10. (d) We have $|(\mathbf{a}\times\mathbf{b}).\mathbf{c}|=|\mathbf{a}||\mathbf{b}||\mathbf{c}|$

$\Rightarrow \left||\mathbf{a}||\mathbf{b}|\sin\theta\,\mathbf{n}.\mathbf{c}\right|=|\mathbf{a}||\mathbf{b}||\mathbf{c}|$

$\Rightarrow \left||\mathbf{a}||\mathbf{b}||\mathbf{c}|\sin\theta\cos\alpha\right|=|\mathbf{a}||\mathbf{b}||\mathbf{c}|$

$\Rightarrow |\sin\theta||\cos\alpha|=1 \Rightarrow \theta=\frac{\pi}{2}$ and $\alpha=0$

$\Rightarrow \mathbf{a}\perp\mathbf{b}$ and $\mathbf{c}||\mathbf{n}$

$\Rightarrow \mathbf{a}\perp\mathbf{b}$ and $\mathbf{c}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$

$\mathbf{a},\mathbf{b},\mathbf{c}$ are mutually perpendicular

Hence, $\mathbf{a}.\mathbf{b}=\mathbf{b}.\mathbf{c}=\mathbf{c}.\mathbf{a}=0$.

11. (a) Vector product is not commutative.

12. (c) The vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&0\\0&1&1\end{vmatrix}=\mathbf{i}-\mathbf{j}+\mathbf{k}$$

Since the length of this vector is $\sqrt{3}$, the unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$ is

$$\pm\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}=\pm\frac{1}{\sqrt{3}}(\mathbf{i}-\mathbf{j}+\mathbf{k})$$

Hence there are two such vectors.

13. (b) Let $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$

But $(\mathbf{i}-\mathbf{j}+\mathbf{k}).(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})=1 \Rightarrow b_1-b_2+b_3=1$ ......(i)

and $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-1&1\\b_1&b_2&b_3\end{vmatrix}$

$=-\mathbf{i}(b_2+b_3)+\mathbf{j}(b_1-b_3)+\mathbf{k}(b_2+b_1)$

$\Rightarrow \mathbf{a}\times\mathbf{b}=\mathbf{c}$

Comparing the coefficients of $\mathbf{i},\mathbf{j}$ and $\mathbf{k}$ respectively,

we get $b_2+b_3=1$ .....(ii)

$b_1-b_3=-1$ .....(iii)

$b_2+b_1=0$ .....(iv)

By solving the equations (i), (ii), (iii) and (iv), we get $b_1=0$, $b_2=0$ and $b_3=1$.

14. (a) Since $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}\neq\mathbf{0}\Rightarrow\mathbf{a}\times\mathbf{b}-\mathbf{b}\times\mathbf{c}=\mathbf{0}$

$\Rightarrow \mathbf{a}\times\mathbf{b}+\mathbf{c}\times\mathbf{b}=\mathbf{0}\Rightarrow(\mathbf{a}+\mathbf{c})\times\mathbf{b}=\mathbf{0}$

$\Rightarrow \mathbf{a}+\mathbf{c}$ is parallel to $\mathbf{b}\Rightarrow\mathbf{a}+\mathbf{c}=k\mathbf{b}$.

15. (c) It is obvious.

16. (a,c) Let angle between $\mathbf{a}$ and $\mathbf{b}$ be θ.

$\mathbf{v}=\mathbf{a}\times\mathbf{b}=|\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$

$\therefore |\mathbf{v}|=\sin\theta$, $\left[\because |\mathbf{a}|=1, |\mathbf{b}|=1, \hat{\mathbf{n}}=\frac{(\mathbf{a}\times\mathbf{b})}{|\mathbf{a}\times\mathbf{b}|}=\frac{\mathbf{v}}{|\mathbf{v}|}\right]$

$\mathbf{u}=\mathbf{a}-(\mathbf{a}.\mathbf{b})\mathbf{b}=\mathbf{a}-\cos\theta\,\mathbf{b}$

$(\because \mathbf{a}.\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta=\cos\theta)$

$\mathbf{u}.\mathbf{u}=|\mathbf{u}|^2=1+\cos^2\theta-2\cos\theta\cos\theta=\sin^2\theta$

$\therefore |\mathbf{u}|=\sin\theta$

$\mathbf{u}.\mathbf{a}=\mathbf{a}.\mathbf{a}-\cos\theta\,\mathbf{a}.\mathbf{b}=1-\cos^2\theta=\sin^2\theta$

$\mathbf{u}.\mathbf{b}=\mathbf{a}.\mathbf{b}-\cos\theta\,\mathbf{b}.\mathbf{b}=\cos\theta-\cos\theta=0$

$\mathbf{u}.(\mathbf{a}+\mathbf{b})=(\mathbf{a}-\cos\theta\,\mathbf{b}).(\mathbf{a}+\mathbf{b})$

$=1+\cos\theta-\cos^2\theta-\cos\theta$

$=1-\cos^2\theta=\sin^2\theta$

Hence (a) and (c) are correct.

17. (b) $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}\Rightarrow(\mathbf{a}+\mathbf{c})\times\mathbf{b}=0$, but $\mathbf{a}+\mathbf{c}\neq0$

$\Rightarrow \mathbf{a}+\mathbf{c}||\mathbf{b}$.

18. (a) $\mathbf{a}=\mathbf{i}+\mathbf{j}-3\mathbf{k}$, $\mathbf{b}=-2\mathbf{i}+2\mathbf{j}+2\mathbf{k}$

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&-3\\-2&2&2\end{vmatrix}=8\mathbf{i}+4\mathbf{j}+4\mathbf{k}$$

Hence unit vector $=\pm\frac{2\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{6}}$.

19. (c) $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&3&-5\\m&n&12\end{vmatrix}$

$=(36+5n)\mathbf{i}-(24+5m)\mathbf{j}+(2n-3m)\mathbf{k}=0$

$\Rightarrow m=\frac{-24}{5}$, $n=\frac{-36}{5}$.

20. (d) Unit vector is equal to $\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}=\frac{1}{\sqrt{5}}(2\mathbf{i}+\mathbf{k})$.

21. (a) $\overrightarrow{AB}=2\mathbf{i}-\mathbf{j}-2\mathbf{k}$, $\overrightarrow{AC}=3\mathbf{i}-3\mathbf{j}+0\mathbf{k}$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-1&-2\\3&-3&0\end{vmatrix}=(-6\mathbf{i}-6\mathbf{j}-3\mathbf{k})$$

Hence unit vector $=\pm\left(\frac{2\mathbf{i}+2\mathbf{j}+\mathbf{k}}{3}\right)$.

22. (c) Unit vector perpendicular to both the given vectors is,

$$\frac{(6\mathbf{i}+2\mathbf{j}+3\mathbf{k})\times(3\mathbf{i}-6\mathbf{j}-2\mathbf{k})}{|(6\mathbf{i}+2\mathbf{j}+3\mathbf{k})\times(3\mathbf{i}-6\mathbf{j}-2\mathbf{k})|}=\frac{2\mathbf{i}+3\mathbf{j}-6\mathbf{k}}{7}.$$

23. (c) $(\mathbf{a}\times\mathbf{b})^2=(\mathbf{a}\times\mathbf{b}).(\mathbf{a}\times\mathbf{b})=(ab\sin\theta\,\hat{\mathbf{n}})(ab\sin\theta\,\hat{\mathbf{n}})$

$=a^2b^2\sin^2\theta=a^2b^2(1-\cos^2\theta)$

$=a^2b^2-a^2b^2\cos^2\theta=a^2b^2-(\mathbf{a}.\mathbf{b})^2$.

24. (c) $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&2&-1\\12&5&-5\end{vmatrix}=-5\mathbf{i}+3\mathbf{j}-9\mathbf{k}$.

Unit vector along $\mathbf{a}\times\mathbf{b}=\frac{-5\mathbf{i}+3\mathbf{j}-9\mathbf{k}}{\sqrt{115}}$.

25. (a) $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&2&-1\\12&5&-5\end{vmatrix}=-5\mathbf{i}+3\mathbf{j}-9\mathbf{k}$

$\Rightarrow \sin\theta=\frac{\sqrt{25+9+81}}{\sqrt{14}.\sqrt{194}}=\frac{\sqrt{115}}{\sqrt{14}.\sqrt{194}}$.

26. (d) $\mathbf{a}\times\mathbf{b}=\mathbf{0}$, even if $\mathbf{a}\neq\mathbf{0},\mathbf{b}\neq\mathbf{0}$ *i.e.*, $\mathbf{a}$ and $\mathbf{b}$ are parallel.

27. (b) $(\mathbf{a}\times\mathbf{b})^2=a^2b^2-(\mathbf{a}.\mathbf{b})^2=\begin{vmatrix}\mathbf{a}.\mathbf{a}&\mathbf{a}.\mathbf{b}\\\mathbf{a}.\mathbf{b}&\mathbf{b}.\mathbf{b}\end{vmatrix}$.

28. (a) It is obvious.

29. (c) $\mathbf{a}.(\mathbf{b}-\mathbf{c})=0$, $\mathbf{a}\times(\mathbf{b}-\mathbf{c})=\mathbf{0}\Rightarrow \mathbf{b}=\mathbf{c}$ or $\mathbf{a}=\mathbf{0}$, but $\mathbf{a}\neq\mathbf{0}$. Hence $\mathbf{b}-\mathbf{c}=\mathbf{0}$. *i.e.*, $\mathbf{b}=\mathbf{c}$.

30. (d) $\mathbf{a}.\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$; But $|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin\theta$

$$\Rightarrow \sin\theta=\frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}=\frac{4}{5}\Rightarrow\cos\theta=\frac{3}{5}$$

Therefore, $\mathbf{a}.\mathbf{b}=2\times5\times\frac{3}{5}=6$.

31. (b) $|\mathbf{a}.\mathbf{b}|=ab\cos\theta=3$ .....(i)

and $|\mathbf{a}\times\mathbf{b}|=ab\sin\theta=4$ .....(ii)

Dividing (ii) by (i),

we get $\tan\theta=\frac{4}{3}\Rightarrow\cos\theta=\frac{3}{5}\Rightarrow\theta=\cos^{-1}\frac{3}{5}$.

32. (c) $\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 2 & -1\\ 6 & -3 & 2\end{vmatrix}=\mathbf{i}-10\mathbf{j}-18\mathbf{k}$.

33. (a) Here $(l\mathbf{a}+m\mathbf{b})\times\mathbf{b}=\mathbf{c}\times\mathbf{b}\Rightarrow l\mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{b}$

$$\Rightarrow l(\mathbf{a}\times\mathbf{b})^2=(\mathbf{c}\times\mathbf{b}).(\mathbf{a}\times\mathbf{b})\Rightarrow l=\frac{(\mathbf{c}\times\mathbf{b}).(\mathbf{a}\times\mathbf{b})}{(\mathbf{a}\times\mathbf{b})^2}$$

Similarly, $m=\frac{(\mathbf{c}\times\mathbf{a}).(\mathbf{b}\times\mathbf{a})}{(\mathbf{b}\times\mathbf{a})^2}$.

34. (b) $|\mathbf{a}\times\mathbf{i}|^2=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ 1 & 0 & 0\end{vmatrix}^2$, (Since $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$)

$$=|a_3\mathbf{j}-a_2\mathbf{k}|^2=a_3^2+a_2^2$$

Similarly, $|\mathbf{a}\times\mathbf{j}|^2=a_1^2+a_3^2$ and $|\mathbf{a}\times\mathbf{k}|^2=a_1^2+a_2^2$

Hence the required result can be given as

$2(a_1^2+a_2^2+a_3^2)=2|\mathbf{a}|^2$.

