(1+\log x) \Rightarrow f'(1) = -1$.

Example: 19 If $y = (1+x)(1+x^2)(1+x^4).......(1+x^{2^n})$ then $\frac{dy}{dx}$ at $x=0$ is

(a) 1 (b) -1 (c) 0 (d) None of these

Solution: (a) $y = \frac{(1-x)(1+x)(1+x^2).....(1+x^{2^n})}{1-x} = \frac{1-x^{2^{n+1}}}{1-x}$

$\therefore \frac{dy}{dx} = \frac{-2^{n+1}.x^{2^{n+1}-1}(1-x)+1-x^{2^{n+1}}}{(1-x)^2}$, $\therefore$ At $x=0$, $\frac{dy}{dx} = \frac{-2^{n+1}\,0.1+1-0}{1^2} = 1$.

Example: 20 If $f(x) = \cos x.\cos 2x.\cos 4x.\cos 8x.\cos 16x$ then $f'\left(\frac{\pi}{4}\right)$ is

(a) $\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) 1 (d) None of these

Solution: (a) $f(x) = \frac{2\sin x.\cos x.\cos 2x.\cos 4x.\cos 8x.\cos 16x}{2\sin x} = \frac{\sin 32x}{2^5 \sin x}$

$\therefore f'(x) = \frac{1}{32}.\frac{32\cos 32x.\sin x - \cos x.\sin 32x}{\sin^2 x}$

$\therefore f'\left(\frac{\pi}{4}\right) = \frac{32.\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}.0}{32.\left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{2}$.

1.4 Relation between dy/dx and dx/dy

Let x and y be two variables connected by a relation of the form $f(x, y) = 0$. Let Δx be a small change in x and let Δy be the corresponding change in y. Then $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ and $\frac{dx}{dy} = \lim_{\Delta y \to 0} \frac{\Delta x}{\Delta y}$.

Now, $\frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta y} = 1 \Rightarrow \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta y}\right) = 1$

$\Rightarrow \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \cdot \lim_{\Delta y \to 0} \frac{\Delta x}{\Delta y} = 1$ $[\because \Delta x \to 0 \Leftrightarrow \Delta y \to 0]$ $\Rightarrow \frac{dy}{dx} \cdot \frac{dx}{dy} = 1$. So, $\frac{dy}{dx} = \frac{1}{dx/dy}$.

1.5 Methods of Differentiation

(1) **Differentiation of implicit functions :** If y is expressed entirely in terms of x, then we say that y is an explicit function of x. For example $y = \sin x$, $y = e^x$, $y = x^2 + x + 1$ etc. If y is related to x but can not be conveniently expressed in the form of $y = f(x)$ but can be expressed in the form $f(x, y) = 0$, then we say that y is an implicit function of x.

(i) *Working rule* 1 : (a) Differentiate each term of $f(x, y) = 0$ with respect to x.

(b) Collect the terms containing dy/dx on one side and the terms not involving dy/dx on the other side.

(c) Express dy/dx as a function of x or y or both.

Note : ❑ In case of implicit differentiation, dy/dx may contain both x and y.

(ii) *Working rule* 2 : If $f(x, y)$= constant, then $\frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}$

where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial x}$ are partial differential coefficients of $f(x, y)$ with respect to x and y respectively.

Note : ❑ Partial differential coefficient of $f(x, y)$ with respect to x means the ordinary differential coefficient of $f(x, y)$ with respect to x keeping y constant.

Example: 21 If $xe^{xy} = y + \sin^2 x$, then at $x = 0$, $\frac{dy}{dx} =$

Solution: (c) We are given that $xe^{xy} = y + \sin^2 x$

When $x = 0$, we get $y = 0$

Differentiating both sides *w.r.t.* x, we get, $e^{xy} + xe^{xy}\left[x\frac{dy}{dx} + y\right] = \frac{dy}{dx} + 2\sin x \cos x$

Putting, $x = 0$, $y = 0$, we get $\frac{dy}{dx} = 1$.

Example: 22 If $\sin(x + y) = \log(x + y)$, then $\frac{dy}{dx} =$

(a) 2 (b) -2 (c) 1 (d) -1

Solution: (d) $\sin(x + y) = \log(x + y)$

Differentiating with respect to x, $\cos(x+y)\left[1+\frac{dy}{dx}\right]=\frac{1}{x+y}\left[1+\frac{dy}{dx}\right]$

$$\left[\cos(x+y)-\frac{1}{x+y}\right]\left[1+\frac{dy}{dx}\right]=0$$

$\because \quad \cos(x+y)\neq\frac{1}{x+y}$ for any x and y. So, $1+\frac{dy}{dx}=0$, $\frac{dy}{dx}=-1$.

Trick: It is an implicit function, so $\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}=-\frac{\cos(x+y)-\frac{1}{x+y}}{\cos(x+y)-\frac{1}{x+y}}=-1$.

Example: 23 If $\ln(x+y)=2xy$, then $y'(0)=$

(a) 1 (b) -1 (c) 2 (d) 0

Solution: (a) $\ln(x+y)=2xy \Rightarrow \frac{(1+dy/dx)}{(x+y)}=2\left(x\frac{dy}{dx}+y\right) \Rightarrow \frac{dy}{dx}=\frac{1-2xy-2y^2}{2x^2+2xy-1} \Rightarrow y'(0)=\frac{1-2}{-1}=1$, at $x=0,\ y=1$.

(2) **Logarithmic differentiation :** If differentiation of an expression or an equation is done after taking log on both sides, then it is called logarithmic differentiation. This method is useful for the function having following forms.

(i) $y=[f(x)]^{g(x)}$

(ii) $y=\frac{f_1(x).f_2(x)........}{g_1(x).g_2(x)........}$ where $g_i(x)\neq 0$ (where i = 1, 2, 3,.....), $f_i(x)$ and $g_i(x)$ both are differentiable

(i) **Case I :** $y=[f(x)]^{g(x)}$ where $f(x)$ and $g(x)$ are functions of x. To find the derivative of this type of functions we proceed as follows:

Let $y=[f(x)]^{g(x)}$. Taking logarithm of both the sides, we have $\log y=g(x).\log f(x)$

Differentiating with respect to x, we get $\frac{1}{y}\frac{dy}{dx}=g(x).\frac{1}{f(x)}\frac{df(x)}{dx}+\log\{f(x)\}.\frac{dg(x)}{dx}$

$$\therefore \frac{dy}{dx}=y\left[\frac{g(x)}{f(x)}.\frac{df(x)}{dx}+\log[f(x)].\frac{dg(x)}{dx}\right]=[f(x)^{g(x)}\left[\frac{g(x)}{f(x)}\frac{df(x)}{dx}+\log[f(x)\frac{dg(x)}{dx}\right]$$

(ii) **Case II :** $y=\frac{f_1(x).f_2(x)}{g_1(x).g_2(x)}$

Taking logarithm of both the sides, we have $\log y=\log[f_1(x)]+\log[f_2(x)]-\log[g_1(x)]-\log[g_2(x)]$

Differentiating with respect to x, we get $\frac{1}{y}\frac{dy}{dx}=\frac{f_1'(x)}{f_1(x)}+\frac{f_2'(x)}{f_2(x)}-\frac{g_1'(x)}{g_1(x)}-\frac{g_2'(x)}{g_2(x)}$

$$\frac{dy}{dx}=y\left[\frac{f_1'(x)}{f_1(x)}+\frac{f_2'(x)}{f_2(x)}-\frac{g_1'(x)}{g_1(x)}-\frac{g_2'(x)}{g_2(x)}\right]=\frac{f_1(x).f_2(x)}{g_1(x).g_2(x)}\left[\frac{f_1'(x)}{f_1(x)}+\frac{f_2'(x)}{f_2(x)}-\frac{g_1'(x)}{g_1(x)}-\frac{g_2'(x)}{g_2(x)}\right]$$

Working rule : (a) To take logarithm of the function (b) To differentiate the function

Example: 24 If $x^m y^n=2(x+y)^{m+n}$, the value of $\frac{dy}{dx}$ is

(a) $x+y$ (b) $\frac{x}{y}$ (c) $\frac{y}{x}$ (d) $x-y$

Solution: (c) $x^m y^n=2(x+y)^{m+n} \Rightarrow m\log x+n\log y=\log 2+(m+n)\log(x+y)$

Differentiating $w.r.t.$ x both sides

$$\frac{m}{x}+\frac{n}{y}\frac{dy}{dx}=\frac{m+n}{x+y}\left[1+\frac{dy}{dx}\right]\Rightarrow \frac{dy}{dx}=\frac{y}{x}.$$

Example: 25 If $y=(\sin x)^{\tan x}$, then $\frac{dy}{dx}$ is equal to

(a) $(\sin x)^{\tan x}.(1+\sec^2 x.\log\sin x)$ (b) $\tan x.(\sin x)^{\tan x-1}.\cos x$

(c) $(\sin x)^{\tan x}.,\sec^2 x\log\sin x$ (d) $\tan x.(\sin x)^{\tan x-1}$

Solution: (a) Given $y=(\sin x)^{\tan x}$

$\log y=\tan x.\log\sin x$

Differentiating *w.r.t.* x, $\frac{1}{y}.\frac{dy}{dx}=\tan x.\cot x+\log\sin x.\sec^2 x$

$\frac{dy}{dx}=(\sin x)^{\tan x}[1+\log\sin x.\sec^2 x]$.

