Chapter
19
Vector Algebra

Introduction

Vectors represent one of the most important mathematical systems, which is used to handle certain types of problems in Geometry, Mechanics and other branches of Applied Mathematics, Physics and Engineering.

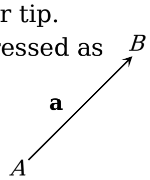
Scalar and vector quantities : Those quantities which have only magnitude and which are not related to any fixed direction in space are called *scalar quantities*, or briefly scalars. *Examples:* Mass, Volume, Density, Work, Temperature etc. Those quantities which have both magnitude and direction, are called vectors. Displacement, velocity, acceleration, momentum, weight, force are examples of vector quantities.

Representation of vectors

Geometrically a vector is represent by a line segment. For example, $\mathbf{a}=\overrightarrow{AB}$. Here A is called the initial point and B, the terminal point or tip.

Magnitude or modulus of $\mathbf{a}$ is expressed as

$$|\mathbf{a}|=|\overrightarrow{AB}|=AB.$$



Types of vector

(1) **Zero or null vector :** A vector whose magnitude is zero is called zero or null vector and it is represented by $\vec{O}$.

(2) **Unit vector :** A vector whose modulus is unity, is called a unit vector. The unit vector in the direction of a vector $\mathbf{a}$ is denoted by $\hat{\mathbf{a}}$, read as "*a cap*". Thus, $|\hat{\mathbf{a}}|=1$.

$$\hat{\mathbf{a}}=\frac{\mathbf{a}}{|\mathbf{a}|}=\frac{\text{Vector } a}{\text{Magnitude of } a}$$

(3) **Like and unlike vectors :** Vectors are said to be like when they have the same sense of direction and unlike when they have opposite directions.

(4) **Collinear or parallel vectors :** Vectors having the same or parallel supports are called collinear or parallel vectors.

(5) **Co-initial vectors :** Vectors having the same initial point are called *co-initial vectors*.

(6) **Coplanar vectors :** A system of vectors is said to be coplanar, if their supports are parallel to the same plane.

Two vectors having the same initial point are always coplanar but such three or more vectors may or may not be coplanar.

(7) **Coterminous vectors :** Vectors having the same terminal point are called coterminous vectors.

(8) **Negative of a vector :** The vector which has the same magnitude as the vector $\mathbf{a}$ but opposite direction, is called the negative of $\mathbf{a}$ and is denoted by $-\mathbf{a}$. Thus, if $\overrightarrow{PQ}=\mathbf{a}$, then $\overrightarrow{QP}=-\mathbf{a}$.

(9) **Reciprocal of a vector :** A vector having the same direction as that of a given vector $\mathbf{a}$ but magnitude equal to the reciprocal of the given vector is known as the reciprocal of $\mathbf{a}$ and is denoted by $\mathbf{a}^{-1}$. Thus, if $|\mathbf{a}|=a, |\mathbf{a}^{-1}|=\frac{1}{a}$.

(10) **Localized and free vectors :** A vector which is drawn parallel to a given vector through a specified point in space is called a localized vector. For example, a force acting on a rigid body is a localized vector as its effect depends on the line of action of the force. If the value of a vector depends only on its length and

direction and is independent of its position in the space, it is called a free vector.

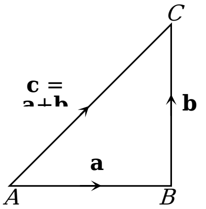
(11) **Position vectors :** The vector $\overrightarrow{OA}$ which represents the position of the point A with respect to a fixed point O (called origin) is called position vector of the point A. If (x, y, z) are co-ordinates of the point A, then $\overrightarrow{OA} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

(12) **Equality of vectors :** Two vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be equal, if (i) $|\mathbf{a}| = |\mathbf{b}|$ (ii) They have the same or parallel support and (iii) The same sense.

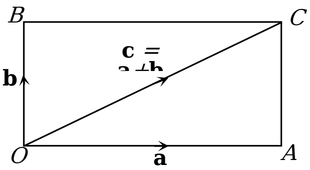
Properties of vectors

(1) **Addition of vectors**

(i) **Triangle law of addition :** If in a ΔABC, $\overrightarrow{AB} = \mathbf{a}$ $\overrightarrow{BC} = \mathbf{b}$ and $\overrightarrow{AC} = \mathbf{c}$, then $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ *i.e.,* **a** + **b** = **c**.



(ii) **Parallelogram law of addition :** If in a parallelogram $OACB$, $\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$



Then $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$ *i.e.,* $\mathbf{a} + \mathbf{b} = \mathbf{c}$, where OC is a diagonal of the parallelogram $OABC$.

(iii) **Addition in component form :** If the vectors are defined in terms of **i**, **j** and **k**, *i.e.,* if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then their sum is defined as $\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}$.

Properties of vector addition : Vector addition has the following properties.

(a) **Binary operation :** The sum of two vectors is always a vector.

(b) **Commutativity :** For any two vectors **a** and **b**, **a** + **b** = **b** + **a.**

(c) **Associativity :** For any three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$.

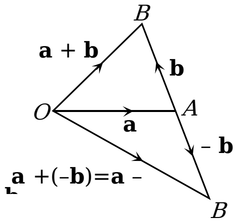
(d) **Identity :** Zero vector is the identity for addition. For any vector $\mathbf{a}$, $\mathbf{0} + \mathbf{a} = \mathbf{a} = \mathbf{a} + \mathbf{0}$

(e) **Additive inverse :** For every vector **a** its negative vector $-\mathbf{a}$ exists such that $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$ *i.e.,* $(-\mathbf{a})$ is the additive inverse of the vector **a**.

(2) **Subtraction of vectors :** If **a** and **b** are two vectors, then their subtraction $\mathbf{a} - \mathbf{b}$ is defined as $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$ where $-\mathbf{b}$ is the negative of **b** having magnitude equal to that of **b** and direction opposite to **b**.

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

Then $\mathbf{a} - \mathbf{b} = (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j} + (a_3 - b_3)\mathbf{k}$.



