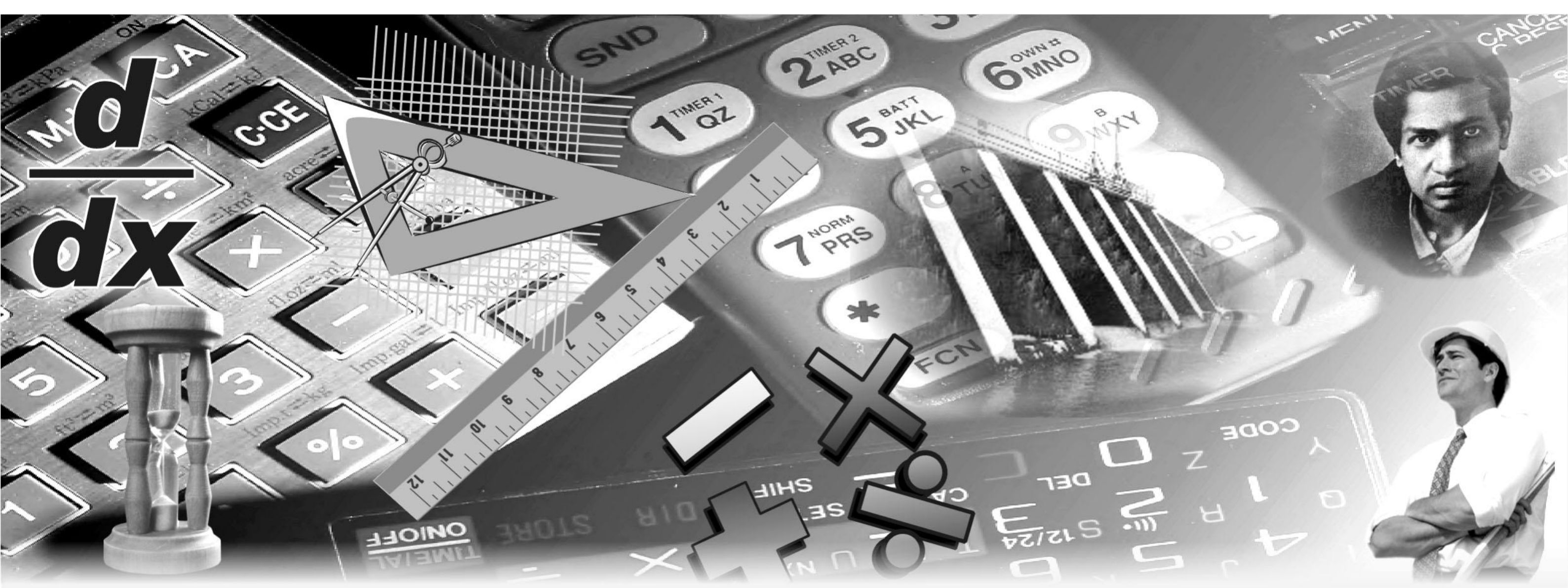


Chapter

11

Trigonometrical Equations and Inequations, Properties of Triangles, Height and Distance

Trigonometrical Equations and Inequations

Definition

An equation involving one or more trigonometrical ratio of an unknown angle is called a trigonometrical equation

i.e., $\sin x + \cos 2x = 1$, $(1-\tan\theta)(1+\sin 2\theta) = 1+\tan\theta$,

$|\sec\left(\theta+\frac{\pi}{4}\right)| = 2$ etc.

A trigonometric equation is different from a trigonometrical identities. An identity is satisfied for every value of the unknown angle *e.g.*, $\cos^2 x = 1-\sin^2 x$ is true $\forall x \in R$, while a trigonometric equation is satisfied for some particular values of the unknown angle.

(1) **Roots of trigonometrical equation :** The value of unknown angle (a variable quantity) which satisfies the given equation is called the root of an equation, *e.g.*, $\cos\theta = \frac{1}{2}$, the root is $\theta = 60^o$ or $\theta = 300^o$ because the equation is satisfied if we put $\theta = 60^o$ or $\theta = 300^o$.

(2) **Solution of trigonometrical equations :** A value of the unknown angle which satisfies the trigonometrical equation is called its solution.

Since all trigonometrical ratios are periodic in nature, generally a trigonometrical equation has more than one solution or an infinite number of solutions. There are basically three types of solutions:

(i) **Particular solution :** A specific value of unknown angle satisfying the equation.

(ii) **Principal solution :** Smallest numerical value of the unknown angle satisfying the equation (Numerically smallest particular solution).

(iii) **General solution :** Complete set of values of the unknown angle satisfying the equation. It contains all particular solutions as well as principal solutions.

Trigonometrical equations with their general solution

Trigonometrical equation	General solution
$\sin\theta = 0$	$\theta = n\pi$
$\cos\theta = 0$	$\theta = n\pi + \pi/2$
$\tan\theta = 0$	$\theta = n\pi$
$\sin\theta = 1$	$\theta = 2n\pi + \pi/2$
$\cos\theta = 1$	$\theta = 2n\pi$
$\sin\theta = \sin\alpha$	$\theta = n\pi + (-1)^n\alpha$
$\cos\theta = \cos\alpha$	$\theta = 2n\pi \pm \alpha$
$\tan\theta = \tan\alpha$	$\theta = n\pi \pm \alpha$
$\sin^2\theta = \sin^2\alpha$	$\theta = n\pi \pm \alpha$
$\tan^2\theta = \tan^2\alpha$	$\theta = n\pi \pm \alpha$
$\cos^2\theta = \cos^2\alpha$	$\theta = n\pi \pm \alpha$
$\left.\begin{matrix}\sin\theta = \sin\alpha \\ \cos\theta = \cos\alpha\end{matrix}\right\vert *$	$\theta = 2n\pi + \alpha$
$\left.\begin{matrix}\sin\theta = \sin\alpha \\ \tan\theta = \tan\alpha\end{matrix}\right\vert *$	$\theta = 2n\pi + \alpha$
$\left.\begin{matrix}\tan\theta = \tan\alpha \\ \cos\theta = \cos\alpha\end{matrix}\right\vert *$	$\theta = 2n\pi + \alpha$

General solution of the form *a* cos + *b* sin = *c*

In $a\cos\theta + b\sin\theta = c$, put $a = r\cos\alpha$ and $b = r\sin\alpha$ where $r = \sqrt{a^2+b^2}$ and $|c| \le \sqrt{a^2+b^2}$

Then, $r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) = c$

$\Rightarrow \cos(\theta - \alpha) = \frac{c}{\sqrt{a^2 + b^2}} = \cos\beta$, (say) .....(i)

$\Rightarrow \theta - \alpha = 2n\pi \pm \beta \Rightarrow \theta = 2n\pi \pm \beta + \alpha$, where $\tan\alpha = \frac{b}{a}$ is the general solution.

Alternatively, putting $a = r\sin\alpha$ and $b = r\cos\alpha$,

where $r = \sqrt{a^2 + b^2}$ $\Rightarrow \sin(\theta + \alpha) = \frac{c}{\sqrt{a^2 + b^2}} = \sin\gamma$, (say)

$\Rightarrow \theta + \alpha = n\pi + (-1)^n\gamma \Rightarrow \theta = n\pi + (-1)^n\gamma - \alpha$,

where $\tan\alpha = \frac{a}{b}$ is the general solution.