35. (b) A vector perpendicular to the plane determined by the points $P(1,-1,2)$; $Q(2,0,-1)$ and $R(0,2,1)$ is given by

$\overrightarrow{QR}\times\overrightarrow{PR}\Rightarrow(-2\mathbf{i}+2\mathbf{j}+2\mathbf{k})\times(-\mathbf{i}+3\mathbf{j}-\mathbf{k})$

Therefore, unit vector $=\frac{2\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{4+1+1}}=\frac{2\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{6}}$.

36. (b) Let $\mathbf{a}=4\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $\mathbf{b}=-2\mathbf{i}+\mathbf{j}-2\mathbf{k}$

Unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$

But $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & -1 & 3\\ -2 & 1 & -2\end{vmatrix}$

$$=\mathbf{i}(2-3)-\mathbf{j}(-8+6)+\mathbf{k}(4-2)=-\mathbf{i}+2\mathbf{j}+2\mathbf{k}$$

$$\therefore\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}=\frac{-\mathbf{i}+2\mathbf{j}+2\mathbf{k}}{\sqrt{1+4+4}}=\frac{-\mathbf{i}+2\mathbf{j}+2\mathbf{k}}{3}.$$

Trick : Check it with the options. Since the vector $\frac{-\mathbf{i}+2\mathbf{j}+2\mathbf{k}}{3}$ is unit and perpendicular to both the given vectors.

37. (c) Obviously, $\mathbf{b}+\mathbf{c}=-2\mathbf{i}+4\mathbf{j}$ and $\mathbf{a}+\mathbf{b}=3\mathbf{j}$.

Hence the unit vector $\mathbf{k}$ is perpendicular to both $\mathbf{b}+\mathbf{c}$ and $\mathbf{a}+\mathbf{b}$.

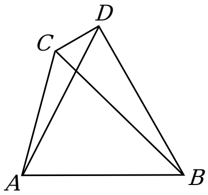
38. (a) Since $\mathbf{a},\mathbf{c},\mathbf{b}$ form a right handed system

$\therefore\mathbf{c}=\mathbf{b}\times\mathbf{a}=\mathbf{j}\times(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$

$=x(\mathbf{j}\times\mathbf{i})+z(\mathbf{j}\times\mathbf{k})=-x\mathbf{k}+z\mathbf{i}=z\mathbf{i}-x\mathbf{k}$.

39. (b) Let A be the origin and let the poisition vectors of B,C and D be $\mathbf{b},\mathbf{c}$ and $\mathbf{d}$ respectively.

Then $\overrightarrow{AB}=\mathbf{b}$, $\overrightarrow{CD}=\mathbf{d}-\mathbf{c}$, $\overrightarrow{BC}=\mathbf{c}-\mathbf{b}$, $\overrightarrow{AD}=\mathbf{d}$, $\overrightarrow{CA}=-\mathbf{c}$ and $\overrightarrow{BD}=\mathbf{d}-\mathbf{b}$.



$\therefore|\overrightarrow{AB}\times\overrightarrow{CD}+\overrightarrow{BC}\times\overrightarrow{AD}+\overrightarrow{CA}\times\overrightarrow{BD}|$

$=|\mathbf{b}\times(\mathbf{d}-\mathbf{c})+(\mathbf{c}-\mathbf{b})\times\mathbf{d}-\mathbf{c}\times(\mathbf{d}-\mathbf{b})|$

$=|\mathbf{b}\times\mathbf{d}-\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{d}-\mathbf{b}\times\mathbf{d}-\mathbf{c}\times\mathbf{d}+\mathbf{c}\times\mathbf{b}|$

$=|-\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{b}|=|-2\mathbf{b}\times\mathbf{c}|=2|\mathbf{b}\times\mathbf{c}|$

$=4$ (area of triangle ABC).

40. (c) We know that $(\mathbf{a}\times\mathbf{b})^2+(\mathbf{a}.\mathbf{b})^2=|\mathbf{a}|^2|\mathbf{b}|^2$

$\therefore\ 144=16|\mathbf{b}|^2\Rightarrow|\mathbf{b}|=3$.

41. (b) $\mathbf{r}\times\mathbf{a}-\mathbf{b}\times\mathbf{a}=0$ and $\mathbf{r}\times\mathbf{b}+\mathbf{b}\times\mathbf{a}=0$

Adding, we get $\mathbf{r}\times(\mathbf{a}+\mathbf{b})=0$

But as we are given $\mathbf{a}\neq\lambda\mathbf{b}$, therefore $\mathbf{r}=\mathbf{a}+\mathbf{b}$.

42. (d) Since $|\mathbf{a}\times\mathbf{r}|^2+|\mathbf{a}.\mathbf{r}|^2=|\mathbf{a}|^2|\mathbf{r}|^2$

$\Rightarrow|\mathbf{j}|^2+(\mathbf{a}.\mathbf{r})^2=|\mathbf{a}|^2|\mathbf{r}|^2\Rightarrow(\mathbf{a}.\mathbf{r})=\pm\sqrt{|\mathbf{a}|^2|\mathbf{r}|^2-1}$

This shows that $\mathbf{a}.\mathbf{r}$ depends on $|\mathbf{r}|$ for given $\mathbf{a}$.

Hence $\mathbf{a}.\mathbf{r}$ is arbitrary scalar.

43. (a) Let $\mathbf{a}=2\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $\mathbf{b}=3\mathbf{i}+4\mathbf{j}-\mathbf{k}$, then a unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$

Here $\mathbf{a}\times\mathbf{b}=-3\mathbf{i}+5\mathbf{j}+11\mathbf{k}$

Unit vector is $\frac{-3\mathbf{i}+5\mathbf{j}+11\mathbf{k}}{\sqrt{155}}$.

44. (a) $\cos\theta=\frac{\vec{A}.\vec{B}}{|\vec{A}||\vec{B}|}=\frac{6-2+8}{\sqrt{14}\times\sqrt{24}}$

$\cos\theta=\frac{12}{\sqrt{14}\times\sqrt{24}}=\sqrt{\frac{3}{7}}$, $\therefore\ \sin\theta=\frac{2}{\sqrt{7}}$.

45. (c) Any vector $(\mathbf{r})$ in plane of $\mathbf{a}, \mathbf{b}$ must be in form of linear combination of $\mathbf{a}$ and $\mathbf{b}$

$\vec{r} = x\mathbf{a} + y\mathbf{b}$

Such combination is possible in alternate (c).

As $\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{c}.\mathbf{b})\mathbf{a} - (\mathbf{c}.\mathbf{a})\mathbf{b}$ .....(i)

Also (i) is perpendicular to $\mathbf{c}$

As $\mathbf{c}.\{(\mathbf{c}.\mathbf{b})\mathbf{a} - (\mathbf{c}.\mathbf{a})\mathbf{b}\} = (\mathbf{c}.\mathbf{a})(\mathbf{c}.\mathbf{b}) - (\mathbf{c}.\mathbf{b})(\mathbf{c}.\mathbf{a}) = 0$

Thus unit vector perpendicular to $\mathbf{c}$ and coplanar with $\mathbf{a}, \mathbf{b}$ is, $\dfrac{\mathbf{c} \times (\mathbf{a} \times \mathbf{b})}{|\mathbf{c} \times (\mathbf{a} \times \mathbf{b})|}$.

Other similar concets :

(1) Unit vector perpendicular to $\mathbf{a}$ and coplanar with $\mathbf{b}$ and $\mathbf{c}$ is $\mathbf{r} = \dfrac{\mathbf{a} \times (\mathbf{b} \times \mathbf{c})}{|\mathbf{a} \times (\mathbf{b} \times \mathbf{c})|}$.

(2) Unit vector perpendicular to $\mathbf{b}$ and coplanar with $\mathbf{c}$ and $\mathbf{a}$ is $\mathbf{r} = \dfrac{\mathbf{b} \times (\mathbf{c} \times \mathbf{a})}{|\mathbf{b} \times (\mathbf{c} \times \mathbf{a})|}$.

46. (b) We know $(\mathbf{a} \times \mathbf{b})^2 + (\mathbf{a}.\mathbf{b})^2 = a^2b^2 = (\mathbf{a}.\mathbf{a})(\mathbf{b}.\mathbf{b})$.

47. (b) Unit vector perpendicular to plane of ΔABC is, $\dfrac{\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|}$,

where $\overrightarrow{AB} = \mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$ and $\overrightarrow{AC} = 6\mathbf{i} + 3\mathbf{j} + 8\mathbf{k}$

$\overrightarrow{AB} \times \overrightarrow{AC} = 31\mathbf{i} - 38\mathbf{j} - 9\mathbf{k}$ and $|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{2486}$

Required vector $= \dfrac{31\mathbf{i} - 38\mathbf{j} - 9\mathbf{k}}{\sqrt{2486}}$.

48. (b) Unit vector perpendicular to plane

$$= \frac{\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|} = \frac{(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})}{|(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})|}$$

$$= \frac{\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}}{|\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|}.$$

49. (a) Angle between the given vectors $\mathbf{a}$ and $\mathbf{b}$ is θ.

We know that, $\dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}.\mathbf{b}|} = \dfrac{|\mathbf{a}||\mathbf{b}|\sin\theta}{|\mathbf{a}||\mathbf{b}|\cos\theta} = \tan\theta$.

50. (b) Here $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$. Take cross products with $\mathbf{a}$ and $\mathbf{b}$ by turn.

51. (c) Vector perpendicular to both of the vectors $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{i} + \mathbf{j}$ is, $\dfrac{(\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (\mathbf{i} + \mathbf{j})}{|(\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (\mathbf{i} + \mathbf{j})|}$

$= \dfrac{-\mathbf{i} + \mathbf{j}}{\sqrt{2}} = \dfrac{-1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$ or $c(\mathbf{i} - \mathbf{j})$, where c is a scalar.

52. (c) Perpendicular vector to $\mathbf{a}$ and $\mathbf{b}$ $= \mathbf{a} \times \mathbf{b}$ and perpendicular unit vector $= \dfrac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix} = 15\mathbf{i} - 10\mathbf{j} + 30\mathbf{k}$$

and $|\mathbf{a} \times \mathbf{b}| = \sqrt{225 + 100 + 900} = 35$

Required vector $= \dfrac{15\mathbf{i} - 10\mathbf{j} + 30\mathbf{k}}{35} = \dfrac{3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}}{7}$.

53. (a) Unit vector $= \dfrac{(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \times (\mathbf{i} + 2\mathbf{j} - \mathbf{k})}{|(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \times (\mathbf{i} + 2\mathbf{j} - \mathbf{k})|} = \dfrac{-\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}$.

54. (c) Vectors $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

We know that $\mathbf{a} \times \mathbf{b} = \mathbf{i}(1 - 3) - \mathbf{j}(-1 - 2) + \mathbf{k}(3 + 2)$

$= -2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$

and $|\mathbf{a} \times \mathbf{b}| = \sqrt{(-2)^2 + (3)^2 + (5)^2} = \sqrt{38}$

Therefore unit vector $\dfrac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|} = \dfrac{-2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}}{\sqrt{38}}$.

55. (b) $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & -1 \\ 1 & 4 & -2 \end{vmatrix} = \mathbf{i}(6 + 4) - \mathbf{j}(-4 + 1) + \mathbf{k}(8 + 3)$

$= 10\mathbf{i} + 3\mathbf{j} + 11\mathbf{k}$.

56. (b) $(\mathbf{a} \times \mathbf{b})^2 = (|\mathbf{a}||\mathbf{b}|\sin\theta)^2$

$= (4.2\sin 30^o)^2 = \left(8.\dfrac{1}{2}\right)^2 = 4^2 = 16$.

57. (a) $(\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -5 \\ 1 & 2 & 3 \end{vmatrix} = 22\mathbf{i} - 11\mathbf{j} + 0\mathbf{k}$

$(\mathbf{a} \times \mathbf{b}) = \sqrt{(22)^2 + (11)^2} = 11\sqrt{2^2 + 1} = 11\sqrt{5}$.

