1. On the weak characterization of $L^p(\mathbb{R}^n,\mu)$

Let $\mathcal{D}$ be the collection of all dyadic cubes in $\mathbb{R}^n$ and $a=\{a_Q\}_{Q\in\mathcal{D}}$ a sequence of numbers indexed by $\mathcal{D}$. For a positive measure μ on $\mathbb{R}^n$, $1\le p<\infty$ and a real number $\gamma\neq 0$, we define $l^{p,\infty}_{\gamma}(\mathbb{R}^n,\mu)$ as the space of all sequences a such that the weighted weak type norm

$$\|a\|_{l^{p,\infty}_{\gamma}(\mathbb{R}^n,\mu)} := \left(\sup_{\lambda>0}\lambda^p\sum_{|a_Q|>\lambda} l(Q)^{-\gamma}\mu(Q)\right)^{\frac{1}{p}} \tag{1}$$

is finite. Also, consider the smaller quantities

$$[f]_{l^{p,\infty}_{\gamma}(\mathbb{R}^n,\mu)} := \left(\liminf_{\lambda\to 0^+}\lambda^p\sum_{|a_Q|>\lambda} l(Q)^{-\gamma}\mu(Q)\right)^{\frac{1}{p}} \qquad \gamma>0, \tag{2}$$

and

$$[f]_{l^{p,\infty}_{\gamma}(\mathbb{R}^n,\mu)} := \left(\limsup_{\lambda\to +\infty}\lambda^p\sum_{|a_Q|>\lambda} l(Q)^{-\gamma}\mu(Q)\right)^{\frac{1}{p}} \qquad \gamma<0. \tag{3}$$

Now, for $f\in L^1_{loc}(\mathbb{R}^n,\mu)$ let

$$a(f)_Q := l(Q)^{\frac{\gamma}{p}} \fint_Q f\,d\mu,$$

then we have the following:

Theorem 1.1. *1. The inequality*

$$\|a(f)\|_{l^{p,\infty}_{\gamma}(\mathbb{R}^n,\mu)} \lesssim \|f\|_{L^p(\mathbb{R}^n,\mu)}, \tag{4}$$

holds if

- *μ is a positive measure and $1<p<\infty$, $\gamma\neq 0$ or $p=1$, $\gamma<0$.*
- *μ is the Lebesgue measure and $\gamma>n$.*

2. The inequality

$$\|f\|_{L^p(\mathbb{R}^n,\mu)} \lesssim [a(f)]_{l^{p,\infty}_{\gamma}(\mathbb{R}^n,\mu)}, \tag{5}$$

holds if

- *the basis of dyadic cubes differentiates $L^1_{loc}(\mathbb{R}^n,\mu)$, $1\le p<\infty$, and $\gamma>0$*
- *the measure μ is doubling, $1\le p<\infty$, and $\gamma<0$.*

Before we give the proof of this theorem we bring the following simple lemma which is needed later.

Lemma 1.2. *Let μ be a measure on $\mathbb{R}^n$, $1<p<\infty$, and $\mathcal{C}$ a finite collection of dyadic cubes in $\mathbb{R}^n$. Then we have*

$$\Big\|\sum_{Q\in\mathcal{C}} l(Q)^{-\frac{\gamma}{p'}}\chi_Q\Big\|_{L^{p'}(\mathbb{R}^n,\mu)} \lesssim \left(\sum_{Q\in\mathcal{C}} l(Q)^{-\gamma}\mu(Q)\right)^{\frac{1}{p'}}$$

Proof. First, we partition the collection $\mathcal{C}$ into a sequence of disjoint generations $\mathcal{C}_1, \mathcal{C}_2, \ldots$ such that for $j < i$, each cube in the i-th generation, $\mathcal{C}_i$, is contained in a unique cube in the j-th generation, $\mathcal{C}_j$. To do this, simply let $\mathcal{C}_1$ be the subcollection of maximal cubes in $\mathcal{C}$ and suppose we have constructed $\mathcal{C}_j$ for $j < i$, then the i-th generation C_i consists of maximal cubes in $\mathcal{C} - \cup_{j<i}\mathcal{C}_j$. Now, let

$$g(x) = \sum_{Q \in \mathcal{C}} l(Q)^{-\frac{\gamma}{p'}} \chi_Q(x),$$

and partition $\cup\mathcal{C}$ into the sets $E_1, E_2, \ldots$ as

$$E_i = \cup\mathcal{C}_i - \cup\mathcal{C}_{i+1}$$

Here, we note that for each $x \in E_i$ there is a unique sequence of cubes $Q_1 \supset Q_2 \supset \ldots \supset Q_i$, with the property that $Q_j \in C_j$ for $1 \leq j \leq i$, and thus

$$g(x) = \sum_{j=1}^{i} l(Q_j)^{-\frac{\gamma}{p'}}. \tag{6}$$

Now, depending on the sign of γ the above series decays or increases geometrically. Therefore if $\gamma < 0$, we have $g(x) \lesssim l(Q_1)^{-\frac{\gamma}{p'}}$, which implies

$$g(x) \lesssim \sum_{Q \in \mathcal{C}_1} l(Q_1)^{-\frac{\gamma}{p'}} \chi_{Q_1}(x).$$

Taking the $L^{p'}(\mathbb{R}^n, \mu)$ norm of the above inequality yields

$$\|g\|_{L^{p'}(\mathbb{R}^n,\mu)} \lesssim \left(\sum_{Q_1 \in \mathcal{C}_1} l(Q_1)^{-\gamma} \mu(Q_1) \right)^{\frac{1}{p'}} \lesssim \left(\sum_{Q \in \mathcal{C}} l(Q)^{-\gamma} \mu(Q) \right)^{\frac{1}{p'}},$$

which proves the claim in this case.

