Chapter 4 Higher Order Differential Equations

Highest differentiation: $\dfrac{d^n y}{dx^n},\ n > 1$

Most of the methods in Chapter 4 are applied for the linear DE.

附錄五 DE 的分類

- DE
  - Linear
    - Constant coefficients
      - Homogeneous
      - Nonhomogeneous
    - Cauchy-Euler
      - Homogeneous
      - Nonhomogeneous
    - Others
      - Homogeneous
      - Nonhomogeneous
  - Nonlinear

附錄六 Higher Order DE 解法

- linear Sec 4-1~4-8
  - associated homogeneous part
    - reduction of order Sec 4-2
    - auxiliary function Sec 4-3
    - Cauchy-Euler Sec 4-7
  - particular solution
    - “guess” method Sec 4-4
    - annihilator Sec 4-5
    - variation of parameters Sec 4-6
    - Fourier series (Sec. 11-3)
  - multiple linear DEs Sec 4-8
- nonlinear Sec 4-10
  - reduction of order
  - Taylor series
  - numerical method
- both (but mainly linear)
  - series solution Chap 6
  - transform
    - Laplace transform Ch 7
    - Fourier series Ch 11
    - Fourier cosine /sine series Ch 11
    - Fourier transform Ch 14

4-1 Linear Differential Equations: Basic Theory

The n^{th} order linear DE

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

$g(x)=0 \longrightarrow$ homogeneous

$g(x)\neq 0 \longrightarrow$ nonhomogeneous

4.1.1 Nonhomogeneous Equations (可和 page 60 相比較)

Nonhomogeneous linear DE

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y=g(x)$$

Part 1

名1 Associated homogeneous DE

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)$$
$$+\cdots+a_1(x)y'(x)+a_0(x)y=0$$

find ***n*** **linearly independent solutions**

$$y_1(x), y_2(x), \cdots\cdots, y_n(x)$$

page 146

名2 **complementary function**

$$c_1y_1(x)+c_2y_2(x)+\cdots\cdots+c_n y_n(x)$$

☆1

Part 2

名3 particular solution y_p

(**any** solution of the nonhomogeneous linear DE)

☆2

superposition principle (sometimes)

$$g(x)=g_1(x)+g_2(x)+\cdots\cdots+g_k(x)$$
$$\downarrow \qquad \downarrow \qquad \qquad \downarrow$$
$$y_p(x)=y_{p_1}(x)+y_{p_2}(x)+\cdots\cdots+y_{p_k}(x)$$

general solution of the nonhomogeneous linear DE

$$y(x)=c_1y_1(x)+c_2y_2(x)+\cdots\cdots+c_ny_n(x)+y_p(x)$$

Non-homogeneous Linear DE 解法的步驟 (Also see page 143)

Step 1: Find the general solution (i.e., the complementary function) of the associated homogeneous DE

(Sections 4-2, 4-3, 4-7)

Step 2: Find the particular solution

(Sections 4-4, 4-5, 4-6)

Step 3: Combine the complementary function and the particular solution

Extra Step: Consider the initial (or boundary) conditions

4.1.2 Homogeneous Equations and Complementary Function

4.1.2.1 Definition

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

$g(x) = 0 \longrightarrow$ homogeneous

$g(x) \neq 0 \longrightarrow$ nonhomogeneous

- 重要名詞：Associated homogeneous equation

名1

The associated homogeneous equation of a nonhomogeneous DE:

Setting $g(x) = 0$

- Review: Section 2-3, pages 58, 60

[Example] $y''' - 6y'' + 11y' - 6y = 3x$

Associated homogeneous equation: $y''' - 6y'' + 11y' - 6y = 0$

4.1.2.2 Solution of the Homogeneous Equation

[Important Theory]: An n^{th} order homogeneous linear DE has n linearly independent solutions.

[Theorem 4.1.5] ☆1

For an n^{th} order homogeneous linear DE, if

① $y_1(t), y_2(t), \ldots\ldots, y_n(t)$ are the solutions of this DE

② $y_1(t), y_2(t), \ldots\ldots, y_n(t)$ are linearly independent

then any solution of the homogeneous linear DE can be expressed as:

$$\boxed{y = c_1 y_1 + c_2 y_2 + \cdots\cdots + c_n y_n}$$

可以和矩陣的概念相比較

From Theorem 4.1.5:

An n^{th} order homogeneous linear DE has n linearly independent solutions.

