CMSC 660 HW II

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1 Chapter 4, Problem 1

1.1 Calculation

Let L be the differentiation operator which takes P_3 to P_2 described in section 4.2.2. Let $f_k = H_k(x)$ for $k = 0, 1, 2, 3$ be the Hermite Polynomial basis of P_3 and $g_k = H_k(x)$ for $k = 0, 1, 2$ be the Hermite Polynomial basis of P_2. What is the matrix A that represents this L in these bases?

1.1.1 Answer

First, we note the values of the Probabilist's Hermite Polynomials [1](pg. 71)[2] relevant to this problem. Specifically:

$$
\begin{aligned}
H_0 &= 1 \\
H_1 &= x \\
H_2 &= x^2 - 1 \\
H_3 &= x^3 - 3x
\end{aligned}
$$

Therefore, we can state that the Hermite Polynomial Bases for P_3 and P_2 are

P_3	P_2
$f_0 = 1$	$g_0 = 1$
$f_1 = x$	$g_1 = x$
$f_2 = x^2 - 1$	$g_2 = x^2 - 1$
$f_3 = (x^3 - 3x)$	

Applying the f_k basis to P_3 yields the expanded polynomial

$$P_3 = p_0 + p_1 x + p_2(x^2 - 1) + p_3(x^3 - 3x) \tag{1}$$

This implies that

$$
\begin{aligned}
AP_3 &= \frac{d}{dx}(p_0 + p_1 x + p_2(x^2 - 1) + p_3(x^3 - 3x)) \\
&= p_1 + p_2(2x) + p_3(3(x^2 - 1))
\end{aligned}
$$

Expressing this polynomial in terms of a vector of coefficients and the basis P_2 yields:

$$
AP_3[f_k] = P_2[g_k] \rightarrow A \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 - 1 \\ x^3 - 3x \end{bmatrix} = \begin{bmatrix} p_1 \\ 2p_2 \\ 3p_3 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 - 1 \end{bmatrix} \tag{2}
$$

Therefore, we deduce that the matrix A must be

$$
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \tag{3}
$$

We can confirm this by applying A to P_3 then applying P_2's basis to verify that the generated polynomial is equivalent to P_2:

$$
\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = p_1 + 2p_2 + 3p_3 \tag{4}
$$

Vectorizing this polynomial and applying the basis for P_2 yields:

$$
\begin{bmatrix} p_1 \\ 2p_2 \\ 3p_3 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 - 1 \end{bmatrix} = p_1 + 2xp_2 + 3(x^2 - 1)p_3 = \frac{d}{dx}(p_0 + p_1 x + p_2(x^2 - 1) + p_3(x^3 - 3x)) \tag{5}
$$

2 Chapter 4, Problem 2

Using the following information from the problem:

- L is a linear transformation from $V \to V$
- $f_1, ..., f_n$ and $g_1, ..., g_n$ are to bases of V
- Any $u \in V$ can be written in a unique was as $u = \sum_{k=1}^{n} v_k f_k$ or as $u = \sum_{k=1}^{n} w_k g_k$
- R is an $n \times n$ matrix that relates the f_k expansion coefficients v_k to the g_k coefficients w_k by $v_j = \sum_{k=1}^{n} r_{jk} w_k$
- $v = \begin{bmatrix} v_1 \\ . \\ . \\ . \\ v_k \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ . \\ . \\ . \\ w_k \end{bmatrix}$ are related by $v = Rw$
- A represents L in the f_k basis and B represents L in the g_k basis

2.1 a. Proof

Show that $B = R^{-1}AR$.

2.1.1 Answer

Because R is an invertible matrix (as evidenced by the problem statement) we can use the property $AA^{-1} = I$ to rewrite the function in a way that eliminates the inversion:

$$B = R^{-1}AR$$
$$RB = RR^{-1}AR$$

$$RB = AR \tag{6}$$

Examining the left side of equation (6), we rewrite the expression using the notation from the HW:

$$RB = R_{\mathcal{G}}[L]_{\mathcal{G}}$$

Since R directly express the expansion coefficients in the f_k basis to the coefficients in the g_k basis, we can write

$$RB = R_{\mathcal{G}}[L]_{\mathcal{G}} =_{\mathcal{F}} [L]_{\mathcal{F}}$$
$$\therefore R^{-1}RB = R_{\mathcal{F}}^{-1}[L]_{\mathcal{F}}$$
$$\therefore B = R_{\mathcal{F}}^{-1}[L]_{\mathcal{F}}$$

Examining the right side of equation (6) and expressing it in the HW notation:

$$AR =_{\mathcal{F}} [L]_{\mathcal{F}}R$$

2.2 b. Calculation

For $V = P_3$, $f_k = x^k$ and $g_k = H_k$ find R.

