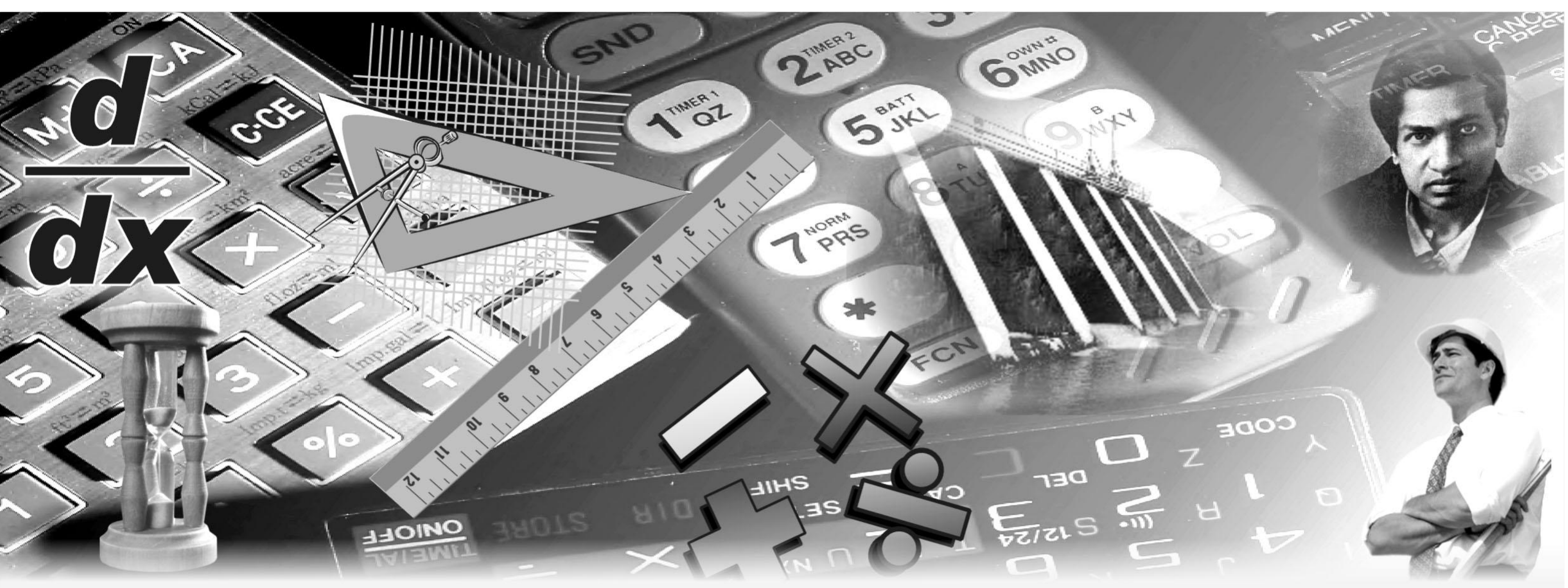

Chapter
6

Permutations and Combinations

Permutations

Introduction

(1) **The Factorial :** Factorial notation: Let n be a positive integer. Then, the continued product of first n natural numbers is called factorial $\lfloor n$, to be denoted by $n!$ or $\lfloor n$.

Also, we define $0! = 1$.

when n is negative or a fraction, $n!$ is not defined.

Thus, $n! = n(n-1)(n-2)......3.2.1.$

(2) **Exponent of Prime p in n ! :** Let p be a prime number and n be a positive integer. Then the last integer amongst 1, 2, 3, ...$(n-1)$, n which is divisible by p is $\left[\frac{n}{p}\right]p$, where $\left[\frac{n}{p}\right]$ denotes the greatest integer less than or equal to $\frac{n}{p}$.

Definition of permutation

The ways of arranging or selecting a smaller or an equal number of persons or objects at a time from a given group of persons or objects with due regard being paid to the order of arrangement or selection are called the (different) *permutations*.

