CMSC 660 HW II

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1 Chapter 4, Problem 1

1.1 Calculation

Let L be the differentiation operator which takes P_3 to P_2 described in section 4.2.2. Let $f_k = H_k(x)$ for $k = 0, 1, 2, 3$ be the Hermite Polynomial basis of P_3 and $g_k = H_k(x)$ for $k = 0, 1, 2$ be the Hermite Polynomial basis of P_2. What is the matrix A that represents this L in these bases?

1.1.1 Answer

First, we note the values of the Probabilist's Hermite Polynomials [1](pg. 71)[2] relevant to this problem. Specifically:

$$
\begin{aligned}
H_0 &= 1 \\
H_1 &= x \\
H_2 &= x^2 - 1 \\
H_3 &= x^3 - 3x
\end{aligned}
$$

Therefore, we can state that the Hermite Polynomial Bases for P_3 and P_2 are

P_3	P_2
$f_0 = 1$	$g_0 = 1$
$f_1 = x$	$g_1 = x$
$f_2 = x^2 - 1$	$g_2 = x^2 - 1$
$f_3 = (x^3 - 3x)$	

Applying the f_k basis to P_3 yields the expanded polynomial

$$P_3 = p_0 + p_1 x + p_2(x^2 - 1) + p_3(x^3 - 3x) \tag{1}$$

This implies that

$$
\begin{aligned}
AP_3 &= \frac{d}{dx}(p_0 + p_1 x + p_2(x^2 - 1) + p_3(x^3 - 3x)) \\
&= p_1 + p_2(2x) + p_3(3(x^2 - 1))
\end{aligned}
$$

Expressing this polynomial in terms of a vector of coefficients and the basis P_2 yields:

$$
AP_3[f_k] = P_2[g_k] \to A \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 - 1 \\ x^3 - 3x \end{bmatrix} = \begin{bmatrix} p_1 \\ 2p_2 \\ 3p_3 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 - 1 \end{bmatrix} \tag{2}
$$

Therefore, we deduce that the matrix A must be

$$
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \tag{3}
$$

We can confirm this by applying A to P_3 then applying P_2's basis to verify that the generated polynomial is equivalent to P_2:

$$
\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = p_1 + 2p_2 + 3p_3 \tag{4}
$$

Vectorizing this polynomial and applying the basis for P_2 yields:

$$
\begin{bmatrix} p_1 \\ 2p_2 \\ 3p_3 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 - 1 \end{bmatrix} = p_1 + 2xp_2 + 3(x^2 - 1)p_3 = \frac{d}{dx}(p_0 + p_1 x + p_2(x^2 - 1) + p_3(x^3 - 3x)) \tag{5}
$$

2 Chapter 4, Problem 2

Using the following information from the problem:

- L is a linear transformation from $V \to V$
- $f_1, ..., f_n$ and $g_1, ..., g_n$ are to bases of V
- Any $u \in V$ can be written in a unique was as $u = \sum_{k=1}^{n} v_k f_k$ or as $u = \sum_{k=1}^{n} w_k g_k$
- R is an $n \times n$ matrix that relates the f_k expansion coefficients v_k to the g_k coefficients w_k by $v_j = \sum_{k=1}^{n} r_{jk} w_k$
- $v = \begin{bmatrix} v_1 \\ . \\ . \\ . \\ v_k \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ . \\ . \\ . \\ w_k \end{bmatrix}$ are related by $v = Rw$
- A represents L in the f_k basis and B represents L in the g_k basis

2.1 a. Proof

Show that $B = R^{-1}AR$.

