Chapter

1

Set Theory and Relations

Set Theory

Introduction

A set is well defined class or collection of objects.

A set is often described in the following two ways.

(1) **Roster method or Listing method** : In this method a set is described by listing elements, separated by commas, within braces {}. The set of vowels of English alphabet may be described as {a, e, i, o, u}.

(2) **Set-builder method or Rule method** : In this method, a set is described by a characterizing property $P(x)$ of its elements x. In such a case the set is described by $\{x : P(x)$ holds$\}$ or $\{x \mid P(x)$ holds$\}$, which is read as 'the set of all x such that $P(x)$ holds'. The symbol '$|$' or ':' is read as 'such that'.

The set $A = \{0, 1, 4, 9, 16, ...\}$ can be written as $A = \{x^2 \mid x \in Z\}$.

❑ Symbols

Symbol	Meaning
$\Rightarrow$	Implies
$\in$	Belongs to
$A \subset B$	A is a subset of B
$\Leftrightarrow$	Implies and is implied by
$\notin$	Does not belong to
s.t.(: or $\mid$)	Such that
$\forall$	For every
$\exists$	There exists
iff	If and only if
&	And
$a \mid b$	a is a divisor of b
N	Set of natural numbers
I or Z	Set of integers
R	Set of real numbers
C	Set of complex numbers
Q	Set of rational numbers

Types of sets

(1) **Null set or Empty set** : The set which contains no element at all is called the null set. This set is sometimes also called the 'empty set' or the 'void set'. It is denoted by the symbol ϕ or {}.

(2) **Singleton set** : A set consisting of a single element is called a singleton set. The set {5} is a singleton set.

(3) **Finite set** : A set is called a finite set if it is either void set or its elements can be listed (counted, labelled) by natural number 1, 2, 3, ... and the process of listing terminates at a certain natural number n (say).

Cardinal number of a finite set **:** The number n in the above definition is called the cardinal number or order of a finite set A and is denoted by $n(A)$ or $O(A)$.

(4) **Infinite set :** A set whose elements cannot be listed by the natural numbers 1, 2, 3, ..., n, for any natural number n is called an infinite set.

(5) **Equivalent set :** Two finite sets A and B are equivalent if their cardinal numbers are same *i.e.* $n(A) = n(B)$.

Example **:** $A = \{1, 3, 5, 7\}$; $B = \{10, 12, 14, 16\}$ are equivalent sets, $[\because O(A) = O(B) = 4]$.

(6) **Equal set :** Two sets A and B are said to be equal *iff* every element of A is an element of B and also every element of B is an element of A. Symbolically, $A = B$ if $x \in A \Leftrightarrow x \in B$.

Example **:** If $A = \{2, 3, 5, 6\}$ and $B = \{6, 5, 3, 2\}$. Then $A = B$, because each element of A is an element of B and vice-versa.

(7) **Universal set :** A set that contains all sets in a given context is called the universal set.

It should be noted that universal set is not unique. It may differ in problem to problem.

(8) **Power set :** If S is any set, then the family of all the subsets of S is called the power set of S.

The power set of S is denoted by $P(S)$. Symbolically, $P(S) = \{T : T \subseteq S\}$. Obviously ϕ and S are both elements of $P(S)$.

Example **:** Let $S = \{a, b, c\}$, then $P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

Power set of a given set is always non-empty.

(9) **Subsets (Set inclusion) :** Let A and B be two sets. If every element of A is an element of B, then A is called a subset of B.

If A is subset of B, we write $A \subseteq B$, which is read as "A is a subset of B" or "A is contained in B".

Thus, $A \subseteq B \Rightarrow a \in A \Rightarrow a \in B$.

Proper and improper subsets **:** If A is a subset of B and $A \neq B$, then A is a proper subset of B. We write this as $A \subset B$.

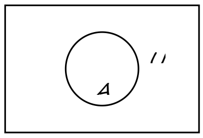
The null set ϕ is subset of every set and every set is subset of itself, *i.e.*, $\phi \subset A$ and $A \subseteq A$ for every set A. They are called improper subsets of A. Thus every non-empty set has two improper subsets. It should be noted that ϕ has only one subset ϕ which is improper.

