CMSC 660 HW I

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9/3/18

1 Chapter 2, Problem 3

1.1 Proof

Show that $f_{j,k} = \sin(x_0 + (j-k)\frac{\pi}{3})$ satisfies the recurrence relation

$$f_{j,k+1} = f_{j,k} - f_{j+1,k}$$

1.1.1 Answer

Applying the trigonometric identity

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{1}$$

Allows $f_{j+1,k}$ to be written as:

$$\begin{aligned} f_{j+1,k} &= \sin(x_0 + (j+1-k)\frac{\pi}{3}) \\ &= \sin(x_0 + (j-k)\frac{\pi}{3} + \frac{\pi}{3})) \\ &= \sin(\frac{\pi}{3})\cos(x_0 + (j-k)\frac{\pi}{3}) + \cos(\frac{\pi}{3})\sin(x_0 + (j-k)\frac{\pi}{3}) \\ &= \frac{\sqrt{3}}{2}\cos(x_0 + (j-k)\frac{\pi}{3}) + \frac{1}{2}\sin(x_0 + (j-k)\frac{\pi}{3}) \end{aligned} \tag{2}$$

Therefore we can write the $f_{j,k} - f_{j+1,k}$ as

$$\begin{aligned} f_{j,k} - f_{j+1,k} &= \sin(x_0 + (j-k)\frac{\pi}{3}) - (\frac{\sqrt{3}}{2}\cos(x_0 + (j-k)\frac{\pi}{3}) + \frac{1}{2}\sin(x_0 + (j-k)\frac{\pi}{3})) \\ &= \frac{1}{2}\sin(x_0 + (j-k)\frac{\pi}{3}) - \frac{\sqrt{3}}{2}\cos(x_0 + (j-k)\frac{\pi}{3}) \end{aligned} \tag{3}$$

Applying the same trigonometric identity to $f_{j,k+1}$ yields

$$\begin{aligned} f_{j,k+1} &= \sin(x_0 + (j-k-1)\frac{\pi}{3}) \\ &= \sin(x_0 + (j-k)\frac{\pi}{3} - \frac{\pi}{3}) \\ &= \sin(-\frac{\pi}{3})\cos(x_0 + (j-k)\frac{\pi}{3}) + \cos(-\frac{\pi}{3})\sin(x_0 + (j-k)\frac{\pi}{3}) \\ &= -\frac{\sqrt{3}}{2}\cos(x_0 + (j-k)\frac{\pi}{3}) + \frac{1}{2}\sin(x_0 + (j-k)\frac{\pi}{3}) \end{aligned} \tag{4}$$

We therefore see

$$\begin{aligned} f_{j,k+1} &= -\frac{\sqrt{3}}{2}\cos(x_0 + (j-k)\frac{\pi}{3}) + \frac{1}{2}\sin(x_0 + (j-k)\frac{\pi}{3}) \\ f_{j,k} - f_{j+1,k} &= \frac{1}{2}\sin(x_0 + (j-k)\frac{\pi}{3}) - \frac{\sqrt{3}}{2}\cos(x_0 + (j-k)\frac{\pi}{3}) \\ \therefore f_{j,k+1} &= f_{j,k} - f_{j+1,k} \end{aligned} \tag{5}$$

1.2 Proof

Show that if $|\hat{f}_{j,k} - f_{j,k}| \leq \epsilon$ for all j then $|\hat{f}_{j,k+1} - f_{j,k+1}| \leq 2\epsilon$ for all j.

1.2.1 Answer

We know from the recurrence relation $f_{j,k+1} = f_{j,k} - f_{j+1,k}$ that we can rewrite the second expression:

$$\begin{aligned} |\hat{f}_{j,k+1} - f_{j,k+1}| &= |(\hat{f}_{j,k} - \hat{f}_{j+1,k}) - (f_{j,k} - f_{j+1,k})| \\ &= |(\hat{f}_{j,k} - f_{j,k}) + (f_{j+1,k} - \hat{f}_{j+1,k})| \\ &= |\hat{f}_{j,k} - f_{j,k}| + |f_{j+1,k} - \hat{f}_{j+1,k}| \\ &= |\hat{f}_{j,k} - f_{j,k}| + |\hat{f}_{j+1,k} - f_{j+1,k}| \end{aligned} \tag{6}$$

