CMSC 660 HW II

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1 Chapter 4, Problem 1

1.1 Calculation

Let L be the differentiation operator which takes P_3 to P_2 described in section 4.2.2. Let $f_k = H_k(x)$ for $k = 0, 1, 2, 3$ be the Hermite Polynomial basis of P_3 and $g_k = H_k(x)$ for $k = 0, 1, 2$ be the Hermite Polynomial basis of P_2. What is the matrix A that represents this L in these bases?

1.1.1 Answer

First, we note the values of the Probabilist's Hermite Polynomials [1](pg. 71)[2] relevant to this problem. Specifically:

$$
\begin{aligned}
H_0 &= 1 \\
H_1 &= x \\
H_2 &= x^2 - 1 \\
H_3 &= x^3 - 3x
\end{aligned}
$$

Therefore, we can state that the Hermite Polynomial Bases for P_3 and P_2 are

P_3	P_2
$f_0 = 1$	$g_0 = 1$
$f_1 = x$	$g_1 = x$
$f_2 = x^2 - 1$	$g_2 = x^2 - 1$
$f_3 = (x^3 - 3x)$	

Applying the f_k basis to P_3 yields the expanded polynomial

$$P_3 = p_0 + p_1 x + p_2(x^2 - 1) + p_3(x^3 - 3x) \tag{1}$$

This implies that

$$\begin{aligned} AP_3 &= \frac{d}{dx}(p_0 + p_1 x + p_2(x^2 - 1) + p_3(x^3 - 3x)) \\ &= p_1 + p_2(2x) + p_3(3(x^2 - 1)) \end{aligned}$$

Expressing this polynomial in terms of a vector of coefficients and the basis P_2 yields:

$$AP_3[f_k] = P_2[g_k] \rightarrow A \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 - 1 \\ x^3 - 3x \end{bmatrix} = \begin{bmatrix} p_1 \\ 2p_2 \\ 3p_3 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 - 1 \end{bmatrix} \tag{2}$$

Therefore, we calculate that the matrix A must be

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \tag{3}$$

We can confirm this by applying A to P_3 then applying P_2's basis to verify that the generated polynomial is equivalent to P_2:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = p_1 + 2p_2 + 3p_3 \tag{4}$$

Vectorizing this polynomial and applying the basis for P_2 yields:

$$\begin{bmatrix} p_1 \\ 2p_2 \\ 3p_3 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 - 1 \end{bmatrix} = p_1 + 2xp_2 + 3(x^2 - 1)p_3 = \frac{d}{dx}(p_0 + p_1 x + p_2(x^2 - 1) + p_3(x^3 - 3x)) \tag{5}$$

2 Chapter 4, Problem 2

Using the following information from the problem:

- L is a linear transformation from $V \to V$
- $f_1, ..., f_n$ and $g_1, ..., g_n$ are to bases of V
- Any $u \in V$ can be written in a unique was as $u = \sum_{k=1}^{n} v_k f_k$ or as $u = \sum_{k=1}^{n} w_k g_k$
- R is an $n \times n$ matrix that relates the f_k expansion coefficients v_k to the g_k coefficients w_k by $v_j = \sum_{k=1}^{n} r_{jk} w_k$
- $v = \begin{bmatrix} v_1 \\ . \\ . \\ . \\ v_k \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ . \\ . \\ . \\ w_k \end{bmatrix}$ are related by $v = Rw$
- A represents L in the f_k basis and B represents L in the g_k basis

2.1 a. Proof

Show that $B = R^{-1}AR$.

2.1.1 Answer

Because R is an invertible matrix (as evidenced by the problem statement) we can use the property $AA^{-1} = I$ to rewrite the function in a way that eliminates the inversion:

$$B = R^{-1}AR$$
$$RB = RR^{-1}AR$$

$$RB = AR \tag{6}$$

Multiplying both sides of the expression by vector $w = [u]_{\mathcal{G}}$ yields:

$$RBw = ARw \tag{7}$$

Examining the right side of equation (7), we note that we can apply the relation $v = Rw$:

$$ARw = A(Rw)$$
$$= Av$$

Examining the left side of equation (7), we note that, because R directly relates coefficients in the w basis to the v basis,

$$\begin{aligned} RBw &= R(Bw) \\ &= Av \end{aligned}$$

We can now see that both sides of equation (7) can be expressed as Av, therefore, $B = R^{-1}AR$.

2.2 b. Calculation

For $V = P_3$, $f_k = x^k$ and $g_k = H_k$ find R.

