Quadratic Equations and Inequations

Contents

Eucli

The Babylonians knew of quadratic equations some 4000 years ago. The Greek mathematician Euclid (300 B.C.) *gives several quadratic equation while solving geometrical problems,.*

Aryabhatta (476 A.D.) *gives a rule to sum the geometric series which involves the solution of the quadratic equations* Brahmagupta (598 A.D.) *provides a rule for the solution of the quadratic equations which is very much the quadratic formula.* Mahavira *around 850 A.D. proposed a problem involving the use of quadratic equation and its solution.*

It was Sridhara, *an Indian mathematician, around* 900 A.D. *who was the first to give an algebraic solution of the general equation* $ax^2 + bx + c = 0$ $a \neq 0$, *showing the roots to be* $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The first important treatment of a quadratic equation, by factoring, is found in Harriot's works in approximately 1631 A.D.

Quadratic Equations and Inequations

1.1 Polynomial

Algebraic expression containing many terms of the form cx^n, n being a non-negative integer is called a polynomial. *i.e.*, $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_{n-1}x^{n-1} + a_nx^n$, where x is a variable, $a_0, a_1, a_2 ....a_n$ are constants and $a_n \neq 0$

Example : $4x^4 + 3x^3 - 7x^2 + 5x + 3$, $3x^3 + x^2 - 3x + 5$.

(1) **Real polynomial :** Let $a_0, a_1, a_2 ....a_n$ be real numbers and x is a real variable.

Then $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n$ is called real polynomial of real variable x with real coefficients.

Example : $3x^3 - 4x^2 + 5x - 4, x^2 - 2x + 1$ etc. are real polynomials.

(2) **Complex polynomial :** If $a_0, a_1, a_2 ....a_n$ be complex numbers and x is a varying complex number.

Then $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n$ is called complex polynomial of complex variable x with complex coefficients.

Example : $3x^2 - (2+4i)x + (5i - 4), x^3 - 5i\,x^2 + (1+2i)x + 4$ etc. are complex polynomials.

(3) **Degree of polynomial :** Highest power of variable x in a polynomial is called degree of polynomial.

Example : $f(x) = a_0 + a_1x + a_2x^2 + ... + a_{n-1}x^{n-1} + a_nx^n$ is a n degree polynomial.

$f(x) = 4x^3 + 3x^2 - 7x + 5$ is a 3 degree polynomial.

$f(x) = 3x - 4$ is single degree polynomial or linear polynomial.

$f(x) = bx$ is an odd linear polynomial.

A polynomial of second degree is generally called a quadratic polynomial. Polynomials of degree 3 and 4 are known as cubic and biquadratic polynomials respectively.

(4) **Polynomial equation :** If $f(x)$ is a polynomial, real or complex, then $f(x) = 0$ is called a polynomial equation.

1.2 Types of Quadratic Equation

A quadratic polynomial $f(x)$ when equated to zero is called quadratic equation.

Example : $3x^2 + 7x + 5 = 0, -9x^2 + 7x + 5 = 0, x^2 + 2x = 0, 2x^2 = 0$

or

An equation in which the highest power of the unknown quantity is two is called quadratic equation.

Quadratic equations are of two types :

(1) **Purely quadratic equation :** A quadratic equation in which the term containing the first degree of the unknown quantity is absent is called a purely quadratic equation.

i.e. $ax^2 + c = 0$ where $a, c \in C$ and $a \neq 0$

(2) **Adfected quadratic equation :** A quadratic equation which contains terms of first as well as second degrees of the unknown quantity is called an adfected quadratic equation.

i.e. $ax^2 + bx + c = 0$ where $a, b, c \in C$ and $a \neq 0, b \neq 0$.

(3) **Roots of a quadratic equation :** The values of variable x which satisfy the quadratic equation is called roots of quadratic equation.

Important Tips

- *An equation of degree n has n roots, real or imaginary.*
- *Surd and imaginary roots always occur in pairs in a polynomial equation with real coefficients i.e. if 2 – 3i is a root of an equation, then 2 + 3i is also its root. Similarly if $2+\sqrt{3}$ is a root of given equation, then $2-\sqrt{3}$ is also its root.*
- *An odd degree equation has at least one real root whose sign is opposite to that of its last term (constant term), provided that the coefficient of highest degree term is positive.*
- *Every equation of an even degree whose constant term is negative and the coefficient of highest degree term is positive has at least two real roots, one positive and one negative.*

1.3 Solution of Quadratic Equation

(1) **Factorization method :** Let $ax^2+bx+c=a(x-\sim)(x-\ \ )=0$. Then $x=\sim$ and $x=$ will satisfy the given equation.

Hence, factorize the equation and equating each factor to zero gives roots of the equation.

Example : $3x^2-2x+1=0 \Rightarrow (x-1)(3x+1)=0$

$x=1,-1/3$

(2) **Hindu method (Sri Dharacharya method) :** By completing the perfect square as

$$ax^2+bx+c=0 \Rightarrow x^2+\frac{b}{a}x+\frac{c}{a}=0$$

Adding and subtracting $\left(\frac{b}{2a}\right)^2$, $\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right]=0$ which gives, $\boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$

Hence the quadratic equation $ax^2+bx+c=0$ $(a\neq 0)$ has two roots, given by

$$\sim=\frac{-b+\sqrt{b^2-4ac}}{2a},\quad =\frac{-b-\sqrt{b^2-4ac}}{2a}$$

Note : ❑ Every quadratic equation has two and only two roots.

1.4 Nature of Roots

In quadratic equation $ax^2+bx+c=0$, the term b^2-4ac is called discriminant of the equation, which plays an important role in finding the nature of the roots. It is denoted by Δ or D.

(1) **If *a*, *b*, *c* Č *R* and *a* ÷ 0, then :** (i) If $D<0$, then equation $ax^2+bx+c=0$ has non-real complex roots.

(ii) If $D>0$, then equation $ax^2+bx+c=0$ has real and distinct roots, namely $\sim=\frac{-b+\sqrt{D}}{2a}$, $\quad =\frac{-b-\sqrt{D}}{2a}$

and then $ax^2+bx+c=a(x-\sim)(x-\ \ )$ .....(i)

(iii) If $D=0$, then equation $ax^2+bx+c=0$ has real and equal roots $\sim = \quad =\frac{-b}{2a}$

and then $ax^2+bx+c=a(x-\sim)^2$ .....(ii)

To represent the quadratic expression ax^2+bx+c in form (i) and (ii), transform it into linear factors.

(iv) If $D \geq 0$, then equation $ax^2 + bx + c = 0$ has real roots.

(2) **If *a, b, c* Č Q, *a* ÷ 0, then :** (i) If $D > 0$ and D is a perfect square $\Rightarrow$ roots are unequal and rational.
(ii) If $D > 0$ and D is not a perfect square $\Rightarrow$ roots are irrational and unequal.

(3) **Conjugate roots :** The irrational and complex roots of a quadratic equation always occur in pairs. Therefore

(i) If one root be $\sim + i$ then other root will be $\sim - i$.

(ii) If one root be $\sim + \sqrt{\ }$ then other root will be $\sim - \sqrt{\ }$.

(4) **If D_1 and D_2 be the discriminants of two quadratic equations,then**

(i) If $D_1 + D_2 \geq 0$, then

(a) At least one of D_1 and $D_2 \geq 0$. (b) If $D_1 < 0$ then $D_2 > 0$

(ii) If $D_1 + D_2 < 0$, then

(a) At least one of D_1 and $D_2 < 0$. (b) If $D_1 > 0$ then $D_2 < 0$.

1.5 Roots Under Particular Conditions

For the quadratic equation $ax^2 + bx + c = 0$.

(1) If $b = 0$ $\Rightarrow$ roots are of equal magnitude but of opposite sign.

(2) If $c = 0$ $\Rightarrow$ one root is zero, other is $-b/a$.

(3) If $b = c = 0$ $\Rightarrow$ both roots are zero.

(4) If $a = c$ $\Rightarrow$ roots are reciprocal to each other.

(5) If $\left.\begin{matrix} a > 0 & c < 0 \\ a < 0 & c > 0 \end{matrix}\right\}$ $\Rightarrow$ roots are of opposite signs.

(6) If $\left.\begin{matrix} a > 0 & b > 0 & c > 0 \\ a < 0 & b < 0 & c < 0 \end{matrix}\right\}$ $\Rightarrow$ both roots are negative, provided $D \geq 0$.