Suppose now that $\gamma > 0$, then from (6) we get $g(x) \lesssim l(Q_i)^{-\frac{\gamma}{p'}}$, and thus when $x \in E_i$ we have

$$g(x) \lesssim \sum_{Q_i \in \mathcal{C}_i} l(Q_i)^{-\frac{\gamma}{p'}} \chi_{Q_i}(x), \qquad x \in E_i.$$

Raising both sides to power p' and integrating the above inequality over E_i gives us

$$\int_{E_i} g^{p'} d\mu \lesssim \sum_{Q_i \in \mathcal{C}_i} l(Q_i)^{-\gamma} \mu(Q_i).$$

Finally we note that the support of g is the disjoint union of the sets E_i and by summing the above inequality over i we obtain

$$\|g\|_{L^{p'}(\mathbb{R}^n,\mu)} = \left(\sum_i \int_{E_i} g^{p'} d\mu \right)^{\frac{1}{p'}} \lesssim \left(\sum_{Q \in \mathcal{C}} l(Q)^{-\gamma} \mu(Q) \right)^{\frac{1}{p'}},$$

which completes the proof. $\square$

Now we turn to the proof of the above theorem.

Proof of Theorem 1.1. Case 1. μ is a positive measure and $1 < p < \infty$.
To prove (4) in this case, fix $\lambda > 0$ and take an arbitrary finite sub-collection $\mathcal{C}$ of dyadic cubes Q such that $|a(f)_Q| > \lambda$. By triangle inequality we have

$$l(Q)^{-\gamma}\mu(Q) < \frac{1}{\lambda}\int |f| l(Q)^{-\frac{\gamma}{p'}}\chi_Q d\mu, \quad Q \in \mathcal{C},$$

and the summation over $\mathcal{C}$ gives us

$$\sum_{Q\in\mathcal{C}} l(Q)^{-\gamma}\mu(Q) < \frac{1}{\lambda}\int |f| \sum_{Q\in\mathcal{C}} l(Q)^{-\frac{\gamma}{p'}}\chi_Q d\mu, \quad Q \in \mathcal{C},$$

which after applying Holder inequality yields

$$\sum_{Q\in\mathcal{C}} l(Q)^{-\gamma}\mu(Q) < \frac{1}{\lambda}\|f\|_{L^p(\mathbb{R}^n,\mu)}\|\sum_{Q\in\mathcal{C}} l(Q)^{-\frac{\gamma}{p'}}\chi_Q\|_{L^{p'}(\mathbb{R}^n,\mu)}.$$

Now, an application of Lemma 1.2 implies

$$\sum_{Q\in\mathcal{C}} l(Q)^{-\gamma}\mu(Q) \lesssim \frac{1}{\lambda}\|f\|_{L^p(\mathbb{R}^n,\mu)}\left(\sum_{Q\in\mathcal{C}} l(Q)^{-\gamma}\mu(Q)\right)^{\frac{1}{p'}},$$

and noting that $\mathcal{C}$ is finite shows that

$$\left(\lambda^p \sum_{Q\in\mathcal{C}} l(Q)^{-\gamma}\mu(Q)\right)^{\frac{1}{p}} \lesssim \|f\|_{L^p(\mathbb{R}^n,\mu)}.$$

To finish the proof in this case it is enough to take the supremum first over $\mathcal{C}$ and then $\lambda > 0$.

Case 2. μ is a positive measure and $\gamma < 0$.
Let $\mathcal{C}$ be as in the above and $\mathcal{C}_{\max}$ be the collection of maximal cubes in $\mathcal{C}$, and note that every cube in $\mathcal{C}$ is contained in some maximal cube. Therefore, we have

$$\sum_{Q\in\mathcal{C}} l(Q)^{-\gamma}\mu(Q) = \sum_{Q\in\mathcal{C}_{\max}}\sum_{k=0}^{\infty}\sum_{Q'\in\mathcal{D}_k(Q)} l(Q')^{-\gamma}\mu(Q') \leq$$
$$\sum_{Q\in\mathcal{C}_{\max}} \mu(Q)l(Q)^{-\gamma}\sum_{k=0}^{\infty} 2^{k\gamma} \lesssim \sum_{Q\in\mathcal{C}_{\max}} \mu(Q)l(Q)^{-\gamma}.$$

Now, it is enough to note that cubes in $\mathcal{C}_{\max}$ are disjoint and use the triangle inequality to obtain

$$\lambda\sum_{Q\in\mathcal{C}} l(Q)^{-\gamma}\mu(Q) \lesssim \lambda\sum_{Q\in\mathcal{C}_{\max}} l(Q)^{\gamma-n} \leq \int_{\cup_{Q\in\mathcal{C}_{\max}} Q} |f| \leq \|f\|_{L^1},$$

which completes the proof of this case.

Case 3. μ is the Lebesgue measure and $\gamma > n$.
Let $\mathcal{C}$ be as in the previous cases and this time let $\mathcal{C}_{\min}$ be the sub-collection of minimal cubes in $\mathcal{C}$. Then, one can see that each cube in $\mathcal{C}$ contains a minimal cube in $\mathcal{C}_{\min}$ and thus

$$\sum_{Q\in\mathcal{C}} l(Q)^{n-\gamma} = \sum_{Q\in\mathcal{C}_{\min}}\sum_{Q\subset Q'} l(Q')^{n-\gamma} \approx \sum_{Q\in\mathcal{C}_{\min}} l(Q)^{n-\gamma}. \tag{7}$$

Now, noting that cubes in $\mathcal{C}_{\min}$ are disjoint and using triangle inequality as in the previous case proves the claim.

Next, we prove the second inequality (5), and we start by assuming the basis of dyadic cubes differentiates $L^1_{loc}(\mathbb{R}^n,\mu)$, $1\le p<\infty$, and $\gamma>0$. So for an arbitrary dyadic cube Q_0 and $m\in\mathbb{N}$ let

$$\lambda=\frac{1}{2}\min\left\{\frac{1}{m},\lambda'\right\},\qquad \lambda'=\min\left\{|a(f)_Q||Q\in D,\quad Q\subset Q_0,\quad l(Q)\ge 2^{-m}l(Q_0)\right\}.$$

Then let $\mathcal{C}_m$ be the collection of maximal dyadic cubes, $Q\subset Q_0$, for which $|a(f)_Q|\le\lambda$. This collection has the following properties:

1. For each $Q\in\mathcal{C}_m$, we have $l(Q)\le 2^{-m}l(Q_0)$, and $|a(f)_Q|\le\lambda$. In addition to this, for $\bar{Q}$ the parent of Q, we have $|a(f)_{\bar{Q}}|>\lambda$, which is a consequence of the maximality of Q.

