# 2-4 Exact Equations

## 2-4-1 方法的條件

任何 first order DE 皆可改寫成

$$M(x,y)dx+N(x,y)dy=0$$ 的型態

(1) 當 $\boxed{\frac{\partial}{\partial y}M(x,y)=\frac{\partial}{\partial x}N(x,y)}$ 成立時，

☆1-1

可以用本節的 Exact Equation 的方法來解

(2) 當 $\boxed{\dfrac{\frac{\partial}{\partial y}M(x,y)-\frac{\partial}{\partial x}N(x,y)}{M(x,y)}}$ ☆1-2 is independent of $x$ page 115

或 $\boxed{\dfrac{\frac{\partial}{\partial y}M(x,y)-\frac{\partial}{\partial x}N(x,y)}{N(x,y)}}$ ☆1-3 is independent of $y$ page 116

可以用 Modified Exact Equation Method 來解 (見講義 2-4-5)

page 117

## 2-4-2 方法的來源

- Review the concept of partial differentiation

$$df(x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

- Specially, when $f(x, y) = c$ where $c$ is some constant,

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

補充：

$$df(x,y,z)=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz$$

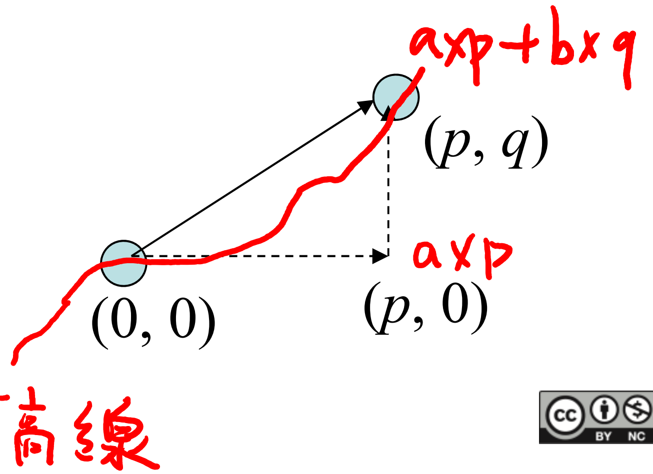
$$df(x_1,x_2,x_3,\cdots,x_k)=\frac{\partial f}{\partial x_1}dx_1+\frac{\partial f}{\partial x_2}dx_2+\frac{\partial f}{\partial x_3}dx_3+\cdots\cdots+\frac{\partial f}{\partial x_k}dx_k$$

(0,0)

思考：假設一個人在山坡的某處。若往東走，每走 1 公尺，高度會增加 $a$ 公尺。若往北走，每走 1 公尺，高度會增加 $b$ 公尺。假設這人現在所在的位置是(0, 0)。那麼這人的東北方，座標為 $(p, q)$ 的地方，高度會比 (0, 0) 高多少？

$$a\times p+b\times q$$

$$df(x,y)=\underbrace{\boxed{\frac{\partial f(x,y)}{\partial x}}}_{a}\underbrace{\boxed{dx}}_{p}+\underbrace{\boxed{\frac{\partial f(x,y)}{\partial y}}}_{b}\underbrace{\boxed{dy}}_{q}$$



**[Definition 2.4.1]** We can express any 1st order DE as

$$\boxed{M(x,y)dx + N(x,y)dy = 0}$$

• If there exists some function $f(x, y)$ that satisfies

☆3 $$\boxed{\frac{\partial f(x,y)}{\partial x} = M(x,y)} \quad \text{and} \quad \boxed{\frac{\partial f(x,y)}{\partial y} = N(x,y)},$$

$\Rightarrow \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$

$d\,f(x,y) = 0$

$f(x,y) = c$

then we call the 1st order DE the **exact equation**.