$(-\sqrt{a^2 + b^2}) \le a\cos\theta + b\sin\theta \le (\sqrt{a^2 + b^2})$

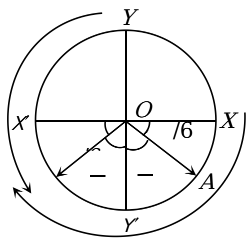
The general solution of $a\cos x + b\sin x = c$ is

$x = 2n\pi + \tan^{-1}\left(\frac{b}{a}\right) \pm \cos^{-1}\left(\frac{c}{\sqrt{a^2 + b^2}}\right)$.

Method for finding principal value

Suppose we have to find the principal value of satisfying the equation $\sin\theta = -\frac{1}{2}$.

Since $\sin\theta$ is negative, will be in 3rd or 4th quadrant. We can approach 3rd or 4th quadrant from two directions. If we take anticlockwise direction the numerical value of the angle will be greater than . If we approach it in clockwise direction the angle will be numerically less than . For principal value, we have to take numerically smallest angle. So for principal value.



(1) If the angle is in 1st or 2nd quadrant we must select anticlockwise direction and if the angle is in 3rd or 4th quadrant, we must select clockwise direction.

(2) Principal value is never numerically greater than .

(3) Principal value always lies in the first circle (*i.e.*, in first rotation). On the above criteria, will be $-\frac{\pi}{6}$ or $-\frac{5\pi}{6}$. Between these two $-\frac{\pi}{6}$ has the least numerical value. Hence $-\frac{\pi}{6}$ is the principal value of satisfying the equation $\sin\theta = -\frac{1}{2}$.

From the above discussion, the method for finding principal value can be summed up as follows :

(i) First draw a trigonometrical circle and mark the quadrant, in which the angle may lie.

(ii) Select anticlockwise direction for 1st and 2nd quadrants and select clockwise direction for 3rd and 4th quadrants.

(iii) Find the angle in the first rotation.

(iv) Select the numerically least angle. The angle thus found will be principal value.

(v) In case, two angles one with positive sign and the other with negative sign qualify for the numerically least angle, then it is the convention to select the angle with positive sign as principal value.

Important points to be taken in case of while solving trigonometrical equations

(1) Check the validity of the given equation, *e.g.* $2\sin\theta - \cos\theta = 4$ can never be true for any θ as the value $(2\sin\theta - \cos\theta)$ can never exceeds $\sqrt{2^2 + (-1)^2} = \sqrt{5}$. So there is no solution of this equation.

(2) Equation involving $\sec\theta$ or $\tan\theta$ can never have a solution of the form, $(2n+1)\frac{\pi}{2}$.

Similarly, equations involving $\operatorname{cosec}\theta$ or $\cot\theta$ can never have a solution of the form $\theta = n\pi$. The corresponding functions are undefined at these values of .

(3) If while solving an equation we have to square it, then the roots found after squaring must be checked whether they satisfy the original equation or not, *e.g.* let $x = 3$. Squaring, we get $x^2 = 9$ $x = 3$ and -3 but $x = -3$ does not satisfy the original equation $x = 3$.

(4) Do not cancel common factors involving the unknown angle on L.H.S. and R.H.S. because it may delete some solutions. In the equation $\sin\theta(2\cos\theta - 1) = \sin\theta\cos^2\theta$ if we cancel $\sin\theta$ on both sides we get $\cos^2\theta - 2\cos\theta + 1 = 0$ $\Rightarrow (\cos\theta - 1)^2 = 0$ $\Rightarrow \cos\theta = 1 \Rightarrow \theta = 2n\pi$. But $\theta = n\pi$ also satisfies the equation because it makes $\sin\theta = 0$. So, the complete solution is $\theta = n\pi, n \in Z$.

(5) Any value of *x* which makes both R.H.S. and L.H.S. equal will be a root but the value of *x* for which $\infty = \infty$ will not be a solution as it is an indeterminate form.

Hence, $\cos x \ne 0$ for those equations which involve $\tan x$ and $\sec x$ whereas $\sin x \ne 0$ for those which involve $\cot x$ and $\operatorname{cosec} x$.

Also exponential function is always $+ve$ and $\log_a x$ is defined if $x > 0$, $x \ne 0$ and $a > 0, a \ne 1$ $\sqrt{f(x)} = +ve$ always and not $\pm$ *i.e.* $\sqrt{(\tan^2 x)} = \tan x$ and not $\pm\tan x$.

(6) Denominator terms of the equation if present should never become zero at any stage while solving for any value of contained in the answer.

(7) Sometimes the equation has some limitations also *e.g.*, $\cot^2\theta + \mathrm{cosec}^2\theta = 1$ can be true only if $\cot^2\theta = 0$ and $\mathrm{cosec}^2\theta = 1$ simultaneously as $\mathrm{cosec}^2\theta \ge 1$. Hence the solution is $\theta = (2n+1)\pi/2$.

(8) If $xy = xz$ then $x(y-z) = 0 \Rightarrow$ either $x = 0$ or $y = z$ or both. But $\frac{y}{x} = \frac{z}{x} \Rightarrow y = z$ only and not $x = 0$, as it will make $\infty = \infty$. Similarly if $ay = az$, then it will also imply $y = z$ only as $a \ne 0$ being a constant.

Similarly $x + y = x + z \Rightarrow y = z$ and $x - y = x - z \Rightarrow y = z$. Here we do not take $x = 0$ as in the above because x is an additive factor and not multiplicative factor.

(9) Student are advised to check whether all the roots obtained by them, satisfy the equation and lie in the domain of the variable of the given equation.

Periodic functions

A function is said to be periodic function if its each value is repeated after a definite interval. So a function $f(x)$ will be periodic if a positive real number T exist such that, $f(x+T) = f(x), \forall x \in$ domain. Here the least positive value of T is called the period of the function. Clearly $f(x) = f(x+T) = f(x+2T) = f(x+3T) = ...$. *e.g.*, $\sin x, \cos x, \tan x$ are periodic functions with period $2\pi, 2\pi$ and π respectively.