2. $\mu\left(Q_0-\cup\mathcal{C}_m\right)=0$, for the reason that if $x\in Q_0-\cup\mathcal{C}_m$, there is a sequence of dyadic cubes $\{Q_i\}$ shrinking to x such that

$$l(Q_i)^{\frac{\gamma}{p}}|\fint_{Q_i}fd\mu|>\lambda.$$

Then, we note that $l(Q_i)^{\frac{\gamma}{p}}\to 0$ since $\gamma>0$, and $|\fint_{Q_i}fd\mu|\to|f(x)|$ for μ-almost every x, so we must have $|f(x)|=\infty$. But since f is locally integrable the measure of such points is zero.

3. $\sum_{Q\in\mathcal{C}_m}l(Q)^{-\gamma}\mu(Q)\lesssim\sum_{|a_{Q'}(f)|>\lambda}l(Q')^{-\gamma}\mu(Q')$, which can be verified by noting that

$$\sum_{Q\in\mathcal{C}_m}l(Q)^{-\gamma}\mu(Q)\le 2^{\gamma+n}\sum_{Q\in\mathcal{C}_m}l(\bar{Q})^{-\gamma}\mu(\bar{Q})\le 2^{\gamma+n}\sum_{|a_{Q'}(f)|>\lambda}l(Q')^{-\gamma}\mu(Q').$$

Now, consider a sequence of functions defined by

$$f_m=\sum_{Q\in\mathcal{C}_m}|\fint_Q d\mu|\chi_Q,\qquad m=1,2,...,$$

which from the Lebesgue differentiation theorem converges to $|f|$ for μ-almost every point in Q_0, and also satisfies

$$\int_{Q_0}|f_m|^pd\mu=\sum_{Q\in\mathcal{C}_m}|\fint_Q d\mu|^p\mu(Q)=\sum_{Q\in\mathcal{C}_m}|a(f)_Q|^pl(Q)^{-\gamma}\mu(Q)\le\lambda^p\sum_{Q\in\mathcal{C}_m}l(Q)^{-\gamma}\mu(Q).$$

Here, from the third property of $\mathcal{C}_m$ and Fatou's lemma we obtain

$$\int_{Q_0}|f|^pd\mu\le\liminf_{m\to\infty}\int_{Q_0}|f_m|^pd\mu\lesssim\liminf_{\lambda\to0^+}\lambda^p\sum_{|a_{Q'}(f)|>\lambda}l(Q')^{-\gamma}\mu(Q'),$$

which completes the proof after noting that Q_0 is arbitrary.

A similar argument works for (??), which we discuss briefly. This time we take

$$\lambda=2\max\{m,\lambda'\},\quad \lambda'=\max\left\{|a(f)_Q||Q\in D,\quad Q\subset Q_0,\quad l(Q)\ge 2^{-m}l(Q_0)\right\},$$

and as in the previous case we let $\mathcal{C}'_m$ be the maximal collection of dyadic cubes Q such that $|a(f)_Q|\ge\lambda$. The cubes in $\mathcal{C}'_m$ enjoy the following properties:

1. For each $Q \in \mathcal{C}'_m$, we have $l(Q) \leq 2^{-m} l(Q)$, and $\lambda \leq |a(f)_Q| \lesssim \lambda$. This follows from the maximality of Q and the doubling assumption on μ.

2. On the set $Q_0 - \cup \mathcal{C}'_m$, f is zero for μ-almost every point. Similar to the previous case this can be seen by noting that for each $x \in Q_0 - \cup \mathcal{C}'_m$, there is a sequence of dyadic cubes $\{Q_i\}$ which contracts to x with the property that

$$l(Q_i)^{\frac{\gamma}{p}} | \fint_{Q_i} f d\mu| < \lambda.$$

Then, since $\gamma < 0$, we have $l(Q_i)^{\frac{\gamma}{p}} \to \infty$, so we must have $|\fint_{Q_i} f d\mu| \to 0$, and then the Lebesgue differentiation theorem implies that $f(x) = 0$, for μ-almost every point in $Q_0 - \cup_{Q \in \mathcal{C}_m} Q$.

The rest is just similar to the previous case and the proof is now complete. $\square$

Next, we present examples of functions in $L^1[0,1]$ for which the inequality (4) fails to be true, in case where μ is the Lebesgue measure, $p = 1$, and $0 < \gamma < 1$.

Example 1. Let $\gamma = 1$ and consider the function

$$f = \sum_{n=1}^{\infty} \frac{2^{n^3}}{n^2} \chi_{[0, 2^{-n^3}]},$$

which belongs to $L^1[0,1]$. Now, note that for each n, and every dyadic interval $I \subset [0,1]$, which contains $[0, 2^{-n^3}]$ we have

$$l(I) \fint_I f \geq \int_{[0, 2^{-n^3}]} f > \frac{1}{n^2},$$

which implies that for $\lambda = \frac{1}{n^2}$ we have

$$\lambda \sum_{|a(f)_I| > \lambda} l(I)^{1-\gamma} = \frac{\#\{I | \int_I f > \frac{1}{n^2}\}}{n^2} \geq n.$$

So in this case $[a(f)]_{l_1^{1,\infty}[0,1]} = \infty$.

Example 2. This time we provide an example of the failure of (4) in the case $0 < \gamma < 1$, and to this aim, first we choose a sequence of natural numbers $\{n_k\}$ such that

$$a_k = [2^{n_k(1-\gamma)}], \quad \frac{a_k}{2^{n_k(1-\gamma)}} = 1 - \varepsilon_k, \quad \varepsilon_k \leq 2^{-k}, \quad k = 1, 2, ... \tag{8}$$

Next, we cut $[0,1]$ into pieces of length 2^{-n_1}, choose a_1 number of these intervals, and we call this collection $\mathcal{C}_1$. Suppose now we have constructed $\mathcal{C}_i$ then $\mathcal{C}_{i+1}$ is obtained by cutting each interval in $\mathcal{C}_i$ into $2^{n_{i+1}}$ pieces and selecting a_{i+1} of them. Therefore, from the construction for C_k we have

$$\#\mathcal{C}_k = a_1 ... a_k, \quad l(I) = 2^{-(n_1 + ... + n_k)}, \quad I \in \mathcal{C}_k. \tag{9}$$

Now, pick $0 < \alpha < 1$ and consider the following function

$$f = \sum_{k=1}^{\infty} \frac{2^{\gamma(n_1 + ... + n_k)}}{k^{1+\alpha}} \sum_{I \in \mathcal{C}_k} \chi_I.$$