All other subsets of A are called its proper subsets. Thus, if $A \subset B$, $A \neq B$, $A \neq \phi$, then A is said to be proper subset of B.

Example **:** Let $A = \{1, 2\}$. Then A has $\phi; \{1\}, \{2\}, \{1, 2\}$ as its subsets out of which ϕ and $\{1, 2\}$ are improper and $\{1\}$ and $\{2\}$ are proper subsets.

Venn-Euler diagrams

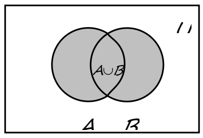
The combination of rectangles and circles are called *Venn-Euler diagrams* or simply **Venn-diagrams**.



If A and B are not equal but they have some common elements, then to represent A and B we draw two intersecting circles. Two disjoints sets are represented by two non-intersecting circles.

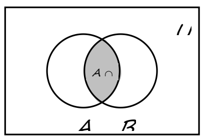
Operations on sets

(1) **Union of sets :** Let A and B be two sets. The union of A and B is the set of all elements which are in set A or in B. We denote the union of A and B by $A \cup B$, which is usually read as "A union B".



Symbolically, $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

(2) **Intersection of sets :** Let A and B be two sets. The intersection of A and B is the set of all those elements that belong to both A and B.



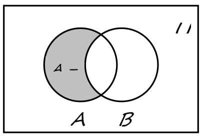
The intersection of A and B is denoted by $A \cap B$ (read as "A intersection B").

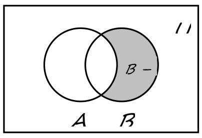
Thus, $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

(3) **Disjoint sets :** Two sets A and B are said to be disjoint, if $A \cap B = \phi$. If $A \cap B \neq \phi$, then A and B are said to be non-intersecting or non-overlapping sets.

Example **:** Sets $\{1, 2\}$; $\{3, 4\}$ are disjoint sets.

(4) **Difference of sets :** Let A and B be two sets. The difference of A and B written as $A - B$, is the set of all those elements of A which do not belong to B.





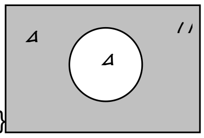
Thus, $A - B = \{x : x \in A \text{ and } x \notin B\}$

Similarly, the difference $B - A$ is the set of all those elements of B that do not belong to A *i.e.,* $B - A = \{x \in B : x \notin A\}$.

Example : Consider the sets $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A - B = \{1, 2\}; B - A = \{4, 5\}$.

(5) **Symmetric difference of two sets :** Let A and B be two sets. The symmetric difference of sets A and B is the set $(A - B) \cup (B - A)$ and is denoted by $A \Delta B$. Thus, $A \Delta B = (A - B) \cup (B - A) = \{x : x \notin A \cap B\}$.

(6) **Complement of a set :** Let U be the universal set and let A be a set such that $A \subset U$. Then, the complement of A with respect to U is denoted by A' or A^c or $C(A)$ or $U - A$ and is defined the set of all those elements of U which are not in A.



Thus, $A' = \{x \in U : x \notin A\}$.

Clearly, $x \in A' \Leftrightarrow x \notin A$

Example : Consider $U = \{1, 2, ..., 10\}$ and $A = \{1, 3, 5, 7, 9\}$.

Then $A' = \{2, 4, 6, 8, 10\}$

Some important results on number of elements in sets

If A, B and C are finite sets and U be the finite universal set, then (1) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(2) $n(A \cup B) = n(A) + n(B) \Leftrightarrow A, B$ are disjoint non-void sets.

