WEEK 1

The first two weeks of tutorials are the most important sessions as they lay the foundation for further active participation during the rest of the semester. As such, the tutor should start with a brief reminder of the concepts of functions and composite functions as well as their domains and ranges.

Question 1

Suppose $f(x) = \sqrt[3]{x-1}$ and $g(x) = 1 + (\frac{x}{1+x^3})^3$. Find $f \circ g$ and $g \circ f$ and their domains.

Question 2

Let $f(x) = \ln(4 - x^2)$ and $g(x) = \sqrt{1-x}$. Find $f \circ g$ and determine its domain.

Question 3

Let $f(x) = x^2 + 2$ and $g(x) = \sqrt{2-x}$. Find $h = f \circ g$, and determine the domain and the range of $h(x)$.

Question 4

Suppose $f(x) = x + \frac{1}{x}$ and $g(x) = \frac{x+1}{x+2}$. Find $f \circ g$ and $g \circ f$. What is the domain of $g \circ f$?

Question 5

One of $\sin x$, $\cos x$, and $\tan x$ is given. Find the other two if x lies in the specified interval.

$\tan x = 2, \quad x \in \left[0, \frac{\pi}{2}\right]$

$\cos x = -\frac{5}{13}, \quad x \in \left[\frac{\pi}{2}, \pi\right]$

$\sin x = -\frac{1}{2}, \quad x \in \left[\pi, \frac{3\pi}{2}\right]$

Question 6

Find the function values in

$\cos^2 \frac{5\pi}{12}$

$\sin^2 \frac{3\pi}{8}$

1)

$$f(x) = \sqrt[3]{x-1} \quad , \quad g(x) = 1 + \left(\frac{x}{1+x^3}\right)^3$$

Find $g \circ f(x)$

$$g(f(x)) = 1 + \left(\frac{\sqrt[3]{x-1}}{\cancel{1} + x - \cancel{1}}\right)^3$$

$$= 1 + \frac{x-1}{x} = \frac{2x-1}{x}$$

$$f(g(x)) = \left(\cancel{1} + \left(\frac{x}{1+x^3}\right)^3 - \cancel{1}\right)^{1/3}$$

$$= \frac{x}{1+x^3}$$

$\mathrm{Dom}(g \circ f) = \mathbb{R} \setminus \{0\}$

$\mathrm{Dom}(f \circ g) = \mathbb{R} \setminus \{-1\}$

2

$$f(x) = \ln(4-x^2) \quad , \quad g(x) = \sqrt{1-x}$$

$$f\circ g(x) = f(g(x)) = \ln\left(4-(1-x)\right)$$

$$= \ln(3+x)$$

$\mathrm{Dom}(f\circ g)$: $3+x>0 \Rightarrow x>-3$

3

$$f(x) = x^2+2 \quad , \quad g(x) = \sqrt{2-x}$$

$$h(x) = f\circ g(x) = f(g(x)) = 2-x+2 = 4-x$$

$$\mathrm{Dom}\, h = \mathbb{R}$$

21

$$f(x) = x + \frac{1}{x} \quad و \quad g(x) = \frac{x+1}{x+2}$$

$$f(g(x)) = \frac{x+1}{x+2} + \frac{x+2}{x+1} = \frac{(x+1)^2 + (x+2)^2}{(x+1)(x+2)}$$

$$\mathrm{Dom}(f \circ g) = \mathbb{R} \setminus \{-1, -2\}$$

$$g(f(x)) = \frac{x + \frac{1}{x} + 1}{x + \frac{1}{x} + 2} = \frac{\frac{x^2+1+x}{\cancel{x}}}{\frac{x^2+1+2x}{\cancel{x}}}$$

$$= \frac{x^2 + x + 1}{(x+1)^2}$$

$$\mathrm{Dom}(g \circ f) = \mathbb{R} \setminus \{-1\}$$

5

$\tan x = 2 \quad , \quad x \in [0, \pi/2]$

$\dfrac{\sin x}{\cos x} = 2 \Longrightarrow \sin x = 2\cos x$

$\sin x = 2\sqrt{1-\sin^2 x}$

$x = 2\sqrt{1-x^2} \Longrightarrow x^2 = 4 - 4x^2$

$5x^2 = 4 \Longrightarrow x^2 = \dfrac{4}{5} \Rightarrow x = \dfrac{2}{\sqrt{5}}$

