影　像　轉　換

壹、傅利葉轉換簡介

貳、離散傅利葉轉換

參、二維傅利葉轉換之性質

肆、快速傅利葉轉換

伍、其他可分性轉換

陸、Hotelling 轉換

影　像　轉　換

壹、傅利葉轉換簡介

一、定義：

$$\Im\{f(x)\} = F(u) = \int_{-\infty}^{\infty} f(x)\exp[-j2\pi ux]dx \qquad (3.1-1)$$

其反轉換(Inverse Fourier Transform)為：

$$\Im^{-1}\{F(u)\} = f(x)$$

$$= \int_{-\infty}^{\infty} F(u)\exp[j2\pi ux]du \qquad (3.1-2)$$

以上兩式稱為傅利葉轉換偶(Fourier Transform Pair)，其中 $j = \sqrt{-1}$，f(x)為連續可積分函數，F(u)為可積分函數。就影像處理而言，f(x)經常為實數函數，F(u)則一般為複數函數：

$$F(u) = R(u) + j\,I(u)$$

依複數性質，一些常用名詞定義如下：

$$F(u) = |F(u)|\,e^{j\phi(u)}$$

傅利葉頻譜(Fourier Spectrum)為：

$$|F(u)| = [R^2(u) + I^2(u)]^{1/2}$$

相角(Phase Angle)為：

$$\phi(u) = \tan^{-1}[I(u)/R(u)]$$

能量頻譜(Power Spectrum)為：

$$P(u) = |F(u)|^2 = R^2(u) + I^2(u)$$

尤拉式(Euler Formula)為：

$$\exp[-j\,2\pi ux] = \cos(2\pi ux) - j\,\sin(2\pi ux)$$

二、範例：

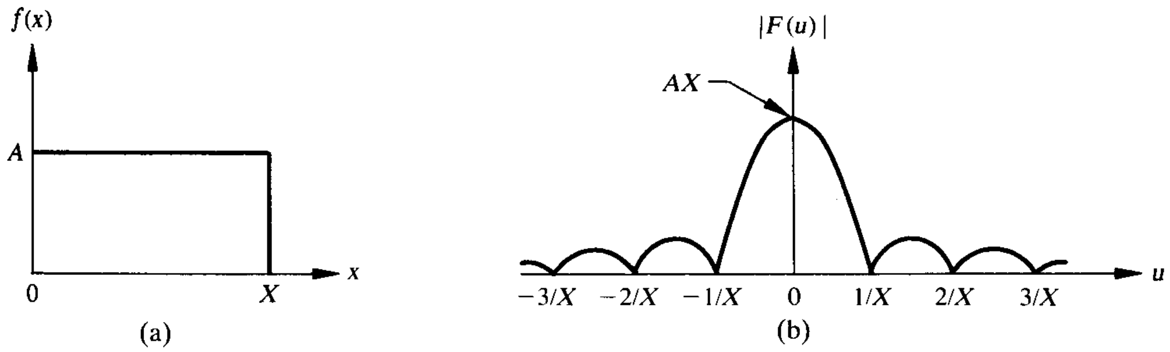


Figure 3.1 *A simple function and its Fourier spectrum.*

$$
\begin{aligned}
F(u) &= \int_{-\infty}^{\infty} f(x)\exp[-j2\pi ux]dx \\
&= \int_{0}^{X} A\exp[-j2\pi ux]dx \\
&= \frac{-A}{j2\pi u}[e^{-j2\pi ux}]_0^X = \frac{-A}{j2\pi u}[e^{-j2\pi uX}-1] \\
&= \frac{A}{j2\pi u}[e^{j\pi uX}-e^{-j\pi uX}]e^{-j\pi uX} \\
&= \frac{A}{\pi u}\sin(\pi uX)e^{-j\pi uX}
\end{aligned}
$$

上式為一個複數函數，其傅利葉頻譜為：

$$
\begin{aligned}
|F(u)| &= \left|\frac{A}{\pi u}\right| |\sin(\pi uX)| \left|e^{-j\pi uX}\right| \\
&= AX\left|\frac{\sin(\pi uX)}{(\pi uX)}\right|
\end{aligned}
$$

常用函數之傅利葉轉換		
Function	f(x)	F(u)
Gaussian	$e^{-\pi x^2}$	$e^{-\pi u^2}$
Rectangular pulse	$\prod(x)$	$\frac{\sin(\pi u)}{\pi u}$
Triangular pulse	$\Lambda(x)$	$\frac{\sin^2(\pi u)}{(\pi u)^2}$
Impulse	$\delta(x)$	1
Unit step	$u(x)$	$\frac{1}{2}[\delta(u) - \frac{j}{\pi u}]$
Cosine	$\cos(2\pi f x)$	$\frac{1}{2}[\delta(u+f) + \delta(u-f)]$
Sine	$\sin(2\pi f x)$	$j\frac{1}{2}[\delta(u+f) - \delta(u-f)]$
Complex exponential	$e^{j2\pi f x}$	$\delta(u-f)$

三、二維傅利葉轉換：

１．傅利葉轉換偶：

$$\Im\{f(x,y)\} = F(u,v) = \int\int_{-\infty}^{\infty} f(x,y)\exp[-j2\pi(ux+vy)]\mathrm{dxdy} \quad (3.1\text{-}9)$$

$$\Im^{-1}\{F(u,v)\} = f(x,y) = \int\int_{-\infty}^{\infty} F(u,v)\exp[j2\pi(ux+vy)]\mathrm{dudv} \quad (3.1\text{-}10)$$

２．傅利葉頻譜，相角與能量頻譜：

$$|F(u,v)| = [R^2(u,v) + I^2(u,v)]^{1/2} \quad (3.1\text{-}11)$$

$$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right] \quad (3.1\text{-}12)$$

$$P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v) \quad (3.1\text{-}13)$$

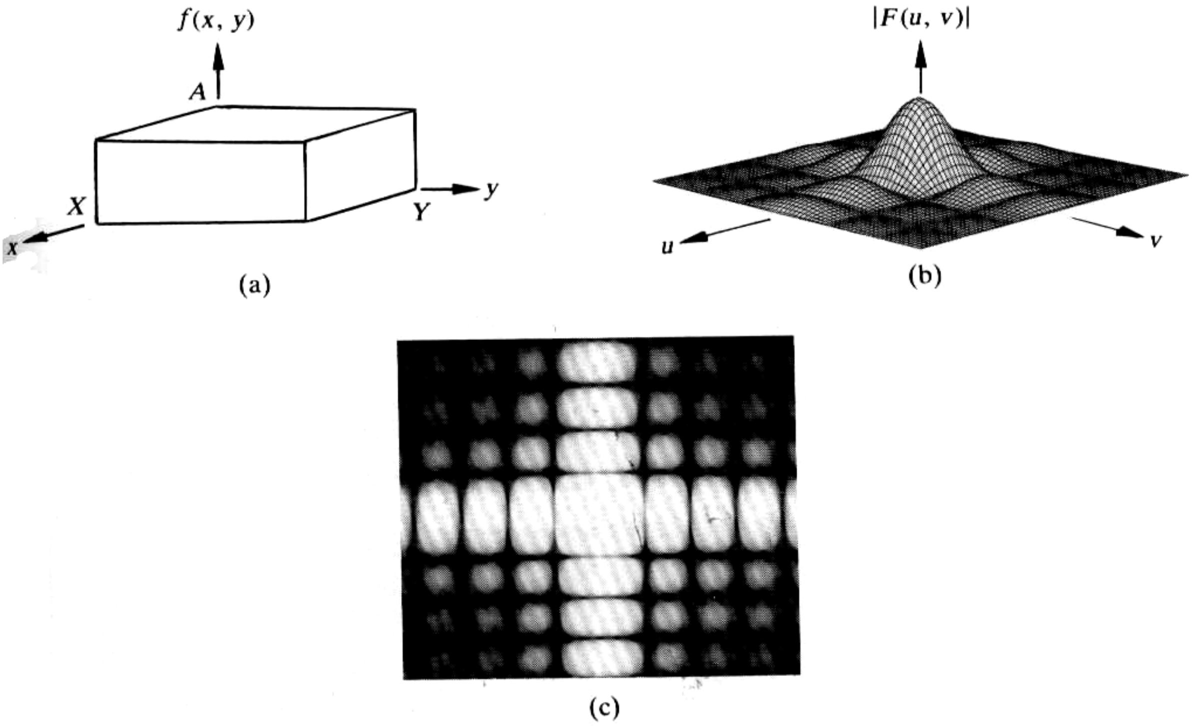


Figure 3.2 (a) A 2-D function; (b) its Fourier spectrum; and (c) the spectrum displayed as an intensity function.