Table : 11.1 Some standard results on periodic functions

Functions	Periods
$\sin^n x, \cos^n x, \sec^n x, \mathrm{cosec}^n x$	$\left.\begin{matrix}\pi; \\ 2\pi;\end{matrix}\right\}\begin{matrix}\text{if } n \text{ is even} \\ \text{if } n \text{ is odd or fraction}\end{matrix}$
$\tan^n x, \cot^n x$	$\pi; n$ is even or odd.
$\sin(ax+b), \cos(ax+b)$ $\sin(ax+b), \cos(ax+b)$	$2\pi/a$
$\tan(ax+b), \cot(ax+b)$	π/a
$\|\sin x\|, \|\cos x\|, \|\tan x\|,$ $\|\cot x\|, \|\sec x\|, \|\mathrm{cosec}\, x\|$	
$\|\sin(ax+b)\|, \|\cos(ax+b)\|,$ $\|\sec(ax+b)\|, \|\mathrm{cosec}(ax+b)\|$ $\|\tan(ax+b)\|, \|\cot(ax+b)\|$	π/a
$x - [x]$	1
Algebraic functions *e.g.*, $\sqrt{x}, x^2, x^3+5, ...\text{etc}$	Period does not exist

Properties of Triangles and Solutions of Triangles

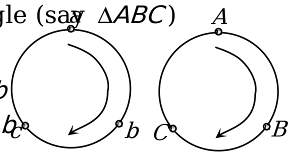
Relation between sides and angles

A triangle has six components, three sides and three angles. The three angles of a ΔABC are denoted by letters A, B, C and the sides opposite to these angles by letters a, b and c respectively. Following are some well known relations for a triangle (say ΔABC)

$A + B + C = 180^o$ (or)

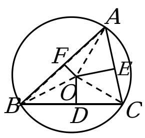
$a + b > c,\ b + c > a,\ c + a > b$

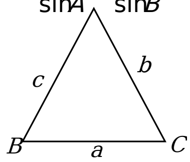
$|a-b| < c, |b-c| < a, |c-a| < b$



Generally, the relations involving the sides and angles of a triangle are cyclic in nature, e.g. to obtain the second similar relation to $a + b > c$, we simply replace a by b, b by c and c by a. So, to write all the relations, follow the cycles given.

The law of sines or sine rule : The sides of a triangle are proportional to the sines of the angles opposite to them *i.e.*, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$, (say)





More generally, if R be the radius of the circumcircle of the triangle ABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$.

The law of cosines or cosine rule

In any triangle ABC, the square of any side is equal to the sum of the squares of the other two sides diminished by twice the product of these sides and the cosine of their included angle, that is for a triangle ABC,

(1) $a^2 = b^2 + c^2 - 2bc\cos A \Rightarrow \cos A = \frac{b^2 + c^2 - a^2}{2bc}$

(2) $b^2 = c^2 + a^2 - 2ca\cos B \Rightarrow \cos B = \frac{c^2 + a^2 - b^2}{2ca}$

(3) $c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow \cos C = \frac{a^2 + b^2 - c^2}{2ab}$

Combining with $\sin A = \frac{a}{2R}, \sin B = \frac{b}{2R}, \sin C = \frac{c}{2R}$

We have by division, $\tan A = \frac{abc}{R(b^2 + c^2 - a^2)}$,

$\tan B = \frac{abc}{R(c^2 + a^2 - b^2)}$, $\tan C = \frac{abc}{R(a^2 + b^2 - c^2)}$

where, R be the radius of the circum-circle of the triangle ABC.

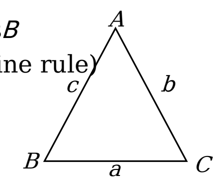
Projection formulae

In any triangle ABC, $b\cos C + c\cos B$

$= k\sin B\cos C + k\sin C\cos B$, (from sine rule)

$= k[\sin(B+C)] = k\sin(\pi - A)$

$= k\sin A = a$



Similarly, we can deduct other projection formulae from sine rule.

(i) $a = b\cos C + c\cos B$

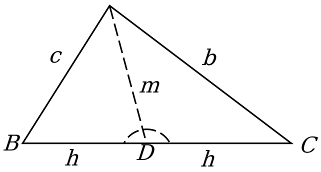
(ii) $b = c\cos A + a\cos C$

(iii) $c = a\cos B + b\cos A$

Theorem of the medians: (Apollonius theorem)

In every triangle the sum of the squares of any two sides is equal to twice the square on half the third side together with twice the square on the median that bisects the third side.

For any triangle *ABC*, $b^2+c^2=2(h^2+m^2)=2\{m^2+(a/2)^2\}$ by use of cosine rule.



If be right angled, the mid point of hypotenuse is equidistant from the three vertices so that $DA=DB=DC$.

$\therefore b^2+c^2=a^2$ which is pythagoras theorem. This theorem is very useful for solving problems of height and distance.

Napier's analogy (Law of tangents)

For any triangle *ABC*, (1) $\tan\left(\frac{A-B}{2}\right)=\left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$

(2) $\tan\left(\frac{B-C}{2}\right)=\left(\frac{b-c}{b+c}\right)\cot\frac{A}{2}$

(3) $\tan\left(\frac{C-A}{2}\right)=\left(\frac{c-a}{c+a}\right)\cot\frac{B}{2}$

Mollweide's formula: For any triangle,

$$\frac{a+b}{c}=\frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C},\ \frac{a-b}{c}=\frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}.$$

Area of triangle

Let three angles of ΔABC are denoted by *A, B, C* and the sides opposite to these angles by letters *a, b, c* respectively.

(1) **When two sides and the included angle be given :**

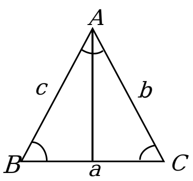
The area of triangle *ABC* is given by,

$$\Delta=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B=\frac{1}{2}ab\sin C$$

i.e., $\Delta=\frac{1}{2}$ (Product of two sides) × sine of included angle

(2) **When three sides are given :**

Area of $\Delta ABC=\Delta=\sqrt{s(s-a)(s-b)(s-c)}$



where semiperimeter of triangle $s=\frac{a+b+c}{2}$

(3) **When three sides and the circum-radius be given :**

Area of triangle $\Delta=\frac{abc}{4R}$, where *R* be the circum-radius of the triangle.

(4) **When two angles and included side be given :**

$$\Delta=\frac{1}{2}a^2\frac{\sin B\sin C}{\sin(B+C)}=\frac{1}{2}b^2\frac{\sin A\sin C}{\sin(A+C)}=\frac{1}{2}c^2\frac{\sin A\sin B}{\sin(A+B)}$$

Half angle formulae

If 2*s* shows the perimeter of a triangle ABC then, *i.e.*, $2s=a+b+c$, then

(1) **Formulae for $\sin\frac{A}{2},\sin\frac{B}{2},\sin\frac{C}{2}$**

(i) $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\sin\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{ca}}$

(iii) $\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$

(2) **Formulae for $\cos\frac{A}{2},\cos\frac{B}{2},\cos\frac{C}{2}$**

(i) $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$ (ii) $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}}$

(iii) $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$

(3) **Formulae for $\tan\frac{A}{2},\tan\frac{B}{2},\tan\frac{C}{2}$**

(i) $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ (ii) $\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

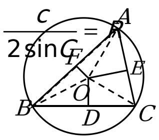
(iii) $\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Circle connected with triangle

(1) **Circumcircle of a triangle and its radius**

(i) **Circumcircle :** The circle which passes through the angular points of a triangle is called its circumcircle. The centre of this circle is the point of intersection of perpendicular bisectors of the sides and is called the circumcentre. Its radius is always denoted by *R*. The circumcentre may lie within, outside or upon one of the sides of the triangle.