$\sin x = \dfrac{2}{\sqrt{5}} \quad , \quad \cos x = \dfrac{1}{\sqrt{5}}$

$\cos x = \dfrac{-5}{13} \qquad x \in (\pi/2, \pi)$

$\sin x = \sqrt{1-\cos^2 x} \Longrightarrow \sin^2 x = 1 - \cos^2 x$

$$\sin^2 x = 1 - \frac{25}{169} = \frac{169 - 25}{169} = \frac{144}{169}$$

$$\sin x = \frac{12}{13} \Rightarrow \tan x = -\frac{12}{5}$$

$$\sin x = -\frac{1}{2} \qquad x \in \left[\frac{3\pi}{2}, 2\pi\right]$$

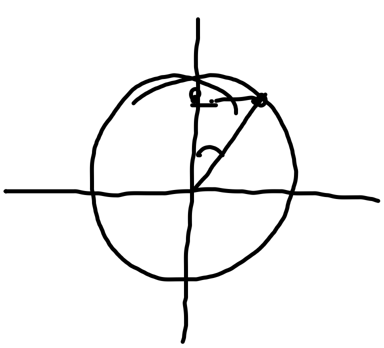
$$\cos^2 x = 1 - \sin^2 x$$

$$\cos^2 x = 1 - \frac{1}{4} = 3/4$$

$$\cos x = -\frac{\sqrt{3}}{2} \rightarrow \tan x = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

6

$$\cos^2 \frac{5\pi}{12} = \cos^2\left(\frac{6\pi}{12} - \frac{\pi}{12}\right)$$



$$= \cos^2\left(\frac{\pi}{2} - \frac{\pi}{12}\right) = \sin^2\left(\frac{\pi}{12}\right)$$

$$\cos(A+B) = \cos(A)\cos(B) - \sin A \sin B$$

$$\cos(2A) = \cos^2 A - \sin^2 A$$

$$\cos(2A) = 1 - 2\sin^2 A \Rightarrow \sin^2 A = \frac{1-\cos 2A}{2}$$

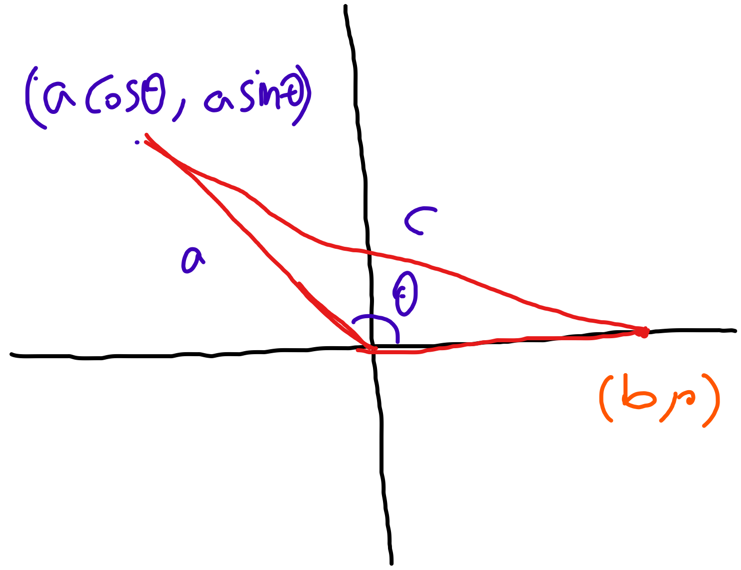
$$\sin^2(\pi/12) = \frac{1-\cos \pi/6}{2} = \frac{1-\sqrt{3}/2}{2} = \frac{2-\sqrt{3}}{4}$$