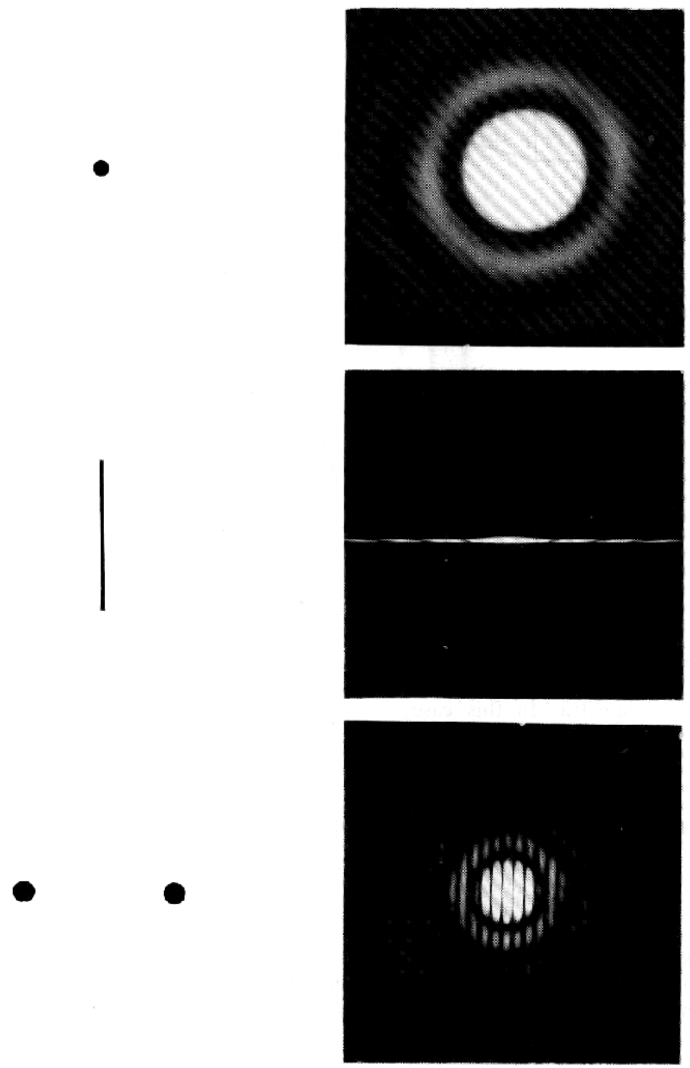


Figure 3.3 Some 2-D functions and their Fourier spectra.

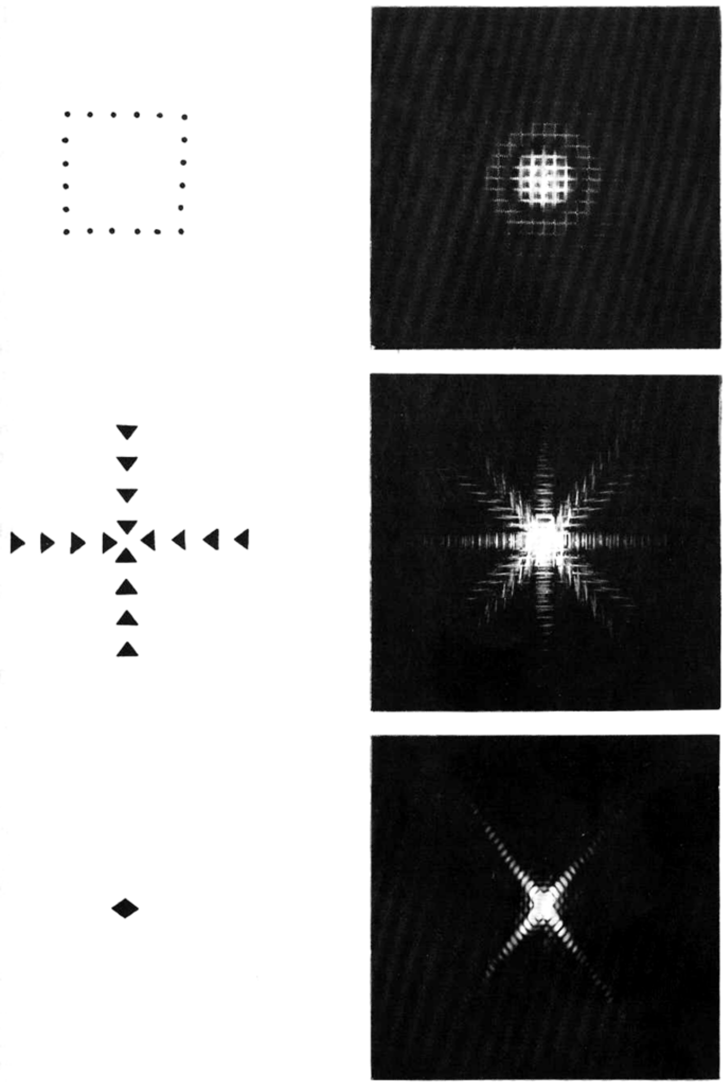


Figure 3.3 (Continued)

貳、離散傅利葉轉換(The Discrete Fourier Transform)

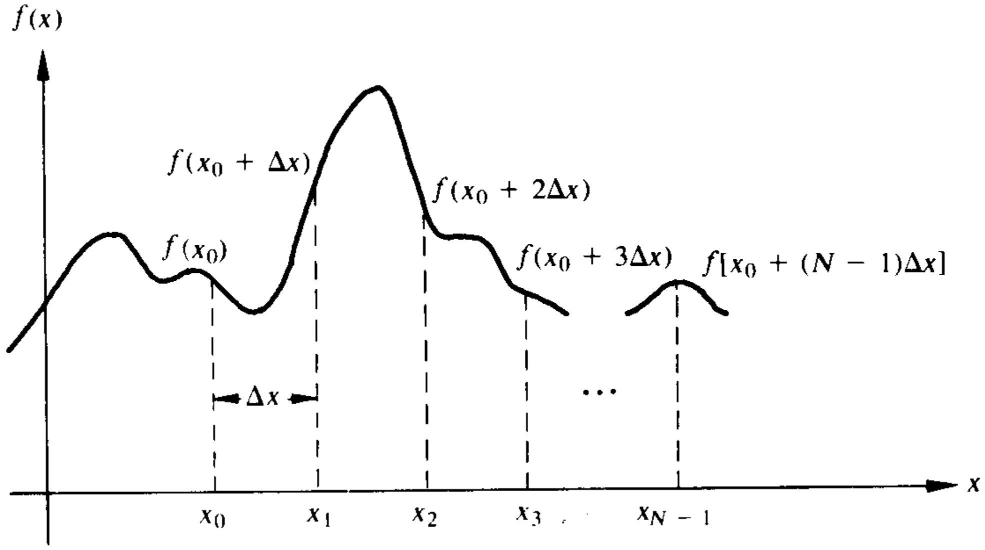


Figure 3.4 Sampling a continuous function.

由一連續函數以△x 間隔取樣 N 次而得到之序列

{ $f(x_0)$, $f(x_0+\triangle x)$, $f(x_0+2\triangle x)$, . . . , $f(x_0+[n-1]\triangle x)$ } ，

可以下式表之：

$$f(x) = f(x_0 + x\Delta x) \qquad (3.2\text{-}1)$$

離散函數 f(x)之傅利葉轉換偶則為：

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\exp[-j2\pi ux/N] \qquad u = 0,1,2,...,N-1 \qquad (3.2\text{-}2)$$

$$f(x) = \sum_{u=0}^{N-1} F(u)\exp[j2\pi ux/N] \qquad x = 0,1,2,...,N-1 \qquad (3.2\text{-}3)$$

△x 和△u 之關係如下：

$$\Delta u = \frac{1}{N\Delta x} \qquad (3.2\text{-}4)$$

二維離散傅利葉轉換：

$$F(u,v)=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi(ux/M+vy/N)]$$

$$f(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}F(u,v)\exp[j2\pi(ux/M+vy/N)]$$

頻率領域(Frequency Domain)與空間領域(Spatial Domain)變數間之關係爲：

$$\Delta u=\frac{1}{M\Delta x} \qquad \Delta v=\frac{1}{N\Delta y}$$

多數影像之 N = M ，則(3.2-5)及(3.2-6)可化爲：

$$F(u,v)=\frac{1}{N}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi(ux/M+vy/N)]$$

$$f(x,y)=\frac{1}{N}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}F(u,v)\exp[j2\pi(ux/M+vy/N)]$$

*(3.2-9)及(3.2-10)均已乘上N)