Properties of vector subtraction

(i) $\mathbf{a} - \mathbf{b} \neq \mathbf{b} - \mathbf{a}$ (ii) $(\mathbf{a} - \mathbf{b}) - \mathbf{c} \neq \mathbf{a} - (\mathbf{b} - \mathbf{c})$

(iii) Since any one side of a triangle is less than the sum and greater than the difference of the other two sides, so for any two vectors a and b, we have

(a) $|\mathbf{a} + \mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$ (b) $|\mathbf{a} + \mathbf{b}| \ge |\mathbf{a}| - |\mathbf{b}|$

(c) $|\mathbf{a} - \mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$ (d) $|\mathbf{a} - \mathbf{b}| \ge |\mathbf{a}| - |\mathbf{b}|$

(3) **Multiplication of a vector by a scalar :** If **a** is a vector and m is a scalar (*i.e.,* a real number) then $m\mathbf{a}$ is a vector whose magnitude is m times that of **a** and whose direction is the same as that of **a**, if m is positive and opposite to that of **a**, if m is negative.

Properties of Multiplication of vectors by a scalar : The following are properties of multiplication of vectors by scalars, for vectors $\mathbf{a}, \mathbf{b}$ and scalars m, n.

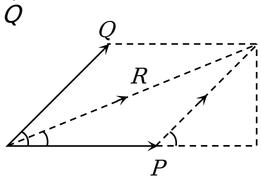
(i) $m(-\mathbf{a}) = (-m)\mathbf{a} = -(m\mathbf{a})$ (ii) $(-m)(-\mathbf{a}) = m\mathbf{a}$

(iii) $m(n\mathbf{a}) = (mn)\mathbf{a} = n(m\mathbf{a})$ (iv) $(m + n)\mathbf{a} = m\mathbf{a} + n\mathbf{a}$

(v) $m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b}$

(4) **Resultant of two forces**

Let $\vec{P}$ and $\vec{Q}$ be two forces and $\vec{R}$ be the resultant of these two forces then, $\vec{R} = \vec{P} + \vec{Q}$



$|\vec{R}| = R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$

where $|\vec{P}| = P$, $|\vec{Q}| = Q$,

Also, $\tan\alpha = \dfrac{Q\sin\theta}{P + Q\cos\theta}$

Deduction : When $|\vec{P}| = |\vec{Q}|$,

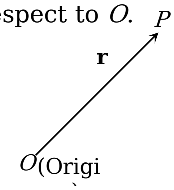
i.e., $P = Q$, $\tan\alpha = \dfrac{P\sin\theta}{P + P\cos\theta} = \dfrac{\sin\theta}{1 + \cos\theta} = \tan\dfrac{\theta}{2}$;

$\therefore \quad \alpha = \dfrac{\theta}{2}$

Hence, the angular bisector of two unit vectors **a** and **b** is along the vector sum $\mathbf{a} + \mathbf{b}$.

Position vector

If a point O is fixed as the origin in space (or plane) and P is any point, then $\overrightarrow{OP}$ is called the position vector of P with respect to O.



If we say that P is the point $\mathbf{r}$, then we mean that the position vector of P is $\mathbf{r}$ with respect to some origin O.

(1) **$\overrightarrow{AB}$ in terms of the position vectors of points A and B :** If $\mathbf{a}$ and $\mathbf{b}$ are position vectors of points A and B respectively. Then, $\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b}$

$\therefore$ $\overrightarrow{AB}$ = (Position vector of B) – (Position vector of A)

$$= \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$$

(2) **Position vector of a dividing point :** The position vectors of the points dividing the line AB in the ratio $m : n$ internally or externally are $\frac{m\mathbf{b}+n\mathbf{a}}{m+n}$ or $\frac{m\mathbf{b}-n\mathbf{a}}{m-n}$.

Linear combination of vectors

A vector $\mathbf{r}$ is said to be a linear combination of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$..... etc, if there exist scalars x, y, z etc., such that $\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + ...$

Examples **:** Vectors $\mathbf{r}_1 = 2\mathbf{a} + \mathbf{b} + 3\mathbf{c}, \mathbf{r}_2 = \mathbf{a} + 3\mathbf{b} + \sqrt{2}\mathbf{c}$ are linear combinations of the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

(1) **Collinear and Non-collinear vectors :** Let $\mathbf{a}$ and $\mathbf{b}$ be two collinear vectors and let $\mathbf{x}$ be the unit vector in the direction of $\mathbf{a}$. Then the unit vector in the direction of $\mathbf{b}$ is $\mathbf{x}$ or $-\mathbf{x}$ according as $\mathbf{a}$ and $\mathbf{b}$ are like or unlike parallel vectors. Now, $\mathbf{a} = | \mathbf{a}| \hat{\mathbf{x}}$ $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ and $\mathbf{b} = \pm | \mathbf{b}| \hat{\mathbf{x}}$.

$\therefore$ $\mathbf{a} = \left(\frac{|\mathbf{a}|}{|\mathbf{b}|}\right) |\mathbf{b}| \hat{\mathbf{x}}$ $\quad \mathbf{a} = \left(\pm\frac{|\mathbf{a}|}{|\mathbf{b}|}\right)\mathbf{b}$ $\quad \mathbf{a} = \lambda\mathbf{b}$, where $\lambda = \pm\frac{|\mathbf{a}|}{|\mathbf{b}|}$. Thus, if $\mathbf{a}, \mathbf{b}$ are collinear vectors, then $\mathbf{a} = \lambda\mathbf{b}$ or $\mathbf{b} = \lambda\mathbf{a}$ for some scalar λ.

(2) **Relation between two parallel vectors**

(i) If $\mathbf{a}$ and $\mathbf{b}$ be two parallel vectors, then there exists a scalar k such that $\mathbf{a} = k\mathbf{b}$ *i.e.,* there exist two non-zero scalar quantities x and y so that $x\mathbf{a} + y\mathbf{b} = \mathbf{0}$.

If $\mathbf{a}$ and $\mathbf{b}$ be two non-zero, non-parallel vectors then $x\mathbf{a} + y\mathbf{b} = \mathbf{0}$ $\quad x = 0$ and $y = 0$.

Obviously $x\mathbf{a} + y\mathbf{b} = \mathbf{0}$ $\begin{cases} \mathbf{a} = \mathbf{0}, \mathbf{b} = \mathbf{0} \\ \text{or} \\ x = 0, y = 0 \\ \text{or} \\ \mathbf{a} \| \mathbf{b} \end{cases}$

(ii) If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then from the property of parallel vectors, we have $\mathbf{a} \| \mathbf{b} \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.

(3) **Test of collinearity of three points :** Three points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are collinear *iff* there exist scalars x, y, z not all zero such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{0}$, where $x + y + z = 0$. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j}$, then the points with position vector $\mathbf{a}, \mathbf{b}, \mathbf{c}$ will be collinear *iff* $\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix} = 0$.