For example : Three different things *a, b* and *c* are given, then different arrangements which can be made by taking two things from three given things are *ab*, *ac, bc, ba, ca, cb.*

Therefore the number of permutations will be 6.

Number of permutations without repetition

(1) Arranging n objects, taken r at a time equivalent to filling r places from n things.

	1	2	3	4		r
r-places :						
Number of choices :	n	$(n-1)$	$(n-2)$	$(n-3)$		$n-(r-1)$

The number of ways of arranging = The number of ways of filling r places.

$$= n(n-1)(n-2).......(n-r+1)$$

$$= \frac{n(n-1)(n-2).....(n-r+1)((n-r)!)}{(n-r)!} = \frac{n!}{(n-r)!} = {}^nP_r$$

(2) The number of arrangements of n different objects taken all at a time $= {}^nP_n = n!$

(i) ${}^nP_0 = \frac{n!}{n!} = 1;\ {}^nP_r = n.{}^{n-1}P_{r-1}$

(ii) $0! = 1;\ \frac{1}{(-r)!} = 0$ or $(-r)! = \infty\ (r \in N)$

Number of permutations with repetition

(1) The number of permutations (arrangements) of n different objects, taken r at a time, when each object may occur once, twice, thrice,........upto r times in any arrangement = The number of ways of filling r places where each place can be filled by any one of n objects.

	1	2	3	4		r
r – places :						
Number of choices :	n	n	n	n		n

The number of permutations = The number of ways of filling r places $= (n)^r$.

(2) The number of arrangements that can be formed using n objects out of which p are identical (and of one kind) q are identical (and of another kind), r are identical (and of another kind) and the rest are distinct is $\frac{n!}{p!q!r!}$.

Conditional permutations

(1) Number of permutations of n dissimilar things taken r at a time when p particular things always occur $= {}^{n-p}C_{r-p}\, r!$.

(2) Number of permutations of n dissimilar things taken r at a time when p particular things never occur $= {}^{n-p}C_r\, r!$.

(3) The total number of permutations of n different things taken not more than r at a time, when each thing may be repeated any number of times, is $\frac{n(n^r-1)}{n-1}$.

(4) Number of permutations of n different things, taken all at a time, when m specified things always come together is $m!\times(n-m+1)!$.

(5) Number of permutations of n different things, taken all at a time, when m specified things never come together is $n!-m!\times(n-m+1)!$.

(6) Let there be n objects, of which m objects are alike of one kind, and the remaining $(n-m)$ objects are alike of another kind. Then, the total number of mutually distinguishable permutations that can be formed from these objects is $\frac{n!}{(m!)\times(n-m)!}$.

The above theorem can be extended further *i.e.,* if there are n objects, of which p_1 are alike of one kind; p_2 are alike of another kind; p_3 are alike of 3rd kind;......; p_r are alike of r^{th} kind such that $p_1+p_2+......+p_r=n$; then the number of permutations of these n objects is $\frac{n!}{(p_1!)\times(p_2!)\times.....\times(p_r!)}$.

Circular permutations

In circular permutations, what really matters is the position of an object relative to the others.

Thus, in circular permutations, we fix the position of the one of the objects and then arrange the other objects in all possible ways.

There are two types of circular permutations :

(i) The circular permutations in which clockwise and the anticlockwise arrangements give rise to different permutations, *e.g.* Seating arrangements of persons round a table.

(ii) The circular permutations in which clockwise and the anticlockwise arrangements give rise to same permutations, *e.g.* arranging some beads to form a necklace.

Difference between clockwise and anti-clockwise arrangement : If anti-clockwise and clockwise order of arrangement are not distinct *e.g.*, arrangement of beads in a necklace, arrangement of flowers in garland etc. then the number of circular permutations of n distinct items is $\frac{(n-1)!}{2}$.