(3) $n(A - B) = n(A) - n(A \cap B)$ *i.e.,* $n(A - B) + n(A \cap B) = n(A)$

(4) $n(A \Delta B)$ = Number of elements which belong to exactly one of A or B $= n((A - B) \cup (B - A)) = n(A - B) + n(B - A)$

[$\because$ $(A - B)$ and $(B - A)$ are disjoint]

$= n(A) - n(A \cap B) + n(B) - n(A \cap B) = n(A) + n(B) - 2n(A \cap B)$

(5) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(6) n (Number of elements in exactly two of the sets A, B, C) $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$

(7) n(Number of elements in exactly one of the sets A, B, C) $= n(A) + n(B) + n(C)$

$- 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$

(8) $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$

(9) $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$

Laws of algebra of sets

(1) **Idempotent laws :** For any set A, we have

(i) $A \cup A = A$ (ii) $A \cap A = A$

(2) **Identity laws :** For any set A, we have

(i) $A \cup \phi = A$ (ii) $A \cap U = A$

i.e., ϕ and U are identity elements for union and intersection respectively.

(3) **Commutative laws :** For any two sets A and B, we have

(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$

(iii) $A \Delta B = B \Delta A$

i.e., union, intersection and symmetric difference of two sets are commutative.

(iv) $A - B \neq B - A$ (v) $A \times B \neq B \times A$

i.e., difference and cartesian product of two sets are not commutative

(4) **Associative laws :** If A, B and C are any three sets, then

(i) $(A \cup B) \cup C = A \cup (B \cup C)$ (ii) $A \cap (B \cap C) = (A \cap B) \cap C$

(iii) $(A \Delta B) \Delta C = A \Delta (B \Delta C)$

i.e., union, intersection and symmetric difference of two sets are associative.

(iv) $(A-B)-C \neq A-(B-C)$ (v) $(A \times B) \times C \neq A \times (B \times C)$

i.e., difference and cartesian product of two sets are not associative.

(5) **Distributive law** : If A, B and C are any three sets, then

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

i.e., union and intersection are distributive over intersection and union respectively.

(iii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(iv) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(v) $A \times (B - C) = (A \times B) - (A \times C)$

(6) **De-Morgan's law** : If A, B and C are any three sets, then

(i) $(A \cup B)' = A' \cap B$

(ii) $(A \cap B)' = A' \cup B$

(iii) $A - (B \cap C) = (A - B) \cup (A - C)$

(iv) $A - (B \cup C) = (A - B) \cap (A - C)$

(7) If A and B are any two sets, then

(i) $A - B = A \cap B$ (ii) $B - A = B \cap A'$

(iii) $A - B = A \Leftrightarrow A \cap B = \phi$ (iv) $(A - B) \cup B = A \cup B$

(v) $(A - B) \cap B = \phi$ (vi) $A \subseteq B \Leftrightarrow B' \subseteq A'$

(vii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

(8) If A, B and C are any three sets, then

(i) $A \cap (B - C) = (A \cap B) - (A \cap C)$

(ii) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

Cartesian product of sets

Cartesian product of sets : Let A and B be any two non-empty sets. The set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called the cartesian product of the sets A and B and is denoted by $A \times B$.

Thus, $A \times B = [(a, b) : a \in A \text{ and } b \in B]$

If $A = \phi$ or $B = \phi$, then we define $A \times B = \phi$.

Example : Let $A = \{a, b, c\}$ and $B = \{p, q\}$.

Then $A \times B = \{(a, p), (a, q), (b, p), (b, q), (c, p), (c, q)\}$

Also $B \times A = \{(p, a), (p, b), (p, c), (q, a), (q, b), (q, c)\}$

Important theorems on cartesian product of sets :

Theorem 1 : For any three sets A, B, C

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Theorem 2 : For any three sets A, B, C

$A \times (B - C) = (A \times B) - (A \times C)$

Theorem 3 : If A and B are any two non-empty sets, then

$A \times B = B \times A \Leftrightarrow A = B$

Theorem 4 : If $A \subseteq B$, then $A \times A \subseteq (A \times B) \cap (B \times A)$

Theorem 5 : If $A \subseteq B$, then $A \times C \subseteq B \times C$ for any set C.

Theorem 6 : If $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$

Theorem 7 : For any sets A, B, C, D

$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

Theorem 8 : For any three sets A, B, C

(i) $A \times (B \cup C)' = (A \times B) \cap (A \times C)$

(ii) $A \times (B \cap C)' = (A \times B) \cup (A \times C)$

Relations

Definition

Let A and B be two non-empty sets, then every subset of $A \times B$ defines a relation from A to B and every relation from A to B is a subset of $A \times B$.