(3) **Differentiation of parametric functions :** Sometimes x and y are given as functions of a single variable, *e.g.*, $x=\phi(t)$, $y=\psi(t)$ are two functions and t is a variable. In such a case x and y are called parametric functions or parametric equations and t is called the parameter. To find $\frac{dy}{dx}$ in case of parametric functions, we first obtain the relationship between x and y by eliminating the parameter t and then we differentiate it with respect to x. But every time it is not convenient to eliminate the parameter. Therefore $\frac{dy}{dx}$ can also be obtained by the following formula

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$$

To prove it, let Δx and Δy be the changes in x and y respectively corresponding to a small change Δt in t.

Since $\frac{\Delta y}{\Delta x}=\frac{\Delta y/\Delta t}{\Delta x/\Delta t}$, $\therefore \frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{\lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t}}{\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t}}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\Psi'(t)}{\phi'(t)}$

Example: 26 If $x=a(\cos\theta+\theta\sin\theta)$, $y=a(\sin\theta-\theta\cos\theta)$, $\frac{dy}{dx}=$

(a) $\cos\theta$ (b) $\tan\theta$ (c) $\sec\theta$ (d) $\text{cosec}\,\theta$

Solution: (b) $\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{a[\cos\theta-\theta(-\sin\theta)-\cos\theta]}{a[-\sin\theta+\theta\cos\theta+\sin\theta]}=\frac{\theta\sin\theta}{\theta\cos\theta}=\tan\theta$.

Example: 27 If $\cos x=\frac{1}{\sqrt{1+t^2}}$ and $\sin y=\frac{t}{\sqrt{1+t^2}}$, then $\frac{dy}{dx}=$

(a) -1 (b) $\frac{1-t}{1+t^2}$ (c) $\frac{1}{1+t^2}$ (d) 1

Solution: (d) Obviously $x=\cos^{-1}\frac{1}{\sqrt{1+t^2}}$ and $y=\sin^{-1}\frac{t}{\sqrt{1+t^2}}$

$\Rightarrow x=\tan^{-1}t$ and $y=\tan^{-1}t\Rightarrow y=x\Rightarrow \frac{dy}{dx}=1$.

Example: 28 If $x=\frac{1-t^2}{1+t^2}$ and $y=\frac{2t}{1+t^2}$, then $\frac{dy}{dx}=$

(a) $\frac{-y}{x}$ (b) $\frac{y}{x}$ (c) $\frac{-x}{y}$ (d) $\frac{x}{y}$

Solution: (c) $x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$

Put $t = \tan\theta$ in both the equations, we get $x = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$ and $y = \frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$. Differentiating both the equations, we get $\frac{dx}{d\theta} = -2\sin 2\theta$ and $\frac{dy}{d\theta} = 2\cos 2\theta$.

Therefore $\frac{dy}{dx} = -\frac{\cos 2\theta}{\sin 2\theta} = -\frac{x}{y}$.

(4) **Differentiation of infinite series :** If y is given in the form of infinite series of x and we have to find out $\frac{dy}{dx}$ then we remove one or more terms, it does not affect the series

(i) If $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + ...\infty}}}$, then $y = \sqrt{f(x) + y} \Rightarrow y^2 = f(x) + y$

$$2y\frac{dy}{dx} = f'(x) + \frac{dy}{dx}, \quad \therefore \frac{dy}{dx} = \frac{f'(x)}{2y-1}$$

(ii) If $y = f(x)^{f(x)^{f(x)^{f(x).....\infty}}}$ then $y = f(x)^y$

$\therefore \log y = y \log f(x)$

$$\frac{1}{y}\frac{dy}{dx} = \frac{y.f'(x)}{f(x)} + \log f(x).\frac{dy}{dx}, \quad \therefore \frac{dy}{dx} = \frac{y^2 f'(x)}{f(x)[1 - y\log f(x)]}$$

(iii) If $y = f(x) + \cfrac{1}{f(x) + \cfrac{1}{f(x) + ...\infty}}$ then $\frac{dy}{dx} = \frac{yf'(x)}{2y - f(x)}$

Example: 29 If $y = \sqrt{x + \sqrt{x + \sqrt{x + ...\text{to } \infty}}}$ then $\frac{dy}{dx} =$

(a) $\frac{x}{2y-1}$ (b) $\frac{2}{2y-1}$ (c) $\frac{-1}{2y-1}$ (d) $\frac{1}{2y-1}$

Solution: (d) $y = \sqrt{x + \sqrt{x + \sqrt{x + ...\text{to } \infty}}} \Rightarrow y = \sqrt{x+y} \Rightarrow y^2 = x + y \Rightarrow 2y\frac{dy}{dx} = 1 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx}(2y-1) = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y-1}$

Example: 30 If $y = x^{x^{x......\infty}}$, then $x(1 - y\log_e x)\frac{dy}{dx}$ is

Solution: (b) $y = x^{x^{x......\infty}} \Rightarrow y = x^y \Rightarrow \log_e y = y\log_e x \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \frac{y}{x} + \log_e x\frac{dy}{dx} \Rightarrow \left(\frac{1}{y} - \log_e x\right)\frac{dy}{dx} = \frac{y}{x} \Rightarrow x(1 - y\log_e x)\frac{dy}{dx} = y^2$

Example: 31 If $y = x^2 + \cfrac{1}{x^2 + \cfrac{1}{x^2 + \cfrac{1}{x^2 + ...\infty}}}$, then $\frac{dy}{dx} =$

(a) $\frac{2xy}{2y - x^2}$ (b) $\frac{xy}{y + x^2}$ (c) $\frac{xy}{y - x^2}$ (d) $\frac{2x}{2 + \frac{x^2}{y}}$

Solution: (a) $y = x^2 + \frac{1}{y} \Rightarrow y^2 = x^2y + 1 \Rightarrow 2y\frac{dy}{dx} = y.2x + x^2\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{2xy}{2y - x^2}$

Example: 32 If $x = e^{y+e^{y+\dots\dots\text{to}\,\infty}}$, then $\frac{dy}{dx}$ is

(a) $\frac{1+x}{x}$ (b) $\frac{1}{x}$ (c) $\frac{1-x}{x}$ (d) $\frac{x}{1+x}$

Solution: (c) $x = e^{y+x}$

Taking log both sides, $\log x = (y+x)\log e = y + x \Rightarrow y + x = \log x \Rightarrow \frac{dy}{dx} + 1 = \frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{1}{x} - 1 = \frac{1-x}{x}$

(5) **Differentiation of composite function :** Suppose function is given in form of $fog(x)$ or $f[g(x)]$

Working rule **:** Differentiate applying chain rule $\frac{d}{dx}f[g(x)] = f'[g(x)].g'(x)$

Example: 33 If $f(x) = |x - 2|$ and $g(x) = f(f(x))$, then for $x>20$, $g'(x)$ equals

(a) -1 (b) 1 (c) 0 (d) None of these

Solution: (b) For $x > 20$, we have

$f(x) = |x-2| = x - 2$ and, $g(x) = f(f(x)) = f(x-2) = x - 2 - 2 = x - 4$

$\therefore g'(x) = 1$

Example: 34 If g is inverse of f and $f'(x) = \frac{1}{1+x^n}$, then $g'(x)$ equals

(a) $1 + x^n$ (b) $1 + [f(x)]^n$ (c) $1 + [g(x)]^n$ (d) None of these

Solution: (c) Since g is inverse of f. Therefore,

$fog(x) = x$ for all $x \Rightarrow \frac{d}{dx}\{fog(x)\} = 1$ for all x

$\Rightarrow f'(g(x)).g'(x) = 1 \Rightarrow f'\{g(x)\} = \frac{1}{g'(x)} \Rightarrow \frac{1}{1+[g(x)]^n} = \frac{1}{g'(x)}$ $\left[\because f'(x) = \frac{1}{1+x^n}\right]$

$\Rightarrow g'(x) = 1 + [g(x)]^n$

1.6 Differentiation of a Function with Respect to Another Function

In this section we will discuss derivative of a function with respect to another function. Let $u = f(x)$ and $v = g(x)$ be two functions of x. Then, to find the derivative of $f(x)$ *w.r.t.* $g(x)$ i.e., to find $\frac{du}{dv}$ we use the following formula

$$\frac{du}{dv} = \frac{du/dx}{dv/dx}$$

Thus, to find the derivative of $f(x)$ *w.r.t.* $g(x)$ we first differentiate both *w.r.t.* x and then divide the derivative of $f(x)$ *w.r.t.* x by the derivative of $g(x)$ *w.r.t.* x.

