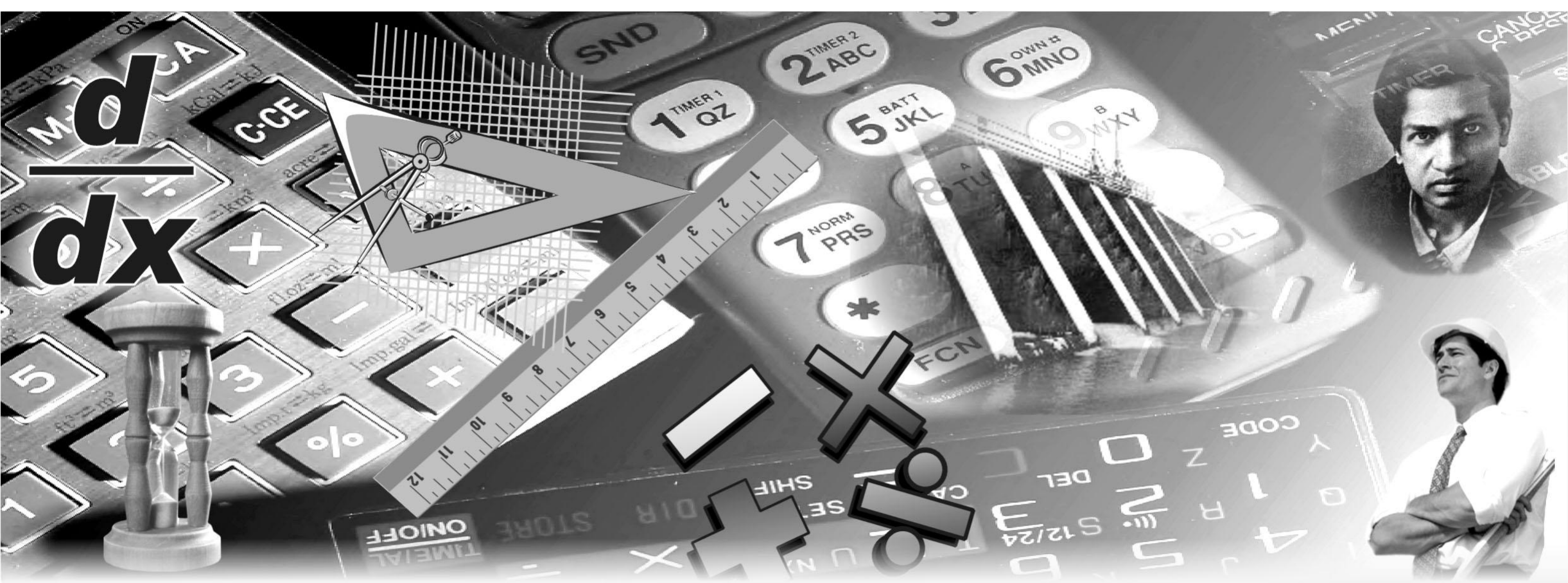


Chapter 28

Probability

Introduction

Numerical study of chances of occurrence of events is dealt in probability theory.

The theory of probability is applied in many diverse fields and the flexibility of the theory provides approximate tools for so great a variety of needs.

Definitions of various terms

(1) **Sample space :** The set of all possible outcomes of a trial (random experiment) is called its sample space. It is generally denoted by S and each outcome of the trial is said to be a sample point.

(2) **Event :** An event is a subset of a sample space.

(i) **Simple event :** An event containing only a single sample point is called an elementary or simple event.

(ii) **Compound events :** Events obtained by combining together two or more elementary events are known as the compound events or decomposable events.

(iii) **Equally likely events :** Events are equally likely if there is no reason for an event to occur in preference to any other event.

(iv) **Mutually exclusive or disjoint events :** Events are said to be mutually exclusive or disjoint or incompatible if the occurrence of any one of them prevents the occurrence of all the others.

(v) **Mutually non-exclusive events :** The events which are not mutually exclusive are known as compatible events or mutually non exclusive events.

