CMSC 660 HW III

Joe Asercion

9/19/18

1 Chapter 4, Problem 11

Given the following information:

- $\frac{1}{2}\delta_x^2 u = f(x)$ for $0 < x < 1$
- Boundary Conditions: $u(0) = u(1) = 0$
- Discretized interval: $[0,1]$ in a uniform grid of points $x_j = j\Delta x$ with $n\Delta x = 1$
- $n-1$ unknowns U_j are approximations to $u(x_j),\ j = 1, ..., n-1$
- Second order approximation: $\frac{1}{2}\frac{1}{\Delta x^2}(U_{j+1} - 2U_j + U_{j-1}) = f(x_j) = F_j$
- These linear equations can be written as $AU = F$

1.1 a. Calculation

Check that there are $n-1$ distinct eigenvectors of A having the form $r_{kj} = \sin(k\pi x_j)$ where r_{kj} is the j component of the eigenvector r_k.

1.1.1 Answer

We are given the expression $AU = F$ and are told the form of both F and U. Therefore we can calculate the form the matrix A must be in:

$$U = \begin{bmatrix} 0 \\ U_1 \\ U_2 \\ U_3 \\ ... \\ U_{n-1} \\ 0 \end{bmatrix}$$

$$F_j = \frac{1}{2}\frac{1}{\Delta x^2}(U_{j+1} - 2U_j + U_{j-1}) = f(x_j)$$

$$\therefore A = \frac{1}{2}\frac{1}{\Delta x^2}\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & ... & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & ... \\ 0 & 1 & -2 & 1 & 0 & 0 & ... \\ 0 & 0 & 1 & -2 & 1 & 0 & ... \\ ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & 0 & 1 & -2 & 1 \\ 0 & ... & ... & ... & 0 & 1 & -2 \end{bmatrix}$$

Where A has dimensions of $(n-1) \times (n-1)$.

Let R_k be the kth vector consisting of elements $r_{kj} = sin(k\pi x_j)$. If R_k is an eigenvector of matrix A then it satisfies

$$AR_k = \lambda_k R_k$$

Where λ_k is the eigenvalue associated with the k^{th} eigenvector. Starting with the $k = 1$ case we can plug in values and expand this expression:

$$\frac{1}{2}\frac{1}{\Delta x^2}\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & ... & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & ... \\ 0 & 1 & -2 & 1 & 0 & 0 & ... \\ 0 & 0 & 1 & -2 & 1 & 0 & ... \\ ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & 0 & 1 & -2 & 1 \\ 0 & ... & ... & ... & 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} \sin(\pi x_1) \\ \sin(\pi x_{1+1}) \\ \sin(\pi x_{1+2}) \\ \sin(\pi x_{1+3}) \\ ... \\ \sin(\pi x_{n-2}) \end{bmatrix} = \lambda_1 \begin{bmatrix} \sin(\pi x_1) \\ \sin(\pi x_{1+1}) \\ \sin(\pi x_{1+2}) \\ \sin(\pi x_{1+3}) \\ ... \\ \sin(\pi x_{n-1}) \end{bmatrix}$$

Noting that $r_{k,j+1} = \sin(k\pi(x + \Delta x))$, and that $n\Delta x = 1$, we can rewrite the above expression as

$$\frac{1}{2}\frac{1}{\Delta x^2}\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & ... & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & ... \\ 0 & 1 & -2 & 1 & 0 & 0 & ... \\ 0 & 0 & 1 & -2 & 1 & 0 & ... \\ ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & 0 & 1 & -2 & 1 \\ 0 & ... & ... & ... & 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} \sin(\pi x_1) \\ \sin(\pi(x_1+\Delta x)) \\ \sin(\pi(x_1+2\Delta x)) \\ \sin(\pi(x_1+3\Delta x)) \\ ... \\ \sin(\pi(x_1+(n-3)\Delta x)) \\ \sin(\pi(x_1+(n-2)\Delta x)) \end{bmatrix} = \lambda_1 \begin{bmatrix} \sin(\pi x_1) \\ \sin(\pi(x_1+\Delta x)) \\ \sin(\pi(x_1+2\Delta x)) \\ \sin(\pi(x_1+3\Delta x)) \\ ... \\ \sin(\pi(x_1+(n-3)\Delta x)) \\ \sin(\pi(x_1+(n-2)\Delta x)) \end{bmatrix}$$

