Recurrence Relation

The general form of a first order linear homogeneous recurrence relation with constant coefficients is

$a_{n+1} = d\,a_n,\ n \geq 0$, where d is a constant. Since a_{n+1} depends only on its immediate predecessor, the relation is said to be first order.

The unique solution of the recurrence relation $a_{n+1} = d\,a_n,\ n \geq 0$, d is a constant with initial condition $a_0 = A$ is

$$\boxed{\begin{array}{l} a_n = A d^n,\ n \geq 0. \\ a_n = a_0 d^n \end{array}}$$

(i) Solve the recurrence relation $a_n = 7a_{n-1}$, where $n \geq 1$ and $a_2 = 98$

Solution

The solution is

$$a_n = a_0 7^n \quad (\text{Here } d = 7)$$

given $a_2 = 98$

$$a_0 7^2 = 98 \implies a_0 = \frac{98}{49} = 2.$$

$$\therefore\ a_n = 2 \cdot 7^n,\ n \geq 0.$$

(2) Solve $a_{n+1} - 1.5\, a_n = 0, \; n \geq 0$

Solution

$$a_{n+1} = 1.5\, a_n$$

$$d = 1.5$$

Solution $a_n = a_0 (1.5)^n$

(3) Solve $3a_{n+1} - 4a_n = 0, \; n \geq 0, \; a_1 = 5$

Solution

$$3a_{n+1} = 4a_n$$

$$a_{n+1} = \frac{4}{3} a_n$$

$$d = \frac{4}{3}.$$

solution $a_n = a_0 (4/3)^n$.

Given $a_1 = 5$

$$a_0 (4/3) = 5$$

$$a_0 = \frac{15}{4}$$

$$\therefore \quad a_n = \left(\frac{15}{4}\right)(4/3)^n$$

(4) Find the unique solution of the recurrence relation $6a_n - 7a_{n-1} = 0, \; n \geq 1, \; a_3 = 343$

Solution

$$6a_n = 7a_{n-1}$$

$$a_n = \left(\frac{7}{6}\right) a_{n-1}$$

$$d = 7/6.$$

Solution is $a_n = a_0\left(\frac{7}{6}\right)^n$

Given $a_3 = 343$

$$a_0\left(\frac{7}{6}\right)^3 = 343$$

$$a_0 = \frac{343 \times 216}{343} = 216$$

$$\therefore a_n = 216\left(\frac{7}{6}\right)^n$$

(5) Find a_{12} if $a_{n+1}^2 = 5a_n^2$, $a_n > 0$ for $n \geq 0$ and $a_0 = 2$

Solution

The recurrence relation is not linear in a_n. Let $b_n = a_n^2$, then the relation becomes

$$b_{n+1} = 5b_n, \quad b_0 = a_0^2 = 4$$

The solution for b_n is $b_n = b_0 5^n$

$$b_n = 4 \cdot 5^n.$$

$$\therefore a_n = 2(\sqrt{5})^n,$$

$$a_{12} = 2(\sqrt{5})^{12}$$

$$a_{12} = 31,250$$

(6) Solve the relation $a_n = n \cdot a_{n-1}$, where $n \geq 1$ and $a_0 = 1$.

Solution

This relation is a recurrence relation with variable coefficient.

$a_0 = 1$

$a_1 = 1 \cdot a_0 = 1$

$a_2 = 2 \cdot a_1 = 2 \cdot 1 = 2!$

$a_3 = 3 \cdot a_2 = 3 \cdot 2 \cdot 1 = 3!$

$a_4 = 4 \cdot a_3 = 4 \cdot 3 \cdot 2 \cdot 1 = 4!$

$\therefore \quad a_n = n!$

Exercises

① Solve the recurrence relations

(a) $4a_n - 5a_{n-1} = 0, \quad n \geq 1$

(b) $2a_n - 3a_{n-1} = 0, \quad n \geq 1, \quad a_4 = 81$

Second Order Linear Homogeneous Recurrence Relation with Constant Coefficients.