Let $R \subseteq A \times B$ and $(a, b) \in R$. Then we say that a is related to b by the relation R and write it as aRb. If $(a, b) \in R$, we write it as aRb.

(1) **Total number of relations** : Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then $A \times B$ consists of mn ordered pairs. So, total number of subset of $A \times B$ is 2^{mn}. Since each subset of $A \times B$ defines relation from A to B, so total number of relations from A to B is 2^{mn}. Among these

2^{mn} relations the void relation ϕ and the universal relation $A \times B$ are trivial relations from A to B.

(2) **Domain and range of a relation** : Let R be a relation from a set A to a set B. Then the set of all first components or coordinates of the ordered pairs belonging to R is called the domain of R, while the set of all second components or coordinates of the ordered pairs in R is called the range of R.

Thus, Dom $(R) = \{a : (a, b) \in R\}$ and Range $(R) = \{b : (a, b) \in R\}$.

Inverse relation

Let A, B be two sets and let R be a relation from a set A to a set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$

Clearly $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$. Also, Dom (R) = Range (R^{-1}) and Range (R) = Dom (R^{-1})

Example : Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$ and $R = \{(a, 1), (a, 3), (b, 3), (c, 3)\}$.

Then, (i) $R^{-1} = \{(1, a), (3, a), (3, b), (3, c)\}$

(ii) Dom $(R) = \{a, b, c\}$ = Range (R^{-1})

(iii) Range $(R) = \{1, 3\}$ = Dom (R^{-1})

Types of relations

(1) **Reflexive relation** : A relation R on a set A is said to be reflexive if every element of A is related to itself.

Thus, R is reflexive $\Leftrightarrow (a, a) \in R$ for all $a \in A$.

Example : Let $A = \{1, 2, 3\}$ and $R = \{(1, 1); (1, 3)\}$

Then R is not reflexive since $3 \in A$ but $(3, 3) \notin R$

A reflexive relation on A is not necessarily the identity relation on A.

The universal relation on a non-void set A is reflexive.

(2) **Symmetric relation** : A relation R on a set A is said to be a symmetric relation *iff* $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

i.e., $aRb \Rightarrow bRa$ for all $a, b \in A$.

it should be noted that R is symmetric iff $R^{-1} = R$

The identity and the universal relations on a non-void set are symmetric relations.

A reflexive relation on a set A is not necessarily symmetric.

(3) **Anti-symmetric relation** : Let A be any set. A relation R on set A is said to be an anti-symmetric relation *iff* $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in A$.

Thus, if $a \neq b$ then a may be related to b or b may be related to a, but never both.

(4) **Transitive relation** : Let A be any set. A relation R on set A is said to be a transitive relation *iff*

$(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$ *i.e.*, aRb and $bRc \Rightarrow aRc$ for all $a, b, c \in A$.

Transitivity fails only when there exists a, b, c such that $a\,R\,b$, $b\,R\,c$ but $a \not R c$.

Example : Consider the set $A = \{1, 2, 3\}$ and the relations

$R_1 = \{(1, 2), (1, 3)\}$; $R_2 = \{(1, 2)\}$; $R_3 = \{(1, 1)\}$;

$R_4 = \{(1, 2), (2, 1), (1, 1)\}$

Then R_1, R_2, R_3 are transitive while R_4 is not transitive since in $R_4, (2, 1) \in R_4; (1, 2) \in R_4$ but $(2, 2) \notin R_4$.

The identity and the universal relations on a non-void sets are transitive.

(5) **Identity relation** : Let A be a set. Then the relation $I_A = \{(a, a) : a \in A\}$ on A is called the identity relation on A.

In other words, a relation I_A on A is called the identity relation if every element of A is related to itself only. Every identity relation will be reflexive, symmetric and transitive.

Example : On the set = $\{1, 2, 3\}$, $R = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on A .

