CMSC 660 HW III

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1 Chapter 4, Problem 11

Given the following information:

- $\frac{1}{2}\delta_x^2 u = f(x)$ for $0 < x < 1$
- Boundary Conditions: $u(0) = u(1) = 0$
- Discretized interval: $[0, 1]$ in a uniform grid of points $x_j = j\Delta x$ with $n\Delta x = 1$
- $n - 1$ unknowns U_j are approximations to $u(x_j)$, $j = 1, ..., n - 1$
- Second order approximation: $\frac{1}{2}\frac{1}{\Delta x^2}(U_{j+1} - 2U_j + U_{j-1}) = f(x_j) = F_j$
- These linear equations can be written as $AU = F$

1.1 a. Calculation

Check that there are $n - 1$ distinct eigenvectors of A having the form $r_{kj} = \sin(k\pi x_j)$ where r_{kj} is the j component of the eigenvector r_k.

1.1.1 Answer

We are given the expression $AU = F$ and are told the form of both F and U. Therefore we can calculate the form the matrix A must be in:

$$U = \begin{bmatrix} 0 \\ U_1 \\ U_2 \\ U_3 \\ ... \\ U_{n-1} \\ 0 \end{bmatrix}$$

$$F_j = \frac{1}{2}\frac{1}{\Delta x^2}(U_{j+1} - 2U_j + U_{j-1}) = f(x_j)$$

$$\therefore A = \frac{1}{2}\frac{1}{\Delta x^2} \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & ... & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & ... \\ 0 & 1 & -2 & 1 & 0 & 0 & ... \\ 0 & 0 & 1 & -2 & 1 & 0 & ... \\ ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & 0 & 1 & -2 & 1 \\ 0 & ... & ... & ... & 0 & 1 & -2 \end{bmatrix}$$

Where A has dimensions of $(n-1) \times (n-1)$.

Let R_k be the kth vector consisting of elements $r_{kj} = sin(k\pi x_j)$. If R_k is an eigenvector of matrix A then it satisfies

$$AR_k = \lambda_k R_k$$

Where λ_k is the eigenvalue associated with the k^{th} eigenvector. Starting with the $k = 1$ case we can plug in values and expand this expression:

$$\frac{1}{2}\frac{1}{\Delta x^2} \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & ... & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & ... \\ 0 & 1 & -2 & 1 & 0 & 0 & ... \\ 0 & 0 & 1 & -2 & 1 & 0 & ... \\ ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & 0 & 1 & -2 & 1 \\ 0 & ... & ... & ... & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} \sin(\pi x_1) \\ \sin(\pi x_{1+1}) \\ \sin(\pi x_{1+2}) \\ \sin(\pi x_{1+3}) \\ ... \\ \sin(\pi x_{n-1}) \end{bmatrix} = \lambda_1 \begin{bmatrix} \sin(\pi x_1) \\ \sin(\pi x_{1+1}) \\ \sin(\pi x_{1+2}) \\ \sin(\pi x_{1+3}) \\ ... \\ \sin(\pi x_{n-1}) \end{bmatrix}$$

Noting that $r_{k,j+1} = \sin(k\pi(x + \Delta x))$, and that $n\Delta x = 1$, we can rewrite the above expression as

$$
\frac{1}{2}\frac{1}{\Delta x^2}\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & ... & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & ... \\ 0 & 1 & -2 & 1 & 0 & 0 & ... \\ 0 & 0 & 1 & -2 & 1 & 0 & ... \\ ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & 0 & 1 & -2 & 1 \\ 0 & ... & ... & ... & 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} \sin(\pi x_1) \\ \sin(\pi(x_1+\Delta x)) \\ \sin(\pi(x_1+2\Delta x)) \\ \sin(\pi(x_1+3\Delta x)) \\ ... \\ \sin(\pi(x_1+(n-3)\Delta x)) \\ \sin(\pi(x_1+(n-2)\Delta x)) \end{bmatrix} = \lambda_1 \begin{bmatrix} \sin(\pi x_1) \\ \sin(\pi(x_1+\Delta x)) \\ \sin(\pi(x_1+2\Delta x)) \\ \sin(\pi(x_1+3\Delta x)) \\ ... \\ \sin(\pi(x_1+(n-3)\Delta x)) \\ \sin(\pi(x_1+(n-2)\Delta x)) \end{bmatrix}
$$