From (8) and (9) it follows that

$$|\cup \mathcal{C}_k| = a_1 ... a_k 2^{-(n_1+...+n_k)} \leq 2^{-\gamma(n_1+...+n_k)},$$

which implies that $f \in L^1[0,1]$. Next, we estimate $a(f)_I$ on each $I \in \mathcal{C}_k$ from below and to do this we note that for each natural number i the total number of intervals of $\mathcal{C}_{k+i}$ inside I is $a_{k+1}...a_{k+i}$. Then from this observation and (8) it follows that

$$a(f)_I = l(I)^{\gamma-1} \int_I f \geq l(I)^{\gamma-1} \sum_{i=1}^{\infty} \frac{a_{k+1}...a_{k+i}}{(k+i)^{1+\alpha}} 2^{-(n_1+...+n_{k+i})(1-\gamma)} = \sum_{i=1}^{\infty} \frac{(1-\varepsilon_{k+1})...(1-\varepsilon_{k+i})}{(k+i)^{1+\alpha}},$$

where the last expression is bounded from below by

$$\prod_{i=1}^{\infty}(1-2^{-i}) \sum_{i=1}^{\infty} \frac{1}{(k+i)^{1+\alpha}} \approx k^{-\alpha}.$$

So for each $I \in \mathcal{C}_k$ we have $a(f)_I \gtrsim k^{-\alpha}$, which implies that for $\lambda \approx k^{-\alpha}$ we get

$$\begin{aligned} \lambda \sum_{|a(f)_I|>\lambda} l(I)^{1-\gamma} \gtrsim k^{-\alpha} \sum_{j=1}^{k} \sum_{I \in \mathcal{C}_j} l(I)^{1-\gamma} &= k^{-\alpha} \sum_{j=1}^{k} a_1 ... a_j 2^{-(n_1+...+n_j)(1-\gamma)} \\ &= k^{-\alpha} \sum_{j=1}^{k} (1-\varepsilon_1)...(1-\varepsilon_j) \gtrsim k^{1-\alpha}, \end{aligned}$$

and this shows that $[f]_{l_\gamma^{1,\infty}[0,1]} = \infty$.

Next, we discuss two counterparts of (4) in the two parameter setting. By a dyadic rectangle in $\mathbb{R}^{n+m}$ we mean the product of two dyadic cubes in $\mathbb{R}^n$ and $\mathbb{R}^m$. Now let $R = Q_1 \times Q_2$ be a dyadic rectangle here we denote $l_1(R) = l(Q_1)$ and $l_2(R) = l(Q_2)$ also by $\mathcal{D}_n \times \mathcal{D}_m$ we mean the collection of all dyadic rectangles in $\mathbb{R}^{n+m}$.

Theorem 1.3. *Let $1 < p < \infty$, $\gamma > 0$, μ be a measure on $\mathbb{R}^{n+m}$, and $f \in L^1_{loc}(\mathbb{R}^{n+m}, \mu)$.*

1. *If for a dyadic rectangle R*

$$a(f)_R := \min\{l_1(R), l_2(R)\}^{\gamma(1+\frac{1}{p})} \max\{l_1(R), l_2(R)\}^{-\gamma} \fint_R f d\mu,$$

we have

$$\left(\sup_{\lambda>0} \lambda^p \sum_{|a(f)_R|>\lambda} \max\{l_1(R), l_2(R)\}^{-\gamma} \mu(R) \right)^{\frac{1}{p}} \lesssim \|f\|_{L^p(\mathbb{R}^n,\mu)},$$

and if

$$a(f)_R := \min\{l_1(R), l_2(R)\}^{\frac{\gamma}{p}} \fint_R f d\mu,$$

then we have

$$\left(\sup_{\lambda>0} \lambda^p \sum_{|a(f)_R|>\lambda} (l_1(R) l_2(R))^{-\frac{\gamma}{2}} \mu(R) \right)^{\frac{1}{p}} \lesssim \|f\|_{L^p(\mathbb{R}^n,\mu)}.$$

Similar to the one parameter setting the above theorem is a consequence of the following lemma.

Lemma 1.4. *Let $\mathcal{C}$ be a finite collection of dyadic rectangles then*
If

$$g(x,y)=\sum_{R\in\mathcal{C}}\min\{l_1(R),l_2(R)\}^{\gamma(1+\frac{1}{p})}\max\{l_1(R),l_2(R)\}^{-2\gamma}\chi_R(x,y),$$

we have

$$\|g\|_{L^{p'}(\mathbb{R}^{n+m},\mu)}\lesssim\left(\sum_{R\in\mathcal{C}}\max\{l_1(R),l_2(R)\}^{-\gamma}\mu(R)\right)^{\frac{1}{p'}},$$

and if

$$g(x,y)=\sum_{R\in\mathcal{C}}\min\{l_1(R),l_2(R)\}^{\gamma(1+\frac{1}{p})}(l_1(R)l_2(R))^{-\frac{\gamma}{2}}\chi_R(x,y),$$

we have

$$\|g\|_{L^{p'}(\mathbb{R}^{n+m},\mu)}\lesssim\left(\sum_{R\in\mathcal{C}}(l_1(R)l_2(R))^{-\frac{\gamma}{2}}\mu(R)\right)^{\frac{1}{p'}}.$$

Proof of Lemma 1.4. We only prove the first inequality since the same argument works for the other one. We begin by partitioning $\mathcal{C}$ into two sub-collections $\mathcal{C}_1$ and $\mathcal{C}_2$ as

$$\mathcal{C}_1=\{R\in\mathcal{C}|l_1(R)\le l_2(R)\},\quad \mathcal{C}_2=\{R\in\mathcal{C}|l_1(R)>l_2(R)\},$$

and then we prove the (1.4) for the corresponding g-functions of $\mathcal{C}_1$ and $\mathcal{C}_2$. Now, since the proof of these two inequalities are exactly the same without loss of generality we may assume that for $R\in\mathcal{C}$ we have $l_1(R)\le l_2(R)$.

