MACM416 Project Presentation

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Introduction

- In the previous project, we set standard 1D Wave Equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad (0 \leq x \leq L,\ t \geq 0)$$

  - u(x,t): wave displacement
  - c: wave speed.

- Goal: Study the behavior and properties of solutions to a modified version of the wave equation with viscous friction.

Modified Wave Equation

- Modified equation includes spatially varying wave speed and damping term:

$$\frac{\partial^2 u}{\partial t^2} = c^2(x)\frac{\partial^2 u}{\partial x^2} - \lambda\frac{\partial u}{\partial t} + S(x,t), \quad (0 \leq x \leq L,\ t \geq 0)$$

- C(x): a function of space represent wave speed $c(x) = 1 + 0.5 \cdot x$
- Damping Term $\lambda\frac{\partial u}{\partial t}$

  λ represents damping coefficient
- External Forcing (S(x,t)):
  - Represents an external influence or source, such as a force applied to the system at specific locations and times.

Discretization of Wave Equation

- Finite Difference Method (FDM):
  - Spatial and Temporal Discretization:
  - Spatial points: $x_i = i\Delta x, i = 0, 1, \ldots, N, \Delta x = \frac{L}{N}$
  - Temporal points: $t_n = n\Delta t, n = 0, 1, 2, \ldots.$
- Centeral Difference approximations:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

$$\frac{\partial^2 u}{\partial t^2} \approx \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2}$$

- Finite Difference Equation:

$$u_i^{n+1} = 2u_i^n - u_i^{n-1} + \frac{c^2(x_i)\Delta t^2}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) - \lambda\Delta t^2 \frac{u_i^n - u_i^{n-1}}{\Delta t} + \Delta t^2 S(x_i, t_n)$$

Grid Size Selection

- Grid size parameter:
  - Spatial grid size: Nx=100
  - Temporal Grid Size: Nt=2000
  - Domain Length: Lx=10
- Step Sizes:
  - Spatial discretization step size: $$\Delta x = \frac{L_x}{N_x - 1}$$
  - Temporal step size: $$\Delta t = \frac{T_{\text{total}}}{N_t}$$
- Convergence Verification:
  - test the solution with different values of Nx and Nt to ensure stability and consistency

Time-Stepping Scheme

- Explicit Central Difference
  - Easy to implement
  - Second-order accurate in both space and time
  - computationally efficient
- Stability Condition:
  - Must satisfy the CFL condition for stability:

$$\Delta t \leq \frac{\Delta x}{c_{\max}}$$

Cmax : Maximum wave speed in the domain

Boundary Conditions

- Dirichelt Boundary condition

$$u(0,t) = 0, \quad u(L,t) = 0, \quad \forall t \geq 0$$

Modelling a string that is fixed at both ends.

In FDM, at each time step, we explicitly set

$$u_0^n = 0, u_N^n = 0, \forall n \geq 0$$

Ensures that the boundary values remain zero throughout the simulation

Matrix Construction

- Pointwise Update Scheme:

  The FDM is implemented with a loop to update each point.

- Implicit Tridiagonal Matrix:
  - The entries represent a tridiagonal structure corresponding to the central difference approximation of the spatial derivative
  - Main diagonal: $2 + \lambda \Delta t$
  - Off-diagonals: $-\dfrac{c^2(x_i)\Delta t^2}{\Delta x^2}$

$$A = \begin{bmatrix} 2+\lambda\Delta t & -\frac{c^2(x_1)\Delta t^2}{\Delta x^2} & 0 & \cdots & 0 \\ -\frac{c^2(x_1)\Delta t^2}{\Delta x^2} & 2+\lambda\Delta t & -\frac{c^2(x_2)\Delta t^2}{\Delta x^2} & \cdots & 0 \\ 0 & -\frac{c^2(x_2)\Delta t^2}{\Delta x^2} & 2+\lambda\Delta t & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2+\lambda\Delta t \end{bmatrix}$$

Problem Description

- Modified 1D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2(x)\frac{\partial^2 u}{\partial x^2} - \lambda\frac{\partial u}{\partial t} + S(x,t), \quad (0 \le x \le L,\ t \ge 0)$$

- Boundary Conitions: $u(0,t) = 0, \quad u(L,t) = 0, \quad \forall t \ge 0$
- Initial Conditions:

$$u(x,0) = u_0(x)$$
$$\frac{\partial u}{\partial t}(x,0) = v_0(x)$$

- Quanlitative Features:

wave speed c(x) varies with position,introducing inhomongenity into the medium.

