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Progressions

CONTENTS

Leonardo Fibonacci

According to Boethius (510 A.D.) *arithmetic, Geometric and Harmonic sequences were known to early Greek writers. Among the Indian mathematician;* Aryabhatta (476 A.D.) *was the first to give the formula for the sum of squares and cubes of natural numbers in his famous work* Aryabhatiyam.

Another special type of sequence having important applications in mathematics, called Fibonacci *sequence, was discovered by Italian Mathematician* Leonardo Fibonacci (1170-1250 A.D.) *The general series was given by* Frenchman Francois-vieta (1540-1603 A.D.)
It was only through the rigorous developed of algebraic and set theoretic tools that the concepts related to sequence and series could be formulated suitably.

PROGRESSIONS

1.1 Introduction

(1) **Sequence :** A sequence is a function whose domain is the set of natural numbers, N.

If $f : N \to C$ is a sequence, we usually denote it by $< f(n) > = < f(1), f(2), f(3), ... >$

It is not necessary that the terms of a sequence always follow a certain pattern or they are described by some explicit formula for the n^{th} term. Terms of a sequence are connected by commas. *Example* : 1, 1, 2, 3, 5, 8, ... is a sequence.

(2) **Series :** By adding or subtracting the terms of a sequence, we get a series.

If $t_1, t_2, t_3, ... t_n, ...$ is a sequence, then the expression $t_1 + t_2 + t_3 + ... + t_n ....$ is a series.

A series is finite or infinite as the number of terms in the corresponding sequence is finite or infinite.

Example : $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ...$ is a series.

(3) **Progression :** A progression is a sequence whose terms follow a certain pattern *i.e.* the terms are arranged under a definite rule.

Example : 1, 3, 5, 7, 9, ... is a progression whose terms are obtained by the rule : $T_n = 2n - 1$, where T_n denotes the n^{th} term of the progression.

Progression is mainly of three types : Arithmetic progression, Geometric progression and Harmonic progression.

However, here we have classified the study of progression into five parts as :

- Arithmetic progression
- Geometric progression
- Arithmetico-geometric progression
- Harmonic progression
- Miscellaneous progressions

Arithmetic progression(A.P)

1.2 Definition

A sequence of numbers $< t_n >$ is said to be in arithmetic progression (A.P.) when the difference $t_n - t_{n-1}$ is a constant for all $n \in N$. This constant is called the common difference of the A.P., and is usually denoted by the letter d.

If 'a' is the first term and 'd' the common difference, then an A.P. can be represented as $a, a + d, a + 2d, a + 3d, ...$

Example : 2, 7, 12, 17, 22, ... is an A.P. whose first term is 2 and common difference 5.

Algorithm to determine whether a sequence is an A.P. or not.

Step I: Obtain a_n (the n^{th} term of the sequence).

Step II: Replace n by $n - 1$ in a_n to get a_{n-1}.

Step III: Calculate $a_n - a_{n-1}$.

If $a_n - a_{n-1}$ is independent of n, the given sequence is an A.P. otherwise it is not an A.P. An arithmetic progression is a linear function with domain as the set of natural numbers N.

$\therefore\ t_n = An + B$ represents the n^{th} term of an A.P. with common difference A.

1.3 General Term of an A.P.

(1) Let 'a' be the first term and 'd' be the common difference of an A.P. Then its n^{th} term is $a+(n-1)d$.

$$\boxed{T_n = a+(n-1)d}$$

(2) **p^{th} term of an A.P. from the end :** Let 'a' be the first term and 'd' be the common difference of an A.P. having n terms. Then p^{th} term from the end is $(n-p+1)^{th}$ term from the beginning.

$$\boxed{p^{th} \text{ term from the end} = T_{(n-p+1)} = a+(n-p)d}$$

Important Tips

- *General term (T_n) is also denoted by l (last term).*
- *Common difference can be zero, +ve or –ve.*
- *n (number of terms) always belongs to set of natural numbers.*
- *If T_k and T_p of any A.P. are given, then formula for obtaining T_n is $\frac{T_n - T_k}{n-k} = \frac{T_p - T_k}{p-k}$.*
- *If $pT_p = qT_q$ of an A.P., then $T_{p+q} = 0$.*
- *If p^{th} term of an A.P. is q and the q^{th} term is p, then $T_{p+q} = 0$ and $T_n = p + q - n$.*
- *If the p^{th} term of an A.P. is $\frac{1}{q}$ and the q^{th} term is $\frac{1}{p}$, then its pq^{th} term is 1.*
- *If $T_n = pn + q$, then it will form an A.P. of common difference p and first term p + q.*

Example: 1 Let T_r be rth term of an A.P. whose first term is a and common difference is d. If for some positive integers $m, n, m \neq n$, $T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$, then $a - d$ equals

(a) $\frac{1}{m}+\frac{1}{n}$ (b) 1 (c) $\frac{1}{mn}$ (d) 0

Solution: (d) $T_m = \frac{1}{n} \Rightarrow a+(m-1)d = \frac{1}{n}$ .....(i)

and $T_n = \frac{1}{m} \Rightarrow a+(n-1)d = \frac{1}{m}$ .....(ii)

Subtract (ii) from (i), we get $(m-n)d = \frac{1}{n}-\frac{1}{m} \Rightarrow (m-n)d = \frac{(m-n)}{mn} \Rightarrow d = \frac{1}{mn}$, as $m-n \neq 0$

$a = \frac{1}{m}-(n-1)d = \frac{1}{m}-\frac{n-1}{mn} = \frac{1}{mn} = d$. Therefore $a - d = 0$

Example: 2 The 19th term from the end of the series 2 + 6 + 10 + …. + 86 is

(a) 6 (b) 18 (c) 14 (d) 10

Solution: (c) $86 = 2+(n-1)4 \Rightarrow n = 22$

19^{th} term from end $= t_{n-19+1} = t_{22-19+1} = t_4 = 2+(4-1)4 = 14$

Example: 3 In a certain A.P., 5 times the 5^{th} term is equal to 8 times the 8^{th} term, then its 13^{th} term is

(a) 0 (b) – 1 (c) – 12 (d) – 13

Solution: (a) We have $5T_5 = 8T_8$

Let a and d be the first term and common difference respectively

$\therefore\ 5\{a+(5-1)d\} = 8\{a+(8-1)d\}$

$\Rightarrow 3a+36d = 0 \Rightarrow a+12d = 0$, *i.e.* $a+(13-1)d = 0$. Hence 13^{th} term = 0

Example: 4 If 7^{th} and 13^{th} term of an A.P. be 34 and 64 respectively, then its 18^{th} term is

(a) 87 (b) 88 (c) 89 (d) 90

Solution: (c) Let a be the first term and d be the common difference of the given A.P., then

$T_7 = 34 \Rightarrow a + 6d = 34$ .....(i)

$T_{13} = 64 \Rightarrow a + 12d = 64$ .....(ii)

From (i) and (ii), $d = 5$, $a = 4$

$\therefore T_{18} = a + 17d = 4 + 17 \times 5 = 89$

Trick: $\frac{T_n - T_k}{n-k} = \frac{T_p - T_k}{p-k} \Rightarrow \frac{T_{18} - T_7}{18-7} = \frac{T_{13} - T_7}{13-7} \Rightarrow \frac{T_{18} - 34}{11} = \frac{64-34}{6} \Rightarrow T_{18} = 89$

Example: 5 If $< a_n >$ is an arithmetic sequence, then $\Delta = \begin{vmatrix} a_m & a_n & a_p \\ m & n & p \\ 1 & 1 & 1 \end{vmatrix}$ equals

(a) 1 (b) –1 (c) 0 (d) None of these

Solution: (c) Let a be the first term and d the common difference. Then $a_r = a + (r-1)d$

$$\Delta = \begin{vmatrix} a+(m-1)d & a+(n-1)d & a+(p-1)d \\ m & n & p \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a & a & a \\ m & n & p \\ 1 & 1 & 1 \end{vmatrix} + d\begin{vmatrix} m-1 & n-1 & p-1 \\ m & n & p \\ 1 & 1 & 1 \end{vmatrix}$$

$$= a\begin{vmatrix} 1 & 1 & 1 \\ m & n & p \\ 1 & 1 & 1 \end{vmatrix} + d\begin{vmatrix} m & n & p \\ m & n & p \\ 1 & 1 & 1 \end{vmatrix} = a.0 + d.0 = 0$$

Example: 6 The n^{th} term of the series 3 + 10 + 17 + ... and 63 + 65 + 67 + ... are equal, then the value of n is

(a) 11 (b) 12 (c) 13 (d) 15

Solution: (c) n^{th} term of 1^{st} series $= 3 + (n-1)7 = 7n - 4$

n^{th} term of 2^{nd} series $= 63 + (n-1) = 2n + 61$

$\therefore$ we have, $7n - 4 = 2n + 61 \Rightarrow n = 13$

1.4 Selection of Terms in an A.P.

When the sum is given, the following way is adopted in selecting certain number of terms :

Number of terms	Terms to be taken
3	$a-d, a, a+d$
4	$a-3d, a-d, a+d, a+3d$
5	$a-2d, a-d, a, a+d, a+2d$

In general, we take $a - rd, a - (r-1)d, ..., a - d, a, a + d, ..., a + (r-1)d, a + rd$, in case we have to take $(2r + 1)$ terms (*i.e.* odd number of terms) in an A.P.

And, $a-(2r-1)d, a-(2r-3)d, ..., a-d, a+d, ..., a+(2r-1)d$, in case we have to take $2r$ terms in an A.P.

When the sum is not given, then the following way is adopted in selection of terms.

Number of terms	Terms to be taken
3	$a, a+d, a+2d$
4	$a, a+d, a+2d, a+3d$
5	$a, a+d, a+2d, a+3d, a+4d$

Sum of *n* terms of an A.P. : The sum of n terms of the series $a+(a+d)+(a+2d)+.......+\{a+(n-1)d\}$ is given by

$$\boxed{S_n = \frac{n}{2}[2a+(n-1)d]}$$

Also, $S_n = \frac{n}{2}(a+l)$, where l = last term $= a+(n-1)d$

Important Tips

- *The common difference of an A.P is given by* $d = S_2 - 2S_1$ *where* S_2 *is the sum of first two terms and* S_1 *is the sum of first term or the first term.*
- *The sum of infinite terms* $= \begin{cases} \infty, & \text{when } d>0 \\ -\infty, & \text{when } d<0 \end{cases}$.
- *If sum of n terms* S_n *is given then general term* $T_n = S_n - S_{n-1}$, *where* S_{n-1} *is sum of (n – 1) terms of A.P.*
- *Sum of n terms of an A.P. is of the form* An^2+Bn *i.e. a quadratic expression in n, in such case, common difference is twice the coefficient of* n^2 *i.e. 2A.*
- • *If for the different A.P's* $\frac{S_n}{S'_n} = \frac{f_n}{¤_n}$, *then* $\frac{T_n}{T'_n} = \frac{f(2n-1)}{¤(2n-1)}$
  • *If for two A.P.'s* $\frac{T_n}{T'_n} = \frac{An+B}{Cn+D}$ *then* $\frac{S_n}{S'_n} = \frac{A\left(\frac{n+1}{2}\right)+B}{C\left(\frac{n+1}{2}\right)+D}$
- *Some standard results*
  - *Sum of first n natural numbers* $= 1+2+3+........+n = \sum_{r=1}^{n} r = \frac{n(n+1)}{2}$
  - *Sum of first n odd natural numbers* $= 1+3+5+.....+(2n-1) = \sum_{r=1}^{n}(2r-1) = n^2$
  - *Sum of first n even natural numbers* $= 2+4+6+......+2n = \sum_{r=1}^{n} 2r = n(n+1)$
- • *If for an A.P. sum of p terms is q and sum of q terms is p, then sum of (p + q) terms is {–(p + q)}.*
  • *If for an A.P., sum of p terms is equal to sum of q terms, then sum of (p + q) terms is zero.*
  • *If the* p^{th} *term of an A.P. is* $\frac{1}{q}$ *and* q^{th} *term is* $\frac{1}{p}$, *then sum of pq terms is given by* $S_{pq} = \frac{1}{2}(pq+1)$

Example: 7 7^{th} term of an A.P. is 40, then the sum of first 13 terms is

(a) 53 (b) 520 (c) 1040 (d) 2080

Solution: (b) $S_{13} = \frac{13}{2}\{2a+12d\} = 13\{a+6d\} = 13\times T_7 = 13\times 40 = 520$

Example: 8 The first term of an A.P. is 2 and common difference is 4. The sum of its 40 terms will be

(a) 3200 (b) 1600 (c) 200 (d) 2800

Solution: (a) $S = \frac{n}{2}[2a+(n-1)d] = \frac{40}{2}[2\times 2+(40-1)4] = 3200$

Example: 9 The sum of the first and third term of an A.P. is 12 and the product of first and second term is 24, the first term is

(a) 1 (b) 8 (c) 4 (d) 6

Solution: (c) Let $a-d, a, a+d, ...$ be an A.P.