(i) Number of circular permutations of n different things, taken r at a time, when clockwise and anticlockwise orders are taken as different is $\frac{{}^nP_r}{r}$.

(ii) Number of circular permutations of n different things, taken r at a time, when clockwise and anticlockwise orders are not different is $\frac{{}^nP_r}{2r}$.

Theorems on circular permutations

Theorem (i) : The number of circular permutations of n different objects is $(n-1)!$.

Theorem (ii) : The number of ways in which n persons can be seated round a table is $(n-1)!$.

Theorem (iii) : The number of ways in which n different beads can be arranged to form a necklace, is $\frac{1}{2}(n-1)!$.

Combinations

Definition

Each of the different groups or selections which can be formed by taking some or all of a number of objects, irrespective of their arrangements, is called a combination.

Notation : The number of all combinations of n things, taken r at a time is denoted by $C(n,r)$ or ${}^nC_r$ or $\binom{n}{r}$.

${}^nC_r$ is always a natural number.

Difference between a permutation and combination :

(i) In a combination only selection is made whereas in a permutation not only a selection is made but also an arrangement in a definite order is considered.

(ii) Each combination corresponds to many permutations. For example, the six permutations *ABC, ACB, BCA, BAC, CBA* and *CAB* correspond to the same combination *ABC*.

Number of combinations without repetition

The number of combinations (selections or groups) that can be formed from n different objects taken $r(0\le r\le n)$ at a time is ${}^nC_r=\frac{n!}{r!(n-r)!}$. Also ${}^nC_r={}^nC_{n-r}$.

Let the total number of selections (or groups) = x. Each group contains r objects, which can be arranged in r ! ways. Hence the number of arrangements of r objects = $x\times(r!)$. But the number of arrangements = ${}^nP_r$.

$$\Rightarrow x(r!)={}^nP_r \Rightarrow x=\frac{{}^nP_r}{r!}\Rightarrow x=\frac{n!}{r!(n-r)!}={}^nC_r.$$

Number of combinations with repetition and all possible selections

(1) The number of combinations of n distinct objects taken r at a time when any object may be repeated any number of times.

= Coefficient of x^r in $(1+x+x^2+......+x^r)^n$

= Coefficient of x^r in $(1-x)^{-n} = {}^{n+r-1}C_r$

(2) The total number of ways in which it is possible to form groups by taking some or all of n things at a time is ${}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n - 1$.

(3) The total number of ways in which it is possible to make groups by taking some or all out of $n = (n_1 + n_2 + ...)$ things, when n_1 are alike of one kind, n_2 are alike of second kind, and so on is $\{(n_1+1)(n_2+1)......\} - 1$.

(4) The number of selections of r objects out of n identical objects is 1.

(5) Total number of selections of zero or more objects from n identical objects is $n+1$.

(6) The number of selections taking at least one out of $a_1 + a_2 + a_3 + ... + a_n + k$ objects, where a_1 are alike (of one kind), a_2 are alike (of second kind) and so on...... a_n are alike (of n[th] kind) and k are distinct

$$= [(a_1+1)(a_2+1)(a_3+1).......(a_n+1)]2^k - 1.$$

Conditional combinations

(1) The number of ways in which r objects can be selected from n different objects if k particular objects are

(i) Always included = ${}^{n-k}C_{r-k}$ (ii) Never included = ${}^{n-k}C_r$

(2) The number of combinations of n objects, of which p are identical, taken r at a time is

$${}^{n-p}C_r + {}^{n-p}C_{r-1} + {}^{n-p}C_{r-2} + ... + {}^{n-p}C_0, \text{ if } r \le p \text{ and}$$

$${}^{n-p}C_r + {}^{n-p}C_{r-1} + {}^{n-p}C_{r-2} + ... + {}^{n-p}C_{r-p}, \text{ if } r > p.$$

Division into groups

Case I : (1) The number of ways in which n different things can be arranged into r different groups is ${}^{n+r-1}P_n$ or $n!\ {}^{n-1}C_{r-1}$ according as blank group are or are not admissible.