Example: 35 The differential coefficient of $\tan^{-1}\frac{2x}{1-x^2}$ *w.r.t.* $\sin^{-1}\frac{2x}{1+x^2}$ is

(a) 1 (b) -1 (c) 0 (d) None of these

Solution: (a) Let $y_1 = \tan^{-1}\frac{2x}{1-x^2}$ and $y_2 = \sin^{-1}\frac{2x}{1+x^2}$

Putting $x = \tan^{-}$

$\therefore\ y_1 = \tan^{-1}\tan 2\bar{} = 2\bar{} = 2\tan^{-1}x$ and $y_2 = \sin^{-1}\sin 2\bar{} = 2\tan^{-1}x$

Again $\frac{dy_1}{dx} = \frac{d}{dx}[2\tan^{-1}x] = \frac{2}{1+x^2}$ ....(i)

and $\frac{dy_2}{dx} = \frac{d}{dx}[2\tan^{-1}x] = \frac{2}{1+x^2}$ ....(ii)

Hence $\frac{dy_1}{dy_2} = 1$

Example: 36 The first derivative of the function $\left[\cos^{-1}\left(\sin\frac{\sqrt{1+x}}{2}\right) + x^x\right]$ with respect to x at $x = 1$ is

(a) $\frac{3}{4}$ (b) 0 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

Solution: (a) $f(x) = \cos^{-1}\left[\cos\left(\frac{\text{®}}{2} - \sqrt{\frac{1+x}{2}}\right)\right] + x^x = \frac{\text{®}}{2} - \sqrt{\frac{1+x}{2}} + x^x$

$\therefore\ f'(x) = -\frac{1}{\sqrt{2}}\cdot\frac{1}{2\sqrt{1+x}} + x^x(1+\log x) \Rightarrow f'(1) = -\frac{1}{4} + 1 = \frac{3}{4}$

1.7 Successive Differentiation or Higher Order Derivatives

(1) **Definition and notation :** If y is a function of x and is differentiable with respect to x, then its derivative $\frac{dy}{dx}$ can be found which is known as derivative of first order. If the first derivative $\frac{dy}{dx}$ is also a differentiable. function, then it can be further differentiated with respect to x and this derivative is denoted by d^2y/dx^2 which is called the second derivative of y with respect to x further if $\frac{d^2y}{dx^2}$ is also differentiable then its derivative is called third derivative of y which is denoted by $\frac{d^3y}{dx^3}$. Similarly n^{th} derivative of y is denoted by $\frac{d^ny}{dx^n}$. All these derivatives are called as successive derivative and this process is known as successive differentiation. We also use the following symbols for the successive derivatives of $y = f(x)$:

$y_1,\ y_2,\ y_3, ..., y_n, ...$ $\qquad y^I,\ y^{II},\ y^{III} ...., y^n, ...$

$Dy,\ D^2y,\ D^3y ...., D^ny, ...$ (where $D = \frac{d}{dx}$) $\qquad \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, ..., \frac{d^ny}{dx^n}, ...$

$f'(x),\ f''(x),\ f'''(x), ..., f^n(x), ...$

If $y = f(x)$, then the value of the n^{th} order derivative at $x = a$ is usually denoted by

$\left(\frac{d^ny}{dx^n}\right)_{x=a}$ or $(y_n)_{x=a}$ or $(y^n)_{x=a}$ or $f^n(a)$

(2) n^{th} Derivatives of some standard functions :

(i) (a) $\frac{d^n}{dx^n}\sin(ax+b) = a^n \sin\left(\frac{n\text{®}}{2} + ax + b\right)$ (b) $\frac{d^n}{dx^n}\cos(ax+b) = a^n \cos\left(\frac{n\text{®}}{2} + ax + b\right)$

(ii) $\frac{d^n}{dx^n}(ax+b)^m = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$, where $m > n$

Particular cases :

(i) (a) When $m = n$

$$D^n\{(ax+b)^n\} = a^n.n!$$

(b) When $m < n, D^n\{(ax+b)^m\} = 0$

(ii) When $a = 1, b = 0$, then $y = x^n$

$$\therefore\ D^n(x^m) = m(m-1).......(m-n+1)x^{m-n} = \frac{m!}{(m-n)!}x^{m-n}$$

(iii) When $a = 1$, $b = 0$ and $m = n$,

then $y = x^n$

$$\therefore D^n(x^n) = n!$$

(iv) When $m = -1$, $y = \dfrac{1}{(ax+b)}$

$$D^n(y) = a^n(-1)(-2)(-3)........(-n)(ax+b)^{-1-n}$$

$$= a^n(-1)^n(1.2.3......n)(ax+b)^{-1-n} = \frac{a^n(-1)^n n!}{(ax+b)^{n+1}}$$

(3) $\dfrac{d^n}{dx^n}\log(ax+b) = \dfrac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$

(4) $\dfrac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$

(5) $\dfrac{d^n(a^x)}{dx^n} = a^x(\log a)^n$

(6) (i) $\dfrac{d^n}{dx^n}e^{ax}\sin(bx+c) = r^n e^{ax}\sin(bx+c+n\text{¤})$

where $r = \sqrt{a^2+b^2}$; $\text{¤} = \tan^{-1}\dfrac{b}{a}$, $y = e^{ax}\sin(bx+c)$

(ii) $\dfrac{d^n}{dx^n}e^{ax}\cos(bx+c) = r^n e^{ax}\cos(bx+c+n\text{¤})$

Example: 37 If $y = \left(x+\sqrt{1+x^2}\right)^n$, then $(1+x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx}$ is

(a) n^2y (b) $-n^2y$ (c) $-y$ (d) $2x^2y$

Solution: (a) $y = (x+\sqrt{1+x^2})^n \Rightarrow \dfrac{dy}{dx} = n(x+\sqrt{1+x^2})^{n-1}\left(1+\dfrac{x}{\sqrt{1+x^2}}\right) \Rightarrow \dfrac{dy}{dx} = \dfrac{n(x+\sqrt{1+x^2})^n}{\sqrt{1+x^2}} \Rightarrow (\sqrt{1+x^2})\dfrac{dy}{dx} = n\left(x+\sqrt{1+x^2}\right)^n$

$$\Rightarrow \frac{d^2y}{dx^2}.\sqrt{1+x^2} + \frac{dy}{dx}\left(\frac{x}{\sqrt{1+x^2}}\right) = n^2\left(x+\sqrt{1+x^2}\right)^{n-1}\left(1+\frac{x}{\sqrt{1+x^2}}\right)$$

$$\Rightarrow (1+x^2).\frac{d^2y}{dx^2} + x.\frac{dy}{dx} = n^2(x+\sqrt{1+x^2})^n \Rightarrow (1+x^2)\frac{d^2y}{dx^2} + x.\frac{dy}{dx} = n^2y.$$

Example: 38 If $f(x) = x^n$, then the value of $f(1) - \dfrac{f'(1)}{1!} + \dfrac{f''(1)}{2!} - \dfrac{f'''(1)}{3!} + ... + \dfrac{(-1)^n f^n(1)}{n!}$ is

(a) 2^n (b) 2^{n-1} (c) 0 (d) 1

Solution: (c) $f(x) = x^n \Rightarrow f(1) = 1$, $f'(x) = nx^{n-1} \Rightarrow f'(1) = n$

$f''(x) = n(n-1)x^{n-2} \Rightarrow f''(1) = n(n-1)$ .....

$f^n(x) = n! \Rightarrow f^n(1) = n!$, $\therefore\ f(1) - \dfrac{f'(1)}{1!} + \dfrac{f''(1)}{2!} ...... + \dfrac{(-1)^n f^n(1)}{n!}$

$$= 1 - \frac{n}{1!} + \frac{n(n-1)}{2!} - \frac{n(n-1)(n-2)}{3!} + ... + (-1)^n\frac{n!}{n!} = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + ... + (-1)^n\ {}^nC_n = 0.$$

Example: 39 If $f(x) = \tan^{-1}\left\{\dfrac{\log\left(\dfrac{e}{x^2}\right)}{\log(ex^2)}\right\} + \tan^{-1}\left(\dfrac{3+2\log x}{1-6\log x}\right)$, then $\dfrac{d^ny}{dx^n}$ is $(n \geq 1)$

(a) $\tan^{-1}\{(\log x)^n\}$ (b) 0 (c) 1/2 (d) None of these

Solution: (b) We have $y=\tan^{-1}\left(\frac{\log e-\log x^2}{\log e+\log x^2}\right)+\tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)=\tan^{-1}\left(\frac{1-2\log x}{1+2\log x}\right)+\tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$

$=\tan^{-1}1-\tan^{-1}(2\log x)+\tan^{-1}3+\tan^{-1}(2\log x) \Rightarrow y=\tan^{-1}1+\tan^{-1}3 \Rightarrow \frac{dy}{dx}=0 \Rightarrow \frac{d^ny}{dx^n}=0$.