Viscous Damping term $\lambda\frac{\partial u}{\partial t}$ introduce energy dissipation affecting the amplitude and stability of wave propagation

External Forcing source term S(x,t) represent complex external influences

Test Problem

- Goal: To validate our numerical implementation, we define a test problem distinct from the 'hard' problem:
- Test PDE:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, (0 \leq x \leq L, t \geq 0)$$

- Boundary condition:

$$u(0,t) = 0, u(L,t) = 0$$

- Initial Condition:

$$u(x,0) = \sin\left(\frac{\pi x}{L}\right), \frac{\partial u}{\partial t}(x,0) = 0$$

- Analytical solution:

$$u(x,t) = \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi c t}{L}\right)$$

Algorithm Steps

- Initialization:

  set initial displacement: $u_i^0 = u_0(x_i)$

  set inital velocity using u_i^1 based on $u_0(x_i)$ and $v_0(x_i)$

$$u_i^1 = u_i^0 + \Delta t v_0(x_i) + \frac{\Delta t^2}{2}\left(c^2(x_i)\frac{u_{i+1}^0 - 2u_i^0 + u_{i-1}^0}{\Delta x^2} - \lambda v_0(x_i) + S(x_i, 0)\right)$$

- Time-Stepping loop: for each time step n=1 to Nt-1

$$u_i^{n+1} = \frac{2u_i^n - u_i^{n-1} + \frac{c^2(x_i)\Delta t^2}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) - \lambda \Delta t u_i^n + \Delta t^2 S(x_i, t_n)}{1 + \lambda \Delta t}$$

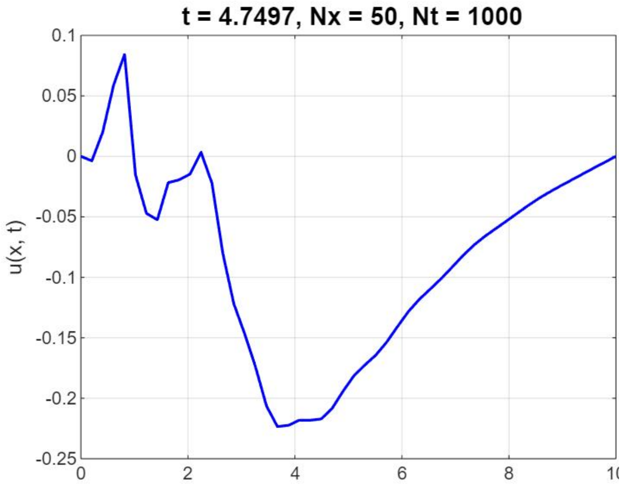
- Apply boundary conditions:

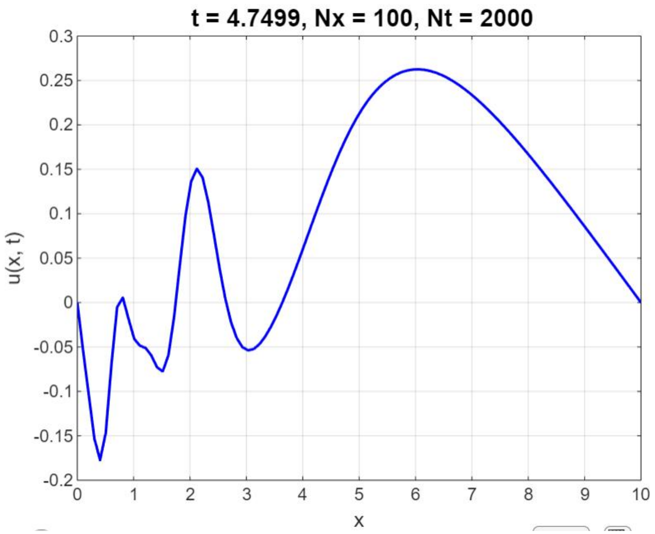
$$u_0^{n+1} = 0$$
$$u_{N_x-1}^{n+1} = 0$$