58. (d) Unit vector perpendicular to both

$= \dfrac{(\mathbf{i} + \mathbf{j}) \times (\mathbf{j} + \mathbf{k})}{|(\mathbf{i} + \mathbf{j}) \times (\mathbf{j} + \mathbf{k})|} = \dfrac{\mathbf{i} - \mathbf{j} + \mathbf{k}}{\sqrt{3}}$.

59. (b) Let a unit vector in the plane of $2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{i} - \mathbf{j} + \mathbf{k}$ be $\hat{\mathbf{a}} = \alpha(2\mathbf{i} + \mathbf{j} + \mathbf{k}) + \beta(\mathbf{i} - \mathbf{j} + \mathbf{k})$

$\hat{\mathbf{a}} = (2\alpha + \beta)\mathbf{i} + (\alpha - \beta)\mathbf{j} + (\alpha + \beta)\mathbf{k}$

As $\hat{\mathbf{a}}$ is unit vector, we have

$(2\alpha + \beta)^2 + (\alpha - \beta)^2 + (\alpha + \beta)^2 = 1$

$6\alpha^2 + 4\alpha\beta + 3\beta^2 = 1$ .....(i)

As $\hat{\mathbf{a}}$ is orthogonal to $5\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$, we get

$5(2\alpha + \beta) + 2(\alpha - \beta) + 6(\alpha + \beta) = 0$

$18\alpha + 9\beta = 0 \Rightarrow \beta = -2\alpha$

From (i), we get $6\alpha^2 - 8\alpha^2 + 12\alpha^2 = 1$

$\alpha = \pm\dfrac{1}{\sqrt{10}} \Rightarrow \beta = \mp\dfrac{2}{\sqrt{10}}$. Thus $\hat{\mathbf{a}} = \pm\left(\dfrac{3}{\sqrt{10}}\mathbf{j} - \dfrac{1}{\sqrt{10}}\mathbf{k}\right)$.

60. (b) If angle between $\mathbf{b}$ and $\mathbf{c}$ is α and $|\mathbf{b} \times \mathbf{c}| = \sqrt{15}$

$|\mathbf{b}|\ |\mathbf{c}|\ \sin\alpha = \sqrt{15}$

$\sin\alpha = \dfrac{\sqrt{15}}{4}$; $\therefore$ $\cos\alpha = \dfrac{1}{4}$

$\mathbf{b} - 2\mathbf{c} = \lambda\mathbf{a} \Rightarrow |\mathbf{b} - 2\mathbf{c}|^2 = \lambda^2|\mathbf{a}|^2$

$|\mathbf{b}|^2 + 4|\mathbf{c}|^2 - 4.\mathbf{b}.\mathbf{c} = \lambda^2|\mathbf{a}|^2$

$16 + 4 - 4\{|\mathbf{b}||\mathbf{c}|\cos\alpha\} = \lambda^2$

$16 + 4 - 4 \times 4 \times 1 \times \dfrac{1}{4} = \lambda^2 \Rightarrow \lambda^2 = 16 \Rightarrow \lambda = \pm 4$.

61. (b) Here $\overrightarrow{OA} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OB} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\overrightarrow{OC} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

These implies $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

and $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 2\mathbf{i}$

Hence required area is given by $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -3 \\ 2 & 0 & 0 \end{vmatrix} = -2(3\mathbf{j} + 2\mathbf{k})$$

$\Rightarrow$ Area of triangle $= \frac{1}{2} \times 2|3\mathbf{j} + 2\mathbf{k}| = \sqrt{13}$.

62. (b) $\Delta = \sqrt{\Delta_x^2 + \Delta_y^2 + \Delta_z^2}$

where $\Delta_x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}$ and so on.

Alter : Form two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\Delta = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{matrix}\right\|$$

$$= \frac{1}{2}\left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{matrix}\right\| = \frac{1}{2}|8\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}|$$

$$= \frac{1}{2}\sqrt{64 + 16 + 16} = \frac{\sqrt{96}}{2} = 2\sqrt{6}.$$

63. (c) Area of triangle $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$$= \frac{1}{2}\left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{matrix}\right\|$$

Here, $(x_1, y_1, z_1) \equiv (1, 2, 3)$, $(x_2, y_2, z_2) \equiv (2, 5, -1)$, $(x_3, y_3, z_3) \equiv (-1, 1, 2)$

$$= \frac{1}{2}\left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -4 \\ -2 & -1 & -1 \end{matrix}\right\| = \frac{1}{2}|(-7\mathbf{i} + 9\mathbf{j} + 5\mathbf{k})|$$

$$= \frac{1}{2}\sqrt{49 + 81 + 25} = \frac{\sqrt{155}}{2} \textit{ sq. unit.}$$

64. (c) The area of parallelogram is given by $= |\overrightarrow{AB} \times \overrightarrow{AD}| = \frac{1}{2}|\overrightarrow{AC} \times \overrightarrow{BD}|$

Here we are given adjacent sides and so

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & -1 \\ 1 & 2 & 0 \end{vmatrix} = 2\mathbf{i} - \mathbf{j} + 6\mathbf{k}$$

Hence required area is $= |2\mathbf{i} - \mathbf{j} + 6\mathbf{k}| = \sqrt{41}$.

65. (b) $\Delta = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$

But $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = -2\mathbf{i} - 14\mathbf{j} - 10\mathbf{k}$.

Hence $\Delta = \frac{1}{2}|\mathbf{a} \times \mathbf{b}| = \frac{1}{2}\sqrt{4 + 196 + 100} = 5\sqrt{3}$.

66. (d) Vector area $= \frac{1}{2}(\overrightarrow{AB} \times \overrightarrow{AC}) = \frac{1}{2}|(-\mathbf{i} + \mathbf{k}) \times (-\mathbf{j} + \mathbf{k})|$

$$= \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} = \frac{1}{2}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

Hence by comparing, $\vec{\alpha} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

67. (c) $\Delta = \frac{1}{2}|\mathbf{a} \times \mathbf{b}| = \frac{1}{2}\left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ 1 & 3 & 1 \end{matrix}\right\| = \frac{1}{2}|(5\mathbf{i} - 4\mathbf{j} + 7\mathbf{k})|$

$$\Delta = \frac{1}{2}\sqrt{25 + 16 + 49} = \frac{1}{2}\sqrt{90} = \frac{3}{2}\sqrt{10}.$$

68. (c) It is obvious.

69. (c) Let the position vectors of the points A, B, C are $\mathbf{0}$, $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$ and $\theta = 90^\circ$.

Area of triangle $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$$= \frac{1}{2}|(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b})| = \frac{1}{2}|2\mathbf{b} \times \mathbf{a}|$$

$$= ba\sin\theta = 3 \times 2\sin 90^\circ = 6.$$

70. (c) $\Delta = |\mathbf{a} \times \mathbf{b}| = \left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & -2 & 1 \end{matrix}\right\| = |8\mathbf{i} + 8\mathbf{j} - 8\mathbf{k}| = 8\sqrt{3}$.

71. (c) Area is given by,

$$\frac{1}{2}|(3\mathbf{i} + 4\mathbf{j}) \times (-5\mathbf{i} + 7\mathbf{j})| = \frac{1}{2}|(41)\mathbf{k}| = \frac{41}{2}.$$

72. (b) Let $\mathbf{a} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 2\mathbf{j}$

$$\Rightarrow \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 2 \\ -1 & 2 & 0 \end{vmatrix} = -4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

Hence area is equal to $\frac{1}{2}|\mathbf{a} \times \mathbf{b}| = \frac{\sqrt{21}}{2}$.

73. (b) Let $\mathbf{p} = 2\mathbf{a} - \mathbf{b}$ and $\mathbf{q} = 4\mathbf{a} - 5\mathbf{b}$.

Then $\mathbf{p} \times \mathbf{q} = (2\mathbf{a} - \mathbf{b}) \times (4\mathbf{a} - 5\mathbf{b}) = -6(\mathbf{a} \times \mathbf{b})$

$$= -6|\mathbf{a}||\mathbf{b}|\sin\frac{\pi}{4}\hat{\mathbf{n}} = -6 \times \frac{1}{\sqrt{2}}\hat{\mathbf{n}} = -3\sqrt{2}\,\hat{\mathbf{n}}.$$

Hence the area of the given parallelogram

$$= \frac{1}{2}|\mathbf{p} \times \mathbf{q}| = \frac{3}{\sqrt{2}}.$$

74. (c) Required area $= \left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 3 \\ 2 & 1 & -4 \end{vmatrix}\right\|$

$= |5\mathbf{i}+10\mathbf{j}+5\mathbf{k}| = \sqrt{150} = 5\sqrt{6}$.

75. (d) Area of parallelogram $= \frac{1}{2}|\mathbf{d}_1 \times \mathbf{d}_2|$

$\mathbf{d}_1 \times \mathbf{d}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 1 & 3 & -4 \end{vmatrix}$; $\mathbf{d}_1 \times \mathbf{d}_2 = 2\mathbf{i}+10\mathbf{j}+8\mathbf{k}$

$\therefore \frac{1}{2}|\mathbf{d}_1 \times \mathbf{d}_2| = \frac{1}{2}\sqrt{4+100+64} = \frac{1}{2}\sqrt{168} = \sqrt{42}$.

76. (a) Adjacent sides of parallelogram are $\mathbf{a} = \mathbf{i}+2\mathbf{j}+3\mathbf{k}$ and $\mathbf{b} = -3\mathbf{i}-2\mathbf{j}+\mathbf{k}$. We know that vector area of parallelogram.

$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -3 & -2 & 1 \end{vmatrix} = \mathbf{i}(2+6)-\mathbf{j}(1+9)+\mathbf{k}(-2+6)$

$= 8\mathbf{i}-10\mathbf{j}+4\mathbf{k}$.

Therefore area of parallelogram

$= |\mathbf{a}\times\mathbf{b}| = \sqrt{(8)^2+(-10)^2+(4)^2} = \sqrt{64+100+16}$

$= \sqrt{180}$ *sq. unit.*

77. (a) $\mathbf{a}+\mathbf{b} = 2\mathbf{i}+4\mathbf{j}+6\mathbf{k}$, $\mathbf{b}+\mathbf{c} = 8\mathbf{i}+12\mathbf{j}+16\mathbf{k}$

Area of parallelogram $= \frac{1}{2}|\vec{A}\times\vec{B}|$, where $\vec{A}$ and $\vec{B}$ are diagonals $= \frac{1}{2}\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 6 \\ 8 & 12 & 16 \end{vmatrix}\right\|$

$= \frac{1}{2}|\mathbf{i}(64-72)-\mathbf{j}(32-48)+\mathbf{k}(24-32)|$

$= \frac{1}{2}|-8\mathbf{i}+16\mathbf{j}-8\mathbf{k}| = |-4\mathbf{i}+8\mathbf{j}-4\mathbf{k}|$

$= \sqrt{16+64+16} = \sqrt{96} = 4\sqrt{6}$.

78. (a) $A = \frac{1}{2}|\mathbf{d}_1\times\mathbf{d}_2| = \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3}{2} & \frac{1}{2} & -1 \\ 2 & -6 & 8 \end{vmatrix}$

$= \frac{1}{2}|-2\mathbf{i}-14\mathbf{j}-10\mathbf{k}|$

$A = \frac{1}{2}\sqrt{4+(14)^2+100} = \frac{1}{2}\sqrt{300} = \frac{1}{2}.10\sqrt{3} = 5\sqrt{3}$.

79. (d) Area of triangle $\Delta = \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

$\Delta = \frac{1}{2}\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2-1 & 3+2 & -1-3 \\ 4-1 & -7+2 & 7-3 \end{vmatrix}\right\|$

$\Delta = \frac{1}{2}\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 5 & -4 \\ 3 & -5 & 4 \end{vmatrix}\right\| = 0$.

80. (c) Area of parallelogram $= |\mathbf{a}\times\mathbf{b}| = \left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 2 & 3 \end{vmatrix}\right\|$

$= |2\mathbf{i}-3\mathbf{j}+2\mathbf{k}| = \sqrt{4+9+4} = \sqrt{17}$.