The general form of second order homogeneous recurrence relation with constant coefficients is

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0, \quad n \geq 2$$

Take $a_n = c r^n$

$$C_0 c r^n + C_1 c r^{n-1} + C_2 c r^{n-2} = 0.$$

$$(C_0 r^2 + C_1 r + C_2) c r^{n-2} = 0.$$

$$C_0 r^2 + C_1 r + C_2 = 0.$$

The equation $C_0 r^2 + C_1 r + C_2 = 0$ is called the characteristic equation. Let r_1 and r_2 be the roots of the equation.

<u>Case (A)</u> <u>Distinct</u> real roots.

If r_1 and r_2 are the distinct real roots, the solution is

$$\cancel{a_n = c_1(r_1^n) + c_2(r_2)(r_2^n)}$$

$$a_n = c_1(r_1^n) + c_2(r_2^n)$$

(1) Solve the recurrence relation

$a_n + a_{n-1} - 6a_{n-2} = 0$, $n \geq 2$, $a_0 = -1$, $a_1 = 8$.

<u>Solution</u>

If $a_n = cr^n$

$$\cancel{a_{n+1} = cr^{n+1}}$$

$$a_{n-1} = cr^{n-1}$$

$$a_{n-2} = cr^{n-2}.$$

Equation becomes $cr^n + cr^{n-1} - 6cr^{n-2} = 0$

$$cr^{n-2}(r^2 + r - 6) = 0$$

$$r^2 + r - 6 = 0$$

$$(r+3)(r-2) = 0$$

$$r = -3, 2.$$

$$\therefore a_n = c_1(2^n) + c_2(-3)^n$$

If $a_0 = -1$,

$$a_0 = c_1(2^0) + c_2(-3)^0 = c_1 + c_2 = -1$$

If $a_1 = 8$

$$a_1 = c_1(2^1) + c_2(-3)^1 = 2c_1 - 3c_2 = 8$$

$$c_1 + c_2 = -1 \quad \text{---}(1)$$

$$2c_1 - 3c_2 = 8 \quad \text{---}(2)$$

$$(1) \times 2 \Rightarrow 2c_1 + 2c_2 = -2 \quad \text{---}(3)$$

$$(2) \Rightarrow 2c_1 - 3c_2 = 8 \quad \text{---}(4)$$

$$(3) - (4) \Rightarrow 5c_2 = -10$$

$$c_2 = -2$$

$$(1) \Rightarrow c_1 - 2 = -1$$

$$c_1 = 1$$

$$\therefore \; a_n = 2^n - 2(-3)^n.$$

(2) Solve the recurrence relation

$$2a_n = 7a_{n-1} - 3a_{n-2} \,;\; a_0 = 2,\; a_1 = 5$$

<u>Solution</u>

If $a_n = c r^n$

$$a_{n-1} = c r^{n-1}$$

$$a_{n-2} = c r^{n-2}$$

Equation becomes $2cr^n = 7cr^{n-1} - 3cr^{n-2}$

$$c r^{n-2}\left[2r^2 - 7r + 3\right] = 0$$

$$2r^2 - 7r + 3 = 0$$

$$r = \frac{7 \pm \sqrt{49-24}}{4}$$

$$r = \frac{7 \pm 5}{4}$$

$$r = 3, \frac{1}{2}$$

$$\therefore \quad a_n = c_1 (3^n) + c_2 \left(\tfrac{1}{2}\right)^n$$

If $a_0 = 2$, $\quad a_0 = c_1 (3^0) + c_2 (1/2)^0 = c_1 + c_2 = 2.$

If $a_1 = 5$, $\quad a_1 = c_1 (3^1) + c_2 (1/2)^1 = 3c_1 + \frac{1}{2} c_2 = 5$

$$c_1 + c_2 = 2 \quad \text{--- (1)}$$

$$3c_1 + \frac{1}{2} c_2 = 5 \quad \text{--- (2)}$$

$(2) \times 2 \quad 6c_1 + c_2 = 10 \quad$ --- (3)

$(3) - (1) \Rightarrow \quad 5c_1 = 8$

$$c_1 = \frac{8}{5}$$

$(1) \Rightarrow \quad c_2 = 2 - \frac{8}{5} = \frac{2}{5}$

$$a_n = \frac{8}{5}(3^n) + \frac{2}{5}(1/2)^n$$

(3) Solve the recurrence relation $F_{n+2} = F_{n+1} + F_n$.

where $n \geq 0$, and $F_0 = 0$, $F_1 = 1$

Solution

If $F_n = cr^n$

$F_{n+1} = cr^{n+1}$

$F_{n+2} = cr^{n+2}$.