(4) **Test of coplanarity of three vectors :** Let $\mathbf{a}$ and $\mathbf{b}$ two given non-zero non-collinear vectors. Then any vectors $\mathbf{r}$ coplanar with $\mathbf{a}$ and $\mathbf{b}$ can be uniquely expressed as $\mathbf{r} = x\mathbf{a} + y\mathbf{b}$ for some scalars x and y.

(5) **Test of coplanarity of Four points :** Four points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are coplanar *iff* there exist scalars x, y, z, u not all zero such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + u\mathbf{d} = \mathbf{0}$, where $x + y + z + u = 0$.

Four points with position vectors

$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, $\mathbf{d} = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$ will be coplanar, *iff* $\begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix} = 0$.

Linear independence and dependence of vectors

(1) **Linearly independent vectors :** A set of non-zero vectors $\mathbf{a}_1, \mathbf{a}_2, ...\mathbf{a}_n$ is said to be linearly independent, if $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + ... + x_n\mathbf{a}_n = \mathbf{0} \Rightarrow x_1 = x_2 = ... = x_n = 0$.

(2) **Linearly dependent vectors :** A set of vectors $\mathbf{a}_1, \mathbf{a}_2, ...\mathbf{a}_n$ is said to be linearly dependent if there exist scalars $x_1, x_2, ..., x_n$ not all zero such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + ... + x_n\mathbf{a}_n = \mathbf{0}$

Three vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ will be linearly dependent vectors *iff* $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$.

Properties of linearly independent and dependent vectors

(i) Two non-zero, non-collinear vectors are linearly independent.

(ii) Any two collinear vectors are linearly dependent.

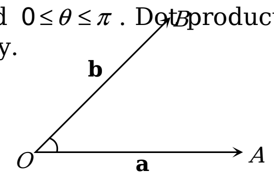
(iii) Any three non-coplanar vectors are linearly independent.

(iv) Any three coplanar vectors are linearly dependent.

(v) Any four vectors in 3-dimensional space are linearly dependent.

Scalar or Dot product

(1) **Scalar or Dot product of two vectors :** If $\mathbf{a}$ and $\mathbf{b}$ are two non-zero vectors and θ be the angle between them, then their scalar product (or dot product) is denoted by $\mathbf{a}.\mathbf{b}$ and is defined as the scalar $|\mathbf{a}||\mathbf{b}|\cos\theta$, where $|\mathbf{a}|$ and $|\mathbf{b}|$ are modulii of $\mathbf{a}$ and $\mathbf{b}$ respectively and $0 \le \theta \le \pi$. Dot product of two vectors is a scalar quantity.



Angle between two vectors : If $\mathbf{a}, \mathbf{b}$ be two vectors inclined at an angle θ, then $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \qquad \theta = \cos^{-1}\left(\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)$$

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$; then

$$\theta = \cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}\right).$$

(2) **Properties of scalar product**

(i) **Commutativity :** The scalar product of two vector is commutative *i.e.,* $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a}$.

(ii) **Distributivity of scalar product over vector addition** The scalar product of vectors is distributive over vector addition *i.e.,* (a) $\mathbf{a}.(\mathbf{b}+\mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$, (Left distributivity)

(b) $(\mathbf{b}+\mathbf{c}).\mathbf{a} = \mathbf{b}.\mathbf{a} + \mathbf{c}.\mathbf{a}$, (Right distributivity)

(iii) Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero vectors $\mathbf{a}.\mathbf{b} = 0 \Leftrightarrow \mathbf{a} \perp \mathbf{b}$.

As $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are mutually perpendicular unit vectors along the co-ordinate axes, therefore, $\mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{i} = 0$; $\mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{j} = 0$; $\mathbf{k}.\mathbf{i} = \mathbf{i}.\mathbf{k} = 0$.

(iv) For any vector $\mathbf{a}$, $\mathbf{a}.\mathbf{a} = |\mathbf{a}|^2$.

As $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along the co-ordinate axes, therefore $\mathbf{i}.\mathbf{i} = |\mathbf{i}|^2 = 1$, $\mathbf{j}.\mathbf{j} = |\mathbf{j}|^2 = 1$ and $\mathbf{k}.\mathbf{k} = |\mathbf{k}|^2 = 1$

(v) If *m, n* are scalars and $\mathbf{a}, \mathbf{b}$ be two vectors, then $m\mathbf{a}.n\mathbf{b} = mn(\mathbf{a}.\mathbf{b}) = (mn\mathbf{a}).\mathbf{b} = \mathbf{a}.(mn\mathbf{b})$

(vi) For any vectors $\mathbf{a}$ and $\mathbf{b}$, we have

(a) $\mathbf{a}.(-\mathbf{b}) = -(\mathbf{a}.\mathbf{b}) = (-\mathbf{a}).\mathbf{b}$ (b) $(-\mathbf{a}).(-\mathbf{b}) = \mathbf{a}.\mathbf{b}$

(vii) For any two vectors $\mathbf{a}$ and $\mathbf{b}$, we have

(a) $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}.\mathbf{b}$

(b) $|\mathbf{a}-\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a}.\mathbf{b}$

(c) $(\mathbf{a}+\mathbf{b}).(\mathbf{a}-\mathbf{b}) = |\mathbf{a}|^2 - |\mathbf{b}|^2$

(d) $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}| + |\mathbf{b}| \quad \mathbf{a} \| \mathbf{b}$

(e) $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 \Rightarrow \mathbf{a} \perp \mathbf{b}$

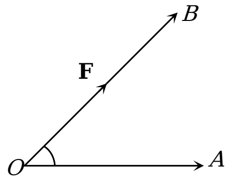
(f) $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}| \Rightarrow \mathbf{a} \perp \mathbf{b}$

(3) **Scalar product in terms of components:** If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then, $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

The components of $\mathbf{b}$ along and perpendicular to $\mathbf{a}$ are $\left(\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$ and $\mathbf{b} - \left(\frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$ respectively.

(4) **Work done by a force :**

Work done $= |\mathbf{F}||\overrightarrow{OA}|\cos\theta = \mathbf{F}.\overrightarrow{OA} = \mathbf{F}.\mathbf{d}$, where $\mathbf{d} = \overrightarrow{OA}$

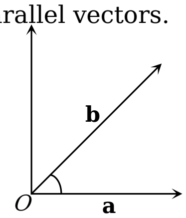


Work done = (Force). (Displacement)

If a number of forces are acting on a particle, then the sum of the works done by the separate forces is equal to the work done by the resultant force.