• The method for checking whether the DE is an exact equation:

$$\boxed{\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}}$$ ☆, 限制

(Proof): If $\frac{\partial f(x,y)}{\partial x} = M(x,y)$ and $\frac{\partial f(x,y)}{\partial y} = N(x,y)$,

then $$\frac{\partial M(x,y)}{\partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}f(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}f(x,y) = \frac{\partial N(x,y)}{\partial x}$$

For the exact equation,

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\downarrow$$

$$\frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy = 0$$

$$\downarrow$$

可改寫成 $df(x,y) = 0,$

$$\downarrow$$

$$f(x,y) = c$$

$f(x,y)$ 是一條等高線

# 2-4-3-1 The Method for Solving the Exact Equation (A)

1-1

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

(0) Check $\dfrac{\partial}{\partial y}M(x,y) = \dfrac{\partial}{\partial x}N(x,y)$

$$\boxed{\frac{\partial f(x,y)}{\partial x} = M(x,y)}$$

(1)

$$f(x,y) = \int M(x,y)\,dx + g(y) = c$$

g(y) is a constant for x

3 4

(3)

$$\boxed{\int M(x,y)\,dx + g(y) = c}$$

(4)

further computation

Solution

$$\boxed{\frac{\partial f(x,y)}{\partial y} = N(x,y)}$$

(2)

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,dx + g'(y) = N(x,y)$$

(2)

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx$$

(2)

$$g(y) = \int g'(y)\,dy$$

(3) 代入

(2)

double N

# 整理

Previous Step: Check whether $\boxed{\dfrac{\partial M(x,y)}{\partial y} = \dfrac{\partial N(x,y)}{\partial x}}$ is satisfied.

Step 1: Solve $\dfrac{\partial f(x,y)}{\partial x} = M(x,y) \longrightarrow f(x,y) = \int M(x,y)\,dx + g(y)$

Step 2: 將 $f(x, y)$ 代入 $\dfrac{\partial f(x,y)}{\partial y} = N(x,y)$ ，以解出 $g(y)$

Step 3: Substitute $g(y)$ into

$$f(x,y) = \int M(x,y)\,dx + g(y) = c$$

Step 4: Further computation and obtain the solution

Extra Steps: (a) Consider the initial value problem

## 2-4-3-2 The Method for Solving the Exact Equation (B):

☆☆ 1-2

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

(0) Check $\dfrac{\partial}{\partial y}M(x,y) = \dfrac{\partial}{\partial x}N(x,y)$

$$\boxed{\frac{\partial f(x,y)}{\partial x} = M(x,y)} \qquad \boxed{\frac{\partial f(x,y)}{\partial y} = N(x,y)}$$

(1)

☆4

$$f(x,y) = \int N(x,y)\,dy + h(x) = \underline{\underline{c}}$$

☆3

$h(x)$ is a constant for $y$

(2)

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\int N(x,y)\,dy + h'(x) = M(x,y)$$

(2)

$$h'(x) = M(x,y) - \frac{\partial}{\partial x}\int N(x,y)\,dy$$

(2)

$$h(x) = \int h'(x)\,dx$$

(3) 代入

(3)

$$\boxed{\int N(x,y)\,dy + h(x) = c}$$

(4) further computation

Solution

double N

## 2-4-4 例子

**[Example 1]** (text page 67)

It can also be solved by Sec. 2-2 and Sec. 2-3.

standard

$$\frac{dy}{dx}+\frac{2x}{x^2-1}y=0$$

$x=\pm 1$ are singular points

$$\boxed{2xydx+(x^2-1)dy=0}$$

$$\underline{M(x,y)}=\underline{2xy} \qquad \underline{N(x,y)}=\underline{x^2-1}$$

**Step 0**: check whether it is exact

$$\frac{\partial M}{\partial y}=2x=\frac{\partial N}{\partial x} \quad ☆_1$$

$$☆☆_{1\text{-}1}\ \frac{\partial f}{\partial x}=2xy \qquad \frac{\partial f}{\partial y}=x^2-1$$

Step 1

$$f(x,y)=x^2y+g(y) \quad ☆_4$$

Step 2

$$\frac{\partial f}{\partial y}=x^2+g'(y)=x^2-1$$

Step 2

$$g'(y)=-1$$

Step 2

$$g(y)=-y$$

+c可忽略

Step 3

$$x^2y-y=\underline{\underline{c}} \quad ☆_3$$

Step 4

$$\boxed{y=c/(x^2-1)}$$

$x\in(-1,1)$

思考: 是否有其他的方法可以解 Example 1?