Example: 40 If $f(x)=(\cos x+i\sin x)(\cos 3x+i\sin 3x).....(\cos(2n-1)x+i\sin(2n-1)x)$, then $f''(x)$ is equal to

(a) $n^2 f(x)$ (b) $-n^4 f(x)$ (c) $-n^2 f(x)$ (d) $n^4 f(x)$

Solution: (b) We have, $f(x)=\cos(x+3x+....+(2n-1)x)+i\sin(x+3x+5x+....+(2n-1)x)=\cos n^2x+i\sin n^2x$

$\Rightarrow f'(x)=-n^2(\sin n^2x)+n^2(i\cos n^2x) \Rightarrow f''(x)=-n^4\cos n^2x-n^4 i\sin n^2x$

$\Rightarrow f''(x)=-n^4(\cos n^2x+i\sin n^2x) \Rightarrow f''(x)=-n^4 f(x)$

1.8 n^{th} Derivative using Partial fractions

For finding n^{th} derivative of fractional expressions whose numerator and denominator are rational algebraic expression, firstly we resolve them into partial fractions and then we find n^{th} derivative by using the formula giving the n^{th} derivative of $\frac{1}{ax+b}$.

Example: 41 If $y=\frac{x^4}{x^2-3x+2}$, then for $n>2$ the value of y_n is equal to

(a) $(-1)^n n![16(x-2)^{-n-1}-(x-1)^{-n-1}]$ (b) $(-1)^n n![16(x-2)^{-n-1}+(x-1)^{-n-1}]$

(c) $n![16(x-2)^{-n-1}+(x-1)^{-n-1}]$ (d) None of these

Solution: (a) $y=\frac{x^4}{x^2-3x+2}=x^2+3x+7+\frac{15x-14}{(x-1)(x-2)}=x^2+3x+7-\frac{1}{(x-1)}+\frac{16}{(x-2)}$

$\therefore y_n=D^n(x^2)+D^n(3x)+D^n(7)-D^n[(x-1)^{-1}]+16D^n[(x-2)^{-1}]$

$=(-1)^n n![-(x-1)^{-n-1}+16(x-2)^{-n-1}]=(-1)^n n![16(x-2)^{-n-1}-(x-1)^{-n-1}]$.

1.9 Differentiation of Determinants

Let $\Delta(x)=\begin{vmatrix} a_1(x) & b_1(x)\\ a_2(x) & b_2(x)\end{vmatrix}$. Then $\Delta'(x)=\begin{vmatrix} a_1'(x) & b_1'(x)\\ a_2(x) & b_2(x)\end{vmatrix}+\begin{vmatrix} a_1(x) & b_1(x)\\ a_2'(x) & b_2'(x)\end{vmatrix}$

If we write $\Delta(x)=|C_1C_2C_3|$. Then $\Delta'(x)=|C_1'C_2C_3|+|C_1C_2'C_3|+|C_1C_2C_3'|$

Similarly, if $\Delta(x)=\begin{vmatrix}R_1\\R_2\\R_3\end{vmatrix}$, then $\Delta'(x)=\begin{vmatrix}R_1'\\R_2\\R_3\end{vmatrix}+\begin{vmatrix}R_1\\R_2'\\R_3\end{vmatrix}+\begin{vmatrix}R_1\\R_2\\R_3'\end{vmatrix}$

Thus, to differentiate a determinant, we differentiate one row (or column) at a time, keeping others unchanged.

Example: 42 If $f_r(x), g_r(x), h_r(x), r=1,2,3$ are polynomials in x such that $f_r(a)=g_r(a)=h_r(a), r=1,2,3$ and

$F(x)=\begin{vmatrix} f_1(x) & f_2(x) & f_3(x)\\ g_1(x) & g_2(x) & g_3(x)\\ h_1(x) & h_2(x) & h_3(x)\end{vmatrix}$, then find $F'(x)$ at $x=a$

(a) 0 (b) $f_1(a)g_2(a)h_3(a)$ (c) 1 (d) None of these

Solution: (a) $F'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1'(x) & g_2'(x) & g_3'(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1'(x) & h_2'(x) & h_3'(x) \end{vmatrix}$

y $F'(a) = \begin{vmatrix} f_1'(a) & f_2'(a) & f_3'(a) \\ g_1(a) & g_2(a) & g_3(a) \\ h_1(a) & h_2(a) & h_3(a) \end{vmatrix} + \begin{vmatrix} f_1(a) & f_2(a) & f_3(a) \\ g_1'(a) & g_2'(a) & g_3'(a) \\ h_1(a) & h_2(a) & h_3(a) \end{vmatrix} + \begin{vmatrix} f_1(a) & f_2(a) & f_3(a) \\ g_1(a) & g_2(a) & g_3(a) \\ h_1'(a) & h_2'(a) & h_3'(a) \end{vmatrix}$

$= .0 + 0 + 0 = 0$ $[\because f_r(a) = g_r(a) = h_r(a), r = 1, 2, 3]$

Example: 43 Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ 1 & p^2 & p^3 \end{vmatrix}$ where p is a constant. Then $\frac{d^3}{dx^3}[f(x)]$ at $x = 0$ is

(a) p (b) $p + p^2$ (c) $p + p^3$ (d) Independent of p

Solution: (d) Given $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ 1 & p^2 & p^3 \end{vmatrix}$, 2nd and 3rd rows are constant, so only 1st row will take part in differentiation

$\therefore \frac{d^3}{dx^3} f(x) = \begin{vmatrix} \frac{d^3}{dx^3}x^3 & \frac{d^3}{dx^3}\sin x & \frac{d^3}{dx^3}\cos x \\ 6 & -1 & 0 \\ 1 & p^2 & p^3 \end{vmatrix}$

We know that $\frac{d^n}{dx^n}x^n = n!, \frac{d^n}{dx^n}\sin x = \sin(x + \frac{n®}{2})$ and $\frac{d^n}{dx^n}\cos x = \cos(x + \frac{n®}{2})$

Using these results, $\frac{d^3}{dx^3} f(x) = \begin{vmatrix} 3! & \sin\left(x + \frac{3®}{2}\right) & \cos\left(x + \frac{3®}{2}\right) \\ 6 & -1 & 0 \\ 1 & p^2 & p^3 \end{vmatrix}$

$\left.\frac{d^3}{dx^3} f(x)\right|_{\text{at } x=0} = \begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ 1 & p^2 & p^3 \end{vmatrix} = 0$ *i.e.*, independent of p.

1.10 Differentiation of Integral Function

If $g_1(x)$ and $g_2(x)$ both functions are defined on $[a, b]$ and differentiable at a point $x \in (a, b)$ and $f(t)$ is continuous for $g_1(a) \le f(t) \le g_2(b)$

Then $\frac{d}{dx}\int_{g_1(x)}^{g_2(x)} f(t)dt = f[g_2(x)]g_2'(x) - f[g_1(x)]g_1'(x) = f[g_2(x)]\frac{d}{dx}g_2(x) - f[g_1(x)]\frac{d}{dx}g_1(x)$.

Example: 44 If $F(x) = \int_{x^2}^{x^3} \log t\, dt\ (x > 0)$, then $F'(x) =$

(a) $(9x^2 - 4x)\log x$ (b) $(4x - 9x^2)\log x$ (c) $(9x^2 + 4x)\log x$ (d) None of these

Solution: (a) Applying formula we get $F'(x) = (\log x^3)3x^2 - (\log x^2)2x$

$= (3\log x)3x^2 - 2x(2\log x) = 9x^2 \log x - 4x\log x = (9x^2 - 4x)\log x$.

Example: 45 If $x = \int_0^y \frac{1}{\sqrt{1 + 4t^2}}dt$, then $\frac{d^2y}{dx^2}$ is

(a) $2y$ (b) $4y$ (c) $8y$ (d) $6\,y$

Solution: (b) $x = \int_0^y \frac{1}{\sqrt{1+4t^2}}dt \Rightarrow \frac{dx}{dy} = \frac{1}{\sqrt{1+4y^2}} \Rightarrow \frac{dy}{dx} = \sqrt{1+4y^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{4y}{\sqrt{1+4y^2}}\frac{dy}{dx} \Rightarrow \frac{d^2y}{dx^2} = \frac{4y}{\sqrt{1+4y^2}}\cdot\sqrt{1+4y^2} = 4y$

1.11 Leibnitz's Theorem

G.W. Leibnitz, a German mathematician gave a method for evaluating the nth differential coefficient of the product of two functions. This method is known as Leibnitz's theorem.

Statement of the theorem – If u and v are two functions of x such that their nth derivative exist then $D^n(u.v.) = {}^nC_0(D^nu)v + {}^nC_1D^{n-1}u.Dv + {}^nC_2D^{n-2}u.D^2v + ... + {}^nC_rD^{n-r}u.D^rv + ... + u.(D^nv)$.

Note : ❒ The success in finding the nth derivative by this theorem lies in the proper selection of first and second function. Here first function should be selected whose nth derivative can be found by standard formulae. Second function should be such that on successive differentiation, at some stage, it becomes zero so that we need not to write further terms.