Vector or Cross product

(1) **Vector product of two vectors :** Let $\mathbf{a}, \mathbf{b}$ be two non-zero, non-parallel vectors.



Then $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta\, \hat{\eta}$, and $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $\hat{\eta}$ is a unit vector perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}, \mathbf{b}, \hat{\eta}$ form a right-handed system.

(2) **Properties of vector product**

(i) Vector product is not commutative *i.e.,* if $\mathbf{a}$ and $\mathbf{b}$ are any two vectors, then $\mathbf{a}\times\mathbf{b} \neq \mathbf{b}\times\mathbf{a}$, however, $\mathbf{a}\times\mathbf{b} = -(\mathbf{b}\times\mathbf{a})$

(ii) If $\mathbf{a}, \mathbf{b}$ are two vectors and *m, n* are scalars, then $m\mathbf{a}\times n\mathbf{b} = mn(\mathbf{a}\times\mathbf{b}) = m(\mathbf{a}\times n\mathbf{b}) = n(m\mathbf{a}\times\mathbf{b})$.

(iii) Distributivity of vector product over vector addition.

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be any three vectors. Then

(a) $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}$ (Left distributivity)

(b) $(\mathbf{b}+\mathbf{c})\times\mathbf{a} = \mathbf{b}\times\mathbf{a} + \mathbf{c}\times\mathbf{a}$ (Right distributivity)

(iv) For any three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ we have $\mathbf{a}\times(\mathbf{b}-\mathbf{c}) = \mathbf{a}\times\mathbf{b} - \mathbf{a}\times\mathbf{c}$.

(v) The vector product of two non-zero vectors is zero vector *iff* they are parallel (Collinear) *i.e.,* $\mathbf{a}\times\mathbf{b} = 0 \Leftrightarrow \mathbf{a} \| \mathbf{b}, \mathbf{a}, \mathbf{b}$ are non-zero vectors.

It follows from the above property that $\mathbf{a}\times\mathbf{a} = 0$ for every non-zero vector $\mathbf{a}$, which in turn implies that $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0$.

(vi) Vector product of orthonormal triad of unit vectors **i, j, k** using the definition of the vector product, we obtain $\mathbf{i}\times\mathbf{j} = \mathbf{k}, \mathbf{j}\times\mathbf{k} = \mathbf{i}, \mathbf{k}\times\mathbf{i} = \mathbf{j}$, $\mathbf{j}\times\mathbf{i} = -\mathbf{k}, \mathbf{k}\times\mathbf{j} = -\mathbf{i}, \mathbf{i}\times\mathbf{k} = -\mathbf{j}$.

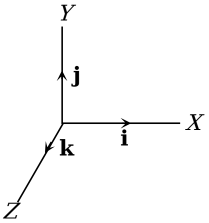
(3) **Vector product in terms of components:** If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$.

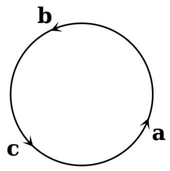
Then, $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

(4) **Angle between two vectors :** If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then $\sin\theta = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$.

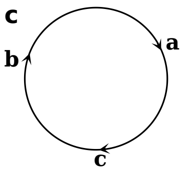
(5) (i) **Right handed system of vectors :** Three mutually perpendicular vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form a right handed system of vector *iff* $\mathbf{a}\times\mathbf{b} = \mathbf{c}$, $\mathbf{b}\times\mathbf{c} = \mathbf{a}$, $\mathbf{c}\times\mathbf{a} = \mathbf{b}$

***Examples*:** The unit vectors $\mathbf{i}, \mathbf{j}$, $\mathbf{k}$ form a right-handed system, $\mathbf{i}\times\mathbf{j} = \mathbf{k}, \mathbf{j}\times\mathbf{k} = \mathbf{i}, \mathbf{k}\times\mathbf{i} = \mathbf{j}$





(ii) **Left handed system of vectors :** The vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ mutually perpendicular to one another form a left handed system of vector *iff* $\mathbf{c}\times\mathbf{b} = \mathbf{a}, \mathbf{a}\times\mathbf{c} = \mathbf{b}, \mathbf{b}\times\mathbf{a} = \mathbf{c}$



(6) **Vector normal to the plane of two given vectors :** If $\mathbf{a}, \mathbf{b}$ be two non-zero, nonparallel vectors and let θ be the angle between them. $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\eta}$ where $\hat{\eta}$ is a unit vector perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}, \mathbf{b}, \eta$ form a right-handed system.

$$(\mathbf{a}\times\mathbf{b}) = |\mathbf{a}\times\mathbf{b}|\,\hat{\eta} \qquad \hat{\eta} = \frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$$

Thus, $\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$ is a unit vector perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$. Note that $-\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$ is also a unit vector perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$. Vectors of magnitude 'λ' normal to the plane of $\mathbf{a}$ and $\mathbf{b}$ are given by $\pm\frac{\lambda(\mathbf{a}\times\mathbf{b})}{|\mathbf{a}\times\mathbf{b}|}$.

(7) **Area of parallelogram and triangle**

(i) The area of a parallelogram with adjacent sides $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a}\times\mathbf{b}|$.

(ii) The area of a parallelogram with diagonals $\mathbf{d}_1$ and $\mathbf{d}_2$ is $\frac{1}{2}|\mathbf{d}_1\times\mathbf{d}_2|$.

(iii) The area of a plane quadrilateral *ABCD* is $\frac{1}{2}|\overrightarrow{AC}\times\overrightarrow{BD}|$, where *AC* and *BD* are its diagonals.

(iv) The area of a triangle with adjacent sides $\mathbf{a}$ and $\mathbf{b}$ is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$

(v) The area of a triangle *ABC* is $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$ or $\frac{1}{2}|\overrightarrow{BC}\times\overrightarrow{BA}|$ or $\frac{1}{2}|\overrightarrow{CB}\times\overrightarrow{CA}|$

(vi) If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are position vectors of vertices of a ΔABC, then its area $= \frac{1}{2}|(\mathbf{a}\times\mathbf{b}) + (\mathbf{b}\times\mathbf{c}) + (\mathbf{c}\times\mathbf{a})|$

(vii) Three points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are collinear if $(\mathbf{a}\times\mathbf{b}) + (\mathbf{b}\times\mathbf{c}) + (\mathbf{c}\times\mathbf{a}) = \mathbf{0}$.