(vi) **Independent events :** Events are said to be independent if the happening (or non-happening) of one event is not affected by the happening (or non-happening) of others.

(vii) **Dependent events :** Two or more events are said to be dependent if the happening of one event affects (partially or totally) other event.

(3) **Exhaustive number of cases :** The total number of possible outcomes of a random experiment in a trial is known as the exhaustive number of cases.

(4) **Favourable number of cases :** The number of cases favourable to an event in a trial is the total number of elementary events such that the occurrence of any one of them ensures the happening of the event.

(5) **Mutually exclusive and exhaustive system of events :** Let S be the sample space associated with a random experiment. Let A_1, A_2, ...A_n be subsets of S such that

(i) $A_i \cap A_j = \phi$ for i j and (ii) $A_1 \cup A_2 \cup .... \cup A_n = S$

Then the collection of events $A_1, A_2, ..., A_n$ is said to form a mutually exclusive and exhaustive system of events.

If $E_1, E_2, ..., E_n$ are elementary events associated with a random experiment, then

(i) $E_i \cap E_j = \phi$ for i j and (ii) $E_1 \cup E_2 \cup .... \cup E_n = S$

So, the collection of elementary events associated with a random experiment always form a system of mutually exclusive and exhaustive system of events.

In this system, $P(A_1 \cup A_2 .... \cup A_n)$

$$= P(A_1) + P(A_2) + ... + P(A_n) = 1.$$

Classical definition of probability

If a random experiment results in n mutually exclusive, equally likely and exhaustive outcomes, out of which m are favourable to the occurrence of an event A, then the probability of occurrence of A is given by

$$P(A) = \frac{m}{n} = \frac{\text{Number of outcomes favourable to } A}{\text{Number of total outcomes}}$$

It is obvious that $0 \le m \le n$. If an event A is certain to happen, then $m = n$, thus $P(A) = 1$.

If A is impossible to happen, then $m = 0$ and so $P(A) = 0$. Hence we conclude that $0 \le P(A) \le 1$.

Further, if $\overline{A}$ denotes negative of A *i.e.* event that A doesn't happen, then for above cases m, n; we shall have

$$P(\overline{A}) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A)\ , \qquad P(A) + P(\overline{A}) = 1.$$

Notations : For two events A and B,

(i) A or $\overline{A}$ or A^C stands for the non-occurrence or negation of A.

(ii) A B stands for the occurrence of at least one of A and B.

(iii) A B stands for the simultaneous occurrence of A and B.

(iv) A B stands for the non-occurrence of both A and B.

(v) A B stands for "the occurrence of A implies occurrence of B".

Problems based on combination and permutation

(1) **Problems based on combination or selection :** To solve such kind of problems, we use $^nC_r = \dfrac{n!}{r!(n-r)!}$.

(2) **Problems based on permutation or arrangement :** To solve such kind of problems, we use $^nP_r = \dfrac{n!}{(n-r)!}$.

Odds in favour and odds against an event

As a result of an experiment if "a" of the outcomes are favourable to an event E and "b" of the outcomes are against it, then we say that odds are a to b in favour of E or odds are b to a against E.

Thus odds in favour of an event E

$$= \frac{\text{Number of favourable cases}}{\text{Number of unfavourable cases}} = \frac{a}{b} = \frac{a/(a+b)}{b/(a+b)} = \frac{P(E)}{P(\overline{E})}.$$

Similarly, odds against an event E

$$= \frac{\text{Number of unfavourable cases}}{\text{Number of favourable cases}} = \frac{b}{a} = \frac{P(\overline{E})}{P(E)}.$$

Addition theorems on probability

Notations : (i) $P(A+B)$ or $P(A \cup B)$ = Probability of happening of A or B

= Probability of happening of the events A or B or both

= Probability of occurrence of at least one event A or B

(ii) $P(AB)$ or $P(A$ $B)$ = Probability of happening of events A and B together.