Example: 46 If $y = x^2e^x$, then value of y_n is

(a) $\{x^2 - 2nx + n(n-1)\}e^x$ (b) $\{x^2 + 2nx + n(n-1)\}e^x$

(c) $\{x^2 + 2nx - n(n-1)\}e^x$ (d) None of these

Solution: (b) Applying Leibnitz's theorem by taking x^2 as second function. We get , $D^ny = D^n(e^x.x^2)$

$$= {}^nC_0D^n(e^x)x^2 + {}^nC_1D^{n-1}(e^x).D(x^2) + {}^nC_2D^{n-2}(e^x).D^2(x^2) + ... = e^x.x^2 + ne^x.2x + \frac{n(n-1)}{2!}e^x.2 + 0 + 0 + ...$$

$y_n = \{x^2 + 2nx + n(n-1)\}e^x$.

Example: 47 If $y = x^2\log x$, then value of y_n is

(a) $\frac{(-1)^{n-1}(n-3)!}{x^{n-2}}$ (b) $\frac{(-1)^{n-1}(n-3)!}{x^{n-2}}.2$ (c) $\frac{(-1)^{n-1}(n-2)!}{x^{n-2}}$ (d) None of these

Solution: (b) Applying Leibnitz's theorem by taking x^2 as second function, we get, $D^ny = D^n(\log x.x^2)$

$$= {}^nC_0D^n(\log x).x^2 + {}^nC_1D^{n-1}(\log x).D(x^2) + {}^nC_2D^{n-2}(\log x)D^2(x^2) + ...$$

$$= \frac{(-1)^{n-1}(n-1)!}{x^n}.x^2 + n.\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}.2x + \frac{n(n-1)}{2!}\frac{(-1)^{n-3}(n-3)!}{x^{n-2}}.2 + 0 + 0.........$$

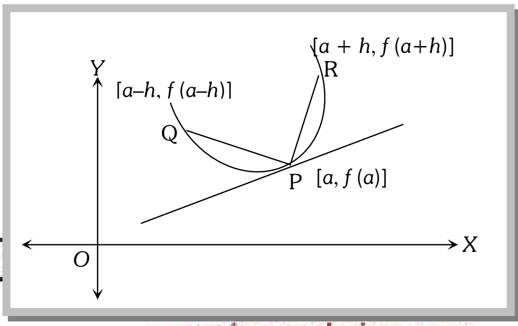
$$= \frac{(-1)^{n-1}(n-1)!}{x^{n-2}} + \frac{2n(-1)^{n-2}(n-2)!}{x^{n-2}} + \frac{n(n-1)(-1)^{n-3}(n-3)!}{x^{n-2}}$$

$$= \frac{(-1)^{n-1}(n-3)!}{x^{n-2}} \times \{(n-1)(n-2) - 2n(n-2) + n(n-1)\} = .\ \frac{(-1)^{n-1}(n-3)!}{x^{n-2}}.2$$

Differentiability

1.1 Differentiability of a Function at a Point

(1) **Meaning of differentiability at a point :** Consider the function $f(x)$ defined on an open interval (b, c) let $P(a, f(a))$ be a point on the curve $y = f(x)$ and let $Q(a-h, f(a-h))$ and $R(a+h, f(a+h))$ be two neighbouring points on the left hand side and right hand side respectively of the point P.



Then slope of chord $PQ = \dfrac{f(a-h)-f(a)}{(a-h)-a} = \dfrac{f(a-h)-f(a)}{-h}$

and, slope of chord $PR = \dfrac{f(a+h)-f(a)}{a+h-a} = \dfrac{f(a+h)-f(a)}{h}$.

$\because$ As $h \to 0$, point Q and R both tends to P from left hand and right hand respectively. Consequently, chords PQ and PR becomes tangent at point P.

Thus, $\lim\limits_{h \to 0} \dfrac{f(a-h)-f(a)}{-h} = \lim\limits_{h \to 0}$ (slope of chord PQ) $= \lim\limits_{Q \to P}$ (slope of chord PQ)

Slope of the tangent at point P, which is limiting position of the chords drawn on the left hand side of point P and $\lim\limits_{h \to 0} \dfrac{f(a+h)-f(a)}{h} = \lim\limits_{h \to 0}$ (slope of chord PR) $= \lim\limits_{R \to P}$ (slope of chord PR).

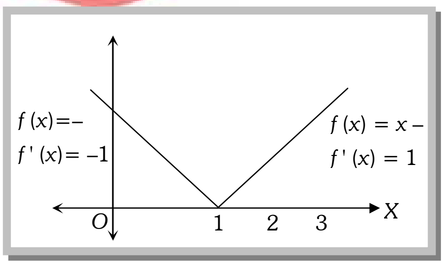
$\Rightarrow$ Slope of the tangent at point P, which is the limiting position of the chords drawn on the right hand side of point P.

Now, $f(x)$ is differentiable at $x = a \Leftrightarrow \lim\limits_{h \to 0} \dfrac{f(a-h)-f(a)}{-h} = \lim\limits_{h \to 0} \dfrac{f(a+h)-f(a)}{h}$

$\Leftrightarrow$ There is a unique tangent at point P.

Thus, $f(x)$ is differentiable at point P, iff there exists a unique tangent at point P. In other words, $f(x)$ is differentiable at a point P iff the curve does not have P as a corner point. *i.e.*, "the function is not differentiable at those points on which function has jumps (or holes) and sharp edges."

Let us consider the function $f(x) = |x-1|$, which can be graphically shown,



Which show $f(x)$ is not differentiable at $x = 1$. Since, $f(x)$ has sharp edge at $x = 1$.

Mathematically : The right hand derivative at $x = 1$ is 1 and left-hand derivative at $x = 1$ is –1. Thus, $f(x)$ is not differentiable at $x = 1$.

(2) **Right hand derivative :** Right hand derivative of $f(x)$ at $x = a$, denoted by $f'(a+0)$ or $f'(a+)$, is the $\lim\limits_{h \to 0} \dfrac{f(a+h)-f(a)}{h}$.

(3) **Left hand derivative :** Left hand derivative of $f(x)$ at $x = a$, denoted by $f'(a-0)$ or $f'(a-)$, is the $\lim\limits_{h \to 0} \dfrac{f(a-h)-f(a)}{-h}$.

(4) A function $f(x)$ is said to be differentiable (finitely) at $x = a$ if $f'(a+0) = f'(a-0) =$ finite

i.e., $\lim\limits_{h \to 0} \dfrac{f(a+h)-f(a)}{h} = \lim\limits_{h \to 0} \dfrac{f(a-h)-f(a)}{-h} =$ finite and the common limit is called the derivative of $f(x)$ at $x = a$, denoted by $f'(a)$. Clearly, $f'(a) = \lim\limits_{x \to a} \dfrac{f(x)-f(a)}{x-a}$ $\{x \to a$ from the left as well as from the right$\}$.

Example: 1 Consider $f(x)=\begin{cases}\dfrac{x^2}{|x|}, & x\neq 0\\ 0 & , x=0\end{cases}$

(a) $f(x)$ is discontinuous everywhere
(b) $f(x)$ is continuous everywhere but not differentiable at $x = 0$
(c) $f'(x)$ exists in (–1, 1)
(d) $f'(x)$ exists in (–2, 2)

Solution: (b) We have, $f(x)=\begin{cases}\dfrac{x^2}{|x|}, & x\neq 0\\ 0 & , x=0\end{cases} = \begin{cases}\dfrac{x^2}{x}=x, & x>0\\ 0 & , x=0\\ \dfrac{x^2}{-x}=-x, & x<0\end{cases}$

$\Rightarrow \lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-} -x=0$, $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+} x=0$ and $f(0) = 0$.

So $f(x)$ is continuous at $x=0$. Also $f(x)$ is continuous for all other values of x. Hence, $f(x)$ is everywhere continuous.

Also, $Rf'(0)=f'(0+0)=\lim\limits_{h\to 0}\dfrac{f(h)-f(0)}{h-0}=\lim\limits_{h\to 0}\dfrac{h-0}{h}=1$

i.e. $Rf'(0)=1$ and $Lf'(0)=f'(0-0)=\lim\limits_{h\to 0}\dfrac{f(-h)-f(0)}{-h}=\lim\limits_{h\to 0}\dfrac{h}{-h}=-1$

i.e. $Lf'(0)=-1$ So, $Lf'(0)\neq Rf'(0)$ i.e., $f(x)$ is not differentiable at $x=0$.

Example: 2 If the function f is defined by $f(x)=\dfrac{x}{1+|x|}$, then at what points f is differentiable

(a) Everywhere (b) Except at $x=\pm 1$ (c) Except at $x=0$ (d) Except at $x=0$ or ± 1

Solution: (a) We have, $f(x)=\dfrac{x}{1+|x|}=\begin{cases}\dfrac{x}{1+x}, & x>0\\ 0, & x=0\\ \dfrac{x}{1-x}, & x<0\end{cases}$; $Lf'(0)=\lim\limits_{h\to 0}\dfrac{f(-h)-f(0)}{-h}=\lim\limits_{h\to 0}\dfrac{\dfrac{-h}{1+h}-0}{-h}=1$

$$Rf'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\frac{\lim\limits_{h\to 0}\dfrac{h}{1+h}-0}{h}=\lim_{h\to 0}\frac{1}{1+h}=1$$

So, $Lf'(0)=Rf'(0)=1$

So, $f(x)$ is differentiable at $x=0$; Also $f(x)$ is differentiable at all other points.