Which gives equations of the form

$$\frac{1}{2}\frac{1}{\Delta x^2}(-2\sin(\pi x_1)+\sin(\pi(x_1+\Delta x))) = \lambda_1\sin(\pi x_1)$$
$$\frac{1}{2}\frac{1}{\Delta x^2}(\sin(\pi x_1)-2\sin(\pi(x_1+\Delta x))+\sin(\pi(x_1+2\Delta x))) = \lambda_1\sin(\pi x_1+\Delta x)$$
$$\frac{1}{2}\frac{1}{\Delta x^2}(\sin(\pi x_1+\Delta x)-2\sin(\pi x_1+2\Delta x)+\sin(\pi(x_1+3\Delta x))) = \lambda_1\sin(\pi(x_1+2\Delta x))$$
$$...$$
$$\frac{1}{2}\frac{1}{\Delta x^2}(\sin(\pi(x_1+(n-4)\Delta x))-2\sin(\pi(x_1+(n-3)\Delta x))+\sin(\pi(x_1+(n-2)\Delta x)))$$
$$= \lambda_1\sin(\pi(x_1+(n-3)\Delta x))$$
$$\frac{1}{2}\frac{1}{\Delta x^2}(\sin(\pi(x_1+(n-3)\Delta x))-2\sin(\pi(x_1+(n-2)\Delta x)))$$
$$= \lambda_1\sin(\pi(x_1+(n-2)\Delta x))$$

Using the given fact that $n\Delta x = 1$, the generated equations can be simplified:

$$\frac{1}{2}\frac{1}{\Delta x^2}(-2\sin(\pi x_1)+\sin(\pi(x_1+\Delta x))) = \lambda_1\sin(\pi x_1)$$
$$\frac{1}{2}\frac{1}{\Delta x^2}(\sin(\pi x_1)-2\sin(\pi(x_1+\Delta x))+\sin(\pi(x_1+2\Delta x))) = \lambda_1\sin(\pi x_1+\Delta x)$$
$$\frac{1}{2}\frac{1}{\Delta x^2}(\sin(\pi x_1+\Delta x)-2\sin(\pi x_1+2\Delta x)+\sin(\pi(x_1+3\Delta x))) = \lambda_1\sin(\pi(x_1+2\Delta x))$$
$$...$$
$$\frac{1}{2}\frac{1}{\Delta x^2}(\sin(\pi(x_1+1-4\Delta x))-2\sin(\pi(x_1+1-3\Delta x))+\sin(\pi(x_1+1-2\Delta x)))$$
$$= \lambda_1\sin(\pi(x_1+1-3\Delta x))$$
$$\frac{1}{2}\frac{1}{\Delta x^2}(\sin(\pi(x_1+1-3\Delta x))-2\sin(\pi(x_1+1-2\Delta x)))$$
$$= \lambda_1\sin(\pi(x_1+1-2\Delta x))$$

Applying $x_j = j\Delta x$...

$$\frac{1}{2}\frac{1}{\Delta x^2}(-2\sin(\pi\Delta x) + \sin(2\pi\Delta x)) = \lambda_1 \sin(\pi\Delta x)$$
$$\frac{1}{2}\frac{1}{\Delta x^2}(\sin(\pi\Delta x) - 2\sin(2\pi\Delta x) + \sin(3\pi\Delta x)) = \lambda_1 \sin(2\pi\Delta x)$$
$$\frac{1}{2}\frac{1}{\Delta x^2}(\sin(2\pi\Delta x) - 2\sin(3\pi\Delta x) + \sin(4\pi\Delta x)) = \lambda_1 \sin(3\pi\Delta x)$$
...
$$\frac{1}{2}\frac{1}{\Delta x^2}(\sin(\pi(1-3\Delta x)) - 2\sin(\pi(1-2\Delta x)) + \sin(\pi(1-\Delta x))) = \lambda_1 \sin(\pi(1-2\Delta x))$$
$$\frac{1}{2}\frac{1}{\Delta x^2}(\sin(\pi(1-2\Delta x)) - 2\sin(\pi(1-\Delta x))) = \lambda_1 \sin(\pi(1-\Delta x))$$

Applying a bit of algebra and the identities of trigonometric functions allows us to isolate the λ_1 eigenvalue:

$$\frac{\cos(\pi\Delta x) - 1}{\Delta x^2} = \lambda_1$$

Generalizing this for all k yields:

$$\frac{\cos(k\pi\Delta x) - 1}{\Delta x^2} = \lambda_k \tag{1}$$

Since (1) is a scalar value solely dependent on the step size Δx and the index k, it shows that the vector R_k is in fact an eigenvector of matrix A. As noted in [1](pg. 77), the $(n-1)\times(n-1)$ matrix A may have up to $n-1$ eigenvectors. We see that (1) implies that iterating the value of k generates a distinct eigenvalue, therefore the set of eigenvectors associated with the eigenvalues defined by (1) are linearly independent and form a basis for $\mathbb{C}^{n-1}$ [1](pg. 77). Therefore, there must be $n-1$ eigenvectors of A in the form of R_k.

1.2 b. Calculation

Use te eigenvalue information from part (a) to show that $||A^{-1}|| \to \frac{2}{\pi^2}$ as $n \to \infty$ and $\kappa(A) = \mathcal{O}(n^2)$ as $n \to \infty$. (Use Euclidian Norms).

