Module - 4

Contents

1 Introduction

A matrix is a rectangular array of numbers.Matrices are useful due to several reasons.System of equations can be represented compactly using matrices.Also it enable us to consider an array of many numbers as a single object and perform calculations in a simple manner.This mathematical short hand so obtained is very elegant and powerful and is suitable for various practical problems. Linear algebra makes systematic use of matrices and vectors.It includes the theory and applicafitions of linear system of equations,linear transformations and eigen value problems.Linear algebra is also used in most science and engineering areas because it allows modelling of many natufiral phenomena and efficiently computing with such models.Coding theory,cryptography,computer graphics and optimisation techniques are some consequences of linear algebra theory.

2 System of Linear Equations

A linear equation in n variables $x_1, x_2, \ldots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b \tag{1}$$

where $a_1, a_2, \ldots, a_n, b$ are the coefficients of the equation which can be real or complex and n is any positive integer.

Note the difference of (1) with the following equations:

$$2x_1 - 3x_2 + x_1x_2 = 15 \tag{2}$$

$$3\sqrt{x_1} - 5x_2 + 10 = 0 \tag{3}$$

Equations (2) and (3) are not linear, since (2) contains an x_1x_2 term and (3) contains a $\sqrt{x}$ term.

A **system of linear equations** or a **Linear system** is a collection of one or more linear equations involving the same variables.
For example,Consider the following system of 3 equations in 3 unknowns.

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_1 + 2x_2 + 3x_3 &= 10 \\ x_1 + 2x_2 + 5x_3 &= 14 \end{aligned} \tag{4}$$

A **solution** is a set of values of $x_1, x_2, \ldots, x_n$ that satisfy each equation completely.
$(4, 0, 2)$ is a solution to system (4) since it satisfy each equation of (4) completely.

3 Geometrical Interpretation of Solutions of Linear System

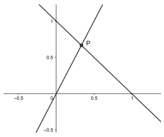
Consider two equations in two unknowns x_1 and x_2.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

If we interpret x_1, x_2 as coordinates in the $x_1 - x_2$ plane, then each of these equations represent a straight line and (x_1, x_2) is a solution if and only if the point (x_1, x_2) lies on both the lines.Hence there are three possibilities:

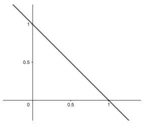
1. exactly one solution if the lines intersect.
2. infinitely many solutions if the lines coincide.
3. no solution if the lines are parallel.

For example:



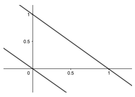
$$x_1 + x_2 = 1$$
$$2x_1 - x_2 = 0$$

with unique solution at P(1/3,2/3)



$$x_1 + x_2 = 1$$
$$2x_1 + 2x_2 = 2$$

with infinite no.of solutions



$$x_1 + x_2 = 1$$
$$x_1 + x_2 = 0$$

with no solution

Hence a linear system can have

1. Exactly one solution
2. Infinite number of solutions
3. No solution

Under conditions (1) or (2), the system is said to be consistent and in situation (3) the system is inconsistent.

4 Matrix representation of linear systems

Consider the following system of m linear equations in n unknowns $x_1, x_2, \dots, x_n$;

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n &= b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n &= b_m \end{aligned} \tag{5}$$

Changing to matrix notation, the system can be written as **AX=B** where the coefficient matrix $\mathbf{A} = [a_{jk}]$ is the $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \text{ and } \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} ; \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}$$

Here **X** and **B** are column vectors.We assume that the coefficients a_{jk} are not all zero, so that **A** is not a zero matrix.Note that **X** has n components where as **B** has m components. The matrix

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & : & b_2 \\ \dots & \dots & \dots & \dots & : & \dots \\ \dots & \dots & \dots & \dots & : & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & : & b_m \end{bmatrix}$$

is called the **augmented matrix** . The augmented matrix **AB** determines the system completely because it contains all the numbers appearing in (7)
If all the b_j's are zero in (7), then $\mathbf{AX} = 0$ is called a **homogeneous system.**

5 Row-echelon form and rank of a matrix

Consider the elementary row operations on matrices:

1. Interchange of two rows.
2. Addition of a constant multiple of one row to another row.
3. Multiplication of a row by a non-zero constant.

They correspond to the following:

1. Interchange of two equations.
2. Addition of a constant multiple of one equation to another equation.
3. Multiplication of a non-zero constant.

Clearly these operations do not alter the solution set.A linear system S_1 is row-equivalent to a linear system S_2 if S_1 can be obtained from S_2 by finitely many elementary row operations.
No column operations on the augmented matrix are permitted because they would generally alter the solution set.