Hence, $f(x)$ is everywhere differentiable.

Example: 3 The value of the derivative of $|x-1|+|x-3|$ at $x=2$ is

(a) –2 (b) 0 (c) 2 (d) Not defined

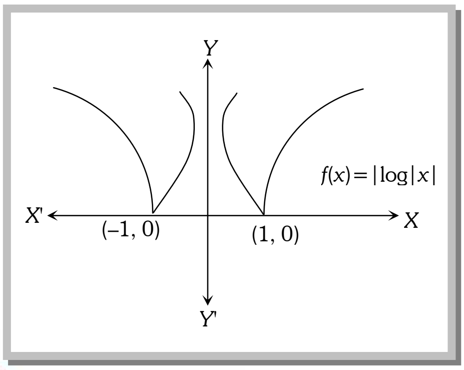
Solution: (b) Let $f(x)=|x-1|+|x-3|=\begin{cases}-(x-1)-(x-3), & x<1\\ (x-1)-(x-3), & 1\le x<3\\ (x-1)+(x-3), & x\ge 3\end{cases}=\begin{cases}-2x+4, & x<1\\ 2, & 1\le x<3\\ 2x-4, & x\ge 3\end{cases}$

Since, $f(x)=2$ for $1\le x<3$. Therefore $f'(x)=0$ for all $x\in(1,3)$.

Hence, $f'(x)=0$ at $x=2$.

Example: 4 The function f defined by $f(x)=\begin{cases}\dfrac{\sin x^2}{x}, & x\neq 0\\ 0, & x=0\end{cases}$

(a) Continuous and derivable at $x=0$
(b) Neither continuous nor derivable at $x=0$
(c) Continuous but not derivable at $x=0$
(d) None of these



Solution: (a) We have, $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \frac{\sin x^2}{x} = \lim\limits_{x\to 0}\left(\frac{\sin x^2}{x^2}\right)x = 1\times 0 = 0 = f(0)$

So, $f(x)$ is continuous at $x = 0$, $f(x)$ is also derivable at $x = 0$, because

$\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0}\frac{\sin x^2}{x^2} = 1$ exists finitely.

Example: 5 If $f(x) = |\log|x||$, then

(a) $f(x)$ is continuous and differentiable for all x in its domain

(b) $f(x)$ is continuous for all x in its domain but not differentiable at $x = \pm 1$.

(c) $f(x)$ is neither continuous nor differentiable at $x = \pm 1$

(d) None of these

Solution: (b) It is evident from the graph of $f(x) = |\log|x||$ that $f(x)$ is everywhere continuous but not differentiable at $x = \pm 1$. .

Example: 6 The left hand derivative of $f(x) = [x]\ \sin(\circledR x)$ at $x = k$ (k is an integer), is

(a) $(-1)^k(k-1)\circledR$ (b) $(-1)^{k-1}(k-1)\circledR$ (c) $(-1)^k k\circledR$ (d) $(-1)^{k-1}k\circledR$

Solution: (a) $f(x) = [x]\ \sin(\circledR\, x)$

If x is just less than k, $[x] = k-1$. $\therefore$ $f(x) = (k-1)\sin(\circledR x)$, when $x < k\quad \forall k\in I$

Now L.H.D. at $x = k$,

$$= \lim_{x\to k}\frac{(k-1)\sin(\circledR\, x) - k\sin(\circledR\, k)}{x-k} = \lim_{x\to k}\frac{(k-1)\sin(\circledR\, x)}{(x-k)} \quad [\text{as } \sin(\circledR\, k) = 0,\ k\in \text{integer}]$$

$$= \lim_{h\to 0}\frac{(k-1)\sin(\circledR(k-h))}{-h} \quad [\text{Let } x = (k-h)]$$

$$= \lim_{h\to 0}\frac{(k-1)(-1)^{k-1}\sin h\circledR}{-h} = \lim_{h\to 0}(k-1)(-1)^{k-1}\frac{\sin h\circledR}{h\circledR}\times(-\circledR) = (k-1)(-1)^k\circledR = (-1)^k(k-1)\circledR\ .$$

Example: 7 The function $f(x) = |x| + |x-1|$ is

(c) Not continuous at $x = 1$ (d) None of these

Solution: (a) We have, $f(x) = |x| + |x-1| = \begin{cases} -2x+1, & x<0 \\ 1, & 0\le x<1 \\ 2x-1, & x\ge 1\end{cases}$

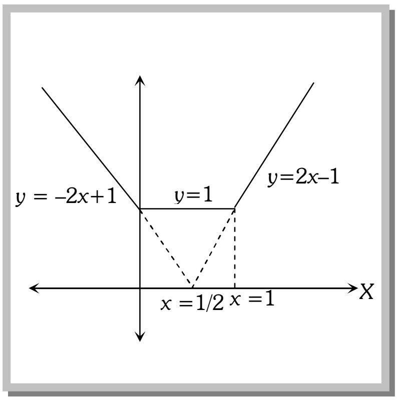
Since, $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} 1 = 1$, $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}(2x-1) = 1$ and $f(1) = 2\times 1 - 1 = 1$

$\therefore$ $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) = f(1)$. So, $f(x)$ is continuous at $x = 1$.

Now, $\lim\limits_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = \lim\limits_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim\limits_{h\to 0}\frac{1-1}{-h} = 0$,

and $\lim\limits_{x\to 1^+}\frac{f(x)-f(1)}{x-1} = \lim\limits_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim\limits_{h\to 0}\frac{2(1+h)-1-1}{h} = 2$.

$\therefore$ (LHD at $x = 1$) $\neq$ (RHD at $x = 1$). So, $f(x)$ is not differentiable at $x = 1$.



Trick : The graph of $f(x) = |x| + |x-1|$ *i.e.* $f(x) = \begin{cases} -2x+1, & x<0 \\ 1, & 0\le x<1 \\ 2x-1, & x\ge 1\end{cases}$ is

By graph, it is clear that the function is not differentiable at $x = 0, 1$ as there it has sharp edges.

Example: 8 Let $f(x) = |x-1| + |x+1|$, then the function is

(a) Continuous (b) Differentiable except $x = \pm 1$

(c) Both (a) and (b) (d) None of these

Solution: (c) Here $f(x) = |x-1| + |x+1| \Rightarrow f(x) = \begin{cases} 2x \ , & \text{when } \ x > 1 \\ 2 \ , & \text{when } -1 \le x \le 1 \\ -2x \ , & \text{when } \ x < -1 \end{cases}$

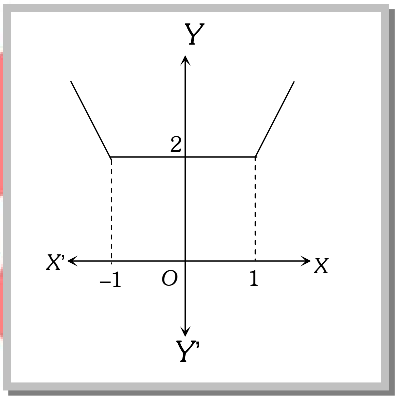
Graphical solution : The graph of the function is shown alongside,

From the graph it is clear that the function is continuous at all real *x*, also differentiable at all real *x* except at $x = \pm 1$; Since sharp edges at $x = -1$ and $x = 1$.

At $x = 1$ we see that the slope from the right *i.e.*, R.H.D. = 2, while slope from the left *i.e.*, L.H.D.= 0

Similarly, at $x = -1$ it is clear that R.H.D. = 0 while L.H.D. = – 2

Trick : In this method, first of all, we differentiate the function and on the derivative equality sign should be removed from doubtful points.



Here, $f'(x) = \begin{cases} -2 \ , & x < -1 \\ 0 \ , & -1 < x < 1 \ \text{(No equality on } -1 \text{ and } +1) \\ 2 \ , & x > 1 \end{cases}$

Now, at $x = 1$, $f'(1^+) = 2$ while $f'(1^-) = 0$ and

at $x = -1$, $f'(-1^+) = 0$ while $f'(-1^-) = -2$

Thus, $f(x)$ is not differentiable at $x = \pm 1$.

Note : ❒ This method is not applicable when function is discontinuous.

Example: 9 If the derivative of the function $f(x) = \begin{cases} ax^2 + b \ , & x < -1 \\ bx^2 + ax + 4 \ , & x \ge -1 \end{cases}$ is everywhere continuous and differentiable at *x* = 1 then

(a) *a* = 2, *b* = 3 (b) *a* = 3, *b* = 2 (c) *a* = –2, *b* = – 3 (d) *a* = – 3, *b* = – 2

Solution: (a) $f(x) = \begin{cases} ax^2 + b \ , & x < -1 \\ bx^2 + ax + 4, & x \ge -1 \end{cases}$

$\therefore f'(x) = \begin{cases} 2ax \ , & x < -1 \\ 2bx + a, & x \ge -1 \end{cases}$

To find a, b we must have two equations in a, b

Since $f(x)$ is differentiable, it must be continuous at $x = -1$.