We are given the relation $|\hat{f}_{j,k} - f_{j,k}| \leq \epsilon$ for all j. Therefore we can state

$$|\hat{f}_{j,k} - f_{j,k}| + |\hat{f}_{j+1,k} - f_{j+1,k}| \leq \epsilon + |\hat{f}_{j+1,k} - f_{j+1,k}| \tag{7}$$

Since the relation applies for all j, we can state

$$|\hat{f}_{j+1,k} - f_{j+1,k}| \leq \epsilon \tag{8}$$

Therefore,

$$
\begin{aligned}
|\hat{f}_{j,k} - f_{j,k}| + |\hat{f}_{j+1,k} - f_{j+1,k}| &\leq \epsilon + \epsilon \\
|\hat{f}_{j,k} - f_{j,k}| + |\hat{f}_{j+1,k} - f_{j+1,k}| &\leq 2\epsilon \\
\therefore |\hat{f}_{j,k+1} - f_{j,k+1}| &\leq 2\epsilon
\end{aligned}
\tag{9}
$$

1.3 Code

Write a program that computes $e_k = \hat{f_{0,k}} - f_{0,k}$ for $1 \leq k \leq 60$ and $x_0 = 1$. Print the e_k and see whether they grow monotonically. Plot the e_k on a linear scale and note what happens around $k = 50$. Plot the e_k on a log scale. For comparison, include a straight line that would represent the error if it were exactly to double each time.

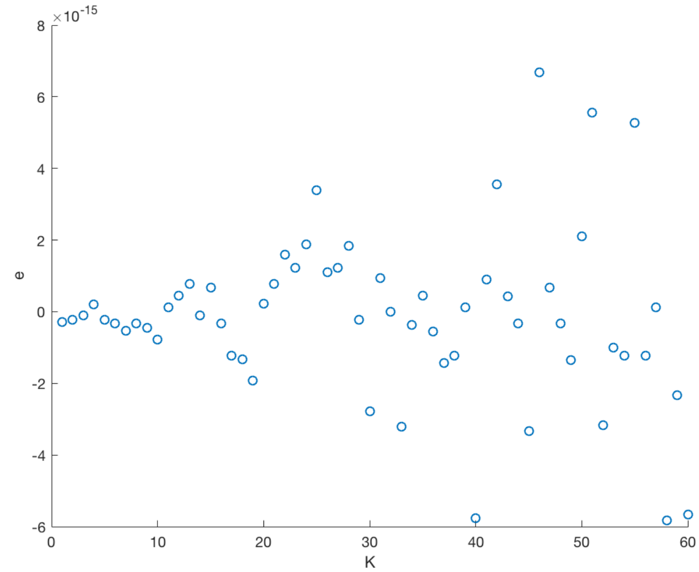
1.3.1 Answer

All code is attached in an appendix. Plots and tables below.