Equation becomes

$$c r^{n+2} = c r^{n+1} + c r^{n}$$

$$c r^{n}\left[r^2 - r - 1\right] = 0$$

$$r^2 - r - 1 = 0$$

$$r = \frac{1 \pm \sqrt{1+4}}{2}$$

$$r = \frac{1 \pm \sqrt{5}}{2}$$

$$F_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$F_0 = c_1 + c_2 = 0 \quad \text{———} (1)$$

$$F_1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 \quad \text{———} (2)$$

$$(1) \times (1+\sqrt{5}) \Rightarrow c_1(1+\sqrt{5}) + c_2(1+\sqrt{5}) = 0 \quad \text{———} (3)$$

$$(2) \times 2 \Rightarrow c_1(1+\sqrt{5}) + c_2(1-\sqrt{5}) = 2 \quad \text{———} (4)$$

$$(3) - (4) \Rightarrow c_2(2\sqrt{5}) = -2.$$

$$c_2 = \frac{-1}{\sqrt{5}}$$

$$(1) \Rightarrow c_1 = \frac{1}{\sqrt{5}}$$

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Repeated Real roots

(1) Solve the recurrence relation

$a_{n+2} = 4a_{n+1} - 4a_n,\quad n \geq 0,\ a_0 = 1,\ a_1 = 3$

Solution

If $a_n = cr^n$

$a_{n+1} = cr^{n+1}$

$a_{n+2} = cr^{n+2}$

$$cr^{n+2} = 4cr^{n+1} - 4cr^n$$

$$cr^n\left[r^2 - 4r + 4\right] = 0$$

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

$$r = 2, 2.$$

$$\therefore a_n = c_1(2^n) + c_2 n(2^n)$$

$$a_0 = c_1 + \cancel{c_2} = 1 \quad \text{——— (1)}$$

$$a_1 = c_1(2) + c_2(2) = 3$$

$$2c_2 = 1$$

$$c_2 = 1/2.$$

$$a_n = (2^n) + \frac{1}{2}n(2^n)$$

(2) Solve the recurrence relation.

$a_n - 6a_{n-1} + 9a_{n-2} = 0,\quad n \geq 2,\ a_0 = 5,\ a_1 = 12.$

Solution

$$\cancel{a_n = cr^n,\ a_{n+1} = cr^{n+1},\ a_{n+2} = cr^{n+2}}$$

$$a_n = cr^n,\ a_{n-1} = cr^{n-1},\ a_{n-2} = cr^{n-2}$$

$$cr^n - 6cr^{n-1} + 9cr^{n-2} = 0$$

$$cr^{n-2}\left[r^2 - 6r + 9\right] = 0$$

$$r^2 - 6r + 9 = 0$$

$$(r-3)^2 = 0$$

$$r = 3, 3$$

$$a_n = c_1(3^n) + c_2 n(3^n)$$

$$a_0 = c_1 = 5 \quad \text{—— (1)}$$

$$a_1 = c_1(3) + c_2(3) = 12 \quad \text{—— (2)}$$

$$15 + 3c_2 = 12$$

$$3c_2 = -3$$

$$c_2 = -1$$

$$a_n = 5(3^n) - n(3^n)$$

Complex Roots

(i) Solve the recurrence relation

$$a_n = 2(a_{n-1} - a_{n-2}),\ n \geq 2,\ a_0 = 1,\ a_1 = 2$$

Solution

$$a_n = cr^n,\ a_{n-1} = cr^{n-1},\ a_{n-2} = cr^{n-2}$$

$$cr^n = 2(cr^{n-1} - cr^{n-2})$$

$$cr^{n-2}(r^2 - 2r + 2) = 0$$

$$r = \frac{2 \pm \sqrt{4-8}}{2}$$

$$r = 1 \pm i$$

$$a_n = c_1 (1+i)^n + c_2 (1-i)^n$$

$$a_0 = c_1 (1+i)^0 + c_2 (1-i)^0 = c_1 + c_2 = 1 \quad \text{---} (1)$$

$$a_1 = c_1 (1+i) + c_2 (1-i) = 2 \quad \text{---} (2)$$

$$(1) \times (1+i) \Rightarrow c_1 (1+i) + c_2 (1+i) = 1+i \quad \text{---} (3)$$