$\therefore$ $(a-d)+(a+d) = 12 \Rightarrow a = 6$. Also, $(a-d)a = 24 \Rightarrow 6-d = \frac{24}{6} = 4 \Rightarrow d = 2$

$\therefore$ First term $= a-d = 6-2 = 4$

Example: 10 If S_r denotes the sum of the first r terms of an A.P., then $\dfrac{S_{3r}-S_{r-1}}{S_{2r}-S_{2r-1}}$ is equal to

(a) $2r-1$ (b) $2r+1$ (c) $4r+1$ (d) $2r+3$

Solution: (b) $\dfrac{S_{3r}-S_{r-1}}{S_{2r}-S_{2r-1}}=\dfrac{\frac{3r}{2}\{2a+(3r-1)d\}-\frac{(r-1)}{2}\{2a+(r-1-1)d\}}{T_{2r}}=\dfrac{(2r+1)a+\frac{d}{2}\{3r(3r-1)-(r-1)(r-2)\}}{a+(2r-1)d}$

$$=\frac{(2r+1)a+\frac{d}{2}\{8r^2-2\}}{a+(2r-1)d}=\frac{(2r+1)a+d(4r^2-1)}{a+(2r-1)d}=2r+1$$

Example: 11 If the sum of the first $2n$ terms of 2, 5, 8…. is equal to the sum of the first n terms of 57, 59, 61…., then n is equal to

(a) 10 (b) 12 (c) 11 (d) 13

Solution: (c) We have, $\frac{2n}{2}\{2\times 2+(2n-1)3\}=\frac{n}{2}\{2\times 57+(n-1)2\}\Rightarrow 6n+1=n+56\Rightarrow n=11$

Example: 12 If the sum of the 10 terms of an A.P. is 4 times to the sum of its 5 terms, then the ratio of first term and common difference is

(a) 1 : 2 (b) 2 : 1 (c) 2 : 3 (d) 3 : 2

Solution: (a) Let a be the first term and d the common difference

Then, $\frac{10}{2}\{\{a+(10-1)d\}=4\times\frac{5}{2}\{2a+(5-1)d\}\Rightarrow 2a+9d=4a+8d\Rightarrow d=2a\Rightarrow\frac{a}{d}=\frac{1}{2}$, $\therefore a:d=1:2$

Example: 13 150 workers were engaged to finish a piece of work in a certain number of days. 4 workers dropped the second day, 4 more workers dropped the third day and so on. It takes eight more days to finish the work now. The number of days in which the work was completed is

(a) 15 (b) 20 (c) 25 (d) 30

Solution: (c) Let the work was to be finished in x days. $\therefore$ Work of 1 worker in a day $=\frac{1}{150x}$

Now the work will be finished in $(x+8)$ days. $\therefore$ Work done = Sum of the fraction of work done

$1=\frac{1}{150x}\times 150+\frac{1}{150x}(150-4)+\frac{1}{150x}(150-8)+\ldots\ldots$ to $(x+8)$ terms

$\Rightarrow 1=\frac{x+8}{2}\left\{2\times\frac{150}{150x}+(x+8-1)\left(\frac{-4}{150x}\right)\right\}\Rightarrow 150x=(x+8)\{150-2(x+7)\}\Rightarrow (x+8)(x+7)-600=0$

$\Rightarrow (x+8)(x+7)=25\times 24$, $\therefore x+8=25$

Hence work completed in 25 days.

Example: 14 If the sum of first p terms, first q terms and first r terms of an A.P. be x, y and z respectively, then $\frac{x}{p}(q-r)+\frac{y}{q}(r-p)+\frac{z}{r}(p-q)$ is

(a) 0 (b) 2 (c) pqr (d) $\dfrac{8xyz}{pqr}$

Solution: (a) We have a, the first term and d, the common difference, $x=\{2a+(p-1)d\}\frac{p}{2}\Rightarrow\frac{x}{p}=a+(p-1)\frac{d}{2}$

Similarly, $\frac{y}{q}=a+(q-1)\frac{d}{2}$ and $\frac{z}{r}=a+(r-1)\frac{d}{2}$

$\therefore\ \frac{x}{p}(q-r)+\frac{y}{q}(r-p)+\frac{z}{r}(p-q)=\left\{a+(p-1)\frac{d}{2}\right\}(q-r)+\left\{a+(q-1)\frac{d}{2}\right\}(r-p)+\left\{a+(r-1)\frac{d}{2}\right\}(p-q)$

$=a\{(q-r)+(r-p)+(p-q)\}+\frac{d}{2}\{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)\}$

$=a..0+\frac{d}{2}[\{pq-pr+rq-pq+pr-qr-\{(q-r)+(r-p)+(p-q)\}=0+\frac{d}{2}\{0-0\}=0$

Example: 15 The sum of all odd numbers of two digits is

(a) 2475 (b) 2530 (c) 4905 (d) 5049

Solution: (a) Required sum, $S = 11 + 13 + 15 + ... + 99$

Let the number of odd terms be n, then $99 = 11 + (n-1)2 \Rightarrow n = 45$

$\therefore\ S = \frac{45}{2}(11 + 99) = 45 \times 55 = 2475$ $\left[\because S = \frac{n}{2}(a + l)\right]$

Example: 16 If sum of n terms of an A.P. is $3n^2 + 5n$ and $T_m = 164$, then $m =$

(a) 26 (b) 27 (c) 28 (d) None of these

Solution: (b) $T_m = S_m - S_{m-1} \Rightarrow 164 = (3m^2 + 5m) - \{3(m-1)^2 + 5(m-1)\} \Rightarrow 164 = 3(2m-1) + 5 \Rightarrow m = 27$

Example: 17 The sum of n terms of the series $\frac{1}{1+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{7}} + ...$ is

(a) $\sqrt{2n+1}$ (b) $\frac{1}{2}\sqrt{2n+1}$ (c) $\sqrt{2n-1}$ (d) $\frac{1}{2}(\sqrt{2n+1} - 1)$

Solution: (d)
$$S_n = \frac{1}{1+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{7}} + ... + \frac{1}{\sqrt{2n-1}+\sqrt{2n+1}}$$
$$= \frac{\sqrt{3}-1}{(\sqrt{3}-1)(\sqrt{3}+1)} + \frac{\sqrt{5}-\sqrt{3}}{2} + \frac{\sqrt{7}-\sqrt{5}}{2} + ... + \frac{\sqrt{2n+1}-\sqrt{2n-1}}{2}$$
$$= \frac{1}{2}[\sqrt{3} - 1 + \sqrt{5} - \sqrt{3} + \sqrt{7} - \sqrt{5} + ... + (\sqrt{2n+1} - \sqrt{2n-1})] = \frac{1}{2}[\sqrt{2n+1} - 1]$$

Example: 18 If $a_1, a_2, ..., a_{n+1}$ are in A.P., then $\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + ... + \frac{1}{a_na_{n+1}}$ is

(a) $\frac{n-1}{a_1a_{n+1}}$ (b) $\frac{1}{a_1a_{n+1}}$ (c) $\frac{n+1}{a_1a_{n+1}}$ (d) $\frac{n}{a_1a_{n+1}}$

Solution: (d)
$$S = \frac{1}{a_1a_2} + \frac{1}{a_2a_3} + ... + \frac{1}{a_na_{n+1}} = \frac{\left(\frac{1}{a_1} - \frac{1}{a_2}\right)}{(a_2 - a_1)} + \frac{\left(\frac{1}{a_2} - \frac{1}{a_3}\right)}{(a_3 - a_2)} + ... + \frac{\left(\frac{1}{a_n} - \frac{1}{a_{n+1}}\right)}{(a_{n+1} - a_n)}$$

As $a_1, a_2, a_3, ..., a_n, a_{n+1}$ are in A.P., *i.e.* $a_2 - a_1 = a_3 - a_2 = ... = a_{n+1} - a_n = d$ (say)
$$\therefore\ S = \frac{1}{d}\left[\left(\frac{1}{a_1} - \frac{1}{a_2}\right) + \left(\frac{1}{a_2} - \frac{1}{a_3}\right) + ... + \left(\frac{1}{a_n} - \frac{1}{a_{n+1}}\right)\right] = \frac{1}{d}\left[\frac{1}{a_1} - \frac{1}{a_{n+1}}\right] = \frac{a_{n+1} - a_1}{d\,.\,a_1\,.\,a_{n+1}} = \frac{[a_1 + (n+1-1)\,d] - a_1}{d\,.\,a_1.\,a_{n+1}}$$
$$S = \frac{nd}{d\,a_1\,a_{n+1}} = \frac{n}{a_1\,a_{n+1}}$$

1.5 Arithmetic Mean

(1) **Definitions**

(i) If three quantities are in A.P. then the middle quantity is called Arithmetic mean (A.M.) between the other two. If a, A, b are in A.P., then A is called A.M. between a and b.

(ii) If $a, A_1, A_2, A_3, ..., A_n, b$ are in A.P., then $A_1, A_2, A_3, ..., A_n$ are called n A.M.'s between a and b.

(2) **Insertion of arithmetic means**

(i) **Single A.M. between a and b :** If a and b are two real numbers then single A.M. between a and b $= \frac{a+b}{2}$

(ii) **n A.M.'s between a and b :** If $A_1, A_2, A_3, ..., A_n$ are n A.M.'s between a and b, then

$$A_1 = a + d = a + \frac{b-a}{n+1},\ A_2 = a + 2d = a + 2\frac{b-a}{n+1},\ A_3 = a + 3d = a + 3\frac{b-a}{n+1},\ ...., A_n = a + nd = a + n\frac{b-a}{n+1}$$

Important Tips

- *Sum of n A.M.'s between a and b is equal to n times the single A.M. between a and b.*

  i.e. $A_1 + A_2 + A_3 + \dots\dots\dots + A_n = n\left(\frac{a+b}{2}\right)$

- *If* A_1 *and* A_2 *are two A.M.'s between two numbers a and b, then* $A_1 = \frac{1}{3}(2a+b), A_2 = \frac{1}{3}(a+2b)$.

- *Between two numbers,* $\frac{\text{Sum of } m \text{ A.M.'s}}{\text{Sum of } n \text{ A.M.'s}} = \frac{m}{n}$.

- *If number of terms in any series is odd, then only one middle term exists which is* $\left(\frac{n+1}{2}\right)^{th}$ *term.*

- *If number of terms in any series is even then there are two middle terms, which are given by* $\left(\frac{n}{2}\right)^{th}$ *and* $\left\{\left(\frac{n}{2}\right)+1\right\}^{th}$ *term.*

Example: 19 After inserting n A.M.'s between 2 and 38, the sum of the resulting progression is 200. The value of n is

(a) 10 (b) 8 (c) 9 (d) None of these

Solution: (b) There will be $(n + 2)$ terms in the resulting A.P. $2, A_1, A_2, \dots\dots, A_n, 38$

Sum of the progression $= \frac{n+2}{2}(2+38) \Rightarrow 200 = (n+2)\times 20 \Rightarrow n = 8$

Example: 20 3 A.M.'s between 3 and 19 are

(a) 7, 11, 15 (b) 4, 6, 10 (c) 6, 10, 14 (d) None of these

Solution: (a) Let A_1, A_2, A_3 be three A.M.'s. Then $3, A_1, A_2, A_3, 19$ are in A.P.