In the definition that follows, a non-zero row in a matrix means a row that contains atleast one non-zero entry; a **leading entry or pivot** of a row refers to the leftmost non-zero entry in a non-zero row.
Echelon Matrix :An $m \times n$ matrix is said to be an echelon matrix if

1. all non-zero rows are above any row of all zeros
2. the number of zeros preceeding this entry is more than the corresponding number in the previous row.That is , all entries in a column below a leading entry(or pivot) are zeros.

Note that echelon matrix has a 'step-like' pattern that moves down and to the right through the matrix.
Eg. $\begin{bmatrix} 2 & -3 & 1 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

Rank of a matrix

Rank of any matrix **A** is the number of non-zero rows in any echelon matrix equivalent to **A**.

Problems

1. Reduce to echelon form and hence find the rank of $\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$

   (U.Q.March 2017)
   Solution:
   Given matrix $\sim \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -15 \end{bmatrix}$ $R_2 \to R_2 + 2R_1; R_3 \to R_3 - 7R_1$

$$\sim \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + 0.5R_2$$

Rank = no. of non zero rows in the echelon form = 2

2. Find the rank of $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

   (U.Q.April 2018)
   Solution:

   Given matrix $\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} R_1 \leftrightarrow R_2$

   $$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} R_2 \to R_2 - 2R_1; R_3 \to R_3 - 3R_1; R_4 \to R_4 - 6R_1$$

   $$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 33 & 22 \end{bmatrix} R_3 \to 5R_3 - 4R_2; R_4 \to 5R_4 - 9R_2$$

   $$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{Rank = no. of non zero rows in the echelon form} = 3$$

6 Linear Independence of vectors

Let $v_1, v_2, \ldots, v_n$ be n vectors,then the expression $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalars is called **linear combination** of these vectors.
Suppose $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$ happens only when $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$, then the vectors $v_1, v_2, \ldots, v_n$ are **linearly independent**.If any $\alpha_i \neq 0$, then $v_1, v_2, \ldots, v_n$ are **linearly dependent.**

6.1 Determining Linear Independence using Matrices

Let $v_1, v_2, \ldots, v_n$ be the n vectors.

- Form a matrix with these vectors as row vectors.
- Find the rank of this matrix.
- If this rank $= n$,then the vectors are linearly independent.
- If the rank is less than n, then the vectors are linearly dependent.

6.2 Rank and Linear Independence

1. The rank of a matrix A is the maximum number of linearly independent row vectors of A

2. The rank of a matrix A is the maximum number of linearly independent column vectors of A
   Hence A and A^T have the same rank

3. Consider p vectors each having n components.If $n < p$, then the vectors are linearly dependent.

Problem
Check whether the vectors $[1, 2, 1], [2, 1, 4], [4, 5, 6]$ and $[1, 8, -3]$ are linearly independent in $\mathbb{R}^3$
Solution:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 4 \\ 4 & 5 & 6 \\ 1 & 8 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & -3 & 2 \\ 0 & 6 & -4 \end{bmatrix} R_2 \to R_2 - 2R_1; R_3 \to R_3 - 4R_1; R_4 \to R_4 - R_1;$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2; R_4 \to R_4 + 2R_2$$

Here rank=2 which is less than the number of vectors.Hence linearly dependent.

7 Solution of Linear system of equations:

7.1 Existence and Uniqueness:Fundamental theorem for Linear Systems

- Existence: A linear system given by (7) is consistent if and only if the coefficient matrix **A** and the augmented matrix **AB** have the same rank.

- Uniqueness:The system (7)has precisely one solution if and only if this common rank r of **A** and **AB** equals n.

- Infinitely many solutions:If this common rank r is less than n, the system (7) has infinitely many solutions.All these solutions can be obtained by determininig r suitable solutions in terms of the remaining $n - r$ unknowns to which arbitrary values can be assigned.

- Gauss elimination:If solutions exist, they can all be obtained by Gauss elimination method.

7.2 Gauss Elimination Method and Back Substitution

Gauss elimination is a standard method for solving linear systems.It is an exact and systematic elimination process.**This method provides an algorithm that performs elementary transformations to bring a system of linear equations to the row-echelon form.** Since a linear system is completely determined by its augmented matrix, the elimination process can be done by merely considering the matrices.

Working principle to check the consistency of Linear system of equations

Consider the system of equations **AX=B**.Reduce the augmented matrix **AB** to echelon form by Gauss elimination method.Then the following cases arise:

- If the rank[**AB**]$\neq$ rank[**A**]; then the system is inconsistent.
- If the rank[**AB**]= rank[**A**]=number of unknowns; then the system is consistent with a unique solution.
- If the rank[**AB**]= rank[**A**]< number of unknowns; then the system is consistent with infinite number of solutions.