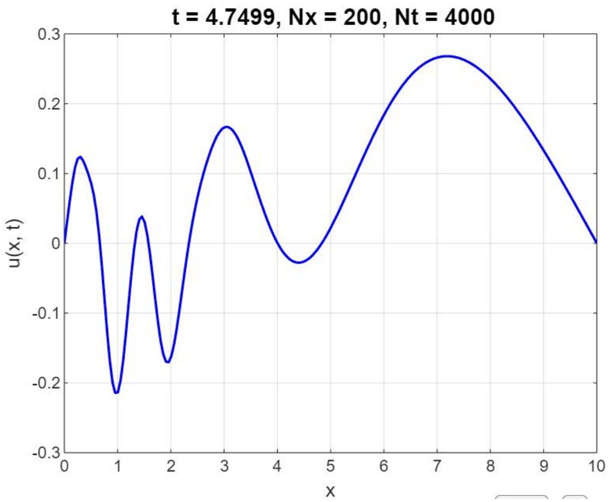
Algorithm Properties

- Stability:
  - The explicit scheme is conditionally stable, requiring adherence to the CFL condition.
- Consistency:
  - The finite difference approximations are consistent with the original PDE.
- Accuracy:
  - Second-order accurate in both space and time.
- Computational Efficiency:
  - The explicit nature allows for straightforward implementation without solving linear systems at each step.

Convergence result







Result:

- ➢ The L2 error decreases as Nx and Nt increase, demonstrating second-order convergence.
- ➢ CFL condtion is satisfied which ensures the stability of the explicit scheme.

Hard Problem setup

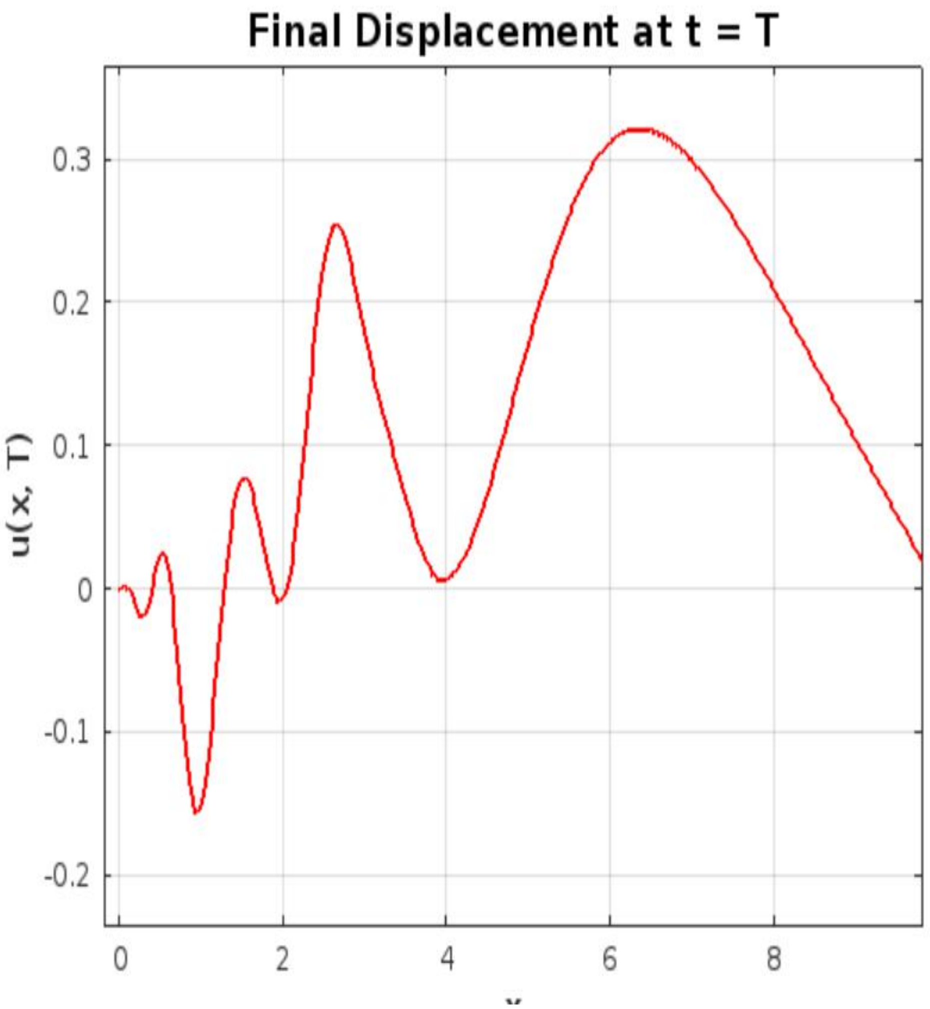
- Problem setup:
  - While keeping wave speed and damping term constant, we alter the external foricing term