$$\sin^2\left(\frac{3\pi}{8}\right) = \sin^2\left(\frac{\pi}{2} - \pi/8\right) = \cos^2(\pi/8)$$

$$\cos^2 A = \frac{1+\cos 2A}{2}$$

$$= \frac{1+\cos \pi/4}{2} = \frac{1+\sqrt{2}/2}{2} = \frac{2+\sqrt{2}}{4}$$

law of cosines

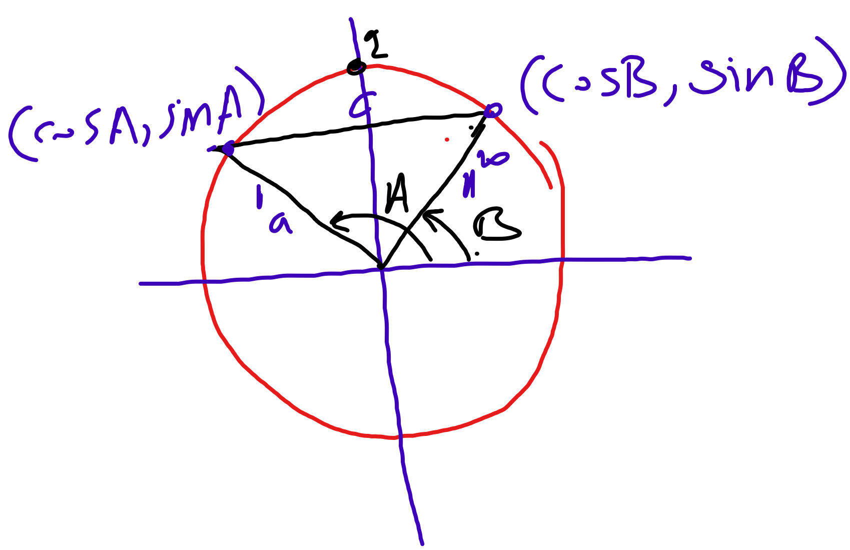


Pythagoros theorem.

$$c^2 = (a\cos\theta - b)^2 + (a\sin\theta)^2$$

$$= a^2\cos^2\theta - 2ab\cos\theta + b^2 + a^2\sin^2\theta$$

$$= a^2 + b^2 - 2ab\cos\theta$$



$\cos(A-B)$?

$c^2 = 1 + 1 - 2\cos(A-B)$

$c^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$

$= \cos^2 A + \cos^2 B - 2\cos A\cos B$

$+ \sin^2 A + \sin^2 B - 2\sin A \sin B$

$= 2 - 2\cos A\cos B - 2\sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\sin(\theta) = \cos(\pi/2 - \theta)$

$\sin(A-B) = \cos(\pi/2 - A + B)$

$= \cos(\pi/2 - A)\cos B - \sin(\pi/2 - A)\sin B$

$$= \sin A \cos B - \cos A \sin B$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

WEEK 2

Question 1

Find the inverse of the function $f(x) = \sqrt{2^x - 2}$. Determine the domain and range of f and f^{-1}.

Question 2

Solve for x : $\log_2(x-4) + \log_2(x+2) = 2\log_2 4$

Given the function $f = 1 + \ln(x^3 - 8)$, find the inverse function f^{-1}, the domain and the range of f and the domain and the range of f^{-1}.

Question 3

Solve for x (find the **exact** value, do not approximate): $e^{2x} = 2e^x + 24$.

Question 4

Find the inverse function f^{-1} of $f(x) = \log_2(4 + 2^x)$, and determine the domain and the range of $f^{-1}(x)$.