Which gives equations of the form

$$
\begin{aligned}
&\frac{1}{2}\frac{1}{\Delta x^2}(-2\sin(\pi x_1)+\sin(\pi(x_1+\Delta x))) = \lambda_1 \sin(\pi x_1) \\
&\frac{1}{2}\frac{1}{\Delta x^2}(\sin(\pi x_1)-2\sin(\pi(x_1+\Delta x))+\sin(\pi(x_1+2\Delta x))) = \lambda_1 \sin(\pi x_1+\Delta x) \\
&\frac{1}{2}\frac{1}{\Delta x^2}(\sin(\pi x_1+\Delta x)-2\sin(\pi x_1+2\Delta x)+\sin(\pi(x_1+3\Delta x))) = \lambda_1 \sin(\pi(x_1+2\Delta x)) \\
&... \\
&\frac{1}{2}\frac{1}{\Delta x^2}(\sin(\pi(x_1+(n-4)\Delta x))-2\sin(\pi(x_1+(n-3)\Delta x))+\sin(\pi(x_1+(n-2)\Delta x))) \\
&= \lambda_1 \sin(\pi(x_1+(n-3)\Delta x)) \\
&\frac{1}{2}\frac{1}{\Delta x^2}(\sin(\pi(x_1+(n-3)\Delta x))-2\sin(\pi(x_1+(n-2)\Delta x))) \\
&= \lambda_1 \sin(\pi(x_1+(n-2)\Delta x))
\end{aligned}
$$

Using the given fact that $n\Delta x = 1$, the generated equations can be simplified:

$$
\begin{aligned}
&\frac{1}{2}\frac{1}{\Delta x^2}(-2\sin(\pi x_1)+\sin(\pi(x_1+\Delta x))) = \lambda_1 \sin(\pi x_1) \\
&\frac{1}{2}\frac{1}{\Delta x^2}(\sin(\pi x_1)-2\sin(\pi(x_1+\Delta x))+\sin(\pi(x_1+2\Delta x))) = \lambda_1 \sin(\pi x_1+\Delta x) \\
&\frac{1}{2}\frac{1}{\Delta x^2}(\sin(\pi x_1+\Delta x)-2\sin(\pi x_1+2\Delta x)+\sin(\pi(x_1+3\Delta x))) = \lambda_1 \sin(\pi(x_1+2\Delta x)) \\
&... \\
&\frac{1}{2}\frac{1}{\Delta x^2}(\sin(\pi(x_1+1-4\Delta x))-2\sin(\pi(x_1+1-3\Delta x))+\sin(\pi(x_1+1-2\Delta x))) \\
&= \lambda_1 \sin(\pi(x_1+1-3\Delta x)) \\
&\frac{1}{2}\frac{1}{\Delta x^2}(\sin(\pi(x_1+1-3\Delta x))-2\sin(\pi(x_1+1-2\Delta x))) \\
&= \lambda_1 \sin(\pi(x_1+1-2\Delta x))
\end{aligned}
$$

Applying $x_j = j\Delta x$...