$$S(x,t) = e^{-\frac{(x-L/2)^2}{2\sigma^2}} \sin(\omega t)$$

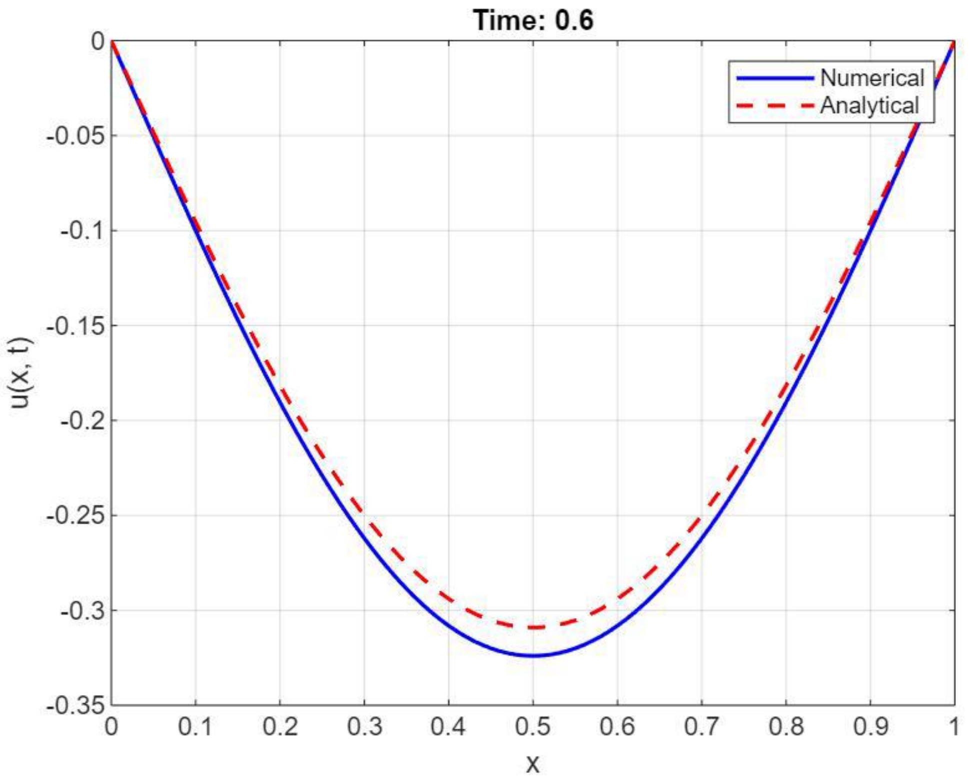
- Simulation Parameters
  - Domain Length:L=10
  - Spatial grid size: Nx=100
  - Temporal grid size: Nt=2000
  - Total simulation time: Ttotal=5

