Stat 521. Examples of models where zeros in precision matrix can be interpreted

The goal of the examples here to understand a few cases where parsimonious dependence models lead to many zeros in the precision matrix.

Let $\mathbf{\Sigma}^{-1} = (\sigma^{jk})$ be the precision matrix (inverse of covariance or correlation matrix). Let $(Y_1, \ldots, Y_d)$ be multivariate normal with this covariance matrix.

G1. (a) If $\sigma^{1k} = 0$ for $k = 2, \ldots, d$ then $Y_1 \perp\!\!\!\perp (Y_2, \ldots, Y_d)$. (b) If $\sigma^{jk} = 0$ for all $2 \le j \le m$ and $k = m+1, \ldots, d$, then $(Y_1, \ldots, Y_m) \perp\!\!\!\perp (Y_{m+1}, \ldots, Y_d)$.

G2. If $\sigma^{jk} = 0$ for all $|k-j| \ge 2$ then $Y_j \perp\!\!\!\perp Y_k | (Y_{j+1}, \ldots, Y_{k-1})$ for all $|k-j| \ge 2$, $1 \le j < k \le d$. Parametrize the correlation matrix in terms of $\rho_{12}, \rho_{23}, \ldots, \rho_{d-1,d}$. Find the closed form for the correlation matrix $\boldsymbol{R}$ and the precision matrix $\boldsymbol{R}^{-1}$.

G3. $(Y_2 \perp\!\!\!\perp \cdots \perp\!\!\!\perp Y_d)|Y_1$ (conditional independence given Y_1) implies $\sigma^{jk} = 0$ for $2 \le j < k \le d$. However the inverse of the correlation matrix of $(Y_2, \ldots, Y_d)$ has no zeros if (Y_j, Y_1) has non-zero correlation for all $2 \le j \le d$. Parametrize the correlation matrix in terms of $\rho_{12}, \rho_{13}, \ldots, \rho_{1d}$. Find the closed form for the correlation matrix $\boldsymbol{R}$.

G4. $(Y_3 \perp\!\!\!\perp \cdots \perp\!\!\!\perp Y_d)|(Y_1, Y_2)$ (conditional independence given $Y_1.Y_2$) implies $\sigma^{jk} = 0$ for $3 \le j < k \le d$. However the inverse of the correlation matrix of $(Y_3, \ldots, Y_d)$ has no zeros if in non-trivial cases. Parametrize the correlation matrix in terms of $\rho_{12}, \rho_{13}, \ldots, \rho_{1d}, \rho_{23;1}, \ldots, \rho_{2d;1}$. Find the closed form for the correlation matrix $\boldsymbol{R}$. It will be shown later than $\rho_{12}, \rho_{13}, \ldots, \rho_{1d}, \rho_{23;1}, \ldots, \rho_{2d;1}$ are algebraically independent in $(-1, 1)$.

G5. Suppose $(Y_1, \ldots, Y_d)$ are jointly multivariate normal with zero means and unit variances. Suppose we can write the joint multivariate density as $f_{1:d} = f_1 f_{2|1} f_{3|12} \cdots f_{d|1:(d-1)}$ in stochastic form:

$$\begin{aligned}
Y_2 &= \beta_{21} Y_1 + \psi_2 \epsilon_2, \\
Y_3 &= \beta_{31} Y_1 + \beta_{32} Y_2 + \psi_3 \epsilon_3, \\
\vdots &\quad \vdots \\
Y_d &= \beta_{d1} Y_1 + \beta_{d2} Y_2 + \cdots + \beta_{d,d-1} Y_{d-1} + \psi_d \epsilon_d,
\end{aligned}$$

where $\epsilon_j \perp\!\!\!\perp (Y_1, \ldots, Y_{j-1})$ for $j = 2, \ldots, d$. Suppose all of the β's are non-zero (no zeros in the Cholesky matrix). Let $\boldsymbol{A}_m = [(\rho_{jk})_{1 \le j,k \le m}]^{-1}$ for $m = 2, \ldots, d$. Then the last row of $\boldsymbol{A}_m$ does not have zeros for $m = 2, \ldots, d$. Also show that $\boldsymbol{A}_d$ does not have any zeros.