1.2.1 Answer

From 4.2.7 Theorem 1 [1](pg. 82), if a matrix is self-adjoint the Rayleigh Quotient of an eigenvector is its corresponding eigenvalue. Plugging in the values found in part a we obtain:

$$Q(x) = \lambda_k = \frac{R_k^* A R_k}{R_k^* R_k}$$
$$\frac{\cos(k\pi\Delta x) - 1}{\Delta x^2} = \frac{(\sin(k\pi x_j))^* A \sin(k\pi x_j)}{(\sin(k\pi x_j))^* \sin(k\pi x_j)}$$

From the definition of the Euclidian Norm of a matrix [1](pg. 76), we see that the maximum of Rayleigh Quotient is the Euclidean Norm of the matrix A squared. i.e.

$$||A||_{l^2} = max\Big(\frac{\cos(k\pi\Delta x) - 1}{\Delta x^2}\Big) = max\Big(n^2(\cos(k\pi\frac{1}{n}) - 1)\Big) \tag{2}$$

Taking the limit of (2) as $n \to \infty$ yields

$$\lim_{n\to\infty} n^2(\cos(k\pi\frac{1}{n}) - 1) = -\frac{k^2\pi^2}{2}$$

To find the norm of $||A^{-1}||$ we need to invert this expression:

$$||A^{-1}||_{l^2} = \frac{2}{\pi^2} \tag{3}$$

Informally, $\kappa(A)$ is on the order of $\mathcal{O}(n^2)$ due to the condition number of an eigenvalue problems's dependence on the the norm of ΔA[1](pg. 92). In (2) the dominant term as $n \to \infty$ will the the leading n^2 term. Therefore the overall condition number of the Eigenvalue problem will be on the order of n^2.

1.3 c. Calculation

Given:

- $\tilde{U_j} = u(x_j)$, where $u(x)$ is the exact but unknown solution to the BVP
- $R = A\tilde{U} - F$ is defined to be the residual

Show that if $u(x)$ is smooth then the residual satisfies $||R|| = \mathcal{O}(\Delta x^2) = \mathcal{O}(\frac{1}{n^2})$

1.3.1 Answer

From the statement of the problem we know $F = AU$. Therefore, we can rewrite the residual as

$$R = A\tilde{U} - AU$$

Therefore,

$$||R|| = ||A\tilde{U} - AU|| \leq ||A\tilde{U}|| - ||AU|| \leq ||A||||\tilde{U}|| - ||A||||U||[1] \text{ (pg. 74)}$$

The Euclidean Norm of a matrix can be calculated as $||A||_2 = \sqrt{Trace(A^TA)}$[2]. A is a symmetric matrix, so $A = A^T$. The Trace of a matrix is simply the sum of the diagonal elements, so only those elements need to be calculated:

$$A^TA = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & ... & 0 \\ 1 & -2 & 1 & 0 & 0 & ... & 0 \\ 0 & 1 & -2 & 1 & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & ... & 0 \\ 1 & -2 & 1 & 0 & 0 & ... & 0 \\ 0 & 1 & -2 & 1 & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 5 & ... & ... & ... & ... & ... & ... \\ ... & 6 & ... & ... & ... & ... & ... \\ ... & ... & 6 & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & 5 \end{bmatrix}$$

Therefore, because A is a $(n-1) \times (n-1)$ square matrix,

$$Trace(A^TA) = ((n-3) * 6) + (2 * 5) = 6n - 8$$
$$\therefore ||A||_{l^2} = \sqrt{6n-8}x$$

1.4 d. Calculation

Show that $A(U - \tilde{U}) = R$. Use part (b) to show that $||U - \tilde{U}|| = \mathcal{O}(\Delta x^2)$

1.4.1 Answer

1.5 f. MATLAB

Write a program in MATLAB to solve $AU = F$ for the second order method.

1.5.1 Answer

Code attached in appendix.

2 Chapter 4, Problem 12

2.1 a. Calculation

Show that if $\dot{S} = SA$, $S(0) = I$, then $p(t) = p(0)S(t)$

2.1.1 Answer

2.2 b. Calculation

Use matrix norms and the fact that $||B^k|| \leq ||B||^k$ to show that the infinite sum of matrices converges

2.2.1 Answer

2.3 c. Calculation

Show that the fundamental solution is given by $S(t) = e^{tA}$.

2.3.1 Answer

2.4 d. Calculation

Show that $e^{tA} = Le^{t\Lambda}R$ and that $e^{t\Lambda}$ is the obvious diagonal matrix

2.4.1 Answer

2.5 e. MATLAB

2.5.1 Answer

2.6 f. MATLAB

2.6.1 Answer

2.7 g. MATLAB

2.7.1 Answer

2.8 h. MATLAB

2.8.1 Answer

3 Chapter 4, Problem 13

4 a. MATLAB

4.1 Answer

5 b. MATLAB

5.1 Answer

References

[1] David Bindel and Johnathan Goodman. *Principles of Scientific Computing.* 2009.

[2] *Wikipedia Page for the Frobenius Norm*

`https://en.wikipedia.org/wiki/Matrix_norm#Frobenius_norm`