Problems

1. Solve the following linear system by Gauss elimination method:

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -x_1 + x_2 - x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 20x_1 + 10x_2 &= 80 \end{aligned} \tag{6}$$

(U.Q.Dece 2016,2018)
Solution:

The augmented matrix $\mathbf{AB} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{bmatrix} R_4 \to R_4 - 20R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 95 & 190 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to 3R_3 - R_2$$

Hence rank[**AB**]= The number of non-zero rows of the final echelon form = 3

Also by the same elementary row transformations: $\mathbf{A} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 30 & -20 \\ 0 & 0 & 95 \\ 0 & 0 & 0 \end{bmatrix}$

and rank[**A**]=3
Thus rank[**AB**]= rank[**A**]= 3 = number of unknowns. Hence the system is consistent with a unique solution which can be obtained by back substitution method:
The given system (7) is equivalent to $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 30 & -20 \\ 0 & 0 & 95 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 80 \\ 190 \end{bmatrix}$ which can be written as :

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 30x_2 - 20x_3 &= 80 \\ 95x_3 &= 190 \end{aligned}$$

Applying back substitution method,from the last equation $x_3 = 2$.Substituting $x_3 = 2$ in the second last equation $x_2 = 4$.Putting x_3, x_2 in the first equation $x_1 = 2$.

Hence the required solution is $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$

2. Test the following system for consistency:

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ 2x_2 + x_2 + x_3 &= 0 \\ 6x_1 + 2x_2 + 4x_3 &= 6 \end{aligned}$$

Solution: $\mathbf{AB} = \begin{bmatrix} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{bmatrix}$

$\sim \begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & -1 & 1 & -6 \\ 0 & -2 & 2 & 0 \end{bmatrix} R_2 \to 3R_2 - 2R_1; R_3 \to R_3 - 2R_1$

$\sim \begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & -1 & 1 & -6 \\ 0 & 0 & 0 & 12 \end{bmatrix} R_3 \to R_3 - 2R_2$

Here rank[$\mathbf{AB}$] $\neq$ rank[$\mathbf{A}$] since rank[$\mathbf{AB}$]=3 and rank[$\mathbf{A}$]=2. Hence the system is inconsistent.

3. Solve the following system of equations by Gauss elimination method:

$$\begin{aligned} 7x - 4y - 2z &= -6 \\ 16x + 2y + z &= 3 \end{aligned}$$

Solution: $\mathbf{AB} = \begin{bmatrix} 7 & -4 & -2 & -6 \\ 16 & 2 & 1 & 3 \end{bmatrix}$

$\sim \begin{bmatrix} 7 & -4 & -2 & -6 \\ 0 & 78 & 39 & 117 \end{bmatrix} R_2 \to 7R_2 - 16R_1$ rank[$\mathbf{AB}$] $=$ rank[$\mathbf{A}$] $= 2 <$ the number of unknowns.

Hence consistent with infinite number of solutions.The solution can be obtained as follows: The reduced system can be written as

$$\begin{aligned} 7x - 4y - 2z &= -6 \\ 78y + 39z &= 117 \end{aligned}$$

or equivalently

$$\begin{aligned} 7x - 4y - 2z &= -6 \\ 2y + z &= 3 \end{aligned}$$

Let $y = a$. Then $z = 3 - 2a$ and $x = 0$.

Hence the required solution is $x = 0, y = a, z = 3 - 2a$.(Since a is arbitrary,we have infinitely many solutions.)

4. Solve the following system of equations by Gauss elimination method:
(U.Q.March 2017)

$$\begin{aligned} y + z - 2w &= 0 \\ 2x - 3y - 3z + 6w &= 2 \\ 4x + y + z - 2w &= 4 \end{aligned}$$

Solution: $\mathbf{AB} = \begin{bmatrix} 0 & 1 & 1 & -2 & 0 \\ 2 & -3 & -3 & 6 & 2 \\ 4 & 1 & 1 & -2 & 4 \end{bmatrix}$

$\sim \begin{bmatrix} 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 7 & 7 & -14 & 0 \end{bmatrix} R_3 \to 2R_1$

$\sim \begin{bmatrix} 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to 7R_2$

rank[$\mathbf{AB}$] = rank[$\mathbf{A}$] $= 2 <$ the number of unknowns.Hence the system is consistent with infinitely many solutions (by assigning arbitrary values to (4-2)unknowns).
Back substitution method:

$$\begin{aligned} 2x - 3y - 3z + 6w &= 2 \\ y + z - 2w &= 0 \end{aligned}$$

Let $w = a, z = b$.Hence $y = 2a - b, x = 1$
Hence the solution is:$x = 1, y = 2a - b, z = b, w = a$