(7) If $\left.\begin{matrix} a > 0 & b < 0 & c > 0 \\ a < 0 & b > 0 & c < 0 \end{matrix}\right\}$ $\Rightarrow$ both roots are positive, provided $D \geq 0$.

(8) If sign of a = sign of $b \neq$ sign of $c \Rightarrow$ greater root in magnitude, is negative.

(9) If sign of b = sign of $c \neq$ sign of $a \Rightarrow$ greater root in magnitude, is positive.

(10) If $a + b + c = 0$ $\Rightarrow$ one root is 1 and second root is c/a.

(11) If $a = b = c = 0$, then equation will become an identity and will be satisfied by every value of x.

(12) If $a = 1$ and $b, c \in I$ and the root of equation $ax^2 + bx + c = 0$ are rational numbers, then these roots must be integers.

Important Tips

☞ *If an equation has only one change of sign, it has one +ve root and no more.*

☞ *If all the terms of an equation are +ve and the equation involves no odd power of x, then all its roots are complex.*

Example: 1 Both the roots of given equation $(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$ are always

(a)Positive (b)Negative (c)Real (d)Imaginary

Solution: (c) Given equation $(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$ can be re-written as $3x^2 - 2(a+b+c)x + (ab+bc+ca) = 0$

$D = 4[(a+b+c)^2 - 3(ab+bc+ca)] = 4[a^2+b^2+c^2-ab-bc-ac] = 2[(a-b)^2+(b-c)^2+(c-a)^2] \geq 0$

Hence both roots are always real.

Example: 2 If the roots of $(b-c)x^2+(c-a)x+(a-b)=0$ are equal then $a+c=$

(a) $2b$ (b) b^2 (c) $3b$ (d) b

Solution: (a) $b-c+c-a+a-b=0$

Hence one root is 1. Also as roots are equal, other root will also be equal to 1.

Also $\sim. \;\; = \dfrac{a-b}{b-c} \Rightarrow 1.1 = \dfrac{a-b}{b-c} \Rightarrow a-b=b-c \Rightarrow 2b=a+c$

Example: 3 If the roots of equation $\dfrac{1}{x+p}+\dfrac{1}{x+q}=\dfrac{1}{r}$ are equal in magnitude but opposite in sign, then $(p+q)=$

(a) $2r$ (b) r (c) $-2r$ (d) None of these

Solution: (a) Given equation can be written as $x^2+(p+q-2r)x+[pq-(p+q)r]=0$

Since the roots are equal and of opposite sign, $\therefore$ Sum of roots = 0

$\Rightarrow -(p+q-2r)=0 \Rightarrow p+q=2r$

Example: 4 If 3 is a root of $x^2+kx-24=0$, it is also a root of

(a) $x^2+5x+k=0$ (b) $x^2-5x+k=0$ (c) $x^2-kx+6=0$ (d) $x^2+kx+24=0$

Solution: (c) Equation $x^2+kx-24=0$ has one root as 3,

$\Rightarrow 3^2+3k-24=0 \Rightarrow k=5$

Put $x=3$ and $k=5$ in option

Only (c) gives the correct answer *i.e.* $\Rightarrow 3^2-15+9=0 \Rightarrow 0=0$

Example: 5 For what values of k will the equation $x^2-2(1+3k)x+7(3+2k)=0$ have equal roots

(a) 1, –10/9 (b) 2, –10/9 (c) 3, –10/9 (d) 4, –10/9

Solution: (b) Since roots are equal then $[-2(1+3k)]^2 = 4.1.7(3+2k) \Rightarrow 1+9k^2+6k=21+14k \Rightarrow 9k^2-8k-20=0$

Solving, we get $k=2,-10/9$

1.6 Relations between Roots and Coefficients

(1) **Relation between roots and coefficients of quadratic equation :** If ~ and   are the roots of quadratic equation $ax^2+bx+c=0$, $(a \neq 0)$ then

Sum of roots $= S = \sim + \;\; = \dfrac{-b}{a} = -\dfrac{\text{coefficient of } x}{\text{coefficient of } x^2}$

Product of roots $= P = \sim. \;\; = \dfrac{c}{a} = \dfrac{\text{constant term}}{\text{coefficient of } x^2}$

If roots of quadratic equation $ax^2+bx+c=0$ $(a \neq 0)$ are ~ and   then

(i) $(\sim - \;\;) = \sqrt{(\sim + \;\;)^2 - 4\sim \;\;} = \pm\dfrac{\sqrt{b^2-4ac}}{a} = \dfrac{\pm\sqrt{D}}{a}$

(ii) $\sim^2 + \;\;^2 = (\sim + \;\;)^2 - 2\sim \;\; = \dfrac{b^2-2ac}{a^2}$

(iii) $\sim^2 - \;\;^2 = (\sim + \;\;)\sqrt{(\sim + \;\;)^2 - 4\sim \;\;} = -\dfrac{b\sqrt{b^2-4ac}}{a^2} = \dfrac{\pm b\sqrt{D}}{a^2}$

(iv) $\sim^3 + \;\;^3 = (\sim + \;\;)^3 - 3\sim \;\;(\sim + \;\;) = -\dfrac{b(b^2-3ac)}{a^3}$

(v) $\alpha^3-\beta^3=(\alpha-\beta)^3+3\alpha\beta(\alpha-\beta)=\sqrt{(\alpha+\beta)^2-4\alpha\beta}\,\{(\alpha+\beta)^2-\alpha\beta\}=\dfrac{\pm(b^2-ac)\sqrt{b^2-4ac}}{a^3}$

(vi) $\alpha^4+\beta^4=\{(\alpha+\beta)^2-2\alpha\beta\}^2-2\alpha^2\beta^2=\left(\dfrac{b^2-2ac}{a^2}\right)^2-2\dfrac{c^2}{a^2}$

(vii) $\alpha^4-\beta^4=(\alpha^2-\beta^2)(\alpha^2+\beta^2)=\dfrac{\pm b(b^2-2ac)\sqrt{b^2-4ac}}{a^4}$

(viii) $\alpha^2+\alpha\beta+\beta^2=(\alpha+\beta)^2-\alpha\beta=\dfrac{b^2-ac}{a^2}$

(ix) $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}=\dfrac{\alpha^2+\beta^2}{\alpha\beta}=\dfrac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\dfrac{b^2-2ac}{ac}$

(x) $\alpha^2\beta+\beta^2\alpha=\alpha\beta(\alpha+\beta)=-\dfrac{bc}{a^2}$

(xi) $\left(\dfrac{\alpha}{\beta}\right)^2+\left(\dfrac{\beta}{\alpha}\right)^2=\dfrac{\alpha^4+\beta^4}{\alpha^2\beta^2}=\dfrac{(\alpha^2+\beta^2)^2-2\alpha^2\beta^2}{\alpha^2\beta^2}=\dfrac{b^2D+2a^2c^2}{a^2c^2}$

(2) **Formation of an equation with given roots :** A quadratic equation whose roots are α and β is given by $(x-\alpha)(x-\beta)=0$

$\therefore\ x^2-(\alpha+\beta)x+\alpha\beta=0$ i.e. $x^2-(\text{sum of roots})x+(\text{product of roots})=0$

$\therefore\ x^2-Sx+P=0$

(3) **Equation in terms of the roots of another equation :** If α,β are roots of the equation $ax^2+bx+c=0$, then the equation whose roots are

(i) $-\alpha,-\beta \Rightarrow ax^2-bx+c=0$ (Replace x by $-x$)

(ii) $1/\alpha, 1/\beta \Rightarrow cx^2+bx+a=0$ (Replace x by $1/x$)

(iii) $\alpha^n,\beta^n;\ n\in N \Rightarrow a(x^{1/n})^2+b(x^{1/n})+c=0$ (Replace x by $x^{1/n}$)

(iv) $k\alpha, k\beta \Rightarrow ax^2+kbx+k^2c=0$ (Replace x by x/k)

(v) $k+\alpha, k+\beta \Rightarrow a(x-k)^2+b(x-k)+c=0$ (Replace x by $(x-k)$)

(vi) $\dfrac{\alpha}{k},\dfrac{\beta}{k} \Rightarrow k^2ax^2+kbx+c=0$ (Replace x by kx)

(vii) $\alpha^{1/n},\beta^{1/n};\ n\in N \Rightarrow a(x^n)^2+b(x^n)+c=0$ (Replace x by x^n)

(4) **Symmetric expressions :** The symmetric expressions of the roots α,β of an equation are those expressions in α and β, which do not change by interchanging α and β. To find the value of such an expression, we generally express that in terms of $\alpha+\beta$ and $\alpha\beta$.