G6. Consider the Cholesky decomposition in G5, where $\beta_{jk} = 0$ for $k = 3, \ldots, j-1$ and $j \ge 3$. Where are the positions with 0s in the precision matrix.

Useful result for getting lower order partial correlation being 0.

R1. Let a, b, c be distinct indices and let T be a subset of the remaining indices. $\rho_{ab;cT} = 0$ and $\rho_{ac;bT} = 0$ implies $\rho_{ab;T} = 0$ and $\rho_{ac;T} = 0$. Assume that the correlation matrix is non-singular.

Proof. From the recursion formula, $\rho_{ab;cT} = 0$ implies

$$\rho_{ab;T} = \rho_{ac;T}\rho_{bc;T}$$

and $\rho_{ac;bT} = 0$ implies

$$\rho_{ac;T} = \rho_{ab;T}\rho_{bc;T}$$

Substitute each of these equations into the other. Get $\rho_{ab;T} = \rho_{ab;T}\rho^2_{bc;T}$ and $\rho_{ac;T} = \rho_{ac;T}\rho^2_{bc;T}$. Non-singularity implies that no partial correlation is ± 1. Hence $\rho_{ab;T} = 0$ and $\rho_{ac;T} = 0$. $\square$

Special cases, mainly with dimensions 4 where some partial correlations are zero.

S1. $\rho_{12;3} = 0$, $\rho_{13;2} = 0$ ($a = 1, b = 2, c = 3, T = \emptyset$) implies $\rho_{12} = 0$, $\rho_{13} = 0$ so that $Y_1 \perp\!\!\!\perp (Y_2, Y_3)$.

S2. $\rho_{12;34} = 0$, $\rho_{13;24} = 0$ ($a = 1, b = 2, c = 3, T = \{4\}$) implies $\rho_{12;4} = 0$, $\rho_{13;4} = 0$ so that $Y_1 \perp\!\!\!\perp (Y_2, Y_3)|Y_4$.

S3. $\rho_{12;34} = 0$, $\rho_{13;24} = 0$, $\rho_{14;23} = 0$ implies $\rho_{12;4} = 0$, $\rho_{13;4} = 0$, $\rho_{12;3} = 0$, $\rho_{14;3} = 0$, $\rho_{13;2} = 0$, $\rho_{13;4} = 0$, $\rho_{12} = 0$, $\rho_{13} = 0$, $\rho_{14} = 0$. Hence $Y_1 \perp\!\!\!\perp (Y_2, Y_3, Y_4)$. This is also a special case of G1.

S4. $\rho_{12;34} = 0$, $\rho_{34;12} = 0$. Not clear what this means stochastically.

S5. $\rho_{12;34} = 0$, $\rho_{13;24} = 0$, $\rho_{23;14} = 0$. This matches a converse of G3 (after permuting indices). It implies $(Y_1 \perp\!\!\!\perp Y_2 \perp\!\!\!\perp Y_3)|Y_4$. Result R1 implies $\rho_{12;4} = 0$, $\rho_{13;4} = 0$, $\rho_{23;4} = 0$, which also leads to $(Y_1 \perp\!\!\!\perp Y_2 \perp\!\!\!\perp Y_3)|Y_4$.

S6. $\rho_{12;34} = 0$, $\rho_{13;24} = 0$, $\rho_{24;13} = 0$. Not clear what this means stochastically.

Stat 521 Homework 1: Write answers in correct mathematical notation for G1, G2, G3, G5 (and G4,G6 if time permits), making use of result R1, the document with results on partial correlations, and hints from the special cases.