Find n linearly independent solutions

== Find all the solutions of an n^{th} order homogeneous linear DE

$y_1(x), y_2(x), \ldots\ldots, y_n(x)$: fundamental set of solutions

$y = c_1 y_1 + c_2 y_2 + \cdots\cdots + c_n y_n$: general solution of the homogenous linear DE

(又稱做 **complementary function**) 名2

也是重要名詞

名 4

[Definition 4.1] Linear Dependence / Independence

If there is no solution other than $c_1 = c_2 = \ldots\ldots = c_n = 0$ for the following equality

$$c_1 y_1(x) + c_2 y_2(x) + \cdots\cdots + c_n y_n(x) = 0 \qquad \text{for all } x$$

恒等式

then $y_1(x)$, $y_2(x)$, ….., $y_n(x)$ are said to be linearly independent.

$x,\ x^2,\ x(x-1) \Rightarrow$ dependent

$1\cdot x + (-1)x^2 + 1\cdot x(x-1) = 0$

$1,\ x,\ x^2 \Rightarrow$ independent for all x

If $c_1 1 + c_2 x + c_3 x^2 = 0$ is satisfied, then $c_1 = c_2 = c_3 = 0$

Otherwise, they are linearly dependent.

判斷是否為 linearly independent 的方法: **Wronskian** wrong scan

[Definition 4.2] Wronskian

名与

$$W(y_1, y_2, \cdots\cdots, y_n) = \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$$

$W(y_1, y_2, \cdots\cdots, y_n) \neq 0 \longrightarrow$ linearly independent

Note: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - afh - bdi - ceg$$

4.1.2.3 Examples

[Example 9] (text page 127)

$e^{0.5x}$ e^{x} $e^{1.5x}$ e^{2x} $\Rightarrow$ indep.

$$y''' - 6y'' + 11y' - 6y = 0$$

$1, \cos^2 x, \cos 2x$ $\quad \cos 2x = 2\cos^2 x - 1$ $\Rightarrow$ dep.

$y_1 = e^x$, $y_2 = e^{2x}$, and $y_3 = e^{3x}$ are three of the solutions

Since

$$e^{x}\,\det\begin{bmatrix} 1 & e^{2x} & e^{3x} \\ 1 & 2e^{2x} & 3e^{3x} \\ 1 & 4e^{2x} & 9e^{3x} \end{bmatrix}$$

$$\det\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} = \det\begin{bmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{bmatrix} = e^{x+2x+3x}\det\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} = 2e^{6x} \neq 0$$

Therefore, y_1, y_2, and y_3 are linear independent for any x

$18+3+4$

$-2-9-12=2$

general solution:

$$\boxed{y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}} \qquad x \in (-\infty, \infty)$$

4.1.3 Particular Solution

Particular solution:

Any solution of the original nonhomogeneous linear DE.

[Example 10] (text page 128)

$$\boxed{y''' - 6y'' + 11y' - 6y = 3x}$$

$$y''' - 6y'' + 11y' - 6y = 0$$

e^{x} e^{2x} e^{3x} Sec 4-3

Complementary function

$$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$$

(Example 9)

Particular solution of

$$y''' - 6y'' + 11y' - 6y = 3x\cancel{)}$$

$$y_p = -\frac{11}{12} - \frac{1}{2}x$$

Sec 4-4

General solution: $\boxed{y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} - \frac{11}{12} - \frac{1}{2}x}$

4.1.3.1 Superposition Principle for Particular Solutions

[Theorem 4.1.7] Superposition Principle

☆2

If $y_{p_1}(x)$ is the particular solution of

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)=g_1(x)$$

$y_{p_2}(x)$ is the particular solution of

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)=g_2(x)$$

$$\vdots$$

$y_{p_k}(x)$ is the particular solution of

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)=g_k(x)$$

then $\boxed{y_{p_1}(x)+y_{p_2}(x)+\cdots+y_{p_k}(x)}$ is the particular solution of

$$\boxed{\begin{aligned}&a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)\\&=g_1(x)+g_2(x)+\cdots\cdots+g_k(x)\end{aligned}}$$

[Example 11] (text page 129)

Sec 4-4

$y_{p_1}(x) = -4x^2$ is a particular solution of $y'' - 3y' + 4y = -16x^2 + 24x - 8$

(子問題一)

$y_{p_2}(x) = e^{2x}$ is a particular solution of $y'' - 3y' + 4y = 2e^{2x}$

(子問題二)

$y_{p_3}(x) = xe^{x}$ is a particular solution of $y'' - 3y' + 4y = 2xe^{x} - e^{x}$

(子問題三)