Some examples of symmetric expressions are :

(i) $\alpha^2+\beta^2$ (ii) $\alpha^2+\alpha\beta+\beta^2$ (iii) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$ (iv) $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}$

(v) $\alpha^2\beta+\beta^2\alpha$ (vi) $\left(\dfrac{\alpha}{\beta}\right)^2+\left(\dfrac{\beta}{\alpha}\right)^2$ (vii) $\alpha^3+\beta^3$ (viii) $\alpha^4+\beta^4$

1.7 Biquadratic Equation

If ~, , ¥, ¢ are roots of the biquadratic equation $ax^4+bx^3+cx^2+dx+e=0$, then

$S_1 = \sim + \ \ + ¥ + ¢ = -b/a$, $S_2 = \sim . \ \ + \sim ¥ + \sim ¢ + \ \ ¥ + \ \ ¢ + ¥¢ = (-1)^2 \frac{c}{a} = \frac{c}{a}$

or $S_2 = (\sim + \ \ )(¥ + ¢) + \sim \ \ + ¥¢ = c/a$, $S_3 = \sim \ \ ¥ + \ \ ¥¢ + ¥¢\sim + \sim \ \ ¢ = (-1)^3 \frac{d}{a} = -d/a$

or $S_3 = \sim \ \ (¥ + ¢) + ¥¢(\sim + \ \ ) = -d/a$ and $S_4 = \sim . \ \ ¥ . ¢ = (-1)^4 \frac{e}{a} = \frac{e}{a}$

Example: 6 If the difference between the corresponding roots of $x^2+ax+b=0$ and $x^2+bx+a=0$ is same and $a \neq b$, then

(a) $a+b+4=0$ (b) $a+b-4=0$ (c) $a-b-4=0$ (d) $a-b+4=0$

Solution: (a) $\sim + \ \ = -a$, $\sim \ \ = b \Rightarrow \sim - \ \ = \sqrt{a^2-4b}$ and $¥ + ¢ = -b$, $¥¢ = a \Rightarrow ¥ - ¢ = \sqrt{b^2-4a}$

According to question, $\sim - \ \ = ¥ - ¢ \Rightarrow \sqrt{a^2-4b} = \sqrt{b^2-4a} \Rightarrow a+b+4=0$

Example: 7 If the sum of the roots of the quadratic equation $ax^2+bx+c=0$ is equal to the sum of the squares of their reciprocals, then $a/c, b/a, c/b$ are in

(a) A.P. (b) G.P. (c) H.P. (d) None of these

Solution: (c) As given, if ~, be the roots of the quadratic equation, then

$\Rightarrow \sim + \ \ = \frac{1}{\sim^2} + \frac{1}{\ \ ^2} = \frac{(\sim + \ \ )^2 - 2\sim \ \ }{\sim^2 \ \ ^2} \Rightarrow -\frac{b}{a} = \frac{b^2/a^2 - 2c/a}{c^2/a^2} = \frac{b^2-2ac}{c^2}$

$\Rightarrow \frac{2a}{c} = \frac{b^2}{c^2} + \frac{b}{a} = \frac{ab^2+bc^2}{ac^2} \Rightarrow 2a^2c = ab^2+bc^2 \Rightarrow \frac{2a}{b} = \frac{b}{c} + \frac{c}{a}$

$\frac{c}{a}, \frac{a}{b}, \frac{b}{c}$ are in A.P. $\Rightarrow \frac{a}{c}, \frac{b}{a}, \frac{c}{b}$ are in H.P.

Example: 8 Let ~, be the roots of $x^2-x+p=0$ and ¥, ¢ be root of $x^2-4x+q=0$. If ~, , ¥, ¢ are in G.P., then the integral value of *p* and *q* respectively are

(a) – 2, – 32 (b) – 2, 3 (c) – 6, 3 (d) – 6, – 32

Solution: (a) $\sim + \ \ = 1$, $\sim \ \ = p$, $¥ + ¢ = 4$, $¥¢ = q$

Since ~, , ¥, ¢ are in G.P.

$r = \ \ /\sim = ¥/\ \ = ¢/¥$

$\sim + \sim r = 1 \Rightarrow \sim(1+r) = 1$, $\sim(r^2+r^3) = 4 \Rightarrow \sim . r^2(1+r) = 4$

So $r^2 = 4 \Rightarrow r = \pm 2$

If $r=2$, $\sim + 2\sim = 1 \Rightarrow \sim = 1/3$ and $r=-2$, $\sim - 2\sim = 1 \Rightarrow \sim = -1$

But $p = \sim \ \ \in I$ $\therefore$ $r=-2, \sim = -1$

$\therefore$ $p=-2$, $q = \sim^2 r^5 = 1(-2)^5 = -32$

Example: 9 If $1-i$ is a root of the equation $x^2+ax+b=0$, then the values of *a* and *b* are

(a) 2, 1 (b) – 2, 2 (c) 2, 2 (d) 2, – 2

Solution: (b) Since $1-i$ is a root of $x^2+ax+b=0$. y $1+i$ is also a root.

Sum of roots $\Rightarrow 1-i+1+i=-a \Rightarrow a=-2$

Product of roots $\Rightarrow (1-i)(1+i)=b \Rightarrow b=2$

Hence $a=-2$, $b=2$

Example: 10 If the roots of the equation $x^2-5x+16=0$ are ~, and the roots of equation $x^2+px+q=0$ are $\sim^2 + \ \ ^2$, $\sim \ \ /2$, then

(a) $p=1, q=-56$ (b) $p=-1, q=-56$ (c) $p=1, q=56$ (d) $p=-1, q=56$

Solution: (b) Since roots of the equation $x^2-5x+16=0$ are $\sim,\ \ $.

$\Rightarrow \sim+\ \ =5, \sim\ \ =16$ and $\sim^2+\ \ ^2+\frac{\sim}{2}=-p \Rightarrow (\sim+\ \ )^2-2\sim\ \ +\frac{\sim}{2}=-p \Rightarrow 25-2(16)+\frac{16}{2}=-p \Rightarrow p=-1$

and $(\sim^2+\ \ ^2)\left(\frac{\sim}{2}\right)=q \Rightarrow [(\sim+\ \ )^2-2\sim\ \ ]\frac{\sim}{2}=q \Rightarrow (25-32)8=q \Rightarrow q=-56$

Example: 11 If $\sim \neq\ \ $, but $\sim^2=5\sim-3,\ \ ^2=5\ \ -3$, then the equation whose roots are $\frac{\sim}{\ }$ and $\frac{\ }{\sim}$ is

(a) $x^2-5x-3=0$ (b) $3x^2-19x+3=0$ (c) $3x^2+12x+3=0$ (d) None of these

Solution: (b) $S=\frac{\sim}{\ }+\frac{\ }{\sim}=\frac{\sim^2+\ \ ^2}{\sim}=\frac{5\sim-3+5\ \ -3}{\sim}$ $\left[\begin{array}{l}\because \sim^2=5\sim-3\\ \ \ ^2=5\ \ -3\end{array}\right]$

$S=\frac{5(\sim+\ \ )-6}{\sim}$, $p=\frac{\sim}{\ }\cdot\frac{\ }{\sim}=1 \Rightarrow p=1$. $\sim,\ \ $ are roots of $x^2-5x+3=0$. Therefore $\sim+\ \ =5, \sim\ \ =3$

$S=\frac{5(5)-6}{3}=\frac{19}{3}$

$\because x^2-\frac{19}{3}x+1=0 \Rightarrow 3x^2-19x+3=0$

Example: 12 Let $\sim,\ \ $ be the roots of the equation $(x-a)(x-b)=c$, $c\neq 0$, then the roots of the equation $(x-\sim)(x-\ \ )+c=0$ are

(a) a, c (b) b, c (c) a, b (d) a, d

Solution: (c) Since $\sim,\ \ $ are the roots of $(x-a)(x-b)=c$ *i.e.* of $x^2-(a+b)x+ab-c=0$

$\therefore \sim+\ \ =a+b \Rightarrow a+b=\sim+\ \ $ and $\sim\ \ =ab-c \Rightarrow ab=\sim\ \ +c$