$\therefore \ R = L = V$ at $x = -1$ for $f(x) \Rightarrow b - a + 4 = a + b$

$\therefore 2a = 4$ *i.e.*, $a = 2$

Again $f'(x)$ is continuous, it must be continuous at $x = -1$.

$\therefore R = L = V$ at $x = -1$ for $f'(x)$

$-2b + a = -2a$. Putting $a = 2$, we get $-2b + 2 = -4$

$\therefore 2b = 6$ or $b = 3$.

Example: 10 Let *f* be twice differentiable function such that $f''(x) = -f(x)$ and $f'(x) = g(x)$, $h(x) = \{f(x)\}^2 + \{g(x)\}^2$. If $h(5) = 11$, then $h(10)$ is equal to

(a) 22 (b) 11 (c) 0 (d) None of these

Solution: (b) Differentiating the given relation $h(x) = [f(x)]^2 + [g(x)]^2$ w.r.t *x*, we get $h'(x) = 2f(x)f'(x) + 2g(x)g'(x)$ ....(i)

But we are given $f''(x) = -f(x)$ and $f'(x) = g(x)$ so that $f''(x) = g'(x)$.

Then (1) may be re-written as $h'(x) = -2f''(x)f'(x) + 2f'(x)f''(x) = 0$. Thus $h'(x) = 0$

Whence by intergrating, we get $h(x) =$ constant $= c$ (say). Hence $h(x) = c$, for all x.

In particular, $h(5) = c$. But we are given $h(5) = 11$.

It follows that $c = 11$ and we have $h(x) = 11$ for all x. Therefore, $h(10) = 11$.

Example: 11 The function $f(x) = \begin{cases} |2x-3|[x], & x \ge 1 \\ \sin\left(\frac{® x}{2}\right), & x < 1 \end{cases}$

(a) Is continuous at $x = 2$
(b) Is differentiable at $x = 1$
(c) Is continuous but not differentiable at $x = 1$
(d) None of these

Solution: (c) $[2+h] = 2, [2-h] = 1, [1+h] = 1, [1-h] = 0$

At $x = 2$, we will check $R = L = V$

$R = \lim_{h\to 0} |4+2h-3|[2+h] = 2, V = 1.2 = 2$

$L = \lim_{h\to 0} |4-2h-3|[2-h] = 1, R \ne L$, $\therefore$ not continuous

At $x = 1, R = \lim |2+2h-3|[1+h] = 1.1 = 1,$

$V = |-1|[1] = 1$

$L = \lim_{h\to 0} \sin\frac{®}{2}(1-h) = 1$

Since $R = L = V$ $\therefore$ continuous at $x = 1$.

R.H.D. $= \lim_{h\to 0} \frac{|2+2h-3|[1+h]-1}{h} = \lim_{h\to 0} \frac{|-1|.1-1}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0$

L.H.D. $= \lim_{h\to 0} \frac{|2-2h-3|[1-h]-1}{-h} = \lim_{h\to 0} \frac{1.0-1}{-h} = \lim_{h\to 0} \frac{1}{h} = \infty$

Since R.H.D. $\ne$ L.H.D. $\therefore$ not differentiable. at $x = 1$.

1.2 Differentiability in an Open Interval

A function $f(x)$ defined in an open interval (a, b) is said to be differentiable or derivable in open interval (a, b) if it is differentiable at each point of (a, b).

Differentiability in a closed interval : A function $f:[a,b] \to R$ is said to be differentiable in $[a, b]$ if

(1) $f'(x)$ exists for every x such that $a < x < b$ *i.e.* f is differentiable in (a, b).

(2) Right hand derivative of f at $x = a$ exists.

(3) Left hand derivative of f at $x = b$ exists.

Everywhere differentiable function : If a function is differentiable at each $x \in R$, then it is said to be everywhere differentiable. *e.g.*, A constant function, a polynomial function, $\sin x, \cos x$ etc. are everywhere differentiable.

Some standard results on differentiability

(1) Every polynomial function is differentiable at each $x \in R$.

(2) The exponential function $a^x, a > 0$ is differentiable at each $x \in R$.

(3) Every constant function is differentiable at each $x \in R$.

(4) The logarithmic function is differentiable at each point in its domain.

(5) Trigonometric and inverse trigonometric functions are differentiable in their domains.

(6) The sum, difference, product and quotient of two differentiable functions is differentiable.

(7) The composition of differentiable function is a differentiable function.

Important Tips

- *If f is derivable in the open interval (a, b) and also at the end points 'a' and 'b', then f is said to be derivable in the closed interval [a, b].*
- *A function f is said to be a differentiable function if it is differentiable at every point of its domain.*
- *If a function is differentiable at a point, then it is continuous also at that point.*
  i.e. Differentiability Ĝ Continuity, but the converse need not be true.
- *If a function 'f' is not differentiable but is continuous at x = a, it geometrically implies a sharp corner or kink at x = a.*
- *If f(x) is differentiable at x = a and g(x) is not differentiable at x =a, then the product function f(x).g(x) can still be differentiable at x = a.*
- *If f(x) and g(x) both are not differentiable at x = a then the product function f(x).g(x) can still be differentiable at x = a.*
- *If f(x) is differentiable at x = a and g(x) is not differentiable at x = a then the sum function f(x) + g(x) is also not differentiable at x = a*
- *If f(x) and g(x) both are not differentiable at x = a, then the sum function may be a differentiable function.*

Example: 12 The set of points where the function $f(x)=\sqrt{1-e^{-x^2}}$ is differentiable

(a) $(-\infty,\infty)$ (b) $(-\infty,0)\cup(0,\infty)$ (c) $(-1,\infty)$ (d) None of these

Solution: (b) Clearly, $f(x)$ is differentiable for all non-zero values of x, For $x\neq 0$, we have $f'(x)=\dfrac{xe^{-x^2}}{\sqrt{1-e^{-x^2}}}$

Now, (L.H.D. at $x = 0$)

$$=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{\sqrt{1-e^{-h^2}}}{-h}=\lim_{h\to 0}-\frac{\sqrt{1-e^{-h^2}}}{h}=-\lim_{h\to 0}\sqrt{\frac{e^{h^2}-1}{h^2}}\times\frac{1}{\sqrt{e^{h^2}}}=-1$$

and, (RHD at $x = 0$) $=\displaystyle\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{h\to 0}\frac{\sqrt{1-e^{-h^2}}-0}{h}=\lim_{h\to 0}\sqrt{\frac{e^{h^2}-1}{h^2}}\times\frac{1}{\sqrt{e^{h^2}}}=1$.

So, $f(x)$ is not differentiable at $x=0$, Hence, the points of differentiability of $f(x)$ are $(-\infty,0)\cup(0,\infty)$.

Example: 13 The function $f(x)=e^{-|x|}$ is

(a) Continuous everywhere but not differentiable at $x=0$
(b) Continuous and differentiable everywhere
(c) Not continuous at $x=0$
(d) None of these

Solution: (a) We have, $f(x)=\begin{cases}e^{-x}, x\geq 0\\ e^{x}, x<0\end{cases}$

Clearly, $f(x)$ is continuous and differentiable for all non-zero x.

Now, $\displaystyle\lim_{x\to 0^-}f(x)=\lim_{x\to 0}e^x=1$ and $\displaystyle\lim_{x\to 0^+}f(x)=\lim_{x\to 0}e^{-x}=1$

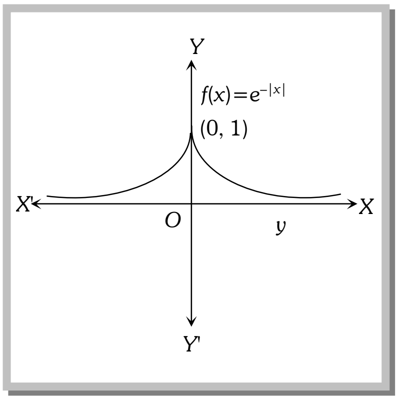
Also, $f(0)=e^0=1$

So, $f(x)$ is continuous for all x.

(LHD at $x=0$) $=\left(\dfrac{d}{dx}(e^x)\right)_{x=0}=[e^x]_{x=0}=e^0=1$

(RHD at $x=0$) $=\left(\dfrac{d}{dx}(e^{-x})\right)_{x=0}=[-e^{-x}]_{x=0}=-1$

So, $f(x)$ is not differentiable at $x=0$.



Hence, $f(x)=e^{-|x|}$ is everywhere continuous but not differentiable at $x=0$. This fact is also evident from the graph of the function.

Example: 14 If $f(x)=\sqrt{1-\sqrt{1-x^2}}$, then $f(x)$ is

(a) Continuous on [–1, 1] and differentiable on (–1, 1)
(b) Continuous on [–1,1] and differentiable on $(-1, 0)\cup(0, 1)$
(c) Continuous and differentiable on [–1, 1]
(d) None of these

Solution: (b) We have, $f(x)=\sqrt{1-\sqrt{1-x^2}}$. The domain of definition of $f(x)$ is [–1, 1].