(2) The number of ways in which n different things can be distributed into r different group is

$$r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - .... + (-1)^{n-1}\ {}^nC_{r-1} \text{ or}$$

Coefficient of x^n is $n!\ (e^x - 1)^r$.

Here blank groups are not allowed.

(3) Number of ways in which $m \times n$ different objects can be distributed equally among n persons (or numbered groups) = (number of ways of dividing into groups) × (number of groups) ! = $\frac{(mn)!n!}{(m!)^n n!} = \frac{(mn)!}{(m!)^n}$.

Case II : (1) The number of ways in which $(m+n)$ different things can be divided into two groups which contain m and n things respectively is,

$${}^{m+n}C_m . {}^nC_n = \frac{(m+n)!}{m!n!}, m \ne n.$$

Corollary: If $m = n$, then the groups are equal size. Division of these groups can be given by two types.

Type I : If order of group is not important : The number of ways in which $2n$ different things can be divided equally into two groups is $\frac{(2n)!}{2!(n!)^2}$.

Type II : If order of group is important : The number of ways in which $2n$ different things can be divided equally into two distinct groups is $\frac{(2n)!}{2!(n!)^2} \times 2! = \frac{2n!}{(n!)^2}$.

(2) The number of ways in which $(m + n + p)$ different things can be divided into three groups which contain m, n and p things respectively is

$${}^{m+n+p}C_m . {}^{n+p}C_n . {}^pC_p = \frac{(m+n+p)!}{m!n!p!}, m \ne n \ne p.$$

Corollary : If $m = n = p$, then the groups are equal size. Division of these groups can be given by two types.

Type I : If order of group is not important : The number of ways in which $3p$ different things can be divided equally into three groups is $\frac{(3p)!}{3!(p!)^3}$.

Type II : If order of group is important : The number of ways in which $3p$ different things can be divided equally into three distinct groups is $\frac{(3p)!}{3!(p!)^3} 3! = \frac{(3p)!}{(p!)^3}$.

(i) If order of group is not important : The number of ways in which mn different things can be divided equally into m groups is $\frac{mn!}{(n!)^m m!}$.

(ii) If order of group is important: The number of ways in which mn different things can be divided equally into m distinct groups is $\frac{(mn)!}{(n!)^m m!} \times m! = \frac{(mn)!}{(n!)^m}$.

Derangement

Any change in the given order of the things is called a derangement.

If n things form an arrangement in a row, the number of ways in which they can be deranged so that no one of them occupies its original place is $n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ... + (-1)^n . \frac{1}{n!}\right)$.

Some important results for geometrical problems

(1) Number of total different straight lines formed by joining the n points on a plane of which m (< n) are collinear is ${}^nC_2 - {}^mC_2 + 1$.

(2) Number of total triangles formed by joining the n points on a plane of which m (< n) are collinear is ${}^nC_3 - {}^mC_3$.

(3) Number of diagonals in a polygon of n sides is ${}^nC_2 - n$.

(4) If m parallel lines in a plane are intersected by a family of other n parallel lines. Then total number of parallelograms so formed is ${}^mC_2 \times {}^nC_2$ i.e., $\frac{mn(m-1)(n-1)}{4}$.

(5) Given n points on the circumference of a circle, then

(i) Number of straight lines = ${}^nC_2$

(ii) Number of triangles = ${}^nC_3$

(iii) Number of quadrilaterals = ${}^nC_4$.

(6) If n straight lines are drawn in the plane such that no two lines are parallel and no three lines are concurrent. Then the number of part into which these lines divide the plane is = $1+\Sigma n$.

(7) Number of rectangles of any size in a square of $n \times n$ is $\sum_{r=1}^{n} r^3$ and number of squares of any size is $\sum_{r=1}^{n} r^2$.