Now, let $\pi_1(\mathcal{C})$ and $\pi_2(\mathcal{C})$ be the collection of cubes obtained by projecting rectangles in $\mathcal{C}$ onto $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Then for $l=1,2$ and $i=1,2,...$ let $\mathcal{C}_i^l$ be the i-th generation of $\pi_l(\mathcal{C})$ and E_i^l be the associated partitioning of $\cup\pi_l(\mathcal{C})$, as constructed in Lemma 1.2. Next, we partition $E=\cup\mathcal{C}$ as

$$E=\cup_{j,k}E_{j,k},\quad E_{j,k}=\left\{(x,y)\in E|x\in E_j^1,\quad y\in E_k^2\right\},$$

and note that for each $(x,y)\in E_{j,k}$ there exists a unique dyadic rectangle $R\in\mathcal{C}$ such that

$$R=Q_1\times Q_2,\quad Q_1\in\mathcal{C}_j^1,Q_2\in\mathcal{C}_k^2.$$

Now, suppose (x,y) belongs a rectangle $R'=Q_1'\times Q_2'$ then we must have

$$Q_1\subset Q_1',\quad Q_2\subset Q_2',$$

and also we have

$$l(Q_1)\le l(Q_2),\quad l(Q_1')\le l(Q_2').$$

From the above observation we have the following estimate for $g(x,y)$:

$$g(x,y)=\sum_{\substack{R'\in\mathcal{C}\\(x,y)\in R}}l_1(R')^{\gamma(1+\frac{1}{p})}l_2(R')^{-2\gamma}\le\sum_{l_2(R)\le 2^t}\sum_{l_1(R)\le 2^s\le l_2(R)}2^{s\gamma(1+\frac{1}{p})}2^{-2t\gamma}\lesssim l_2(R)^{-\frac{\gamma}{p'}}.$$

Noting that for each (x,y) the rectangle R is unique and integrating g over $E_{j,k}$ by using the above bound we obtain

$$\int_{E_{j,k}}g^{p'}d\mu\lesssim\sum_{\substack{R\in\mathcal{C},R=Q_1\times Q_2\\Q_1\in C_j^1,Q_2\in C_k^2}}l_2(R)^{-\gamma}\mu(R)$$

Now it is enough to sum over (j,k) and by doing so we get

$$\int_{\mathbb{R}^{n+m}} g^{p'} d\mu = \sum_{j,k} \int_{E_{j,k}} g^{p'} d\mu \lesssim \sum_{R\in\mathcal{C}} l_2(R)^{-\gamma}\mu(R),$$

which completes the proof. $\square$

2. On the weak quasi-norms and the dyadic John-Nirenberg space

In this section we replace the average in (1) with the mean oscillation and investigate properties of the resulting function space. So for $1 \le p < \infty$, and $\gamma \neq 0$ let $J_\gamma^{p,\infty}(\mathbb{R}^n)$ be the space of locally integrable function f for which

$$\|f\|_{J_\gamma^{p,\infty}(\mathbb{R}^n)} := \|b(f)\|_{l_\gamma^{p,\infty}(\mathbb{R}^n)} \quad b(f)_Q := l(Q)^{\frac{\gamma}{p}} O(f,Q), \tag{10}$$

is finite. For $1 < p < \infty$, the dyadic John-Nirenberg space, $JN_p^d(\mathbb{R}^n)$, consists of all locally integrable functions f such that

$$\|f\|_{JN_p^d(\mathbb{R}^n)} := \left(\sup_{\mathcal{C}} \sum_{Q\in\mathcal{C}} |Q| O(f,Q)^p\right)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all disjoint collections of dyadic cubes $\mathcal{C}$.

Theorem 2.1. *For $\gamma > n$ and $\gamma < 0$ we have*

$$\|f\|_{J_\gamma^{p,\infty}(\mathbb{R}^n)} \lesssim \|f\|_{JN_p^d(\mathbb{R}^n)}. \tag{11}$$

Proof. Let $\mathcal{C}$ be a finite collection of dyadic cubes, Q, with $b(f)_Q > \lambda$, and $\mathcal{C}_{\min}$, and $\mathcal{C}_{\max}$ be its sub-collection of minimal and maximal cubes, respectively. As we have seen before for $\gamma > n$ we have

$$\sum_{Q\in\mathcal{C}} l(Q)^{n-\gamma} \approx \sum_{Q\in\mathcal{C}_{\min}} l(Q)^{n-\gamma},$$

and for $\gamma < 0$ we have

$$\sum_{Q\in\mathcal{C}} l(Q)^{n-\gamma} \approx \sum_{Q\in\mathcal{C}_{\max}} l(Q)^{n-\gamma}.$$

This together with the fact that cubes in $\mathcal{C}_{\min}$ and $\mathcal{C}_{\max}$ are disjoint, and $b(f)_Q > \lambda$ implies

$$\lambda^p \sum_{Q\in\mathcal{C}} l(Q)^{n-\gamma} \lesssim \sum_{Q\in\mathcal{C}_{\min}} |Q| O(f,Q)^p \le \|f\|_{JN_p^d(\mathbb{R}^n)}.$$

Now, taking the supremum over $\mathcal{C}$ and then λ completes the proof. $\square$

The next example shows that these two quasi-norms are not equivalent.

Example 3. Let $p > 1$, and consider the function $f(x) = x^{-\frac{1}{p}}\chi_{[0,1]}$. It is well known that $f \notin JN_p^d[0,1]$ but as we will see f belongs to $J_\gamma^{p,\infty}[0,1]$. In order to show this, let $I = [m2^{-k}, (m+1)2^{-k}]$ be an arbitrary dyadic interval in $[0,1]$ with $0 \le m \le 2^k - 1$ and $k = 0,1,2,...$ Now if $m = 0$ we have

$$b_I(f) = 2^{-k\frac{\gamma}{p}} \fint_{[0,2^{-k}]} x^{-\frac{1}{p}} dx \lesssim 2^{k\frac{1-\gamma}{p}},$$

and if $m \ge 1$ Poincare inequality gives us

$$b_I(f) \lesssim 2^{-k(1+\frac{\gamma}{p})} \fint_I x^{-(1+\frac{1}{p})} dx \lesssim 2^{k\frac{1-\gamma}{p}} m^{-(1+\frac{1}{p})}.$$

So for $\lambda > 0$ we have $b_I(f) > \lambda$ only if $2^{k\frac{1-\gamma}{p}}(m+1)^{-(1+\frac{1}{p})} \gtrsim \lambda$, and this implies

$$\sum_{b(f)_Q > \lambda} l(I)^{1-\gamma} \lesssim \sum_{m=0}^{\infty} \sum_{\substack{2^{k(\gamma-1)} \lesssim \\ \lambda^{-p}(m+1)^{-(p+1)}}} 2^{k(\gamma-1)} \lesssim \lambda^{-p} \sum_{m=0}^{\infty} (m+1)^{-(p+1)} \lesssim \lambda^{-p},$$

and this shows that $f \in J_\gamma^{p,\infty}[0,1]$.