81. (d) Torque $= \mathbf{r}\times\mathbf{F}$ or $\overrightarrow{CP}\times F$.

82. (b) Let $\mathbf{F}_1 = \mathbf{i}+2\mathbf{j}-3\mathbf{k}$, $\mathbf{F}_2 = 2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$, $\mathbf{F}_3 = \mathbf{i}-\mathbf{j}+\mathbf{k}$.

$O(0,1,2)$ and $P(1,-2,0) \Rightarrow \overrightarrow{OP} = \mathbf{i}-3\mathbf{j}-2\mathbf{k}$

Resultant force $(\mathbf{F}) = \mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3 = 4\mathbf{i}+4\mathbf{j}+2\mathbf{k}$

Hence moment of force is $= \overrightarrow{OP}\times\mathbf{F}$

$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & -2 \\ 4 & 4 & 2 \end{vmatrix} = 2\mathbf{i}-10\mathbf{j}+16\mathbf{k}$

Magnitude of moment of force is

$|\overrightarrow{OP}\times\mathbf{F}| = \sqrt{4+100+256} = 6\sqrt{10}$.

83. (b) $\mathbf{F} = \overrightarrow{AB} = 4\mathbf{i}+4\mathbf{j}-\mathbf{k}$ and $\overrightarrow{AP} = 3\mathbf{i}-3\mathbf{k}$

Moment of the force is $\overrightarrow{AP}\times\overrightarrow{AB}$

$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & -3 \\ 4 & 4 & -1 \end{vmatrix} = 12\mathbf{i}-9\mathbf{j}+12\mathbf{k}$

$\therefore$ Magnitude is, $\sqrt{144+81+144} = 3\sqrt{41}$.

84. (d) $\mathbf{r}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -5\mathbf{i}-5\mathbf{j}+5\mathbf{k}$.

85. (a) $\overrightarrow{OA} = 3\mathbf{i}+2\mathbf{j}-9\mathbf{k}$; $\mathbf{F} = (9\mathbf{i}+6\mathbf{j}-2\mathbf{k})\times\frac{6}{11}$

$\therefore$ Moment $= \overrightarrow{OA}\times\mathbf{F} = \frac{6}{11}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -9 \\ 9 & 6 & -2 \end{vmatrix}$

$= \frac{6}{11}(50\mathbf{i}-75\mathbf{j}) = \frac{150}{11}(2\mathbf{i}-3\mathbf{j})$.

86. (c) Force $(\vec{F}) = 2\mathbf{i}+\mathbf{j}-\mathbf{k}$ and its position vector $= 2\mathbf{i}-\mathbf{j}$. We know that the position vector of a force about origin $(\mathbf{r}) = (2\mathbf{i}-\mathbf{j})-(0\mathbf{i}+0\mathbf{j}+0\mathbf{k})$ or $\mathbf{r} = 2\mathbf{i}-\mathbf{j}$.

Therefore, moment of the force about origin

$= \mathbf{r}\times\vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 0 \\ 2 & 1 & -1 \end{vmatrix} = \mathbf{i}+2\mathbf{j}+4\mathbf{k}$.

87. (c) n is perpendicular to $\mathbf{a}$ and $\mathbf{b}$

$\mathbf{n} = \mathbf{a}\times\mathbf{b}$; $\mathbf{n} = \dfrac{\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix}}{\sqrt{2}\times\sqrt{2}} = \frac{2\mathbf{k}}{2} = \mathbf{k}$

$|\mathbf{c}.\mathbf{n}| = |(\mathbf{i}+3\mathbf{j}+5\mathbf{k}).(\mathbf{k})| = |5| = 5$.

88. (d) Unit vectors perpendicular to the plane of **a** and **b** $= \pm\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$

$\therefore$ Required vector is $\pm\frac{(\mathbf{i}-\mathbf{j}+\mathbf{k})\times(-\mathbf{i}+\mathbf{j}+\mathbf{k})}{|(\mathbf{i}-\mathbf{j}+\mathbf{k})\times(-\mathbf{i}+\mathbf{j}+\mathbf{k})|}$

$= \pm\frac{(-(\mathbf{i}+\mathbf{j}))}{\sqrt{2}}$ *i.e.*, $\frac{-(\mathbf{i}+\mathbf{j})}{\sqrt{2}}$ and $\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$.

Scalar triple product and their applications

1. (a) $\frac{\mathbf{a}.\mathbf{b}\times\mathbf{c}}{\mathbf{c}\times\mathbf{a}.\mathbf{b}}+\frac{\mathbf{b}.\mathbf{a}\times\mathbf{c}}{\mathbf{c}.\mathbf{a}\times\mathbf{b}}=\frac{[\mathbf{a}\mathbf{b}\mathbf{c}]}{[\mathbf{c}\mathbf{a}\mathbf{b}]}+\frac{[\mathbf{b}\mathbf{a}\mathbf{c}]}{[\mathbf{c}\mathbf{a}\mathbf{b}]}=\frac{[\mathbf{a}\mathbf{b}\mathbf{c}]}{[\mathbf{c}\mathbf{a}\mathbf{b}]}-\frac{[\mathbf{a}\mathbf{b}\mathbf{c}]}{[\mathbf{c}\mathbf{a}\mathbf{b}]}=0.$

2. (b) $[\mathbf{a}+\mathbf{b}\ \ \mathbf{b}+\mathbf{c}\ \ \mathbf{c}+\mathbf{a}]=(\mathbf{a}+\mathbf{b}).\{(\mathbf{b}+\mathbf{c})\times(\mathbf{c}+\mathbf{a})\}$

$=(\mathbf{a}+\mathbf{b}).(\mathbf{b}\times\mathbf{c}+\mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{c}+\mathbf{c}\times\mathbf{a})$

$=(\mathbf{a}+\mathbf{b}).(\mathbf{b}\times\mathbf{c}+\mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{a})$, $\{\because \mathbf{c}\times\mathbf{c}=0\}$

$=\mathbf{a}.\mathbf{b}\times\mathbf{c}+\mathbf{a}.\mathbf{b}\times\mathbf{a}+\mathbf{a}.\mathbf{c}\times\mathbf{a}+\mathbf{b}.\mathbf{b}\times\mathbf{c}+\mathbf{b}.\mathbf{b}\times\mathbf{a}+\mathbf{b}.\mathbf{c}\times\mathbf{a}$

$=[\mathbf{a}\mathbf{b}\mathbf{c}]+[\mathbf{b}\mathbf{c}\mathbf{a}]=2[\mathbf{a}\mathbf{b}\mathbf{c}]$.

3. (c) Here, $\overrightarrow{OA}=2\mathbf{i}-3\mathbf{j}=\mathbf{a}$ (say)

$\overrightarrow{OB}=\mathbf{i}+\mathbf{j}-\mathbf{k}=\mathbf{b}$ (say)

and $\overrightarrow{OC}=3\mathbf{i}-\mathbf{k}=\mathbf{c}$ (say)

Hence volume is

$[\mathbf{a}\mathbf{b}\mathbf{c}]=\mathbf{a}.(\mathbf{b}\times\mathbf{c})=\begin{vmatrix}2&-3&0\\1&1&-1\\3&0&-1\end{vmatrix}=4$.

4. (c) $\mathbf{a}.(\mathbf{b}\times\mathbf{c})=0$ or $(\mathbf{a}\times\mathbf{b}).\mathbf{c}=0$.

5. (a) $\mathbf{a}.(\mathbf{c}\times\mathbf{b})=\mathbf{c}.(\mathbf{b}\times\mathbf{a})=0$, (Since **a** and **b** are parallel)

6. (d) If the given vectors are coplanar, then their scalar triple product is zero.

$\begin{vmatrix}2&-1&1\\1&2&-3\\3&\lambda&5\end{vmatrix}=0\Rightarrow\lambda=-4.$

7. (d) We have $\mathbf{p}.(\mathbf{a}+\mathbf{b})=\mathbf{p}.\mathbf{a}+\mathbf{p}.\mathbf{b}$

$=\frac{(\mathbf{b}\times\mathbf{c}).\mathbf{a}}{[\mathbf{a}\mathbf{b}\mathbf{c}]}+\frac{(\mathbf{b}\times\mathbf{c}).\mathbf{b}}{[\mathbf{a}\mathbf{b}\mathbf{c}]}=\frac{[\mathbf{b}\mathbf{c}\mathbf{a}]}{[\mathbf{a}\mathbf{b}\mathbf{c}]}+\frac{[\mathbf{b}\ \mathbf{c}\ \mathbf{b}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$

$=1+0=1$, $\{\because[\mathbf{b}\mathbf{c}\mathbf{a}]=[\mathbf{a}\mathbf{b}\mathbf{c}]\text{and}[\mathbf{b}\mathbf{c}\mathbf{b}]=0\}$

Similarly, $\mathbf{q}.(\mathbf{b}+\mathbf{c})=1$ and $\mathbf{r}.(\mathbf{a}+\mathbf{c})=1$

Thus, required result is 1+1+1=3.

8. (a) Let $\mathbf{a}=3\mathbf{i}-2\mathbf{j}-\mathbf{k}$, $\mathbf{b}=2\mathbf{i}+3\mathbf{j}-4\mathbf{k}$, $\mathbf{c}=-\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $\mathbf{d}=4\mathbf{i}+5\mathbf{j}+\lambda\mathbf{k}$.

Since the points are coplanar,

So, $[\mathbf{d}\mathbf{b}\mathbf{c}]+[\mathbf{d}\mathbf{c}\mathbf{a}]+[\mathbf{d}\mathbf{a}\mathbf{b}]=[\mathbf{a}\mathbf{b}\mathbf{c}]$

$\Rightarrow\begin{vmatrix}4&5&\lambda\\2&3&-4\\-1&1&2\end{vmatrix}+\begin{vmatrix}4&5&\lambda\\-1&1&2\\3&-2&-1\end{vmatrix}+\begin{vmatrix}4&5&\lambda\\3&-2&-1\\2&3&-4\end{vmatrix}$

$=\begin{vmatrix}3&-2&-1\\2&3&-4\\-1&1&2\end{vmatrix}$

$\Rightarrow 40+5\lambda+37-\lambda+94+13\lambda=25\Rightarrow\lambda=\frac{-146}{17}.$

9. (a) $\mathbf{p}+\mathbf{q}+\mathbf{r}=\frac{\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}}{[\mathbf{a}\mathbf{b}\mathbf{c}]}$

$(\mathbf{a}+\mathbf{b}+\mathbf{c}).(\mathbf{p}+\mathbf{q}+\mathbf{r})=\frac{[\mathbf{a}\mathbf{b}\mathbf{c}]+[\mathbf{b}\mathbf{c}\mathbf{a}]+[\mathbf{c}\mathbf{a}\mathbf{b}]}{[\mathbf{a}\mathbf{b}\mathbf{c}]}=3$.

10. (c) Since $\begin{vmatrix}-12&0&\alpha\\0&3&-1\\2&1&-15\end{vmatrix}=546\Rightarrow\alpha=-3.$

11. (b) $\begin{vmatrix}a&a&c\\1&0&1\\c&c&b\end{vmatrix}=0\Rightarrow\begin{vmatrix}a&0&c\\1&-1&1\\c&0&b\end{vmatrix}=0$

{Applying $C_2\to C_2-C_1$}

$\Rightarrow a(-b)+c(c)=0\Rightarrow c^2=ab$

Hence c is the geometric mean of a and b.