$\boxed{y = y_{p_1} + y_{p_2} + y_{p_3} = -4x^2 + e^{2x} + xe^{x}}$ is a particular solution of

$$\boxed{y'' - 3y' + 4y = -16x^2 + 24x - 8 + 2e^{2x} + 2xe^{x} - e^{x}}$$

4.1.4 New Notations

Notation: $D^n y = \dfrac{d^n y}{dx^n}$ ☆3-1

$$\frac{d^2 y}{dx^2} + 5\frac{dy}{dx} + 6y \xrightarrow{\text{可改寫成}} D^2 y + 5Dy + 6y \xrightarrow{\text{可改寫成}} (D^2 + 5D + 6)y$$

可再改寫成 ☆3-2

$$\boxed{\begin{array}{c} L(y) \\ L = D^2 + 5D + 6 \end{array}}$$

☆3-3

(Useful for the linear DE with constant coefficients)

4.1.5 Initial-Value and Boundary Value Problems

4.1.5.1 The n^{th} Order Initial Value Problem

i.e., the n^{th} order linear DE with the constraints at the same point

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

$$y(x_0)=y_0,\qquad y'(x_0)=y_1,\qquad y''(x_0)=y_2,\qquad \ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ y^{(n-1)}(x_0)=y_{n-1}$$

n initial conditions

(given at the same point)

[Theorem 4.1.1] ☆4

only applied for linear DE
initial conditions

For an interval I that contains the point x_0

① If $a_0(x)$, $a_1(x)$, $a_2(x)$, ……., $a_{n-1}(x)$, $a_n(x)$ are continuous at $x = x_0$

② $a_n(x_0) \neq 0$

(很像Section 2-3 當中x_0 不是singular point 的條件)

then for the problem on page 155, the solution $y(x)$ exists and is unique on the interval I that contains the point x_0

(Interval I 的範圍，取決於何時 $a_n(x) = 0$ 以及 何時 $a_k(x)$ $(k = 0 \sim n)$ 不為continuous)

Otherwise, the solution is either non-unique or does not exist.

(infinite number of solutions) (no solution)

[Example 1] (text page 119)

$$3y''' + 5y'' - y' + 7y = 0 \qquad y(1) = 0 \qquad y'(1) = 0 \qquad y''(1) = 0$$

unique solution

[Example 2] (text page 120)

$$y'' - 4y' = 12x \qquad y(0) = 4 \qquad y'(0) = 1$$

unique solution

- $x^2 y'' - 2xy' + 2y = 6 \qquad y(0) = 3 \qquad y'(0) = 1$

  $x_0 : 0$

  有無限多組解

  $y = cx^2 + x + 3$ $\qquad c$ 為任意之常數

- 比較：

$$x^2 y'' - 2xy' + 2y = 6 \qquad y(1) = 3 \qquad y'(1) = 1$$

$x_0 = 1$

There is only one solution

$$y = x^2 - x + 3$$

$$x \in (0, \infty)$$

- Note:

The initial value can also be the form as:

$$\alpha y(x_0) + \beta y'(x_0) = y_0$$

$$\sum_{n=0}^{N-1} \alpha_n y^{(n)}(x_0) = y_0$$ (general initial condition)

4.1.5.2 n^{th} Order Boundary Value Problem

Boundary conditions are specified at different points

比較：Initial conditions are specified at the same points

例子：$a_2(x)y'' + a_1(x)y' + a_0(x) = g(x)$

subject to $\quad y(a) = y_0, \quad y(b) = y_1 \qquad a \neq b$

或 $\quad y'(a) = y_0, \quad y(b) = y_1$

或 $$\begin{cases} \alpha_1 y(a) + \beta_1 y'(a) = \gamma_1 \\ \alpha_2 y(b) + \beta_2 y'(b) = \gamma_2 \end{cases}$$

An n^{th} order linear DE with n boundary conditions may have a <u>unique</u> solution, <u>no</u> solution, or <u>infinite</u> number of solutions.

[Example 3] (text page 120)

$$y'' + 16y = 0$$

solution: $y = c_1 \cos(4x) + c_2 \sin(4x)$

$c_1 = 0$ $c_1 = 0$

(1) $y(0) = 0$ $y(\pi/2) = 0$

$y = c_2 \sin(4x)$ c_2 is any constant (infinite number of solutions)

$c_1 = 0$ $c_2 = 0$

(2) $y(0) = 0$ $y(\pi/8) = 0$

$y = 0$ (unique solution)