(1) **When events are not mutually exclusive :** If A and B are two events which are not mutually exclusive, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

or $\quad P(A+B) = P(A) + P(B) - P(AB)$

For any three events A, B, C

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

or $\quad P(A+B+C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC)$

(2) **When events are mutually exclusive :** If A and B are mutually exclusive events, then $n(A \cap B) = 0$

$$P(A \cap B) = 0$$

$$P(A \cup B) = P(A) + P(B).$$

For any three events A, B, C which are mutually exclusive,

$$P(A \cap B) = P(B \cap C) = P(C \cap A) = P(A \cap B \cap C) = 0$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C).$$

The probability of happening of any one of several mutually exclusive events is equal to the sum of their probabilities, *i.e.* if $A_1, A_2 \ldots\ldots A_n$ are mutually exclusive events, then

$$P(A_1 + A_2 + \ldots + A_n) = P(A_1) + P(A_2) + \ldots\ldots + P(A_n)$$

i.e. $P(\sum A_i) = \sum P(A_i)$.

(3) **When events are independent :** If A and B are independent events, then $P(A \cap B) = P(A).P(B)$

$$P(A \cup B) = P(A) + P(B) - P(A).P(B).$$

(4) **Some other theorems**

(i) Let A and B be two events associated with a random experiment, then

(a) $P(\overline{A} \cap B) = P(B) - P(A \cap B)$

(b) $P(A \cap \overline{B}) = P(A) - P(A \cap B)$

If B A, then

(a) $P(A \cap \overline{B}) = P(A) - P(B)$ (b) $P(B) \le P(A)$

Similarly if A B, then

(a) $(\overline{A} \cap B) = P(B) - P(A)$ (b) $P(A) \le P(B)$

Probability of occurrence of neither A nor B is

$$P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$$

(ii) **Generalization of the addition theorem :** If $A_1, A_2, \ldots\ldots, A_n$ are n events associated with a random experiment, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{\substack{i,j=1 \\ i \ne j}}^{n} P(A_i \cap A_j) + \sum_{\substack{i,j,k=1 \\ i \ne j \ne k}}^{n} P(A_i \cap A_j \cap A_k) + \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots\ldots \cap A_n).$$

If all the events $A_i (i = 1, 2 \ldots, n)$ are mutually exclusive, then $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$

i.e. $P(A_1 \cup A_2 \cup \ldots\ldots \cup A_n) = P(A_1) + P(A_2) + \ldots\ldots + P(A_n)$.

(iii) **Booley's inequality :** If $A_1, A_2, \ldots A_n$ are n events associated with a random experiment, then

(a) $P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$ (b) $P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$

These results can be easily established by using the Principle of mathematical induction.

Conditional probability

Let A and B be two events associated with a random experiment. Then, the probability of occurrence of A under the condition that B has already occurred and $P(B)$ 0, is called the conditional probability and it is denoted by $P(A/B)$.

Thus, $P(A/B)$ = Probability of occurrence of A, given that B has already happened.

$$= \frac{P(A \cap B)}{P(B)} = \frac{n(A \cap B)}{n(B)}.$$

Similarly, $P(B/A)$ = Probability of occurrence of B, given that A has already happened.

$$= \frac{P(A \cap B)}{P(A)} = \frac{n(A \cap B)}{n(A)}.$$

Sometimes, $P(A/B)$ is also used to denote the probability of occurrence of A when B occurs. Similarly, $P(B/A)$ is used to denote the probability of occurrence of B when A occurs.

(1) **Multiplication theorems on probability**

(i) If A and B are two events associated with a random experiment, then $P(A \cap B) = P(A).P(B/A)$, if $P(A)$ 0 or $P(A \cap B) = P(B).P(A/B)$, if $P(B)$ 0.