Question 5

Given the function $f = \dfrac{3}{6 + 4^{x+1}}$, find the inverse function f^{-1}, and determine the domain and the range of f^{-1}.

Question 6

Find the exact value of each of the following expressions.

a. $\sin^{-1}\left(\frac{-1}{2}\right)$ **b.** $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ **c.** $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

a. $\cos^{-1}\left(\frac{1}{2}\right)$ **b.** $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ **c.** $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

a. $\arccos(-1)$ **b.** $\arccos(0)$

a. $\arcsin(-1)$ **b.** $\arcsin\left(-\frac{1}{\sqrt{2}}\right)$

WEEK 3

Question 1

Evaluate the limits:

a) $\lim_{x \to 5} \dfrac{\sqrt{2x-1}-3}{x^3-125}$

b) $\lim_{x \to \infty} \dfrac{(x^3+1)(2x-3)^2}{(x+1)^2(3x+2)^3}$.

Question 2

Consider the function $f(x) = \dfrac{|x^2+4x-5|}{x^2-25}$. Calculate both one-sided limits at the point(s) where the function is undefined.

Question 3

Find all horizontal and all vertical asymptotes of the function

$$f(x) = \frac{x\sqrt{4x^2+2x+1}+2x^2}{x^2+25}$$

Question 4

Find the limit if it exists, otherwise explain why it does not exist:

Do not use L'Hôpital's Rule

(a) $\lim_{x \to 2} \dfrac{|x-2|(x+3)}{x^2+x-6}$ **(b)** $\lim_{x \to 1} \dfrac{x-1}{3-\sqrt{x^2+8}}$

Find all horizontal and vertical asymptotes of the function

$$f(x) = \frac{\sqrt{16x^2+1}}{x^2-16} \cdot \frac{x^2}{x+4}$$

WEEK 4

Question 1

Find the value of a and b so that the function

$$f(x) = \begin{cases} 5 + x^2 & \text{if} \quad x \le 0 \\ ax + b & \text{if} \quad 0 < x \le 1 \\ \dfrac{25}{x} & \text{if} \quad x > 1 \end{cases}$$

is continuous everywhere. Sketch the graph of this function.

Question 2

Calculate both one-sided limits at $x = 2$ where the function

$f(x) = \dfrac{x^2 + x - 6}{|x - 2|}$ is undefined.

Is it possible to define f at $x = 2$ to make $f(x)$ continuous everywhere? Explain.

Question 3

Consider the following piecewise-defined function:

$$f(x) = \begin{cases} 4 - a\,x^2 & \text{if} \quad x \le 1 \\ b - 2\,x & \text{if} \quad x > 1 \end{cases}$$

Find the values of a and b so that the function $f(x)$ is differentiable at every x. Sketch the graph of $f(x)$ on the interval $[-3, 3]$.

WEEK 5

Question 1

Let $f\left(x\right)=\dfrac{4-3x^{7}}{x^{5}}$. Find $f'''\left(x\right)$.

Question 2

Find the value of the 10-th derivative $f^{(10)}(0)$ of the function $f(x)=(e^{2x}+e^{-2x})$ at the point $x=0$.
(HINT: study the pattern of higher-order derivatives of $f(x)$.)

Question 3

Calculate the second derivative $f''(x)$ of the function

$f(x)=(x+a)\,(x-a)\,x^{2}$ where a is a parameter, and find $f''(1)$.

Question 4

Find the derivatives of the following functions (you do not need to simplify the answers)

(a) $f\left(x\right)=\dfrac{1+\sqrt{x}-3x-2x\sqrt[3]{x^{2}}-2x\sqrt{x}}{2x\sqrt{x}}$;

$f(x)=x^{1/2}(\sqrt{x}-x^{-3/2})\,2^{x}$

$f(x)=(x^{2}-xe^{\pi})2^{x}$;

$f(x)=x^{e}\,e^{x}+e^{2}$

WEEK 6

Question 1

Given the function $f(x) = \sqrt{x^2 + 8}$

(a) Use appropriate differentiation rules to find the derivative $f'(x)$.