$\therefore$ a, b are the roots of $x^2-(\sim+\ \ )x+\sim\ \ +c=0 \Rightarrow (x-\sim)(x-\ \ )+c=0$

Hence (c) is the correct answer

Example: 13 If $\sim$ and $\ \ $ are roots of the equation $x^2-ax+b=0$ and $V_n=\sim^n+\ \ ^n$, then

(a) $V_{n+1}=aV_n-bV_{n-1}$ (b) $V_{n+1}=bV_n-aV_{n-1}$ (c) $V_{n+1}=aV_n+bV_{n-1}$ (d) $V_{n+1}=bV_n+aV_{n-1}$

Solution: (a) Since $\sim$ and $\ \ $ are roots of equation, $x^2-ax+b=0$, therefore $\sim+\ \ =a$, $\sim\ \ =b$

Now, $V_{n+1}=\sim^{n+1}+\ \ ^{n+1}=(\sim+\ \ )(\sim^n+\ \ ^n)-\sim\ \ (\sim^{n-1}+\ \ ^{n-1}) \Rightarrow V_{n+1}=a.V_n-b.V_{n-1}$

Example: 14 If one root of the equation $x^2+px+q=0$ is the square of the other, then

(a) $p^3+q^2-q(3p+1)=0$ (b) $p^3+q^2+q(1+3p)=0$

(c) $p^3+q^2+q(3p-1)=0$ (d) $p^3+q^2+q(1-3p)=0$

Solution: (d) Let $\sim$ and $\sim^2$ be the roots then $\sim+\sim^2=-p$, $\sim.\sim^2=q$

Now $(\sim+\sim^2)^3=\sim^3+\sim^6+3\sim^3(\sim+\sim^2) \Rightarrow -p^3=q+q^2-3pq \Rightarrow p^3+q^2+q(1-3p)=0$

Example: 15 Let $\sim$ and $\ \ $ be the roots of the equation $x^2+x+1=0$, the equation whose roots are $\sim^{19},\ \ ^7$ is

(a) $x^2-x-1=0$ (b) $x^2-x+1=0$ (c) $x^2+x-1=0$ (d) $x^2+x+1=0$

Solution: (d) Roots of $x^2+x+1=0$ are $x=\frac{-1\pm\sqrt{1-4}}{2}, =\frac{-1\pm\sqrt{3}i}{2}=\mu,\mu^2$

Take $\sim=\mu$, $\ \ =\mu^2$

$\therefore \sim^{19}=w^{19}=w$, $\ \ ^7=(w^2)^7=w^{14}=w^2$

$\therefore$ Required equation is $x^2+x+1=0$

Example: 16 If one root of a quadratic equation is $\frac{1}{2+\sqrt{5}}$, then the equation is

(a) $x^2+4x+1=0$ (b) $x^2+4x-1=0$ (c) $x^2-4x+1=0$ (d) None of these

Solution: (b) Given root $=\frac{1}{2+\sqrt{5}}=\frac{2-\sqrt{5}}{-1}=-2+\sqrt{5}$, $\therefore$ other root $=-2-\sqrt{5}$

Again, sum of roots = – 4 and product of roots = – 1. The required equation is $x^2+4x-1=0$

1.8 Condition for Common Roots

(1) **Only one root is common :** Let $\sim$ be the common root of quadratic equations $a_1x^2+b_1x+c_1=0$ and $a_2x^2+b_2x+c_2=0$.

$\therefore\ a_1\sim^2+b_1\sim+c_1=0$, $a_2\sim^2+b_2\sim+c_2=0$

By Crammer's rule : $\dfrac{\sim^2}{\begin{vmatrix}-c_1 & b_1\\ -c_2 & b_2\end{vmatrix}}=\dfrac{\sim}{\begin{vmatrix}a_1 & -c_1\\ a_2 & -c_2\end{vmatrix}}=\dfrac{1}{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}}$ or $\dfrac{\sim^2}{b_1c_2-b_2c_1}=\dfrac{\sim}{a_2c_1-a_1c_2}=\dfrac{1}{a_1b_2-a_2b_1}$

$\therefore\ \sim=\dfrac{a_2c_1-a_1c_2}{a_1b_2-a_2b_1}=\dfrac{b_1c_2-b_2c_1}{a_2c_1-a_1c_2}$, $\sim\neq 0$

$\therefore$ The condition for only one root common is $(c_1a_2-c_2a_1)^2=(b_1c_2-b_2c_1)(a_1b_2-a_2b_1)$

(2) **Both roots are common:** Then required condition is $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{c_1}{c_2}$.

Important Tips

- *To find the common root of two equations, make the coefficient of second degree term in the two equations equal and subtract. The value of x obtained is the required common root.*
- *Two different quadratic equations with rational coefficient can not have single common root which is complex or irrational as imaginary and surd roots always occur in pair.*

Example: 17 If one of the roots of the equation $x^2+ax+b=0$ and $x^2+bx+a=0$ is coincident. Then the numerical value of $(a+b)$ is

(a) 0 (b) – 1 (c) 2 (d) 5

Solution: (b) If $\sim$ is the coincident root, then $\sim^2+a\sim+b=0$ and $\sim^2+b\sim+a=0$

$\Rightarrow \dfrac{\sim^2}{a^2-b^2}=\dfrac{\sim}{b-a}=\dfrac{1}{b-a}$

$\sim^2=-(a+b)$, $\sim=1 \Rightarrow -(a+b)=1 \Rightarrow (a+b)=-1$

Example: 18 If a, b, c are in G.P. then the equations $ax^2+2bx+c=0$ and $dx^2+2ex+f=0$ have a common root if $\frac{d}{a},\frac{e}{b},\frac{f}{c}$ are in

(a) A.P. (b) G.P. (c) H.P. (d) None of these

Solution: (a) As given, $b^2=ac \Rightarrow ax^2+2bx+c=0$ can be written as $ax^2+2\sqrt{ac}\,x+c=0 \Rightarrow (\sqrt{a}x+\sqrt{c})^2=0 \Rightarrow x=-\sqrt{\frac{c}{a}}$

This must be common root by hypothesis

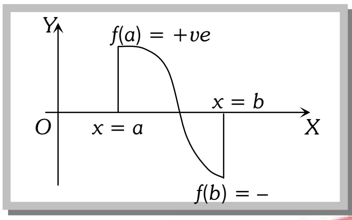
So it must satisfy the equation, $dx^2+2ex+f=0 \Rightarrow d\left(\frac{c}{a}\right)-2e\sqrt{\frac{c}{a}}+f=0$

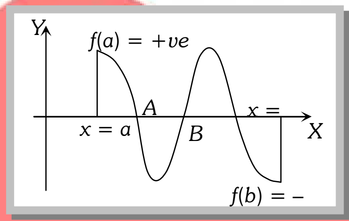
$$\frac{d}{a}+\frac{f}{c}=\frac{2e}{c}\sqrt{\frac{c}{a}}=\frac{2e}{\sqrt{c}.\sqrt{a}} \Rightarrow \frac{d}{a}+\frac{f}{c}=\frac{2e}{b}$$

Hence $\frac{d}{a},\frac{e}{b},\frac{f}{c}$ are in A.P.

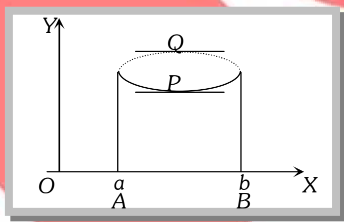
1.9 Properties of Quadratic Equation

(1) If $f(a)$ and $f(b)$ are of opposite signs then at least one or in general odd number of roots of the equation $f(x)=0$ lie between a and b.





(2) If $f(a)=f(b)$ then there exists a point c between a and b such that $f'(c)=0$, $a<c<b$.



As is clear from the figure, in either case there is a point P or Q at $x=c$ where tangent is parallel to *x-axis* *i.e.* $f'(x)=0$ at $x=c$.

(3) If ~ is a root of the equation $f(x)=0$ then the polynomial $f(x)$ is exactly divisible by $(x-\sim)$ or $(x-\sim)$ is factor of $f(x)$.

(4) If the roots of the quadratic equations $ax^2+bx+c=0$, $a_2x^2+b_2x+c_2=0$ are in the same ratio $\left(i.e.\ \frac{\sim_1}{_1}=\frac{\sim_2}{_2}\right)$ then $b_1^2/b_2^2=a_1c_1/a_2c_2$.