(8) In a rectangle of $n \times p\,(n<p)$ number of rectangles of any size is $\frac{np}{4}(n+1)(p+1)$ and number of squares of any size is $\sum_{r=1}^{n}(n+1-r)(p+1-r)$.

Multinomial theorem

Let $x_1, x_2, \ldots\ldots, x_m$ be integers. Then number of solutions to the equation $x_1 + x_2 + \ldots\ldots + x_m = n$ .....(i)

Subject to the condition

$a_1 \le x_1 \le b_1, a_2 \le x_2 \le b_2, \ldots\ldots, a_m \le x_m \le b_m$ .....(ii)

is equal to the coefficient of x^n in

$(x^{a_1} + x^{a_1+1} + \ldots\ldots + x^{b_1})(x^{a_2} + x^{a_2+1} + \ldots.. + x^{b_2})\ldots\ldots(x^{a_m} + x^{a_m+1} + \ldots.. + x^{b_m})$ .....(iii)

This is because the number of ways, in which sum of m integers in (i) equals n, is the same as the number of times x^n comes in (iii).

(1) **Use of solution of linear equation and coefficient of a power in expansions to find the number of ways of distribution :** (i) The number of integral solutions of $x_1 + x_2 + x_3 + \ldots\ldots + x_r = n$ where $x_1 \ge 0, x_2 \ge 0, \ldots\ldots x_r \ge 0$ is the same as the number of ways to distribute n identical things among r persons.

This is also equal to the coefficient of x^n in the expansion of $(x^0 + x^1 + x^2 + x^3 + \ldots\ldots)^r$

= coefficient of x^n in $\left(\frac{1}{1-x}\right)^r$

= coefficient of x^n in $(1-x)^{-r}$

= coefficient of x^n in

$$\left\{1 + rx + \frac{r(r+1)}{2!}x^2 + \ldots. + \frac{r(r+1)(r+2)\ldots.(r+n-1)}{n!}x^n + \ldots\ldots\right.$$

$$= \frac{r(r+1)(r+2)\ldots.(r+n-1)}{n!} = \frac{(r+n-1)!}{n!(r-1)!} = {}^{n+r-1}C_{r-1}.$$

(2) The number of integral solutions of $x_1 + x_2 + x_3 + \ldots.. + x_r = n$ where $x_1 \ge 1, x_2 \ge 1, \ldots\ldots x_r \ge 1$ is same as the number of ways to distribute n identical things among r persons each getting at least 1. This also equal to the coefficient of x^n in the expansion of $(x^1 + x^2 + x^3 + \ldots\ldots)^r$.

= coefficient of x^n in $\left(\frac{x}{1-x}\right)^r$

= coefficient of x^n in $x^r(1-x)^{-r}$

= coefficient of x^n in

$$x^r\left\{1 + rx + \frac{r(r+1)}{2!}x^2 + \ldots + \frac{r(r+1)(r+2)\ldots(r+n-1)}{n!}x^n + \ldots\right\}$$

= coefficient of x^{n-r} in

$$\left\{1 + rx + \frac{r(r+1)}{2!}x^2 + \ldots + \frac{r(r+1)(r+2)\ldots.(r+n-1)}{n!}x^n + \ldots..\right\}$$

$$= \frac{r(r+1)(r+2)\ldots\ldots(r+n-r-1)}{(n-r)!} = \frac{r(r+1)(r+2)\ldots..(n-1)}{(n-r)!}$$

$$= \frac{(n-1)!}{(n-r)!(r-1)!} = {}^{n-1}C_{r-1}.$$

Number of divisors

Let $N = p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3}\ldots\ldots p_k^{\alpha_k}$, where $p_1, p_2, p_3, \ldots\ldots p_k$ are different primes and $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\alpha_k$ are natural numbers then :

(1) The total number of divisors of N including 1 and N is = $(\alpha_1+1)(\alpha_2+1)(\alpha_3+1)\ldots.(\alpha_k+1)$.