**[Example 2]** (text page 67)

$$\boxed{(e^{2y} - y\cos xy)dx + (2xe^{2y} - x\cos xy + 2y)dy = 0}$$

$$M(x,y) = e^{2y} - y\cos xy \qquad N(x,y) = 2xe^{2y} - x\cos xy + 2y$$

Step 0:

check for exact

$$\boxed{\frac{\partial M}{\partial y}} = 2e^{2y} - \cos xy + xy\sin xy = \boxed{\frac{\partial N}{\partial x}}$$

Step 1

$$\frac{\partial f}{\partial y} = 2xe^{2y} - x\cos xy + 2y$$

$$f(x,y) = xe^{2y} - \sin xy + y^2 + \underline{h(x)}$$

*4

Step 2

$$\frac{\partial f}{\partial x} = e^{2y} - y\cos xy$$

$$e^{2y} - y\cos xy + h'(x) = e^{2y} - y\cos xy$$

$$h'(x) = 0$$

Step 2

$$h(x) = \cancel{c_1}\ 0$$

Step 3

Step 3

$$f(x,y) = xe^{2y} - \sin xy + y^2$$

Step 4

$$c = xe^{2y} - \sin xy + y^2$$

$$xe^{2y} - \sin xy + y^2 + c = 0$$

implicit solution

要注意

(a) 自行由另一個方向 $f(x,y)=\int M(x,y)dx+g(y)$ 來練習，

看是否得出同樣的結果。

(b) 得出的解 $xe^{2y}-\sin xy+y^2+c=0$ 為 implicit solution

(c) 思考：何時用 $f(x,y)=\int M(x,y)dx+g(y)$

何時用 $f(x,y)=\int N(x,y)dy+h(x)$

**[Example 3]** (text page 68)

$$\boxed{\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y\left(1 - x^2\right)}}$$

$y(0) = 2$ ☆6

$$\left(xy^2 - \cos x \sin x\right)dx + y\left(x^2 - 1\right)dy = 0$$

Exact!

$M = xy^2 - \cos x \sin x, \quad N = y\left(x^2 - 1\right)$

Step 0 $\quad \dfrac{\partial M}{\partial y} = 2xy = \dfrac{\partial N}{\partial x}$

$$\frac{\partial f}{\partial x} = xy^2 - \cos x \sin x, \qquad \frac{\partial f}{\partial y} = y\left(x^2 - 1\right)$$

$$\Downarrow$$

$$f = \frac{y^2}{2}\left(x^2 - 1\right) + h(x) = c_1$$

$$\Downarrow$$

$$xy^2 + h'(x) = xy^2 - \cos x \sin x$$

$$\Downarrow$$

☆5

$$h'(x) = -\cos x \sin x = -\frac{\sin(2x)}{2}$$

$$\Downarrow$$

$$h(x) = \cos(2x)/4$$

$\cos(2x) = \cos^2 x - \sin^2 x$

$= 2\cos^2 x - 1$

$c_2 = 4c_1$

$$2y^2\left(x^2 - 1\right) + \cos(2x) = c_2$$

$2 \times 2^2 \times (-1) + 2 \times 1^2 - 1 = c_2$

$$\Downarrow$$

$$2y^2\left(x^2 - 1\right) + 2\cos^2(x) - 1 = c_2$$

from $y(0) = 2, \quad c_2 = -7$

$y^2 = \dfrac{-\cos^2 x - 3}{(x^2 - 1)}$

$y = \pm\sqrt{\dfrac{-\cos^2 x - 3}{x^2 - 1}}$

得出的 implicit solution 為 $y^2\left(1-x^2\right)-\cos^2 x=3$, 範圍：$x \in (-1, 1)$

而 explicit solution 為 $\boxed{y=\sqrt{\dfrac{3+\cos^2 x}{1-x^2}}}$, 範圍： $x \in (-1, 1)$

$$為何\ y=-\sqrt{\frac{3+\cos^2 x}{1-x^2}}\ 不為解？$$

注意

(a) 何時用 $f(x,y)=\int M(x,y)\,dx+g(y)$

何時用 $f(x,y)=\int N(x,y)\,dy+h(x)$

(b) Initial value problem

(c) Singular points and the interval of the solution

## 2-4-5 Modified Exact Equation Method

Technique: Use the integrating factor $\mu(x, y)$ to convert the 1st order DE into the exact equation.