(ii) **Circum-radius :** The circum-radius of a ΔABC is given by (a) $\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}=$

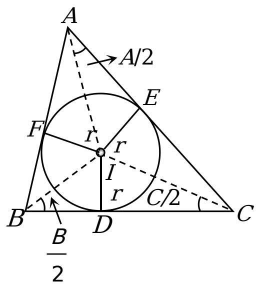


(b) $R=\frac{abc}{4\Delta}$ [$\Delta=$ area of ΔABC]

(2) **Inscribed circle or in-circle of a triangle and its radius**

(i) **In-circle or inscribed circle :** The circle which can be inscribed within a triangle so as to touch each of its sides is called its inscribed circle or in circle. The centre of this circle is the point of intersection of the bisectors of the angles of the triangle. The radius of this circle is always denoted by *r* and is equal to the length of the perpendicular from its centre to any one of the sides of triangle.

(ii) **In-radius :** The radius *r* of the inscribed circle of a triangle *ABC* is given by



(a) $r=\frac{\Delta}{s}$

(b) $r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(c) $r=(s-a)\tan\dfrac{A}{2}$,

$r=(s-b)\tan\dfrac{B}{2}$, $r=(s-c)\tan\dfrac{C}{2}$

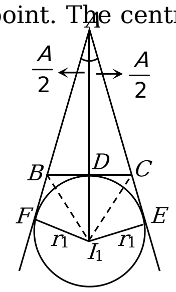
(d) $r=\dfrac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}, r=\dfrac{b\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}}, r=\dfrac{c\sin\frac{B}{2}\sin\frac{A}{2}}{\cos\frac{C}{2}}$

(e) $\cos A+\cos B+\cos C=1+\dfrac{r}{R}$

(3) **Escribed circles of a triangle and their radii**

(i) **Escribed circle :** The circle which touches the side BC and two sides AB and AC produced of a triangle ABC is called the escribed circle opposite to the angle A. Its radius is denoted by r_1. Similarly, r_2 and r_3 denote the radii of the escribed circles opposite to the angles B and C respectively.

The centres of the escribed circles are called the ex-centres. The centre of the escribed circle opposite to the angle A is the point of intersection of the external bisectors of angles B and C. The internal bisectors of angle A also passes through the same point. The centre is generally denoted by I_1.



(ii) **Radii of ex-circles**

In any ΔABC, we have

(a) $r_1=\dfrac{\Delta}{s-a}, r_2=\dfrac{\Delta}{s-b}, r_3=\dfrac{\Delta}{s-c}$

(b) $r_1=s\tan\dfrac{A}{2}$, $r_2=s\tan\dfrac{B}{2}$,

$r_3=s\tan\dfrac{C}{2}$

(c) $r_1=\dfrac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}, r_2=\dfrac{b\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}}, r_3=\dfrac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$

(d) $r_1+r_2+r_3-r=4R$ (e) $\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}=\dfrac{1}{r}$

(f) $\dfrac{1}{r^2}+\dfrac{1}{r_1^2}+\dfrac{1}{r_2^2}+\dfrac{1}{r_3^2}=\dfrac{a^2+b^2+c^2}{\Delta^2}$

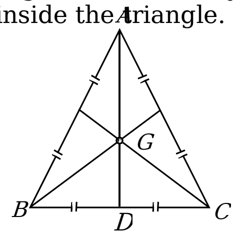
(g) $\dfrac{1}{bc}+\dfrac{1}{ca}+\dfrac{1}{ab}=\dfrac{1}{2Rr}$ (h)

$r_1r_2+r_2r_3+r_3r_1=s^2$

(i) $\Delta=2R^2\sin A.\sin B.\sin C=4Rr\cos\dfrac{A}{2}.\cos\dfrac{B}{2}.\cos\dfrac{C}{2}$

(j) $r_1=4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}; r_2=4R\cos\dfrac{A}{2}.\sin\dfrac{B}{2}.\cos\dfrac{C}{2}$

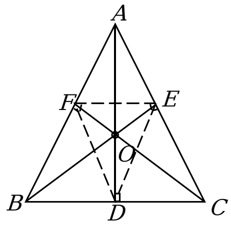
(4) **Centroid (G) :** Common point of intersection of medians of a triangle. Divides every median in the ratio 2:1. Always lies inside the triangle.



(5) **Orthocentre of a triangle :** The point of intersection of perpendicular drawn from the vertices on the opposite sides of a triangle is called its orthocentre.

Pedal triangle

Let the perpendiculars AD, BE and CF from the vertices A, B and C on the opposite sides BC, CA and AB of ΔABC respectively, meet at O. Then O is the orthocentre of the ΔABC. The triangle DEF is called the pedal triangle of the ΔABC.



Othocentre of the triangle is the incentre of the pedal triangle.

If O is the orthocentre and DEF the pedal triangle of the ΔABC, where AD, BE, CF are the perpendiculars drawn from A, B, C on the opposite sides BC, CA, AB respectively, then

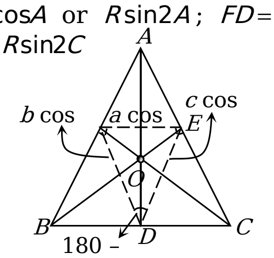
(i) $OA=2R\cos A, OB=2R\cos B$ and $OC=2R\cos C$

(ii) $OD=2R\cos B\cos C, OE=2R\cos C\cos A$

and $OF=2R\cos A\cos B$

(1) **Sides and angles of a pedal triangle:** The angles of pedal triangle DEF are: $180-2A, 180-2B, 180-2C$ and sides of pedal triangle are:

$EF=a\cos A$ or $R\sin 2A$; $FD=b\cos B$ or $R\sin 2B$; $DE=c\cos C$ or $R\sin 2C$



If given ΔABC is obtuse, then angles are represented by $2A$, $2B$, $2C-180^o$ and the sides are $a\cos A, b\cos B, -c\cos C$.

(2) **Area and circum-radius and in-radius of pedal triangle :** Area of pedal triangle $=\dfrac{1}{2}$(Product of the sides $\times$ (sine of included angle)

$\Delta=\dfrac{1}{2}R^2.\sin 2A.\sin 2B.\sin 2C$

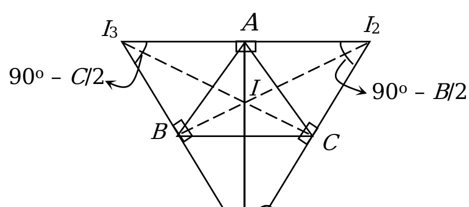
Circum-radius of pedal triangle $=\dfrac{EF}{2\sin FDE}=\dfrac{R\sin 2A}{2\sin(180^o-2A)}=\dfrac{R}{2}$

In-radius of pedal triangle $=\dfrac{\text{area of }\Delta DEF}{\text{semi-perimeter of }\Delta DEF}$

$=\dfrac{\frac{1}{2}R^2\sin 2A.\sin 2B.\sin 2C}{2R\sin A.\sin B.\sin C}=2R\cos A.\cos B.\cos C$

Ex-central triangle

Let ABC be a triangle and I be the centre of incircle. Let I_1, I_2 and I_3 be the centres of the escribed circles



which are opposite to A, B, C respectively then $I_1I_2I_3$ is called the Ex-central triangle of ΔABC.