$\Rightarrow$ common difference $d = \frac{19-3}{3+1} = 4$ .Therefore $A_1 = 3 + d = 7$, $A_2 = 3 + 2d = 11$, $A_3 = 3 + 3d = 15$

Example: 21 If a, b, c, d, e, f are A.M.'s between 2 and 12, then $a+b+c+d+e+f$ is equal to

(a) 14 (b) 42 (c) 84 (d) None of these

Solution: (b) Since, a, b, c, d, e, f are six A.M.'s between 2 and 12

Therefore, $a+b+c+d+e+f = \frac{6}{2}(a+f) = \frac{6}{2}(2+12) = 42$

1.6 Properties of A.P.

(1) If $a_1, a_2, a_3 \dots\dots$ are in A.P. whose common difference is d, then for fixed non-zero number $K \in R$.

(i) $a_1 \pm K, a_2 \pm K, a_3 \pm K, \dots\dots$ will be in A.P., whose common difference will be d.

(ii) $Ka_1, Ka_2, Ka_3 \dots\dots\dots$ will be in A.P. with common difference $= Kd$.

(iii) $\frac{a_1}{K}, \frac{a_2}{K}, \frac{a_3}{K} \dots\dots$ will be in A.P. with common difference $= d/K$.

(2) The sum of terms of an A.P. equidistant from the beginning and the end is constant and is equal to sum of first and last term. *i.e.* $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \dots.$

(3) Any term (except the first term) of an A.P. is equal to half of the sum of terms equidistant from the term *i.e.* $a_n = \frac{1}{2}(a_{n-k} + a_{n+k})$, $k < n$.

(4) If number of terms of any A.P. is odd, then sum of the terms is equal to product of middle term and number of terms.

(5) If number of terms of any A.P. is even then A.M. of middle two terms is A.M. of first and last term.

(6) If the number of terms of an A.P. is odd then its middle term is A.M. of first and last term.

(7) If $a_1, a_2, ...a_n$ and $b_1, b_2, ...b_n$ are the two A.P.'s. Then $a_1 \pm b_1, a_2 \pm b_2, ...a_n \pm b_n$ are also A.P.'s with common difference $d_1 \neq d_2$, where d_1 and d_2 are the common difference of the given A.P.'s.

(8) Three numbers *a*, *b*, *c* are in A.P. iff $2b = a + c$.

(9) If T_n, T_{n+1} and T_{n+2} are three consecutive terms of an A.P., then $2T_{n+1} = T_n + T_{n+2}$.

(10) If the terms of an A.P. are chosen at regular intervals, then they form an A.P.

Example: 22 If $a_1, a_2, a_3, ..., a_{24}$ are in arithmetic progression and $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$, then $a_1 + a_2 + a_3 + ... + a_{23} + a_{24} =$

(a) 909 (b) 75 (c) 750 (d) 900

Solution: (d) $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225 \Rightarrow (a_1 + a_{24}) + (a_5 + a_{20}) + (a_{10} + a_{15}) = 225 \Rightarrow 3(a_1 + a_{24}) = 225 \Rightarrow a_1 + a_{24} = 75$

($\because$ In an A.P. the sum of the terms equidistant from the beginning and the end is same and is equal to the sum of first and last term)

$a_1 + a_2 + ... + a_{24} = \frac{24}{2}(a_1 + a_{24}) = 12 \times 75 = 900$

Example: 23 If *a*, *b*, *c* are in A.P., then $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ will be in

(a) A.P. (b) G.P. (c) H.P. (d) None of these

Solution: (a) *a*, *b*, *c* are in A.P., $\Rightarrow \frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ will be in A.P. [Dividing each term by *abc*]

Example: 24 If $\log 2, \log(2^n - 1)$ and $\log(2^n + 3)$ are in A.P., then $n =$

(a) 5/2 (b) $\log_2 5$ (c) $\log_3 5$ (d) $\frac{3}{2}$

Solution: (b) As, $\log 2, \log(2^n - 1)$ and $\log(2^n + 3)$ are in A.P. Therefore

$2\log(2^n - 1) = \log 2 + \log(2^n + 3) \Rightarrow (2^n - 5)(2^n + 1) = 0$

As 2^n cannot be negative, hence $2^n - 5 = 0 \Rightarrow 2^n = 5$ or $n = \log_2 5$

Geometric progression(G.P.)

1.7 Definition

A progression is called a G.P. if the ratio of its each term to its previous term is always constant. This constant ratio is called its common ratio and it is generally denoted by *r*.

Example: The sequence 4, 12, 36, 108, ….. is a G.P., because $\frac{12}{4} = \frac{36}{12} = \frac{108}{36} = ... = 3$, which is constant.

Clearly, this sequence is a G.P. with first term 4 and common ratio 3.

The sequence $\frac{1}{3}, -\frac{1}{2}, \frac{3}{4}, -\frac{9}{8}, ...$ is a G.P. with first term $\frac{1}{3}$ and common ratio $\left(-\frac{1}{2}\right) / \left(\frac{1}{3}\right) = -\frac{3}{2}$

1.8 General Term of a G.P.

(1) We know that, $a, ar, ar^2, ar^3, ...ar^{n-1}$ is a sequence of G.P.

Here, the first term is '*a*' and the common ratio is '*r*'.

The general term or n^{th} term of a G.P. is $\boxed{T_n = ar^{n-1}}$

It should be noted that,

$$\boxed{r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = ...}$$

(2) **p^{th} term from the end of a finite G.P. :** If G.P. consists of 'n' terms, p^{th} term from the end $=(n-p+1)^{th}$ term from the beginning $=ar^{n-p}$.

Also, the p^{th} term from the end of a G.P. with last term l and common ratio r is $l\left(\frac{1}{r}\right)^{n-1}$

Important Tips

- *If a, b, c are in G.P.* $\Rightarrow \frac{b}{a}=\frac{c}{b}$ *or* $\boxed{b^2=ac}$
- *If T_k and T_p of any G.P. are given, then formula for obtaining T_n is*

$$\boxed{\left(\frac{T_n}{T_k}\right)^{\frac{1}{n-k}}=\left(\frac{T_p}{T_k}\right)^{\frac{1}{p-k}}}$$

- *If a, b, c are in G.P. then*
  $\Rightarrow \frac{b}{a}=\frac{c}{b} \Rightarrow \frac{a+b}{a-b}=\frac{b+c}{b-c}$ *or* $\frac{a-b}{b-c}=\frac{a}{b}$ *or* $\frac{a+b}{b+c}=\frac{a}{b}$
- *Let the first term of a G.P be positive, then if r > 1, then it is an increasing G.P., but if r is positive and less than 1, i.e. 0< r < 1, then it is a decreasing G.P.*
- *Let the first term of a G.P. be negative, then if r > 1, then it is a decreasing G.P., but if 0< r < 1, then it is an increasing G.P.*
- *If a, b, c, d,… are in G.P., then they are also in continued proportion i.e.* $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=....=\frac{1}{r}$

Example: 25 The numbers $(\sqrt{2}+1), 1, (\sqrt{2}-1)$ will be in

(a) A.P. (b) G.P. (c) H.P. (d) None of these

Solution: (b) Clearly $(1)^2=(\sqrt{2}+1).(\sqrt{2}-1)$

$\therefore\ \sqrt{2}+1, 1, \sqrt{2}-1$ are in G.P.

Example: 26 If the p^{th}, q^{th} and r^{th} term of a G.P. are a, b, c respectively, then $a^{q-r}\cdot b^{r-p}\cdot c^{p-q}$ is equal to

(a) 0 (b) 1 (c) abc (d) pqr

Solution: (b) Let $x, xy, xy^2, xy^3,....$ be a G.P.

$\therefore\ a=xy^{p-1}, b=xy^{q-1}, c=xy^{r-1}$

Now, $a^{q-r}.b^{r-p}.c^{p-q}=(xy^{p-1})^{q-r}(xy^{q-1})^{r-p}(xy^{r-1})^{p-q}=x^{(q-r)+(r-p)+(p-q)}.y^{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)}$

$=x^0.y^{p(q-r)+q(r-p)+r(p-q)-(q-r+r-p+p-q)}=x^0.y^{0-0}=(xy)^0=1$

Example: 27 If the third term of a G.P. is 4 then the product of its first 5 terms is

(a) 4^3 (b) 4^4 (c) 4^5 (d) None of these

Solution: (c) Given that $ar^2=4$

Then product of first 5 terms $=a(ar)(ar^2)(ar^3)(ar^4)=a^5r^{10}=[ar^2]^5=4^5$

Example: 28 If $x, 2x+2, 3x+3$ are in G.P., then the fourth term is

(a) 27 (b) – 27 (c) 13.5 (d) – 13.5

Solution: (d) Given that $x, 2x+2, 3x+3$ are in G.P.

Therefore, $(2x+2)^2=x(3x+3) \Rightarrow x^2+5x+4=0 \Rightarrow (x+4)(x+1)=0 \Rightarrow x=-1,-4$

Now first term $a=x$, second term $ar=2(x+1)$

$\Rightarrow r=\frac{2(x+1)}{x}$, then 4^{th} term $=ar^3=x\left[\frac{2(x+1)}{x}\right]^3=\frac{8}{x^2}(x+1)^3$

Putting $x=-4$, we get

$T_4=\frac{8}{16}(-3)^3=-\frac{27}{2}=-13.5$

1.9 Sum of First '*n*' Terms of a G.P.

If a be the first term, r the common ratio, then sum S_n of first n terms of a G.P. is given by

$$S_n = \frac{a(1-r^n)}{1-r}, \qquad |r| < 1$$

$$S_n = \frac{a(r^n-1)}{r-1}, \qquad |r| > 1$$

$$S_n = na, \qquad r = 1$$

1.10 Selection of Terms in a G.P.

(1) When the product is given, the following way is adopted in selecting certain number of terms :

Number of terms	Terms to be taken
3	$\frac{a}{r}, a, ar$
4	$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$
5	$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

(2) When the product is not given, then the following way is adopted in selection of terms

Number of terms	Terms to be taken
3	a, ar, ar^2
4	a, ar, ar^2, ar^3
5	a, ar, ar^2, ar^3, ar^4

Example: 29 Let a_n be the n^{th} term of the G.P. of positive numbers. Let $\sum_{n=1}^{100} a_{2n} = \sim$ and $\sum_{n=1}^{100} a_{2n-1} =$, such that $\sim \neq$, then the common ratio is

(a) $\frac{\sim}{\ }$ (b) $\frac{\ }{\sim}$ (c) $\sqrt{\frac{\sim}{\ }}$ (d) $\sqrt{\frac{\ }{\sim}}$

Solution: (a) Let x be the first term and y, the common ratio of the G.P.