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\downarrow$$

$$\mu(x,y)M(x,y)\,dx + \mu(x,y)N(x,y)\,dy = 0$$

such that $\dfrac{\partial \mu(x,y)M(x,y)}{\partial y} = \dfrac{\partial \mu(x,y)N(x,y)}{\partial x}$

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

$$\mu_x N - \mu_y M = (M_y - N_x)\mu$$

It is hard to find $\mu$.

Simplification: To make $\mu_x = 0$ or $\mu_y = 0$.

$$\boxed{\mu_x N - \mu_y M = (M_y - N_x)\mu}$$

假設1

(1) If we want $\mu_x = 0$, i.e, $\mu$ is dependent on $y$ alone, then

$$-\mu_y M = (M_y - N_x)\mu$$

$$\frac{\mu_y}{\mu} = -\frac{M_y - N_x}{M}$$

限制 ☆ 1-2

$\Longrightarrow$ $\boxed{(M_y - N_x)/M}$ should be a function of $y$ alone.

$$\frac{1}{\mu}\frac{d\mu}{dy} = -\frac{M_y - N_x}{M}$$ 用 separable variable 的方法

Sec 2-2

$$\frac{d\mu}{\mu} = \frac{N_x - M_y}{M}dy$$

$$\ln|\mu| = \int \frac{N_x - M_y}{M} dy$$

$$\boxed{\mu(y) = e^{\int \frac{(N_x - M_y)}{M}dy}}$$

☆☆ 2

$$\mu = \cancel{\pm}\, e^{\int \frac{N_x - M_y}{M} dy}$$

(注意正負號)

$$\boxed{\mu_x N - \mu_y M = (M_y - N_x)\mu}$$

假設2

(1) If we want $\mu_y = 0$, i.e, $\mu$ is dependent on $x$ alone, then

$$\mu_x N = (M_y - N_x)\mu$$

$$\frac{\mu_x}{\mu} = \frac{M_y - N_x}{N}$$

限制 ☆1-3

$\Longrightarrow$ $\boxed{(M_y - N_x)/N}$ should be a function of $x$ alone.

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{M_y - N_x}{N}$$

用 separable variable 的方法

Sec 2-2

$$\frac{d\mu}{\mu} = \frac{M_y - N_x}{N}dx$$

$$\boxed{\mu(x) = e^{\int \frac{(M_y - N_x)}{N}dx}}$$

☆☆3

## 前面 2-4-3 (pages 106, 108) 的解法流程再加一個步驟： ☆☆

**Step 0:**

Whether $M_y = N_x$ ☆1-1

- Yes → **Case 1** ☆☆1
  Use page 106 or 108
- No → Whether $\dfrac{M_y - N_x}{M}$ is independent of $x$ ☆1-2
  - Yes → Case 2 ☆7 (小心易背錯)
    $\mu(y) = e^{\int \frac{(N_x - M_y)}{M} dy}$ ☆☆2
  - No → Whether $\dfrac{M_y - N_x}{N}$ is independent of $y$ ☆1-3
    - Yes → Case 3
      $\mu(x) = e^{\int \frac{(M_y - N_x)}{N} dx}$ ☆☆3
    - No → Case 4
      Use other methods

**In Cases 2 and 3,**

$$M(x,y)dx + N(x,y)dy = 0$$

$$\downarrow$$

$$\mu M(x,y)dx + \mu N(x,y)dy = 0$$

$$\downarrow$$

Use the process of page 106 or 108 but $M(x, y)$ is modified as $\mu M(x, y)$

$N(x, y)$ is modified as $\mu N(x, y)$

**[Example 4]** (text page 69)

$$\boxed{xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0}$$

Step 0: $M = xy$ $\quad N = 2x^2 + 3y^2 - 20$

Not exact!