2.1.1 Answer

Because R is an invertible matrix (as evidenced by the problem statement) we can use the property $AA^{-1} = I$ to rewrite the function in a way that eliminates the inversion:

$$B = R^{-1}AR$$
$$RB = RR^{-1}AR$$

$$RB = AR \tag{6}$$

Examining the left side of equation (6), we rewrite the expression using the notation from the HW:

$$RB = R_{\mathcal{G}}[L]_{\mathcal{G}}$$

Since R directly express the expansion coefficients in the f_k basis to the coefficients in the g_k basis, we can write

$$RB = R_{\mathcal{G}}[L]_{\mathcal{G}} =_{\mathcal{F}} [L]_{\mathcal{F}}$$
$$\therefore R^{-1}RB = R_{\mathcal{F}}^{-1}[L]_{\mathcal{F}}$$
$$\therefore B = R_{\mathcal{F}}^{-1}[L]_{\mathcal{F}}$$

Examining the right side of equation (6) and expressing it in the HW notation:

$$AR =_{\mathcal{F}} [L]_{\mathcal{F}} R$$

2.2 b. Calculation

For $V = P_3$, $f_k = x^k$ and $g_k = H_k$ find R.

2.2.1 Answer

In this case, we are examining an arbitrary vector in V, expressed as $\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$, in two different bases:

$$\begin{bmatrix} p_0 \\ p_1 x \\ p_2 x^2 \\ p_3 x^3 \end{bmatrix} \text{(in the f basis)}$$

&

$$\begin{bmatrix} p_0 \\ p_1 x \\ p_2(x^2 - 1) \\ p_3(x^3 - 3x) \end{bmatrix} \text{(in the g basis)}$$

We are given the relation that $v = Rw$ from the problem, therefore

$$\begin{bmatrix} p_0 \\ p_1 x \\ p_2 x^2 \\ p_3 x^3 \end{bmatrix} = R \begin{bmatrix} p_0 \\ p_1 x \\ p_2(x^2 - 1) \\ p_3(x^3 - 3x) \end{bmatrix}$$

Which allows us to deduce that R must be

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

2.3 c. Calculation

If L is the linear transformation $Lp = q$ with $q(x) = \delta_x(xp(x))$ find the matrix A that represents L in the monomial basis f_k.

2.3.1 Answer

Inserting the given values into the given transformation expression yields

$$A(p_0 + p_1x + p_2x^2 + p_3x^3) = \delta_x(xp(x))$$
$$A(p_0 + p_1x + p_2x^2 + p_3x^3) = \delta_x(p_0x + p_1x^2 + p_2x^3 + p_3x^4)$$
$$A(p_0 + p_1x + p_2x^2 + p_3x^3) = p_0 + 2p_1x + 3p_2x^2 + 4p_3x^3$$

Therefore we deduce that the matrix A must be

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

2.4 d. Calculation

Find the matrix B that represents L in the Hermite polynomial basis H_k

2.4.1 Answer

We perform the same steps as in part c:

$$B(p_0 + p_1x + p_2(x^2 - 1) + p_3(x^3 - 3x)) = \delta_x(xp(x))$$
$$B(p_0 + p_1x + p_2(x^2 - 1) + p_3(x^3 - 3x)) = \delta_x(p_0x + p_1x^2 + p_2(x^3 - x) + p_3(x^4 - 3x^2))$$
$$B(p_0 + p_1x + p_2(x^2 - 1) + p_3(x^3 - 3x)) = p_0 + 2p_1x + p_2(3x^2 - 1) + p_3(4x^3 - 6x)$$

Therefore we deduce that the matrix B must be

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 6 & 0 & 4 \end{bmatrix}$$

2.5 e. Calculation

Multiply the matrices to check explicitly that $B = R^{-1}AR$ in this case.

2.5.1 Answer

$$R^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}$$

$$\therefore R^{-1}AR = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 6 & 0 & 4 \end{bmatrix}$$

Which is equal to our calculated value for matrix B.

References

[1] David Bindel and Johnathan Goodman. *Principles of Scientific Computing.* 2009.

[2] Hermite Polynomials Wikipedia Page
`https://en.wikipedia.org/wiki/Hermite_polynomials`