$$
\begin{aligned}
&\frac{1}{2}\frac{1}{\Delta x^2}(-2\sin(\pi\Delta x) + \sin(2\pi\Delta x)) = \lambda_1 \sin(\pi\Delta x) \\
&\frac{1}{2}\frac{1}{\Delta x^2}(\sin(\pi\Delta x) - 2\sin(2\pi\Delta x) + \sin(3\pi\Delta x)) = \lambda_1 \sin(2\pi\Delta x) \\
&\frac{1}{2}\frac{1}{\Delta x^2}(\sin(2\pi\Delta x) - 2\sin(3\pi\Delta x) + \sin(4\pi\Delta x)) = \lambda_1 \sin(3\pi\Delta x) \\
&... \\
&\frac{1}{2}\frac{1}{\Delta x^2}(\sin(\pi(1-3\Delta x)) - 2\sin(\pi(1-2\Delta x)) + \sin(\pi(1-\Delta x))) = \lambda_1 \sin(\pi(1-2\Delta x)) \\
&\frac{1}{2}\frac{1}{\Delta x^2}(\sin(\pi(1-2\Delta x)) - 2\sin(\pi(1-\Delta x))) = \lambda_1 \sin(\pi(1-\Delta x))
\end{aligned}
$$

Applying a bit of algebra and the identities of trigonometric functions allows us to isolate the λ_1 eigenvalue:

$$\frac{\cos(\pi\Delta x) - 1}{\Delta x^2} = \lambda_1$$

Generalizing this for all k yields:

$$\frac{\cos(k\pi\Delta x) - 1}{\Delta x^2} = \lambda_k \tag{1}$$

Since (1) is a scalar value solely dependent on the step size Δx and the index k, it shows that the vector R_k is in fact an eigenvector of matrix A. As noted in [1](pg. 77), the $(n-1)\times(n-1)$ matrix A may have up to $n-1$ eigenvectors. We see that (1) implies that iterating the value of k generates a distinct eigenvalue, therefore the set of eigenvectors associated with the eigenvalues defined by (1) are linearly independent and form a basis for $\mathbb{C}^{n-1}$ [1](pg. 77). Therefore, there must be $n-1$ eigenvectors of A in the form of R_k.

1.2 b. Calculation

Use te eigenvalue information from part (a) to show that $||A^{-1}|| \to \frac{2}{\pi^2}$ as $n \to \infty$ and $\kappa(A) = \mathcal{O}(n^2)$ as $n \to \infty$. (Use Euclidian Norms).

1.2.1 Answer

From 4.2.7 Theorem 1 [1](pg. 82), if a matrix is self-adjoint the Rayleigh Quotient of an eigenvector is its corresponding eigenvalue. Plugging in the values found in part a we obtain:

$$Q(x) = \lambda_k = \frac{R_k^* A R_k}{R_k^* R_k}$$
$$\frac{\cos(k\pi\Delta x) - 1}{\Delta x^2} = \frac{(\sin(k\pi x_j))^* A \sin(k\pi x_j)}{(\sin(k\pi x_j))^* \sin(k\pi x_j)}$$

From the definition of the Euclidian Norm of a matrix [1](pg. 76), we see that the maximum of Rayleigh Quotient is the Euclidean Norm of the matrix A squared. i.e.

$$||A||_{l^2} = max\Big(\frac{\cos(k\pi\Delta x) - 1}{\Delta x^2}\Big) = max\Big(n^2(\cos(k\pi\frac{1}{n}) - 1)\Big) \tag{2}$$

Taking the limit of (2) as $n \to \infty$ yields

$$\lim_{n\to\infty} n^2(\cos(k\pi\frac{1}{n}) - 1) = -\frac{k^2\pi^2}{2}$$

To find the norm of $||A^{-1}||$ we need to invert this expression:

$$||A^{-1}||_{l^2} = \frac{2}{\pi^2} \tag{3}$$

Informally, $\kappa(A)$ is on the order of $\mathcal{O}(n^2)$ due to the condition number of an eigenvalue problems's dependence on the the norm of ΔA[1](pg. 92). In (2) the dominant term as $n \to \infty$ will the the leading n^2 term. Therefore the overall condition number of the Eigenvalue problem will be on the order of n^2.