Then, $\sim = \sum_{n=1}^{100} a_{2n} = a_2 + a_4 + a_6 + ... + a_{200}$ and $= \sum_{n=1}^{100} a_{2n-1} = a_1 + a_3 + a_5 + ... + a_{199}$

$$\Rightarrow \quad \sim = xy + xy^3 + xy^5 + ... + xy^{199} = xy\,\frac{1-(y^2)^{100}}{1-y^2} = xy\left(\frac{1-y^{200}}{1-y^2}\right)$$

$$= x + xy^2 + xy^4 + ... + xy^{198} = x \cdot \frac{1-(y^2)^{100}}{1-y^2} = x \cdot \left(\frac{1-y^{200}}{1-y^2}\right)$$

$\therefore \quad \frac{\sim}{\ } = y$. Thus, common ratio $= \frac{\sim}{\ }$

Example: 30 The sum of first two terms of a G.P. is 1 and every term of this series is twice of its previous term, then the first term will be

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

Solution: (b) We have, common ratio $r = 2$; $\left[\because \frac{a_n}{a_{n-1}} = 2\right]$

Let a be the first term, then $a + ar = 1 \Rightarrow a(1+r) = 1 \Rightarrow a = \frac{1}{1+r} = \frac{1}{1+2} = \frac{1}{3}$

1.11 Sum of Infinite Terms of a G.P.

(1) When $|r| < 1$, (or $-1 < r < 1$)

$$\boxed{S_\infty = \frac{a}{1-r}}$$

(2) If $r \geq 1$, then S_∞ doesn't exist

Example: 31 The first term of an infinite geometric progression is x and its sum is 5. Then

(a) $0 \leq x \leq 10$ (b) $0 < x < 10$ (c) $-10 < x < 0$ (d) $x > 10$

Solution: (b) According to the given conditions, $5 = \frac{x}{1-r}$, r being the common ratio $\Rightarrow r = 1 - \frac{x}{5}$

Now, $|r| < 1$ *i.e.* $-1 < r < 1 \Rightarrow -1 < 1 - \frac{x}{5} < 1 \Rightarrow -2 < -\frac{x}{5} < 0 \Rightarrow 2 > \frac{x}{5} > 0$ *i.e.* $0 < \frac{x}{5} < 2$, $\therefore 0 < x < 10$

Example: 32 $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} e^{\frac{r}{n}}$ is

(a) $e + 1$ (b) $e - 1$ (c) $1 - e$ (d) e

Solution: (b) $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} e^{r/n} = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} e^{r/n} = \lim_{n\to\infty} \frac{1}{n} \cdot (e^{1/n} + e^{2/n} + e^{3/n} + \ldots + e^{n/n}) = \lim_{n\to\infty} \frac{1}{n} \cdot [e^{1/n} + (e^{1/n})^2 + (e^{1/n})^3 + \ldots + (e^{1/n})^n]$

$$= \lim_{n\to\infty} \frac{1}{n} e^{1/n} \frac{1-(e^{1/n})^n}{1-e^{1/n}} = \lim_{n\to\infty} \frac{1}{n} e^{1/n} \frac{1-e}{1-e^{1/n}} = \lim_{n\to\infty} \frac{(1-e)(e^{1/n}-1+1)}{n(1-e^{1/n})} = \lim_{n\to\infty} \frac{(e-1)}{n} + \lim_{n\to\infty} \frac{(e-1)\cdot\frac{1}{n}}{e^{1/n}-1}$$

Put $\frac{1}{n} = h$, we get $h \to 0$

$$= 0 + (e-1) \lim_{h\to 0} \frac{h}{e^h - 1} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= (e-1) \lim_{h\to 0} \frac{1}{e^h} = (e-1).1 = e-1.$$

Example: 33 The value of $.2\dot{3}\dot{4}$ .234 is

(a) $\frac{232}{990}$ (b) $\frac{232}{9990}$ (c) $\frac{0.232}{990}$ (d) $\frac{232}{9909}$

Solution: (a) $.2\dot{3}\dot{4} = .234343434\ldots = \frac{2}{10} + \frac{34}{1000} + \frac{34}{100000} + \frac{34}{10^7} + \ldots = \frac{2}{10} + \frac{34}{1000}\left(1 + \frac{1}{100} + \frac{1}{(100)^2} + \ldots\right)$

$$= \frac{1}{5} + \frac{17}{500}\left(\frac{1}{1-\frac{1}{100}}\right) = \frac{1}{5} + \frac{17}{500} \times \frac{100}{99} = \frac{1}{5}\left\{1 + \frac{17}{99}\right\} = \frac{116}{495} = \frac{232}{990}$$

Example: 34 If a, b, c are in A.P. and $|a|, |b|, |c| < 1$, and

$x = 1 + a + a^2 + ...\infty$

$y = 1 + b + b^2 + ...\infty$

$z = 1 + c + c^2 + ...\infty$

Then x, y, z shall be in

(a) A.P. (b) G.P. (c) H.P. (d) None of these

Solution: (c) $x = 1 + a + a^2 + ...\infty = \dfrac{1}{1-a}$

$y = 1 + b + b^2 + ...\infty = \dfrac{1}{1-b}$

$z = 1 + c + c^2 + ...\infty = \dfrac{1}{1-c}$

Now, a, b, c are in A.P.

$\Rightarrow$ $1 - a$, $1 - b$, $1 - c$ are in A.P. $\Rightarrow \dfrac{1}{1-a}, \dfrac{1}{1-b}, \dfrac{1}{1-c}$ are in H.P. Therefore x, y, z are in H.P.

1.12 Geometric Mean

(1) **Definition :** (i) If three quantities are in G.P., then the middle quantity is called geometric mean (G.M.) between the other two. If a, G, b are in G.P., then G is called G.M. between a and b.

(ii) If $a, G_1, G_2, G_3, ... G_n, b$ are in G.P. then $G_1, G_2, G_3, ... G_n$ are called n G.M.'s between a and b.

(2) **Insertion of geometric means :** (i) **Single G.M. between *a* and *b* :** If a and b are two real numbers then single G.M. between a and $b = \sqrt{ab}$

(ii) ***n* G.M.'s between *a* and *b* :** If $G_1, G_2, G_3, ..., G_n$ are n G.M.'s between a and b, then

$$G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}},\ G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}},\ G_3 = ar^3 = a\left(\frac{b}{a}\right)^{\frac{3}{n+1}},\ ....,\ G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Important Tips

- *Product of n G.M.'s between a and b is equal to nth power of single geometric mean between a and b.*

  i.e. $G_1 G_2 G_3 ...... G_n = (\sqrt{ab})^n$

- *G.M. of* $a_1 a_2 a_3 ...... a_n$ *is* $(a_1 a_2 a_3 ..... a_n)^{1/n}$

- *If* G_1 *and* G_2 *are two G.M.'s between two numbers a and b is* $G_1 = (a^2 b)^{1/3}, G_2 = (ab^2)^{1/3}$.

- *The product of n geometric means between a and* $\dfrac{1}{a}$ *is 1.*

- *If n G.M.'s inserted between a and b then* $r = \left(\dfrac{b}{a}\right)^{\frac{1}{n+1}}$

1.13 Properties of G.P.

(1) If all the terms of a G.P. be multiplied or divided by the same non-zero constant, then it remains a G.P., with the same common ratio.

(2) The reciprocal of the terms of a given G.P. form a G.P. with common ratio as reciprocal of the common ratio of the original G.P.

(3) If each term of a G.P. with common ratio r be raised to the same power k, the resulting sequence also forms a G.P. with common ratio r^k .

(4) In a finite G.P., the product of terms equidistant from the beginning and the end is always the same and is equal to the product of the first and last term.

i.e., if $a_1, a_2, a_3, ... a_n$ be in G.P. Then $a_1\, a_n = a_2\, a_{n-1} = a_3\, a_{n-2} = a_n\, a_{n-3} = ... = a_r \,.\, a_{n-r+1}$

(5) If the terms of a given G.P. are chosen at regular intervals, then the new sequence so formed also forms a G.P.

(6) If $a_1, a_2, a_3, ..., a_n ......$ is a G.P. of non-zero, non-negative terms, then $\log a_1, \log a_2, \log a_3, ... \log a_n, ...$ is an A.P. and vice-versa.

(7) Three non-zero numbers a, b, c are in G.P. iff $b^2 = ac$.

(8) Every term (except first term) of a G.P. is the square root of terms equidistant from it.

i.e. $T_r = \sqrt{T_{r-p} \cdot T_{r+p}}$; $[r > p]$

(9) If first term of a G.P. of n terms is a and last term is l, then the product of all terms of the G.P. is $(al)^{n/2}$.

(10) If there be n quantities in G.P. whose common ratio is r and S_m denotes the sum of the first m terms, then the sum of their product taken two by two is $\frac{r}{r+1} S_n\, S_{n-1}$.

Example: 35 The two geometric mean between the number 1 and 64 are

(a) 1 and 64 (b) 4 and 16 (c) 2 and 16 (d) 8 and 16

Solution: (b) Let G_1 and G_2 are two G.M.'s between the number $a = 1$ and $b = 64$

$G_1 = (a^2 b)^{\frac{1}{3}} = (1.64)^{\frac{1}{3}} = 4$, $G_2 = (ab^2)^{\frac{1}{3}} = (1.64^2)^{\frac{1}{3}} = 16$

Example: 36 The G.M. of the numbers $3, 3^2, 3^3, ...3^n$ is

(a) $3^{\frac{2}{n}}$ (b) $3^{\frac{n+1}{2}}$ (c) $3^{\frac{n}{2}}$ (d) $3^{\frac{n-1}{2}}$

Solution: (b) G.M. of $(3.3^2.3^3......3^n) = (3.3^2.3^3......3^n)^{1/n} = (3)^{\frac{1+2+3+....+n}{n}} = 3^{\frac{n(n+1)}{2n}} = 3^{\frac{n+1}{2}}$

Example: 37 If a, b, c are in A.P. $b - a$, $c - b$ and a are in G.P., then $a : b : c$ is

(a) 1 : 2 : 3 (b) 1 : 3 : 5 (c) 2 : 3 : 4 (d) 1 : 2 : 4

Solution: (a) Given, a, b, c are in A.P. $\Rightarrow 2b = a + c$

$b - a$, $c - b$, a are in G.P. So $(c - b)^2 = a(b - a)$

$\Rightarrow (b - a)^2 = (b - a)\, a$ $\left[\begin{array}{l} \because\ 2b = a + c \\ \Rightarrow b + b = a + c \\ \Rightarrow b - a = c - b \end{array}\right]$

$\Rightarrow b = 2a$ $[\because b \neq a]$

Put in $2b = a + c$, we get $c = 3a$. Therefore $a : b : c = 1 : 2 : 3$

Harmonic Progression(H.P.)

1.14 Definition

A progression is called a harmonic progression (H.P.) if the reciprocals of its terms are in A.P.

Standard form : $\frac{1}{a} + \frac{1}{a+d} + \frac{1}{a+2d} + ...$

Example: The sequence $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, ...$ is a H.P., because the sequence 1, 3, 5, 7, 9, ….. is an A.P.

1.15 General Term of an H.P.

If the H.P. be as $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, ...$ then corresponding A.P. is $a, a+d, a+2d, ...$

T_n of A.P. is $a+(n-1)d$

$\therefore T_n$ of H.P. is $\frac{1}{a+(n-1)d}$

In order to solve the question on H.P., we should form the corresponding A.P.

Thus, General term : $\boxed{T_n = \frac{1}{a+(n-1)d}}$ or $\boxed{T_n \text{ of H.P.} = \frac{1}{T_n \text{ of A.P.}}}$

Example: 38 The 4^{th} term of a H.P. is $\frac{3}{5}$ and 8^{th} term is $\frac{1}{3}$ then its 6^{th} term is

(a) $\frac{1}{6}$ (b) $\frac{3}{7}$ (c) $\frac{1}{7}$ (d) $\frac{3}{5}$

Solution: (b) Let $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, ...$ be an H.P.

$\therefore$ 4^{th} term $= \frac{1}{a+3d} \Rightarrow \frac{3}{5} = \frac{1}{a+3d}$

$\Rightarrow \frac{5}{3} = a+3d$ .....(i)

Similarly, $3 = a+7d$ .....(ii)

From (i) and (ii), $d = \frac{1}{3}$, $a = \frac{2}{3}$

$\therefore$ 6^{th} term $= \frac{1}{a+5d} = \frac{1}{\frac{2}{3}+\frac{5}{3}} = \frac{3}{7}$

Example: 39 If the roots of $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ be equal, then a, b, c are in

(a) A.P. (b) G.P. (c) H.P. (d) None of these

Solution: (c) As the roots are equal, discriminate $= 0$

$\Rightarrow \{b(c-a)\}^2 - 4a(b-c)c(a-b) = 0 \Rightarrow b^2c^2 + a^2b^2 - 2ab^2c - 4a^2bc + 4a^2c^2 + 4ab^2c - 4abc^2 = 0$

$\Rightarrow (b^2c^2 + 2ab^2c + a^2b^2) = 4ac\{ab+bc-ac\} \Rightarrow (ab+bc)^2 = 4ac(ab+bc-ac) \Rightarrow \{b(a+c)\}^2 = 4abc(a+c) - 4a^2c^2$

$\Rightarrow b^2(a+c)^2 - 2b(a+c)\cdot 2ac + (2ac)^2 = 0 \Rightarrow [b(a+c) - 2ac]^2 = 0$

$\therefore \quad b = \frac{2ac}{a+c}$

Thus, a, b, c are in H.P.