12. (c) $\mathbf{a}^{-1}=\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a}\mathbf{b}\mathbf{c}]}$, $\mathbf{c}^{-1}=\frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{a}\mathbf{b}\mathbf{c}]}$, $\mathbf{b}^{-1}=\frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a}\mathbf{b}\mathbf{c}]}$

$\Rightarrow[\mathbf{a}^{-1}\,\mathbf{b}^{-1}\,\mathbf{c}^{-1}]=\frac{(\mathbf{b}\times\mathbf{c})}{[\mathbf{a}\mathbf{b}\mathbf{c}]}.\left(\frac{(\mathbf{c}\times\mathbf{a})}{[\mathbf{a}\mathbf{b}\mathbf{c}]}\times\frac{(\mathbf{a}\times\mathbf{b})}{[\mathbf{a}\mathbf{b}\mathbf{c}]}\right)$

$=\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a}\mathbf{b}\mathbf{c}]}.\left[\frac{\mathbf{a}}{[\mathbf{a}\mathbf{b}\mathbf{c}]}\right]=\frac{1}{[\mathbf{a}\mathbf{b}\mathbf{c}]}\neq 0$.

13. (a) $\begin{vmatrix}1&-1&1\\1&2&-1\\3&p&5\end{vmatrix}=0\Rightarrow p=-6.$

14. (b) $|\,\mathbf{i}\,\mathbf{k}\,\mathbf{j}|=\mathbf{i}.(\mathbf{k}\times\mathbf{j})=\mathbf{i}.(-\mathbf{i})=-1.$

15. (d) Volume of cube $=[\mathbf{a}\mathbf{b}\mathbf{c}]$

$=\begin{vmatrix}12&4&3\\8&-12&-9\\33&-4&-24\end{vmatrix}=12\begin{vmatrix}12&1&1\\8&-3&-3\\33&-1&-8\end{vmatrix}=3696$

16. (c) $\mathbf{a}.(\mathbf{b}\times\mathbf{c})=\mathbf{a}.\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&1\\1&-1&2\end{vmatrix}=\mathbf{a}.(5\mathbf{i}-\mathbf{j}-3\mathbf{k})$

$=(2\mathbf{i}+\mathbf{j}-\mathbf{k}).(5\mathbf{i}-\mathbf{j}-3\mathbf{k})=10-1+3=12$.

17. (a) Vol. of parallelopiped

$=\begin{vmatrix}\mathbf{a}_1&\mathbf{b}_1&\mathbf{c}_1\\\mathbf{a}_2&\mathbf{b}_2&\mathbf{c}_2\\\mathbf{a}_3&\mathbf{b}_3&\mathbf{c}_3\end{vmatrix}=\begin{vmatrix}2&1&-1\\1&2&3\\-3&-1&1\end{vmatrix}$

$=2(5)-1(1+9)-1(5)=|-5|=5$ *cubic unit.*

18. (a) Since **x** is a non-zero vector, the given conditions will be satisfied, if either (i) at least one of the vectors **a**, **b**, **c** is zero or (ii) **x** is perpendicular to all the vectors **a**, **b**, **c**. In case (ii), **a**, **b**, **c** are coplanar and so $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=0$.

19. (b) Accordingly, $\begin{vmatrix}-bc&b^2+bc&c^2+bc\\a^2+ac&-ac&c^2+ac\\a^2+ab&b^2+ab&-ab\end{vmatrix}=0$

$\Rightarrow(ab+bc+ca)^3=0\Rightarrow ab+bc+ca=0.$

20. (d) $[\mathbf{a}+\mathbf{b}\ \mathbf{b}+\mathbf{c}\ \mathbf{c}+\mathbf{a}]=[\mathbf{abc}]+[\mathbf{aba}]+[\mathbf{acc}]$
$+[\mathbf{aca}]+[\mathbf{bbc}]+[\mathbf{bba}]+[\mathbf{bcc}]+[\mathbf{bca}]$
$=[\mathbf{abc}]+[\mathbf{bca}]=2[\mathbf{abc}]=0$, ($\because$ $\mathbf{a},\mathbf{b},\mathbf{c}$ are coplanar).

21. (b) We have $[\mathbf{a}\ \mathbf{b}\ \mathbf{a}\times\mathbf{b}]=(\mathbf{a}\times\mathbf{b}).(\mathbf{a}\times\mathbf{b})=|\mathbf{a}\times\mathbf{b}|^2$.

22. (d) $(\mathbf{a}\times\mathbf{j}).(2\mathbf{j}-3\mathbf{k})=\mathbf{a}.\{\mathbf{j}\times(2\mathbf{j}-3\mathbf{k})\}$
$=\mathbf{a}.\{-3(\mathbf{j}\times\mathbf{k})\}=-3(\mathbf{a}.\mathbf{i})=-12.$

23. (a) Given vectors are coplanar,
$$\therefore \begin{vmatrix}2 & -3 & 4\\ 1 & 2 & -1\\ x & -1 & 2\end{vmatrix}=0\Rightarrow x=\frac{8}{5}.$$

24. (c) $V=\begin{vmatrix}2 & -3 & 4\\ 1 & 2 & -2\\ 3 & -1 & 1\end{vmatrix}=|-7|=7$ *cubic unit.*

25. (d) $\mathbf{a}\times(\mathbf{a}.\mathbf{b})$ is not meaningful.

26. (b) $\mathbf{i}.(\mathbf{j}\times\mathbf{k})+\mathbf{j}.(\mathbf{k}\times\mathbf{i})+\mathbf{k}.(\mathbf{i}\times\mathbf{j})=\mathbf{i}.\mathbf{i}+\mathbf{j}.\mathbf{j}+\mathbf{k}.\mathbf{k}=3.$

27. (c,d) Since the two vectors given in option (c) and (d) are unit as well as perpendicular to both the given vectors.

28. (c) Volume $=\begin{vmatrix}-3 & 7 & 5\\ -3 & 7 & -3\\ 7 & -5 & -3\end{vmatrix}$
$=-3(-21-15)-7(9+21)+5(15-49)$
$=180-210-170=272.$

29. (c) Since $\mathbf{a}\times\mathbf{b}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$ both, therefore $\mathbf{a}.(\mathbf{a}\times\mathbf{b})=0.$

($\because$ Scalar product of two perpendicular vector is zero)

30. (d) $(\mathbf{a}-\mathbf{b}).(\mathbf{b}-\mathbf{c})\times(\mathbf{c}-\mathbf{a})=(\mathbf{a}-\mathbf{b}).(\mathbf{b}\times\mathbf{c}-\mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{a})$
$=\mathbf{a}.(\mathbf{b}\times\mathbf{c})-\mathbf{a}.(\mathbf{b}\times\mathbf{a})+\mathbf{a}.(\mathbf{c}\times\mathbf{a})$
$-\mathbf{b}.(\mathbf{b}\times\mathbf{c})+\mathbf{b}.(\mathbf{b}\times\mathbf{a})-\mathbf{b}.(\mathbf{c}\times\mathbf{a})=0.$

31. (c) options (a), (b) and (d) $=[\mathbf{u},\mathbf{v},\mathbf{w}]$ while option (c) $=-[\mathbf{u},\mathbf{v},\mathbf{w}]$.

32. (a,c) Since $(\mathbf{u}.\mathbf{v})$ is scalar and scalar dot or cross product are not defined for options (b) and (d). So answers are (a) and (c).

33. (b) Since $\mathbf{d}=\lambda\mathbf{a}+\mu\mathbf{b}+\nu\mathbf{c}$
$\mathbf{d}.(\mathbf{b}\times\mathbf{c})=\lambda\,\mathbf{a}.(\mathbf{b}\times\mathbf{c})+\mu\,\mathbf{b}.(\mathbf{b}\times\mathbf{c})+\mu\,\mathbf{c}.(\mathbf{b}\times\mathbf{c})$
$=\lambda[\mathbf{abc}]$
$$\lambda=\frac{[\mathbf{dbc}]}{[\mathbf{abc}]}=\frac{[\mathbf{bcd}]}{[\mathbf{bca}]}.$$

34. (b) $\vec{A}=2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$ and $\vec{B}=\mathbf{i}+\mathbf{j}+5\mathbf{k}$

$\vec{A},\vec{B}$ and $\vec{C}$ form a left or right handed system according as $[\vec{A}\vec{B}\vec{C}]<0$ or >0 respectively.

Here, $[\vec{A}\vec{B}\vec{C}]<0$, $\vec{C}=-11\mathbf{i}+6\mathbf{j}+\mathbf{k}.$

35. (c) Volume of parallelopiped $=[\mathbf{abc}]$
$$=\begin{vmatrix}1 & -1 & 1\\ 1 & -3 & 4\\ 2 & -5 & 3\end{vmatrix}=(-9+20)-(8-3)+(-5+6)=7\ \textit{unit}.$$

36. (b) $\dfrac{(\mathbf{b}\times\mathbf{c}).(\mathbf{a}+\mathbf{b}+\mathbf{c})}{\lambda}=\dfrac{(\mathbf{b}\times\mathbf{c}).\mathbf{a}+(\mathbf{b}\times\mathbf{c}).\mathbf{b}+(\mathbf{b}\times\mathbf{c}).\mathbf{c}}{\lambda}$
$=\dfrac{(\mathbf{b}\times\mathbf{c}).\mathbf{a}+0+0}{\lambda}=\dfrac{\lambda}{\lambda}=1,$
($\because$ Given $\mathbf{a}.(\mathbf{b}\times\mathbf{c})=\lambda=(\mathbf{b}\times\mathbf{c}).\mathbf{a}$).

37. (a) Here $[\mathbf{abc}]=0$
The given scalar triple product = $k[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=0$.

38. (b) Vectors $\mathbf{i}+3\mathbf{j}-2\mathbf{k}$; $2\mathbf{i}-\mathbf{j}+4\mathbf{k}$ and $3\mathbf{i}+2\mathbf{j}+x\mathbf{k}$. We know that as the vectors are coplanar, therefore $\begin{vmatrix}1 & 3 & -2\\ 2 & -1 & 4\\ 3 & 2 & x\end{vmatrix}=0$
$1(-x-8)-3(2x-12)-2(4+3)=0$
$-x-8-6x+36-14=0$ $\quad 7x=14 \quad x=2.$

39. (a) $[\mathbf{a}-\mathbf{b}\ \mathbf{b}-\mathbf{c}\ \mathbf{c}-\mathbf{a}]=\{(\mathbf{a}-\mathbf{b})\times(\mathbf{b}-\mathbf{c})\}.(\mathbf{c}-\mathbf{a})$
$=(\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}-\mathbf{b}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}).(\mathbf{c}-\mathbf{a})$
$=(\mathbf{a}\times\mathbf{b}+\mathbf{c}\times\mathbf{a}+\mathbf{b}\times\mathbf{c}).(\mathbf{c}-\mathbf{a})$
$=(\mathbf{a}\times\mathbf{b}).\mathbf{c}-(\mathbf{a}\times\mathbf{b}).\mathbf{a}+(\mathbf{c}\times\mathbf{a}).\mathbf{c}-(\mathbf{c}\times\mathbf{a}).\mathbf{a}$
$=(\mathbf{a}\times\mathbf{b}).\mathbf{c}-(\mathbf{a}\times\mathbf{b}).\mathbf{a}+(\mathbf{c}\times\mathbf{a}).\mathbf{c}-(\mathbf{c}\times\mathbf{a}).\mathbf{a}$
$+(\mathbf{b}\times\mathbf{c}).\mathbf{c}-(\mathbf{b}\times\mathbf{c}).\mathbf{a}$
$=[\mathbf{abc}]-[\mathbf{aba}]+[\mathbf{cac}]-[\mathbf{caa}]+[\mathbf{bcc}]-[\mathbf{bca}]=0.$

40. (d) $\because$ $\mathbf{a}=\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{b}=2\mathbf{i}-4\mathbf{k}, \mathbf{c}=\mathbf{i}+\lambda\mathbf{j}+3\mathbf{k}$ arec oplanar.
$$[\mathbf{abc}]=0\Rightarrow\begin{vmatrix}1 & 1 & 1\\ 2 & 0 & -4\\ 1 & \lambda & 3\end{vmatrix}=0$$
$\Rightarrow 4\lambda-(6+4)+2\lambda=0$
$6\lambda=10\Rightarrow\lambda=\dfrac{5}{3}.$

41. (d) To make three vectors coplanar $[\vec{A}\vec{B}\vec{C}]=0$
The value of $[\vec{A}\vec{B}\vec{C}]$ is independent of C_1, hence no value of C_1 can be found.