(ii) **Extension of multiplication theorem :** If $A_1, A_2, ..., A_n$ are n events related to a random experiment, then

$$P(A_1 \cap A_2 \cap A_3 \cap .... \cap A_n) = P(A_1)P(A_2/A_1)P(A_3/A_1 \cap A_2) \\ ....P(A_n/A_1 \cap A_2 \cap ... \cap A_{n-1}),$$

where $P(A_i/A_1 \cap A_2 \cap ... \cap A_{i-1})$ represents the conditional probability of the event A_i, given that the events $A_1, A_2, ..., A_{i-1}$ have already happened.

(iii) **Multiplication theorems for independent events :** If A and B are independent events associated with a random experiment, then $P(A \cap B) = P(A).P(B)$ *i.e.,* the probability of simultaneous occurrence of two independent events is equal to the product of their probabilities. By multiplication theorem, we have $P(A \cap B) = P(A).P(B/A)$. Since A and B are independent events, therefore $P(B/A) = P(B)$. Hence, $P(A \cap B) = P(A).P(B)$.

(iv) **Extension of multiplication theorem for independent events :** If $A_1, A_2, ..., A_n$ are independent events associated with a random experiment, then

$$P(A_1 \cap A_2 \cap A_3 \cap ... \cap A_n) = P(A_1)P(A_2)...P(A_n).$$

By multiplication theorem, we have

$$P(A_1 \cap A_2 \cap A_3 \cap ... \cap A_n) = P(A_1)P(A_2/A_1)P(A_3/A_1 \cap A_2) \\ ...P(A_n/A_1 \cap A_2 \cap ... \cap A_{n-1})$$

Since $A_1, A_2, ..., A_{n-1}, A_n$ are independent events, therefore

$$P(A_2/A_1) = P(A_2), P(A_3/A_1 \cap A_2) = P(A_3), ..., \\ P(A_n/A_1 \cap A_2 \cap ... \cap A_{n-1}) = P(A_n)$$

Hence, $P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1)P(A_2)....P(A_n)$.

(2) **Probability of at least one of the *n* independent events :** If $p_1, p_2, p_3, ..., p_n$ be the probabilities of happening of n independent events $A_1, A_2, A_3, ..., A_n$ respectively, then

(i) Probability of happening none of them

$$= P(\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 ...... \cap \overline{A}_n) = P(\overline{A}_1).P(\overline{A}_2).P(\overline{A}_3).....P(\overline{A}_n).$$

$$= (1-p_1)(1-p_2)(1-p_3).....(1-p_n)$$

(ii) Probability of happening at least one of them

$$= P(A_1 \cup A_2 \cup A_3 .... \cup A_n) = 1 - P(\overline{A}_1)P(\overline{A}_2)P(\overline{A}_3)....P(\overline{A}_n).$$

$$= 1 - (1-p_1)(1-p_2)(1-p_3)...(1-p_n)$$

(iii) Probability of happening of first event and not happening of the remaining $= P(A_1)P(\overline{A}_2)P(\overline{A}_3).....P(\overline{A}_n)$

$$= p_1(1-p_2)(1-p_3).......(1-p_n)$$

Total probability and Baye's rule

(1) **The law of total probability :** Let S be the sample space and let $E_1, E_2, ...E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with E_1 or E_2 or ...or E_n, then

$$P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + ... + P(E_n)\,P(A/E_n).$$

(2) **Baye's rule :** Let S be a sample space and $E_1, E_2, ...E_n$ be n mutually exclusive events such that $\bigcup_{i=1}^{n} E_i = S$ and $P(E_i) > 0$ for i = 1, 2, ……, n. We can think of (E_i's as the causes that lead to the outcome of an experiment. The probabilities $P(E_i)$, i = 1, 2, ….., n are called prior probabilities. Suppose the experiment results in an outcome of event A, where $P(A) > 0$. We have to find the probability that the observed event A was due to cause E_i, that is, we seek the conditional probability $P(E_i/A)$. These probabilities are called posterior probabilities, given by Baye's rule as

$$P(E_i/A) = \frac{P(E_i).P(A/E_i)}{\sum_{k=1}^{n} P(E_k)\,P(A/E_k)}.$$

Binomial distribution

(1) **Geometrical method for probability :** When the number of points in the sample space is infinite, it becomes difficult to apply classical definition of probability. For instance, if we are interested to find the probability that a point selected at random from the interval [1, 6] lies either in the interval [1, 2] or [5, 6], we cannot apply the classical definition of probability. In this case we define the probability as follows:

$$P\{x \in A\} = \frac{\text{Measure of region } A}{\text{Measure of the sample space } S}$$