(8) **Moment of a force :** Moment of a force **F** about a point *O* is $\overrightarrow{OP}\times\mathbf{F}$, where *P* is any point on the line of action of the force **F**.

Scalar triple product

(1) **Scalar triple product of three vectors :** If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are three vectors, then their scalar triple product is defined as the dot product of two vectors **a** and **b** × **c**. It is generally denoted by **a** . (**b** × **c**) or [**a b c**].

(2) **Properties of scalar triple product**

(i) If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are cyclically permuted, the value of scalar triple product remains the same. *i.e.,* $(\mathbf{a}\times\mathbf{b}).\mathbf{c} = (\mathbf{b}\times\mathbf{c}).\mathbf{a} = (\mathbf{c}\times\mathbf{a}).\mathbf{b}$ or $[\mathbf{a}\mathbf{b}\mathbf{c}] = [\mathbf{b}\mathbf{c}\mathbf{a}] = [\mathbf{c}\mathbf{a}\mathbf{b}]$

(ii) The change of cyclic order of vectors in scalar triple product changes the sign of the scalar triple product but not the magnitude *i.e.,* $[\mathbf{a}\mathbf{b}\mathbf{c}] = -[\mathbf{b}\mathbf{a}\mathbf{c}] = -[\mathbf{c}\mathbf{b}\mathbf{a}] = -[\mathbf{a}\mathbf{c}\mathbf{b}]$

(iii) In scalar triple product the positions of dot and cross can be interchanged provided that the cyclic order of the vectors remains same *i.e.,* $(\mathbf{a}\times\mathbf{b}).\mathbf{c} = \mathbf{a}.(\mathbf{b}\times\mathbf{c})$

(iv) The scalar triple product of three vectors is zero if any two of them are equal.

(v) For any three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and scalar λ, $[\lambda\,\mathbf{a}\mathbf{b}\mathbf{c}] = \lambda[\mathbf{a}\mathbf{b}\mathbf{c}]$

(vi) The scalar triple product of three vectors is zero if any two of them are parallel or collinear.

(vii) If $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are four vectors, then $[(\mathbf{a}+\mathbf{b})\;\mathbf{c}\;\mathbf{d}] = [\mathbf{a}\mathbf{c}\mathbf{d}] + [\mathbf{b}\mathbf{c}\mathbf{d}]$

(viii) The necessary and sufficient condition for three non-zero non-collinear vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ to be coplanar is that $[\mathbf{a}\mathbf{b}\mathbf{c}] = 0$.

(ix) Four points with position vectors **a**, **b**, **c** and **d** will be coplanar, if $[\mathbf{a}\;\mathbf{b}\;\mathbf{c}] + [\mathbf{d}\;\mathbf{c}\;\mathbf{a}] + [\mathbf{d}\;\mathbf{a}\;\mathbf{b}] = [\mathbf{a}\;\mathbf{b}\;\mathbf{c}]$.

(x) Volume of parallelopiped whose coterminous edges are **a**, **b**, **c** is $[\mathbf{a\,b\,c}]$ or $\mathbf{a}(\mathbf{b} \times \mathbf{c})$.

(3) **Scalar triple product in terms of components**

(i) If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ be three vectors

then, $[\mathbf{a\,b\,c}] = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

(ii) If $\mathbf{a} = a_1\mathbf{l} + a_2\mathbf{m} + a_3\mathbf{n}, \mathbf{b} = b_1\mathbf{l} + b_2\mathbf{m} + b_3\mathbf{n}$

and $\mathbf{c} = c_1\mathbf{l} + c_2\mathbf{m} + c_3\mathbf{n}$, then

$[\mathbf{a\,b\,c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\mathbf{l\,m\,n}]$

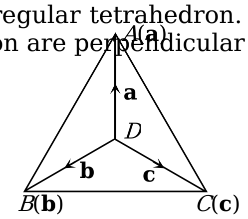
(iii) For any three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$

(a) $[\mathbf{a}+\mathbf{b}\ \ \mathbf{b}+\mathbf{c}\ \ \mathbf{c}+\mathbf{a}] = 2[\mathbf{a\ b\ c}]$

(b) $[\mathbf{a}-\mathbf{b}\ \ \mathbf{b}-\mathbf{c}\ \ \mathbf{c}-\mathbf{a}] = 0$

(c) $[\mathbf{a}\times\mathbf{b}\ \ \mathbf{b}\times\mathbf{c}\ \ \mathbf{c}\times\mathbf{a}] = [\mathbf{a\ b\ c}]^2$

(4) **Tetrahedron :** A tetrahedron is a three-dimensional figure formed by four triangle $OABC$ is a tetrahedron with ΔABC as the base. OA, OB, OC, AB, BC and CA are known as edges of the tetrahedron. $OA, BC; OB, CA$ and OC, AB are known as the pairs of opposite edges. A tetrahedron in which all edges are equal, is called a regular tetrahedron. Any two edges of regular tetrahedron are perpendicular to each other.



Volume of tetrahedron

(i) The volume of a tetrahedron

$= \frac{1}{3}$(area of the base)(corresponding altitude

$= \frac{1}{6}[\overrightarrow{AB}\ \overrightarrow{BC}\ \overrightarrow{AD}]$

(ii) If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are position vectors of vertices A, B and C with respect to O, then volume of tetrahedron $OABC$ $= \frac{1}{6}[\mathbf{a\ b\ c}]$.

(iii) If $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are position vectors of vertices A, B, C, D of a tetrahedron $ABCD$, then its volume $= \frac{1}{6}[\mathbf{b}-\mathbf{a}\ \ \mathbf{c}-\mathbf{a}\ \ \mathbf{d}-\mathbf{a}]$.

(5) **Reciprocal system of vectors :** Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three non-coplanar vectors, and let $\mathbf{a}' = \frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}$, $\mathbf{b}' = \frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]}$, $\mathbf{c}' = \frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{abc}]}$. $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are said to form a reciprocal system of vectors for the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ form a reciprocal system of vectors, then (i) $\mathbf{a}.\mathbf{a}' = \mathbf{b}.\mathbf{b}' = \mathbf{c}.\mathbf{c}' = 1$

(ii) $\mathbf{a}.\mathbf{b}' = \mathbf{a}.\mathbf{c}' = 0; \mathbf{b}.\mathbf{c}' = \mathbf{b}.\mathbf{a}' = 0; \mathbf{c}.\mathbf{a}' = \mathbf{c}.\mathbf{b}' = 0$

(iii) $[\mathbf{a}'\mathbf{b}'\mathbf{c}'] = \frac{1}{[\mathbf{a\ b\ c}]}$

(iv) $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-coplanar *iff* so are $\mathbf{a}', \mathbf{b}', \mathbf{c}'$.