離散傅利葉轉換之傅利葉頻譜，相角，能量頻譜（一維或二維）均和連續傅利葉轉換者類同，唯一的差別爲式中之變數由連續變數改爲離散變數。

●離散傅利葉轉換之範例：

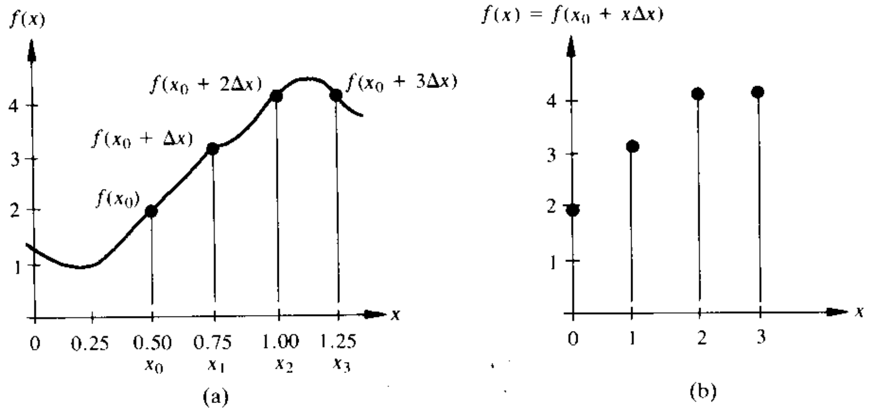


Figure 3.5 *A simple function and samples in the x domain. In (a) x is a continuous variable; in (b) x is discrete.*

$$
\begin{aligned}
F(0) &= \frac{1}{4}\sum_{x=0}^{3} f(x)\exp[0] \\
&= \frac{1}{4}[f(0)+f(1)+f(2)+f(3)] \\
&= \frac{1}{4}[2+3+4+4] \\
&= 3.25
\end{aligned}
$$

$$
\begin{aligned}
F(1) &= \frac{1}{4}\sum_{x=0}^{3} f(x)\exp[-j2\pi x/4] \\
&= \frac{1}{4}[2e^{0}+3e^{-j\pi/2}+4e^{-j\pi}+4e^{-j3\pi/2}] \\
&= \frac{1}{4}[-2+j]
\end{aligned}
$$

同理

$$
F(2) = -\frac{1}{4}[1+j0]
$$

$$
F(3) = -\frac{1}{4}[2+j]
$$

其傅利葉頻譜則分別爲：

$$
\begin{aligned}
|F(0)| &= 3.25 \\
|F(1)| &= [(2/4)^2+(1/4)^2]^{1/2} = \sqrt{5}/4 \\
|F(2)| &= [(1/4)^2+(0/4)^2]^{1/2} = 1/4 \\
|F(3)| &= [(2/4)^2+(1/4)^2]^{1/2} = \sqrt{5}/4
\end{aligned}
$$

參、二維傅利葉轉換之性質

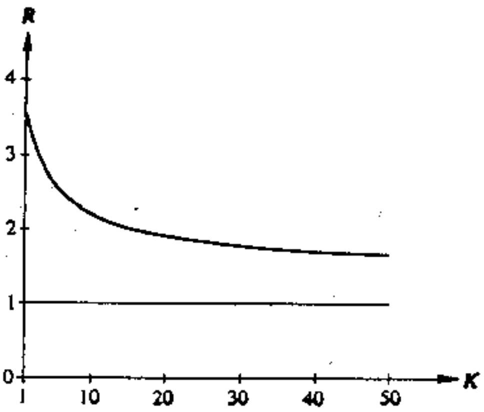
一、頻譜之顯示：

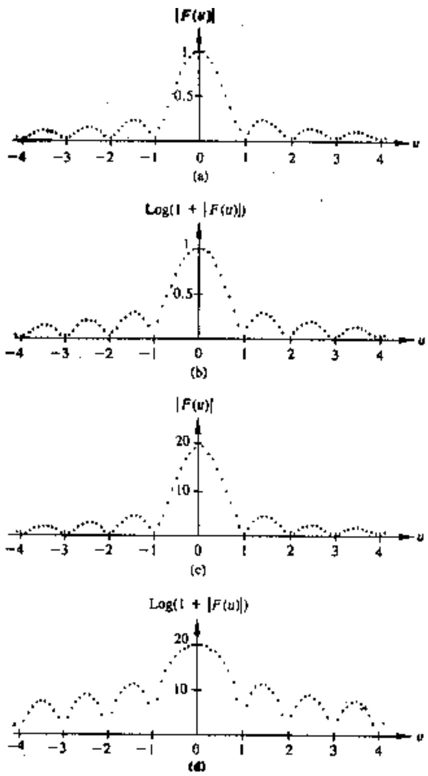
１．對數轉換：

●以對數轉換來突顯頻譜變化之對比。

$$D(u,v) = c\ \log(\ 1 + |F(u,v)|\ )$$

或 $$D(u,v) = c\ \log(\ 1 + K|F(u,v)|\ )$$





２．比值Ｒ與Ｋ之關係：

$$R = \frac{\log(1 + KF_{\max})}{\log(1 + KF_{\min})}$$

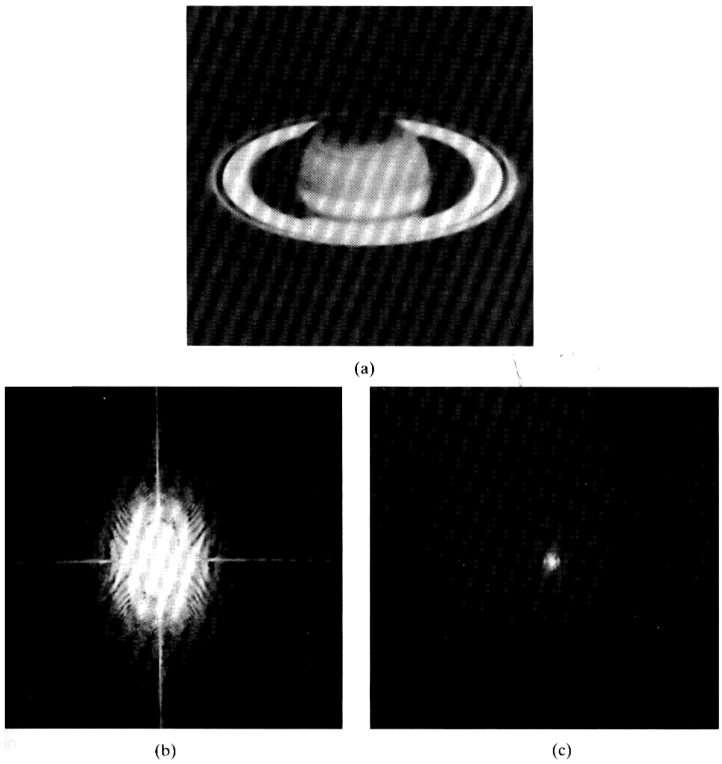
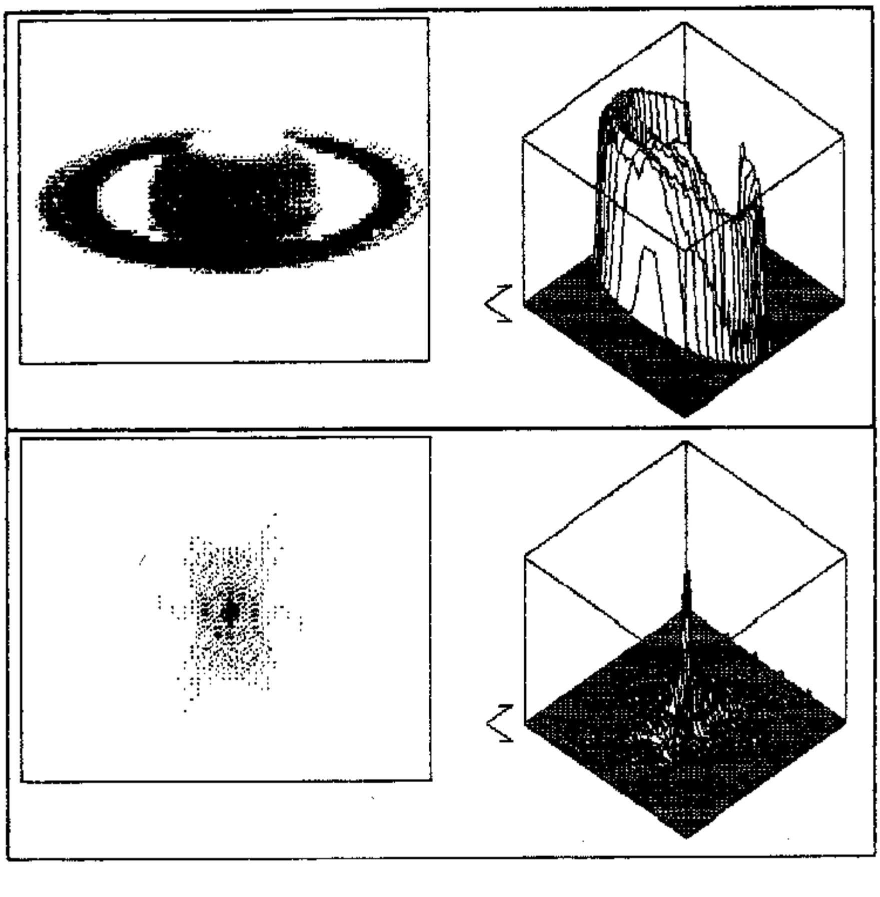