5. Solve the following system of 3 equations in 4 unknowns whose augmented matrix is
$\begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{bmatrix}$
Solution:
Given matrix $\sim \begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & 11.1 & 4.4 & -1.1 \end{bmatrix} R_2 \to 0.2R_1; R_3 \to 0.4R_1$

$\sim \begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_2 \to 0.2R_1; R_3 \to 0.4R_1$

Back substitution method:

$$\begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \end{aligned}$$

Let $x_4 = a, x_3 = b$.Then $x_2 = 1 - b + 4a$ and $x_1 = 2 - a$
If $a = 0, b = 1$ then one solution is
$x_1 = 2, x_2 = 0, x_3 = 1, x_4 = 0$

6. Find the values of a and b for which the system of equations $x + y + 2z = 2, 2x - y + 3z = 10, 5x - y + az = b$ has (i)no solution (ii)unique solution (iii)infinite number of solutions. (U.Q.Dece. 2017)
Solution:$\mathbf{AB} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & -1 & 3 & 10 \\ 5 & -1 & a & b \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & 6 \\ 0 & -6 & a-10 & b-10 \end{bmatrix} R_2 \to R_2 - 2R_1; R_3 \to R_3 - 5R_1$

$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & 6 \\ 0 & 0 & a-8 & b-22 \end{bmatrix} R_3 \to R_3 - 2R_2$

(1)If $a = 8, b \neq 22$ then rank[$\mathbf{AB}$] $\neq$ rank[$\mathbf{A}$].Hence no solution.
(2)If $a \neq 8$ and b any value,then rank[$\mathbf{AB}$] = rank[$\mathbf{A}$]= no.of unknowns.Hence the system

has unique solution.
(3)If $a=8, b=22$,then rank[**AB**] = rank[**A**] $<$ no.of unknowns.Hence the system has infinite number of solutions.

7. Find the values of μ for which the system of equations
$x+y+z=1, x+2y+3z=\mu, x+5y+9z=\mu^2$ will be consistent.For each value of μ so obtained, find the solution of the system.
(U.Q.July 2017)
Solution:AB= $\begin{bmatrix}1&1&1&1\\1&2&3&\mu\\1&5&9&\mu^2\end{bmatrix}$
$\sim \begin{bmatrix}1&1&1&1\\0&1&2&\mu-1\\0&4&8&\mu^2-1\end{bmatrix} R_2\to R_2-R_1; R_3\to R_3-R_1$
$\sim \begin{bmatrix}1&1&1&1\\0&1&2&\mu-1\\0&0&0&(\mu-1)(\mu-3)\end{bmatrix} R_3\to R_3-4R_2$
If $\mu=1$ or 3 for the system to be consistent with infinite number of solutions.
When $\mu=1$, **AB**$\sim \begin{bmatrix}1&1&1&1\\0&1&2&0\\0&0&0&0\end{bmatrix}$
Solution of this system is $x=1+a, y=-2a, z=a$.
When $\mu=3$, **AB**$\sim \begin{bmatrix}1&1&1&1\\0&1&2&2\\0&0&0&0\end{bmatrix}$
Solution of this system is $x=a-1, y=2-2a, z=a$.

8 Homogeneous Linear System of Equations

A homogeneous linear system of equations:

$$\begin{aligned} a_{11}x_1+a_{12}x_2+\ldots\ldots+a_{1n}x_n&=0\\ a_{21}x_1+a_{22}x_2+\ldots\ldots+a_{2n}x_n&=0\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots&\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots&\\ a_{m1}x_1+a_{m2}x_2+\ldots\ldots+a_{mn}x_n&=0 \end{aligned} \tag{7}$$

always has the trivial solution $x_1=0, x_2=0,\ldots,x_n=0$.Hence a homogeneous linear system is always CONSISTENT
Non-trivial solution exist if and only if $rank(A)<n$

Some important results:
Theorem 1: A homogeneous linear system with fewer equations than unknowns has non-trivial solutions.
Theorem 2: A homogeneous linear system $AX=0$ where A is a square matrix will have a non-trivial solution if and only if $|A|=0$ (singular)

Problems

1. Do the equations

$$
\begin{aligned}
x - 3y - 8z &= 0 \\
3x + y &= 0 \\
2x + 5y + 6z &= 0
\end{aligned}
\tag{8}
$$

have a non-trivial solution?Why?
Solution: The matrix corresponding to the above system (8) is

$$
\mathbf{A} = \begin{bmatrix} 1 & -3 & -8 \\ 3 & 1 & 0 \\ 2 & 5 & 6 \end{bmatrix}
$$

$$
\sim \begin{bmatrix} 1 & -3 & -8 \\ 0 & 10 & 24 \\ 0 & 11 & 22 \end{bmatrix} \; R_2 \to R_2 - 3R_1; R_3 \to R_3 - 2R_1
$$

$$
\sim \begin{bmatrix} 1 & -3 & -8 \\ 0 & 10 & 24 \\ 0 & 0 & -44 \end{bmatrix} \; R_3 \to 10R_3 - 11R_2
$$

$Rank(A) = 3 =$ number of unknowns.hence the system (8) has only trivial solution.
Note:Here $|A| = -44 \neq 0$ .Hence also the same conclusion can be reached.