(5) If one root is k times the other root of the quadratic equation $ax^2+bx+c=0$ then $\frac{(k+1)^2}{k}=\frac{b^2}{ac}$.

Example: 19 The value of 'a' for which one root of the quadratic equation $(a^2-5a+3)x^2+(3a-1)x+2=0$ is twice as large as the other is

(a) 2/3 (b) – 2/3 (c) 1/3 (d) – 1/3

Solution: (a) Let the roots are ~ and 2~

Now, $\sim+2\sim=\frac{1-3a}{a^2-5a+3}$, $\sim.2\sim=\frac{2}{a^2-5a+3} \Rightarrow 3\sim=\frac{1-3a}{a^2-5a+3}$, $2\sim^2=\frac{2}{a^2-5a+3}$

$$\Rightarrow 2\left[\frac{1}{9}\frac{(1-3a)^2}{(a^2-5a+3)^2}\right]=\frac{2}{a^2-5a+3} \Rightarrow \frac{(1-3a)^2}{a^2-5a+3}=9 \Rightarrow 9a^2-45a+27=1+9a^2-6a \Rightarrow 39a=26 \Rightarrow a=2/3$$

1.10 Quadratic Expression

An expression of the form $ax^2 + bx + c$, where a, b, $c \in R$ and $a \neq 0$ is called a quadratic expression in x. So in general, quadratic expression is represented by $f(x) = ax^2 + bx + c$ or $y = ax^2 + bx + c$.

(1) **Graph of a quadratic expression :** We have $y = ax^2 + bx + c = f(x)$

$$y = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right] \Rightarrow y + \frac{D}{4a} = a\left(x + \frac{b}{2a}\right)^2$$

Now, let $y + \frac{D}{4a} = Y$ and $X = x + \frac{b}{2a}$

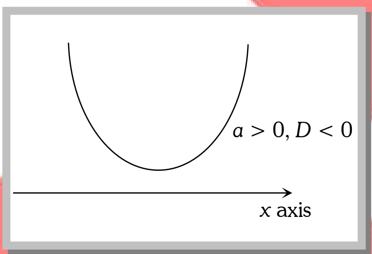
$$Y = a.X^2 \Rightarrow X^2 = \frac{1}{a}Y$$

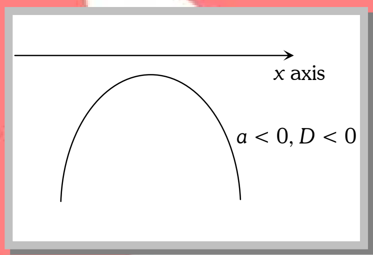
(i) The graph of the curve $y = f(x)$ is parabolic.

(ii) The axis of parabola is $X = 0$ or $x + \frac{b}{2a} = 0$ *i.e.* (parallel to y-axis).

(iii) (a) If $a > 0$, then the parabola opens upward.

(b) If $a < 0$, then the parabola opens downward.



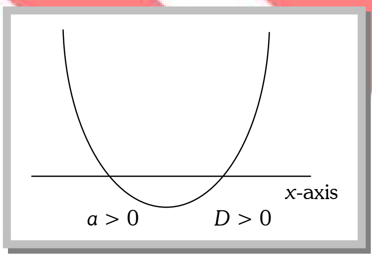


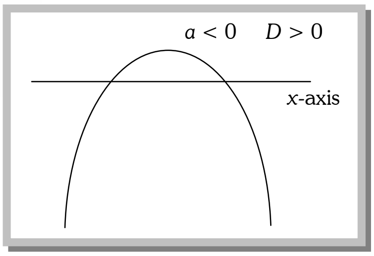
(iv) **Intersection with axis**

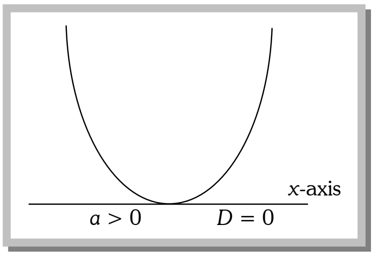
(a) ***x-axis***: For x axis, $y = 0 \Rightarrow ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{D}}{2a}$

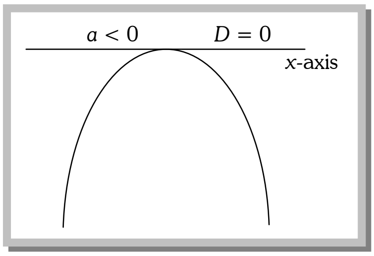
For $D > 0$, parabola cuts x-axis in two real and distinct points *i.e.* $x = \frac{-b \pm \sqrt{D}}{2a}$.

For $D = 0$, parabola touches x-axis in one point, $x = -b/2a$.

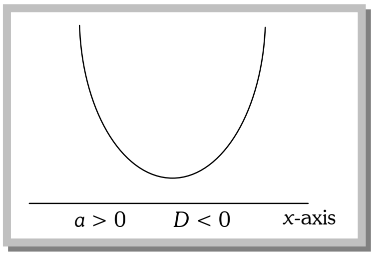


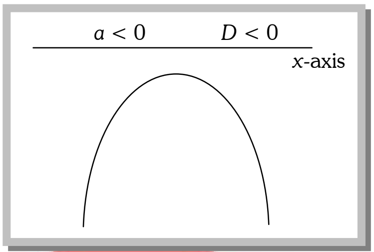






For $D < 0$, parabola does not cut *x-axis*(*i.e.* imaginary value of *x*).





(b) **y-axis :** For y axis $x = 0$, $y = c$

(2) **Maximum and minimum values of quadratic expression :** Maximum and minimum value of quadratic expression can be found out by two methods :

(i) **Discriminant method :** In a quadratic expression $ax^2 + bx + c$.

(a) If $a > 0$, quadratic expression has least value at $x = -b/2a$. This least value is given by $\frac{4ac - b^2}{4a} = -\frac{D}{4a}$.

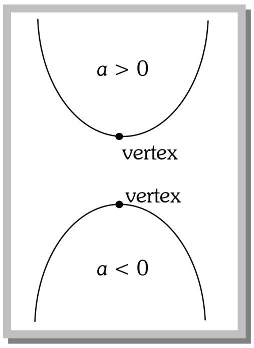
(b) If $a < 0$, quadratic expression has greatest value at $x = -b/2a$. This greatest value is given by $\frac{4ac - b^2}{4a} = -\frac{D}{4a}$.

(ii) **Graphical method :** Vertex of the parabola $Y = aX^2$ is $X = 0$, $Y = 0$

i.e., $x + \frac{b}{2a} = 0$, $y + \frac{D}{4a} = 0 \Rightarrow x = -b/2a$, $y = -D/4a$

Hence, vertex of $y = ax^2 + bx + c$ is $(-b/2a, -D/4a)$

(a) For $a > 0$, $f(x)$ has least value at $x = -\frac{b}{2a}$. This least value is given by $f\left(-\frac{b}{2a}\right) = -\frac{D}{4a}$.



(b) For $a < 0$, $f(x)$ has greatest value at $x = -b/2a$. This greatest value is given by $f\left(-\frac{b}{2a}\right) = -\frac{D}{4a}$.

(3) **Sign of quadratic expression :** Let $f(x) = ax^2 + bx + c$ or $y = ax^2 + bx + c$

Where a, b, $c \in R$ and $a \neq 0$, for some values of x, $f(x)$ may be positive, negative or zero. This gives the following cases :

(i) $a > 0$ and $D < 0$, so $f(x) > 0$ for all $x \in R$ *i.e.*, $f(x)$ is positive for all real values of x.

(ii) $a < 0$ and $D < 0$, so $f(x) < 0$ for all $x \in R$ *i.e.*, $f(x)$ is negative for all real values of x.

(iii) $a > 0$ and $D = 0$ so, $f(x) \geq 0$ for all $x \in R$ *i.e.*, $f(x)$ is positive for all real values of x except at vertex, where $f(x) = 0$.

(iv) $a < 0$ and $D = 0$ so, $f(x) \leq 0$ for all $x \in R$ *i.e.* $f(x)$ is negative for all real values of x except at vertex, where $f(x) = 0$.

(v) $a > 0$ and $D > 0$

Let $f(x) = 0$ have two real roots ~ and $\quad$ (~ < $\quad$), then $f(x) > 0$ for all $x \in (-\infty, \sim) \cup (\quad ,\infty)$ and $f(x) < 0$ for all $x \in (\sim, \quad)$.