Figure 3.6 *(a) A picture of the planet Saturn; (b) display of $|F(u, v)|$; (c) display of $\log[1 + |F(u, v)|]$ scaled to 8 bits (i.e., 0 to 255 gray levels).*



二、分離性(Separability)

$$F(u,v)=\frac{1}{N}\sum_{x=0}^{M-1}\exp[-j2\pi ux/N]\sum_{y=0}^{N-1}f(x,y)\exp[-j2\pi vy/N]$$

$$\text{for u, v} = 0,1,\ldots, \text{N - 1, and}$$

$$f(u,v)=\frac{1}{N}\sum_{x=0}^{M-1}\exp[j2\pi ux/N]\sum_{y=0}^{N-1}F(u,v)\exp[j2\pi vy/N]$$

$$\text{for x, y} = 0,1,\ldots, \text{N - 1}$$

亦即二維傅利葉轉換可以由連續兩個一維傅利葉轉換求得，逆轉換亦然，如下圖所示：

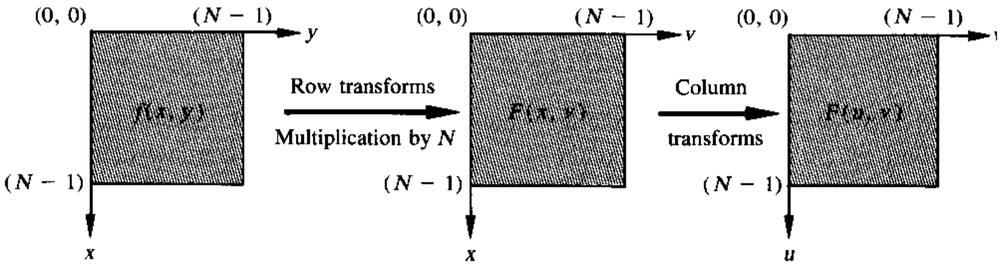


Figure 3.7 *Computation of the 2-D Fourier transform as a series of 1-D transforms.*

三、平移(Translation)

$f(x,y)\ \exp[\ j2\pi(u_0x + v_0y)/N\ ] \quad \leftrightarrow \quad F(\ u-u_0,\ \ v-v_0\ )$

$f(x - x_0,\ \ y - y_0) \quad \leftrightarrow \quad F(u,v)\ \exp[\ -j2\pi(ux_0+vy_0)/N\ ]$

１．將傅利葉轉換後之原點移至 NxN 頻率方陣之中心點的方法：

令 $u_0 = v_0 = N/2$

$f(x,y)(-1)^{x+y} \leftrightarrow F(u-N/2,\ \ v-N/2)$

２．平移對傅利葉頻譜沒有影響：

$|F(u,v)\ \exp[-\ j2\pi(ux_0+vy_0)/N]| \ = \ |F(u,v)|$

四、週期性與共軛對稱性(Periodicity and Conjugate Symmetry)

１．週期性：

$$F(u,v) = F(u+N,v) = F(u,v+N) = F(u+N,v+N)$$

離散傅利葉轉換與其反轉換之週期為 N，這個性質意涵了只需要確定一個週期之 F(u,v)便可以完全確定頻率領域中之 F(u,v)，也因此可以確定空間領域之 f(x,y)。

２．共軛對稱性：

$$F(u,v) = F^*(-u,-v)$$

$$|F(u,v)| = |F(-u,-v)|$$

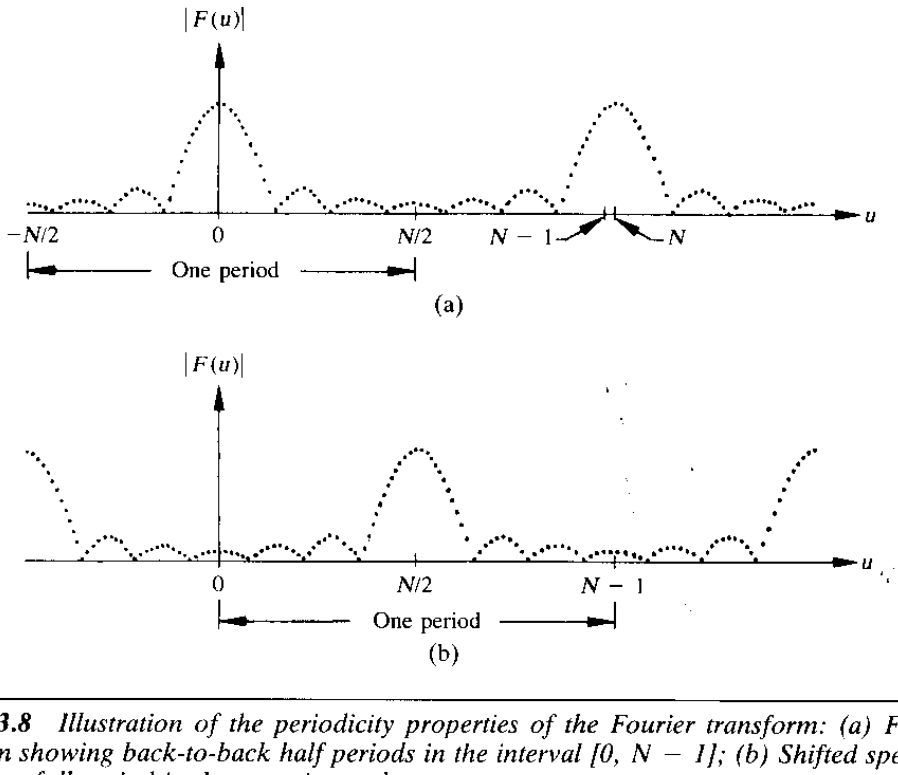


Figure 3.8 Illustration of the periodicity properties of the Fourier transform: (a) Fourier spectrum showing back-to-back half periods in the interval [0, N − 1]; (b) Shifted spectrum showing a full period in the same interval.

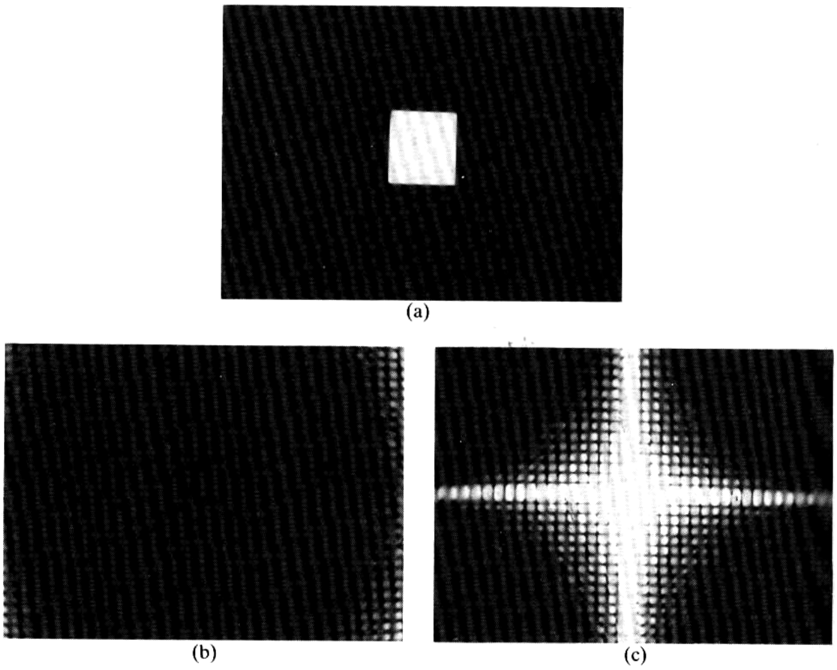


Figure 3.9 *(a) A simple image; (b) Fourier spectrum without shifting; (c) Fourier spectrum shifted to the center of the frequency square.*