(2) The total number of divisors of N excluding 1 and N is = $(\alpha_1+1)(\alpha_2+1)(\alpha_3+1)\ldots..(\alpha_k+1)-2$.

(3) The total number of divisors of N excluding 1 or N is = $(\alpha_1+1)(\alpha_2+1)(\alpha_3+1)\ldots..(\alpha_k+1)-1$.

(4) The sum of these divisors is

$$= (p_1^0 + p_1^1 + p_1^2 + \ldots\ldots + p_1^{\alpha_1})(p_2^0 + p_2^1 + p_2^2 + \ldots + p_2^{\alpha_2})\ldots..(p_k^0 + p_k^1 + p_k^2 + \ldots. + p_k^{\alpha_k})$$

(5) The number of ways in which N can be resolved as a product of two factors is

$$\begin{cases} \frac{1}{2}(\alpha_1+1)(\alpha_2+1)\ldots.(\alpha_k+1), \text{If } N \text{ is not a perfect square} \\ \frac{1}{2}[(\alpha_1+1)(\alpha_2+1)\ldots..(\alpha_k+1)+1], \text{If } N \text{ is a perfect square} \end{cases}$$

(6) The number of ways in which a composite number N can be resolved into two factors which are relatively prime (or co-prime) to each other is equal to 2^{n-1} where n is the number of different factors in N.

Tips & Tricks

$^nC_0 = {^nC_n} = 1, {^nC_1} = n$.

$^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$.

$^nC_x = {^nC_y} \Leftrightarrow x = y \text{ or } x + y = n$.

$n.{^{n-1}C_{r-1}} = (n-r+1)\,{^nC_{r-1}}$.

If n is even then the greatest value of ^nC_r is $^nC_{n/2}$.

If n is odd then the greatest value of ^nC_r is $^nC_{\frac{n+1}{2}}$ or $^nC_{\frac{n-1}{2}}$.

$^nC_r = \frac{n}{r}.{^{n-1}C_{r-1}}$.

Number of selections of zero or more things out of n different things is, $^nC_0 + {^nC_1} + {^nC_2} + ... + {^nC_n} = 2^n$.

The number of ways of answering one or more questions when each question has an alternative is, $3^n - 1$.

The number of ways of answering all of n questions when each question has an alternative is 2^n.

$^nC_0 + {^nC_2} + {^nC_4} + ... = {^nC_1} + {^nC_3} + {^nC_5} + ... = 2^{n-1}$.

$^{2n+1}C_0 + {^{2n+1}C_1} + {^{2n+1}C_2} + ... + {^{2n+1}C_n} = 2^{2n}$.

$^nC_n + {^{n+1}C_n} + {^{n+2}C_n} + {^{n+3}C_n} + ... + {^{2n-1}C_n} = {^{2n}C_{n+1}}$.

Number of combinations of n dissimilar things taken all at a time $^nC_n = \frac{n!}{n!(n-n)!} = \frac{1}{0!} = 1, \ (\because 0! = 1)$.

Gap method : Suppose 5 males *A, B, C, D, E* are arranged in a row as × *A* × *B* × *C* × *D* × *E* ×. There will be six gaps between these five. Four in between and two at either end. Now if three females *P, Q,R* are to be arranged so that no two are together we shall use gap method *i.e.*, arrange them in between these 6 gaps. Hence the answer will be $^6P_3$.

Together : Suppose we have to arrange 5 persons in a row which can be done in 5 ! = 120 ways. But if two particular persons are to be together always, then we tie these two particular persons with a string. Thus we have 5 – 2 + 1 (1 corresponding to these two together) = 3 +1 = 4 units, which can be arranged in 4! ways. Now we loosen the string and these two particular can be arranged in 2! ways. Thus total arrangements = 24 × 2 = 48.