$M_y = \frac{\partial M}{\partial y} = x \quad N_x = \frac{\partial M}{\partial x} = 4x$

$$\boxed{M_y - N_x = x - 4x = -3x}$$

$$\frac{M_y - N_x}{N} = \frac{-3x}{2x^2 + 3y^2 - 20}$$

$$\boxed{\frac{M_y - N_x}{M} = -\frac{3}{y}}$$

(independent of $x$)
(Case 2) ★1-2

$-\frac{3}{y} \Rightarrow \frac{3}{y}$ ★7

$$\mu(y) = e^{\int \frac{3}{y} dy} = e^{3\ln|y|} = y^3$$

Q: 為何 $c$ 以及 $\pm$ 可省略?

If $M = xy^4$, $N = 2x^2y^3 + 3y^5 - 20y^3$

$\frac{\partial M}{\partial y} = 4xy^3$, $\frac{\partial N}{\partial x} = 4xy^3$

Exact!

$$\boxed{xy^4 dx + (2x^2y^3 + 3y^5 - 20y^3)\,dy = 0}$$

double $N$ pages 106, 108

Steps 1~4: $\boxed{\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = c}$

## 2-4-6 本節需要注意的地方

(1) 使本節方法時，要先將 DE 改成如下的型態

$$M(x,y)dx+N(x,y)dy=0$$

並且假設

$$\frac{\partial}{\partial x}f(x,y)=M(x,y), \qquad \frac{\partial}{\partial y}f(x,y)=N(x,y)$$

(2) 對 $x$ 而言，$g(y)$ 是個常數；對 $y$ 而言，$h(x)$ 是個常數

(3) 本節很少有 singular solution 的問題，

但是可能有 singular point 的問題

(4) 背熟三個判別式，二種情況的 integrating factor (小心勿背錯)

並熟悉解法的流程

# 2-5 Solutions by Substitutions

介紹 3 個特殊解法

Question: 尚有不少的 1st order DE 無法用 Sections 2-2~2-4 的方法來解

本節所提到的特殊解法的共通點：

用新的變數 $u$ 來取代 $y$

對 症 下 藥

## 2-5-1 特殊解法 1: Homogeneous Equations

If $g(tx, ty) = t^{\alpha} g(x, y)$,

then $g(x, y)$ is a **homogeneous function of degree α**. ☆,

Which one is homogeneous?

homogeneous $g(x, y) = x^3 + y^3$ $g(tx, ty) = t^3x^3 + t^3y^3 = t^3(x^3 + y^3) = t^3 g(x,y)$

not homogeneous $g(x, y) = x^3 + y^3 + 1$ $g(tx, ty) = t^3x^3 + t^3y^3 + 1 \neq t^3 g(x,y)$

(key: The sum of powers should be the same).

Note：課本中，homogeneous 有兩種定義

一種是 Section 2-3 的定義 (較常用)

一種是這裡的定義

兩者並不相同

- For a 1st order DE:

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

If $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree,

then the 1st order DE is homogeneous.

☆2 解法的限制條件 ☆2

It can by solved by setting $u = \frac{y}{x}$

☆☆1 $\boxed{y = xu, \qquad dy = udx + xdu,}$ (from page 103)

and use the separable value method.

$$dy = \frac{\partial y}{\partial x}dx + \frac{\partial y}{\partial u}du$$
$$= u\,dx + x\,du$$

If $M(x,y)dx + N(x,y)dy = 0$ is homogeneous

$$M(tx,ty) = t^{\alpha} M(x,y) \qquad N(tx,ty) = t^{\alpha} N(x,y)$$

以 $t = 1/x$ 得出

$ty = \frac{y}{x} = u$

then

$$M(1,u) = x^{-\alpha} M(x,y) \qquad N(1,u) = x^{-\alpha} N(x,y)$$

$$M(x,y) = x^{\alpha} M(1,u) \qquad N(x,y) = x^{\alpha} N(1,u)$$

where $\underline{u = y/x}$, $\boldsymbol{y = xu}$

$$\boxed{dy = udx + xdu}$$

$$\cancel{x^{\alpha}} M(1,u)dx + \cancel{x^{\alpha}} N(1,u)(udx + xdu) = 0$$

$$[M(1,u) + uN(1,u)]dx + xN(1,u)du = 0$$

$$\boxed{\frac{dx}{x} = -\frac{N(1,u)du}{M(1,u) + uN(1,u)}} \qquad \text{(separable)}$$