(vi) $a < 0$ and $D > 0$

Let $f(x) = 0$ have two real roots ~ and $\quad$ (~ < $\quad$),

Then $f(x) < 0$ for all $x \in (-\infty, \sim) \cup (\quad ,\infty)$ and $f(x) > 0$ for all $x \in (\sim, \quad)$

Example: 20 If x be real, then the minimum value of $x^2 - 8x + 17$ is

(a) – 1 (b) 0 (c) 1 (d) 2

Solution: (c) Since $a = 1 > 0$ therefore its minimum value is $= \dfrac{4ac - b^2}{4a} = \dfrac{4(1)(17) - 64}{4} = \dfrac{4}{4} = 1$

Example: 21 If x is real, then greatest and least values of $\dfrac{x^2 - x + 1}{x^2 + x + 1}$ are

(a) 3, –1/2 (b) 3, 1/3 (c) – 3, –1/3 (d) None of these

Solution: (b) Let $y = \dfrac{x^2 - x + 1}{x^2 + x + 1}$ $\qquad x^2(y-1) + (y+1)x + (y-1) = 0$

$\because$ x is real, therefore $b^2 - 4ac \ge 0$

$\Rightarrow (y+1)^2 - 4(y-1)(y-1) \ge 0 \Rightarrow 3y^2 - 10y + 3 \le 0 \Rightarrow (3y-1)(y-3) \le 0 \Rightarrow \left(y - \dfrac{1}{3}\right)(y-3) \le 0 \Rightarrow \dfrac{1}{3} \le y \le 3$

Thus greatest and least values of expression are 3, 1/3 respectively.

Example: 22 If $f(x)$ is quadratic expression which is positive for all real value of x and $g(x) = f(x) + f'(x) + f''(x)$. Then for any real value of x

(a) $g(x) < 0$ (b) $g(x) > 0$ (c) $g(x) = 0$ (d) $g(x) \ge 0$

Solution: (b) Let $f(x) = ax^2 + bx + c$, then $g(x) = ax^2 + bx + c + 2ax + b + 2a = ax^2 + (b+2a)x + (b+c+2a)$

$\because$ $f(x) > 0$. Therefore $b^2 - 4ac < 0$ and $a > 0$

Now for $g(x)$, Discriminant $= (b+2a)^2 - 4a(b+c+2a) = b^2 + 4a^2 + 4ab - 4ab - 4ac - 8a^2 = (b^2 - 4ac) - 4a^2 < 0$ as $b^2 - 4ac < 0$

Therefore sign of $g(x)$ and a are same i.e. $g(x) > 0$.

Example: 23 If ~, $\quad$ (~ < $\quad$) are roots of the equation $x^2 + bx + c = 0$ where $(c < 0 < b)$ then

(a) 0 < ~ < (b) ~ < 0 < $\quad$ < |~ | (c) ~ < $\quad$ < 0 (d) ~ < 0 < |~ | <

Solution: (b) Since $f(0) = 0 + 0 + c = c < 0$

$\therefore$ Roots will be of opposite sign, ~ + $\quad$ = $-b = -ve$ $\qquad (b > 0)$

It is given that ~ <

So, ~ + $\quad$ = $-ve$ is possible only when |~ | >

$\Rightarrow$ ~ < 0, $\quad$ > 0, |~ | > $\quad \Rightarrow$ ~ < 0 < $\quad$ < |~ |

1.11 Wavy Curve Method

Let $f(x) = (x - a_1)^{k_1}(x - a_2)^{k_2}(x - a_3)^{k_3} \ldots\ldots (x - a_{n-1})^{k_{n-1}}(x - a_n)^{k_n}$ $\qquad$ .....(i)

Where $k_1, k_2, k_3 \ldots, k_n \in N$ and $a_1, a_2, a_3, \ldots\ldots, a_n$ are fixed natural numbers satisfying the condition

$$a_1 < a_2 < a_3 \ldots\ldots < a_{n-1} < a_n$$

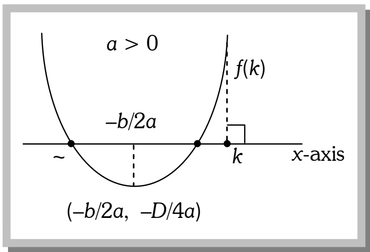
First we mark the numbers $a_1, a_2, a_3, \ldots\ldots, a_n$ on the real axis and the plus sign in the interval of the right of the largest of these numbers, *i.e.* on the right of a_n. If k_n is even then we put plus sign on the left of a_n and if k_n is odd then we put minus sign on the left of a_n. In the next interval we put a sign according to the following rule :

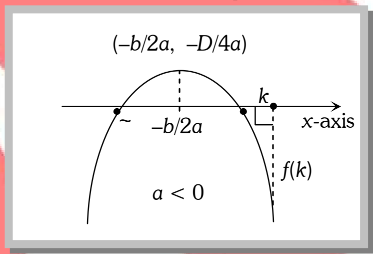
When passing through the point a_{n-1} the polynomial $f(x)$ changes sign if k_{n-1} is an odd number and the polynomial $f(x)$ has same sign if k_{n-1} is an even number. Then, we consider the next interval and put a sign in it using the same rule. Thus, we consider all the intervals. The solution of $f(x) > 0$ is the union of all intervals in which we have put the plus sign and the solution of $f(x) < 0$ is the union of all intervals in which we have put the minus sign.

1.12 Position of Roots of a Quadratic Equation

Let $f(x) = ax^2 + bx + c$, where a, b, $c \in R$ be a quadratic expression and k, k_1, k_2 be real numbers such that $k_1 < k_2$. Let $\sim$, be the roots of the equation $f(x) = 0$ *i.e.* $ax^2 + bx + c = 0$. Then $\sim = \dfrac{-b+\sqrt{D}}{2a}$, $= \dfrac{-b-\sqrt{D}}{2a}$ where D is the discriminant of the equation.

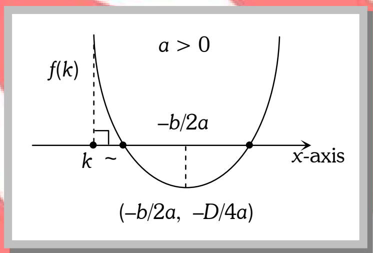
(1) **Condition for a number *k* (If both the roots of *f*(*x*) = 0 are less than *k*)**

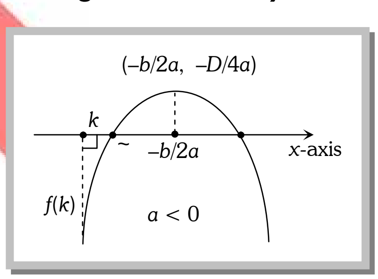




(i) $D \geq 0$ (roots may be equal) (ii) $a\,f(k) > 0$ (iii) $k > -b/2a$, where $\sim \leq$

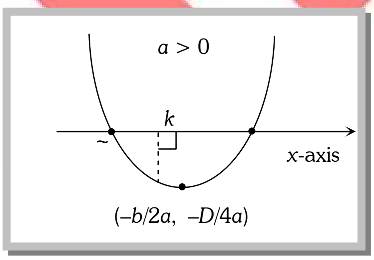
(2) **Condition for a number *k* (If both the roots of *f*(*x*) = 0 are greater than *k*)**

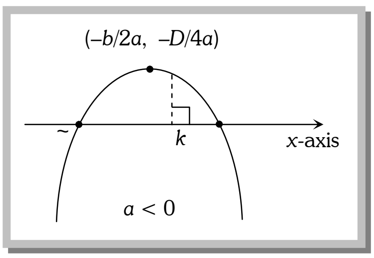




(i) $D \geq 0$ (roots may be equal) (ii) $a\,f(k) > 0$ (iii) $k < -b/2a$, where $\sim \leq$

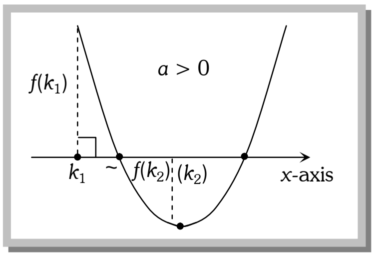
(3) **Condition for a number *k* (If *k* lies between the roots of *f*(*x*) = 0)**

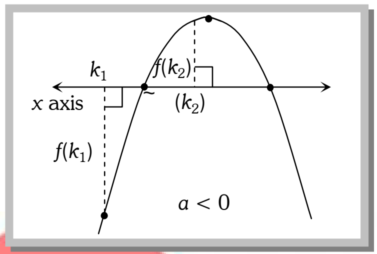




(i) $D > 0$ (ii) $a\,f(k) < 0$, where $\sim <$

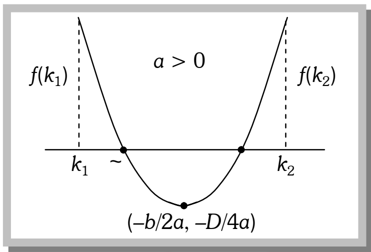
(4) **Condition for numbers k_1 and k_2 (If exactly one root of $f(x) = 0$ lies in the interval (k_1, k_2))**

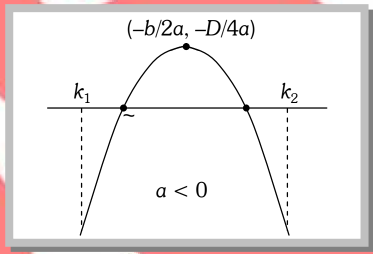




(i) $D > 0$ (ii) $f(k_1)f(k_2) < 0$, where ~ < .