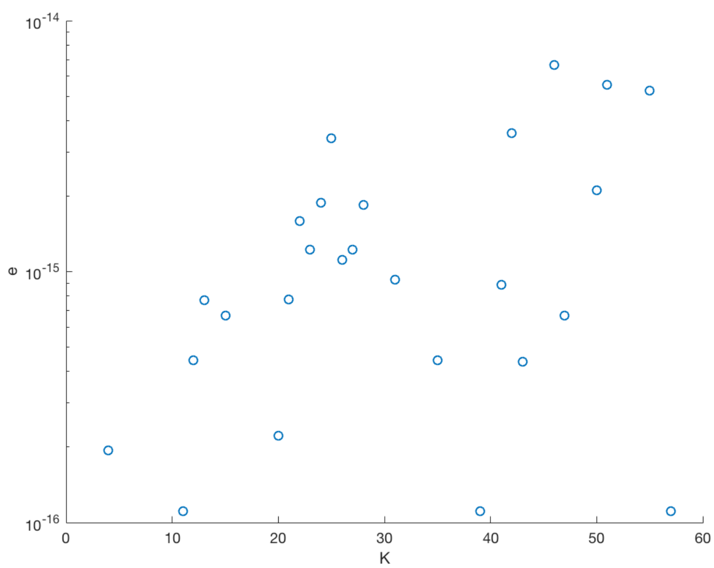
ϵ_k **Table:**

ϵ_k, $k = [1, 60]$
-2.91433543964104e-16
-2.22044604925031e-16
-1.11022302462516e-16
1.94289029309402e-16
-2.22044604925031e-16
-3.33066907387547e-16
-5.34294830600857e-16
-3.33066907387547e-16
-4.44089209850063e-16
-7.84095011141517e-16
1.11022302462516e-16
4.44089209850063e-16
7.70217223333702e-16
-1.11022302462516e-16
6.66133814775094e-16
-3.19189119579733e-16
-1.22124532708767e-15
-1.33226762955019e-15
-1.91513471747840e-15
2.22044604925031e-16
7.77156117237610e-16
1.58900670399476e-15
1.22124532708767e-15
1.88737914186277e-15
3.39311911901063e-15
1.11022302462516e-15
1.22124532708767e-15
1.83880688453542e-15
-2.22044604925031e-16
-2.77555756156289e-15
9.29811783123569e-16
0
-3.21964677141295e-15
-3.60822483003176e-16
4.44089209850063e-16
-5.55111512312578e-16
-1.42247325030098e-15
-1.22124532708767e-15
1.11022302462516e-16
-5.77315972805081e-15
8.88178419700125e-16
3.55271367880050e-15
4.37150315946155e-16
-3.33066907387547e-16
-3.33066907387547e-15
6.67521593555875e-15
6.66133814775094e-16
-3.33066907387547e-16
-1.36002320516582e-15
2.10942374678780e-15
5.55111512312578e-15
-3.17801340798951e-15
-9.99200722162641e-16
-1.22124532708767e-15
5.27355936696949e-15
-1.22124532708767e-15
1.11022302462516e-16
-5.82173198537817e-15
-2.33146835171283e-15
-5.66213742558830e-15

Linear Plot:



Log Plot:



2 Chapter 2, Problem 7

2.1 a. Arithmetic

Do these two expressions result in the same floating point number (down to the last bit) as long as no NaN or inf values or denormalized numbers are produced?

$(x * y) + (z - w)$
$(z - w) + (y * x)$

2.1.1 Answer

From Bindle and Goodman pg.10, we know that floating point addition and multiplication following the IEEE standard for arithmetic operation is commutative. Therefore, we can can state

$$(x * y) + (z - w) = (x * y) + (z - w) \tag{10}$$

So long as floating point calculation produces no underflow or overflow exception, both of these expressions will return the same floating point number down to the last bit.

2.2 b. Arithmetic

Do these two expressions result in the same floating point number (down to the last bit) as long as no NaN or inf values or denormalized numbers are produced?

$(x + y) + z$
$x + (y + z)$

2.2.1 Answer

This operation is susceptible to rounding error. Modeling the error of the first expression yields:

$$\begin{aligned} fl((x+y)+z) &= ((x+y)(1+\epsilon_1)+z)(1+\epsilon_2) \\ &= (x + x\epsilon_1 + y + y\epsilon_1 + z)(1+\epsilon_2) \\ &= (x+y+z+x\epsilon_1+y\epsilon_1)(1+\epsilon_2) \\ &= x+y+z+x\epsilon_1+y\epsilon_1+x\epsilon_2+y\epsilon_2+z\epsilon_2+x\epsilon_1\epsilon_2+y\epsilon_1\epsilon_2 \\ &= (x+y+z)+(x+y)\epsilon_1+(x+y+z)\epsilon_2+(x+y)\epsilon_1\epsilon_2 \end{aligned} \tag{11}$$

As noted on Bindle and Goodman pg. 11, we can drop the mixed ϵ term as it will be smaller than either ϵ_1 or ϵ_2 by a factor of ϵ_{mach}. This is true because $|\epsilon| \leq \epsilon_{mach}$ (B&G pg.10). Therefore we can state

$$fl((x+y)+z) \approx (x+y+z) + (x+y)\epsilon_1 + (x+y+z)\epsilon_2 \tag{12}$$

Performing the same error estimation on the second expression yields:

$$\begin{aligned} fl(x+(y+z)) &= (y+z)+x \text{ since, as previously stated, floating point addition is commutative} \\ &= ((y+z)(1+\epsilon_{1'})+x)(1+\epsilon_{2'}) \\ &= (y+z+y\epsilon_{1'}+z\epsilon_{1'}+x)(1+\epsilon_{2'}) \\ &= (x+y+z)+(y+z)\epsilon_{1'}+(x+y+z)\epsilon_{2'}+(y+z)\epsilon_{1'}\epsilon_{2'} \end{aligned} \tag{13}$$