## Procedure for solving the homogeneous 1st order DE ☆☆₁

Previous Step: Conclude whether the DE is homogeneous

(快速判斷法：看 powers (指數) 之和是否一致)

Step 1: Set $\boldsymbol{u = y/x}$ $(y = ux)$, $dy = udx + xdu$ ☆☆₁

並化簡

Step 2: Convert into the separable 1st order DE

Step 3: Solve the separable 1st order DE (用 Sec. 2-2 的方法)

Step 4: Substitute $u = y/x$ (別忘了這個步驟)

☆₃

**[Example 1]** (text page 73) 指數和=2

$$\underbrace{(x^2+y^2)}_{M(x,y)}dx+\underbrace{\left(x^2-xy\right)}_{N(x,y)}dy=0$$

$M(tx, ty) = t^2M(x, y)$ $N(tx, ty) = t^2N(x, y)$ order=2

homogeneous DE

Previous Step:

Conclude whether the DE is homogeneous

Step 1: Set $y = ux$, $dy = udx + xdu$ ☆☆1

原式

$$(x^2+u^2x^2)dx+\left(x^2-ux^2\right)(udx+xdu)=0$$

divided by $x^2$

$$(1+u^2)dx+\left(1-u\right)(udx+xdu)=0$$

$$(1+u)dx+\left(1-u\right)xdu=0 \longrightarrow \boxed{\left[\frac{1-u}{1+u}\right]du+\frac{dx}{x}=0}$$

Step 2: Convert into the separable 1st order DE

不需分為2邊，只要……=0

Step 3: Solve the separable 1st order DE

Sec 2-2

$$\left[\frac{1-u}{1+u}\right]du+\frac{dx}{x}=0$$

$$\downarrow$$

$$\int\left[-1+\frac{2}{1+u}\right]du+\int\frac{dx}{x}=0$$

$$\downarrow$$

$$-u+2\ln|1+u|+\ln|x|+c_1=0$$

$$\downarrow$$

$$\ln[|(1+u)|^2|x|]=u-c_1$$

$$\downarrow$$

$$(1+u)^2|x|=e^{u-c_1}$$

$$\downarrow$$

$$(1+u)^2x=c_2e^u \qquad (c_2=\pm e^{-c_1})$$

★3

Step 4 代回 $u=y/x$

$$\downarrow$$

$$(1+y/x)^2x=c_2e^{y/x} \longrightarrow \boxed{(x+y)^2=c_2xe^{y/x}}$$

## 2-5-2 特殊解法 2: Bernoulli's Equations

伯努利

【定義】 Bernoulli's equation:

★4 限制條件

$$\boxed{\frac{dy}{dx}+P(x)y=f(x)y^n}$$

so $\underbrace{y^{-n}\frac{dy}{dx}}_{\frac{1}{1-n}\frac{du}{dx}}+P(x)y^{1-n}=f(x)$

u

If n=0 → linear

We can set $\boxed{u=y^{1-n}\ ,\quad \frac{dy}{dx}=\frac{1}{1-n}u^{\frac{n}{1-n}}\frac{du}{dx}}$ , and the method of solving the 1st order linear DE to solve the Bernoulli's equation.

★★

$y=u^{\frac{1}{1-n}}$ $\quad y^{-n}=u^{\frac{-n}{1-n}}$

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\frac{du^{\frac{1}{1-n}}}{du}\frac{du}{dx}=\frac{1}{1-n}u^{\frac{n}{1-n}}\frac{du}{dx}$$

★5 (Chain rule)

# Procedure for solving the Bernoulli's equation

Previous Step : Conclude whether the DE is a Bernoulli's equation

☆ ☆₂

Step 1: Set $\boxed{u = y^{1-n}}$ $\boxed{y = u^{\frac{1}{1-n}}}$ , $\boxed{\dfrac{dy}{dx} = \dfrac{1}{1-n} u^{\frac{n}{1-n}} \dfrac{du}{dx}}$ ☆ ☆₂

Step 2: Convert the Bernoulli's equation into the 1st order linear DE

Step 3: Solve the 1st order linear DE (use the method in Sec. 2-3)

Step 4: Substitute $u = y^{1-n}$ (別忘了)