(5) **Condition for numbers k_1 and k_2 (If both roots of $f(x) = 0$ are confined between k_1 and k_2)**

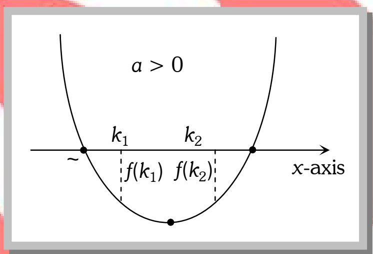


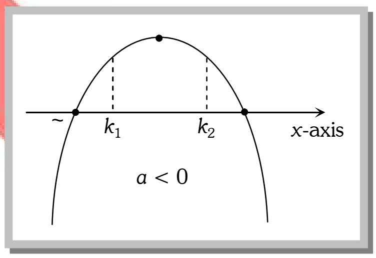


(i) $D \geq 0$ (roots may be equal) (ii) $a\,f(k_1) > 0$ (iii) $a\,f(k_2) > 0$

(iv) $k_1 < -b/2a < k_2$, where ~ ≤ and $k_1 < k_2$

(6) **Condition for numbers k_1 and k_2 (If k_1 and k_2 lie between the roots of $f(x) = 0$)**





(i) $D > 0$ (ii) $a\,f(k_1) < 0$ (iii) $a\,f(k_2) < 0$, where ~ <

Example: 24 If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real and less than 3, then

(a) $a < 2$ (b) $2 \leq a \leq 3$ (c) $3 < a \leq 4$ (d) $a > 4$

Solution: (a) Given equation is $x^2 - 2ax + a^2 + a - 3 = 0$

If roots are real, then $D \geq 0$

$\Rightarrow 4a^2 - 4(a^2 + a - 3) \geq 0 \Rightarrow -a + 3 \geq 0 \Rightarrow a - 3 \leq 0 \Rightarrow a \leq 3$

As roots are less than 3, hence $f(3) > 0$

$9 - 6a + a^2 + a - 3 > 0 \Rightarrow a^2 - 5a + 6 > 0 \Rightarrow (a-2)(a-3) > 0 \Rightarrow a < 2, a > 3$. Hence $a < 2$ satisfy all the conditions.

Example: 25 The value of a for which $2x^2 - 2(2a+1)x + a(a+1) = 0$ may have one root less than a and another root greater than a are given by

(a) $1 > a > 0$ (b) $-1 < a < 0$ (c) $a \geq 0$ (d) $a > 0$ or $a < -1$

Solution: (d) The given condition suggest that a lies between the roots. Let $f(x) = 2x^2 - 2(2a+1)x + a(a+1)$

For 'a' to lie between the roots we must have Discriminant ≥ 0 and $f(a) < 0$

Now, Discriminant ≥ 0

$4(2a+1)^2 - 8a(a+1) \geq 0 \Rightarrow 8(a^2 + a + 1/2) \geq 0$ which is always true.

Also $f(a) < 0 \Rightarrow 2a^2 - 2a(2a+1) + a(a+1) < 0 \Rightarrow -a^2 - a < 0 \Rightarrow a^2 + a > 0 \Rightarrow a(1+a) > 0 \Rightarrow a > 0$ or $a < -1$

1.13 Descarte's Rule of Signs

The maximum number of positive real roots of a polynomial equation $f(x) = 0$ is the number of changes of sign from positive to negative and negative to positive in $f(x)$.

The maximum number of negative real roots of a polynomial equation $f(x) = 0$ is the number of changes of sign from positive to negative and negative to positive in $f(-x)$.

Example: 26 The maximum possible number of real roots of equation $x^5 - 6x^2 - 4x + 5 = 0$ is

(a) 0 (b) 3 (c) 4 (d) 5

Solution: (b) $f(x) = x^5 - 6x^2 - 4x + 5 = 0$

+ − − +

2 changes of sign $\Rightarrow$ maximum two positive roots.

$f(-x) = -x^5 - 6x^2 + 4x + 5$

− − + +

1 changes of sign $\Rightarrow$ maximum one negative roots.

$\Rightarrow$ total maximum possible number of real roots = 2 + 1 = 3.

1.14 Rational Algebraic Inequations

(1) **Values of rational expression $P(x)/Q(x)$ for real values of x, where $P(x)$ and $Q(x)$ are quadratic expressions :** To find the values attained by rational expression of the form $\frac{a_1x^2 + b_1x + c_1}{a_2x^2 + b_2x + c_2}$ for real values of x, the following algorithm will explain the procedure :

Algorithm

Step I: Equate the given rational expression to y.

Step II: Obtain a quadratic equation in x by simplifying the expression in step I.

Step III: Obtain the discriminant of the quadratic equation in Step II.

Step IV: Put Discriminant ≥ 0 and solve the inequation for y. The values of y so obtained determines the set of values attained by the given rational expression.

(2) **Solution of rational algebraic inequation:** If $P(x)$ and $Q(x)$ are polynomial in x, then the inequation $\frac{P(x)}{Q(x)} > 0, \frac{P(x)}{Q(x)} < 0, \frac{P(x)}{Q(x)} \geq 0$ and $\frac{P(x)}{Q(x)} \leq 0$ are known as rational algebraic inequations.

To solve these inequations we use the sign method as explained in the following algorithm.

Algorithm

Step I: Obtain $P(x)$ and $Q(x)$.

Step II: Factorize $P(x)$ and $Q(x)$ into linear factors.

Step III: Make the coefficient of x positive in all factors.

Step IV: Obtain critical points by equating all factors to zero.

Step V: Plot the critical points on the number line. If there are n critical points, they divide the number line into $(n + 1)$ regions.

Step VI: In the right most region the expression $\frac{P(x)}{Q(x)}$ bears positive sign and in other regions the expression bears positive and negative signs depending on the exponents of the factors.

1.15 Algebraic Interpretation of Rolle's Theorem

Let $f(x)$ be a polynomial having α and β as its roots, such that $\alpha < \beta$. Then, $f(\alpha) = f(\beta) = 0$. Also a polynomial function is everywhere continuous and differentiable. Thus $f(x)$ satisfies all the three conditions of Rolle's theorem. Consequently there exists $\gamma \in (\alpha, \beta)$ such that $f'(\gamma) = 0$ *i.e.* $f'(x) = 0$ at $x = \gamma$. In other words $x = \gamma$ is a root of $f'(x) = 0$. Thus algebraically Rolle's theorem can be interpreted as follows.

Between any two roots of polynomial $f(x)$, there is always a root of its derivative $f'(x)$.