$$(2)-(3) \Rightarrow c_2 (-2i) = (1-i)$$

$$c_2 = \frac{1-i}{-2i} = \frac{1}{2}(i+1)$$

$$c_1 = 1 - c_2$$

$$= 1 - \frac{(i+1)}{2}$$

$$= \frac{1}{2}(1-i)$$

$$a_n = \frac{1}{2}(1-i)(1+i)^n + \frac{1}{2}(1+i)(1-i)^n.$$

Exercises

Solve the recurrence relations

(1) $a_n = 5a_{n-1} + 6a_{n-2}, \; n \geq 2, \; a_0 = 1, \; a_1 = 3$

(2) $2a_{n+2} - 11a_{n+1} + 5a_n = 0, \; n \geq 0, \; a_0 = 2, \; a_1 = -8$

(3) $a_{n+3} + a_n = 0, \; n \geq 0, \; a_0 = 0, \; a_1 = 3$

(4) $a_n + 2a_{n-1} + 2a_{n-2} = 0, \; n \geq 2, \; a_0 = 1, \; a_1 = 3.$

Non homogeneous recurrence relation

Consider the non homogeneous first order relation $a_n + C_1 a_{n-1} = k r^n$, where k is a constant and $n \in Z^+$. If r^n is not a solution of the associated homogeneous relation $a_n + C_1 a_{n-1} = 0$, then $a_n^{(P)} = A r^n$ where A is a constant. If r^n is a solution of the corresponding homogeneous relation, then $a_n^{(P)} = Bn r^n$, B is a constant.

Consider the non homogeneous second order relation $a_n + C_1 a_{n-1} + C_2 a_{n-2} = k r^n$, where k is a constant.

(a) If r^n is not a solution of the homogeneous relation $a_n^{(P)} = A r^n$

(b) If $a_n^{(h)} = C_1 r^n + C_2 r_1^n$ where $r_1 \neq r$,

$a_n^{(P)} = Bn r^n$ where B is a constant

(c) If $a_n^{(h)} = (C_1 + C_2 n) r^n$

$a_n^{(P)} = Cn^2 r^n$, C is a constant.

The solution is $a_n = a_n^{(h)} + a_n^{(P)}$

1) Solve the recurrence relation

$$a_n - 3a_{n-1} = 5(7^n), \quad n \geq 1, \text{ and } a_0 = 2$$

Solution

$$a_n = 3a_{n-1}$$

$$a_n^{(h)} = c_0(3^n)$$

Since $f(n) = 5(7^n)$, take $a_n^{(p)} = A(7^n)$

$\cancel{a_n^{(p)} = A}$

$$a_{n-1}^{(p)} = A(7^{n-1})$$

Equation becomes $A(7^n) - 3(A(7^{n-1})) = 5(7^n)$

$$A\,7^{n-1}(7-3) = 5\,7^{n-1}.7$$

$$4A = 35$$

$$A = \frac{35}{4}$$

$$a_n^{(p)} = \frac{35}{4}(7^n)$$

$$\therefore \; a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = c_0(3^n) + \frac{35}{4}(7^n)$$