Example: 40 If the first two terms of an H.P. be $\frac{2}{5}$ and $\frac{12}{23}$ then the largest positive term of the progression is the

(a) 6th term (b) 7th term (c) 5th term (d) 8th term

Solution: (c) For the corresponding A.P., the first two terms are $\frac{5}{2}$ and $\frac{23}{12}$ *i.e.* $\frac{30}{12}$ and $\frac{23}{12}$

Common difference $= -\frac{7}{12}$

$\therefore$ The A.P. will be $\frac{30}{12}, \frac{23}{12}, \frac{16}{12}, \frac{9}{12}, \frac{2}{12}, -\frac{5}{12}, ...$

The smallest positive term is $\frac{2}{12}$, which is the 5th term. $\therefore$ The largest positive term of the H.P. will be the 5th term.

1.16 Harmonic Mean

(1) **Definition :** If three or more numbers are in H.P., then the numbers lying between the first and last are called harmonic means (H.M.'s) between them. For example 1, 1/3, 1/5, 1/7, 1/9 are in H.P. So 1/3, 1/5 and 1/7 are three H.M.'s between 1 and 1/9.

Also, if a, H, b are in H.P., then H is called harmonic mean between a and b.

(2) **Insertion of harmonic means :**

(i) Single H.M. between a and b $= \frac{2ab}{a+b}$

(ii) H, H.M. of n non-zero numbers $a_1, a_2, a_3, ..., a_n$ is given by $\frac{1}{H} = \frac{\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}}{n}$.

(iii) Let a, b be two given numbers. If n numbers $H_1, H_2, ... H_n$ are inserted between a and b such that the sequence $a, H_1, H_2, H_3 ...... H_n, b$ is an H.P., then $H_1, H_2, ... H_n$ are called n harmonic means between a and b.

Now, $a, H_1, H_2, ... H_n, b$ are in H.P. $\Rightarrow \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, ... \frac{1}{H_n}, \frac{1}{b}$ are in A.P.

Let D be the common difference of this A.P. Then,

$$\frac{1}{b} = (n+2)^{th} \text{ term} = T_{n+2}$$

$$\frac{1}{b} = \frac{1}{a} + (n+1)D \Rightarrow D = \frac{a-b}{(n+1)ab}$$

Thus, if n harmonic means are inserted between two given numbers a and b, then the common difference of the corresponding A.P. is given by $D = \frac{a-b}{(n+1)ab}$

Also, $\frac{1}{H_1} = \frac{1}{a} + D$, $\frac{1}{H_2} = \frac{1}{a} + 2D$, ..., $\frac{1}{H_n} = \frac{1}{a} + nD$ where $D = \frac{a-b}{(n+1)ab}$

Important Tips

☞ *If a, b, c are in H.P. then* $b = \frac{2ac}{a+c}$.

☞ *If* H_1 *and* H_2 *are two H.M.'s between a and b, then* $H_1 = \frac{3ab}{a+2b}$ *and* $H_2 = \frac{3ab}{2a+b}$

1.17 Properties of H.P.

(1) No term of H.P. can be zero.

(2) If a, b, c are in H.P., then $\frac{a-b}{b-c} = \frac{a}{c}$.

(3) If H is the H.M. between a and b, then

(i) $\dfrac{1}{H-a}+\dfrac{1}{H-b}=\dfrac{1}{a}+\dfrac{1}{b}$ (ii) $(H-2a)(H-2b)=H^2$ (iii) $\dfrac{H+a}{H-a}+\dfrac{H+b}{H-b}=2$

Example: 41 The harmonic mean of the roots of the equation $(5+\sqrt{2})x^2-(4+\sqrt{3})x+8+2\sqrt{3}=0$ is

(a) 2 (b) 4 (c) 6 (d) 8

Solution: (b) Let ~ and be the roots of the given equation

$$\therefore \quad a+ \quad =\frac{4+\sqrt{3}}{5+\sqrt{2}},\ \sim \quad =\frac{8+2\sqrt{3}}{5+\sqrt{2}}$$

Hence, required harmonic mean $=\dfrac{2\sim}{\sim +}=\dfrac{2\left(\dfrac{8+2\sqrt{3}}{5+\sqrt{2}}\right)}{\dfrac{4+\sqrt{3}}{5+\sqrt{2}}}=\dfrac{2(8+2\sqrt{3})}{4+\sqrt{3}}=\dfrac{4(4+\sqrt{3})}{4+\sqrt{3}}=4$

Example: 42 If a, b, c are in H.P., then the value of $\left(\dfrac{1}{b}+\dfrac{1}{c}-\dfrac{1}{a}\right)\left(\dfrac{1}{c}+\dfrac{1}{a}-\dfrac{1}{b}\right)$ is

(a) $\dfrac{2}{bc}+\dfrac{1}{b^2}$ (b) $\dfrac{3}{c^2}+\dfrac{2}{ca}$ (c) $\dfrac{3}{b^2}-\dfrac{2}{ab}$ (d) None of these

Solution: (c) a, b, c are in H.P. $\Rightarrow \dfrac{1}{a},\dfrac{1}{b},\dfrac{1}{c}$ are in A.P.

$$\therefore \frac{1}{a}+\frac{1}{c}=\frac{2}{b}$$

Now, $\left(\dfrac{1}{b}+\dfrac{1}{c}-\dfrac{1}{a}\right)\left(\dfrac{1}{c}+\dfrac{1}{a}-\dfrac{1}{b}\right)=\left\{\dfrac{1}{b}+\left(\dfrac{1}{a}+\dfrac{1}{c}\right)-\dfrac{2}{a}\right\}\left(\dfrac{2}{b}-\dfrac{1}{b}\right)=\left(\dfrac{1}{b}+\dfrac{2}{b}-\dfrac{2}{a}\right)\left(\dfrac{1}{b}\right)=\dfrac{1}{b}\left(\dfrac{3}{b}-\dfrac{2}{a}\right)=\dfrac{3}{b^2}-\dfrac{2}{ab}$

Example: 43 If a, b, c are in H.P., then which one of the following is true

(a) $\dfrac{1}{b-a}+\dfrac{1}{b-c}=\dfrac{1}{b}$ (b) $\dfrac{ac}{a+c}=b$ (c) $\dfrac{b+a}{b-a}+\dfrac{b+c}{b-c}=1$ (d) None of these

Solution: (d) a, b, c are in H.P. $\Rightarrow b=\dfrac{2ac}{a+c}$, $\therefore$ option (b) is false

$$b-a=\frac{2ac}{a+c}-a=\frac{a(c-a)}{c+a}\Rightarrow b-c=\frac{c(a-c)}{a+c}$$

$$\therefore \frac{1}{b-a}+\frac{1}{b-c}=\frac{a+c}{a-c}\left\{-\frac{1}{a}+\frac{1}{c}\right\}=\frac{a+c}{a-c}\cdot\frac{a-c}{ac}=\frac{a+c}{ac}=\frac{a+c}{2ac}\cdot 2=\frac{2}{b},\ \therefore \text{ option (a) is false}$$

$$\frac{b+a}{b-a}+\frac{b+c}{b-c}=\frac{(c+a)(b+a)}{a(c-a)}+\frac{(b+c)(a+c)}{c(a-c)}=\frac{a+c}{a-c}\left\{-\left(\frac{b+a}{a}\right)+\frac{b+c}{c}\right\}=\frac{a+c}{a-c}\left(\frac{b}{c}-\frac{b}{a}\right)=\frac{a+c}{a-c}\cdot\frac{(a-c)b}{ac}$$

$$=\frac{a+c}{ac}\cdot b=\frac{a+c}{2ac}\cdot 2b=\frac{1}{b}\cdot 2b=2 \therefore \text{ option (c) is false.}$$

Arithmetico-Geometric Progression(A.G.P.)

1.18 n^{th} Term of A.G.P.

If $a_1,a_2,a_3,......,a_n,......$ is an A.P. and $b_1,b_2,......,b_n,......$ is a G.P., then the sequence $a_1b_1,a_2b_2,a_3b_3,......,a_nb_n,.....$ is said to be an arithmetico-geometric sequence.

Thus, the general form of an arithmetico geometric sequence is $a,(a+d)r,(a+2d)r^2,(a+3d)r^3,.....$

From the symmetry we obtain that the nth term of this sequence is $[a+(n-1)d]r^{n-1}$

Also, let $a,(a+d)r,(a+2d)r^2,(a+3d)r^3,.....$ be an arithmetico-geometric sequence. Then, $a+(a+d)r$ $+(a+2d)r^2+(a+3d)r^3+...$ is an arithmetico-geometric series.

1.19 Sum of A.G.P.

(1) **Sum of *n* terms :** The sum of n terms of an arithmetico-geometric sequence $a,(a+d)r,(a+2d)r^2,$ $(a+3d)r^3,.....$ is given by

$$S_n=\begin{cases}\dfrac{a}{1-r}+dr\dfrac{(1-r^{n-1})}{(1-r)^2}-\dfrac{\{a+(n-1)d\}r^n}{1-r}, \text{when } r\neq 1\\ \dfrac{n}{2}[2a+(n-1)d], \text{when } r=1\end{cases}$$

(2) **Sum of infinite sequence :** Let $|r|<1$. Then $r^n, r^{n-1}\to 0$ as $n\to\infty$ and it can also be shown that $n.r^n\to 0$ as $n\to\infty$. So, we obtain that $S_n\to\dfrac{a}{1-r}+\dfrac{dr}{(1-r)^2}$, as $n\to\infty$.

In other words, when $|r|<1$ the sum to infinity of an arithmetico-geometric series is $\boxed{S_\infty=\dfrac{a}{1-r}+\dfrac{dr}{(1-r)^2}}$

1.20 Method for Finding Sum

This method is applicable for both sum of n terms and sum of infinite number of terms.

First suppose that sum of the series is S, then multiply it by common ratio of the G.P. and subtract. In this way, we shall get a G.P., whose sum can be easily obtained.

1.21 Method of Difference

If the differences of the successive terms of a series are in A.P. or G.P., we can find n^{th} term of the series by the following steps :

Step I: Denote the n^{th} term by T_n and the sum of the series upto n terms by S_n.

Step II: Rewrite the given series with each term shifted by one place to the right.

Step III: By subtracting the later series from the former, find T_n.

Step IV: From T_n, S_n can be found by appropriate summation.

Example: 44 $1+\dfrac{3}{2}+\dfrac{5}{2^2}+\dfrac{7}{2^3}+......\infty$ is equal to

(a) 3 (b) 6 (c) 9 (d) 12

Solution: (b)

$$S=1+\frac{3}{2}+\frac{5}{2^2}+\frac{7}{2^3}+.......\infty$$
$$\frac{1}{2}S=\quad\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+.......\infty$$
$$\frac{1}{2}S=1+\frac{2}{2}+\frac{2}{2^2}+\frac{2}{2^3}+........ \quad \text{(on subtracting)}$$

$\Rightarrow \dfrac{S}{2}=1+2\left(\dfrac{1}{2}+\dfrac{1}{2^2}+\dfrac{1}{2^3}+....\infty\right)\Rightarrow \dfrac{S}{2}=1+2\times\left(\dfrac{1/2}{1-1/2}\right)=3$. Hence $S=6$

Example: 45 Sum of the series $1+2.2+3.2^2+4.2^3+.....+100.2^{99}$ is

(a) $100.2^{100}+1$ (b) $99.2^{100}+1$ (c) $99.2^{100}-1$ (d) $100.2^{100}-1$

Solution: (b) Let $S = 1 + 2.2 + 3.2^2 + 4.2^3 + ... + 100.2^{99}$ .....(i)

$2S = \quad 1.2 + 2.2^2 + 3.2^3 + ... + 99.2^{99} + 100.2^{100}$ .....(ii)

Equation (i) – Equation (ii) gives,

$$-S = 1 + (1.2 + 1.2^2 + 1.2^3 + ... \text{ upto 99 terms}) - 100.2^{100} = 1 + \frac{2(2^{99}-1)}{2-1} - 100.2^{100}$$

$$\Rightarrow \quad S = -1 - 2^{100} + 2 + 100.2^{100} = 1 + 99.2^{100}$$

Example: 46 The sum of the series 3 + 33 + 333 + ... + n terms is

(a) $\frac{1}{27}(10^{n+1} + 9n - 28)$ (b) $\frac{1}{27}(10^{n+1} - 9n - 10)$ (c) $\frac{1}{27}(10^{n+1} + 10n - 9)$ (d) None of these

