16/Nov/21
Tuesday

Linear Programming Problem

Linear programming problem is one of the most widely used method in operational's research. We can define the general form of a linear programming problem as to find the values of decision variable $x_1, x_2, x_3 \ldots x_n$ which optimizes (either maximizes or minimizes) the objective function

$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ subject to the linear constraints.

(cap)

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n \le, =, \ge b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n \le, =, \ge b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n \le, =, \ge b_m$$

with nonnegative restrictions $x_1, x_2, x_3 \ldots x_n \ge 0$

Matrix form of LPP

Find X which optimizes Max/Min $Z = CX$ subject to the constraints $AX \le, =, \ge B$ with nonnegative restrictions $X \ge 0$

Standard form of LPP

Finding $x_1, x_2, x_3 \ldots x_n$ which maximizes

$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

~~where $x_i \geq 0$~~ ; with nonnegativity condition $x_i \geq 0$

such that $i = 1$ to n.

Slack variable

If the constraints of LPP is of the form

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \leq b_i$$

then the nonnegative variable x_{n+1} such that

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n + x_{n+1} = b_i$$

is called the slack variable

Surplus variable

If the constraint of the LPP of the form

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \geq b_i$$

then a nonnegative variable x_{n+1} such that

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n + - x_{n+1} = b_i$$

is called a surplus variable.

23/11/21 Tuesday ① Convert the following LPP to standard form

Minimize $z = x_1 + 2x_2 - 4x_3$

Subject to the constraint,

$$2x_1 + x_2 + 3x_3 \leq 16$$

$$x_1 + x_2 + x_3 = 8$$

$$-x_1 + 2x_2 - x_3 \geq -7$$

$$x_1 + x_2 \leq 2$$

$$x_1, x_2, x_3 \geq 0$$

ans: Standard form is,

Introduce 3 slack variable say x_4 x_5 and x_6

Maximize $z = \text{Max}(-z)$

$= \text{Maximize}(z^*) = \text{Minimize}(-z)$

$= -x_1 - 2x_2 + 4x_3 + 0x_4 + 0x_5 + 0x_6$.

Subject to

$$2x_1 + x_2 + 3x_3 + x_4 = 16$$

$$x_1 + x_2 + x_3 = 8$$

$$x_1 - 2x_2 + x_3 + x_5 = 7$$

$$x_1 + x_2 + x_6 = 2$$

with $x_i > 0$ where $i = 1$ to 6

(2) Rewrite the following LPP in the standard form

Maximize $z = 2x_1 + x_2 + 4x_3$

Subject to $-2x_1 + 4x_2 \leq 4$

$$x_1 + 2x_2 + x_3 \geq 5$$

$$2x_1 + 3x_3 \leq 2$$

where $x_1, x_2 \geq 0$ and x_3 unrestricted.

ans: Standard form,

Let $x_3 = x_3' - x_3''$ where $x_3', x_3'' \geq 0$

Introduce slack variables x_4 and x_6 and surplus variable x_5

Max $z = 2x_1 + x_2 + 4x_3 + 0\cdot(x_4) + 0x_5 + 0x_6$

$$= 2x_1 + x_2 + 4x_3' - 4x_3'' + 0x_4 + 0x_5 + 0x_6$$

Subject to.

$$-2x_1 + 4x_2 + x_4 = 4$$

$$x_1 + 2x_2 + x_3' - x_3'' - x_5 = 5$$

$$2x_1 + 3(x_3' - x_3'') + x_6 = 2.$$

Different solutions of LPP

1. Feasible solution.
~~2. Basic~~ solution.
2. Basic feasible solution.
3. Optimum Basic feasible solution.

1. Feasible solution

A feasible solution is any set of non negative values of the unknown variables which satisfy the constraints

2. Basic solution

Consider the standard LPP

Max $Z = CX$

Subject to the constraints

$AX = B$.

with $X \geq 0$

Here A is an $m \times n$ matrix ($A_{m \times n}$) with $n > m$

Now suppose $n-m$ variables are set to zero then the resulting system of m equations and m unknowns has a unique solution called the basic solution. denoted by X_B

Basic Variables

The m variables associated with X_B are called basic variables and the remaining $n-m$ variables are called non basic variable

The maximum no. of basic solution is ${}^nC_m$

3. Basic feasible solution

The basic solution which satisfy the non negative restriction is called a basic feasible solution.