It is interesting to note that every identity relation is reflexive but every reflexive relation need not be an identity relation.

(6) **Equivalence relation** : A relation R on a set A is said to be an equivalence relation on A *iff*

(i) It is reflexive *i.e.* $(a, a) \in R$ for all $a \in A$

(ii) It is symmetric *i.e.* $(a, b) \in R \Rightarrow (b, a) \in R$, for all $a, b \in A$

(iii) It is transitive *i.e.* $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

Congruence modulo (m) : Let m be an arbitrary but fixed integer. Two integers a and b are said to be congruence modulo m if $a-b$ is divisible by m and we write $a \equiv b$ (mod m).

Thus $a \equiv b$ (mod m) $\Leftrightarrow a-b$ is divisible by m. For example, $18 \equiv 3$ (mod 5) because 18 – 3 = 15 which is divisible by 5. Similarly, $3 \equiv 13$ (mod 2) because 3 – 13 = –10 which is divisible by 2. But $25 \not\equiv 2$ (mod 4) because 4 is not a divisor of 25 – 3 = 22.

The relation "Congruence modulo m" is an equivalence relation.

Equivalence classes of an equivalence relation

Let R be equivalence relation in $A(\neq \phi)$. Let $a \in A$. Then the equivalence class of a, denoted by $[a]$ or $\{\bar{a}\}$ is defined as the set of all those points of A which are related to a under the relation R. Thus $[a] = \{x \in A : x \, R \, a\}$.

It is easy to see that

(1) $b \in [a] \Rightarrow a \in [b]$

(2) $b \in [a] \Rightarrow [a] = [b]$

(3) Two equivalence classes are either disjoint or identical.

Composition of relations

Let R and S be two relations from sets A to B and B to C respectively. Then we can define a relation SoR from A to C such that $(a, c) \in SoR \Leftrightarrow \exists\, b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

This relation is called the composition of R and S.

For example, if $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$, $C=\{p, q, r, s\}$ be three sets such that $R = \{(1, a), (2, b), (1, c), (2, d)\}$ is a relation from A to B and $S = \{(a, s), (b, r), (c, r)\}$ is a relation from B to C. Then SoR is a relation from A to C given by $SoR = \{(1, s)\ (2, r)\ (1, r)\}$

In this case RoS does not exist.

In general $RoS \neq SoR$. Also $(SoR)^{-1} = R^{-1}oS^{-1}$.

Tips & Tricks

✍....Equal sets are always equivalent but equivalent sets may need not be equal set.

✍....If A has n elements, then $P(A)$ has 2^n elements.

✍....The total number of subset of a finite set containing n elements is 2^n.

✍....If $A_1, A_2, \ldots\ldots, A_n$ is a finite family of sets, then their union is denoted by $\bigcup_{i=1}^{n} A_i$ or $A_1 \cup A_2 \cup A_3 \ldots\ldots \cup A_n$.

✍....If $A_1, A_2, A_3 \ldots\ldots, A_n$ is a finite family of sets, then their intersection is denoted by $\bigcap_{i=1}^{n} A_i$ or $A_1 \cap A_2 \cap A_3 \cap \ldots\ldots \cap A_n$.

✍.... $R - Q$ is the set of all irrational numbers.

✍....Let A and B two non-empty sets having n elements in common, then $A \times B$ and $B \times A$ have n^2 elements in common.

✍....The identity relation on a set A is an anti-symmetric relation.

✍....The universal relation on a set A containing at least two elements is not anti-symmetric, because if $a \neq b$ are in A, then a is related to b and b is related to a under the universal relation will imply that $a = b$ but $a \neq b$.

✍....The set $\{(a, a) : a \in A\} = D$ is called the diagonal line of $A \times A$. Then "the relation R in A is antisymmetric iff $R \cap R^{-1} \subseteq D$".

✍ The relation 'is congruent to' on the set T of all triangles in a plane is a transitive relation.

✍....If R and S are two equivalence relations on a set A, then $R \cap S$ is also an equivalence relation on A.

✍....The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.

✍....The inverse of an equivalence relation is an equivalence relation.