1.3 c. Calculation

Given:

- $\tilde{U_j} = u(x_j)$, where $u(x)$ is the exact but unknown solution to the BVP
- $R = A\tilde{U} - F$ is defined to be the residual

Show that if $u(x)$ is smooth then the residual satisfies $||R|| = \mathcal{O}(\Delta x^2) = \mathcal{O}(\frac{1}{n^2})$

1.3.1 Answer

From the statement of the problem we know $F = AU$. Therefore, we can rewrite the residual as

$$R = A\tilde{U} - AU$$

Therefore,

$||R|| = ||A\tilde{U} - AU|| \leq ||A\tilde{U}|| - ||AU|| \leq ||A||||\tilde{U}|| - ||A||||U||$[1] (pg. 74)

From part B we know the norm of $||A||$, therefore

$$\begin{aligned}
||R|| &= \frac{\cos(k\pi\Delta x) - 1}{\Delta x^2}||\tilde{U}|| - \frac{\cos(k\pi\Delta x) - 1}{\Delta x^2}||U|| \\
&= \frac{\cos(k\pi\Delta x) - 1}{\Delta x^2}(||\tilde{U}|| - ||U||) \\
&= \frac{\cos(k\pi\Delta x) - 1}{\Delta x^2}(||\Delta x \sum_{k=1}^{n} \tilde{U_j^2}|| - ||\Delta x \sum_{k=1}^{n} U_j^2||) \\
||R|| &= \frac{\cos(k\pi\Delta x) - 1}{\Delta x}(|| \sum_{k=1}^{n} \tilde{U_j^2}|| - || \sum_{k=1}^{n} U_j^2||)
\end{aligned}$$

1.4 d. Calculation

Show that $A(U - \tilde{U}) = R$. Use part (b) to show that $||U - \tilde{U}|| = \mathcal{O}(\Delta x^2)$

1.4.1 Answer

1.5 f. MATLAB

Write a program in MATLAB to solve $AU = F$ for the second order method.

1.5.1 Answer

```
%****************************
% CMSC660 HW3 Problem 1f
% Joe Asercion
%****************************

% Set iteration step size
delta_x=.001;

% Total number of steps
n=(1/delta_x);

% Vector to hold F(x) function values
F=ones(n-1,1);

% Constant for Eigenvector F(x) case
k=1;

% Populate F(x) vector
for j = 1:(n-1)
% Define function to populate F(x) with here:
F(j)=sin(k*pi*(j*delta_x));
end

% 'A' Transformation Matrix
A = (1/2)*(1/(delta_x)^2)*gallery('tridiag',n-1,1,-2,1);

% Solution for AU=F using MATLAB backslash operator
U = A\F;

% Vector consisting of steps for plot
X = 0+delta_x:delta_x:1-delta_x;

% Plot delta_x vs U(x)
scatter(X,U)
```

2 Chapter 4, Problem 12

2.1 a. Calculation

Show that if $\dot{S} = SA$, $S(0) = I$, then $p(t) = p(0)S(t)$

2.1.1 Answer

2.2 b. Calculation

Use matrix norms and the fact that $||B^k|| \leq ||B||^k$ to show that the matrix exponential $e^B = \sum_{k=0}^{\infty} \frac{1}{k!} B^K$ converges.

2.2.1 Answer

Using the given fact that $||B^k|| \leq ||B||^k$, we can state that, for each individual term in sum,

$$||\frac{B^k}{k!}|| \leq \frac{||B||^k}{k!} \tag{4}$$

From the definition of the matrix norm [1](pg. 76), it can be noted that $||B||$ is the sharp constant in the inequality above. Therefore, norm of each individual term of the matrix exponential sum is upper bounded by the norm of the matrix itself.