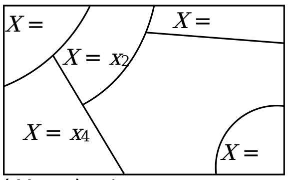
where measure stands for length, area or volume depending upon whether S is a one-dimensional, two-dimensional or three-dimensional region.

(2) **Probability distribution :** Let S be a sample space. A random variable X is a function from the set S to R, the set of real numbers.

For example, the sample space for a throw of a pair of dice is

$$S = \begin{Bmatrix} 11, & 12, & \cdots, & 16 \\ 21, & 22, & \cdots, & 26 \\ \vdots & \vdots & \ddots & \vdots \\ 61, & 62, & \cdots, & 66 \end{Bmatrix}$$

Let X be the sum of numbers on the dice. Then $X(12) = 3, X(43) = 7$, etc. Also, $\{X = 7\}$ is the event {61, 52, 43, 34, 25, 16}. In general, if X is a random variable defined on the
sample space S and r is a real number, then $\{X = r\}$ is an event.
If the random variable X takes n distinct values $x_1, x_2, \ldots, x_n$, then $\{X = x_1\}$, $\{X = x_2\}, \ldots, \{X = x_n\}$ are mutually exclusive and exhaustive events.



Now, since $(X = x_i)$ is an event, we can talk of $P(X = x_i)$. If $P(X = x_i) = P_i (1 \le i \le n)$, then the system of numbers.

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}$$ is said to be the probability distribution of the random variable X.

The expectation (mean) of the random variable X is defined as $E(X) = \sum_{i=1}^{n} p_i x_i$ and the variance of X is defined as

$$\text{var}(X) = \sum_{i=1}^{n} p_i (x_i - E(X))^2 = \sum_{i=1}^{n} p_i x_i^2 - (E(X))^2 \;.$$

(3) **Binomial probability distribution :** A random variable X which takes values 0, 1, 2, …, n is said to follow binomial distribution if its probability distribution function is given by $P(X = r) = {}^nC_r p^r q^{n-r}, r = 0, 1, 2, \ldots, n$

where $p, q > 0$ such that $p + q = 1$.

The notation $X \sim B(n, p)$ is generally used to denote that the random variable X follows binomial distribution with parameters n and p.

We have $P(X = 0) + P(X = 1) + \ldots + P(X = n)$.

$$= {}^nC_0 p^0 q^{n-0} + {}^nC_1 p^1 q^{n-1} + \ldots + {}^nC_n p^n q^{n-n} = (q + p)^n = 1^n = 1$$

Now probability of

(a) Occurrence of the event exactly r times

$$P(X = r) = {}^nC_r q^{n-r} p^r \;.$$

(b) Occurrence of the event at least r times

$$P(X \ge r) = {}^nC_r q^{n-r} p^r + \ldots + p^n = \sum_{X=r}^{n} {}^nC_X p^X q^{n-X} \;.$$

(c) Occurrence of the event at the most r times

$$P(0 \le X \le r) = q^n + {}^nC_1 q^{n-1} p + \ldots + {}^nC_r q^{n-r} p^r = \sum_{X=0}^{r} p^X q^{n-X} \;.$$

If the probability of happening of an event in one trial be p, then the probability of successive happening of that event in r trials is p^r.

If n trials constitute an experiment and the experiment is repeated N times, then the frequencies of 0, 1, 2, …, n successes are given by $N.P(X = 0), N.P(X = 1), N.P(X = 2), \ldots, N.P(X = n)$.