2. Show that the equations

$$
\begin{aligned}
x + 2y - z &= 0 \\
3x + y - z &= 0 \\
2x - y &= 0
\end{aligned}
\tag{9}
$$

have non-trivial solutions and hence find them
Solution: The matrix corresponding to the above system (9) is

$$
\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}
$$

$$
\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 2 \\ 0 & -5 & 2 \end{bmatrix} \; R_2 \to R_2 - 3R_1; R_3 \to R_3 - 2R_1
$$

$$
\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \; R_3 \to R_3 - R_2
$$

$Rank(A) = 2 <$ number of unknowns.hence the system (9) has non-trivial solution which can be obtained as follows:

The given system of equations (9) is equivalent to $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

or equivalently

$$
\begin{aligned}
x + 2y - z &= 0 \\
-5y + 2z &= 0
\end{aligned}
$$

. Hence by assigning arbitrary value to one unknown (n-r=3-2=1),we have a solution as follows:Let $z = a$.then $y = 2a/5$ and $x = a/5$.

$$
\therefore x = \frac{a}{5}, y = \frac{2a}{5}, z = a
$$

where a is arbitrary

Exercise Questions

1. If the matrix $\begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ 6 & -2 & a & b \end{bmatrix}$ is of rank 2,find the values of a and b
(U.Q.Dece. 2019)

   Ans: $a = 4$ and $b = 6$

2. Find the rank of the matrix: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$
(U.Q.Dece. 2017)

   Ans:rank $= 2$

3. Find the rank of the matrix: $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix}$
(U.Q.Dece. 2016)

   Ans:rank $= 2$

4. Show that the system is inconsistent:
$2x + 6y = -11, 6x + 20y - 6z = -3, 6y - 18z = -1$
(U.Q.Dece. 2018)

5. Find the values of λ and $\mu$for which the system of equations
$2x + 3y + 5z = 9, \quad 7x + 3y - 2z = 8 \quad 2x + 3y + \lambda z = \mu$ has (i)no solution (ii)a unique solution(iii)infinite solution.
(U.Q.Dece. 2019)

   Ans:(i)$\lambda = 5, \mu \neq 9$(ii)$\lambda \neq 5, \mu$, any value.(iii)$\lambda = 5, \mu = 9$

6. Solve by Gauss elimination method:
$x + y + z = 6, x + 2y - 3z = -4, -x - 4y + 9z = 18$
(U.Q.Dece. 2017)

   Ans:$x = 1, y = 2, z = 3$

7. Using Gauss elimination method, find the solutions of the system of equations:
$x + 2y - z = 3, 3x - y + 2z = 1, 2x - 2y + 3z = 2, x - y + z = -1$
(U.Q.July 2017)

   Ans:$x = -1, y = 4, z = 4$

8. Solve the system using Gauss elimination method:
$3x + 3y + 2z = 1, x + 2y = 4, 10y + 3z = -2, 2x - 3y - z = 5$
(U.Q.April 2018)

   Ans:$x = 2, y = 1, z = -4$

9. Solve the system of equations using Gauss elimination method:
$x + 2y + 3z = 1, 2x + 3y + 2z = 2, 3x + 3y + 4z = 1$
(U.Q.Dece 2019)

   Ans:$x = -3/7, y = 8/7, z = -2/7$

9 Eigen Values and Eigen Vectors

Consider a square matrix $\mathbf{A}$ of order n.The problem of finding solution to the equation

$$\mathbf{Ax} = \lambda\mathbf{x} \tag{10}$$

where $\mathbf{x}$ is a vector and λ is a scalar, is called an **eigen value problem** .
Here $X = 0$ is a solution to (9).But this trivial solution is of no interest and hence the problem is to find a solution of the form $\mathbf{x} \neq 0$ called eigen vectors of $\mathbf{A}$.The values of λ for which an eigen vector exists are called eigen values of $\mathbf{A}$.
Geometrically, finding solution to eigen value problem implies finding vectors $\mathbf{x}$ for which the multiplication by $\mathbf{A}$ has the same effect as the multiplication by a scalar.In other words $\mathbf{Ax}$ should be proportional to $\mathbf{x}$.
A value of λ for which (9) has a solution $\mathbf{x} \neq 0$ is called an **eigen value**(or **characteristic value or latent root**) of the matrix $\mathbf{A}$.The corresponding solutions $\mathbf{x} \neq 0$ of (9) are called the **eigen vectors** (or **characteristic vectors**)of the matrix $\mathbf{A}$ corresponding to the eigen value λ.
Hence the problem of determinig the eigen values and eigen vectors of a matrix is called an eigen value problem.
The set of all eigen values of $\mathbf{A}$ is called the **spectrum** of $\mathbf{A}$.
The largest of the absolute values of the eigen values of $\mathbf{A}$ is called the **spectral radius** of $\mathbf{A}$.