4. Optimum basic feasible solution

A basic feasible solution which optimizes the objective function of LPP is called optimum basic feasible solution

Degenerate Solution

A basic solution is called a degenerate solution if one or more of the basic variable becomes equal zero

If all the basic variables are positive then the basic solution is called a nondegenerate solution

① Find all the basic solution of.

$2x_1 + x_2 + 4x_3 = 11$

$3x_1 + x_2 + 5x_3 = 14$

ans: $n = 3$

$m = 2$

Max. possible solution, $= {}^{n}C_{m} = {}^{3}C_{2} = 3$

No. of variables to be set zero $= n - m = 3 - 2 = 1$

Let $x_1 = 0$

$x_2 + 4x_3 = 11$

$x_2 + 5x_3 = 14$.

$x_3 = 3$

$x_2 + 4 \times 3 = 11$

$\underline{\underline{x_2 = -1}}$

$X_1 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$ is a basic sol^n but not feasible sol^n.

let $x_2 = 0$

$2x_1 + 4x_3 = 11$

$2x_1 + 5x_3 = 14$.

$-x_3 = -3$

$\underline{\underline{x_3 = 3}}$.

$2x_1 + 4 \times 3 = 11$

$2x_1 = -1$

$x_1 = -1/2$.

$X_2 = \begin{bmatrix} -1/2 \\ 0 \\ 3 \end{bmatrix}$ is a basic sol^n but not feasible

$6x_1 + 12x_3 = 33$

$6x_1 + 10x_3 = 28$

$$2x_3 = 5$$

$$x_3 = 5/2$$

and $2x_1 = 11 - 4 \times \frac{5}{2}$

$$x_1 = 1/2$$

$\therefore$ $X_2 = \begin{bmatrix} 1/2 \\ 0 \\ 5/2 \end{bmatrix}$ feasible solution

Let $x_3 = 0$

$$2x_1 + x_2 = 11$$

$$3x_1 + x_2 = 14.$$

$$-x_1 = -3$$

$$x_1 = 3$$

$\therefore$ $x_2 = 11 - 6 = 5.$

$\therefore$ $X_3 = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}$ feasible solution.

(2) Obtain all the basic feasible solutions of.

$$x_1 + 2x_2 + x_3 = 4$$

$$2x_1 + x_2 + 5x_3 = 5.$$

ans: $n = 3.$

$m = 2.$

$x_1 + 2x_2 + x_3 = 4$

$2x_1 + x_2 + 5x_3 = 5.$

Maximum possible solution $= {}^{n}C_m = 3.$

No: of variable to set zero $= n - m = 3 - 2 = 1$

Let $x_1 = 0$

$2x_2 + x_3 = 4$

$x_2 + 5x_3 = 5$

$2x_2 + x_3 = 4$

$\overset{-}{}2x_2 \overset{-}{+} 10x_3 \overset{-}{=} 10.$

$-9x_3 = -6$

$x_3 = 2/3$

$2x_2 = 4 - \frac{2}{3}$

$x_2 = \frac{10/3}{2} = 5/3$

$\therefore X = \begin{bmatrix} 0 \\ 5/3 \\ 2/3 \end{bmatrix}$ is a feasible solution

Let $x_2 = 0$

$$x_1 + x_3 = 4.$$

$$2x_1 + 5x_3 = 5.$$

$$2x_1 + 2x_3 = 8$$

$$\overset{-}{2}x_1 + \overset{-}{5}x_3 = 5$$

$$-3x_3 = 3$$

$$x_3 = -1$$

$$x_1 + x_3 = 4$$

$$x_1 = 5$$

$$x = \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}$$ is a basic solution but not feasible solution.

Let $x_3 = 0$

$$x_1 + 2x_2 = 4$$

$$2x_1 + x_2 = 5$$

$$2x_1 + 4x_2 = 8$$

$$\overset{-}{2}x_1 \overset{-}{+} x_2 \overset{-}{=} 5$$

$$3x_2 = 3$$

$$x_2 = 1$$

$x_1 + 2x_2 = 4$

$x_1 = 2$

$$x = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$ is a feasible solution.

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Tuesday

Simplex Method.