(i) **Mean and variance of the binomial distribution**

The binomial probability distribution is

X	0	1	2	…..	n
$P(X)$	${}^nC_0 q^n p^0$	${}^nC_1 q^{n-1} p$	${}^nC_2 q^{n-2} p^2$	…..	${}^nC_n q^0 p^n$

The mean of this distribution is

$$\sum_{i=1}^{n} X_i p_i = \sum_{X=1}^{n} X . {}^nC_X q^{n-X} p^X = np,$$

The variance of the Binomial distribution is $\sigma^2 = npq$ and the standard deviation is $\sigma = \sqrt{(npq)}$.

(ii) **Use of multinomial expansion :** If a die has m faces marked with the numbers 1, 2, 3, ….m and if such n dice are thrown, then the probability that the sum of the numbers exhibited on the upper faces equal to p is given by the coefficient of x^p in the expansion of $\frac{(x + x^2 + x^3 + \ldots + x^m)^n}{m^n}$.

(4) **The poisson distribution :** Let X be a discrete random variable which can take on the values 0, 1, 2,... such that the probability function of X is given by

$$f(x) = P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$$

where λ is a given positive constant. This distribution is called the Poisson distribution and a random variable having this distribution is said to be Poisson distributed.

Tips & Tricks

Independent events are always taken from different experiments, while mutually exclusive events are taken from a single experiment.

Independent events can happen together while mutually exclusive events cannot happen together.

Independent events are connected by the word "and" but mutually exclusive events are connected by the word "or".

Number of exhaustive cases of tossing n coins simultaneously (or of tossing a coin n times) = 2^n.

Number of exhaustive cases of throwing n dice simultaneously (or throwing one dice n times) = 6^n.

Probability regarding *n* letters and their envelopes : If n letters corresponding to n envelopes are placed in the envelopes at random, then

(i) Probability that all letters are in right envelopes $= 1/n!$.

(ii) Probability that all letters are not in right envelopes $= 1 - \frac{1}{n!}$.

(iii) Probability that no letter is in right envelopes

$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^n \frac{1}{n!}.$$

(iv) Probability that exactly r letters are in right envelopes $= \frac{1}{r!}\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots.. + (-1)^{n-r}\frac{1}{(n-r)!}\right]$.

If odds in favour of an event are $a : b$, then the probability of the occurrence of that event is $\frac{a}{a+b}$ and the probability of non-occurrence of that event is $\frac{b}{a+b}$.

If odds against an event are $a : b$, then the probability of the occurrence of that event is $\frac{b}{a+b}$ and the probability of non-occurrence of that event is $\frac{a}{a+b}$.

Let A, B, and C are three arbitrary events. Then

Verbal description of event	**Equivalent set theoretic notation**
(i) Only A occurs	(i) $A \cap \overline{B} \cap \overline{C}$
(ii) Both A and B, but not C occur	(ii) $A \cap B \cap \overline{C}$
(iii) All the three events occur	(iii) $A \cap B \cap C$
(iv) At least one occurs	(iv) $A \cup B \cup C$
(v) At least two occur	(v) $(A \cap B) \cup (B \cap C) \cup (A \cap C)$
(vi) One and no more occurs	(vi) $(A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C) \cup (\overline{A} \cap \overline{B} \cap C)$
(vii) Exactly two of A, B and C occur	(vii) $(A \cap B \cap \overline{C}) \cup (\overline{A} \cap B \cap C) \cup (A \cap \overline{B} \cap C) \cup (A \cap \overline{B} \cap C)$
(viii) None occurs	(viii) $\overline{A} \cap \overline{B} \cap \overline{C} = \overline{A \cup B \cup C}$
(ix) Not more than two occur	(ix) $(A \cap B) \cup (B \cap C) \cup (A \cap C) - (A \cap B \cap C)$
(x) Exactly one of A and B occurs	(x) $(A \cap \overline{B}) \cup (\overline{A} \cap B)$