$I_1I_2I_3$ is a triangle, thus the triangle ABC is the pedal triangle of its ex-central triangle $I_1I_2I_3$. The angles of ex-central triangle $I_1I_2I_3$ are $90^o - \frac{A}{2}$, $90^o - \frac{B}{2}$, $90^o - \frac{C}{2}$ and sides are $I_1I_3 = 4R\cos\frac{B}{2}$; $I_1I_2 = 4R\cos\frac{C}{2}$; $I_2I_3 = 4R\cos\frac{A}{2}$.

Area and circum-radius of the ex-central triangle

Area of triangle

$= \frac{1}{2}$ (Product of two sides) × (sine of included angles)

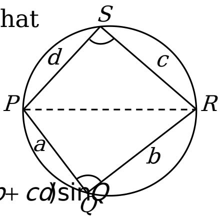
$$\Delta = \frac{1}{2}\left(4R\cos\frac{B}{2}\right).\left(4R\cos\frac{C}{2}\right)\times\sin\left(90^o - \frac{A}{2}\right)$$

$$\Delta = 8R^2\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}$$

$$\text{Circum-radius} = \frac{I_2I_3}{2\sin I_2I_1I_3} = \frac{4R\cos\frac{A}{2}}{2\sin\left(90^o - \frac{A}{2}\right)} = 2R.$$

Cyclic quadrilateral

A quadrilateral $PQRS$ is said to be cyclic quadrilateral if there exists a circle passing through all its four vertices P, Q, R and S.



Let a cyclic quadrilateral be such that $PQ = a, QR = b, RS = c$ and $SP = d$.

Then $\angle Q + \angle S = 180^o$, $\angle A + \angle C = 180^o$

Let $2s = a + b + c + d$

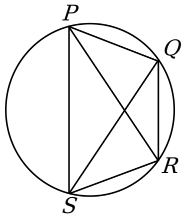
Area of cyclic quadrilateral $= \frac{1}{2}(ab + cd)\sin Q$

Also, area of cyclic quadrilateral $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$, where $2s = a + b + c + d$ and $\cos Q = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$.

(1) **Circumradius of cyclic quadrilateral :** Circum circle of quadrilateral $PQRS$ is also the circumcircle of ΔPQR.

$$R = \frac{1}{4\Delta}\sqrt{(ac+bd)(ad+bc)(ab+cd)} = \frac{1}{4}\sqrt{\frac{(ac+bd)(ad+bc)(ab+cd)}{(s-a)(s-b)(s-c)(s-d)}}.$$

(2) **Ptolemy's theorem :** In a cyclic quadrilateral $PQRS$, the product of diagonals is equal to the sum of the products of the length of the opposite sides *i.e.*, According to Ptolemy's theorem, for a cyclic quadrilateral $PQRS$ $PR.QS = PQ.RS + RQ.PS$.



Regular polygon

A regular polygon is a polygon which has all its sides equal and all its angles equal.

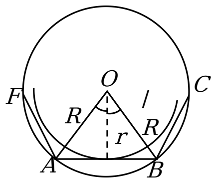
(1) Each interior angle of a regular polygon of n sides is $\left(\frac{2n-4}{n}\right)\times$ right angles $= \left[\frac{2n-4}{n}\right]\times\frac{\pi}{2}$ radians.

(2) The circle passing through all the vertices of a regular polygon is called its *circumscribed* circle.

If a is the length of each side of a regular polygon of n sides, then the radius R of the circumscribed circle, is given by $R = \frac{a}{2}.\text{cosec}\left(\frac{\pi}{n}\right)$

(3) The circle which can be inscribed within the regular polygon so as to touch all its sides is called its inscribed circle.

Again if a is the length of each side of a regular polygon of n sides, then the radius r of the inscribed circle is given by $r = \frac{a}{2}.\cot\left(\frac{\pi}{n}\right)$



(4) The area of a regular polygon is given by $\Delta = n$ × area of triangle OAB

$= \frac{1}{4}na^2\cot\left(\frac{\pi}{n}\right)$, (in terms of side)

$= nr^2.\tan\left(\frac{\pi}{n}\right)$, (in terms of in-radius)

$= \frac{n}{2}.R^2\sin\left(\frac{2\pi}{n}\right)$, (in terms of circum-radius)

Solutions of triangles

Different formulae will be used in different cases and sometimes the same problem may be solved in different ways by different formulae. We should, therefore, look for that formula which will suit the problem best.

(1) Solution of a right angled triangle

(2) Solution of a triangle in general

(1) **Solution of a right angled triangle**

(i) **When two sides are given:** Let the triangle be right angled at C. Then we can determine the remaining elements as given in the following table.

Table : 11.2

Given	Required
a, b	$\tan A = \frac{a}{b}$, $B = 90^o - A$, $c = \frac{a}{\sin A}$
a, c	$\sin A = \frac{a}{c}$, $b = c\cos A, B = 90^o - A$

(ii) **When a side and an acute angle are given :** In this case, we can determine the remaining elements as given in the following table

Table : 11.3

Given	Required
a, A	$B = 90^o - A, b = a\cot A, c = \frac{a}{\sin A}$
c, A	$B = 90^o - A,\ a = c\sin A, b = c\cos A$

(2) **Solution of a triangle in general**

(i) **When three sides *a, b* and *c* are given** : In this case, the remaining elements are determined by using the following formulae, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, where $2s = a + b + c =$ perimeter of triangle

$$\sin A = \frac{2\Delta}{bc}, \quad \sin B = \frac{2\Delta}{ac}, \quad \sin C = \frac{2\Delta}{ab}$$

$$\tan\frac{A}{2} = \frac{\Delta}{s(s-a)}, \ \tan\frac{B}{2} = \frac{\Delta}{s(s-b)}, \ \tan\frac{C}{2} = \frac{\Delta}{s(s-c)}.$$

(ii) **When two sides *a, b* and the included angle *C* are given:** In this case, we use the following formulae $\Delta = \frac{1}{2}ab\sin C$; $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$; $\frac{A+B}{2} = 90^o - \frac{C}{2}$ and $c = \frac{a\sin C}{\sin A}$.

(iii) **When one sides *a* and two angles *A* and *B* are given :** In this case, we use the following formulae to determine the remaining elements $A + B + C = 180^o$

$$C = 180^o - A - B$$

$$b = \frac{a\sin B}{\sin A} \text{ and } c = \frac{a\sin C}{\sin A} \qquad \Delta = \frac{1}{2}ca\sin B.$$

(iv) **When two sides *a, b* and the angle *A* opposite to one side is given :** In this case, we use the following formulae

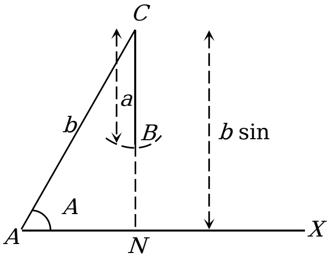
$$\sin B = \frac{b}{a}\sin A \qquad .....(i)$$

$$C = 180^o - (A+B), c = \frac{a\sin C}{\sin A}.$$

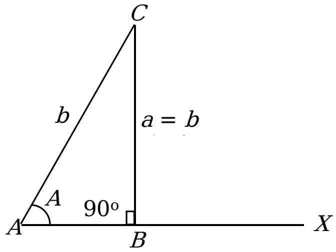
Special Cases

Case I : When *A* is an acute angle

(a) If $a < b\sin A$, there is no triangle. When $a < b\sin A$, then (i), $\sin B > 1$, which is impossible.