4.1.6 本節要注意的地方

(1) Most of the theories in Section 4.1 are applied to the linear DE

(2) 注意 initial conditions 和 boundary conditions 之間的不同

(3) 快速判斷 linear independent

4.1.6.1 名詞

- general solution of the nonhomogenous linear DE (page 143)

$$y(x)=c_1y_1(x)+c_2y_2(x)+\cdots\cdots+c_ny_n(x)+y_p(x)$$

- associated homogeneous equation, (page 145)
  (重要名詞)
- fundamental set of solutions (page 147)
- complementary function (general solution of the homogenous linear DE)
  (重要名詞) (page 147)
- Wronskian (page 149)
- particular solution (page 151)
- initial conditions, boundary conditions (pages 155, 159)
  (重要名詞)

(補充 1) Theorem 4.1.1 的解釋

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

$$y(x_0)=y_0 \qquad y'(x_0)=y_1 \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad y^{(n-1)}(x_0)=y_{n-1}$$

When $a_n(x_0) \neq 0$

$$y^{(n)}(x_0)+\frac{a_{n-1}(x_0)}{a_n(x_0)}y^{(n-1)}(x_0)+\cdots+\frac{a_1(x_0)}{a_n(x_0)}y'(x_0)+\frac{a_0(x)}{a_n(x_0)}y(x_0)=\frac{g(x_0)}{a_n(x_0)}$$

find $y^{(n)}(x_0)$

$$y^{(n-1)}(x_0+\Delta)=y^{(n-1)}(x_0)+y^{(n)}(x_0)\Delta$$

find $y^{(n-1)}(x_0+\Delta)$

(根據 $f'(t)=\dfrac{f(t+\Delta)-f(t)}{\Delta}$ ， $f(t+\Delta)=f(t)+f'(t)\Delta$)

以此類推

$$y^{(n-2)}(x_0+\Delta)=y^{(n-2)}(x_0)+y^{(n-1)}(x_0)\Delta \longrightarrow \text{find } y^{(n-2)}(x_0+\Delta)$$

$$y^{(n-3)}(x_0+\Delta)=y^{(n-3)}(x_0)+y^{(n-2)}(x_0)\Delta \longrightarrow \text{find } y^{(n-3)}(x_0+\Delta)$$

$$\vdots$$

$$y(x_0+\Delta)=y(x_0)+y'(x_0)\Delta \longrightarrow \boxed{\text{find } y(x_0+\Delta)}$$

$$y^{(n)}(x_0+\Delta)+\frac{a_{n-1}(x_0+\Delta)}{a_n(x_0+\Delta)}y^{(n-1)}(x_0+\Delta)+\cdots+\frac{a_1(x_0+\Delta)}{a_n(x_0+\Delta)}y'(x_0+\Delta)$$

$$+\frac{a_0(x+\Delta)}{a_n(x_0+\Delta)}y(x_0+\Delta)=\frac{g(x_0+\Delta)}{a_n(x_0+\Delta)} \longrightarrow \text{find } y^{(n)}(x_0+\Delta)$$

$$y^{(n-1)}(x_0+2\Delta)=y^{(n-1)}(x_0+\Delta)+y^{(n)}(x_0+\Delta)\Delta \longrightarrow \text{find } y^{(n-1)}(x_0+2\Delta)$$

$$y^{(n-2)}(x_0+2\Delta)=y^{(n-2)}(x_0+\Delta)+y^{(n-1)}(x_0+\Delta)\Delta \longrightarrow \text{find } y^{(n-2)}(x_0+2\Delta)$$

$$\vdots$$

$$y\left(x_0+2\Delta\right)=y\left(x_0+\Delta\right)+y'\left(x_0+\Delta\right)\Delta$$

$\boxed{\text{find } y(x_0+2\Delta)}$

$$y^{(n)}\left(x_0+2\Delta\right)+\frac{a_{n-1}\left(x_0+2\Delta\right)}{a_n\left(x_0+2\Delta\right)}y^{(n-1)}\left(x_0+2\Delta\right)+\cdots+\frac{a_1\left(x_0+2\Delta\right)}{a_n\left(x_0+2\Delta\right)}y'\left(x_0+2\Delta\right)$$
$$+\frac{a_0\left(x+2\Delta\right)}{a_n\left(x_0+2\Delta\right)}y\left(x_0+2\Delta\right)=\frac{g\left(x_0+2\Delta\right)}{a_n\left(x_0+2\Delta\right)}$$

以此類推，可將 $y(x_0+3\Delta)$, $y(x_0+4\Delta)$, $y(x_0+5\Delta)$, ...