By lifting the above function to higher dimensions via $F(x_1, ..., x_n) = f(x_1)$ we obtain an example of a function which belongs to $J_\gamma^{p,\infty}[0,1]^n$ which is not in $JN_p^d[0,1]^n$, for $p > 1$.

The next two examples shows that the inequality (11) fails, when $0 < \gamma \leq n$.

Example 4. Here we show that the inequality (11) fails to be true when $\gamma = n$ and $1 < p < \infty$. So, let $Q_0 = [0,1]^n$ and consider the function

$$f = \sum_{k=1}^{\infty} 2^{k\frac{n}{p}} \chi_{[0,2^{-k}]^n},$$

which is in $L^1(Q_0)$ since $p > 1$. Next we note that f is in $JN_p^d(Q_0)$ for the reason that for any collection of disjoint dyadic cubes there is at most one cube that the function f is not constant on it and also that cube must be of the form $[0, 2^{-k}]^n$. Therefore, it is enough to show that

$$|Q|O(f,Q)^p \lesssim 1, \quad Q = [0, 2^{-k}]^n, \quad k = 1, 2, ... \tag{12}$$

Now for such a cube Q, we have

$$f\chi_Q = \sum_{l=k+1}^{\infty} 2^{l\frac{n}{p}} \chi_{[0,2^{-l}]^n} + \text{const},$$

and this implies

$$O(f,Q) \approx \frac{1}{|Q|} \sum_{l=k+1}^{\infty} 2^{l(\frac{n}{p}-n)} \approx 2^{k\frac{n}{p}},$$

which shows that (12) holds and thus $f \in JN_p^d(Q_0)$. Moreover, we have $b(f)_Q \approx 1$ for all $Q = [0, 2^{-k}]^n$, and thus for $\lambda \approx 1$ we get

$$\lambda^p \#\{Q | b(f)_Q > \lambda\} = \infty,$$

which proves that $\|f\|_{J_n^{p,\infty}(Q_0)} = \infty$.

By modifying the above example with

$$f = \sum_{k=1}^{\infty} k^{-\frac{\alpha}{p}} 2^{k\frac{n}{p}} \chi_{[0,2^{-k}]^n}, \quad 0 < \alpha < 1,$$

we obtain a function $f \in JN_p^d(Q_0)$ for which $[b(f)]_{l_n^{p,\infty}(Q_0)} = \infty$.

Example 5. Here, we assume $0 < \gamma < n$, $1 < p < \infty$, and $Q_0 = [0,1]^n$ then we show that the inequality (11) fails also in this case. The construction follows the same line of the Example 2. So let $\{n_k\}$ be a sequence of natural numbers with

$$a_k = [2^{n_k(n-\gamma)}], \quad \frac{a_k}{2^{n_k(n-\gamma)}} = 1 - \varepsilon_k, \quad \varepsilon_k \leq 2^{-k}, \quad n_k > \frac{n}{\gamma}, \quad k = 1, 2, ... \tag{13}$$

We start by cutting Q_0 into pieces of length 2^{-n_1} and choose a_1 number of them such that they have different parents. This is possible since the number of parents is $2^{(n_1-1)n}$ yet the number of cubes that we need to choose is $2^{n_1(n-\gamma)}$, and $2^{n_1(n-\gamma)} < 2^{(n_1-1)n}$, since $n_1 > \frac{n}{\gamma}$. Then we repeat the process inside each of the selected cubes but this time with n_2 and a_2 and so on. So at the k-step we have a collection of cubes $\mathcal{C}_k$ such that

$$\#\mathcal{C}_k = a_1 ... a_k, \quad l(Q) = 2^{-(n_1+...n_k)}, \quad Q \in \mathcal{C}_k, \quad k = 1, 2, ... \tag{14}$$

Now let

$$f = \sum_{k=1}^{\infty} 2^{(n_1+...+n_k)\frac{\gamma}{p}} \sum_{Q\in\mathcal{C}_k} \chi_Q,$$

and note that $f \in L^1(Q_0)$ for the reason that

$$\int_{Q_0} f = \sum_{k=1}^{\infty} 2^{(n_1+...+n_k)\frac{\gamma}{p}} |\cup \mathcal{C}_k| = \sum_{k=1}^{\infty} a_1...a_k 2^{(\frac{\gamma}{p}-n)(n_1+...+n_k)} \leq \sum_{k=1}^{\infty} 2^{-\frac{\gamma}{p'}(n_1+...+n_k)} < \infty.$$

Our next task is to show that $f \notin J^{p,\infty}_{\gamma}(Q_0)$ and in order to do so, let $Q \in \mathcal{C}_k$ and $\bar{Q}$ be its parent. Then, note that from the construction the only cube in $\mathcal{C}_k$ which is inside $\bar{Q}$ is Q itself. Therefore,

$$f\chi_{\bar{Q}} = g + \text{const}, \quad g = \sum_{l=k}^{\infty} 2^{(n_1+...+n_k)\frac{\gamma}{p}} \sum_{\substack{Q'\in\mathcal{C}_l \\ Q'\subset Q}} \chi_{Q'},$$

which implies that

$$O(f, \bar{Q}) \geq \frac{|\bar{Q} - Q|}{\bar{Q}} \fint_{\bar{Q}} g \approx \fint_{Q} g \approx 2^{(n_1+...+n_k)\frac{\gamma}{p}},$$

and this shows

$$b_{\bar{Q}}(f) > c, \quad Q \in \mathcal{C}_k, \quad k = 1, 2, ...,$$

for some constant c. So by choosing $\lambda = c$ we obtain

$$\sum_{b_{Q'}(f)>\lambda} l(Q')^{n-\gamma} \geq \sum_{k=1}^{\infty} \sum_{Q\in\mathcal{C}_k} l(\bar{Q})^{n-\gamma} = 2^{n-\gamma} \sum_{k=1}^{\infty} 2^{-(n-\gamma)(n_1+...+n_k)} a_1...a_k = \sum_{k=1}^{\infty} (1-\varepsilon_1)...(1-\varepsilon_k) = \infty,$$

which shows that $\|f\|_{J^{p,\infty}_{\gamma}(Q_0)} = \infty$.