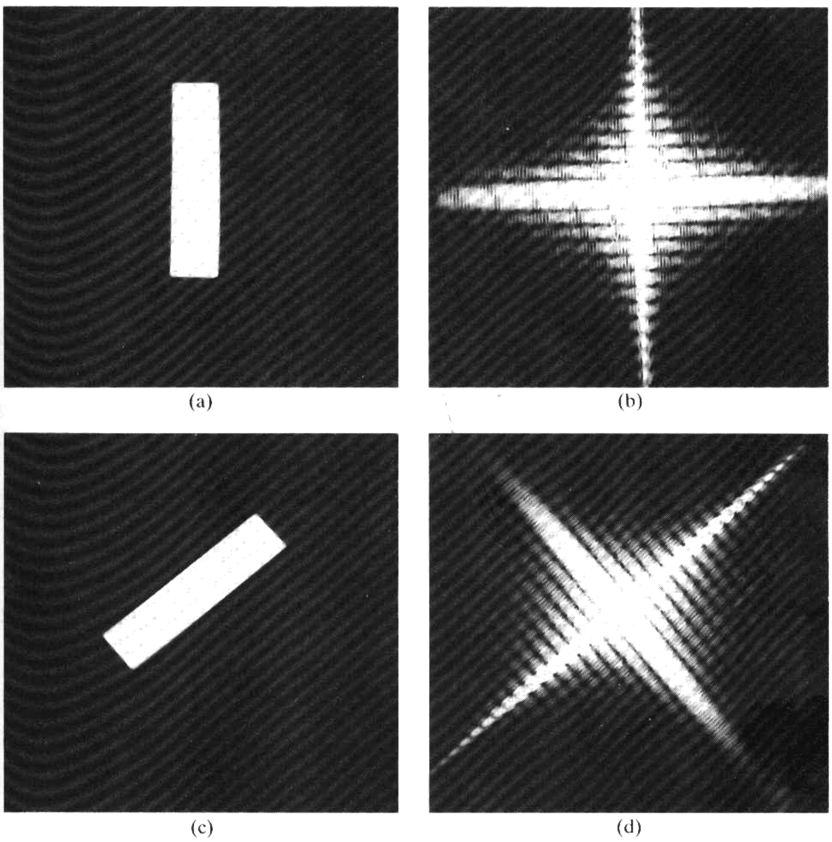


Figure 3.10 *Rotational properties of the Fourier transform: (a) a simple image; (b) spectrum; (c) rotated image; (d) resulting spectrum.*

３．旋轉(Rotation)

若 f(x, y)旋轉θ_0角度，則 F(u, v)亦旋轉相同角度θ_0，反之亦然。

$$f(r, \theta+\theta_0) \leftrightarrow F(\omega, \phi+\theta_0)$$

４．分配律與更改比例(Distributivity and Scaling)

(1).分配律：

$$\Im\{f_1(x,y)+f_2(x,y)\}=\Im\{f_1(x,y)\}+\Im\{f_2(x,y)\}$$
$$\Im\{f_1(x,y)\cdot f_2(x,y)\}\neq\Im\{f_1(x,y)\}\cdot\Im\{f_2(x,y)\}$$

(2).更改比例：

$$a\ f_1(x,y) \quad \leftrightarrow \quad a\,F(u,v)$$
$$f(ax,by) \quad \leftrightarrow \quad \frac{1}{|ab|}\,F(u/a,v/b)$$

５．平均值(Average Value)

$$\bar{f}(x,y)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)=\frac{1}{N}F(0,0)$$

６．Laplacian：

$$\nabla^2 f(x,y)=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}$$
$$\Im\{\nabla^2 f(x,y)\} \ \Leftrightarrow \ -(2\pi)^2(u^2+v^2)F(u,v)$$

７．捲積(Convolution)

(1).定義：

$$f(x)*g(x)=\int_{-\infty}^{\infty} f(\alpha)g(x-\alpha)d\alpha$$

α 爲虛擬變數(Dummy variable)

(2).圖例一：

$$f(x)*g(x)=\begin{cases} x/2 & 0\le \mathrm{x} \le 1 \\ 1-x/2 & 1< \mathrm{x} \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

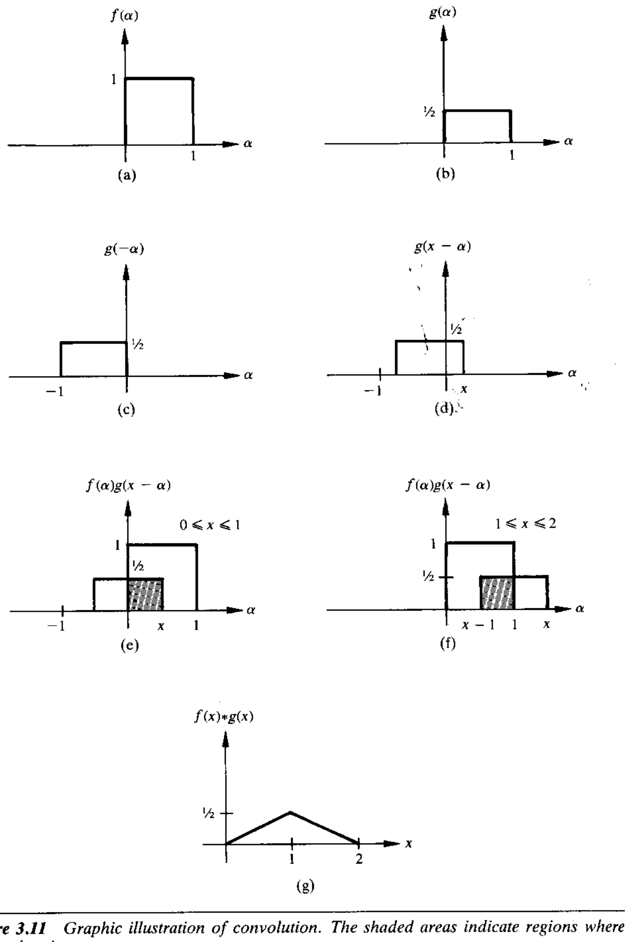


Figure 3.11 Graphic illustration of convolution. The shaded areas indicate regions where the product is not zero.

(3).圖例二：(脈衝函數 Impulse Function)

脈衝函數之定義如下：

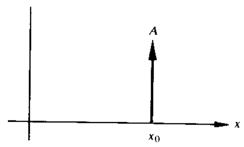


Figure 3.12 Graphic representation of $A\delta(x-x_0)$.

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0)$$

$$\int_{-\infty}^{\infty} \delta(x-x_0)dx = \int_{x_0^-}^{x_0^+} \delta(x-x_0)dx = 1$$

令　g(x) = δ (x+T) + δ (x) + δ (x-T)

$$g(x) = \delta(x+T) + \delta(x) + \delta(x-T)$$

$$f(x) = \begin{cases} A & \text{when } 0 \le x \le a \\ 0 & \text{elsewhere} \end{cases}$$

則　f(x)∗g(x) 爲下圖所示：

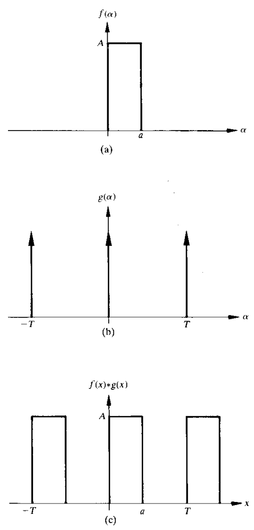


Figure 3.13 Convolution involving impulse functions.

(4).捲積定理(Convolution Theorem)：

f(x)g(x) ←→ F(u)∗G(u)

f(x)∗g(x) ←→ F(u)G(u)

(5).離散函數之捲積：

連續函數 f(x)與 g(x)經取樣成兩序列後：

{ f(0), f(1), f(2), ... , f(A-1) }

{ g(0), g(1), g(2), ... , g(B-1) }

進行捲積時為避免折疊錯誤(Wraparound Error)，

其條件為：

M ≧ A + B - 1

而 M 為離散函數 f(x)及 g(x)之週期

$$f_e(x)=\begin{cases} f(x) & 0\le x\le A-1 \\ 0 & A\le x\le M-1 \end{cases} \qquad \mathrm{g}_e(x)=\begin{cases} g(x) & 0\le x\le B-1 \\ 0 & B\le x\le M-1 \end{cases}$$

$$f_e(x)*g_e(x)=\frac{1}{M}\sum_{m=0}^{M-1} f_e(m)g_e(x-m) \qquad \text{for x} = 0,1,2,...,\text{M-1}$$

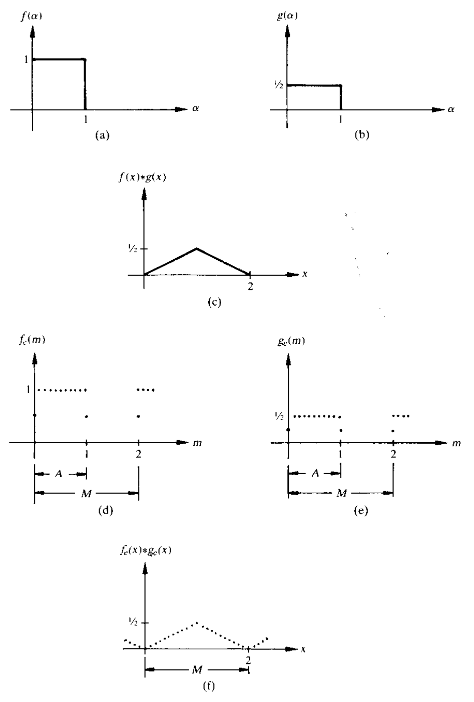


Figure 3.14 *Comparison between continuous and discrete convolution.*

(6).二維捲積(Two-dimensional Convolution)：

$$f(x,y)*g(x,y)=\int_{-\infty}^{\infty}\int f(\alpha,\beta)g(x-\alpha,y-\beta)d\alpha d\beta$$

$$f(x,y)*g(x,y) \quad \Leftrightarrow \quad F(u,v)G(u,v)$$
$$f(x,y)g(x,y) \quad \Leftrightarrow \quad F(u,v)*G(u,v)$$
$$M \geq A+C-1 \qquad N \geq B+D-1$$