Solve the LPP, Minimize $z = x_1 - 3x_2 + 2x_3$

Subject to $3x_1 - x_2 + 2x_3 \leq 7$

$-2x_1 + 4x_2 \leq 12$

$-4x_1 + 3x_2 + 8x_3 \leq 10$

$x_1\ x_2\ x_3 \geq 0$

ans: Introduce slack variable and write LPP in std. form

$3x_1 - x_2 + 2x_3 + x_4 = 7$

$-2x_1 + 4x_2 + x_5 = 12$

$-4x_1 + 3x_2 + 8x_3 + x_6 = 10$

Max $z^* = -x_1 + 3x_2 - 2x_3 + 0x_4 + 0x_5 + 0x_6$.

$x_i \geq 0$ where $i = 1$ to 6.

Now we have to find initial basic feasible soln.

$n = 6$ (no: of unknown)

$m = 3$ (no: of equation)

$n - m = 6 - 3 = 3$

Let $x_1 = 0$ $x_2 = 0$ $x_3 = 0$

then, $x_4 = 7$

$x_5 = 12$

$x_6 = 10$.

This forms the initial basic feasible solution.

Here $z^* = 0$

1st Simplex table.

C_B (cost)	X_B basic var	x_1	x_2	x_3	x_4	x_5	x_6	B	Ratio B/x_j
		coe: of constraint			unit matrix				
0	x_4	3	-1	2	1	0	0	7	—
0	x_5	-2	4	0	0	1	0	12	$\frac{12}{4} = 3$ →
0	x_6	-4	3	8	0	0	1	10	$\frac{10}{3} = 3.3$
	$Z_j = C_B x_j$	0	0	0	0	0	0		
coe: of objective fun z^* →	C_j	-1	3	-2	0	0	0		
	$\Delta_j = Z_j - C_j$	1	-3 ↑	2	0	0	0		

[Δ_j - net evaluation

If $\Delta_j \geq 0$ then optimum solution.

If $\Delta_j < 0$ for any then consider, vector correspond.. the most negative value. and the column is called a pivotal column.

This vectore is called incoming vector.

If all Δ_j are negative it is called unbounded sol.

lecture note.

Now draw the column Ratio B/x_j and the minimum value among this is outgoing vector.]

x_2 is incoming vector

x_5 is outgoing vector.

IInd simplex table.

[In the new table we make entry (x_2) to be 1 by dividing - ie; the second row to make pivotal element 1.

and makes other elements zero].

$R_2 \rightarrow R_2/4$

$R_1 \rightarrow R_1 + R_4$

C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	B	Ratio B/x_j
0	x_4	$5/2$	0	2	1	$1/4$	0	10	$\frac{10}{5/2} = 4$
3	x_2	$-1/2$	1	0	0	$1/4$	0	3	–
0	x_6	$-5/2$	0	8	0	$-3/4$	1	1	–
$z_j = C_B x_j$		$-3/2$	3	0	0	$3/4$	0		
c_j		-1	3	-2	0	0	0		
$\Delta_j = z_j - c_j$		$-1/2$	0	2	0	$3/4$	0		

↑

not optimum solution.

x_1 is incoming vector.

$-\frac{3}{2}+1$

x_4 leaves the basis and x_1 enters the basis

x_1 is incoming vector

x_4 is outgoing vector.

3rd simple table

$R_1 \longrightarrow \frac{2}{5} R_1$

$R_2 \longrightarrow \frac{R_1}{2} + R_2$

$R_3 \longrightarrow R_3 + \frac{5}{2} \cdot R_1$

C_B	x_B	x_1	x_2	x_3	x_4	x_5	x_6	B
-1	$x_{4.1}$	1	0	4/5	2/5	1/10	0	4
3	x_2	0	1	2/5	1/5	3/10	0	5
0	x_6	0	0	10	1	-1/2	1	11
$Z_j = C_B x_j$		-1	3	2/5	1/5	4/5	0	
C_j		-1	3	-2	0	0	0	
$\Delta_j = Z_j - C_j$		0	0	12/5	1/5	4/5	0	

Here all the net evaluations are possitive Hence the current basic feasible solution is optimum.

Solutions.

$x_1 = 4$

$x_2 = 5$

$x_3 = 0$

$\therefore$ Max $z^* = -4 + 3\times5 - 2\times0$

$= \underline{\underline{11}}$

$\therefore$ Min. $z = -\text{Max } z^*$

$= -(11) = \underline{\underline{-11}}$

So the solution of LPP is, min $(z) = -11$ at

$x_1 = 4$, $x_2 = 5$, $x_3 = 0$