Our final goal is to show that $f \in JN^d_p(Q_0)$, so to this aim, let $\mathcal{E}$ be an arbitrary but finite collection of disjoint dyadic cubes in Q_0. We have to show that

$$\sum_{Q\in\mathcal{E}} |Q| O(f, Q)^p \lesssim 1$$

where the above bound is independent of $\mathcal{E}$. In order to do this, first we partition $\mathcal{E}$ into some disjoint sub-collections $\mathcal{E}_l$ as follows. For $Q \in \mathcal{E}$ suppose l is the smallest natural number such that Q contains a cube from $\mathcal{C}_l$, and call the sub-collection of such cubes $\mathcal{E}_l$. This gives us the partition of $\mathcal{E}$ and thus we have

$$\sum_{Q\in\mathcal{E}} |Q| O(f, Q)^p = \sum_{l} \sum_{Q\in\mathcal{E}_l} |Q| O(f, Q)^p. \tag{15}$$

Next, for $Q \in \mathcal{E}_l$ let us estimate the mean oscillation of f on Q. First we note that

$$f\chi_Q = g + \text{const}, \quad g = \sum_{\substack{Q' \subset Q \\ Q' \in \mathcal{C}_l}} \sum_{k=l}^{\infty} 2^{(n_1+\ldots+n_k)\frac{\gamma}{p}} \sum_{\substack{Q'' \subset Q' \\ Q'' \in \mathcal{C}_k}} \chi_{Q''},$$

which implies that

$$O(f,Q) \lesssim \fint_Q g \leq \frac{1}{|Q|} \sum_{\substack{Q' \subset Q \\ Q' \in \mathcal{C}_l}} \sum_{k=l}^{\infty} 2^{(n_1+\ldots+n_k)\frac{\gamma}{p}} |\bigcup_{\substack{Q'' \subset Q' \\ Q'' \in \mathcal{C}_k}} Q''|.$$

Now from the construction we get

$$O(f,Q) \lesssim \frac{\#\{Q'|Q' \subset Q, Q' \in \mathcal{C}_l\}}{|Q|} \left(2^{(\frac{\gamma}{p}-n)(n_1+\ldots+n_l)} + \sum_{k=l+1}^{\infty} 2^{(\frac{\gamma}{p}-n)(n_1+\ldots+n_k)} a_{l+1} \ldots a_k \right),$$

which after using (13) is simplified to

$$O(f,Q) \lesssim \frac{\#\{Q'|Q' \subset Q, Q' \in \mathcal{C}_l\}}{|Q|} 2^{(\frac{\gamma}{p}-n)(n_1+\ldots+n_l)} = \frac{1}{|Q|} |\bigcup_{\substack{Q' \subset Q \\ Q' \in \mathcal{C}_l}} Q'| 2^{(n_1+\ldots+n_l)\frac{\gamma}{p}}. \tag{16}$$

Now, we can bound the inner sum in (15) by

$$\sum_{Q \in \mathcal{E}_l} |Q| O(f,Q)^p \lesssim 2^{(n_1+\ldots+n_l)\gamma} \sum_{Q \in \mathcal{E}_l} |Q| \left(\frac{1}{|Q|} |\bigcup_{\substack{Q' \subset Q \\ Q' \in \mathcal{C}_l}} Q'| \right)^p \leq 2^{(n_1+\ldots+n_l)\gamma} \sum_{Q \in \mathcal{E}_l} |\bigcup_{\substack{Q' \subset Q \\ Q' \in \mathcal{C}_l}} Q'|,$$

and since cubes in $\mathcal{E}_l$ are disjoint we obtain

$$\sum_{Q \in \mathcal{E}_l} |Q| O(f,Q)^p \lesssim 2^{(n_1+\ldots+n_l)\gamma} |\bigcup_{\substack{Q' \subset \cup\mathcal{E}_l \\ Q' \in \mathcal{C}_l}} Q'| = 2^{(n_1+\ldots+n_l)(\gamma-n)} \#\{Q'|Q' \subset \cup\mathcal{E}_l, Q' \in \mathcal{C}_l\} \tag{17}$$

where in the last equality (14) is used. At this point let us denote

$$a'_l = \#\{Q'|Q' \subset \cup\mathcal{E}_l, Q' \in \mathcal{C}_l\} \tag{18}$$

and note that from (13) we have

$$a_1 \ldots a_l \approx 2^{(n_1+\ldots+n_l)(n-\gamma)},$$

which together with (17) and (18) implies that

$$\sum_{Q \in \mathcal{E}_l} |Q| O(f,Q)^p \lesssim \frac{a'_l}{a_1 \ldots a_l}, \quad l = 1, 2, \ldots. \tag{19}$$

Now we bound the numbers a'_l as follows. First we note that $a'_1 \leq a_1$ and then since cubes in $\mathcal{E}_2$ does not contain any cube from $\mathcal{C}_1$, the total number of cubes from $\mathcal{C}_2$ that is possible to be found in $\cup\mathcal{E}_2$ is $(a_1 - a'_1)a_2$, and the same reasoning shows that

$$a'_3 \leq ((a_1 - a'_1)a_2 - a'_2)\, a_3.$$

Then by induction we get

$$\frac{a'_l}{a_1 \ldots a_l} \leq 1 - \sum_{k=1}^{l-1} \frac{a'_k}{a_1 \ldots a_k}, \quad l = 1, 2, \ldots \tag{20}$$

and this together with (15) and (19) shows that

$$\sum_{Q\in\mathcal{E}} |Q| O(f,Q)^p \lesssim 1,$$

which implies that $\|f\|_{JN_p^d(Q_0)} \lesssim 1$.

Similar to the previous example one can check that the function

$$f = \sum_{k=1}^{\infty} k^{-\frac{\alpha}{p}} 2^{(n_1+\ldots+n_k)\frac{\gamma}{p}} \sum_{Q\in\mathcal{C}_k} \chi_Q, \quad 0<\alpha<1,$$

belongs to $JN_p^d(Q_0)$ but $[b(f)]_{l_\gamma^{p,\infty}(Q_0)} = \infty$.