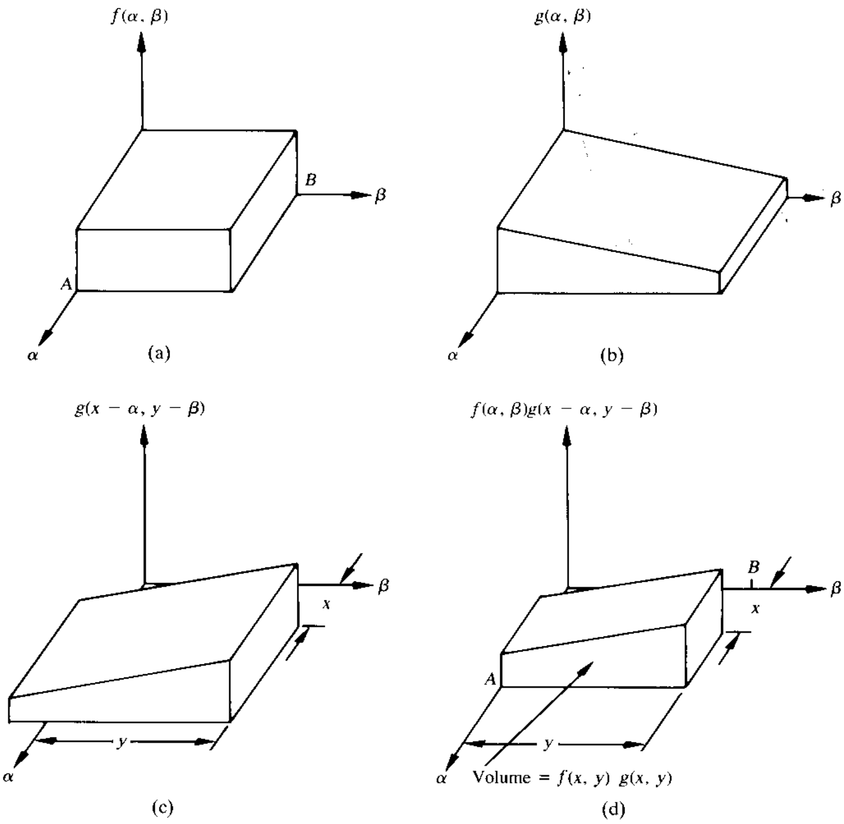


Figure 3.15 *Illustration of the folding, displacement, and multiplication steps needed to perform two-dimensional convolution.*

(7).離散二維捲積：

$$f_e(x,y)*g_e(x,y)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f_e(m,n)g_e(x-m,y-n)$$
$$\text{for } x=0,1,2,\ldots,M-1 \quad \text{and } y=0,1,2,\ldots,N-1$$

８ ．相關(Correlation)：

(1).定義：

$$f(x)\circ g(x)=\int_{-\infty}^{\infty} f^{*}(\alpha)g(x+\alpha)d\alpha$$

$f^*(x)$為 f(x)之共軛複數。

相關之觀念由一維連續函數推導至離散函數及至二維函數之方法均與前述之捲積類同。

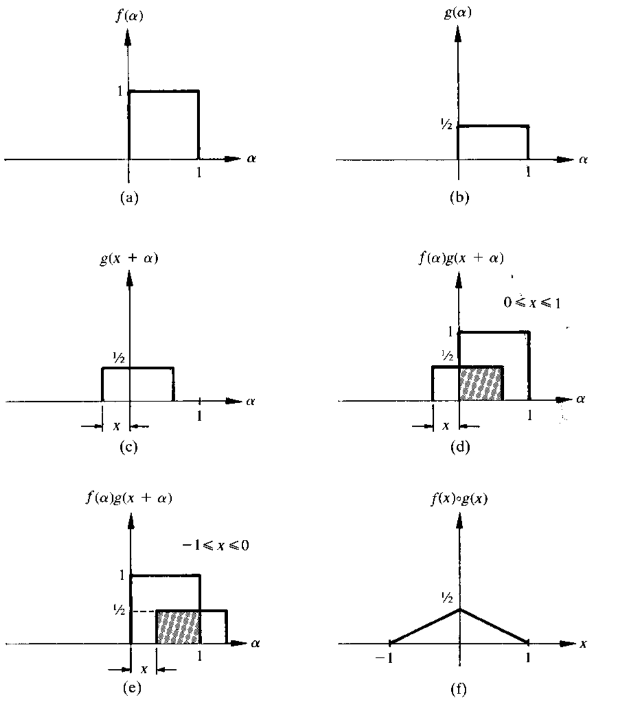


Figure 3.16 *Graphic illustration of correlation. The shaded areas indicate regions where the product is not zero.*

(2).相關定理(Correlation Theorem)：

$f(x,y)\circ g(x,y) \quad \leftarrow\rightarrow \quad F^*(u,v)G(u,v)$

$f^*(x,y)g(x,y) \quad \leftarrow\rightarrow \quad F(u,v)\circ G(u,v)$

(3).模型比對(Prototype Matching)

９．取樣(Sampling)

(1).一維函數之取樣：

A.　有限頻帶函數(Band-limited Function)

B.Whittaker-Shannon 取樣定理：

有限頻帶函數之取樣若能符合下式之條件，則函數之表現可以在取樣後完全復原：

$$\triangle x \leqq 1/(2W)$$

C.假合(Aliasing)(虛假)

D.視窗(Window)

E.有限區間取樣

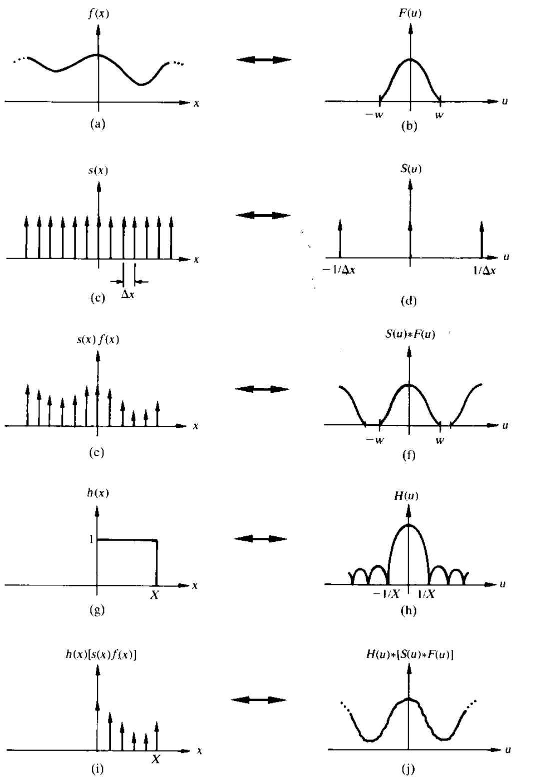


Figure 3.18 Graphic illustration of finite-sampling concepts.