(b) If $a = b\sin A$, then only one triangle is possible which is right angled at *B*. When $a = b\sin A$, then from sine rule. $\sin B = 1$, $\therefore\ \angle B = 90^o$ from fig. It is clear that $CB = a = b\sin A$



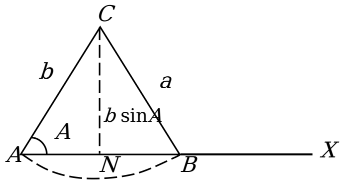
Thus, in this case, only one triangle is possible which is right angled at *B*.

(c) If $a > b\sin A$, then three possibilities will arise:

(i) $a = b$ In this case, from sine rule $\sin B = \sin A$

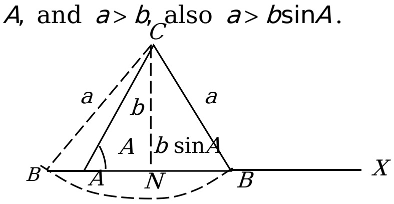
$\therefore B = A$ or $B = 180^o - A$.

But $B = 180^o - A$ $\quad A + B = 180^o$, which is not possible in a triangle. $\therefore$ In this case, we get $\angle A = \angle B$.



Hence, if $b = a > b\sin A$ then only one isosceles triangle *ABC* is possible in which $\angle A = \angle B$.

(ii) $a > b$ In the following figure, Let $AC = b, \angle CAX = A$, and $a > b$, also $a > b\sin A$.

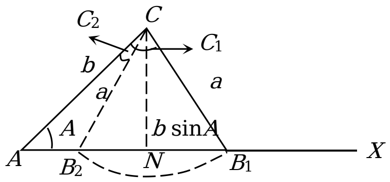


Now taking *C* as centre, if we draw an arc of radius *a*, it will intersect *AX* at one point *B* and hence only one ΔABC is constructed. Also this arc will intersect *XA* produced at B' and $\Delta AB'C$ is also formed but this Δ is inadmissible (because $\angle CAB'$ is an obtuse angle in this triangle)

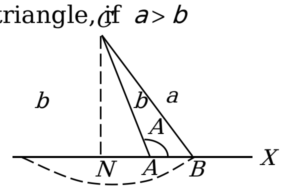
Hence, if $a > b$ and $a > b\sin A$, then only one triangle is possible.

(iii) $b > a$ (*i.e.*, $b > a > b\sin A$)

In fig. let $AC = b, \angle CAX = A$. Now taking *C* as centre, if we draw an arc of radius *a*, then it will intersect *AX* at two points B_1 and B_2. Hence if $b > a > \sin A$, then there are two triangles.

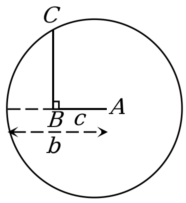


Case II : When A is an obtuse angle: In this case, there is only one triangle, if $a > b$



Case III: $b > c$ and $B = 90^o$

Again the circle with A as centre and b as radius will cut the line only in one point. So, only one triangle is possible.

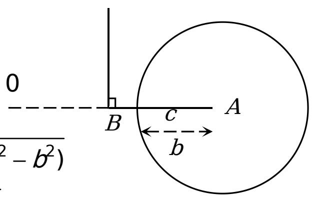


Case IV: $b \le c$ and $B = 90^o$

The circle with A as centre and b as radius will not cut the line in any point. So, no triangle is possible.

This is, sometimes called an ambiguous case.

Alternative method: By applying cosine rule, we have $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$



$$a^2 - (2c\cos B)a + (c^2 - b^2) = 0$$

$$a = c\cos B \pm \sqrt{(c\cos B)^2 - (c^2 - b^2)}$$

$$a = c\cos B \pm \sqrt{b^2 - (c\sin B)^2}$$

This equation leads to following cases:

Case I : If $b < c\sin B$, no such triangle is possible.

Case II : Let $b = c\sin B$, there are further following case.

(a) B is an obtuse angle

$\cos B$ is negative. There exists no such triangle.

(b) B is an acute angle

$\cos B$ is positive. There exists only one such triangle.

Case III : Let $b > c\sin B$. There are further following cases :

(a) B is an acute angle $\cos B$ is positive.

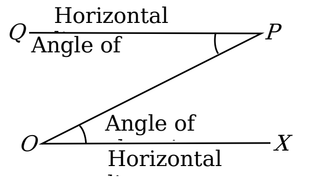
In this case two values of a will exists if and only if $c\cos B > \sqrt{b^2 - (c\sin B)^2}$ or $c > b$. Two such triangle is possible. If $c < b$, only one such triangle is possible.

(b) B is an obtuse angle $\cos B$ is negative. In this case triangle will exist if and only if $\sqrt{b^2 - (c\sin B)^2} > |c\cos B|$ $b > c$. So, in this case only one such triangle is possible. If $b < c$ there exists no such triangle.

Height and Distance

Some terminology related to height and distance

(1) **Angle of elevation and depression:** Let O and P be two points such that P is at higher level than O. Let PQ, OX be horizontal lines through P and O, respectively. If an observer (or eye) is at O and the object is at P, then $\angle XOP$ is called the angle of elevation of P as seen from O. This angle is also called the angular height of P from O.



If an observer (or eye) is at P and the object is at O, then $\angle QPO$ is called the angle of depression of O as seen from P.

(2) **Method of solving a problem of height and distance**

(i) Draw the figure neatly showing all angles and distances as far as possible.

(ii) Always remember that if a line is perpendicular to a plane then it is perpendicular to every line in that plane.

(iii) In the problems of heights and distances we come across a right angled triangle in which one (acute) angle and a side is given. Then to find the remaining sides, use trigonometrical ratios in which known (given) side is used, *i.e.*, use the formula.

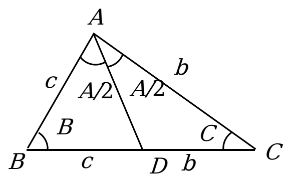
(iv) In any triangle other than right angled triangle, we can use 'the sine rule'.

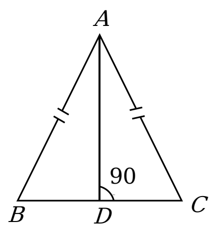
i.e., formula, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, or cosine formula *i.e.,* $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ etc.

(v) Find the length of a particular side from two different triangles containing that side common and then equate the two values thus obtained.

(3) **Geometrical properties and formulae for a triangle**

(i) In a triangle the internal bisector of an angle divides the opposite side in the ratio of the arms of the angle. $\frac{BD}{DC} = \frac{c}{b}$.





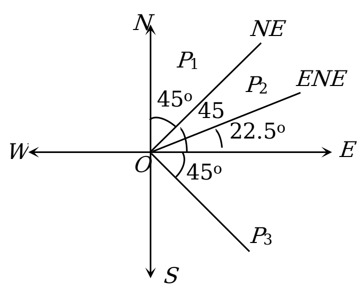
(ii) In an isosceles triangle the median is perpendicular to the base *i.e.,* $AD \perp BC$.