Vector triple product

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be any three vectors, then the vectors $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ and $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ are called vector triple product of **a,b,c** .

Thus, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$

Properties of vector triple product

(i) The vector triple product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is a linear combination of those two vectors which are within brackets.

(ii) The vector $\mathbf{r} = \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is perpendicular to $\mathbf{a}$ and lies in the plane of $\mathbf{b}$ and $\mathbf{c}$.

(iii) The formula $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$ is true only when the vector outside the bracket is on the left most side. If it is not, we first shift on left by using the properties of cross product and then apply the same formula.

Thus, $(\mathbf{b}\times\mathbf{c})\times\mathbf{a} = -\{\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\} = -\{(\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}\}$

$= (\mathbf{a}.\mathbf{b})\mathbf{c} - (\mathbf{a}.\mathbf{c})\mathbf{b}$

(iv) Vector triple product is a vector quantity.

(v) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$

Scalar product of four vectors

$(\mathbf{a}\times\mathbf{b}).(\mathbf{c}\times\mathbf{d})$ is a scalar product of four vectors. It is the dot product of the vectors $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$.

It is a scalar triple product of the vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$ as well as scalar triple product of the vectors $\mathbf{a}\times\mathbf{b}, \mathbf{c}$ and $\mathbf{d}$.

$(\mathbf{a}\times\mathbf{b}).(\mathbf{c}\times\mathbf{d}) = \begin{vmatrix} \mathbf{a}.\mathbf{c} & \mathbf{a}.\mathbf{d} \\ \mathbf{b}.\mathbf{c} & \mathbf{b}.\mathbf{d} \end{vmatrix}$

Vector product of four vectors

(1) $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ is a vector product of four vectors.

It is the cross product of the vectors $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$.

(2) $\mathbf{a}\times\{\mathbf{b}\times(\mathbf{c}\times\mathbf{d})\}, \{(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\}\times\mathbf{d}$ are also different vector products of four vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$.

Rotation of a vector about an axis

Let $\mathbf{a} = (a_1, a_2, a_3)$. If system is rotated about

(i) x-axis through an angle α, then the new components of $\mathbf{a}$ are $(a_1, a_2\cos\alpha + a_3\sin\alpha, -a_2\sin\alpha + a_3\cos\alpha)$.

(ii) y-axis through an angle α, then the new components of $\mathbf{a}$ are $(-a_3\sin\alpha + a_1\cos\alpha, a_2, a_3\cos\alpha + a_1\sin\alpha)$.

(iii) z-axis through an angle α, then the new components of $\mathbf{a}$ are $(a_1\cos\alpha + a_2\sin\alpha, -a_1\sin\alpha + a_2\cos\alpha, a_3)$.

Application of vectors in 3-dimensional geometry

(1) Direction cosines of $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ are $\frac{a}{|\mathbf{r}|}, \frac{b}{|\mathbf{r}|}, \frac{c}{|\mathbf{r}|}$.

(2) **Incentre formula :** The position vector of the incentre of ΔABC is $\frac{a\mathbf{a} + b\mathbf{b} + c\mathbf{c}}{a+b+c}$.

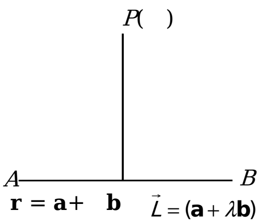
(3) **Orthocentre formula :** The position vector of the orthocentre of ΔABC is $\frac{\mathbf{a}\tan A + \mathbf{b}\tan B + \mathbf{c}\tan C}{\tan A + \tan B + \tan C}$.

(4) Vector equation of a straight line passing through a fixed point with position vector **a** and parallel to a given vector **b** is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$.

(5) The vector equation of a line passing through two points with position vectors **a** and **b** is $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$.

(6) If the lines $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \lambda\mathbf{b}_2$ are coplanar, then $[\mathbf{a}_1\,\mathbf{b}_1\,\mathbf{b}_2] = [\mathbf{a}_2\,\mathbf{b}_1\,\mathbf{b}_2]$ and the equation of the plane containing them is $[\mathbf{r}\,\mathbf{b}_1\,\mathbf{b}_2] = [\mathbf{a}_1\,\mathbf{b}_1\,\mathbf{b}_2]$ or $[\mathbf{r}\,\mathbf{b}_1\,\mathbf{b}_2] = [\mathbf{a}_2\,\mathbf{b}_1\,\mathbf{b}_2]$.

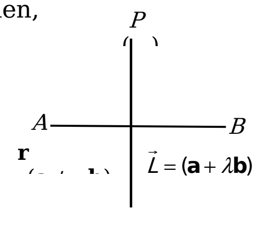
(7) **Perpendicular distance of a point from a line :** Let L is the foot of perpendicular drawn from $P(\vec{\alpha})$ on the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. Since **r** denotes the position vector of any point on the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. So, let the position vector of $\vec{L}$ be $\mathbf{a} + \lambda\mathbf{b}$.



Then $\overrightarrow{PL} = \mathbf{a} - \vec{\alpha} + \lambda\mathbf{b} = (\mathbf{a} - \vec{\alpha}) - \left(\frac{(\mathbf{a} - \vec{\alpha})\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}$

The length PL, is the magnitude of $\overrightarrow{PL}$, and required length of perpendicular.

(8) **Image of a point in a straight line :** Let $Q(\vec{\beta})$ is the image of P in $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, then,



$$\vec{\beta} = 2\mathbf{a} - \left(\frac{2(\mathbf{a} - \vec{\alpha}).\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b} - \vec{\alpha}$$

(9) **Shortest distance between two parallel lines :** Let l_1 and l_2 be two lines whose equations are $l_1 : \mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $l_2 : \mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}_2$ respectively.

Then, shortest distance

$$PQ = \left|\frac{(\mathbf{b}_1 \times \mathbf{b}_2).(\mathbf{a}_2 - \mathbf{a}_1)}{|\mathbf{b}_1 \times \mathbf{b}_2|}\right| = \left|\frac{[\mathbf{b}_1\,\mathbf{b}_2\,(\mathbf{a}_2 - \mathbf{a}_1)]}{|\mathbf{b}_1 \times \mathbf{b}_2|}\right|$$

Shortest distance between two parallel lines : The shortest distance between the parallel lines $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}$ is given by $d = \frac{|(\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b}|}{|\mathbf{b}|}$.