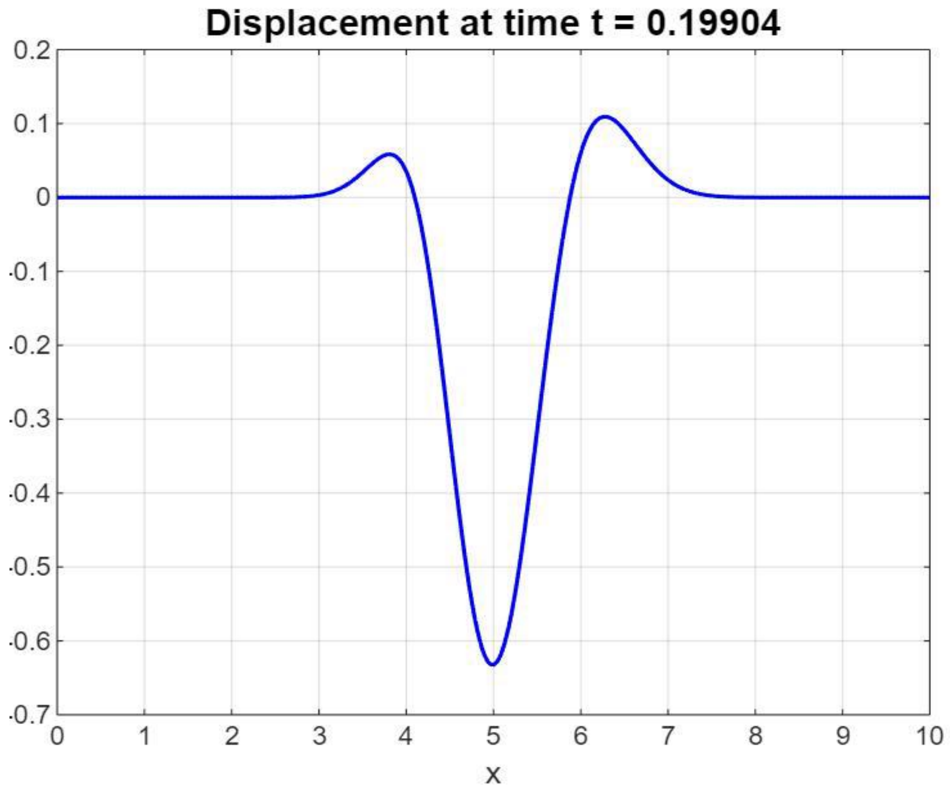
Hard Problem result

- **Discussion:**
  - The external forcing term acts as a continuous source that maintains oscillatory behavior around the region where the source is applied.
  - The effect of the external forcing is localized, which creates a persistent region of oscillation even when the surrounding areas experience decay due to damping.



Visulaization (Movies)





Comparison with Neural Network Approach

- Accuracy
- Finite Difference Method (FDM):
  - Higher accuracy with 2nd order convergence
- Neural Network (Physics-Informed Neural Networks – PINNs):
  - Approximate solutions with reasonable accuracy
  - May require extensive training
  - May struggle with capturing fine-scale features
  - If network complexity is not sufficient

- Efficiency
- FDM:
  - Computationally efficient for structured grids and explicit schemes
- PINN:
  - Training these can be computationally intensive and time-consuming
  - Especially if complexity of PDE increases.

Conclusion

- Finite Difference Scheme Implementation:
  - Developed and implemented a finite difference scheme to solve a 1D wave equation.
  - Incorporated spatially varying wave speed to capture the effect of non-uniform media.
  - Added viscous damping to simulate realistic energy dissipation over time.
- Convergence Study:
  - Conducted a comprehensive convergence analysis using a test problem with a known analytical solution.
  - Verified the numerical accuracy and stability of the finite difference scheme.