(iii) In similar triangles the corresponding sides are proportional.

(iv) The exterior angle is equal to sum of interior opposite angles.

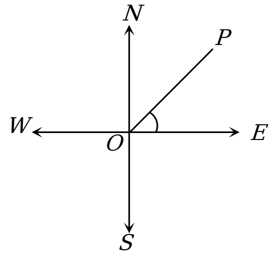
(4) **North-east :**

North-east means



equally inclined to north and east, south-east means equally inclined to south and east. *ENE* means equally inclined to east and north-east.

(5) **Bearing :** In the figure, if the observer and the object *i.e., O* and *P* be on the same level then bearing is defined. To measure the 'Bearing', the four standard directions East, West, North and South are taken as the cardinal directions.



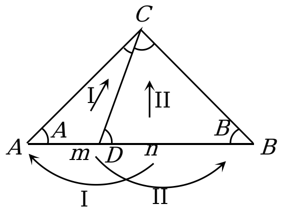
Angle between the line of observation *i.e.*, *OP* and any one standard direction– east, west, north or south is measured.

Thus, $\angle POE = \theta$ is called the bearing of point *P* with respect to *O* measured from east to north. In other words the bearing of *P* as seen from *O* is the direction in which *P* is seen from *O*.

(6) ***m-n* cot theorem of trigonometry :** $(m+n)\cot\theta$

$= m\cot\alpha - n\cot\beta = n\cot A - m\cot$

(on the right)

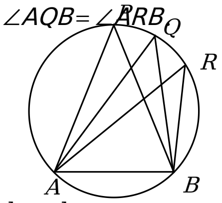


If θ is on the left then angle in the right is $\pi - \theta$ and $\cot(\pi - \theta) = -\cot\theta$. Hence in this case *m*- *n* theorem becomes

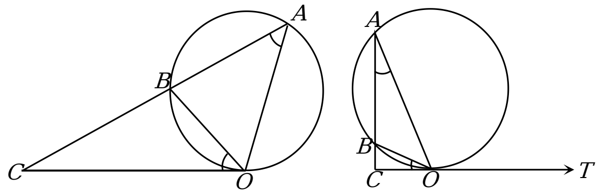
$-(m+n)\cot\theta = m\cot\alpha - n\cot\beta \;\; = n\cot A - m\cot B$ (on the left).

Some properties related to circle

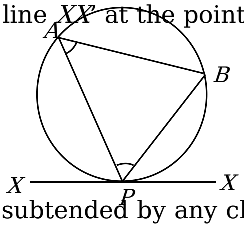
(1) Angles in the same segment of a circle are equal *i.e.*, $\angle APB = \angle AQB = \angle ARB$



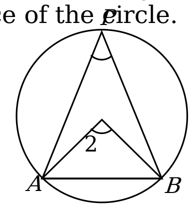
(2) Angles in the alternate segments of a circle are equal.



(3) If the line joining two points *A* and *B* subtends the greatest angle at a point *P* then the circle, will touch the straight line *XX'* at the point *P*.

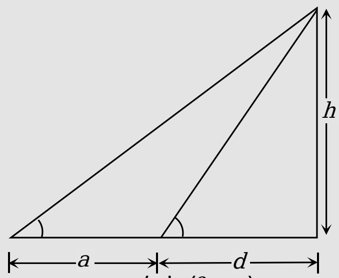


(4) The angle subtended by any chord at the centre is twice the angle subtended by the same on any point on the circumference of the circle.



Some important results

(1)

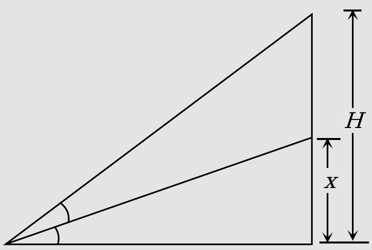


$a = h(\cot\alpha - \cot\beta) = \dfrac{h\sin(\beta - \alpha)}{\sin\alpha.\sin\beta}$

$\therefore h = a\sin\alpha\sin\beta\,\text{cosec}(\beta - \alpha)$ and

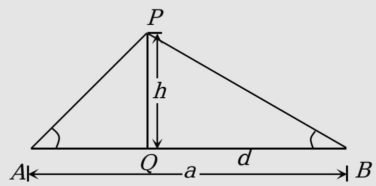
$d = h\cot\beta = a\sin\alpha.\cos\beta.\text{cosec}(\beta - \alpha)$

(2)



$H = x\cot\alpha\tan(\alpha + \beta)$

(3)

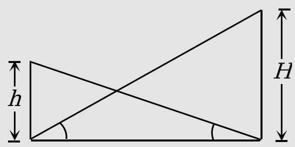


$a = h(\cot\alpha + \cot\beta)$, where by

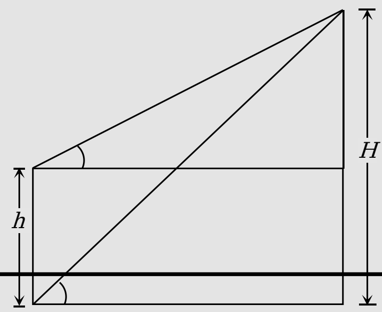
$h = a\sin\alpha.\sin\beta.\text{cosec}(\alpha + \beta)$ and

$d = h\cot\beta = a\sin\alpha.\cos\beta.\text{cosec}(\alpha + \beta)$

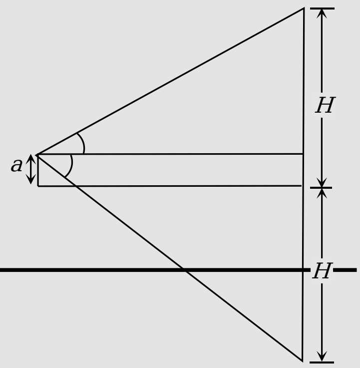
(4)

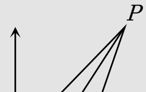


(5)



(6)



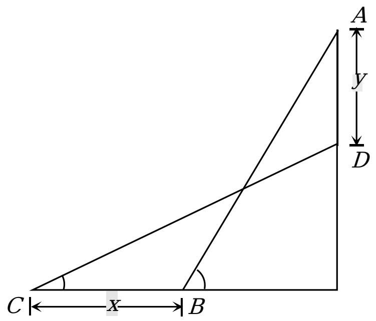


$H=\dfrac{h\cot\beta}{\cot\alpha}$

$h=\dfrac{H\sin(\beta-\alpha)}{\cos\alpha\sin\beta}$ or $H=\dfrac{h\cot\alpha}{\cot\alpha-\cot\beta}$

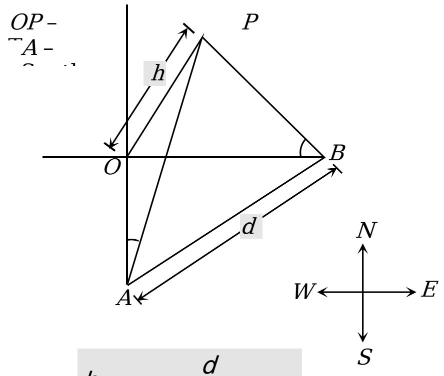
$H=\dfrac{a\sin(\alpha+\beta)}{\sin(\beta-\alpha)}$