Never together = Total – Together = 120 – 48 = 72.

The number of ways in which n (one type of different) things and n (another type of different) things can be arranged in a row alternatively is $2.n!.n!$.

The number of ways in which m things of one type and n things of another type can be arranged in the form of a garland so that all the second type of things come together $= \frac{m!n!}{2}$ and no two things of second type come together $= \frac{(m-1)!\,{^mP_n}}{2}$.

All the numbers whose last digit is an even number 0, 2, 4, 6 or 8 are divisible by 2.

All the numbers sum of whose digits are divisible by 3,is divisible by 3 *e.g.*, 534. Sum of the digits is 12, which are divisible by 3, and hence 534 is also divisible by 3.

All those numbers whose last two-digit number is divisible by 4 are divisible by 4 *e.g.*, 7312, 8936, are such that 12, 36 are divisible by 4 and hence the given numbers are also divisible by 4.

All those numbers, which have either 0 or 5 as the last digit, are divisible by 5.

All those numbers, which are divisible by 2 and 3 simultaneously, are divisible by 6. *e.g.,* 108, 756 etc.

All those numbers whose last three-digit number is divisible by 8 are divisible by 8.

All those numbers sum of whose digit is divisible by 9 are divisible by 9.

All those numbers whose last two digits are divisible by 25 are divisible by 25 *e.g.,* 73125, 2400 etc.

If we are given n different digits $(a, a_2, a_3.........a_n)$ then sum of the digits in the unit place of all numbers formed without repetition is $(n-1)!(a_1 + a_2 + a_3 + ... + a_n)$. Sum of the total numbers in this case can be obtained by applying the formula $(n-1)!(a_1 + a_2 + a_3 + ... + a_n)$. (1111......... n times).

Ordinary Thinking

Objective Questions

Definition of permutation, Number of permutations with or without repetition, Conditional permutations

1. If the best and the worst paper never appear together, then six examination papers can be arranged in how many ways

(a) 120 (b) 480

(c) 240 (d) None of these

2. How many numbers divisible by 5 and lying between 3000 and 4000 can be formed from the digits 1, 2, 3, 4, 5, 6 (repetition is not allowed)

(a) ${}^6P_2$ (b) ${}^5P_2$

(c) ${}^4P_2$ (d) ${}^6P_3$

3. The number of ways in which 6 rings can be worn on the four fingers of one hand is

(a) 4^6 (b) ${}^6C_4$

(c) 6^4 (d) None of these

4. How many numbers can be formed from the digits 1, 2, 3, 4 when the repetition is not allowed

(a) ${}^4P_4$ (b) ${}^4P_3$

(c) ${}^4P_1 + {}^4P_2 + {}^4P_3$ (d) ${}^4P_1 + {}^4P_2 + {}^4P_3 + {}^4P_4$

5. There are 3 candidates for a post and one is to be selected by the votes of 7 men. The number of ways in which votes can be given is

(a) 7^3 (b) 3^7

(c) ${}^7C_3$ (d) None of these

6. 4 buses runs between Bhopal and Gwalior. If a man goes from Gwalior to Bhopal by a bus and comes back to Gwalior by another bus, then the total possible ways are

(a) 12 (b) 16

(c) 4 (d) 8

7. If ${}^nP_5 = 20 . {}^nP_3$, then $n =$

(a) 4 (b) 8

(c) 6 (d) 7

8. How many words comprising of any three letters of the word UNIVERSAL can be formed

(a) 504 (b) 405

(c) 540 (d) 450

9. If ${}^nP_4 : {}^nP_5 = 1 : 2$, then $n =$ **[MP PET 1987; RPET 1996]**

(a) 4 (b) 5

(c) 6 (d) 7

10. In how many ways can mn letters be posted in n letter-boxes

(a) $(mn)^n$ (b) m^{mn}

(c) n^{mn} (d) None of these