(b) Use the definition of derivative to verify the answer in part a.

Question 2

Consider the function $y = \sqrt{1 + 2x}$.

(a) Use the **definition of derivative** to find the formula for dy/dx.

Question 3

Given the function $f(x) = \dfrac{2x}{x+2}$,

(a) Find the derivative $f'(x)$ using the definition of the derivative as the limit of the difference quotient.

Question 4

Find the derivatives of the following functions:

$$f(x) = \sin[\sqrt{x^2 + 1} \cdot \cos(e^x)]$$

$$f(x) = \frac{\sin^2(x) + \sin(x^2)}{\ln \sqrt{1 + x}}$$

$$f(x) = \cos[x \sin(x) + \sin(x + \cos x)]$$

$$f(x) = \frac{e^x + e^{-x}}{\cos(x) + \sin(x)}$$

WEEK 7

Question 1

The equation of a curve is $y^4 \tan x = xy^3 + y - 1$ and defines y implicitly as a function of x. Verify that the point $(0, 1)$ belongs to this curve and find an equation of the tangent line to the curve at this point.

Question 2

(a) Verify that the point (2,1) belongs to the curve defined by the equation

$y^2 + x\sqrt{3+y} = 1 + x^2$, and find the implicit derivative $\dfrac{dy}{dx}$ at this point.

(b) Write the equation of the tangent line to the curve at the point (2,1).

Question 3

(a) Verify that the point (2,1) belongs to the curve defined by the equation $x^2 + 2y^2 + 2 = x^3\,y^3$, and find an equation of the tangent line to the curve at this point.

WEEK 8

Question 1

Find the derivatives of the following functions (you do not need to simplify the answers)

(a) $f(x) = \dfrac{\sqrt{x} + 3\sqrt[3]{x^2} + x^5}{2x\sqrt[3]{x}}$

(b) $f(x) = (x^3 + ex - \sin\pi)(\cos 2x)$

(c) $f(x) = \ln^3(x^2 + \tan(3x))$

(d) $f(x) = \dfrac{\arcsin^2 x}{\sqrt{1-x^2}}$

(e) $f(x) = (3x^2 + 5)^{\arctan x}$

Question 2

A particle is moving along the plane curve $2x^2 + 5y^2 = 22$. At the moment when $x = -1$ the x-coordinate is increasing at the rate of 5 cm/sec. If the y-coordinate is positive at this moment, is it increasing or decreasing? How fast?

Question 3

The length of a rectangle is increasing at the rate of 8 cm/s and its width is increasing at the rate of 5 cm/s. When the length is 20 cm and the width is 12 cm, how fast is the area of the rectangle increasing at that instant?

Question 4

(b) A 13 ft ladder is leaning against a vertical wall of a house when its base starts to slide away (in horizontal direction) from the wall. By the time when the base is at the distance $x = 5$ ft from the wall the base is moving at the rate of $dx/dt = 2$ ft/sec. How fast is the top of the ladder sliding down the wall at that instant?

Question 5

Find the derivatives of the following functions (for **full marks** you have to show at least **one intermediate step** of your calculations):

(a) $f(x) = \ln\left(\dfrac{e^2x^3}{\sqrt{x+3}}\right)$

(b) $f(x) = \dfrac{e^{-x}\sec x}{1+e^x}$

(c) $f(x) = \ln[e^{x\sin x} + x\sin(e^x)]$

(d) $f(x) = (1+\cos x)^{x\cos x}$

WEEK 9

Question 1

Given the function $f(x) = \sqrt{x^2 + 8}$

Find the differential of the function.

Use the differential above, or (equivalently) use the linear approximation at $a = 1$ (with the appropriate choice of Δx) to find the approximate value of $\sqrt{8.49}$. Check the approximation with your calculator.