(7)



$AB=CD$. Then, $x=y\tan\left(\dfrac{\alpha+\beta}{2}\right)$

(8)

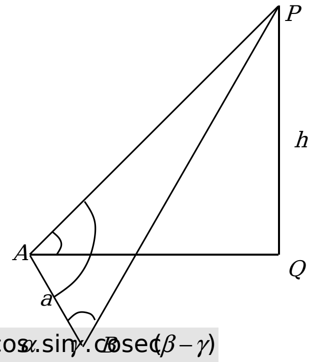


$h=\dfrac{d}{\sqrt{\cot^2\beta+\cot^2\alpha}}$

(9)

$h=\dfrac{AB}{\sqrt{\cot^2\beta-\cot^2\alpha}}$

(10)

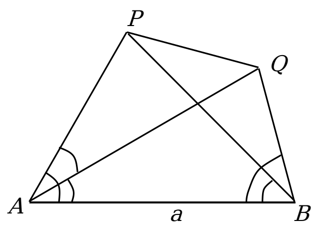


$h=AP\sin\alpha$

$=a\sin\alpha.\sin\gamma.\operatorname{cosec}(\beta-\gamma)$ and

if $AQ=d$, then $d=AP\cos\alpha=a\cos\alpha.\sin\gamma.\operatorname{cosec}(\beta-\gamma)$

(11)



$AP=a\sin\gamma.\operatorname{cosec}(\alpha-\gamma)$,

$AQ=a\sin\delta.\operatorname{cosec}(\beta-\delta)$

and apply, $PQ^2=AP^2+AQ^2-2AP.AQ\cos\theta$

Tips & Tricks

$\tan\dfrac{A}{2}\tan\dfrac{B}{2}=\dfrac{s-c}{s}$ $\quad\cot\dfrac{A}{2}\cot\dfrac{B}{2}=\dfrac{s}{s-c}$

$\tan\dfrac{A}{2}+\tan\dfrac{B}{2}=\dfrac{c}{s}\cot\dfrac{C}{2}$

$\tan\dfrac{A}{2}-\tan\dfrac{B}{2}=\dfrac{a-b}{s}(s-c)$

$\cot\dfrac{A}{2}+\cot\dfrac{B}{2}=\dfrac{c}{s-c}\cot\dfrac{C}{2}$

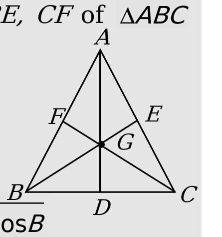
Circum-centre, Centroid and Orthocentre are collinear.

In any right angled triangle, the orthocentre coincides with the vertex containing the right angled.

The mid-point of the hypotenuse of a right angled triangle is equidistant from the three vertices of the triangle.

The mid-point of the hypotenuse of a right angled triangle is the circumcentre of the triangle.

The length of the medians AD, BE, CF of ΔABC are given by $AD=\dfrac{1}{2}\sqrt{2b^2+2c^2-a^2}$,



$=\dfrac{1}{2}\sqrt{b^2+c^2+2bc\cos A}$

$BE=\dfrac{1}{2}\sqrt{2c^2+2a^2-b^2}=\dfrac{1}{2}\sqrt{c^2+a^2+2ca\cos B}$

$CF=\dfrac{1}{2}\sqrt{2a^2+2b^2-c^2}=\dfrac{1}{2}\sqrt{a^2+b^2+2ab\cos C}$

The distance between the circumcentre O and centroid G of ΔABC is given by

$OG=\dfrac{1}{3}OH=\dfrac{1}{3}R\sqrt{1-8\cos A.\cos B.\cos C}$, where H is the orthocentre of ΔABC.

The distance between the orthocentre H and centroid G of ΔABC is given by $HG=\dfrac{2}{3}R\sqrt{1-8\cos A.\cos B.\cos C}$.

The distance between the circumcentre O and the incentre I of ΔABC given by

$OI=R\sqrt{1-8\sin\dfrac{A}{2}.\sin\dfrac{B}{2}.\sin\dfrac{C}{2}}$

If I_1 is the centre of the escribed circle opposite to the angle B, then $OI_1 = R\sqrt{1+8\sin\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}}$

Similarly, $OI_2 = R\sqrt{1+8\cos\frac{A}{2}.\sin\frac{B}{2}.\cos\frac{C}{2}}$,

$OI_3 = R\sqrt{1+8\cos\frac{A}{2}.\cos\frac{B}{2}.\sin\frac{C}{2}}$

$\sin A+\sin B+\sin C$ is maximum, when $A=B=C$.

$\cos A+\cos B+\cos C$ is maximum, when $A=B=C$.

$\tan A+\tan B+\tan C$ is minimum, when $A=B=C$.

$\cot A+\cot B+\cot C$ is minimum, when $A=B=C$

If $\cos A+\cos B+\cos C=\frac{3}{2}$, then the triangle is equilateral.

If $\sin A+\sin B+\sin C=\frac{3\sqrt{3}}{2}$, then the triangle is equilateral.

If $\tan A+\tan B+\tan C=3\sqrt{3}$, then the triangle is equilateral.

If $\cot A+\cot B+\cot C=\sqrt{3}$, then the triangle is equilateral.

If $\cos^2 A+\cos^2 B+\cos^2 C=1$, then the triangle is right angled.

Circle circumscribing the pedal triangle of a given triangle bisects the sides of the given triangle and also the lines joining the vertices of the given triangle to the orthocentre of the given triangle. This circle is known as "Nine point circle".

Circumcentre of the pedal triangle of a given triangle bisects the line joining the circum-centre of the triangle to the orthocentre.

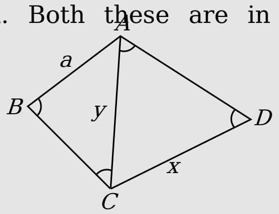
$\Sigma(p-q)=(p-q)+(q-r)+(r-p)=0$,

$\Sigma p(q-r)=p(q-r)+q(r-p)+r(p-q)=0$.

$\Sigma(p+a)(q-r)=\Sigma p(q-r)+a\Sigma(q-r)=0$.

In the application of sine rule, the following point be noted. We are given one side a and some other side x is to be found. Both these are in different triangles.

We choose a common side y of these triangles. Then apply sine rule for a and y in one triangle and for x and y for the other triangle and eliminate y.



Thus, we will get unknown side x in terms of a.

In the adjoining figure a is known side of ABC and x is unknown is side of triangle ACD. The common side of these triangle is $AC = y$ (say). Now apply sine rule

$$\frac{a}{\sin\alpha}=\frac{y}{\sin\beta} \quad .....(i) \qquad \text{and} \qquad \frac{x}{\sin\theta}=\frac{y}{\sin\gamma} \quad .....(ii)$$

Dividing (ii) by (i) we get,

$$\frac{x\sin\alpha}{a\sin\theta}=\frac{\sin\beta}{\sin\gamma}; \quad \therefore \quad x=\frac{a\sin\beta\sin\theta}{\sin\alpha\sin\gamma}$$