2.2.1 Answer

In this case, we are examining an arbitrary vector in V, expressed as $\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$, in two different bases:

$$\begin{bmatrix} p_0 \\ p_1 x \\ p_2 x^2 \\ p_3 x^3 \end{bmatrix} \text{(in the f basis)}$$

&

$$\begin{bmatrix} p_0 \\ p_1 x \\ p_2(x^2 - 1) \\ p_3(x^3 - 3x) \end{bmatrix} \text{(in the g basis)}$$

We are given the relation that $v = Rw$ from the problem, therefore

$$\begin{bmatrix} p_0 \\ p_1 x \\ p_2 x^2 \\ p_3 x^3 \end{bmatrix} = R \begin{bmatrix} p_0 \\ p_1 x \\ p_2(x^2 - 1) \\ p_3(x^3 - 3x) \end{bmatrix}$$

Which allows us to deduce that R must be

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

2.3 c. Calculation

If L is the linear transformation $Lp = q$ with $q(x) = \delta_x(xp(x))$ find the matrix A that represents L in the monomial basis f_k.

2.3.1 Answer

Inserting the given values into the given transformation expression yields

$$
\begin{aligned}
A(p_0 + p_1x + p_2x^2 + p_3x^3) &= \delta_x(xp(x)) \\
A(p_0 + p_1x + p_2x^2 + p_3x^3) &= \delta_x(p_0x + p_1x^2 + p_2x^3 + p_3x^4) \\
A(p_0 + p_1x + p_2x^2 + p_3x^3) &= p_0 + 2p_1x + 3p_2x^2 + 4p_3x^3
\end{aligned}
$$

Therefore we deduce that the matrix A must be

$$
A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}
$$

2.4 d. Calculation

Find the matrix B that represents L in the Hermite polynomial basis H_k

2.4.1 Answer

We perform the same steps as in part c:

$$
\begin{aligned}
B(p_0 + p_1x + p_2(x^2-1) + p_3(x^3-3x)) &= \delta_x(xp(x)) \\
B(p_0 + p_1x + p_2(x^2-1) + p_3(x^3-3x)) &= \delta_x(p_0x + p_1x^2 + p_2(x^3-x) + p_3(x^4-3x^2)) \\
B(p_0 + p_1x + p_2(x^2-1) + p_3(x^3-3x)) &= p_0 + 2p_1x + p_2(3x^2-1) + p_3(4x^3-6x)
\end{aligned}
$$

Therefore we deduce that the matrix B must be

$$
B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 6 & 0 & 4 \end{bmatrix}
$$

2.5 e. Calculation

Multiply the matrices to check explicitly that $B = R^{-1}AR$ in this case.

2.5.1 Answer

$$R^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}$$

$$\therefore R^{-1}AR = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 6 & 0 & 4 \end{bmatrix}$$

Which is equal to our calculated value for matrix B.

3 Chapter 4, Problem 3

3.1 Proof

Let A be an $n \times m$ matrix and B be an $m \times l$ matrix. Then AB is an $n \times l$ matrix. Show that

$$(AB)^* = B^* A^* \tag{7}$$

3.1.1 Answer

To find the adjoint of a matrix we must first calculate the matrix of cofactors then take the inverse of that matrix.

Starting with the left hand side of equation (7), we can express matrix AB generally as:

$$AB = \begin{bmatrix} (AB)_{0,0} & . & . & . & (AB)_{0,l} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ (AB)_{n,0} & . & . & . & (AB)_{n,l} \end{bmatrix}$$

4 Chapter 4: Questions

4.1 Definition

Define a 'real vector space.

4.1.1 Answer

4.2 Proof

Is $V' := u \in \mathbb{R}^n, \sum_{k=1}^n u_k = 0$ a subspace of $\mathbb{R}^n$? Provide a proof of your assertion.

4.2.1 Answer

4.3 Proof

Is $V' := u \in \mathbb{R}^n, \sum_{k=1}^n u_k = 1$ a subspace of $\mathbb{R}^n$? Provide a proof of your assertion.

4.3.1 Answer

4.4 Definition

What is the standard basis of $\mathbb{R}^n$?

4.4.1 Answer

4.5 Definition

What is the standard inner product of $\mathbb{R}^n$?

4.5.1 Answer

4.6 Proof

Probe that the space of polynomials with real coefficients is a vector space.

4.6.1 Answer

4.7 Concept

If A is a real matrix, what is the difference between its transpose and its adjoint?

4.7.1 Answer

References

[1] David Bindel and Johnathan Goodman. *Principles of Scientific Computing.* 2009.

[2] Hermite Polynomials Wikipedia Page
`https://en.wikipedia.org/wiki/Hermite_polynomials`