Consider an eigen value problem;

$$\begin{aligned}
\mathbf{Ax} &= \lambda\mathbf{x}\\
\mathbf{Ax} - \lambda\mathbf{x} &= 0\\
(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} &= 0
\end{aligned}$$

This is a system of linear homogeneous equations and has a non-trivial solution if $|\mathbf{A}-\lambda\mathbf{I}| = 0$.This equation is called the **characteristic equation**.The roots of the characteristic equation are called eigen values and corresponding to each eigen value, eigen vectors can be computed.
Note:Eigen values are unique for a matrix.But eigen vectors are not unique.Since any scalar multiple of an eigen vector is also an eigen vector.

Let us discuss the method of finding eigen values and eigen vectors with the help of an example:
Consider $\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$
Solution: Step I:Finding the eigen values
The characteristic equation corresponding to $\mathbf{A}$ is $|\mathbf{A} - \lambda\mathbf{I}| = 0$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$
$$\lambda^2 + 7\lambda + 6 = 0$$

The solution of this quadratic equation is :$\lambda = -1, -6$
Step II:Finding eigen vectors corresponding to each eigen value:
(a)Eigen vector corresponding to $\lambda = -1$

$$\begin{aligned}
\mathbf{Ax} &= -\mathbf{x}\\
(\mathbf{A} + \mathbf{I})\mathbf{x} &= 0
\end{aligned}$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Corresponding equations are: $-4x + 2y = 0; 2x - y = 0$ (which gives just one equation)
Hence a solution is : $x = 1, y = 2$

$\therefore X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (2)Eigen vector corresponding to $\lambda = -6$

$$\mathbf{Ax} = -6\mathbf{x}$$
$$(\mathbf{A} + 6\mathbf{I})\mathbf{x} = 0$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ Corresponding equations are: $x + 2y = 0; 2x + 4y = 0$ which gives a solution as $x = -2, y = 1$

$\therefore X_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

More problems

1. Find the eigen values and eigen vectors of $\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

   (*U.Q.March 17*)

   Solution: Characteristic equation is given by $|\mathbf{A} - \lambda\mathbf{I}| = 0$

   $$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$
   $$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

   <u>Aliter</u>:Shortcut method for finding the characteristic equation corresponding to any 3×3 matrix $\mathbf{A}$:

   $$\lambda^3 - (\text{sum of main diagonal elements})\lambda^2 + (\text{sum of minors of main diagonal elements})\lambda - |\mathbf{A}| = 0$$

   Hence CE using this alternate way:

   $$\lambda^3 - (-2 + 1)\lambda^2 + \left(\begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} \right) - 45 = 0$$

   $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$ Solution of this cubic equation gives $\lambda = 5, -3, -3$

   (a)Eigen vector corresponding to $\lambda = 5$

   $[\mathbf{A} - 5\mathbf{I}]\, X = 0$

   $$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (1)$$

$$\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix} \sim \begin{bmatrix}-1 & -2 & -5\\ 2 & -4 & -6\\ -7 & 2 & -3\end{bmatrix} R_3 \leftrightarrow R_1$$

$$\sim \begin{bmatrix}-1 & -2 & -5\\ 0 & -8 & -16\\ 0 & 16 & 32\end{bmatrix} R_2 \to R_2 + 2R_1; R_3 \to R_3 - 7R_1$$

$$\sim \begin{bmatrix}-1 & -2 & -5\\ 0 & -8 & -16\\ 0 & 0 & 0\end{bmatrix} R_3 \to R_3 + R_2$$

Corresponding system of equations are

$$-x + -2y - 5z = 0$$
$$-8y - 16z = 0$$

Let $z = a$.Then $y = -2a$ and $x = -1$.Hence one solution is $X_1 = \begin{bmatrix}-1\\ -2\\ 1\end{bmatrix}$

Aliter:Rule of cross multiplication to find the solution:
Since the three equations given by (1) are different,consider any two equations,

$$-7x + 2y - 3z = 02x - 4y - 6z = 0$$

By taking the coefficients of these two equations

-7	2	-3	-7	2
2	-4	-6	2	-4

we have

$$\frac{x}{(2\times -6) - (-3\times -4)} = \frac{y}{(-3\times 2) - (-7\times -6)} = \frac{z}{(-7\times -4) - (2\times 2)}$$