☆₃

**[Example 2]** (text page 74)

$$\boxed{x\frac{dy}{dx}+y=x^2y^2}$$

$\downarrow$

Previous Step: 判斷 (Bernoulli, $n=2$)

Step 1: set $\boldsymbol{u=y^{-1}}$ ($y=u^{-1}$) $\dfrac{dy}{dx}=\dfrac{dy}{du}\dfrac{du}{dx}=-u^{-2}\dfrac{du}{dx}$

$u=y^{1-n}=y^{-1}$

(Chain rule)

Step 2: Convert into the 1st order linear DE (standard form)

$$\text{原式}\longrightarrow -xu^{-2}\frac{du}{dx}+u^{-1}=x^2u^{-2}\longrightarrow \frac{du}{dx}-\frac{1}{x}u=-x$$

$\times\dfrac{-u^2}{x}$

Step 3: Obtain the solution of the 1st order DE

$$u=-x^2+cx$$

integrating factor

$e^{-\int\frac{1}{x}dx}=e^{-\ln|x|}=|x|^{-1}$

$x^{-1}$

$(x^{-1}u)'=-1$ $\quad x^{-1}u=-x+c$

Step 4: 代回 $u=y^{-1}$ $\boxed{y=\dfrac{1}{-x^2+cx}}$

☆3

## 2-5-3 特殊解法 3

If the 1st order DE has the form,

$$\boxed{\frac{dy}{dx} = f\left(Ax + By + C\right)} \quad (B \neq 0)$$

(解法的限制條件) ☆6 限制條件

☆☆3

we can set $\boxed{u = Ax + By + C}$ to solve it.

$$\boxed{\frac{dy}{dx} = \frac{1}{B}\frac{du}{dx} - \frac{A}{B}}$$

divided by B dx

Since $\boxed{du = Adx + Bdy}$ (這式子也許較好記)

$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = Ax + By$

**Procedure for solving** $\frac{dy}{dx} = f\left(Ax + By + C\right)$ ☆6

Previous Step: Conclude

Step 1: Set $\boxed{u = Ax + By + C}$ $\longrightarrow$ $\boxed{du = Adx + Bdy}$ $\longrightarrow$ $\boxed{\frac{dy}{dx} = \frac{1}{B}\frac{du}{dx} - \frac{A}{B}}$

☆ ☆3

Step 2: Converting (轉化成用其他方法可以解出來的 DE
未必一定是轉化成 separable variable DE)

Step 3: Solving

Step 4: Substitute $u = Ax + By + C$ (別忘了)

☆3

**[Example 3]** (text page 74)

$$\boxed{\frac{dy}{dx} = (-2x+y)^2 - 7}, \qquad y(0) = 0$$

Previous Step: 判斷

Step 1: Set $\boxed{u = -2x + y}$ $\boxed{du = -2dx + dy}$ $\boxed{\frac{dy}{dx} = \frac{du}{dx} + 2}$

☆☆3

Step 2: Converting

$$\text{原式} \longrightarrow \frac{du}{dx} + 2 = u^2 - 7 \longrightarrow \boxed{\frac{du}{u^2 - 9} = dx}$$

Step 3: Obtain the solution (別忘了在運算過程中，代回 $u = Ax + By$)

$$\frac{1}{6}\left(\frac{1}{u-3} - \frac{1}{u+3}\right) du = dx$$

$$\ln|u-3| - \ln|u+3| = 6x + c_1$$

$$\frac{u-3}{u+3} = c_2 e^{6x} \qquad c_2 = \pm e^{c_1}$$

$$\frac{u-3}{u+3} = c_2 e^{6x}$$

$$\boxed{u = \frac{3c_2 e^{6x} + 3}{1 - c_2 e^{6x}}}$$

Step 4: 代回 $u = Ax + By + C$ ★3

$$\boxed{-2x + y = \frac{3c_2 e^{6x} + 3}{1 - c_2 e^{6x}}} \qquad y = 2x + \frac{3c_2 e^{6x} + 3}{1 - c_2 e^{6x}}$$