Solution: (b)

$$S = 3 + 33 + 333 + ... \text{ to } n \text{ terms}$$
$$S = \quad 3 + 33 + ...$$
$$0 = 3 + 30 + 300 + ... \text{ to } n \text{ terms} - T_n \quad \text{(on subtracting)}$$

$$\therefore \quad T_n = 3(1 + 10 + 100 + ... \text{ to } n \text{ terms}) = 3 \times 1 \cdot \frac{10^n - 1}{10 - 1} = \frac{1}{3}(10^n - 1)$$

$$S_n = \sum_{n=1}^{n} \frac{1}{3}(10^n - 1) = \frac{1}{3}\sum_{n=1}^{n} 10^n - \frac{1}{3}\sum_{n=1}^{n} 1 = \frac{1}{3}\left(10 \cdot \frac{10^n - 1}{10 - 1}\right) - \frac{1}{3}n$$

$$S = \frac{1}{27}(10^{n+1} - 9n - 10)$$

Example: 47 The sum of n terms of the following series $1 + (1+x) + (1+x+x^2) + ...$ will be

(a) $\frac{1-x^n}{1-x}$ (b) $\frac{x(1-x^n)}{1-x}$ (c) $\frac{n(1-x) - x(1-x^n)}{(1-x)^2}$ (d) None of these

Solution: (c)

$$S = 1 + (1+x) + (1+x+x^2) + ...$$
$$S = \quad 1 + (1+x) + ...$$
$$0 = (1 + x + x^2 + ... \text{ to } n \text{ terms}) - T_n \quad \text{(on subtracting)}$$

$$\therefore \quad T_n = \frac{1-x^n}{1-x}$$

$$S_n = \sum_{n=1}^{n} T_n = \sum_{n=1}^{n} \frac{1-x^n}{1-x} = \frac{1}{1-x}\sum_{n=1}^{n} 1 - \frac{1}{1-x}\sum_{n=1}^{n} x^n = \frac{1}{1-x} \cdot n - \frac{1}{1-x} \cdot x \cdot \left(\frac{1-x^n}{1-x}\right)$$

$$= \frac{n}{1-x} - \frac{x(1-x^n)}{(1-x)^2} = \frac{n(1-x) - x(1-x^n)}{(1-x)^2}$$

Example: 48 The sum to n terms of the series $1 + 3 + 7 + 15 + 31 + ...$ is

(a) $2^{n+1} - n$ (b) $2^{n+1} - n - 2$ (c) $2^n - n - 2$ (d) None of these

Solution: (b)

$$S = 1 + 3 + 7 + 15 + 31 + ...$$
$$S = \quad 1 + 3 + 7 + 15 + ...$$
$$0 = (1 + 2 + 4 + 8 + 16 + ... \text{ to } n \text{ terms}) - T_n \quad \text{(on subtracting)}$$

$$\therefore \quad T_n = 1 + 2 + 4 + 8 + ... \text{ to } n \text{ terms} = 1 \cdot \frac{2^n - 1}{2 - 1} = 2^n - 1$$

$$S_n = \sum_{n=1}^{n} T_n = \sum_{n=1}^{n} (2^n - 1) = \sum_{n=1}^{n} 2^n - \sum_{n=1}^{n} 1 = 2 \cdot \left(\frac{2^n - 1}{2 - 1}\right) - n = 2^{n+1} - n - 2$$

Miscellaneous series

1.22 Special Series

There are some series in which n^{th} term can be predicted easily just by looking at the series.

If $T_n = \sim n^3 + \quad n^2 + ¥\, n + ¢$

Then $S_n = \sum_{n=1}^{n} T_n = \sum_{n=1}^{n} (\sim n^3 + \quad n^2 + ¥\, n + ¢) = \sim \sum_{n=1}^{n} n^3 + \quad \sum_{n=1}^{n} n^2 + ¥ \sum_{n=1}^{n} n + ¢ \sum_{n=1}^{n} 1$

$$= \sim \left(\frac{n(n+1)}{2}\right)^2 + \quad \left(\frac{n(n+1)(2n+1)}{6}\right) + ¥ \left(\frac{n(n+1)}{2}\right) + ¢\, n$$

Note : ❑ Sum of squares of first n natural numbers $= 1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

❑ Sum of cubes of first n natural numbers $= 1^3 + 2^3 + 3^3 + 4^3 + \ldots\ldots + n^3 = \sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2$

1.23 V_n Method

(1) To find the sum of the series $\frac{1}{a_1 a_2 a_3 \ldots . a_r} + \frac{1}{a_2 a_3 \ldots . a_{r+1}} + \ldots . + \frac{1}{a_n a_{n+1} \ldots . a_{n+r-1}}$

Let d be the common difference of A.P. Then $a_n = a_1 + (n-1)d$.

Let S_n and T_n denote the sum to n terms of the series and n^{th} term respectively.

$$S_n = \frac{1}{a_1 a_2 \ldots . a_r} + \frac{1}{a_2 a_3 \ldots . a_{r+1}} + \ldots . + \frac{1}{a_n a_{n+1} \ldots . a_{n+r-1}}$$

$$\therefore\ T_n = \frac{1}{a_n a_{n+1} \ldots . a_{n+r-1}}$$

Let $V_n = \frac{1}{a_{n+1} a_{n+2} \ldots . a_{n+r-1}}$; $V_{n-1} = \frac{1}{a_n a_{n+1} \ldots . a_{n+r-2}}$

$$\Rightarrow V_n - V_{n-1} = \frac{1}{a_{n+1} a_{n+2} \ldots . a_{n+r-1}} - \frac{1}{a_n a_{n+1} \ldots . a_{n+r-2}} = \frac{a_n - a_{n+r-1}}{a_n a_{n+1} \ldots . a_{n+r-1}}$$

$$= \frac{[a_1 + (n-1)d] - [a_1 + \{(n+r-1)-1\}d]}{a_n a_{n+1} \ldots . a_{n+r-1}} = d(1-r)T_n$$

$$\therefore\ T_n = \frac{1}{d(r-1)}\{V_{n-1} - V_n\},\quad \therefore\ S_n = \sum_{n=1}^{n} T_n = \frac{1}{d(r-1)}(V_0 - V_n)$$

$$S_n = \frac{1}{(r-1)(a_2 - a_1)}\left\{\frac{1}{a_1 a_2 \ldots . a_{r-1}} - \frac{1}{a_{n+1} a_{n+2} \ldots\ldots a_{n+r-1}}\right\}$$

Example: If $a_1, a_2, \ldots\ldots a_n$ are in A.P., then $\frac{1}{a_1 a_2 a_3} + \frac{1}{a_2 a_3 a_4} + \ldots + \frac{1}{a_n a_{n+1} a_{n+2}} = \frac{1}{2(a_2 - a_1)}\left\{\frac{1}{a_1 a_2} - \frac{1}{a_{n+1} a_{n+2}}\right\}$

(2) If $S_n = a_1 a_2 \ldots . a_r + a_2 a_3 \ldots . a_{r+1} \ldots . + a_n a_{n+1} \ldots a_{n+r-1}$

$T_n = a_n a_{n+1} \ldots . a_{n+r-1}$

Let $V_n = a_n a_{n+1} \ldots . a_{n+r-1} a_{n+r}$, $\therefore\ V_{n-1} = a_{n-1} a_{n+1} \ldots\ldots a_{n+r-1}$

$$\Rightarrow V_n - V_{n-1} = a_n a_{n+1} a_{n+2} \ldots . a_{n+r-1}(a_{n+r} - a_{n-1}) = T_n\{[a_1 + (n+r-1)d] - [a_1 + (n-2)d]\} = T_n(r+1)d$$

$$\therefore\ T_n = \frac{V_n - V_{n-1}}{(r+1)d}$$

$$S_n = \sum_{n=1}^{n} T_n = \frac{1}{(r+1)d}\sum_{n=1}^{n}(V_n - V_{n-1}) = \frac{1}{(r+1)d}(V_n - V_0) = \frac{1}{(r+1)d}\{(a_n a_{n+1}....a_{n+r}) - (a_0 a_1 ....a_r)\}$$

$$= \frac{1}{(r+1)(a_2 - a_1)}\{a_n a_{n+1}....a_{n+r} - a_0 a_1 .....a_r\}$$

Example: $1.2.3.4 + 2.3.4.5 + ... + n(n+1)(n+2)(n+3) = \frac{1}{5.1}\{n(n+1)(n+2)(n+3) - 0.1.2.3\}$

$$= \frac{1}{5}\{n(n+1)(n+2)(n+3)\}$$

Example: 49 The sum of $1^3 + 2^3 + 3^3 + 4^3 + ... + 15^3$ is

(a) 22000 (b) 10000 (c) 14400 (d) 15000

Solution: (c) $S = 1^3 + 2^3 + 3^3 + ... + 15^3$; For $n = 15$, the value of $\left(\frac{n(n+1)}{2}\right)^2 = \left(\frac{15\times 16}{2}\right)^2 = 14400$

Example: 50 A series whose n^{th} term is $\left(\frac{n}{x}\right) + y$, the sum of r terms will be

(a) $\left\{\frac{r(r+1)}{2x}\right\} + ry$ (b) $\left\{\frac{r(r-1)}{2x}\right\}$ (c) $\left\{\frac{r(r-1)}{2x}\right\} - ry$ (d) $\left\{\frac{r(r+1)}{2x}\right\} - rx$

Solution: (a) $S_r = \sum_{n=1}^{r} t_n = \sum_{n=1}^{r}\left(\frac{n}{x} + y\right) = \frac{1}{x}\sum_{n=1}^{r} n + y\sum_{n=1}^{r} 1 = \frac{1}{x}\frac{r(r+1)}{2} + yr = \frac{r(r+1)}{2x} + ry$

Example: 51 If $(1^2 - t_1) + (2^2 - t_2) + ... + (n^2 - t_n) = \frac{1}{3}n(n^2 - 1)$, then t_n is

(a) $\frac{n}{2}$ (b) $n-1$ (c) $n+1$ (d) n

Solution: (d) $\frac{1}{3}n(n^2 - 1) = (1^2 + 2^2 + ... + n^2) - (t_1 + t_2 + ... + t_n)$

$$\Rightarrow t_1 + t_2 + ... + t_n = 1^2 + 2^2 + 3^2 + ... + n^2 - \frac{1}{3}n(n^2 - 1) = \frac{n(n+1)(2n+1)}{6} - \frac{1}{3}n(n^2 - 1) = \frac{n(n+1)}{6}[2n + 1 - (2n - 2)]$$

$$\therefore t_1 + t_2 + t_3 + ... + t_n = \frac{n(n+1)}{2} \Rightarrow S_n = \frac{n(n+1)}{2}$$

$$t_n = S_n - S_{n-1} = \frac{n(n+1)}{2} - \frac{(n-1)n}{2} = n$$

Example: 52 The sum of the series $\frac{1}{3\times 7} + \frac{1}{7\times 11} + \frac{1}{11\times 15} + ...$ is

(a) $\frac{1}{3}$ (b) $\frac{1}{6}$ (c) $\frac{1}{9}$ (d) $\frac{1}{12}$

Solution: (d) $S = \left(\frac{1}{3\times 7} + \frac{1}{7\times 11} + \frac{1}{11\times 15} + ...\right) = \frac{1}{4}\left[\left(\frac{1}{3} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{11}\right) + \left(\frac{1}{11} - \frac{1}{15}\right) + ... \frac{1}{\infty}\right] = \frac{1}{4}\left[\frac{1}{3} - \frac{1}{\infty}\right] = \frac{1}{4}\left[\frac{1}{3} - 0\right] = \frac{1}{12}$