42. (c) $[\mathbf{abc}]=\begin{vmatrix}1 & 0 & -1\\ x & 1 & 1-x\\ y & x & 1+x-y\end{vmatrix}=\begin{vmatrix}1 & 0 & 0\\ x & 1 & 1\\ y & x & 1+x\end{vmatrix}=\begin{vmatrix}1 & 1\\ x & 1+x\end{vmatrix}=1.$

43. (b) $\mathbf{a}.(\mathbf{b}\times\mathbf{c})=\begin{vmatrix}3 & -2 & 2\\ 6 & 4 & -2\\ 3 & -2 & -4\end{vmatrix}$
$=3[-16-4]+2[-24+6]+2[-12-12]$
$=-60-36-48=-144.$

44. (b) $(\mathbf{a}+\mathbf{b}).(\mathbf{b}+\mathbf{c})\times(\mathbf{a}+\mathbf{b}+\mathbf{c})$

$= (\mathbf{a}+\mathbf{b}).\{-\mathbf{a}\times\mathbf{b}+\mathbf{c}\times\mathbf{a}+\mathbf{b}\times\mathbf{b}+\mathbf{c}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{c}\}$

$= (\mathbf{a}+\mathbf{b}).\{-\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{c}\times\mathbf{b}\}$

$[\because\ \mathbf{b}\times\mathbf{b}=0 \text{ and } \mathbf{c}\times\mathbf{c}=0]$

$= (\mathbf{a}+\mathbf{b}).(-\mathbf{a}\times\mathbf{b}+\mathbf{c}\times\mathbf{a})$

$= -[\mathbf{a\,a\,b}]+[\mathbf{a\,c\,a}]-[\mathbf{b\,a\,b}]+[\mathbf{b\,c\,a}]$

$= 0+0-0+[\mathbf{b\,c\,a}]=[\mathbf{a\,b\,c}]$.

45. (c) It is obvious.

46. (a) $[\mathbf{a}\times\mathbf{b}\ \ \mathbf{b}\times\mathbf{c}\ \ \mathbf{c}\times\mathbf{a}] = (\mathbf{a}\times\mathbf{b}).[(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})]$

$= (\mathbf{a}\times\mathbf{b}).([\mathbf{b\,c\,a}]\mathbf{c}-[\mathbf{b\,c\,c}]\mathbf{a}) = (\mathbf{a}\times\mathbf{b}).([\mathbf{b\,c\,a}]\mathbf{c}-0)$

$= [\mathbf{b\,c\,a}][\mathbf{a\,b\,c}] = [\mathbf{a\,b\,c}][\mathbf{a\,b\,c}] = 4.4 = 16.$

47. (d) Volume of parallelopiped $V=[\mathbf{a\,b\,c}]$

$$\therefore\ V=\begin{vmatrix} 1 & -1 & 1 \\ 2 & -4 & 5 \\ 3 & -5 & 2 \end{vmatrix}$$

$V = 1(-8+25)+1(4-15)+1(-10+12)$

$V = 17-11+2 = 8$ *unit.*

48. (d) $[\mathbf{i\,k\,j}]+[\mathbf{k\,j\,i}]+[\mathbf{j\,k\,i}]=[\mathbf{i\,k\,j}]+[\mathbf{i\,k\,j}]-[\mathbf{i\,k\,j}] = [\mathbf{i\,k\,j}] = -1$.

49. (b) $(\mathbf{u}+\mathbf{v}-\mathbf{w}).(\mathbf{u}\times\mathbf{v}-\mathbf{u}\times\mathbf{w}-\mathbf{v}\times\mathbf{v}+\mathbf{v}\times\mathbf{w})$

$= (\mathbf{u}+\mathbf{v}-\mathbf{w})(\mathbf{u}\times\mathbf{v}-\mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w})$

$$= \frac{\mathbf{u}.(\mathbf{u}\times\mathbf{v})}{0}-\frac{\mathbf{u}.(\mathbf{u}\times\mathbf{w})}{0}+\mathbf{u}.(\mathbf{v}\times\mathbf{w})+\frac{\mathbf{v}.(\mathbf{u}\times\mathbf{v})}{0}$$

$$-\mathbf{v}.(\mathbf{u}\times\mathbf{w})+\frac{\mathbf{v}.(\mathbf{v}\times\mathbf{w})}{0}-\mathbf{w}.(\mathbf{u}\times\mathbf{v})+\frac{\mathbf{w}.(\mathbf{u}\times\mathbf{w})}{0}$$

$$-\frac{\mathbf{w}.(\mathbf{u}\times\mathbf{w})}{0} = \mathbf{u}.(\mathbf{v}\times\mathbf{w})-\mathbf{v}.(\mathbf{u}\times\mathbf{w})-\mathbf{w}.(\mathbf{u}\times\mathbf{v})$$

$= [\mathbf{u\,v\,w}]+[\mathbf{v\,w\,u}]-[\mathbf{w\,u\,v}] = \mathbf{u}.(\mathbf{v}\times\mathbf{w})$.

50. (d) $\mathbf{a}.[(\mathbf{b}+\mathbf{c})\times(\mathbf{a}+\mathbf{b}+\mathbf{c})]$

$= \mathbf{a}.(\mathbf{b}\times\mathbf{a}+\mathbf{b}\times\mathbf{b}+\mathbf{b}\times\mathbf{c})+\mathbf{a}.(\mathbf{c}\times\mathbf{a}+\mathbf{c}\times\mathbf{b}+\mathbf{c}\times\mathbf{c})$

$= [\mathbf{a\,b\,a}]+[\mathbf{a\,b\,b}]+[\mathbf{a\,b\,c}]+[\mathbf{a\,c\,a}]+[\mathbf{a\,c\,b}]+[\mathbf{a\,c\,c}]$

$0+0+[\mathbf{a\,b\,c}]+0-[\mathbf{a\,b\,c}]+0=0.$

51. (c) $\because$ The vectors $4\mathbf{i}+11\mathbf{j}+m\mathbf{k}, 7\mathbf{i}+2\mathbf{j}+6\mathbf{k}$ and $\mathbf{i}+5\mathbf{j}+4\mathbf{k}$ are coplanar.

$$\therefore \begin{vmatrix} 4 & 11 & m \\ 7 & 2 & 6 \\ 1 & 5 & 4 \end{vmatrix} = 0$$

$\Rightarrow 4(8-30)-11(28-6)+m(35-2)=0$

$\Rightarrow -88-11\times 22+33m=0 \Rightarrow -8-22+3m=0$

$\Rightarrow 3m=30\ \Rightarrow m=10.$

52. (b) Let vector be $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$.

$\because a\mathbf{i}+b\mathbf{j}+c\mathbf{k}, \mathbf{i}+\mathbf{j}, \mathbf{j}+\mathbf{k}$ are coplanar.

$$\begin{vmatrix} a & b & c \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 0\ \Rightarrow a-b+c=0$$

Also, since $(a\mathbf{i}+b\mathbf{j}+c\mathbf{k}) \| (2\mathbf{i}-2\mathbf{j}-4\mathbf{k})$

$(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\times(2\mathbf{i}-2\mathbf{j}-4\mathbf{k})=0$

i.e., $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ 2 & -2 & -4 \end{vmatrix} = 0$

$\mathbf{i}(-4b+2c)-\mathbf{j}(-4a-2c)+\mathbf{k}(-2a-2b)=0$

$-4b+2c=0,\ 4a+2c=0,\ 2a+2b=0$

$\frac{c}{2}=\frac{b}{1},\ \frac{c}{2}=\frac{a}{-1},\ \frac{a}{-1}=\frac{b}{1}$

i.e., $\frac{a}{-1}=\frac{b}{1}=\frac{c}{2}$ or $\frac{a}{1}=\frac{b}{-1}=\frac{c}{-2}$

Required vector is $\mathbf{i}-\mathbf{j}-2\mathbf{k}$.

53. (c) Let four points A, B, C, D represent the given points

So, $\overrightarrow{AB}=-\mathbf{i}-\mathbf{j}+4\mathbf{k},\ \overrightarrow{BC}=2\mathbf{i}+2\mathbf{j}-5\mathbf{k},$

$\overrightarrow{CD}=-2\mathbf{i}-(\lambda+4)\mathbf{j}+3\mathbf{k}$

From the condition, $[\overrightarrow{AB}\ \overrightarrow{BC}\ \overrightarrow{CD}]=0$

$$\begin{vmatrix} -1 & -1 & 4 \\ 2 & 2 & -5 \\ -2 & -(\lambda+4) & 3 \end{vmatrix} = 0$$

$-1[2.3-5(\lambda+4)]+1[6-10]+4[-2(\lambda+4)+4]=0$

$\Rightarrow \lambda=-2$.

54. (c) As $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non coplanar vectors

$[\mathbf{a\,b\,c}]\neq 0$.

Now $\mathbf{a}+2\mathbf{b}+3\mathbf{c},\ \lambda\mathbf{b}+4\mathbf{c}$ and $(2\lambda-1)\mathbf{c}$ will be non- coplanor, *iff* $(\mathbf{a}+2\mathbf{b}+3\mathbf{c}).\{\lambda\mathbf{b}+4\mathbf{c})\times(2\lambda-1)\mathbf{c}\}\neq 0$

i.e., $(\mathbf{a}+2\mathbf{b}+3\mathbf{c})\{\lambda(2\lambda-1)(\mathbf{b}\times\mathbf{c})\}\neq 0$

i.e., $\lambda(2\lambda-1)[\mathbf{a\,b\,c}]\neq 0$, . $\lambda\neq 0, \frac{1}{2}$

Thus given vectors will be non-coplanar for all values of λ except two values, $\lambda=0$ and $\lambda=1/2$.

Trick : For coplanarity,

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & \lambda & 4 \\ 0 & 0 & 2\lambda-1 \end{vmatrix} = 0 \Rightarrow \lambda=0, \frac{1}{2}$$

$\therefore$ All values except two values of $\lambda = 0, \frac{1}{2}$.

55. (d) It is obvious.

56. (c) We have $\mathbf{a}.\mathbf{b}=\mathbf{b}.\mathbf{c}=\mathbf{c}.\mathbf{a}=0$ and the scalar triple product of three vectors is that, $[\mathbf{a\,b\,c}]=(\mathbf{a}\times\mathbf{b}).\mathbf{c}$

$\because \mathbf{a}.\mathbf{b}=0,\ \ \therefore \mathbf{a}\perp\mathbf{b}$

So, angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta=90^\circ$.

Similarly, $[\mathbf{a\,b\,c}]=|\mathbf{a}||\mathbf{b}|\hat{\mathbf{n}}.\mathbf{c}$, where $\hat{\mathbf{n}}$ is a normal vector

$[\mathbf{a\,b\,c}]=|\mathbf{a}||\mathbf{b}|\hat{\mathbf{n}}\ \mathbf{c}$

$\because \hat{\mathbf{n}}$ and $\mathbf{c}$ are parallel to each other

$[\mathbf{a\,b\,c}]=|\mathbf{a}||\mathbf{b}||\hat{\mathbf{n}}|\ |\mathbf{c}|\cos\theta=|\mathbf{a}||\mathbf{b}||\mathbf{c}|$.

57. (c) Since $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are coplanar vectors.

$$[\mathbf{a}\,\mathbf{b}\,\mathbf{c}] = 0 \qquad \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 1 & \alpha & 0 \end{vmatrix} = 0$$

$$1[0-\alpha] - 1[0-1] - 1[2\alpha - 3] = 0$$

$$-3\alpha + 4 = 0 \Rightarrow \alpha = \frac{4}{3}.$$

58. (d) It is obvious.