以至於將 $y(x)$ 所有的值都找出來。

(求 $y(x)$ for $x > x_0$ 時, 用正的 Δ 值，

求 $y(x)$ for $x < x_0$ 時, 用負的 Δ 值)

Requirement 1: $a_0(x)$, $a_1(x)$, $a_2(x)$, ……., $a_{n-1}(x)$, $a_n(x)$ are continuous 是為了讓 $a_k(x_0+m\Delta)$ 皆可以定義

Requirement 2: $a_n(x) \neq 0$ 是為了讓 $a_k(x_0+m\Delta)/a_n(x_0+m\Delta)$ 不為無限大

4-2 Reduction of Order

2nd order ⇒ 1st order

4.2.1 適用情形

限制 ☆:

Suitable for the 2^{nd} order linear homogeneous DE
(1) (2) (3)

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

$y_1(x) \neq 0$

(4) One of the nontrivial solution $y_1(x)$ has been known.

trivial solution : $y=0$
nontrivial $y \neq 0$

4.2.2 解法

假設 $\boxed{y_2(x)=u(x)y_1(x)}$

將 DE 變成 Standard form ☆2

$$a_2(x)y''+a_1(x)y'+a_0(x)y=0$$

$$\Downarrow$$

$$y''+\frac{a_1(x)}{a_2(x)}y'+\frac{a_0(x)}{a_2(x)}y=0$$

$$P(x)=\frac{a_1(x)}{a_2(x)}$$

☆2

先將DE 變成 Standard form

P(x) = a₁(x)/a₂(x)

$$y''+P(x)y'+Q(x)y=0$$

If $\boxed{y(x)=u(x)\,y_1(x)}$, (比較 Section 2-3)

$$y'=uy_1'+u'y_1 \qquad y''=uy_1''+2u'y_1'+u''y_1$$

✓ ✓ ✓

$$uy_1''+2u'y_1'+u''y_1+P(x)uy_1'+P(x)u'y_1+Q(x)uy_1=0$$

$$u\underbrace{(y_1''+P(x)y_1'+Q(x)y_1)}_{\text{zero}}+2u'y_1'+u''y_1+P(x)u'y_1=0$$

$$u''y_1 + u'(2y_1' + P(x)y_1) = 0$$

set $w = u'$

$$\downarrow$$

$$\frac{dw}{dx}y_1 + w(2\frac{dy_1}{dx} + P(x)y_1) = 0$$

multiplied by $dx/(y_1 w)$

$$\downarrow$$

$$\frac{dw}{w} + 2\frac{dy_1}{y_1} + P(x)dx = 0$$

separable variable
(with 3 variables)

$$\downarrow$$

$$\int\frac{dw}{w} + 2\int\frac{dy_1}{y_1} + \int P(x)dx = 0$$

$$\downarrow$$

$$\ln|w| + c_3 + 2\ln|y_1| + c_4 = -\int P(x)dx$$

$$\ln|w| + 2\ln|y_1| = \ln|w| + \ln|y_1|^2 = \ln|w||y_1|^2 = \ln|wy_1^2|$$

$$\downarrow$$

$$\ln|wy_1^2| = -\int P(x)dx + c$$

$c = -c_3 - c_4$

$$\ln\left|wy_1^2\right| = -\int P(x)\,dx + c$$

$$wy_1^2 = \pm e^{-\int P(x)dx + c}$$

$$w = c_1 e^{-\int P(x)dx} / y_1^2$$

$$\boxed{u = \int w\,dx = c_1 \int \frac{e^{-\int P(x)dx}}{y_1^2}\,dx + c_2}$$

We can set $c_1 = 1$ and $c_2 = 0$

(因為我們算 $u(x)$ 的目的，只是為了要算出與 $y_1(x)$ 互相independent的另一個解)

☆☆

$$\boxed{y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)}\,dx}$$

where $P(x) = \dfrac{a_1(x)}{a_2(x)}$ (the coefficient of $y'(x)$ in the standard form)

☆2

4.2.3 例子

[Example 1] (text page 132)

$$\boxed{y'' - y = 0}$$

We have known that $y_1 = e^x$ is one of the solution

$e^{-\int p(x)dx} = e^{c_1}$

$= c$

$\underline{P(x) = 0}$ $\quad y_2(x) = e^x \int ce^{-2x} dx = -\frac{1}{2}ce^{-x}$

(no y')