Example: 53 The sum of the series 1.2.3 + 2.3.4 + 3.4.5 + ….. to n terms is

(a) $n(n+1)(n+2)$ (b) $(n+1)(n+2)(n+3)$

(c) $\frac{1}{4}n(n+1)(n+2)(n+3)$ (d) $\frac{1}{4}(n+1)(n+2)(n+3)$

Solution: (c) $T_n = n(n+1)(n+2) = n^3 + 3n^2 + 2n$

$$\therefore \quad S = 1.2.3 + 2.3.4 + 3.4.5 + \ldots\ldots \text{ to } n \text{ terms} = \sum_{n=1}^{n}(n^3+3n^2+2n) = \sum_{n=1}^{n} n^3 + 3\sum_{n=1}^{n} n^2 + 2\sum_{n=1}^{n} n$$

$$S = \left(\frac{n(n+1)}{2}\right)^2 + 3\frac{n(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2} = \frac{1}{4}n(n+1)[n(n+1)+2(2n+1)+4]$$

$$= \frac{1}{4}n(n+1)[n^2+5n+6] = \frac{1}{4}n(n+1)(n+2)(n+3)$$

1.24 Properties of Arithmetic, Geometric and Harmonic means between Two given Numbers

Let A, G and H be arithmetic, geometric and harmonic means of two numbers a and b. Then, $A = \frac{a+b}{2}, G = \sqrt{ab}$ and $H = \frac{2ab}{a+b}$

These three means possess the following properties :

(1) $A \geq G \geq H$

$$A = \frac{a+b}{2}, G = \sqrt{ab} \text{ and } H = \frac{2ab}{a+b}$$

$$\therefore \; A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{(\sqrt{a}-\sqrt{b})^2}{2} \geq 0$$

$\Rightarrow A \geq G$ .....(i)

$$G - H = \sqrt{ab} - \frac{2ab}{a+b} = \sqrt{ab}\left(\frac{a+b-2\sqrt{ab}}{a+b}\right) = \frac{\sqrt{ab}}{a+b}(\sqrt{a}-\sqrt{b})^2 \geq 0$$

$\Rightarrow G \geq H$ .....(ii)

From (i) and (ii), we get $A \geq G \geq H$

Note that the equality holds only when $a = b$

(2) A, G, H from a G.P., *i.e.* $G^2 = AH$

$$AH = \frac{a+b}{2} \times \frac{2ab}{a+b} = ab = (\sqrt{ab})^2 = G^2$$

Hence, $G^2 = AH$

(3) The equation having a and b as its roots is $x^2 - 2Ax + G^2 = 0$

The equation having a and b its roots is $x^2 - (a+b)x + ab = 0$

$\Rightarrow x^2 - 2Ax + G^2 = 0$ $\left[\because A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}\right]$

The roots a, b are given by $A \pm \sqrt{A^2 - G^2}$

(4) If A, G, H are arithmetic, geometric and harmonic means between three given numbers a, b and c, then the equation having a, b, c as its roots is $x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$

$$A = \frac{a+b+c}{3}, G = (abc)^{1/3} \text{ and } \frac{1}{H} = \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}$$

$\Rightarrow a+b+c=3A, abc=G^3$ and $\frac{3G^3}{H}=ab+bc+ca$

The equation having a, b, c as its roots is $x^3-(a+b+c)x^2+(ab+bc+ca)x-abc=0$

$\Rightarrow x^3-3Ax^2+\frac{3G^3}{H}x-G^3=0$

1.25 Relation between A.P., G.P. and H.P.

(1) If A, G, H be A.M., G.M., H.M. between a and b, then $\boxed{\frac{a^{n+1}+b^{n+1}}{a^n+b^n}=\begin{cases} A \text{ when } n=0 \\ G \text{ when } n=-1/2 \\ H \text{ when } n=-1 \end{cases}}$

(2) If A_1, A_2 be two A.M.'s; G_1, G_2 be two G.M.'s and H_1, H_2 be two H.M.'s between two numbers a and b then $\boxed{\frac{G_1G_2}{H_1H_2}=\frac{A_1+A_2}{H_1+H_2}}$

(3) **Recognization of A.P., G.P., H.P. :** If a, b, c are three successive terms of a sequence.

Then if, $\frac{a-b}{b-c}=\frac{a}{a}$, then a, b, c are in A.P.

If, $\frac{a-b}{b-c}=\frac{a}{b}$, then a, b, c are in G.P.

If, $\frac{a-b}{b-c}=\frac{a}{c}$, then a, b, c are in H.P.

(4) If number of terms of any A.P./G.P./H.P. is odd, then A.M./G.M./H.M. of first and last terms is middle term of series.

(5) If number of terms of any A.P./G.P./H.P. is even, then A.M./G.M./H.M. of middle two terms is A.M./G.M./H.M. of first and last terms respectively.

(6) If p^{th}, q^{th} and r^{th} terms of a G.P. are in G.P. Then p, q, r are in A.P.

(7) If a, b, c are in A.P. as well as in G.P. then $a=b=c$.

(8) If a, b, c are in A.P., then x^a, x^b, x^c will be in G.P. $(x \neq \pm 1)$

Example: 54 If the A.M., G.M. and H.M. between two positive numbers a and b are equal, then

(a) $a=b$ (b) $ab=1$ (c) $a>b$ (d) $a<b$

Solution: (a) $\because$ A.M. = G.M.

$\Rightarrow \frac{a+b}{2}=\sqrt{ab} \Rightarrow \frac{(\sqrt{a})^2-2\sqrt{a}\sqrt{b}+(\sqrt{b})^2}{2}=0 \Rightarrow \frac{(\sqrt{a}-\sqrt{b})^2}{2}=0 \Rightarrow a=b$

$\because$ G.M. = H.M.

$\Rightarrow \sqrt{ab}=\frac{2ab}{a+b} \Rightarrow a+b-2\sqrt{ab}=0 \Rightarrow (\sqrt{a}-\sqrt{b})^2=0 \Rightarrow \sqrt{a}=\sqrt{b} \ \therefore\ a=b$

Thus A.M. =(G.M.) (H.M.) So $a=b$

Example: 55 Let two numbers have arithmetic mean 9 and geometric mean 4. Then these numbers are the roots of the quadratic equation

(a) $x^2-18x-16=0$ (b) $x^2-18x+16=0$ (c) $x^2+18x-16=0$ (d) $x^2+18x+16=0$

Solution: (b) $A=9$, $G=4$ are respectively the A.M. and G.M. between two numbers, then the quadratic equation having its roots as the two numbers, is given by $x^2-2Ax+G^2=0$ i.e. $x^2-18x+16=0$

Example: 56 If $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$ are in H.P., then

(a) a^2b, c^2a, b^2c are in A.P. (b) a^2b, b^2c, c^2a are in H.P.

(c) a^2b, b^2c, c^2a are in G.P. (d) None of these

Solution: (a) $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$ are in H.P.

$\Rightarrow \frac{b}{a}, \frac{c}{b}, \frac{a}{c}$ are in A.P. $\Rightarrow abc \times \frac{b}{a}, abc \times \frac{c}{b}, abc \times \frac{a}{c}$ are in A.P. $\Rightarrow b^2c, ac^2, a^2b$ are in A.P.

$\therefore\ a^2b, c^2a, b^2c$ are in A.P.

Example: 57 If a, b, c are in G.P., then $\log_a x, \log_b x, \log_c x$ are in

(a) A.P. (b) G.P. (c) H.P. (d) None of these

Solution: (c) a, b, c are in G.P.

$\Rightarrow \log_x a, \log_x b, \log_x c$ are in A.P. $\Rightarrow \frac{1}{\log_a x}, \frac{1}{\log_b x}, \frac{1}{\log_c x}$ are in A.P.

$\therefore\ \log_a x, \log_b x, \log_c x$ are in H.P.

Example: 58 If A_1, A_2 ; G_1, G_2 and H_1, H_2 be two A.M.'s, G.M.'s and H.M.'s between two quantities, then the value of $\frac{G_1G_2}{H_1H_2}$ is

(a) $\frac{A_1+A_2}{H_1+H_2}$ (b) $\frac{A_1-A_2}{H_1+H_2}$ (c) $\frac{A_1+A_2}{H_1-H_2}$ (d) $\frac{A_1-A_2}{H_1-H_2}$

Solution: (a) Let a and b be the two numbers

$$\therefore\ A_1 = a + \left(\frac{b-a}{3}\right) = \frac{2a+b}{3},\ A_2 = a + 2\left(\frac{b-a}{3}\right) = \frac{a+2b}{3}$$

$$G_1 = a\left(\frac{b}{a}\right)^{1/3} = a^{2/3}b^{1/3},\ G_2 = a\left(\left(\frac{b}{a}\right)^{1/3}\right)^2 = a^{1/3}b^{2/3}$$

$$H_1 = \frac{1}{\frac{1}{a} + \left(\frac{1}{b} - \frac{1}{a}\right)\frac{1}{3}} = \frac{3}{\frac{2}{a} + \frac{1}{b}} = \frac{3ab}{a+2b},\ H_2 = \frac{3ab}{2a+b}$$

$$\therefore\ \frac{G_1G_2}{H_1H_2} = \frac{(a^{2/3}b^{1/3})(a^{1/3}b^{2/3})}{\frac{3ab}{a+2b} \cdot \frac{3ab}{2a+b}} = \frac{(a+2b)(2a+b)}{9ab}$$

$$A_1 + A_2 = \frac{2a+b}{3} + \frac{a+2b}{3} = a+b$$

$$H_1 + H_2 = \frac{3ab}{a+2b} + \frac{3ab}{2a+b} = 3ab\left(\frac{2a+b+a+2b}{(a+2b)(2a+b)}\right) = \frac{9ab\,(a+b)}{(a+2b)(2a+b)}$$

$$\therefore\ \frac{A_1+A_2}{H_1+H_2} = \frac{(a+2b)(2a+b)}{9ab} = \frac{G_1G_2}{H_1H_2}$$

Example: 59 If the ratio of H.M. and G.M. of two quantities is 12 : 13, then the ratio of the numbers is

(a) 1 : 2 (b) 2 : 3 (c) 3 : 4 (d) None of these

Solution: (d) Let x and y be the numbers

$$\therefore\ \text{H.M.} = \frac{2xy}{x+y},\ \text{G.M.} = \sqrt{xy}$$

$$\therefore\ \frac{\text{H.M.}}{\text{G.M.}} = \frac{2\sqrt{xy}}{x+y} = \frac{2\sqrt{x/y}}{\frac{x}{y}+1} \Rightarrow \frac{12}{13} = \frac{2r}{r^2+1},\ (\because\ r = \sqrt{\frac{x}{y}}) \Rightarrow 12r^2 - 26r + 12 = 0 \Rightarrow 6r^2 - 13r + 6 = 0$$

$$\therefore\ r = \frac{13 \pm \sqrt{13^2 - 4.6.6}}{2 \times 6} = \frac{13 \pm 5}{12} = \frac{18}{12}, \frac{8}{12} = \frac{3}{2}, \frac{2}{3}$$

$$\therefore \text{ Ratio of numbers } = \frac{x}{y} = r^2 : 1 = \frac{9}{4} : 1 \text{ or } \frac{4}{9} : 1 = 9 : 4 \text{ or } 4 : 9$$

Example: 60 If the A.M. of two numbers is greater than G.M. of the numbers by 2 and the ratio of the numbers is 4 : 1, then the numbers are

(a) 4, 1 (b) 12, 3 (c) 16, 4 (d) None of these

Solution: (c) Let x and y be the numbers $\therefore$ A.M. = G.M. + 2 $\Rightarrow \frac{x+y}{2} = \sqrt{xy} + 2$

Also, $\frac{x}{y} = 4 : 1 \Rightarrow x = 4y$

$$\therefore\ \frac{4y+y}{2} = \sqrt{4y.y} + 2 \Rightarrow \frac{5y}{2} = 2y + 2 \Rightarrow y = 4 \Rightarrow x = 4 \times 4 = 16$$

$\therefore$ The numbers are 16, 4.