For $x\neq 0, x\neq 1,\ x\neq -1$ we have $f'(x)=\frac{1}{\sqrt{1-\sqrt{1-x^2}}}\times\frac{x}{\sqrt{1-x^2}}$

Since $f(x)$ is not defined on the right side of $x=1$ and on the left side of $x=-1$. Also, $f'(x)\to\infty$ when $x\to -1^+$ or $x\to 1^-$. So, we check the differentiability at $x = 0$.

Now, (LHD at $x=0$) $=\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim\limits_{h\to 0}\frac{f(0-h)-f(0)}{-h}$

$$=\lim_{h\to 0}\frac{\sqrt{1-\sqrt{1-h^2}}-0}{-h}=-\lim_{h\to 0}\frac{\sqrt{1-\{1-(1/2)h^2+(3/8)h^4+....\}}}{h}=-\lim_{h\to 0}\sqrt{\frac{1}{2}-\frac{3}{8}h^2+.....}=-\frac{1}{\sqrt{2}}$$

Similarly, (RHD at $x = 0$) $=\frac{1}{\sqrt{2}}$

Hence, $f(x)$ is not differentiable at $x = 0$.

Example: 15 Let $f(x)$ be a function differentiable at $x=c$. Then $\lim\limits_{x\to c} f(x)$ equals

(a) $f'(c)$ (b) $f''(c)$ (c) $\frac{1}{f(c)}$ (d) None of these

Solution: (d) Since $f(x)$ is differentiable at $x=c$, therefore it is continuous at $x=c$. Hence, $\lim\limits_{x\to c} f(x)=f(c)$.

Example: 16 The function $f(x)=(x^2-1)|x^2-3x+2|+\cos(|x|)$ is not differentiable at

(a) -1 (b) 0 (c) 1 (d) 2

Solution: (d) $(x^2-3x+2)=(x-1)(x-2)=+ive$

When $x<1$ or >2, $-ive$ when $1\leq x\leq 2$

Also $\cos|x|=\cos x$ (since $\cos(-x)=\cos x$)

$\therefore\ f(x)=-(x^2-1)(x^2-3x+2)+\cos x,\quad 1\leq x\leq 2$

$\therefore\ f(x)=(x^2-1)(x^2-3x+2)+\cos x,\quad x>2$ ....(i)

Evidently $f(x)$ is not differentiable at $x=2$ as $L'\neq R'$

Note : ❒ For all other values like $x<0,\ 0\leq x<1,\ f(x)$ is same as given by (i).

Example: 17 If $f(x)=\begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x\neq 0\\ 0, & x=0\end{cases}$, then $f(x)$ is

(a) Continuous as well as differentiable for all x (b) Continuous for all x but not differentiable at $x=0$
(c) Neither differentiable nor continuous at $x=0$ (d) Discontinuous every where

Solution: (b) $f(0)=0$ and $f(x)=xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}$

R.H.L. $=\lim\limits_{h\to 0}(0+h)e^{-2/h}=\lim\limits_{h\to 0}\frac{h}{e^{2/h}}=0$

L.H.L. $=\lim\limits_{h\to 0}(0-h)e^{-\left(\frac{1}{h}-\frac{1}{h}\right)}=0$

$\therefore\ f(x)$ is continuous.

$Rf'(x)$ at $(x=0)=\lim\limits_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0}\frac{he^{-2/h}}{h}=e^{-\infty}=0$

$Lf'(x)$ at $(x=0)=\lim\limits_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim\limits_{h\to 0}\frac{-he^{-\left(\frac{1}{h}-\frac{1}{h}\right)}}{-h}=+1\Rightarrow Lf'(x)\neq Rf'(x)$

$f(x)$ is not differentiable at $x=0$.

Example: 18 The function $f(x)=x^2\sin\dfrac{1}{x}, x\neq 0,\ f(0)=0$ at $x=0$

(a) Is continuous but not differentiable (b) Is discontinuous
(c) Is having continuous derivative (d) Is continuous and differentiable

Solution: (d) $\lim\limits_{x\to 0} f(x)=x^2\sin\left(\dfrac{1}{x}\right)$ but $-1\le \sin\left(\dfrac{1}{x}\right)\le 1$ and $x\to 0$

$\therefore \lim\limits_{x\to 0^+} f(x)=0=\lim\limits_{x\to 0^-} f(x)=f(0)$

Therefore $f(x)$ is continuous at $x=0$. Also, the function $f(x)=x^2\sin\dfrac{1}{x}$ is differentiable because

$Rf'(x)=\lim\limits_{h\to 0}\dfrac{h^2\sin\frac{1}{h}-0}{h}=0$, $Lf'(x)=\lim\limits_{h\to 0}\dfrac{h^2\sin\left(\frac{1}{-h}\right)}{-h}=0$.

Example: 19 Which of the following is not true

(a) A polynomial function is always continuous (b) A continuous function is always differentiable
(c) A differentiable function is always continuous (d) e^x is continuous for all x

Solution: (b) A continuous function may or may not be differentiable. So (b) is not true.

Example: 20 If $f(x)=\text{sgn}(x^3)$, then

(a) f is continuous but not derivable at $x=0$ (b) $f'(0^+)=2$
(c) $f'(0^-)=1$ (d) f is not derivable at $x=0$

Solution: (d) Here, $f(x)=\text{sgn}\,x^3=\begin{cases}\begin{cases}\dfrac{x^3}{|x^3|} & \text{for } x^3\neq 0\\ 0 & \text{for } x^3=0\end{cases}\\ \begin{cases}\dfrac{x}{|x|} & \text{for } x\neq 0\\ 0 & \text{for } x=0\end{cases}\\ \begin{cases}-1, & x<0\\ 0, & x=0\\ 1, & x>0\end{cases}\end{cases}$. Thus, $f(x)=\text{sgn}\,x^3=\text{sgn}\,x$, which is neither continuous nor derivable at 0.

Note : ❒ $f'(0^+)=\lim\limits_{h\to 0^+}\dfrac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0^+}\dfrac{1-0}{h}\to\infty$ and $f'(0^-)=\lim\limits_{h\to 0^-}\dfrac{f(0-h)-f(0)}{h}=\lim\limits_{h\to 0^-}\dfrac{-1-0}{h}\to\infty$.

$\therefore f'(0^+)\neq f'(0^-)$, $\therefore f$ is not derivable at $x=0$.

Example: 21 A function $f(x)=\begin{cases}1+x, & x\le 2\\ 5-x, & x>2\end{cases}$ is

(a) Not continuous at $x=2$ (b) Differentiable at $x=2$
(c) Continuous but not differentiable at $x=2$ (d) None of the above

Solution: (c) $\lim\limits_{h\to 0^-} 1+(2-h)=3$, $\lim\limits_{h\to 0^+} 5-(2+h)=3$, $f(2)=3$

Hence, f is continuous at $x=2$

Now $Rf'(x)=\lim\limits_{h\to 0}\dfrac{5-(2+h)-3}{h}=-1$

$Lf'(x)=\lim\limits_{h\to 0}\dfrac{1+(2-h)-3}{-h}=1$

$\because Rf'(x) \neq Lf'(x)$

$\therefore$ *f* is not differentiable at $x = 2$.

Example: 22 Let $f : R \to R$ be a function. Define $g : R \to R$ by $g(x) = |f(x)|$ for all *x*. Then *g* is

(a) Onto if *f* is onto

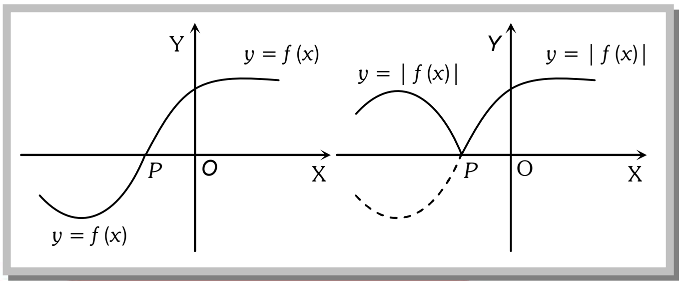
(b) One-one if *f* is one-one

(c) Continuous if *f* is continuous

(d) Differentiable if *f* is differentiable

Solution: (c) $g(x) = |f(x)| \geq 0$. So $g(x)$ cannot be onto. If $f(x)$ is one-one and $f(x_1) = -f(x_2)$ then $g(x_1) = g(x_2)$. So, '$f(x)$ is one-one' does not ensure that $g(x)$ is one-one.

If $f(x)$ is continuous for $x \in R, |f(x)|$ is also continuous for $x \in R$. This is obvious from the following graphical consideration.



So the answer (c) is correct. The fourth answer (d) is not correct from the above graphs $y = f(x)$ is differentiable at *P* while $y = |f(x)|$ has two tangents at *P*, *i.e.* not differentiable at *P*.

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