Given $a_0 = 2.$

$$a_0 = c_0(3^0) + \frac{35}{4}(7^0)$$

$$a_0 = c_0 + \frac{35}{4} = 2$$

$$c_0 = 2 - \frac{35}{4} = \frac{-27}{4}$$

$$a_n = \frac{-27}{4}(3^n) + \frac{35}{4}(7^n)$$

(2) Solve the recurrence relation

$$a_n - 3a_{n-1} = 5(3^n), \quad n \geq 1, \quad a_0 = 2$$

Solution

$$a_n - 3a_{n-1} = 0$$

$$a_n = 3a_{n-1}$$

$$a_n^{(h)} = c_0(3^n)$$

Since $f(n) = 5(3^n)$. take $a_n^{(p)} = Bn(3^n)$

$$a_{n-1}^{(p)} = B(n-1)(3^{n-1})$$

Equation becomes $Bn(3^n) - 3B(n-1)(3^{n-1}) = 5(3^n)$

$$3^{n-1}(Bn3 - 3B(n-1)) = 5 3^{n-1} 3$$

$$3B(n-n+1) = 15$$

$$B = 5$$

$$a_n^{(p)} = 5n(3^n)$$

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = c_0(3^n) + 5n(3^n).$$

Given $a_0 = 2$

$$a_0 = c_0(3^0) + 0 = 2$$

$$c_0 = 2$$

$$a_n = 2(3^n) + 5n(3^n)$$

$$a_n = (2+5n)(3^n)$$

(3) (~~5~~) Solve the recurrence relation

$$a_{n+2} + 3a_{n+1} + 2a_n = 3^n, \quad n \geq 0, \quad a_0 = 0, \; a_1 = 1$$

Solution

Let $a_n = cr^n$ be the solution of the homogeneous equation $a_{n+2} + 3a_{n+1} + 2a_n = 0$.

$$cr^{n+2} + 3cr^{n+1} + 2cr^n = 0$$

$$r^n(cr^2 + 3cr + 2c) = 0$$

$$r^2 + 3r + 2 = 0$$

$$(r+1)(r+2) = 0$$

$$r = -1, -2$$

$$\therefore a_n^{(h)} = A(-1)^n + B(-2)^n$$

Since $f(n) = 3^n$, take $a_n^{(p)} = D(3^n)$

$$D(3^{n+2}) + 3D(3^{n+1}) + 2D(3^n) = 3^n$$

$$3^n[9D + 9D + 2D] = 3^n$$

$$20D = 1$$

$$D = \frac{1}{20}$$

$$\therefore a_n^{(p)} = \frac{1}{20}(3^n)$$

$$\therefore a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = A(-1)^n + B(-2)^n + \frac{1}{20}(3^n).$$

(4) Solve the recurrence relation

$$a_{n+2} - 8a_{n+1} + 16a_n = 8(5^n) + 6(4^n), \quad n \geq 0$$

$a_0 = 12, \ a_1 = 5$

<u>Solution</u>

Let $a_n = c\gamma^n$ be the solution of the homogeneous equation $a_{n+2} - 8a_{n+1} + 16a_n = 0$, Then $a_{n+1} = c\gamma^{n+1}$ and $a_{n+2} = c\gamma^{n+2}$

$$\therefore \quad c\gamma^{n+2} - 8c\gamma^{n+1} + 16c\gamma^n = 0$$

$$c\gamma^n[\gamma^2 - 8\gamma + 16] = 0$$

$$\gamma^2 - 8\gamma + 16 = 0$$

$$(\gamma - 4)^2 = 0$$

$$\gamma = 4, 4$$

$$\therefore \ a_n^{(h)} = A(4^n) + Bn(4^n)$$

Here $f(n) = 8(5^n) + 6(4^n)$, take

$$a_n^{(P)} = c(5^n) + Dn^2(4^n)$$

$$a_{n+1}^{(P)} = c(5^{n+1}) + D(n+1)^2(4^{n+1})$$

$$a_{n+2}^{(P)} = c(5^{n+2}) + D(n+2)^2(4^{n+2})$$

Substituting in the equation.

$$c(5^{n+2}) + D(n+2)^2(4^{n+2}) - 8\left[c(5^{n+1}) + D(n+1)^2(4^{n+1})\right]$$

$$+16\left[c(5^n) + Dn^2(4^n)\right] = 8(5^n) + 6(4^n)$$

Comparing coefficients of (5^n) and (4^n)