We again drop the mixed term for the reason stated above and write:

$$fl(x+(y+z)) \approx (x+y+z) + (y+z)\epsilon_{1'} + (x+y+z)\epsilon_{2'} \tag{14}$$

Comparing equation (12) and (14) shows that, in fact, $fl(x+(y+z))$ does not necessarily yield the same bitwise result as $fl((x+y)+z)$. Therefore, floating point addition is not associative.

2.3 c. Arithmetic

Do these two expressions result in the same floating point number (down to the last bit) as long as no NaN or inf values or denormalized numbers are produced?

$x * \text{oneHalf} + y * \text{oneHalf}$
$(x+y) * \text{oneHalf}$

2.3.1 Answer

From B&G pg. 11, we know that division by powers of 2 is exact so long as the result is a normalized number. Therefore, the only error introduced in these expressions is due to rounding in the floating point addition step. Starting with the first expression,

$$\begin{aligned} &fl(x*\text{oneHalf}+y*\text{oneHalf}) = (x*\text{oneHalf}+y*\text{oneHalf})(1+\epsilon_1) \\ &= (x*\text{oneHalf}+y*\text{oneHalf}) + (x*\text{oneHalf}+y*\text{oneHalf})\epsilon_1 \end{aligned} \tag{15}$$

Because division by 2 is exact, we can rewrite the above expression as

$$fl(x * \text{oneHalf} + y * \text{oneHalf}) = ((x + y) + (x + y)(\epsilon_1)) * \text{oneHalf} \tag{16}$$

Doing the same for the second expression yields

$$\begin{aligned} fl((x + y) * \text{oneHalf}) &= (x + y)(1 + \epsilon_{1'}) * \text{oneHalf} \\ fl((x + y) * \text{oneHalf}) &= ((x + y) + (x + y)(\epsilon_{1'})) * \text{oneHalf} \end{aligned} \tag{17}$$

Comparing the results in equations (17) and (16) shows that both expressions yield the same bitwise floating point number.

2.4 d. Arithmetic

Do these two expressions result in the same floating point number (down to the last bit) as long as no NaN or inf values or denormalized numbers are produced?

$x * \text{oneThird} + y * \text{oneThird}$
$(x + y) * \text{oneThird}$

2.4.1 Answer

As noted in part c, division by three must not be exact as 3 is not a power of 2. Because we know from part b that floating point addition is not associative and division by 3 introduces its own error we can immediately come to the conclusion that these two expressions will not produce the same bitwise floating point number.

3 Chapter 2, Problem 6

3.1 Proof

Show that the absolute error in the Sum S computed by the code in the text is no worse than $(n-1)\epsilon_{mach} \sum_{k=0}^{n-1} |x_i|$ if the floating point sum $x+y$ has relative error $\epsilon < \epsilon_{mach}$.

3.1.1 Answer

Performing the error analysis and expanding the summation function out yields:

$$\begin{aligned} fl(\sum_{i=0}^{n-1} x[i]) &= (((x[0] + x[1])(1 + \epsilon_0) + x[2])(1 + \epsilon_1) + x[3])(1 + \epsilon_3) + ... \\ &\approx ((x[0] + x[1]) + (x[0] + x[1])\epsilon_0 + (x[0] + x[1] + x[2]) + (x[0] + x[1] + x[2])\epsilon_1 + ... \end{aligned} \tag{18}$$

Noting that in this case the array containing the summation terms is indexed at zero. Above we also neglect error cross terms in the expansion, as those errors will be much smaller that the individual addition operation errors (as noted in previous problems).

We can express the absolute error of this operation as:

$$(\sum_{i=0}^{n-1} \epsilon_i)(\sum_{i=0}^{n-1} |x_i|) \tag{19}$$

There are a total of $n-1$ error values in the operation, as each term in the summation contribute one ϵ value except for $x[0]$.