Example: 61 If the ratio of A.M. between two positive real numbers a and b to their H.M. is $m : n$, then $a : b$ is

(a) $\frac{\sqrt{m-n}+\sqrt{n}}{\sqrt{m-n}-\sqrt{n}}$ (b) $\frac{\sqrt{n}+\sqrt{m-n}}{\sqrt{n}-\sqrt{m-n}}$ (c) $\frac{\sqrt{m}+\sqrt{m-n}}{\sqrt{m}-\sqrt{m-n}}$ (d) None of these

Solution: (c) We have, $\frac{m}{n} = \frac{(a+b)/2}{2ab/(a+b)} \Rightarrow \frac{m}{n} = \frac{(a+b)^2}{4ab} = \frac{\left(\frac{a}{b}+1\right)^2}{4\frac{a}{b}} \Rightarrow 4\frac{m}{n}\left(\frac{a}{b}\right) = \left(\frac{a}{b}+1\right)^2 \Rightarrow 2\frac{\sqrt{m}}{\sqrt{n}}\sqrt{\frac{a}{b}} = \left(1+\frac{a}{b}\right)$

Let $\frac{a}{b} = r^2$, $\therefore\ \frac{2\sqrt{m}}{\sqrt{n}} r = (1+r^2) \Rightarrow 2\sqrt{m}\,r = \sqrt{n} + \sqrt{n}\,r^2 \Rightarrow \sqrt{n}\,r^2 - 2\sqrt{m}\,r + \sqrt{n} = 0$

$$\therefore\ r = \frac{2\sqrt{m} \pm \sqrt{4m-4n}}{2\sqrt{n}} = \frac{\sqrt{m} \pm \sqrt{m-n}}{\sqrt{n}}$$

Considering $+ve$ sign, $r = \frac{\sqrt{m}+\sqrt{m-n}}{\sqrt{n}} = \frac{(\sqrt{m}+\sqrt{m-n})(\sqrt{m}-\sqrt{m-n})}{\sqrt{n}(\sqrt{m}-\sqrt{m-n})} = \frac{m-(m-n)}{\sqrt{n}(\sqrt{m}-\sqrt{m-n})} = \frac{\sqrt{n}}{\sqrt{m}-\sqrt{m-n}}$

$\therefore\ r^2 = \frac{\sqrt{m}+\sqrt{m-n}}{\sqrt{n}} \cdot \frac{\sqrt{n}}{\sqrt{m}-\sqrt{m-n}}$. Hence, $\frac{a}{b} = \frac{\sqrt{m}+\sqrt{m-n}}{\sqrt{m}-\sqrt{m-n}}$.

1.26 Applications of Progressions

There are many applications of progressions is applied in science and engineering. Properties of progressions are applied to solve problems of inequality and maximum or minimum values of some expression can be found by the relation among A.M., G.M. and H.M.

Example: 62 If $x = \log_5 3 + \log_7 5 + \log_9 7$ then

(a) $x \geq \frac{3}{2}$ (b) $x \geq \frac{1}{\sqrt[3]{2}}$ (c) $x \geq \frac{3}{\sqrt[3]{2}}$ (d) None of these

Solution: (c) $x = \log_5 3 + \log_7 5 + \log_9 7$

$$\frac{\log_5 3 + \log_7 5 + \log_9 7}{3} \geq (\log_5 3 . \log_7 5 . \log_9 7)^{1/3} \qquad \text{[A.M.} \geq \text{G.M.]}$$

$$\Rightarrow \frac{x}{3} \geq (\log_9 3)^{1/3} \Rightarrow x \geq 3(\log_9 9^{1/2})^{1/3} \Rightarrow x \geq 3\left(\frac{1}{2}\right)^{1/3}. \text{ Hence } x \geq \frac{3}{\sqrt[3]{2}}$$

Example: 63 If a, b, c, d are four positive numbers then

(a) $\left(\frac{a}{b}+\frac{b}{c}\right)\left(\frac{c}{d}+\frac{d}{e}\right) \geq 4 \cdot \sqrt{\frac{a}{e}}$ (b) $\left(\frac{a}{b}+\frac{c}{d}\right)\left(\frac{b}{c}+\frac{d}{e}\right) \geq 4 \cdot \sqrt{\frac{a}{e}}$

(c) $\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}+\frac{e}{a} \geq 5$ (d) $\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{e}{d}+\frac{a}{e} \geq \frac{1}{5}$

Solution: (a,b,c) We have $\dfrac{\frac{a}{b}+\frac{b}{c}}{2} \geq \left(\frac{a}{b}\cdot\frac{b}{c}\right)^{1/2}$; $(\because$ A.M. $\geq$ G.M.)

$\Rightarrow \frac{a}{b}+\frac{b}{c} \geq 2\sqrt{\frac{a}{c}}$ .....(i)

Similarly, $\frac{c}{d}+\frac{d}{e} \geq 2\sqrt{\frac{c}{e}}$ .....(ii)

Multiplying (i) by (ii),

$\left(\frac{a}{b}+\frac{b}{c}\right)\left(\frac{c}{d}+\frac{d}{e}\right) \geq 4\sqrt{\frac{a}{c}}\sqrt{\frac{c}{e}} \Rightarrow \left(\frac{a}{b}+\frac{b}{c}\right)\left(\frac{c}{d}+\frac{d}{e}\right) \geq 4\sqrt{\frac{a}{e}}$, $\therefore$ (a) is true

Next, $\left(\frac{a}{b}+\frac{c}{d}\right)\left(\frac{b}{c}+\frac{d}{e}\right) \geq 2\left(\frac{a}{b}\cdot\frac{c}{d}\right)^{1/2} \cdot 2\left(\frac{b}{c}\cdot\frac{d}{e}\right)^{1/2} \Rightarrow \left(\frac{a}{b}+\frac{c}{d}\right)\left(\frac{b}{c}+\frac{d}{e}\right) \geq 4\sqrt{\frac{a}{e}}$, $\therefore$ (b) is true

$\dfrac{\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}+\frac{e}{a}}{5} \geq \left(\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{d}\cdot\frac{d}{e}\cdot\frac{e}{a}\right)^{1/5} \Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}+\frac{e}{a} \geq 5$, $\therefore$ (c) is true

Now, $\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{e}{d}+\frac{a}{e} \geq 5\left(\frac{b}{a}\cdot\frac{c}{b}\cdot\frac{d}{c}\cdot\frac{e}{d}\cdot\frac{a}{e}\right)^{1/5} \Rightarrow \frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{e}{d}+\frac{a}{e} \geq 5$, $\therefore$ (d) is false

Example: 64 Let a_n = product of first n natural numbers. Then for all $n \in N$

(a) $n^n \geq a_n$ (b) $\left(\frac{n+1}{2}\right)^n \geq n!$ (c) $n^n \geq a_n + 1$ (d) None of these

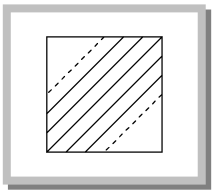
Solution: (a,b) We have $a_n = 1.2.3............n = n!$, $n^n = n.n.n........$ to n times

$\Rightarrow n^n \geq n.(n-1)(n-2)....\{n-(n-1)\} \Rightarrow n^n \geq n.(n-1)(n-2).........2.1 \Rightarrow n^n \geq n! \;\therefore\; n^n \geq a_n$. So (a) is true

$n^n \ngeq (n+1)! \Rightarrow n^n \ngeq a_n + 1$. So (c) is false

$\frac{1+2+3+.....+n}{n} \geq (1.2.3......n)^{1/n} \Rightarrow \frac{n(n+1)}{2n} \geq (n!)^{1/n} \Rightarrow \frac{n+1}{2} \geq (n!)^{1/n}$. $\therefore \left(\frac{n+1}{2}\right)^n \geq n!$. So (b) is true.

Example: 65 In the given square, a diagonal is drawn and parallel line segments joining points on the adjacent sides are drawn on both sides of the diagonal. The length of the diagonal is $n\sqrt{2}$ cm. If the distance between consecutive line segments be $\frac{1}{\sqrt{2}}$ cm then the sum of the lengths of all possible line segments and the diagonal is



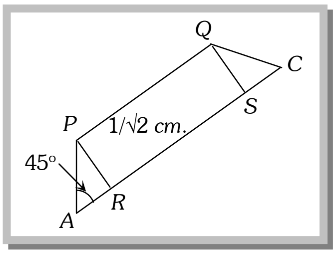
(a) $n(n+1)\sqrt{2}\,cm$ (b) $n^2\,cm$ (c) $n(n+2)\,cm$ (d) $n^2\sqrt{2}\,cm$

Solution: (d) Let us consider the diagonal and an adjacent parallel line

Length of the line $PQ = RS = AC - (AR + SC) = AC - 2AR \quad (\because AR = SC)$

$= AC - 2.PR \quad (\because AR = PR)$

$$= n\sqrt{2} - 2\cdot\frac{1}{\sqrt{2}} = n\sqrt{2} - \sqrt{2} = (n-1)\sqrt{2}\ cm$$



Length of line adjacent to PQ, other than AC, will be $((n-1)-1)\sqrt{2} = (n-2)\sqrt{2}\ cm$

$\therefore$ Sum of the lengths of all possible line segments and the diagonal

$$= 2\times[n\sqrt{2} + (n-1)\sqrt{2} + (n-2)\sqrt{2} + ...] - n\sqrt{2}, \qquad n \in N$$

$$= 2\times\sqrt{2}[n + (n-1) + (n-2) + ... + 1] - n\sqrt{2} = 2\sqrt{2}\times\frac{n(n+1)}{2} - n\sqrt{2} = n\sqrt{2}\{n+1-1\} = n^2\sqrt{2}\ cm$$

Example: 66 Let $f(x) = \dfrac{1-x^{n+1}}{1-x}$ and $g(x) = 1 - \dfrac{2}{x} + \dfrac{3}{x^2} - .... + (-1)^n\dfrac{n+1}{x^n}$. Then the constant term in $f'(x)\times g(x)$ is equal to

(a) $\dfrac{n(n^2-1)}{6}$ when n is even (b) $\dfrac{n(n+1)}{2}$ when n is odd (c) $-\dfrac{n}{2}(n+1)$ when n is even (d) $-\dfrac{n(n-1)}{2}$ when n is odd

Solution: (b,c) $f(x) = \dfrac{1-x^{n+1}}{1-x} = \dfrac{(1-x)(1+x+x^2+.....+x^n)}{(1-x)} = 1+x+x^2+.......+x^n;\quad f'(x) = 1+2x+3x^2+......+nx^{n-1}$

$$f'(x).g(x) = (1+2x+3x^2+....+nx^{n-1})\times\left(1-\frac{2}{x}+\frac{3}{x^2}-......+(-1)^n\frac{n+1}{x^n}\right)$$

$\therefore$ constant term in $f'(x)\times g(x)$ is

$$c = 1^2 - 2^2 + 3^2 - 4^2 + ... + n^2(-1)^{n-1} = [1^2+2^2+3^2+4^2+......+n^2] - 2[2^2+4^2+6^2+....]$$

when n is odd,

$$c = [1^2+2^2+.....+n^2] - 2[2^2+4^2+6^2+......+(n-1)^2] = \left[\frac{n(n+1)(2n+1)}{6} - 2.2^2[1^2+2^2+3^2+......+\left(\frac{n-1}{2}\right)^2\right]$$

$$= \frac{n(n+1)(2n+1)}{6} - 8\frac{\left(\frac{n-1}{2}\right)\cdot\left(\frac{n-1}{2}+1\right)\left(2\frac{n-1}{2}+1\right)}{6} = \frac{n(n+1)(2n+1)}{6} - \frac{n(n-1)(n+1)}{3}$$

$$= \frac{n(n+1)}{6}(2n+1-2(n-1)) = \frac{n(n+1)}{6}\times 3 = \frac{n(n+1)}{2}$$

when n is even, $c = [1^2+2^2+.....+n^2] - 2[2^2+4^2+.....+n^2] = \dfrac{n(n+1)(2n+1)}{6} - 2.2^2\left[1^2+2^2+....+\left(\dfrac{n}{2}\right)^2\right]$

$$= \frac{n(n+1)(2n+1)}{6} - 8\frac{\left(\frac{n}{2}\right)\cdot\left(\frac{n}{2}+1\right)\left(2\cdot\frac{n}{2}+1\right)}{6} = \frac{n(n+1)(2n+1)}{6} - \frac{1}{3}n(n+1)(n+2)$$

$$= \frac{1}{6}n(n+1)(2n+1-2(n+2)) = -\frac{1}{2}n(n+1)$$

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