$(5^n)\left[\cancel{c\,25 + D(n+2)^2}\right.$

$$(5^n)\left[25C\cancel{*}-40C+16C\right]=8(5^n)$$

$$25C-40C+16C=8$$

$$C=8$$

$$(4^n)\left[D(n+2)^2 16-8(n+1)^2 D(4)+16Dn^2\right]=6(4^n)$$

$$D\left[16n^2+64n+64-32n^2-64n-32+16n^2\right]=6$$

$$D(32)=6$$

$$D=\frac{6}{32}=\frac{3}{16}$$

$$a_n=a_n^{(h)}+a_n^{(P)}$$

$$a_n=A\,4^n+Bn(4^n)+8(5^n)+\frac{3}{16}n^2(4^n)$$

Given $a_0=12$

$$a_0=A+8=12$$

$$A=4$$

Given $a_1=5$

$$a_1=4A+4B+40+\frac{3}{4}=5$$

$$4B+56+\frac{3}{4}=5$$

$$4B=-51+\frac{3}{4}=\frac{-207}{4}$$

$$B=-\frac{207}{16}$$

$$\therefore a_n=4(4^n)-\frac{207}{16}n(4^n)+8(5^n)+\frac{3}{16}n^2(4^n)$$

(5) (b) Solve the recurrence relation

$a_{n+2} + 4a_{n+1} + 4a_n = 7, \quad n \geq 0, \quad a_0 = 1, a_1 = 2.$

Solution

Let $a_n = cr^n$ be the solution of the homogeneous equation $a_{n+2} + 4a_{n+1} + 4a_n = 0$.

$$cr^{n+2} + 4cr^{n+1} + 4cr^n = 0$$

$$cr^n(r^2 + 4r + 4) = 0$$

$$r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0$$

$$r = -2, -2$$

$$a_n^{(h)} = \cancel{c_1 +} A(-2)^n + Bn(-2)^n$$

Since $f(n) = 7$, take $a_n^{(p)} = \cancel{[illegible]}\ C$

$$\cancel{C(n+2) + 4C(n+1) + 4Cn = 7}$$

$$C + 4C + 4C = 7$$

$$9C = 7$$

$$C = 7/9$$

$$\therefore a_n = a_n^{(h)} + a_n^{(p)}.$$

$$a_n = A(-2)^n + Bn(-2)^n + 7/9.$$

(6) ~~(3)~~ Solve $a_{n+2} - 4a_{n+1} + 3a_n = -200$, $n \geq 0$, given

$a_0 = 3000$, $a_1 = 3300$.

Solution

If $a_n^{(h)} = c\gamma^n$ be the solution of the homogeneous ~~equation~~ relation $a_{n+2} - 4a_{n+1} + 3a_n = 0$

$$a_{n+1}^{(h)} = c\gamma^{n+1}, \quad a_{n+2}^{(h)} = c\gamma^{n+2}.$$

$$c\gamma^{n+2} - 4c\gamma^{n+1} + 3c\gamma^n = 0$$

$$c\gamma^n(\gamma^2 - 4\gamma + 3) = 0$$

$$\gamma = 3, 1.$$

Hence $a_n^{(h)} = c_1(3^n) + c_2(1^n)$

$= c_1(3^n) + c_2$

Since $f(n) = -200 = -200(1^n)$ is a solution of the homogeneous ~~equation~~ relation, take

$a_n^{(p)} = An$ for some constant A.

$A(n+2) - 4A(n+1) + 3An = -200$

$-2A = -200$

$A = 100$

Hence $a_n = c_1(3^n) + c_2 + 100n$.

$a_0 = c_1 + c_2 = 3000$ —— (1)

$a_1 = 3c_1 + c_2 + 100 = 3300$

$3c_1 + c_2 = 3200$ —— (2)

$(2) - (1) \Rightarrow \quad 2c_1 = 200$

$c_1 = 100$

$c_2 = 3000 - 100 = 2900$

$a_n = 100(3^n) + 2900 + 100n.$

Exercises

Solve the recurrence relation

(1) $a_{n+1} - a_n = 2n+3, \quad n \geq 0, \; a_0 = 1$

(2) $a_{n+1} - a_n = 3n^2 - n, \quad n \geq 0, \; a_0 = 3$

(3) $a_{n+1} - 2a_n = 5, \quad n \geq 0, \; a_0 = 1$

(4) $a_{n+1} - 2a_n = 2^n, \quad n \geq 0, \; a_0 = 1.$

(5) $\cancel{a_{n+2} +} \; a_{n+2} - 6a_{n+1} + 9a_n = 3(2^n) + 7(3^n)$