That is

$$\frac{x}{-24} = \frac{y}{-48} = \frac{z}{24} = \frac{1}{24}(say)$$

Hence $x = -1, y = -2, z = 1$
(b)Eigen vector corresponding to $\lambda = -3$

$[\mathbf{A} + 3\mathbf{I}]\, X = 0$

$$\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

This corresponds to just one equation in 3 unknowns:$x + 2y - 3z = 0$
Two independent solutions can be produced from this equation .One pair of such independent solutions is obtained by putting $z = 1, y = 0$ and $z = 0, y = 1$.In the first case $x = 3$ and in the second case $x = -2$ Hence the solutions are $X_2 = \begin{bmatrix}3 & 0 & 1\end{bmatrix}^T$ and $X_3 = \begin{bmatrix}-2 & 1 & 0\end{bmatrix}^T$

Note: The problem has 3 eigen values of which 2 are equal.Here the similar eigen values have linearly independent eigen vectors.(Property (b))

2. Find the eigen values and eigen vectors of $\begin{bmatrix}13 & 5 & 2\\ 2 & 7 & -8\\ 5 & 4 & 7\end{bmatrix}$

Solution: The characteristic equation of matrix $\mathbf{A}$ is given by $|\mathbf{A} - \lambda I| = 0$.

For the matrix given CE is obtained as $\lambda^3 - 27\lambda^2 + 243\lambda - 729 = 0$ .The solution of which gives the Eigen values as $\lambda = 9, 9, 9$
Eigen vector corresponding to $\lambda = 9$ can be evaluated from $[A - 9I]X = 0$

$$\begin{bmatrix} 4 & 5 & 2 \\ 2 & -2 & -8 \\ 5 & 4 & -2 \end{bmatrix} X = 0$$

By the rule of cross multiplication:By taking the coefficients of second and third equations

2	-2	-8	2	-2
5	4	-2	5	4

$$\frac{x}{(-2 \times -2) - (4 \times -8)} = \frac{y}{(-8 \times 5) - (-2 \times 2)} = \frac{z}{(2 \times 4) - (5 \times -2)}$$

That is

$$\frac{x}{36} = \frac{y}{-36} = \frac{z}{18} = \frac{1}{18}$$

Hence $x = 2, y = -2, z = 1$ and we write the eigen vector as $X = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

Note:In this problem, Corresponding to equal eigen values,eigen vectors are linearly dependent.(Property (b))

3. Find the eigen values and eigen vectors of $\begin{bmatrix} 6 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix}$

Solution: The characteristic equation of matrix $\mathbf{A}$ is given by $|\mathbf{A} - \lambda I| = 0$.
For the matrix given CE is obtained as $\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$ .The solution of which gives the Eigen values as $\lambda = 3, 6, 9$
(a) Eigen vetor corresponding to $\lambda = 3$ can be evaluated from $[A - 3I]X = 0$
which is $\begin{bmatrix} 3 & 2 & -2 \\ 2 & 2 & 0 \\ -2 & 0 & 4 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
By the rule of cross multiplication(by taking coefficients of 2nd and third equations)

$$\frac{x}{8} = \frac{y}{-8} = \frac{z}{4} = \frac{1}{4}(\text{say})$$

Hence $X_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$
(b) Eigen vetor corresponding to $\lambda = 6$ can be evaluated from $[A - 6I]X = 0$
which is $\begin{bmatrix} 0 & 2 & -2 \\ 2 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
By the rule of cross multiplication(by taking coefficients of 1st and 2nd equations)

$$\frac{x}{-2} = \frac{y}{-4} = \frac{z}{-4} = \frac{1}{-2}(\text{say})$$

Hence $X_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$
(c) Eigen vetor corresponding to $\lambda = 9$ can be evaluated from $[A - 9I]X = 0$

which is $\begin{bmatrix} -3 & 2 & -2 \\ 2 & -4 & 0 \\ -2 & 0 & -2 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By the rule of cross multiplication(by taking coefficients of 1st and 2nd equations)

$$\frac{x}{-8} = \frac{y}{-4} = \frac{z}{8} = \frac{1}{-4}(\text{say})$$

Hence $X_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

Note: The problem has 3 distinct eigen values and the corresponding eigen vectors are linearly independent.(Property (a))