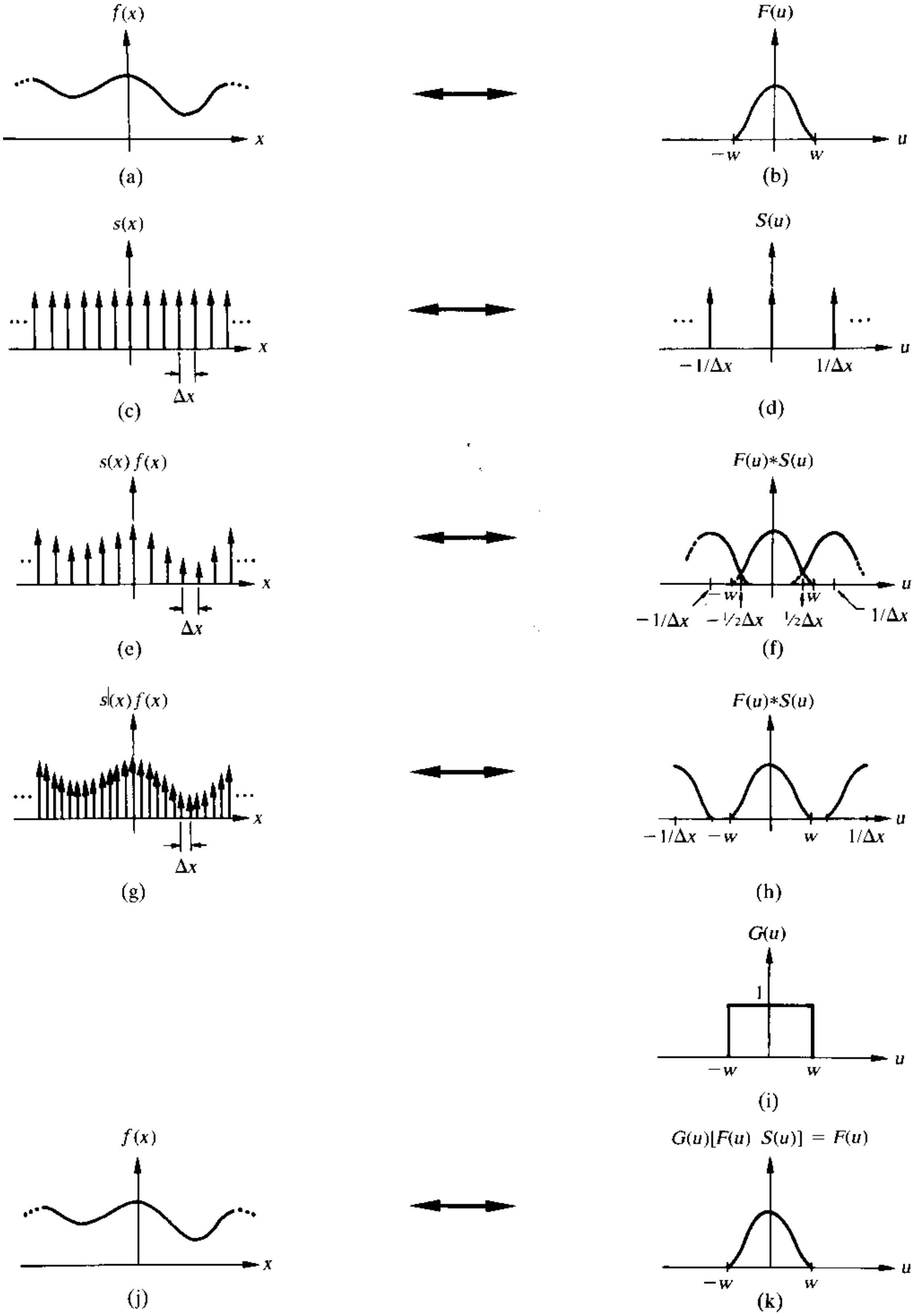


Figure 3.17 *Graphic development of sampling concepts.*

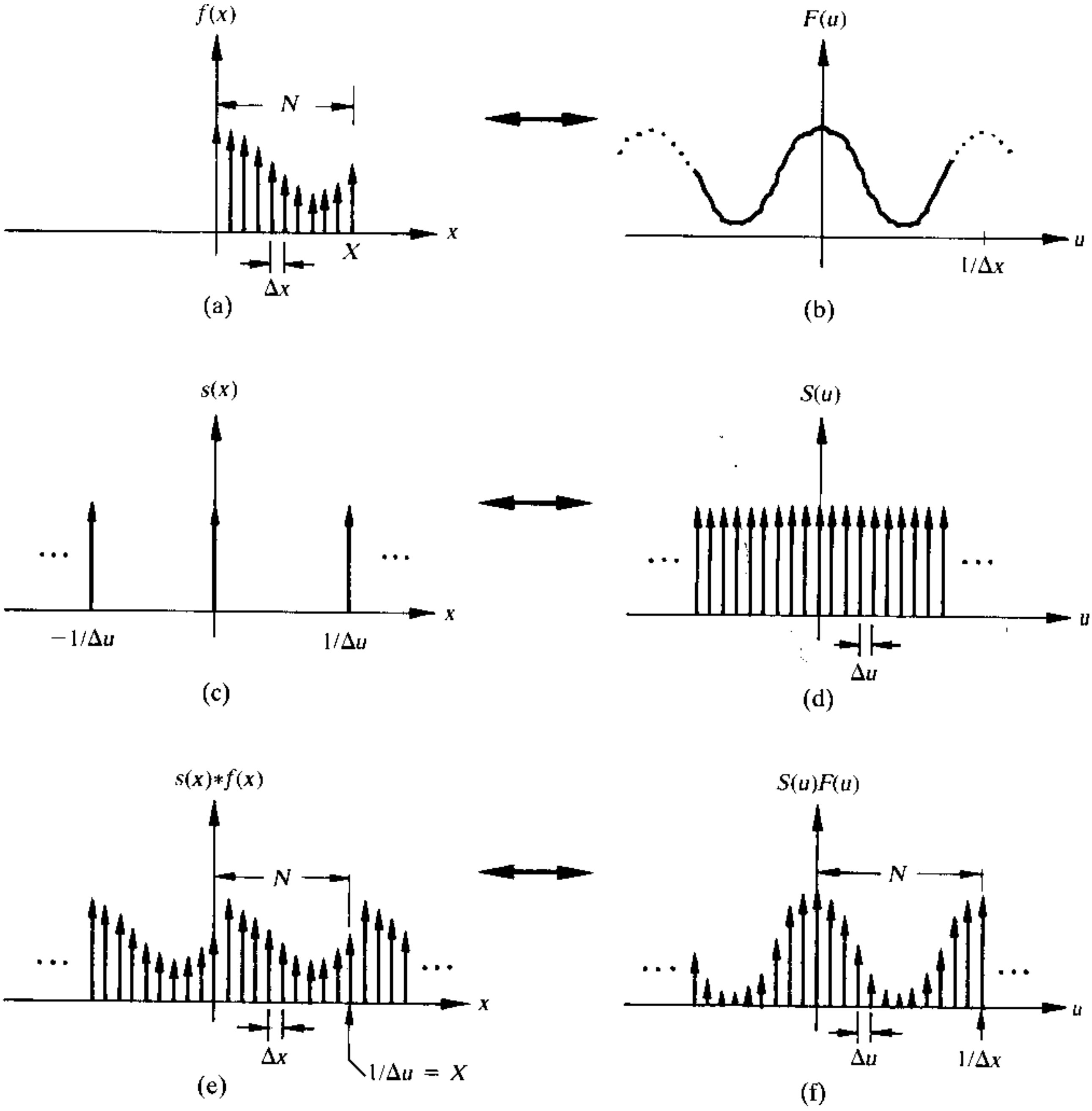


Figure 3.19 *Graphic illustration of the discrete Fourier transform.*

(2).二維函數之取樣

A. 二維脈衝函數：

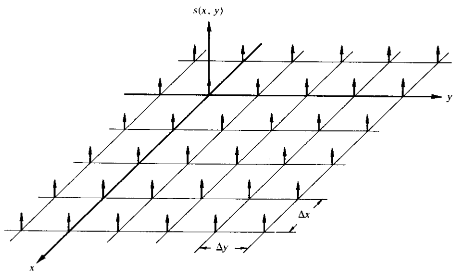


Figure 3.20 A 2-D sampling function.

$$\int_{-\infty}^{\infty}\int f(x,y)\delta(x-x_0,y-y_0)dxdy = f(x_0,y_0)$$

B.取樣：

$s(x,y)f(x,y) \quad \leftrightarrow \quad S(u,v)*F(u,v)$

$$G(u,v)=\begin{cases}1 & \text{(u, v) inside one of the rectangle} \\ & \text{enclosing R} \\ 0 & \text{elsewhere}\end{cases}$$

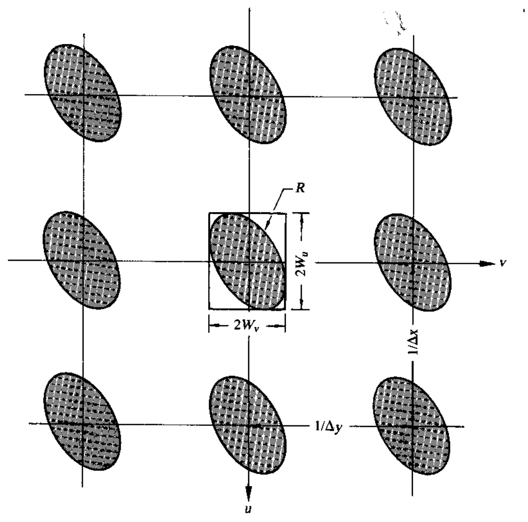


Figure 3.21 Frequency domain representation of a sampled 2-D, band-limited function.

$\triangle x \leqq 1/(2W_u)$ $\qquad$ $\triangle y \leqq 1/(2W_v)$

肆、快速傅利葉轉換(FFT)

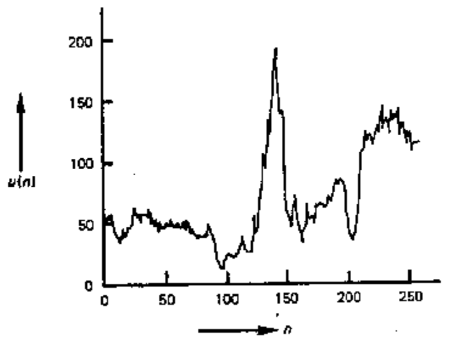


Figure 5.4 A 256-sample scan line of an image.