The largest error contribution in the summation will be provided by the largest ϵ value due to a summation operation in the above expansion. Therefore, if we let $\epsilon_{largest}$ represent the largest ϵ_i value we can write

$$\sum_{i=0}^{n-1} \epsilon_i \leq (n-1)\epsilon_{largest} \tag{20}$$

From B&G pg 10, we know that any individual error value is upper bounded by the machine precision. i.e. $|\epsilon| \leq \epsilon_{mach}$. Therefore, we can state

$$(n-1)\epsilon_{largest} \leq (n-1)\epsilon_{mach} \tag{21}$$

Which implies

$$(\sum_{i=0}^{n-1} \epsilon_i)(\sum_{i=0}^{n-1} |x_i|) \leq (n-1)\epsilon_{mach} \sum_{i=0}^{n-1} |x_i| \tag{22}$$

4 Chapter 2, Problem 9

4.1 a. Code

Using the recurrence relation $a_{n,k+1} = \frac{n-k}{k+1} a_{n,k}$, compute $a_{n,k}$ for a range of n values.

4.1.1 Answer

Code found in the appendix. Computed values for $n = [1, 10]$ below. Computations run from $a_{n,0}$ to $a_{n,n+1}$ for each value n.

Single Precision Computed Values

n=1	n=2	n=3	n=4	n=5	n=6	n=7	n=8	n=9	n=10
1	1	1	1	1	1	1	1	1	1
0	0.5	1	1.5	2	2.5	3	3.5	4	4.5
	0	0.333333343267441	1	2	3.33333325386047	5	7	9.33333301544189	12
		0	0.25	1	2.5	5	8.75	14	21
			0	0.200000002980232	1	3	7	14	25.2000007629395
				0	0.16666667163372	1	3.5	9.33333301544189	21
					0	0.142857149243355	1	3.99999976158142	12
						0	0.125	0.999999940395355	4.5
							0	0.111111104488373	1
								0	0.100000001490116
									0

Double Precision Computed Values

n=1	n=2	n=3	n=4	n=5	n=6	n=7	n=8	n=9	n=10
1	1	1	1	1	1	1	1	1	1
0	0.5	1	1.5	2	2.5	3	3.5	4	4.5
	0	0.333333333333333	1	2	3.33333333333333	5	7	9.33333333333333	12
		0	0.25	1	2.5	5	8.75	14	21
			0	0.2	1	3	7	14	25.2
				0	0.166666666666667	1	3.5	9.33333333333333	21
					0	0.142857142857143	1	4	12
						0	0.125	1	4.5
							0	0.111111111111111	1
								0	0.1
									0

4.2 a. Reasoning

Why is roundoff not a problem here? Comparing values where $\hat{a_{n,n}} \approx 1$ in double precision but not in single precision explain how this is possible given roundoff not being a problem.

4.2.1 Answer

The value of n for which the calculated $\hat{a}_{n,n} \approx 1$ in double precision but not single precision in this range is

$n = 9$

Single Precision: .999999940395355
Double Precision: 1

4.3 b. Code

Use the algorithm from part a to compute:

$E(k) = \frac{1}{2^n} \sum_{k=0}^{n} k a_{n,k} = \frac{n}{2}$

Do not safeguard against overflow or zero divide. Show in both single and double precision that the computed answer has high accuracy as long as the intermediate results are within the range of floating point numbers.

4.3.1 Answer

Code is in the appendix. Calculated values for $n = [1, 10]$ in both single and double precision:

Single Precision Computed Values

n=1	n=2	n=3	n=4	n=5	n=6	n=7	n=8	n=9	n=10
0	0.125	0.208333343267441	0.265625	0.306250005960464	0.3359375	0.358258932828903	0.37548828125	0.389105886220932	0.400097638368607

Double Precision Computed Values

n=1	n=2	n=3	n=4	n=5	n=6	n=7	n=8	n=9	n=10
0	0.125	0.208333333333333	0.265625	0.30625	0.3359375	0.358258928571429	0.37548828125	0.389105902777778	0.40009765625

4.4 b. Reasoning

Explain how the computer gets an accurate, small answer when the intermediate numbers have such a wide range of values. Why is cancellation not a problem?