Because the norm of B, $||B||$, is a scalar function we can note that $\sum_{k=0}^{\infty} \frac{||B||^k}{k!}$ is in the form of the Exponential Series, which is known to be convergent. Due to (4),

$$\sum_{k=0}^{\infty} ||\frac{B^k}{k!}|| \leq \sum_{k=0}^{\infty} \frac{||B||^k}{k!}$$

Therefore, by the Direct Comparison convergence test[3] we can state that $\sum_{k=0}^{\infty} ||\frac{B^k}{k!}||$ must converge as well. Because the norm of a matrix is, by definition, the largest amount it can stretch a vector[1](pg. 76),

$$\sum_{k=0}^{\infty} \frac{1}{k!} B^K \leq \sum_{k=0}^{\infty} ||\frac{B^k}{k!}$$

By the Direct Comparison test once again we can state that $\sum_{k=0}^{\infty} \frac{1}{k!} B^k$ must converge.

2.3 c. Calculation

Show that the fundamental solution is given by $S(t) = e^{tA}$.

2.3.1 Answer

First Condition: Is $\frac{d}{dt}S(t) = S(t)A$?

Expressing $S(t)$ as an infinite sum,

$$S(t) = \sum_{k=0}^{\infty} \frac{1}{k!}(tA)^k = 1 + tA + \frac{(tA)^2}{2} + \frac{(tA)^3}{6} + \frac{(tA)^4}{24} + ...$$

$$\therefore \frac{d}{dt}(S(t)) = \frac{d}{dt}(\sum_{k=0}^{\infty} \frac{(tA)^k}{k!}) = \sum_{k=1}^{\infty} \frac{k(t^{k-1})A^k}{k!} = A + tA^2 + \frac{t^2A^3}{2} + \frac{t^3A^4}{6} + \frac{t^4A^5}{24} + ...$$

$$\therefore \frac{d}{dt}S(t) = S(t)A$$

Second Condition: Is $S(0) = I$?

Again expressing $S(t)$ as an infinite sum,

$$S(0) = \sum_{k=0}^{\infty} \frac{(0A)^k}{k!} = \sum_{k=0}^{\infty} \frac{\mathbf{0}^k}{k!}$$

Where $\mathbf{0}$ is a zero matrix with the dimensions of A. $\mathbf{0}^n = I$ for $n = 0$ and $\mathbf{0}^n = \mathbf{0}$ for $n \neq 0$. Therefore $S(0) = I$.

2.4 d. Calculation

Show that $e^{tA} = Re^{t\Lambda}L$ and that $e^{t\Lambda}$ is the obvious diagonal matrix

2.4.1 Answer

Let $A = R\Lambda L$. Then

$$\begin{aligned} e^{tA} &= \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(tR\Lambda L)^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k(R\Lambda L)^k}{k!} \end{aligned}$$

Examining the $(R\Lambda L)^k$ term,

$$(R\lambda L)^k = (R\lambda L)(R\lambda L)(R\lambda L)....$$

However, $L = R^{-1}$. Therefore, the only two R and L matrices that persist at the end of the product will be the R at the head of the term and the L at the tail. The sum can thusly be rewritten as

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k R \Lambda^k L}{k!}$$

Since R and L are no longer subject to the index k (as they appear once in each term of the sum), they can be pulled out of the summation and the expression can be simplified:

$$\begin{aligned} e^{tA} &= \sum_{k=0}^{\infty} \frac{t^k R \Lambda^k L}{k!} \\ &= R(\sum_{k=0}^{\infty} \frac{t^k \Lambda^k}{k!})L \\ &= Re^{t\Lambda}L \end{aligned}$$

$e^{t\Lambda}$ is the obvious diagonal matrix for e^{tA} as it simply takes the matrix exponential of the scaled Λ. The original Λ matrix is only being modified along the diagonal, since nothing is being added to it, and the scaled eigenvalues in the $e^{t\Lambda}$ matrix still correspond with the R and L eigenvectors, as shown above.

2.5 e. MATLAB

Calculate the eigenvalues and right eigenvector matrix of A using the Matlab eig(A) function.