Extra: From $y(0) = 0$ $\qquad 0 = 0 + \frac{3c_2 + 3}{1 - c_2} \qquad c_2 = -1$

$$\boxed{y = 2x + \frac{3(1 - e^{6x})}{1 + e^{6x}}}$$

## 2-5-4 本節要注意的地方

(1) 對症下藥，先判斷 DE 符合什麼樣的條件，再決定要什麼方法來解 (部分的 DE 可以用兩個以上的方法來解)

(2) 別忘了，寫出解答時，要將 $u$ 用 $y/x$, $y^{1-n}$, 或 $Ax + By + C$ 代回來

(3) 本節方法皆有五大步驟

Previous Step: 判斷用什麼方法
Step 1: Set $u = \ldots$, $du/dx = \ldots$
Step 2: Converting，
Step 3: Solving，
Step 4: 將 $u$ 用 $x, y$ 代回來

# 附錄四　整理：Methods of solving the 1st order DE

(1)Direct integral

條件：$\dfrac{dy}{dx}=f(x)$

破解法：直接積分

$$y=\int f(x)dx+c$$

(2) Separable variable

條件：$\dfrac{dy}{dx}=g(x)h(y)$

破解法：$x, y$ 各歸一邊後積分

$$\int\frac{dy}{h(y)}=\int g(x)dx$$

(3) Linear DE

條件：$\dfrac{dy}{dx}=-P(x)y+f(x)$

破解法：算 $e^{\int P(x)dx}$

$$\boxed{\left(e^{\int P(x)dx}y(x)\right)'=e^{\int P(x)dx}f(x)}$$

(4) Exact equation

條件：$\dfrac{dy}{dx}=-\dfrac{M(x,y)}{N(x,y)}$

$$\frac{\partial M(x,y)}{\partial y}=\frac{\partial N(x,y)}{\partial x}$$

破解法：pages 106, 108 (double *N*)

先處理 $\dfrac{\partial f}{\partial x}=M(x,y)$

(或反過來)

再處理 $\dfrac{\partial f}{\partial y}=N(x,y)$

## (4-1) Exact equation 變型

條件：$\frac{dy}{dx}=-\frac{M(x,y)}{N(x,y)}$

$(M_y-N_x)/M$ independent of $x$

破解法：$\mu(y)=e^{\int\frac{(N_x-M_y)}{M}dy}$

$$\mu(y)M(x,y)dx+\mu(y)N(x,y)dy=0$$

is exact

## (4-2) Exact equation 變型

條件：$\frac{dy}{dx}=-\frac{M(x,y)}{N(x,y)}$

$(M_y-N_x)/N$ independent of $y$

破解法：$\mu(x)=e^{\int\frac{(M_y-N_x)}{N}dx}$

$$\mu(x)M(x,y)dx+\mu(x)N(x,y)dy=0$$

is exact

## (5) Homogeneous equation

條件：$\frac{dy}{dx}=-\frac{M(x,y)}{N(x,y)}$

$$M(tx,ty)=t^{\alpha}M(x,y)$$

$$N(tx,ty)=t^{\alpha}N(x,y)$$

破解法：$u=y/x,\quad (y=xu)$

$$dy=udx+xdu$$

再用 separable variable method

## (6) Bernoulli's Equation

條件：$\dfrac{dy}{dx} = -P(x)y + f(x)y^n$

破解法： $u = y^{1-n}$

$$\frac{dy}{dx} = \frac{1}{1-n} u^{\frac{n}{1-n}} \frac{du}{dx}$$

再用 linear DE 的方法

## (7) $Ax + By + C$

條件：$\dfrac{dy}{dx} = f(Ax + By + C)$

破解法： $u = Ax + By + c$

$$\frac{dy}{dx} = \frac{1}{B}\frac{du}{dx} - \frac{A}{B}$$

注意 (a) 速度的訓練

(b) Exercises in Review 2 多練習

(c) 行有餘力，觀察 singular solution 和 singular point

練習題

Section 2-4: 3, 8, 13, 17, 20, 25, 29, 32, 34, 35, 38, 42

Section 2-5: 3, 5, 10, 13, 14, 17, 20, 22, 24, 25, 29, 31

Chapter 2 Review: 2, 13 , 16, 17, 18, 19, 22, 23, 24, 26, 33