2.2.1 Answer

In this case, we are examining an arbitrary vector in V, expressed as $\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$, in two different bases:

$$\begin{bmatrix} p_0 \\ p_1 x \\ p_2 x^2 \\ p_3 x^3 \end{bmatrix} \text{(in the f basis)}$$

$$\&$$

$$\begin{bmatrix} p_0 \\ p_1 x \\ p_2(x^2 - 1) \\ p_3(x^3 - 3x) \end{bmatrix} \text{(in the g basis)}$$

We are given the relation that $v = Rw$ from the problem, therefore

$$\begin{bmatrix} p_0 \\ p_1 x \\ p_2 x^2 \\ p_3 x^3 \end{bmatrix} = R \begin{bmatrix} p_0 \\ p_1 x \\ p_2(x^2 - 1) \\ p_3(x^3 - 3x) \end{bmatrix}$$

Which allows us to calculate that R must be

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

2.3 c. Calculation

If L is the linear transformation $Lp = q$ with $q(x) = \delta_x(xp(x))$ find the matrix A that represents L in the monomial basis f_k.

2.3.1 Answer

Inserting the given values into the given transformation expression yields

$$\begin{aligned}
A(p_0 + p_1x + p_2x^2 + p_3x^3) &= \delta_x(xp(x)) \\
A(p_0 + p_1x + p_2x^2 + p_3x^3) &= \delta_x(p_0x + p_1x^2 + p_2x^3 + p_3x^4) \\
A(p_0 + p_1x + p_2x^2 + p_3x^3) &= p_0 + 2p_1x + 3p_2x^2 + 4p_3x^3
\end{aligned}$$

Therefore we calculate that the matrix A must be

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

2.4 d. Calculation

Find the matrix B that represents L in the Hermite polynomial basis H_k

2.4.1 Answer

We perform the same steps as in part c:

$$\begin{aligned}
B(p_0 + p_1x + p_2(x^2 - 1) + p_3(x^3 - 3x)) &= \delta_x(xp(x)) \\
B(p_0 + p_1x + p_2(x^2 - 1) + p_3(x^3 - 3x)) &= \delta_x(p_0x + p_1x^2 + p_2(x^3 - x) + p_3(x^4 - 3x^2)) \\
B(p_0 + p_1x + p_2(x^2 - 1) + p_3(x^3 - 3x)) &= p_0 + 2p_1x + p_2(3x^2 - 1) + p_3(4x^3 - 6x)
\end{aligned}$$

Therefore we calculate that the matrix B must be

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 6 & 0 & 4 \end{bmatrix}$$

2.5 e. Calculation

Multiply the matrices to check explicitly that $B = R^{-1}AR$ in this case.

2.5.1 Answer

$$R^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}$$

$$\therefore R^{-1}AR = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 6 & 0 & 4 \end{bmatrix}$$

Which is equal to our calculated value for matrix B.

3 Chapter 4, Problem 3

3.1 Proof

Let A be an $n \times m$ matrix and B be an $m \times l$ matrix. Then AB is an $n \times l$ matrix. Show that

$$(AB)^* = B^* A^* \tag{8}$$

3.1.1 Answer

To find the adjoint of a matrix we first take the complex conjugate of each element then take the transpose of the resulting matrix.[1](pg. 74)

Starting with the left hand side of equation (8), we calculate the general form of matrix $C := AB$:

$$C = \begin{bmatrix} \sum_{k=1}^{m} a_{1,k} b_{k,1} & \dots & \sum_{k=1}^{m} a_{1,k} b_{k,l} \\ \dots & \dots & \dots \\ sum_{k=1}^{m} a_{n,k} b_{k,1} & \dots & \sum_{k=1}^{m} a_{n,k} b_{k,l} \end{bmatrix}$$

Therefore,

$$C^* = \left(\begin{bmatrix} \overline{\sum_{k=1}^{m} a_{1,k} b_{k,1}} & \dots & \overline{\sum_{k=1}^{m} a_{1,k} b_{k,l}} \\ \dots & \dots & \dots \\ \overline{\sum_{k=1}^{m} a_{n,k} b_{k,1}} & \dots & \overline{\sum_{k=1}^{m} a_{n,k} b_{k,l}} \end{bmatrix} \right)^T$$

$$= \begin{bmatrix} \overline{\sum_{k=1}^{m} a_{1,k} b_{k,1}} & \dots & \overline{\sum_{k=1}^{m} a_{n,k} b_{k,1}} \\ \dots & \dots & \dots \\ \overline{\sum_{k=1}^{m} a_{1,k} b_{k,l}} & \dots & \overline{\sum_{k=1}^{m} a_{n,k} b_{k,l}} \end{bmatrix}$$