$\left(-\frac{1}{2}(ce^{-2x} + c_0\right)e^x$

Specially, set $c = -2$, ($y_2(x)$ 只要 independent of $y_1(x)$ 即可 所以 c 的值可以任意設)

set $c = -2$
$c_0 = 0$

$$\boxed{y_2(x) = e^{-x}}$$

General solution: $\boxed{y(x) = c_1 e^x + c_2 e^{-x}}$

$x=0$ singular

[Example 2] (text page 133) (將課本 x 的範圍做更改)

$$\boxed{x^2y''-3xy'+4y=0} \qquad \text{when } x \in (-\infty, 0)$$

We have known that $y_1 = x^2$ is one of the solution

Note: the interval of x

If $x \in (0, \infty)$ $(x > 0)$, $\int dx/x = \ln x$ 如課本

If $x < 0$, $\int dx/x = \ln(-x)$

$$y_2(x)=x^2\int\frac{e^{3\ln(-x)}}{x^4}dx = x^2\int\frac{(-x)^3}{(-x)^4}dx$$

constant

$$=-x^2\int\frac{1}{x}dx = -x^2\ln|x|$$

$$\boxed{y(x)=c_1x^2+c_2x^2\ln|x|}$$

standard

$y''-\frac{3}{x}y'+\frac{4}{x^2}y=0$

$P(x)=-\frac{3}{x}$

$e^{\int -P(x)dx}$

$=e^{\int\frac{3}{x}dx}$

$=e^{3\ln|x|}=|x|^3$

$=(-x)^3$

$x\in(-\infty,0)$

4.2.4 本節需注意的地方

(1) 記住公式 $\boxed{y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx}$

(2) 若不背公式 (不建議)，在計算過程中別忘了對 $w(x)$ 做積分

(3) 別忘了 $P(x)$ 是 "standard form" 一次微分項的 coefficient term

(4) 同樣有 singular point 的問題

(5) 因為 $y_2(x)$ 是 homogeneous linear DE 的 "任意解" ，所以計算時，常數的選擇以方便為原則

(6) 由於 $\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$ 的計算較複雜且花時間，所以要多加練習

多算習題

附錄七： Hyperbolic Function

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2}$$

☆1

比較： $\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

☆2

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$e^{jx} = \cos x + j\sin x$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

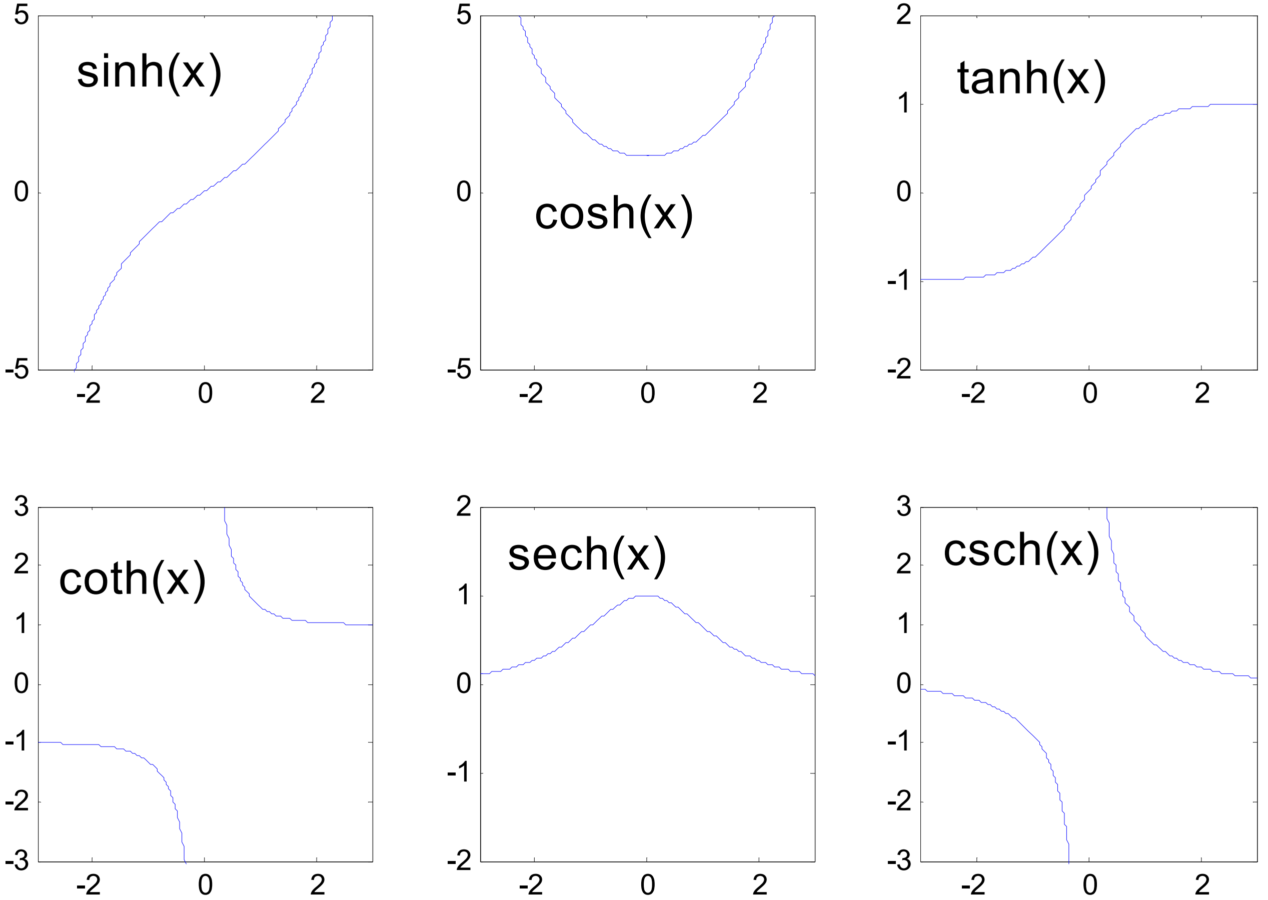
$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$$