Question 2

Consider the function $y = \sqrt{1 + 2x}$.

Find the linearization $L(x)$ of the function $y(x)$ at $a = 4$

Use this linearization to approximate the exact value of $\sqrt{10}$.

Question 3

Find the absolute extrema of $f(x) = \dfrac{x}{x^2 - x + 1}$ on the interval $[0, 3]$.

Question 4

Find the absolute maximum and absolute minimum of $f(x) = \dfrac{x+2}{x^2+5}$ on the interval $[-1, 3]$.

Question 5

Calculate the differential df of the function $f = \tan(x)$. Use it to find an approximation for the change of f in the vicinity of $x = \pi/4$, i.e. $\Delta f = f(x) - f(\pi/4)$, in the form of differential df at $x = \pi/4$.

Use this differential to approximate $\tan(\pi/3)$ given that $\frac{\pi}{3} = \frac{\pi}{4} + \frac{\pi}{12}$.

WEEK 10

Question 1

Use L'Hôpital's rule to evaluate $\lim_{x\to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$

Question 2

Let $f(x) = x^3 - 2x + 3$.

(a) Find the slope m of the secant line joining the points $(-2, f(-2))$ and $(0, f(0))$.

(b) Find all points $x = c$ (if any) on the interval [-2,0] such that $f'(c) = m$.

Question 3

Use l'Hôpital's rule to evaluate the $\lim_{x\to 0} \dfrac{\ln(1+x^2)}{1-\cos(2x)}$.

Question 4

Use l'Høpital's rule to evaluate the $\lim_{x\to 0} \dfrac{x\sin(3x)}{(e^{2x}-1)^2}$.

WEEK 11

Question 1

Given the function $f(x) = \dfrac{2x}{x^2 - 9}$,

(a) Find the domain and check for symmetry. Find all asymptotes (if there are any).

(b) Calculate $f'(x)$ and use it to determine interval(s) where the function is increasing, interval(s) where the function is decreasing, and local extrema (if there are any).

(c) Calculate $f''(x)$ and use it to determine interval(s) where the function is concave upward, interval(s) where the function is concave downward, and points of inflection (if there are any).

(d) Sketch the graph of the function.

Question 2

Given the function $f(x) = x^4 - 6x^2$.

(a) Find the domain of $f(x)$, check for symmetry, and also find asymptotes (if any).

(b) Calculate $f'(x)$ and use it to determine intervals where the function is increasing, intervals where it is decreasing, and the local extrema (if any).

(c) Calculate $f''(x)$ and use it to determine intervals where the function is concave upward, intervals where the function is concave downward, and the inflection points (if any).

(d) Sketch the graph of the function $f(x)$ using the information obtained above.

Question 3

Given the function $f(x) = 2x^3 - 21x^2 + 36x - 9$.

(a) Calculate $f'(x)$ and use it to determine intervals where the function is increasing, intervals where it is decreasing, and all critical numbers on the x-axis where $f(x)$ has local maximum or local minimum.

(b) Calculate $f''(x)$ and use it to determine intervals where the function is concave upward, intervals where the function is concave downward, and the inflection points (if any).

(c) Sketch the graph of the function $f(x)$ using the information obtained above.

WEEK 12

Question 1

A rectangle $ABCD$ has sides parallel to the coordinate axes and point A is located at the origin. Point B is on the positive x-axis and point C is on the graph of the function $y = e^{-2x}$ and has positive x and y coordinates. Find the coordinates of the point C that maximizes the area of the rectangle.

Question 2

Find the point (x_0, y_0) on the line $y = 2 - 2x$ that is closest to the point (5,2).

If $A = 1200\,\mathrm{cm}^2$ of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

Question 3

Find the radius r and the height h of a cylindrical can closed on both top and bottom that has a given volume V, but has the smallest possible surface area A (first express h as a function of r).