If the lines $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}_2$ intersect, then the shortest distance between them is zero.

Therefore, $[\mathbf{b}_1\,\mathbf{b}_2\,(\mathbf{a}_2 - \mathbf{a}_1)] = 0$

$$[(\mathbf{a}_2 - \mathbf{a}_1)\;\mathbf{b}_1\mathbf{b}_2] = 0 \qquad (\mathbf{a}_2 - \mathbf{a}_1).(\mathbf{b}_1 \times \mathbf{b}_2) = 0.$$

(10) If the lines $\mathbf{r} = \mathbf{a}_1 + \lambda\,\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \lambda\,\mathbf{b}_2$ are coplanar, then $[\mathbf{a}_1\mathbf{b}_1\mathbf{b}_2]$ = $[\mathbf{a}_2\mathbf{b}_1\mathbf{b}_2]$ and the equation of the plane containing them is $[\mathbf{r}\,\mathbf{b}_1\,\mathbf{b}_2]$ = $[\mathbf{a}_1\,\mathbf{b}_1\,\mathbf{b}_2]$ or $[\mathbf{r}\,\mathbf{b}_1\,\mathbf{b}_2]$ = $[\mathbf{a}_2\,\mathbf{b}_1\,\mathbf{b}_2]$.

(11) Vector equation of a plane through the point $A(\mathbf{a})$ and perpendicular to the vector **n** is $(\mathbf{r} - \mathbf{a}).\mathbf{n} = 0$ or $\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$ or $\mathbf{r}.\mathbf{n} = d$, where $d = \mathbf{a}.\mathbf{n}$. This is known as the scalar product form of a plane.

(12) Vector equation of a plane normal to unit vector $\hat{\mathbf{n}}$ and at a distance d from the origin is $\mathbf{r}.\hat{\mathbf{n}} = d$.

If **n** is not a unit vector, then to reduce the equation $\mathbf{r}.\mathbf{n} = d$ to normal form we divide both sides by $|\mathbf{n}|$ to obtain $\mathbf{r}\cdot\frac{\mathbf{n}}{|\mathbf{n}|} = \frac{d}{|\mathbf{n}|}$ or $\mathbf{r}.\hat{\mathbf{n}} = \frac{d}{|\mathbf{n}|}$.

(13) The equation of the plane passing through a point having position vector **a** and parallel to **b** and **c** is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ or $[\mathbf{r}\,\mathbf{b}\,\mathbf{c}] = [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$, where and are scalars.

(14) Vector equation of a plane passing through a point **a,b,c** is $\mathbf{r} = (1 - s - t)\mathbf{a} + s\mathbf{b}\,t + \mathbf{c}$

or $\mathbf{r}.(\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b}) = [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$.

(15) The equation of any plane through the intersection of planes $\mathbf{r}.\mathbf{n}_1 = d_1$ and $\mathbf{r}.\mathbf{n}_2 = d_2$ is $\mathbf{r}.(\mathbf{n}_1 + \lambda\mathbf{n}_2) = d_1 + \lambda d_2$, where is an arbitrary constant.

(16) The perpendicular distance of a point having position vector $\mathbf{a}$ from the plane $\mathbf{r}.\mathbf{n} = d$ is given by $p = \frac{|\mathbf{a}.\mathbf{n} - d|}{|\mathbf{n}|}$.

(17) An angle between the planes $\mathbf{r}_1.\mathbf{n}_1 = d_1$ and $\mathbf{r}_2.\mathbf{n}_2 = d_2$ is given by $\cos\theta = \pm\frac{\mathbf{n}_1.\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$.

(18) Perpendicular distance of a point $P(\mathbf{r})$ from a line passing through $\mathbf{a}$ and parallel to $\mathbf{b}$ is given by

$$PM = \frac{|(\mathbf{r}-\mathbf{a})\times\mathbf{b}|}{|\mathbf{b}|} = \left[(\mathbf{r}-\mathbf{a})^2 - \left\{\frac{(\mathbf{r}-\mathbf{a}).\mathbf{b}}{|\mathbf{b}|}\right\}^2\right]^{1/2}.$$

(19) The equation of the planes bisecting the angles between the planes $\mathbf{r}_1.\mathbf{n}_1 = d_1$ and $\mathbf{r}_2.\mathbf{n}_2 = d_2$ are

$$\frac{|\mathbf{r}.\mathbf{n}_1 - d_1|}{|\mathbf{n}_1|} = \frac{|\mathbf{r}.\mathbf{n}_2 - d_2|}{|\mathbf{n}_2|}$$

or $$\frac{\mathbf{r}.\mathbf{n}_1 - d_1}{|\mathbf{n}_1|} = \pm\frac{\mathbf{r}.\mathbf{n}_2 - d_2}{|\mathbf{n}_2|}$$

or $$\mathbf{r}.(\mathbf{n}_1 \pm \mathbf{n}_2) = \frac{d_1}{|\mathbf{n}_1|} \pm \frac{d_2}{|\mathbf{n}_2|}.$$

(20) Perpendicular distance of a point $P(\mathbf{r})$ from a plane passing through a point $\mathbf{a}$ and parallel to $\mathbf{b}$ and $\mathbf{c}$ is given by $PM = \frac{(\mathbf{r}-\mathbf{a}).(\mathbf{b}\times\mathbf{c})}{|\mathbf{b}\times\mathbf{c}|}$.

(21) Perpendicular distance of a point $P(\mathbf{r})$ from a plane passing through the points $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ is given by

$$PM = \frac{(\mathbf{r}-\mathbf{a}).(\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b})}{|\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}|}.$$

(22) **Angle between line and plane :** If is the angle between a line $\mathbf{r} = (\mathbf{a}+\lambda\mathbf{b})$ and the plane $\mathbf{r}.\mathbf{n} = d$, then $\sin\theta = \frac{\mathbf{b}.\mathbf{n}}{|\mathbf{b}||\mathbf{n}|}$.

(i) **Condition of perpendicularity:** If the line is perpendicular to the plane, then it is parallel to the normal to the plane. Therefore **b** and **n** are parallel.

So, $\mathbf{b}\times\mathbf{n} = 0$ or **b** = **n** for some scalar .

(ii) **Condition of parallelism :** If the line is parallel to the plane, then it is perpendicular to the normal to the plane. Therefore **b** and **n** are perpendicular. So, **b**.**n** = 0.