Note: $\sinh(x) = -\sinh(-x)$ ☆3

$\cosh(x) = \cosh(-x)$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^{x} + e^{-x}}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^{x} - e^{-x}}$$


sinh(x)
cosh(x)
tanh(x)
coth(x)
sech(x)
csch(x)
5
0
-5
2
1
-1
-2
3
-3
-2
0
2

☆4
$$\frac{d}{dx}\sinh(ax) = a\cosh(ax)$$
$$\frac{d}{dx}\cosh(ax) = a\sinh(ax)$$

$$\frac{d}{dx}\tanh(ax) = a\,\mathrm{sech}^2(ax)$$

$$\frac{d}{dx}\coth(ax) = -a\,\mathrm{csch}^2(ax)$$

$$\frac{d}{dx}\mathrm{sech}(ax) = -a\,\mathrm{sech}(ax)\tanh(ax)$$

$$\frac{d}{dx}\mathrm{csch}(ax) = -a\,\mathrm{csch}(ax)\coth(ax)$$

☆5
$$\sinh(0) = 0$$
$$\cosh(0) = 1$$

☆6
$$\sinh'(0) = 1$$
$$\cosh'(0) = 0$$

$$\sin(ix) = i\sinh(x)$$

$$\cos(ix) = \cosh(x)$$

$$\int \sinh(ax)\,dx = \frac{\cosh(ax)}{a} + c$$

$$\int \cosh(ax)\,dx = \frac{\sinh(ax)}{a} + c$$

$$\int \tanh(ax)\,dx = \frac{\ln\left|\cosh(ax)\right|}{a} + c$$

$$\int \coth(ax)\,dx = \frac{\ln\left|\sinh(ax)\right|}{a} + c$$

$$\int \operatorname{sech}(ax)\,dx = \frac{2\tan^{-1}\left(\tanh(\frac{a}{2}x)\right)}{a} + c$$

$$\int \operatorname{csch}(ax)\,dx = \frac{\ln\left|\tanh(\frac{a}{2}x)\right|}{a} + c$$

4-3 Homogeneous Linear Equations with Constant Coefficients

附屬

本節使用 auxiliary equation 的方法來解 homogeneous DE

KK: [ɔgˋzɪljərɪ]

4-3-1 限制條件

限制條件: (1) homogeneous

☆1

(2) linear

(3) constant coefficients

$$\boxed{a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = 0}$$

a_0, a_1, a_2, …. , a_n are constants   independent of x

(the simplest case of the higher order DEs)

4-3-2 解法

解法核心：

☆2

Suppose that the solutions has the form of e^{mx}

Example: $y''(x) - 3\,y'(x) + 2\,y(x) = 0$

complementary function

Set $y(x) = e^{mx}$, $m^2 \cancel{e^{mx}} - 3m\, \cancel{e^{mx}} + 2\, \cancel{e^{mx}} = 0$

$c_1 e^{x} + c_2 e^{2x}$

$$m^2 - 3m + 2 = 0 \longrightarrow \text{solve } m$$

$(m-1)(m-2)=0$ $m=1,2$ e^{x}, e^{2x}

可以直接把 n 次微分用 m^n 取代，變成一個多項式

這個多項式被稱為 auxiliary equation

名 1

- 解法流程 ☆☆

$$\boxed{a_n y^{(n)}(x)+a_{n-1}y^{(n-1)}(x)+\cdots+a_1 y'(x)+a_0 y=0}$$

$$\downarrow\; y^{(k)}(x)\to m^k$$

$y^{(k)}(x) = m^k$

Step 1-1
auxiliary function

名1

$$\boxed{a_n m^n+a_{n-1}m^{n-1}+\cdots+a_1 m+a_0=0}$$

Step 1-1 Find n roots , $m_1, m_2, m_3, \ldots., m_n$

(If $m_1, m_2, m_3, \ldots., m_n$ are distinct)