59. (c) $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}] = \mathbf{a}.(\mathbf{b}\times\mathbf{c}) = \mathbf{a}.(|\mathbf{b}||\mathbf{c}|\sin\theta\,\hat{\mathbf{n}})$

$$= \mathbf{a}(3\times 4\sin\frac{2\pi}{3}.\hat{\mathbf{n}}) = \mathbf{a}.(12\times\frac{\sqrt{3}}{2}\hat{\mathbf{n}})$$

$$= 6\sqrt{3}|\mathbf{a}||\hat{\mathbf{n}}| = 6\sqrt{3}\times 2\times 1 \Rightarrow 12\sqrt{3}.$$

60. (d) $[\lambda(\mathbf{a}+\mathbf{b})\;\lambda^2\mathbf{b}\;\lambda\mathbf{c}] = [\mathbf{a}\;\;\mathbf{b}+\mathbf{c}\;\;\mathbf{b}]$

$$\lambda(\mathbf{a}+\mathbf{b}).(\lambda^2\mathbf{b}\times\lambda\mathbf{c}) = \mathbf{a}.((\mathbf{b}+\mathbf{c})\times\mathbf{b})$$

$$\lambda(\mathbf{a}+\mathbf{b})\lambda^3(\mathbf{b}\times\mathbf{c}) = \mathbf{a}.(\mathbf{b}\times\mathbf{b}+\mathbf{c}\times\mathbf{b})$$

$$\lambda^4[\mathbf{a}.(\mathbf{b}\times\mathbf{c}) + \mathbf{b}.(\mathbf{b}\times\mathbf{c})] = \mathbf{a}.(\mathbf{c}\times\mathbf{b})$$

$$\lambda^4[\mathbf{a}\,\mathbf{b}\,\mathbf{c}] = -[\mathbf{a}\,\mathbf{b}\,\mathbf{c}] \qquad [\mathbf{a}\,\mathbf{b}\,\mathbf{c}](\lambda^4+1) = 0$$

Since **a, b, c** are non-coplanar, so $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}] \neq 0$

$\therefore\;\; \lambda^4 = -1$. Hence no real value of λ.

61. (c) Given vectors are coplanar

$$\therefore \begin{vmatrix} 2 & 1 & -1 \\ -1 & 2 & \lambda \\ -5 & 2 & -1 \end{vmatrix} = 0$$

$$-4 - 4\lambda - 5\lambda - 1 - 8 = 0$$

$$-9\lambda - 13 = 0 \qquad \lambda = \frac{-13}{9}.$$

62. (b) We have, $\mathbf{a}.\mathbf{b}_1 = 0$, $\mathbf{b}_1.\mathbf{c}_2 = 0$, $\mathbf{a}.\mathbf{c}_2 = 0$

$\therefore$ Set of orthogonal vectors, $[\mathbf{a}\;\mathbf{b}_1\;\mathbf{c}_2] = 0$

$\therefore$ Option (b) is the correct answer.

63. (a) As vector $\boldsymbol{\alpha}$ lies in the plane of $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$

$\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are coplanar $\quad [\alpha\;\beta\;\gamma] = 0$.

Vector triple product

1. (a) $\mathbf{b}\times\mathbf{c}$ is a vector perpendicular to $\mathbf{b}, \mathbf{c}$. Therefore, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is a vector again in plane of $\mathbf{b}, \mathbf{c}$.

2. (c) Let $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\mathbf{i}\times(\mathbf{a}\times\mathbf{i}) + \mathbf{j}\times(\mathbf{a}\times\mathbf{j}) + \mathbf{k}\times(\mathbf{a}\times\mathbf{k})$$

$$= (\mathbf{i}.\mathbf{i})\mathbf{a} - \mathbf{i}(\mathbf{a}.\mathbf{i}) + (\mathbf{j}.\mathbf{j})\mathbf{a} - \mathbf{j}(\mathbf{a}.\mathbf{j}) + (\mathbf{k}.\mathbf{k})\mathbf{a} - \mathbf{k}(\mathbf{a}.\mathbf{k})$$

$$= 3\mathbf{a} - \mathbf{a} = 2\mathbf{a}.$$

3. (a) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\times\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{vmatrix} = \mathbf{a}\times(-2\mathbf{i}+3\mathbf{j}+7\mathbf{k})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ -2 & 3 & 7 \end{vmatrix} = 20\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}.$$

4. (d) $\boldsymbol{\alpha}.[\boldsymbol{\gamma}\times(\boldsymbol{\alpha}\times\boldsymbol{\beta})] = \boldsymbol{\alpha}.[(\boldsymbol{\gamma}.\boldsymbol{\beta})\boldsymbol{\alpha} - (\boldsymbol{\gamma}.\boldsymbol{\alpha})\boldsymbol{\beta}]$

$$= (\boldsymbol{\alpha})^2(\boldsymbol{\gamma}.\boldsymbol{\beta}) - (\boldsymbol{\gamma}.\boldsymbol{\alpha})(\boldsymbol{\beta}.\boldsymbol{\alpha}) = 14(-3) - (4)(8) = -74.$$

5. (b) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{0} \Rightarrow \mathbf{a}\,||\,(\mathbf{b}\times\mathbf{c})$ or $\mathbf{b}\times\mathbf{c} = \mathbf{0}$

i.e., $\mathbf{b}\,||\,\mathbf{c}$ or $\mathbf{a} = \mathbf{0}$.

6. (c) It is a fundamental concept.

7. (d) $\mathbf{a} = \mathbf{b}\times\mathbf{c}$ and $\mathbf{a}\times\mathbf{b} = \mathbf{c}$

$\therefore$ **a** is perpendicular to both **b** and **c** and **c** is perpendicular to both **a** and **b.**

$\mathbf{a}, \mathbf{b}, \mathbf{c}$ are mutually perpendicular

Now, $\mathbf{a} = \mathbf{b}\times\mathbf{c} = \mathbf{b}\times(\mathbf{a}\times\mathbf{b}) = (\mathbf{b}.\mathbf{b})\mathbf{a} - (\mathbf{b}.\mathbf{a})\mathbf{b}$

or $\mathbf{a} = b^2\mathbf{a} - (\mathbf{b}.\mathbf{a})\mathbf{b} = b^2\mathbf{a}$, $\{\because \mathbf{a}\perp\mathbf{b}\}$

$\Rightarrow 1 = b^2$, $\therefore \mathbf{c} = \mathbf{a}\times\mathbf{b} = ab\sin 90^\circ\,\hat{\mathbf{n}}$

Take moduli of both sides, then $c = ab$, but $b = 1 \Rightarrow c = a$.

8. (a) $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{b}.\mathbf{c})\mathbf{a}$

$$= (3+2+4)(2\mathbf{i}+\mathbf{j}-\mathbf{k}) - (2-2-2)(3\mathbf{i}-\mathbf{j}+2\mathbf{k})$$

$$= 18\mathbf{i} + 9\mathbf{j} - 9\mathbf{k} + 6\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} = 24\mathbf{i} + 7\mathbf{j} - 5\mathbf{k}.$$

9. (b) $\mathbf{i}\times\mathbf{j}\times\mathbf{k} = \mathbf{k}\times\mathbf{k} = 0$.

10. (d) As we know, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$ ......(i)

$\because\; \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \frac{\mathbf{b}}{2}$ (Given)

From equation (i),

$(\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c} = \frac{\mathbf{b}}{2}$ or $\left(\mathbf{a}.\mathbf{c} - \frac{1}{2}\right)\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c} = \mathbf{c}$

Comparison on both sides of **b** and **c**

$$\mathbf{a}.\mathbf{c} - \frac{1}{2} = 0, \;\; \mathbf{a}.\mathbf{b} = 0$$

$$\Rightarrow |\mathbf{a}||\mathbf{c}|\cos\theta = \frac{1}{2} \Rightarrow (1)(1)\cos\theta = \frac{1}{2} \Rightarrow \theta = 60^\circ$$

or $\mathbf{a}.\mathbf{b} = 0$, $\therefore\; \theta = 90^\circ$.

So the angle between **a** with **b** and **c** are 90° and 60° respectively.

11. (b) $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

$$\therefore |\mathbf{a}\times\mathbf{b}| = \sqrt{4+4+1} = 3$$

$$|\mathbf{c}-\mathbf{a}| = 2\sqrt{2} \Rightarrow |\mathbf{c}-\mathbf{a}|^2 = (\mathbf{c}-\mathbf{a})^2 = 8$$

$$\Rightarrow |\mathbf{c}|^2 - 2\mathbf{a}.\mathbf{c} + |\mathbf{a}|^2 = 8 \Rightarrow |\mathbf{c}|^2 - 2|\mathbf{c}| + 9 = 8$$

$$\Rightarrow |\mathbf{c}|^2 - 2|\mathbf{c}| + 1 = 0 \Rightarrow |\mathbf{c}| = 1$$

$$\therefore |(\mathbf{a}\times\mathbf{b})\times\mathbf{c}| = |\mathbf{a}\times\mathbf{b}||\mathbf{c}|\sin 30^\circ = \frac{3}{2}.$$

12. (d) $(\mathbf{i}\times\mathbf{i}) + (\mathbf{j}\times\mathbf{j}) + (\mathbf{k}\times\mathbf{k}) = 0.$

13. (c) $[\mathbf{b}\times\mathbf{c}\;\;\mathbf{c}\times\mathbf{a}\;\;\mathbf{a}\times\mathbf{b}] = (\mathbf{b}\times\mathbf{c}).[(\mathbf{c}\times\mathbf{a})\times(\mathbf{a}\times\mathbf{b})]$

Let $\mathbf{a}\times\mathbf{b} = \mathbf{d}$

so , $(\mathbf{b}\times\mathbf{c})[(\mathbf{c}\times\mathbf{a})\times\mathbf{d}] = (\mathbf{b}\times\mathbf{c})[(\mathbf{d}.\mathbf{a})\mathbf{c} - (\mathbf{d}.\mathbf{c}).\mathbf{a}]$

$= (\mathbf{b}\times\mathbf{c})[\mathbf{a}.(\mathbf{a}\times\mathbf{b}).\mathbf{c} - (\mathbf{a}\times\mathbf{b}).\mathbf{c}.\mathbf{a}]$

$= (\mathbf{b}\times\mathbf{c})[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]\mathbf{a} = \mathbf{a}.[\mathbf{b}\times\mathbf{c}].[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$

$= [\mathbf{a}\mathbf{b}\mathbf{c}][\mathbf{a}\mathbf{b}\mathbf{c}] = [\mathbf{a}\mathbf{b}\mathbf{c}]^2$.

14. (b) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$

$\because\ \mathbf{a}\perp\mathbf{b}$, $\therefore\ \mathbf{a}.\mathbf{b} = 0$

$\because\ \mathbf{a}\,||\,\mathbf{c}$, $\therefore\ \mathbf{a}.\mathbf{c} = 1$ ($\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are unit vectors)

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (1)\mathbf{b} - (0)\mathbf{c} = \mathbf{b}$.

15. (b) $\because\ \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$

$\because\ \mathbf{a}.\mathbf{c} = (\mathbf{i}+\mathbf{j}-\mathbf{k}).(\mathbf{i}-\mathbf{j}-\mathbf{k}) = 1-1+1 = 1$

$\mathbf{a}.\mathbf{b} = (\mathbf{i}+\mathbf{j}-\mathbf{k}).(\mathbf{i}-\mathbf{j}+\mathbf{k}) = 1-1-1 = -1$

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (1)\mathbf{b} - (-1)\mathbf{c}$

$= \mathbf{b}+\mathbf{c} = \mathbf{i}-\mathbf{j}+\mathbf{k}+\mathbf{i}-\mathbf{j}-\mathbf{k} = 2\mathbf{i}-2\mathbf{j}$.

