Lectures on Scientific Computing

Lise-Marie Imbert-Gerard

9/11/18

Finding a complete set (m of them) of eigenvalues and eigenvectors is called **diagonalizing** A.

A matrix with m linearly independent eigenvectors is said to be **diagonalizable**.

Counter Example:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

note that $J^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. If $J = R\Lambda R^{-1}$, then $J^2 = R\Lambda^2 R^{-1}$ so $\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

A **Jordan Block** of size $h > 1$ with eigenvalues λ is a square matrix with

- λ on its diagonal
- 1 on its super diagonal
- 0 everywhere else

As a reminder, if a matrix does not have a diagonal form there exists a basis in which A has a Jordan form, i.e. block diagonal with diagonal blocks being either Jordan blocks or 1×1 blocks.

A matrix Q is said to be **orthogonal** if $Q^*Q = I$. In otherwords, if a matrix's adjoint is its inverse it is orthogonal.

Proposition:
If a matrix A has a Jordan Block then there is no basis of eigenvectors.

Even if A is diagonalizable, computing the eigenvectors may be very sensitive. Often in practice we can use the Schur Form $AQ = QT$ with T upper triangular and Q orthogonal.

Returning to the symmetric case, we propose that the eigenvalues of a self-adjoint matrix are real.
Proof:

$$\begin{aligned}
Ax &= \lambda x \\
x^* Ax &= \lambda x^* x \\
|Ax|^* x &= \lambda x^* x \\
\bar{\lambda} x^* x &= \lambda x^* x \\
(\bar{\lambda} - \lambda) x^* x &= 0 \\
\rightarrow \bar{\lambda} &= \lambda \text{ since } x \neq 0
\end{aligned}$$

The left eigenvectors are adjoint of the right eigenvector. There are no Jordan Blocks, so the matrix is diagonizable and a complete set of its eigenvectors forms a basis.

1 Differentiation and Perturbation Theory

Suppose we look at a function f which is a function of n variables $(x = (x_1 ... x_n \in \mathbb{R}^n)$ and matrix valued $(f_1 ... f_n \in \mathbb{R}^n)$. The **Jacobian Matrix** $f'(x)$ is defined as $(\delta_{x1} f | ... | \delta_{xn} f) \in \mathbb{R}^{nx^n}$.

If f is differentiable and f' is Lipschitz then

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x + \mathcal{O}(||\Delta x||^2) \tag{1}$$

Δx is a vector $\in \mathbb{R}^n$ and Δf is a vector $\in \mathbb{R}^n$.

Imagine we have a parameter s and we move along the differentiable curve defined by x, i.e. $\Delta \rightarrow x(s) \in \mathbb{R}^n$ with $x(0) = x_0$. Then $s \rightarrow f(x(s))$ defines a differentiable curve in $\mathbb{R}^n$ with $f(x(0)) = f(x_0)$. We define the **Virtual Perturbation** by $\dot{x} := \frac{d}{ds}|_s x_s$ and $\dot{f} := \frac{d}{ds}|_{s=0} f(x(s))$. Therefore,

$$\dot{f} = f'(x_0)\dot{x} \tag{2}$$

The virtual perturbation theory is to compute equation (2) to find the general relation (1). (2) stands for any $x(s)$. In particular we can use $x(s) = x_0 + s\dot{x}$.

Example:

$$f(A, B) \in \mathbb{R}^{m \times m} \times \mathbb{R}^{m \times p} \to AB \in \mathbb{R}^{m \times p}$$

with differentiable curves $A(s)$ and $B(s)$. We can write

$$(AB)_j = \sum_{k=1}^{n} ...$$

REVIEW SECTION ON PERTURBATION THEORY

When $m = 1$ we write the Jacobian as ∇f.
Example:

$$f(x) = x^* Ax$$

for a given matrix A, then $\dot{f} = x^*(Ax) + x^*(\dot{Ax}) = \dot{x}^*(Ax) + x^* A\dot{x}$

$$\begin{aligned} \dot{x}^*(Ax) + x^* A\dot{x} &= x^*(A^* + A)\dot{x} \\ &=> \nabla(x^* Ax) = x^*(A^* + A) \end{aligned}$$

2 Variational Principles for the Symmetric Eigenvalue Problem

A **Variational Principle** means that you can express the problem of finding one point in a space by isolating it as the minimum or maximum of a function. i.e. you are creating a new problem in which the thing that you are looking for is the solution as a maximum or a minimum.

The **Rayleigh Quotient** for a matrix $A \in \mathbb{C}^{m \times m}$ is
$\forall x \in \mathbb{C}^m (x \neq 0)$ $Q(x) = \frac{x^* Ax}{x^* x}$

A vector r is a stationary point of Q if $\nabla Q(r) = 0$ and $\lambda := Q(r)$ which is a stationary value.

2.1 Theorem

Suppose A is self-adjoint. Then $x \neq 0$ is an eigenvector if and only if it is a stationary point of Q. In this case the corresponding eigenvalue is the stationary value.