Step 1-2 n linearly independent solutions $e^{m_1 x}, e^{m_2 x}, e^{m_3 x}, \ldots\ldots, e^{m_n x}$

(有三個 Cases)

Step 1-3 Complementary function

$$\boxed{y=c_1 e^{m_1 x}+c_2 e^{m_2 x}+c_3 e^{m_3 x}+\ldots\ldots+c_n e^{m_n x}}$$

4-3-3 Three Cases for Roots (2nd Order DE)

$$\boxed{a_2 y''(x) + a_1 y'(x) + a_0 y(x) = 0}$$

$$\boxed{a_2 m^2 + a_1 m + a_0 = 0}$$

roots $m_1 = \dfrac{-a_1 + \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$ $\qquad m_2 = \dfrac{-a_1 - \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$

solutions

$a_1^2 - 4a_2a_0 > 0$

Case 1 $m_1 \neq m_2$, m_1, m_2 are <u>real</u> (其實 m_1, m_2 不必限制為 real)

☆3-1

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

☆3-2 $a_1^2 - 4a_2 a_0 = 0$ $m_1 = m_2 = -\frac{a_1}{2a_2}$

Case 2 $m_1 = m_2$ (m_1 and m_2 are of course real)

First solution: $\boxed{y_1 = e^{m_1 x}}$

Second solution: using the method of "Reduction of Order"

Sec 4-2

standard

$y'' + \frac{a_1}{a_2} y' + \frac{a_0}{a_2} = 0$

$P(x)$

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$$

$$= e^{m_1 x} \int e^{-2m_1 x} e^{-\int a_1/a_2 dx} dx$$

$$= e^{m_1 x} \int e^{(-2m_1 - \frac{a_1}{a_2})x} dx \qquad m_1 = \frac{-a_1}{2a_2}$$

$$= e^{m_1 x} \int dx = e^{m_1 x}(x + \cancel{c})$$

$P(x) = -2m_1$

$e^{-\int P(x)dx}$

$= e^{2m_1 x}$

$= y_1^2(x)$

$$\boxed{y_2(x) = xe^{m_1 x}}$$

$$\boxed{y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}}$$

重根 → 乘 x

☆3-3

Case 3 $m_1 \neq m_2$, m_1 and m_2 are conjugate and complex

$a_1^2-4a_2a_0<0$

$$m_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2a_0}}{2a_2} = \alpha + j\beta \qquad m_2 = \alpha - j\beta$$

$$\alpha = -a_1/2a_2, \qquad \beta = \sqrt{4a_2a_0 - a_1^2}/2a_2$$

Solution: $\boxed{y = C_1e^{\alpha x + j\beta x} + C_2e^{\alpha x - j\beta x}}$

Another form:

$$y = e^{\alpha x}\left(C_1e^{j\beta x} + C_2e^{-j\beta x}\right)$$

$$= e^{\alpha x}\left(C_1\cos\beta x + jC_1\sin\beta x + C_2\cos\beta x - jC_2\sin\beta x\right)$$

set $c_1 = C_1 + C_2$ and $c_2 = jC_1 - jC_2$

$\boxed{y = e^{\alpha x}\left(c_1\cos\beta x + c_2\sin\beta x\right)}$ c_1 and c_2 are some constant

☆3-3

[Example 1] (text page 137)

(a) $\boxed{2y'' - 5y' - 3y = 0}$

$2m^2 - 5m - 3 = 0, \quad m_1 = -1/2, \; m_2 = 3$

$\boxed{y = c_1 e^{-x/2} + c_2 e^{3x}}$

(b) $\boxed{y'' - 10y' + 25y = 0}$ $\rightarrow$ $(m-5)^2 = 0$

$m^2 - 10m + 25 = 0, \quad m_1 = 5, \; m_2 = 5$

$\boxed{y = c_1 e^{5x} + c_2 x e^{5x}}$

(c) $\boxed{y'' + 4y' + 7y = 0}$ $\rightarrow$ $(m+2)^2 = -3$ $\quad m+2 = \pm i\sqrt{3}$

$m^2 + 4m + 7 = 0, \quad m_1 = -2 + i\sqrt{3}, \quad m_2 = -2 - i\sqrt{3}$

$\boxed{y = e^{-2x}\left(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right)}$ $\quad \alpha = -2, \; \beta = \sqrt{3}$