Examining the right hand side of equation (8),

$$B^* = \left(\begin{bmatrix} \overline{b_{1,1}} & ... & \overline{b_{1,l}} \\ ... & ... & ... \\ \overline{b_{m,1}} & ... & \overline{b_{m,l}} \end{bmatrix} \right)^T$$
$$= \begin{bmatrix} \overline{b_{1,1}} & ... & \overline{b_{m,1}} \\ ... & ... & ... \\ \overline{b_{1,l}} & ... & \overline{b_{m,l}} \end{bmatrix}$$

$$A^* = \left(\begin{bmatrix} \overline{a_{1,1}} & ... & \overline{a_{1,m}} \\ ... & ... & ... \\ \overline{a_{n,1}} & ... & \overline{a_{n,m}} \end{bmatrix} \right)^T$$
$$= \begin{bmatrix} \overline{a_{1,1}} & ... & \overline{a_{n,1}} \\ ... & ... & ... \\ \overline{a_{1,m}} & ... & \overline{a_{n,l}} \end{bmatrix}$$

$$\therefore B^*A^* = \begin{bmatrix} \sum_{k=1}^{m} \overline{a_{1,k}}\overline{b_{k,1}} & ... & \sum_{k=1}^{m} \overline{a_{n,k}}\overline{b_{k,1}} \\ ... & ... & ... \\ \sum_{k=1}^{m} \overline{a_{1,k}}\overline{b_{k,l}} & ... & \sum_{k=1}^{m} \overline{a_{n,k}}\overline{b_{k,l}} \end{bmatrix}$$

Because $\overline{ab} = \overline{a}\overline{b}$, we see that, in fact, $(AB)^* = A^*B^*$.

4 Chapter 4: Questions

4.1 Definition

Define a 'real vector space'.

4.1.1 Answer

As noted in B&G, a real vector space denoted by $\mathbb{R}^n$ consists of one or more column vectors with n components: $u = \begin{bmatrix} u_0 \\ . \\ . \\ . \\ u_n \end{bmatrix}$ with each arbitrary $u_n \in \mathbb{R}$. As with Complex Vector Spaces, these vectors can be scaled or added component-wise [1](pg.69).

4.2 Proof

Is $V' := u \in \mathbb{R}^n, \sum_{k=1}^{n} u_k = 0$ a subspace of $\mathbb{R}^n$? Provide a proof of your assertion.

4.2.1 Answer

Yes. As noted in [1](pg.69), this vector space is closed under vector addition or scalar multiplication and is therefore a subspace of $\mathbb{R}^n$.

4.3 Proof

Is $V' := u \in \mathbb{R}^n, \sum_{k=1}^{n} u_k = 1$ a subspace of $\mathbb{R}^n$? Provide a proof of your assertion.

4.3.1 Answer

No. As noted in [1](pg. 69) this vector space is not closed under vector addition or scalar multiplication and is therefore not a subspace of $\mathbb{R}^n$. This can be proved by noting that the by the definition of the subspace the addition of any two vectors within the subspace must equal 2. This is outside the vector space, therefore $V' := u \in \mathbb{R}^n, \sum_{k=1}^{n} u_k = 1$ is not closed under addition.

4.4 Definition

What is the standard basis of $\mathbb{R}^n$?

4.4.1 Answer

The standard basis of $\mathbb{R}^n$ expressed as an $n \times n$ matrix is $u = \begin{bmatrix} 1 & 0 & . & . & . & u_{0,n} = 0 \\ 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & 0 & . & 0 \\ 0 & . & 0 & 1 & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ u_{n,0} = 0 & 0 & . & . & 0 & u_{n,n} = 1 \end{bmatrix}$

i.e. a diagonal matrix where all diagonal components are equal to one and all off diagonal components are equal to zero.[1](pg. 70)

4.5 Definition

What is the standard inner product of $\mathbb{R}^n$?

4.5.1 Answer

From equation (4.2)[1], the standard inner product of two vectors in $\mathbb{R}^n$ is

$$< u, v >= \sum_{k=1}^{n} u_k v_k$$

4.6 Proof

Prove that the space of polynomials with real coefficients is a vector space.

4.6.1 Answer

Because $\mathbb{R}$ is closed under multiplication and addition, the space of polynomials with coefficients$\in \mathbb{R}$ must also be closed. Therefore this set can be considered a vector space.

4.7 Concept

If A is a real matrix, what is the difference between its transpose and its adjoint?

4.7.1 Answer

There is no difference. As defined in [1] (pg.74), the adjoint of a matrix is the transpose of the complex conjugate of its components. For matrices where every element $\in \mathbb{R}^n$, there is no imaginary component of the element. Therefore, $A^T = A^*,\ A \in \mathbb{R}^n$

References

[1] David Bindel and Johnathan Goodman. *Principles of Scientific Computing*. 2009.

[2] Hermite Polynomials Wikipedia Page
`https://en.wikipedia.org/wiki/Hermite_polynomials`