Lagrange's theorem : Let $f(x)$ be a function defined on $[a\ b]$ such that

(i) $f(x)$ is continuous on $[a\ b]$ and

(ii) $f(x)$ is differentiable on (a, b), then $c \in (a, b)$, such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

Lagrange's identity : If $a_1, a_2, a_3, b_1, b_2, b_3 \in R$ then :

$$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 = (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2$$

Example: 27 If $\frac{2x}{2x^2 + 5x + 2} > \frac{1}{x+1}$, then

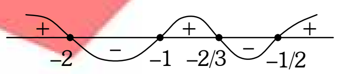
(a) $-2 > x > -1$ (b) $-2 \geq x \geq -1$ (c) $-2 < x < -1$ (d) $-2 < x \leq -1$

Solution: (c) Given $\frac{2x}{2x^2 + 5x + 2} - \frac{1}{x+1} > 0 \Rightarrow \frac{2x^2 + 2x - 2x^2 - 5x - 2}{(2x+1)(x+2)(x+1)} > 0 \Rightarrow \frac{-3x - 2}{(2x+1)(x+2)(x+1)} > 0$

$\Rightarrow \frac{-3(x + 2/3)}{(x+1)(x+2)(2x+1)} > 0 \Rightarrow \frac{(x + 2/3)}{(x+1)(x+2)(2x+1)} < 0$

Equating each factor equal to 0,

We get $x = -2, -1, -2/3, -1/2$



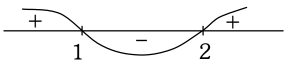
$\therefore x \in]-2, -1[\cup]-2/3, -1/2[\Rightarrow -2/3 < x < -1/2$ or $-2 < x < -1$

Example: 28 If for real values of x, $x^2 - 3x + 2 > 0$ and $x^2 - 3x - 4 \leq 0$, then

(a) $-1 \leq x < 1$ (b) $-1 \leq x < 4$ (c) $-1 \leq x < 1$ or $2 < x \leq 4$ (d) $2 < x \leq 4$

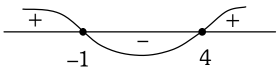
Solution: (c) $x^2 - 3x + 2 > 0$ or $(x-1)(x-2) > 0$

y $x \in (-\infty, 1) \cup (2, \infty)$ .....(i)



Again $x^2 - 3x - 4 \leq 0$ or $(x-4)(x+1) \leq 0$

$\therefore x \in [-1, 4]$ .....(ii)



From eq. (i) and (ii), $x \in [-1, 1) \cup (2\ 4] \Rightarrow -1 \leq x < 1$ or $2 < x \leq 4$

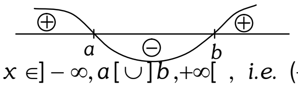
Example: 29 If $b > a$, then the equation $(x-a)(x-b) - 1 = 0$, has

(a) Both roots in $[a\ b]$ (b) Both roots in $(-\infty, a)$

(c) Both roots in (b, ∞) (d) One root in $(-\infty, a)$ and other in $(b, +\infty)$

Solution: (d) We have, $(x-a)(x-b)-1=0$

$(x-a)(x-b)=1>0 \Rightarrow (x-a)(x-b)>0 \quad [\because b>a]$



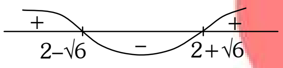
$x \in]-\infty, a[\cup]b, +\infty[$, *i.e.* $(-\infty, a)$ and (b, ∞).

Example: 30 The number of integral solution of $\dfrac{x+1}{x^2+2} > \dfrac{1}{4}$ is

(a) 1 (b) 2 (c) 5 (d) None of these

Solution: (c) $\dfrac{x+1}{x^2+2} - \dfrac{1}{4} > 0 \Rightarrow \dfrac{x^2-4x-2}{x^2+2} < 0 \Rightarrow (x-(2+\sqrt{6}))(x-(2-\sqrt{6})) < 0$

$\Rightarrow 2-\sqrt{6} < x < 2+\sqrt{6}$



Approximately, $-0.4 < x < 4.4$

Hence, integral values of x are 0, 1, 2, 3, 4

Hence, number of integral solution = 5

Example: 31 If $2a+3b+6c=0$ then at least one root of the equation $ax^2+bx+c=0$ lies in the interval

(a) (0, 1) (b) (1, 2) (c) (2, 3) (d) (3, 4)

Solution: (a) Let $f'(x) = ax^2+bx+c$

$\therefore \ f(x) = \int f'(x)\,dx = \dfrac{ax^3}{3} + \dfrac{bx^2}{2} + cx$

Clearly $f(0)=0$, $f(1) = \dfrac{a}{3} + \dfrac{b}{2} + c = \dfrac{2a+3b+6c}{6} = \dfrac{0}{6} = 0$

Since, $f(0)=f(1)=0$. Hence, there exists at least one point c in between 0 and 1, such that $f'(x)=0$, by Rolle's theorem.

Trick: Put the value of $a=-3, b=2, c=0$ in given equation

$-3x^2+2x=0 \Rightarrow 3x^2-2x=0 \Rightarrow x(3x-2)=0$

$x=0, x=2/3$, which lie in the interval (0, 1)

1.16 Equation and Inequation containing Absolute Value

(1) **Equations containing absolute values**

By definition, $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Important forms containing absolute value :

Form I: The equation of the form $|f(x)+g(x)| = |f(x)| + |g(x)|$ is equivalent of the system $f(x).g(x) \geq 0$.

Form II: The equation of the form $|f_1(x)| + |f_2(x)| + |f_3(x)| + \ldots\ldots |f_n(x)| = g(x)$ .....(i)

Where $f_1(x), f_2(x), f_3(x) \ldots\ldots f_n(x), g(x)$ are functions of x and $g(x)$ may be a constant.

Equations of this form can be solved by the method of interval. We first find all critical points of $f_1(x), f_2(x) \ldots\ldots f_n(x)$. If coefficient of x is $+ve$, then graph starts with $+ve$ sign and if it is negative, then graph starts with negative sign. Then using the definition of the absolute value, we pass form equation (i) to a collection of system which do not contain the absolute value symbols.

(2) **Inequations containing absolute value**

By definition, $|x| < a \Rightarrow -a < x < a \ (a > 0)$, $|x| \le a \Rightarrow -a \le x \le a$,

$|x| > a \Rightarrow x < -a$ or $x > a$ and $|x| \ge a \Rightarrow x \le -a$ or $x \ge a$

Example: 32 The roots of $|x-2|^2 + |x-2| - 6 = 0$ are

(a) 0, 4 (b) –1, 3 (c) 4, 2 (d) 5, 1

Solution: (a) We have $|x-2|^2 + |x-2| - 6 = 0$

Let $|x-2| = X$

$X^2 + X - 6 = 0$

$\Rightarrow X = \dfrac{-1 \pm \sqrt{1+24}}{2} = 2, -3 \Rightarrow X = 2$ and $X = -3$

$\therefore |x-2| = 2$ and $|x-2| = -3$, which is not possible.

$\Rightarrow x - 2 = 2$ or $x - 2 = -2$

$\therefore x = 4$ or $x = 0$

Example: 33 The set of all real numbers x for which $x^2 - |x+2| + x > 0$, is

(a) $(-\infty, -2) \cup (2, \infty)$ (b) $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$ (c) $(-\infty, -1) \cup (1, \infty)$ (d) $(\sqrt{2}, \infty)$

Solution: (b) **Case I:** If $x + 2 \ge 0$ *i.e.* $x \ge -2$, we get

$x^2 - x - 2 + x > 0 \Rightarrow x^2 - 2 > 0 \Rightarrow (x - \sqrt{2})(x + \sqrt{2}) > 0$

$\Rightarrow x \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

But $x \ge -2$

$\therefore x \in [-2, -\sqrt{2}) \cup (\sqrt{2}\ \infty)$ .....(i)

Case II: $x + 2 < 0$ *i.e.* $x < -2$, then

$x^2 + x + 2 + x > 0 \Rightarrow x^2 + 2x + 2 > 0 \Rightarrow (x+1)^2 + 1 > 0$. Which is true for all x

$\therefore x \in (-\infty, -2)$ .....(ii)

From (i) and (ii), we get, $x \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

Example: 34 Product of real roots of the equation $t^2x^2 + |x| + 9 = 0$ $(t \ne 0)$

(a) Is always $+ve$ (b) Is always $-ve$ (c) Does not exist (d) None of these

Solution: (c) Expression is always $+ve$, so $t^2x^2 + |x| + 9 \ne 0$. Hence roots of given equation does not exist.

Example: 35 The number of solution of $\log_4(x-1) = \log_2(x-3)$

(a) 3 (b) 1 (c) 2 (d) 0

Solution: (b) We have $\log_4(x-1) = \log_2(x-3)$

$(x-1) = (x-3)^2 \Rightarrow x - 1 = x^2 + 9 - 6x \Rightarrow x^2 - 7x + 10 = 0 \Rightarrow (x-5)(x-2) = 0$

$x = 5$ or $x = 2$

But $x - 3 < 0$, when $x = 2$. $\therefore$ Only solution is $x = 5$.

Hence number of solution is one.

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