3. On the weak-type characterization of $\dot W^{1,p}(\mathbb{R}^n)$

For $1 \le p < \infty$, and $f \in L^1_{loc}(\mathbb{R}^n)$ consider the function $m(f)$ and the measure ν_p which are defined on the upper half space $\mathbb{R}^{n+1}_+$ with

$$m(f)(x,r) := O(f, B(x,r)), \quad d\nu_p(x,r) := \frac{dxdr}{r^{p+1}}, \quad x\in\mathbb{R}^n, \quad r>0.$$

In Frank paper the author has shown that for $1<p<\infty$, the following equivalence of norms holds:

$$\|\nabla f\|_{L^p(\mathbb{R}^n)} \approx \|m(f)\|_{L^{p,\infty}(\mathbb{R}^{n+1}_+,\nu_p)}. \tag{21}$$

Similar to case of $L^p(\mathbb{R}^n)$, one can define the following quasi-norm

$$\|f\|_{O^{p,\infty}(\mathbb{R}^n)} := \|O(f)\|_{l_p^{p,\infty}(\mathbb{R}^n)}, \quad O(f)_Q = O(f,Q).$$

which can be think of as the dyadic version of the weak norm on the right hand side (21). An argument similar to the case of $L^p(\mathbb{R}^n)$ shows directly that

$$\|f\|_{O^{p,\infty}(\mathbb{R}^n)} \lesssim \|m(f)\|_{L^{p,\infty}(\mathbb{R}^{n+1}_+)}, \quad 1\le p<\infty,$$

even tough the reverse inequality does not hold as can be seen by the function $f = \chi_{\mathbb{R}_+\times\mathbb{R}^{n-1}}$. Nevertheless, using a well-known dyadic technique we can prove that for $O^{p,\infty}(\mathbb{R}^n)$, the weak Sobolev-Poincare inequality holds.

Theorem 3.1. *Let $1\le p<n$, $p^* = \frac{np}{n-p}$, and $|Q_0|=1$. Then we have*

$$\|f - f_{Q_0}\|_{L^{p^*,\infty}(Q_0)} \lesssim \|f\|_{O^{p,\infty}(Q_0)}. \tag{22}$$

Proof. To simplify the matter assume $\|f\|_{O^{p,\infty}(Q_0)} = 1$ then let M be the dyadic maximal function localized in Q_0 and for $t>0$ let

$$E_t = \{x\in Q_0 | M(f-f_{Q_0}) > t\}.$$

Then since the level set of $f - f_{Q_0}$ at height t is contained in E_t it is enough to show that

$$|E_t| \lesssim t^{-p^*}.$$

In order to do this, note that E_t is the union of maximal dyadic cubes, $\mathcal{C} = \{Q\}$, with

$$⨍_Q |f - f_{Q_0}| > t,$$

and also the maximality of Q implies

$$⨍_Q |f-f_{Q_0}| \le 2^n ⨍_{\bar Q} |f - f_{Q_0}| \le 2^n t.$$

Next, let $\{Q_j\}$ be the collection of maximal dyadic cubes of the height $2^{n+1}t$, containing in Q, and note that

$$2^{n+1}t < \fint_{Q_j} |f - f_{Q_0}| \leq \fint_{Q_j} |f - f_Q| + \fint_Q |f - f_{Q_0}| \leq \fint_{Q_j} |f - f_Q| + 2^n t,$$

which gives us

$$|Q_j| < \frac{1}{2^n t} \int_{Q_j} |f - f_Q|.$$

This, together with $E_{2^{n+1}t} \cap Q = \cup_j Q_j$, yields

$$\frac{|E_{2^{n+1}t} \cap Q|}{|Q|} < \frac{1}{2^n t} O(f, Q). \tag{23}$$

Now let $\lambda > 0$, and partition $\mathcal{C}$ into two sub-collections as

$$\mathcal{C}_1 = \{Q | O(f,Q) \leq \lambda\}, \quad \mathcal{C}_2 = \{Q | O(f,Q) > \lambda\},$$

which together with (23) implies

$$|E_{2^{n+1}t} \cap Q| < \begin{cases} \frac{\lambda}{2^n t} |Q| & Q \in \mathcal{C}_1 \\ |Q| & Q \in \mathcal{C}_2. \end{cases} \tag{24}$$

Therefore, we can estimate $|E_{2^{n+1}t}|$ with

$$|E_{2^{n+1}t}| = \sum_{Q \in \mathcal{C}} |E_{2^{n+1}t} \cap Q| = \sum_{Q \in \mathcal{C}_1} |E_{2^{n+1}t} \cap Q| + \sum_{Q \in \mathcal{C}_2} |E_{2^{n+1}t} \cap Q|$$

$$\leq \frac{\lambda}{2^n t} \sum_{Q \in \mathcal{C}_1} |Q| + \sum_{Q \in \mathcal{C}_2} |Q| \leq \frac{\lambda}{2^n t} |E_t| + \left(\sum_{O(f,Q) > \lambda} l(Q)^{n-p} \right)^{\frac{n}{n-p}} \leq \frac{\lambda}{2^n t} |E_t| + \lambda^{-p^*},$$

and after optimizing the last term with

$$\lambda = \left(\frac{|E_t|}{2^n t} \right)^{-\frac{1}{p^*+1}},$$

we obtain

$$|E_{2^{n+1}t}| \leq 2 \left(\frac{|E_t|}{2^n t} \right)^{\frac{p^*}{p^*+1}}. \tag{25}$$

At this point choose $t = 1$, and to simplify the notation let $\alpha = \frac{p^*}{p^*+1}$ and $b = 2^{n+1}$. Then from the above inequality we have

$$|E_{b^{k+1}}| \leq \left(\frac{|E_{b^k}|}{b^k} \right)^{\alpha},$$

which its successive application give us

$$|E_{b^{k+1}}| \leq b^{-\sum_{l=1}^{k+1}(k+1-l)\alpha^l} |E_1| \lesssim b^{(k+1)\frac{\alpha}{\alpha - 1}} = b^{-(k+1)p^*},$$

This shows that

$$|E_t| \lesssim t^{-p^*}, \quad t > 0,$$

which completes the proof. $\square$