2.5.1 Answer

```
%****************************
% CMSC660 HW3 Problem 2e
% Joe Asercion
%****************************

% Set Markov chain transition matrix constants
lambda = 1;
mu = 4;

% Time constant
t = 1;

% Iterations
n = 4;
k = 20;

% Preallocate matrices to store the eigenvalues and eigenvectors
% by eig[A]
R = zeros(n);
Lam = zeros(n);

% Build the MCT matrix.  Call a test tridiag matrix of size n-1
% elements [1, 1] and [n, n] to their correct values.
A = gallery('tridiag',n,mu,-(lambda+mu),lambda);
A(1, 1) = -lambda;
A(n, n) = -mu;

% Convert sparse matrix A to a full matrix so that eig can be ca
A = full(A);

% Call eig to calculate eigenvalues/right eigenvectors
[R,Lam] = eig(A);

for k = 1:n
l = Lam(:,k);
r = R(:,k);
n = l.*r-A*r;
k = num2str(k);
disp([newline,'Ar_{k}, k=',k newline]);
```

```
disp(A*r);
disp([newline,'lambda_{k}r_{k}, k=',k newline]);
disp(l.*r);
disp([newline,'Norm lambda_{k}r_{k}-Ar_{k}, k=',k newline]);
disp(norm(n));
end
```

2.6 f. MATLAB

2.6.1 Answer

```
%*****************************
% CMSC660 HW3 Problem 2f
% Joe Asercion
%*****************************

% Set Markov chain transition matrix constants
lambda = 1;
mu = 4;

% Time constant
t = 1;

% Iterations
n = 4;

% Preallocate matrices to store the eigenvalues and eigenvectors
% by eig[A]
R = zeros(n);
Lam = zeros(n);

% Build the MCT matrix.  Call a test tridiag matrix of size n an
% elements [1, 1] and [n, n] to their correct values.
A = gallery('tridiag',n,mu,-(lambda+mu),lambda);
A(1, 1) = -lambda;
A(n, n) = -mu;

% Convert sparse matrix A to a full matrix so that eig can be ca
A = full(A);

% Call eig to calculate eigenvalues/right eigenvectors
[R,Lam] = eig(A);

L = R^(-1);

for k = 1:n
lam = Lam(:,k);
l = L(:,k);
n = l.*lam-A*l;
k = num2str(k);
disp([newline,'Al_{k}, k=',k newline]);
disp(A*l);
disp([newline,'l_{k}lambda_{k}, k=',k newline]);
```

```
disp(l.*lam);
disp([newline,'Norm lambda_{k}-Al_{l}, k=',k newline]);
disp(norm(n));
end
```

2.7 g. MATLAB

2.7.1 Answer

2.8 h. MATLAB

2.8.1 Answer

3 Chapter 4, Problem 13

3.1 a. MATLAB

3.1.1 Answer

```
%****************************
% CMSC660 HW3 Problem 3a
% Joe Asercion
%****************************

% Constants
n = 20;
s = .1;
lambda = 1;
mu = 4;

% Create matrix B
B = zeros(n);
B(1, 1) = -1;
B(1, n) = 1;

% Create MCT matrix
lambda = 1;
A = gallery('tridiag',n,mu,-(lambda+mu),lambda);
A(1, 1) = -lambda;
A(n, n) = -mu;

A(1, 1) = -lambda;
A(n, n) = -mu;
A = full(A);

% Compute the eigenvalues of the A(s) function
Y = A + s*B;
Lam = eig(Y);
```

3.2 b. MATLAB

3.2.1 Answer

References

[1] David Bindel and Johnathan Goodman. *Principles of Scientific Computing*. 2009.

[2] *Wikipedia Page for the Frobenius Norm*

`https://en.wikipedia.org/wiki/Matrix_norm#Frobenius_norm`

[3] *Wikipedia Page for Convergence Tests*

`https://en.wikipedia.org/wiki/Convergence_tests`