(iii) If the line $\mathbf{r} = \mathbf{a}+\lambda\mathbf{b}$ lies in the plane **r**.**n** = *d*, then

(a) **b**.**n** = 0

(b) **a**.**n** = *d*.

(23) The equation of sphere with centre at $C(\mathbf{c})$ and radius '*a*' is $|\mathbf{r}-\mathbf{c}| = a$.

(24) The plane $\mathbf{r}.\mathbf{n} = d$ touches the sphere $|\mathbf{r}-\mathbf{a}| = R$, if $\frac{|\mathbf{a}.\mathbf{n} - d|}{|\mathbf{n}|} = R$.

(25) If the position vectors of the extremities of a diameter of a sphere are **a** and **b**, then its equation is $(\mathbf{r}-\mathbf{a}).(\mathbf{r}-\mathbf{b}) = 0$ or $|\mathbf{r}|^2 - \mathbf{r}.(\mathbf{a}-\mathbf{b}) + \mathbf{a}.\mathbf{b} = 0$.

Tips & Tricks

Unit vectors parallel to *x*-axis, *y*-axis and *z*-axis are denoted by **i**, **j** and **k** respectively.

Two unit vectors may not be equal unless they have the same direction.

A unit vector is self reciprocal.

The internal bisector of the angle between any two vectors is along the vector sum of the corresponding unit vectors.

The external bisector of the angle between two vectors is along the vector difference of the corresponding unit vectors.

If $\mathbf{a},\mathbf{b},\mathbf{c}$ are position vectors of vertices of a triangle, then position vector of its centroid is $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$.

If $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$ are position vectors of vertices of a tetrahedron, then position vector of its centroid is $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}}{4}$.

Lagrange's identity: If **a**, **b** are any two vectors, then $|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}.\mathbf{b})^2$ or $|\mathbf{a}\times\mathbf{b}|^2 + (\mathbf{a}.\mathbf{b})^2 = |\mathbf{a}|^2|\mathbf{b}|^2$

$\mathbf{a}.\mathbf{b} \le |\mathbf{a}||\mathbf{b}|$.

$\mathbf{a}.\mathbf{b} > 0$ Angle between **a** and **b** is acute.

$\mathbf{a}.\mathbf{b} < 0$ Angle between $\mathbf{a}$ and $\mathbf{b}$ is obtuse.

The dot product of a zero and non-zero vector is a scalar zero.

Centre of the sphere is the centroid of tetrahedron.

The angle between any two plane faces of a regular tetrahedron is $\cos^{-1}\frac{1}{3}$.

The distance of any vertex from the opposite face of regular tetrahedron is $\sqrt{\frac{2}{3}}k$, k being the length of any edge.

Ordinary Thinking

Objective Questions

Modulus of vector, Algebra of vectors

1. The perimeter of a triangle with sides $3\mathbf{i}+4\mathbf{j}+5\mathbf{k}$, $4\mathbf{i}-3\mathbf{j}-5\mathbf{k}$ and $7\mathbf{i}+\mathbf{j}$ is **[MP PET 1991]**

(a) $\sqrt{450}$ (b) $\sqrt{150}$

(c) $\sqrt{50}$ (d) $\sqrt{200}$

2. If the position vectors of the vertices of a triangle be $2\mathbf{i}+4\mathbf{j}-\mathbf{k}$, $4\mathbf{i}+5\mathbf{j}+\mathbf{k}$ and $3\mathbf{i}+6\mathbf{j}-3\mathbf{k}$, then the triangle is

(a) Right angled (b) Isosceles

(c) Equilateral (d) Right angled isosceles

3. If one side of a square be represented by the vector $3\mathbf{i}+4\mathbf{j}+5\mathbf{k}$, then the area of the square is

(a) 12 (b) 13

(c) 25 (d) 50

4. If $\mathbf{a}=2\mathbf{i}+2\mathbf{j}-\mathbf{k}$ and $|x\mathbf{a}|=1$, then $x=$

(a) $\pm\frac{1}{3}$ (b) $\pm\frac{1}{4}$

(c) $\pm\frac{1}{5}$ (d) $\pm\frac{1}{6}$

5. Which of the following is not a unit vector for all values of

(a) $(\cos\theta)\mathbf{i}-(\sin\theta)\mathbf{j}$ (b) $(\sin\theta)\mathbf{i}+(\cos\theta)\mathbf{j}$

(c) $(\sin 2\theta)\mathbf{i}-(\cos\theta)\mathbf{j}$ (d) $(\cos 2\theta)\mathbf{i}-(\sin 2\theta)\mathbf{j}$

6. If $\mathbf{a}+\mathbf{b}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a}$ and $\mathbf{b}$ are

(a) Mutually perpendicular (b) Unlike vectors

(c) Equal in magnitude (d) None of these

7. If $\mathbf{a}=\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ and $\mathbf{b}=3\mathbf{i}+6\mathbf{j}+2\mathbf{k}$, then a vector in the direction of $\mathbf{a}$ and having magnitude as $|\mathbf{b}|$ is **[IIT 1983]**

(a) $7(\mathbf{i}+\mathbf{j}+\mathbf{k})$ (b) $\frac{7}{3}(\mathbf{i}+2\mathbf{j}+2\mathbf{k})$

(c) $\frac{7}{9}(\mathbf{i}+2\mathbf{j}+2\mathbf{k})$ (d) None of these

8. If $\mathbf{p}=7\mathbf{i}-2\mathbf{j}+3\mathbf{k}$ and $\mathbf{q}=3\mathbf{i}+\mathbf{j}+5\mathbf{k}$, then the magnitude of $\mathbf{p}-2\mathbf{q}$ is **[MP PET 1987]**

(a) $\sqrt{29}$ (b) 4

(c) $\sqrt{62}-2\sqrt{35}$ (d) $\sqrt{66}$

9. Let $\mathbf{a}=\mathbf{i}$ be a vector which makes an angle of 120^o with a unit vector $\mathbf{b}$. Then the unit vector $(\mathbf{a}+\mathbf{b})$ is

[MP PET 1991]

(a) $-\frac{1}{2}\mathbf{i}+\frac{\sqrt{3}}{2}\mathbf{j}$ (b) $-\frac{\sqrt{3}}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}$

(c) $\frac{1}{2}\mathbf{i}+\frac{\sqrt{3}}{2}\mathbf{j}$ (d) $\frac{\sqrt{3}}{2}\mathbf{i}-\frac{1}{2}\mathbf{j}$