Properties of eigen values and eigen vectors

1. If all the eigen values of a matrix $\mathbf{A}$ are distinct, then the corresponding eigen vectors will be linearly independent.
2. If 2 or more eigen values are equal, then the eigen vectors may be either linearly dependent or linearly independent.
3. A square matrix and its transpose have the same eigen values.
4. The eigen values of a triangular matrix are the same as its diagonal elements.
5. The eigen values of a diagonal matrix are the same as its diagonal elements of the matrix.
6. The sum of eigen values of a matrix $\mathbf{A}$ is equal to the sum of elements on its diagonal(or the trace of $\mathbf{A}$)
7. The product of eigen values of a matrix $\mathbf{A}$ is equal to its determinant.
8. If λ is an eigen value $\mathbf{A}$,then $\lambda - k$ is an eigen value of $\mathbf{A} - kI$,for any scalar k.
9. If λ is an eigen value $\mathbf{A}$,then λ^m is an eigen value of $\mathbf{A}^m$
10. If $\lambda \neq 0$ is an eigen value $\mathbf{A}$,then $\dfrac{1}{\lambda}$ is an eigen value of $\mathbf{A}^{-1}$
11. If $\lambda \neq 0$ is an eigen value $\mathbf{A}$,then $\dfrac{|\mathbf{A}|}{\lambda}$ is an eigen value of $adj\mathbf{A}$

Problems:

1. If 2 is an eigen value of $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$,without using characteistic equation find the other eigen values.Also find the eigen values of $A^3, A^{-1}, A^T, 5A, A - 3I$ and $Adj(A)$

   (UQ.July 17,Dece 19)

   Solution: The sum of eigen values of a matrix $\mathbf{A}$ is equal to the sum of elements on its diagonal(or the trace of $\mathbf{A}$).
   Hence $\lambda_1 + \lambda_2 + \lambda_3 = 11$
   Given one of the eigen values as 2. So let $\lambda_1 = 2$.

$$\lambda_2 + \lambda_3 = 9 \tag{11}$$

Since, The product of eigen values of a matrix **A** is equal to its determinant.

$$\lambda_1 \times \lambda_2 \times \lambda_3 = 2 \times \lambda_2 \times \lambda_3 = 36$$
$$\therefore \lambda_2 \times \lambda_3 = 18 \tag{12}$$

From (11) and (12) $\lambda_2, \lambda_3 = 3, 6$
Hence the eigen values are 2,3,6

Eigen values of A:	2	3	6	
Eigen values of A^3:	8	27	216	
Eigen values of A^T:	2	3	6	(Refer properties 7-11)
Eigen values of 5A:	10	15	30	
Eigen values of A-3I:	-1	0	3	
Eigen values of adj A:	18	12	6	

2. Find the eigen values of $A = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix}$,without using the characteistic equation and hence find the eigen vectors.
Solution: The eigen values of A are 3,4,1 (Since,the eigen values of a triangular matrix are the same as its diagonal elements.)
(a) Eigen vector corresponding to 3 can be obtained from the equations given by:

$$[A - 3I]X = 0$$

Solving we will get $X_1 = [0 \quad 0 \quad 1]^T$ (Verify!!!)
(b) Eigen vector corresponding to 4 can be obtained from the equations given by:

$$[A - 4I]X = 0$$

Solving we will get $X_2 = [1 \quad -5 \quad -27/2]^T$ (Verify!!!)
(c) Eigen vector corresponding to 1 can be obtained from the equations given by:

$$[A - I]X = 0$$

Solving we will get $X_1 = [0 \quad 1 \quad 2]^T$ (Verify!!!)

Exercise questions(Previous Uty.Questions)

1. Find eigen values and eigen vectors of $\begin{bmatrix} 4 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 3 \end{bmatrix}$ (UQ.Dece 19)
Ans.EV are 1,4,7,and eigen vectors $[2 \quad -1 \quad 2]^T, [-1 \quad 2 \quad 2]^T, [-2 \quad -2 \quad 1]^T$
2. Find eigen values and eigen vectors of $A = \begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ (UQ.Dece 18)
Ans.EV are 3,4,1,and eigen vectors $[7 \quad -4 \quad 2]^T, [5 \quad 1 \quad 0]^T, [1 \quad 0 \quad 0]^T$
3. Find eigen values and eigen vectors of $\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ (UQ.April 18)
Ans.EV are 1,-1,2,and eigen vectors $[3 \quad 2 \quad 1]^T, [1 \quad 0 \quad 1]^T, [1 \quad 3 \quad 1]^T$

4. Find eigen values and eigen vectors of $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ (UQ.July 17)

Ans.ev are 1,2,3 and eigen vectors $[1 \quad -1 \quad 0]^T, [-2 \quad 1 \quad 2]^T, [-1 \quad 1 \quad 2]^T$

5. Find eigen values and eigen vectors of $\begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ (UQ.Dece 16)

Ans.ev are 1,2,3 and eigen vectors $[-1 \quad -2 \quad 1]^T, [1 \quad -1 \quad 1]^T, [1 \quad 0 \quad 1s]^T$