16. (a) $\vec{A}\times\vec{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -3 \\ 2 & 1 & -1 \end{vmatrix} = \mathbf{i}(2+3) - \mathbf{j}(-1+6) + \mathbf{k}(1+4)$

$= 5\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$

Now $(\vec{A}\times\vec{B})\times\vec{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -5 & 5 \\ 1 & 3 & -2 \end{vmatrix}$

$= \mathbf{i}(10-15) - \mathbf{j}(-10-5) + \mathbf{k}(15+5)$

$= -5\mathbf{i} + 15\mathbf{j} + 20\mathbf{k} = 5(-\mathbf{i}+3\mathbf{j}+4\mathbf{k})$.

17. (a) $\because\ \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{b}.\mathbf{c})\mathbf{a} - (\mathbf{a}.\mathbf{b})\mathbf{c}$

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b})$

$= (\mathbf{b}.\mathbf{c})\mathbf{a} - (\mathbf{a}.\mathbf{b})\mathbf{c} + (\mathbf{b}.\mathbf{a})\mathbf{c} - (\mathbf{b}.\mathbf{c})\mathbf{a} + (\mathbf{b}.\mathbf{c})\mathbf{a} - (\mathbf{a}.\mathbf{c})\mathbf{b}$

$= 0$, $\{\because\ \mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a}\ \ etc\}$.

18. (a) We have $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$

$\Rightarrow (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c} = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{b}.\mathbf{c})\mathbf{a}$

$\Rightarrow -(\mathbf{a}.\mathbf{b})\mathbf{c} = -(\mathbf{b}.\mathbf{c})\mathbf{a} \Rightarrow (\mathbf{b}.\mathbf{c})\mathbf{a} - (\mathbf{b}.\mathbf{a})\mathbf{c} = 0$

$\Rightarrow \mathbf{b}\times(\mathbf{a}\times\mathbf{c}) = 0$.

19. (a) $\mathbf{a}.\mathbf{c} = 1$ and $\mathbf{b}.\mathbf{c} = 1$

Given that $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{c}.\mathbf{a})\mathbf{b} - (\mathbf{c}.\mathbf{b})\mathbf{a} = \mu\,\mathbf{b} + \lambda\mathbf{a}$

where $\mu = \mathbf{c}.\mathbf{a} = 1$, $\lambda = -(\mathbf{c}.\mathbf{b}) = -1$

$\Rightarrow \mu + \lambda = 1 - 1 = 0$.

20. (c) $\mathbf{a}\times[\mathbf{a}\times\{\mathbf{a}\times(\mathbf{a}\times\mathbf{b})\}] = \mathbf{a}\times[\mathbf{a}\times\{\mathbf{a}\times ab\hat{\mathbf{n}}\}]$

$= \mathbf{a}\times[\mathbf{a}\times a^2\mathbf{b}] = \mathbf{a}\times a^3 b\hat{\mathbf{n}} = |\mathbf{a}|^4\ \mathbf{b}$.

21. (d) $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = [\mathbf{a}\,\mathbf{b}\,\mathbf{d}]\mathbf{c} - [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]\mathbf{d}$

$\because\ \mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$ are coplanar vectors

$\therefore\ [\mathbf{a}\,\mathbf{b}\,\mathbf{d}] = [\mathbf{a}\,\mathbf{b}\,\mathbf{c}] = 0$. So, $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = \mathbf{0}$.

22. (d) $\mathbf{a}\times[\mathbf{a}\times(\mathbf{a}\times\mathbf{b})] = \mathbf{a}\times\{(\mathbf{a}.\mathbf{b})\mathbf{a} - (\mathbf{a}.\mathbf{a})\mathbf{b}\}$

$= (\mathbf{a}.\mathbf{b})(\mathbf{a}\times\mathbf{a}) - (\mathbf{a}.\mathbf{a})(\mathbf{a}\times\mathbf{b}) = (\mathbf{a}.\mathbf{b})0 + (\mathbf{a}.\mathbf{a})(\mathbf{b}\times\mathbf{a})$

$= (\mathbf{a}.\mathbf{a})\,(\mathbf{b}\times\mathbf{a})$.

23. (d) Multiplying (i) scalarly by $\mathbf{a}$, we get $\mathbf{a}.\mathbf{x} + \mathbf{a}.\mathbf{y} = \mathbf{a}^2$

$\therefore \mathbf{a}.\mathbf{y} = \mathbf{a}^2 - 1$ .....(iv), {By (iii)}

Again $\mathbf{a}\times(\mathbf{x}\times\mathbf{y}) = \mathbf{a}\times\mathbf{b}$ or $(\mathbf{a}.\mathbf{y})\mathbf{x} - (\mathbf{a}.\mathbf{x})\mathbf{y} = \mathbf{a}\times\mathbf{b}$

$(\mathbf{a}^2-1)\mathbf{x} - \mathbf{y} = \mathbf{a}\times\mathbf{b}$ .....(v), {By (iii) and (iv)}

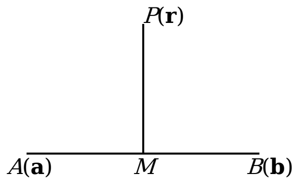
Adding and subtracting (i) and (v), we get

$x = \dfrac{\mathbf{a} + (\mathbf{a}\times\mathbf{b})}{\mathbf{a}^2}$ and $\mathbf{y} = \mathbf{a} - \mathbf{x}$ etc.

24. (c) $(\mathbf{b}\times\mathbf{c}.\mathbf{a})\mathbf{c} - (\mathbf{b}\times\mathbf{c}.\mathbf{c})\mathbf{a} = [\mathbf{b}\,\mathbf{c}\,\mathbf{a}]\mathbf{c} - \mathbf{0} = [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]\mathbf{c}$.

Application of vectors in three dimensional geometry

1. (a) Let $P(\mathbf{r})$ be equidistant from $A(\mathbf{a})$ and $B(\mathbf{b})$ and PM be perpendicular to AB.



Then M is the mid point of AB.

Position vector of M is $\frac{1}{2}(\mathbf{a}+\mathbf{b})$.

$\overrightarrow{PM}.\overrightarrow{BA} = 0$ or $\left[\mathbf{r} - \frac{1}{2}(\mathbf{a}+\mathbf{b})\right].(\mathbf{a}-\mathbf{b}) = 0$.

2. (a) Since $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$ are non-coplanar, hence $\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z(\mathbf{a}\times\mathbf{b})$ for some scalars x, y and z.

Now, $\mathbf{b} = \mathbf{r}\times\mathbf{a} = \{x\mathbf{a} + y\mathbf{b} + z(\mathbf{a}\times\mathbf{b})\}\times\mathbf{a}$

$= y(\mathbf{b}\times\mathbf{a}) + z[(\mathbf{a}\times\mathbf{b})\times\mathbf{a}] = -y(\mathbf{a}\times\mathbf{b}) - z[\mathbf{a}\times(\mathbf{a}\times\mathbf{b})]$

$= -y(\mathbf{a}\times\mathbf{b}) - z[(\mathbf{a}.\mathbf{b})\mathbf{a} - (\mathbf{a}.\mathbf{a})\mathbf{b}]$

$= -y(\mathbf{a}\times\mathbf{b}) + z(\mathbf{a}.\mathbf{a})\mathbf{b}$, $\{\because \mathbf{a}.\mathbf{b} = 0\}$

$\Rightarrow y = 0$ and $z = \dfrac{1}{(\mathbf{a}.\mathbf{a})} \Rightarrow \mathbf{r} = x\mathbf{a} + \dfrac{1}{\mathbf{a}.\mathbf{a}}(\mathbf{a}\times\mathbf{b})$.

3. (d) It is a fundamental property.

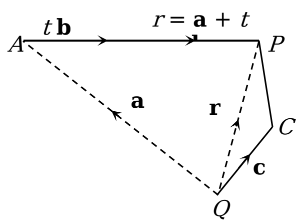
4. (a) The plane is $2x - y + z = 4$ and the line is $\dfrac{x-1}{1} = \dfrac{y-2}{-1} = \dfrac{z+1}{1}$

$\therefore\ \sin\theta = \dfrac{2+1+1}{\sqrt{6}\sqrt{3}} = \dfrac{4}{\sqrt{18}} = \dfrac{2\sqrt{2}}{3} \Rightarrow \theta = \sin^{-1}\left(\dfrac{2\sqrt{2}}{3}\right)$.

5. (b) For point P on the line $r = \mathbf{a} + t\mathbf{b}$

$\therefore\ \overrightarrow{PC} = (\mathbf{c}-\mathbf{a}) - t\mathbf{b}$, $\because\ \overrightarrow{PC}\perp\mathbf{b}$

$\therefore\ |(\mathbf{c}-\mathbf{a}) - t\mathbf{b}|.\mathbf{b} = 0$ or $t = \dfrac{(\mathbf{c}-\mathbf{a}).\mathbf{b}}{\mathbf{b}^2}$ .....(i)



Distance of $\mathbf{c}$ from line $|\overrightarrow{PC}| = d = |\mathbf{c} - \mathbf{a} - t\mathbf{b}|$

$$d = \left| \mathbf{c} - \mathbf{a} - \frac{(\mathbf{c}-\mathbf{a}).\mathbf{b}\mathbf{b}}{\mathbf{b}^2} \right| = \left| \frac{(\mathbf{c}-\mathbf{a})\mathbf{b}.\mathbf{b} - (\mathbf{c}-\mathbf{a}).\mathbf{b}\mathbf{b}}{\mathbf{b}^2} \right|$$

$$d = \left| \frac{\mathbf{b} \times (\mathbf{c}-\mathbf{a}) \times \mathbf{b}}{\mathbf{b}^2} \right| = \frac{|\mathbf{b}|| (\mathbf{c}-\mathbf{a}) \times \mathbf{b}| \sin 90^\circ}{|\mathbf{b}|^2},$$

$(\because \mathbf{b} \perp (\mathbf{c}-\mathbf{a}) \times \mathbf{b})$

$$d = \frac{|(\mathbf{c}-\mathbf{a}) \times \mathbf{b}|}{|\mathbf{b}|}.$$

6. (c) Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\mathbf{r} \times \mathbf{a} = \mathbf{b} \times \mathbf{a}$

$$(\mathbf{r} - \mathbf{b}) \times \mathbf{a} = 0 \Rightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x-2 & y & z+1 \\ 1 & 1 & 0 \end{vmatrix} = 0,$$

$\therefore z = -1,\ x - y = 2$

Now $\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$ $\quad (\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$

$\therefore y = 1,\ x + 2z = 1 \Rightarrow x = 3,\ y = 1,\ z = -1$

$\mathbf{r} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$.

7. (b) If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-coplanar vectors then the vector equation $\mathbf{r} = (1 - p - q)\mathbf{a} + p\mathbf{b} + q\mathbf{c}$ represents a plane.

8. (c) Vector equation of a straight line passing through two points $\mathbf{a}$ and $\mathbf{b}$ is, $\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$

$\mathbf{r} = (\mathbf{i} - 2\mathbf{j} + \mathbf{k}) + t(2\mathbf{k} - \mathbf{i})$.

9. (c) It is obvious.

10. (a) Angle between two plane faces is equal to the angle between the normals $\mathbf{n}_1$ and $\mathbf{n}_2$ to the planes. $\mathbf{n}_1$ the normal of face OAB is given by

$$\overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k} \quad .....(i)$$

$\mathbf{n}_2$ the normal of face ABC is given by $\overrightarrow{AB} \times \overrightarrow{AC}$ $\quad 2-1, 1-2, 3-1$ and $-1-1, 1-2, 2-1$ *i.e.,* $1, -1, 2$ and $-2, -1, 1$.

$$\therefore \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ -2 & -1 & 1 \end{vmatrix} = \mathbf{i} - 5\mathbf{j} - 3\mathbf{k} \quad .....(ii)$$

If be the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$, then

$$\cos\theta = \frac{\mathbf{n}_1 . \mathbf{n}_2}{|\mathbf{n}_1| . |\mathbf{n}_2|} = \frac{5+5+9}{\sqrt{35}.\sqrt{35}} = \frac{19}{35}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{19}{35}\right).$$