4.4.1 Answer

5 Code Appendix

5.1 Problem 1

```
%***************************
% CMSC660 HW1 Problem 1
% Joe Asercion
%***************************

%Set used variables
x=1;
j=0;
kmin = 1;
kmax = 60;
len = kmax-kmin;
f1 = zeros(len,1);
f2 = zeros(len,1);
e = zeros(len,1);

%Calculate approximation
f1(1) = double(sin(x)-sin(x+pi/3));
for k = kmin:kmax
%Second term generated using trig identity
%sin(a+b)=sin(a)cos(b)+cos(a)sin(b) in order to use the recursive f1(k)
%term
```

```
f1(k+1) = double(f1(k)-((sqrt(3)/2)*cos(x-k*pi/3)+(1/2)*f1(k)));
%f1(k+1) = double(f1(k)-sin(x+(1-k)*pi/3));
end

%Calculate exact
for k = kmin:kmax
f2(k)= sin(x-k*pi/3);
end

%Calculate absolute error
for i = kmin:kmax
e(i) = f1(i)-f2(i);
end

%Plot
scatter(kmin:kmax,e);
set(gca,'yscale','log');
xlabel('K'); % x-axis label
ylabel('e') % y-axis label
```

5.2 Problem 4.a

```
%****************************
% CMSC660 HW1 Problem 4 Part a
% Joe Asercion
%****************************


%**********************
%Single Precision Case
%**********************

% Range of 'n' to calculate over

n=1:10;
len=range(n);

% Array to hold output values from loops
singleArr=zeros(len,len);


for j=(1:10) % Outer loop iterates over n values
for k=(0:j) % Inner loop iterates over k from 0 to the current n value
if k == 0
singleArr(k+1,j)=single(1); % k=0 case
else
```

```
singleArr(k+1,j)=single(singleArr(k,j)*(j-k)/(k+1)); % Recurrance relatio
end
end
end

SingleT = array2table(singleArr);
writetable(SingleT,'SingleTable.txt');
%**********************
%Double Precision Case
%**********************

% Range of 'n' to calculate over

n=1:10;
len=range(n);

% Array to hold output values from loops
doubleArr=zeros(len,len);


for j=(1:10) % Outer loop iterates over n values
for k=(0:j) % Inner loop iterates over k from 0 to the current n value
if k == 0
doubleArr(k+1,j)=double(1); % k=0 case
else
doubleArr(k+1,j)=double(doubleArr(k,j)*(j-k)/(k+1)); % Recurrance relatio
end
end
end

DoubleT = array2table(doubleArr);
writetable(DoubleT,'DoubleTable.txt');
```

5.3 Problem 4.b

```
%****************************
% CMSC660 HW1 Problem 4 Part b
% Joe Asercion
%****************************


%**********************
%Single Precision Case
%**********************

% Range of 'n' to calculate over
```

```
n=1:10;
len=range(n);

% Array to hold output values from loops
kArr1=zeros(len,len);
singleArr=zeros(len,1);
sum1=0;


for j=(1:10) % Outer loop iterates over n values
for k=(0:j) % Inner loop iterates over k from 0 to the current n value
if k == 0
kArr1(k+1,j)=single(1); % k=0 case
else
kArr1(k+1,j)=single(kArr1(k,j)*(j-k)/(k+1)); % Recurrance relation calcu
end
end
for k=(0:j)
sum1=single(sum1+k*kArr1(k+1,j));
end
singleArr(j)=single((1/(2^j))*sum1);
sum1 = 0;
end

SingleT = array2table(singleArr);
writetable(SingleT,'SingleTable.txt');
%**********************
%Double Precision Case
%**********************

% Range of 'n' to calculate over

n=1:10;
len=range(n);

% Array to hold output values from loops
doubleArr=zeros(len,1);
kArr2=zeros(len,len);
sum2=0;

for j=(1:10) % Outer loop iterates over n values
for k=(0:j) % Inner loop iterates over k from 0 to the current n value
if k == 0
kArr2(k+1,j)=double(1); % k=0 case
else
```

```
kArr2(k+1,j)=double(kArr2(k,j)*(j-k)/(k+1)); % Recurrance relation calcu
end
end
for k=(0:j)
sum2=double(sum2+k*kArr2(k+1,j));
end
doubleArr(j)=double((1/(2^j))*sum2);
sum2 = 0;
end

DoubleT = array2table(doubleArr);
writetable(DoubleT,'DoubleTable.txt');
```