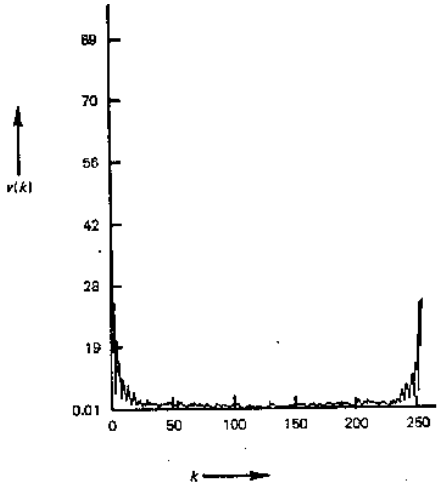


Figure 5.5 Unitary discrete Fourier transform of Fig. 5.4.

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\cdot \exp[-j2\pi ux/N$$

◎在上式中的乘法與加法次數可以參考下表

Table 3.1 A Comparison of N^2 versus $N\ \mathrm{Log}_2 N$ for Various Values of N

N	N^2 (*Direct* FT)	$N \log_2 N$ (FFT)	*Computational Advantage* ($N/\log_2 N$)
2	4	2	2.00
4	16	8	2.00
8	64	24	2.67
16	256	64	4.00
32	1,024	160	6.40
64	4,096	384	10.67
128	16,384	896	18.29
256	65,536	2,048	32.00
512	262,144	4,608	56.89
1024	1,048,576	10,240	102.40
2048	4,194,304	22,528	186.18
4096	16,777,216	49,152	341.33
8192	67,108,864	106,496	630.15

一、FFT 演算法(FFT Algorithm)

１．Kernel Expression：

$$F(u)=\frac{1}{N}\sum_{x=0}^{N-1} f(x)\cdot W_N^{ux}$$

其中　$W_N = \exp[-j2\pi/N]$,

$N = 2^n$

２．Sucessive Doubling Method：

令　$N = 2M$,

則由 Kernel Expression 可得：

$$F(u)=\frac{1}{2M}\sum_{x=0}^{2M-1} f(x)\cdot W_{2M}^{ux}$$

$$=\frac{1}{2}\{\frac{1}{M}\sum_{x=0}^{M-1} f(2x)\cdot W_{2M}^{u(2x)}+\frac{1}{M}\sum_{x=0}^{M-1} f(2x+1)\cdot W_{2M}^{u(2x+1)}\}$$

$$F(u)=\frac{1}{2}\{\frac{1}{M}\sum_{x=0}^{M-1} f(2x)\cdot W_{M}^{ux}+\frac{1}{M}\sum_{x=0}^{M-1} f(2x+1)\cdot W_{M}^{ux}W_{2M}^{u}\}$$

let

$$F_{odd}(u)=\frac{1}{M}\sum_{x=0}^{M-1} f(2x+1)\cdot W_{M}^{ux}$$

$$F_{even}(u)=\frac{1}{M}\sum_{x=0}^{M-1} f(2x)\cdot W_{M}^{ux}$$

Then

$$F(u)=\tfrac{1}{2}\{F_{even}(u)+F_{odd}(u)W_{2M}^{u}\}$$

$$F(u+M)=\tfrac{1}{2}\{F_{even}(u)-F_{odd}(u)W_{2M}^{u}\}$$

二、運算次數(Number of Operation)

１．遞迴式(Recursive Expression)：

n = 1　m(1) = 1 multiplication and a(1) = 2 additions.

n = 2　m(2) = 2m(1) + 2 and a(2) = 2a(1) + 4.

n = 3　m(3) = 2m(2) + 4 and a(3) = 2a(2) + 8.

.　　　　　　　　　　　　　　.

.　　　　　　　　　　　　　　.

$m(n) = 2m(n-1) + 2^{n-1}$　　　$n \geqq 1$

$a(n) = 2a(n-1) + 2^{n}$　　　$n \geqq 1$

２．對數表示式：

應用歸納證法可以推導出運算次數與 n 之關係式如下：

$$
\begin{aligned}
m(n) &= \frac{1}{2} 2^n \log_2 2^n \\
&= \frac{1}{2} N \log_2 N \\
&= \frac{1}{2} Nn, \qquad \mathrm{n} \geq 1
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{a(n)} &= 2^n \log_2 2^n \\
&= N \log_2 N \\
&= Nn, \qquad \mathrm{n} \geq 1
\end{aligned}
$$

三、快速傅利葉反轉換(Inverse FFT)

任何離散傅利葉轉換之運算法亦適用於其反轉換。此可由下列各式應用共軛複數之代換來說明：

傅利葉轉換偶之表示式爲

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\cdot \exp[-j2\pi ux/N$$

$$f(x) = \sum_{x=0}^{N-1} F(u)\cdot \exp[j2\pi ux/N]$$

上面第二式中 F(u)若以其共軛複數 $F^*(u)$代換，同時原式除以 N，則可得應用 FFT 運算之形式：

$$\frac{1}{N} f^*(x) = \frac{1}{N}\sum_{u=0}^{N-1} F^*(u)\cdot \exp[-j2\pi ux/N$$

故應用 FFT 進行反轉換之步驟可簡述如下：

(1).取 F(u)之共軛複數 $F^*(u)$。

(2).利用 FFT 演算法以 $F^*(u)$求得 $1/N \cdot f^*(x)$。

(3).乘 N 以求得 $f^*(x)$。

(4).取 $f^*(x)$之共軛複數以求得 f(x)。

二維轉換之原理亦同：

$$f^*(x,y) = \frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} F^*(u,v)\cdot \exp[-j2\pi(ux+vy)/N]$$

四、快速傅利葉轉換之程式：

１．輸入陣列之排序與重組原理：

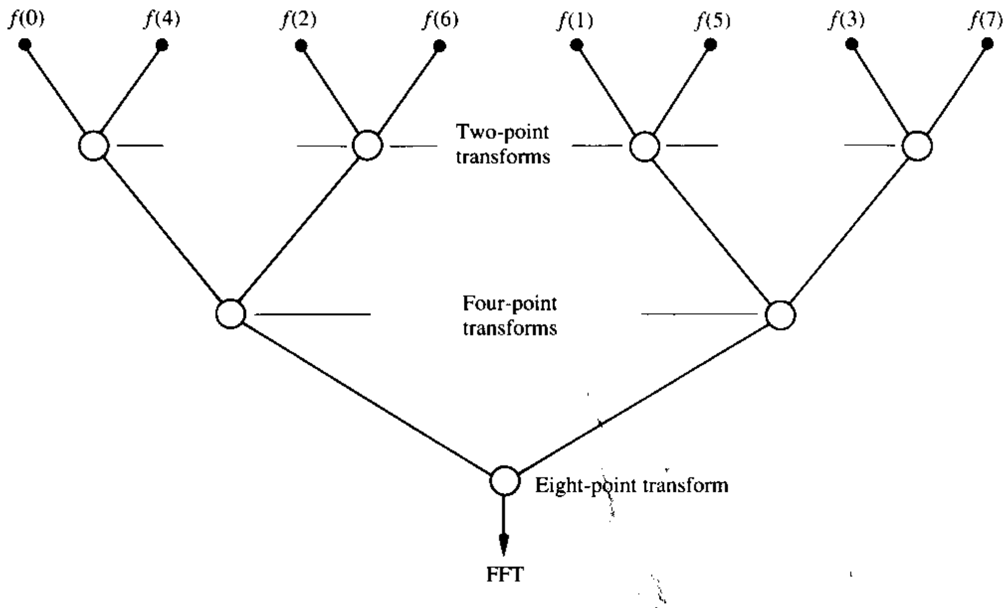


Figure 3.22 *Ordered input array and its use in the successive-doubling method.*

Table 3.2 Example of Bit Reversal and Reordering of Array for Input into FFT Algorithm

Original Argument			*Original Array*	*Bit-Reversed Argument*			*Reordered Array*
0	0	0	$f(0)$	0	0	0	$f(0)$
0	0	1	$f(1)$	1	0	0	$f(4)$
0	1	0	$f(2)$	0	1	0	$f(2)$
0	1	1	$f(3)$	1	1	0	$f(6)$
1	0	0	$f(4)$	0	0	1	$f(1)$
1	0	1	$f(5)$	1	0	1	$f(5)$
1	1	0	$f(6)$	0	1	1	$f(3)$
1	1	1	$f(7)$	1	1	1	$f(7)$

２．Successive-doubling Mothod 複數運算。

３．FFT base-three Formulation。

４．FORTRAN 程式範例。

● FFT FORTRAN 程式範例

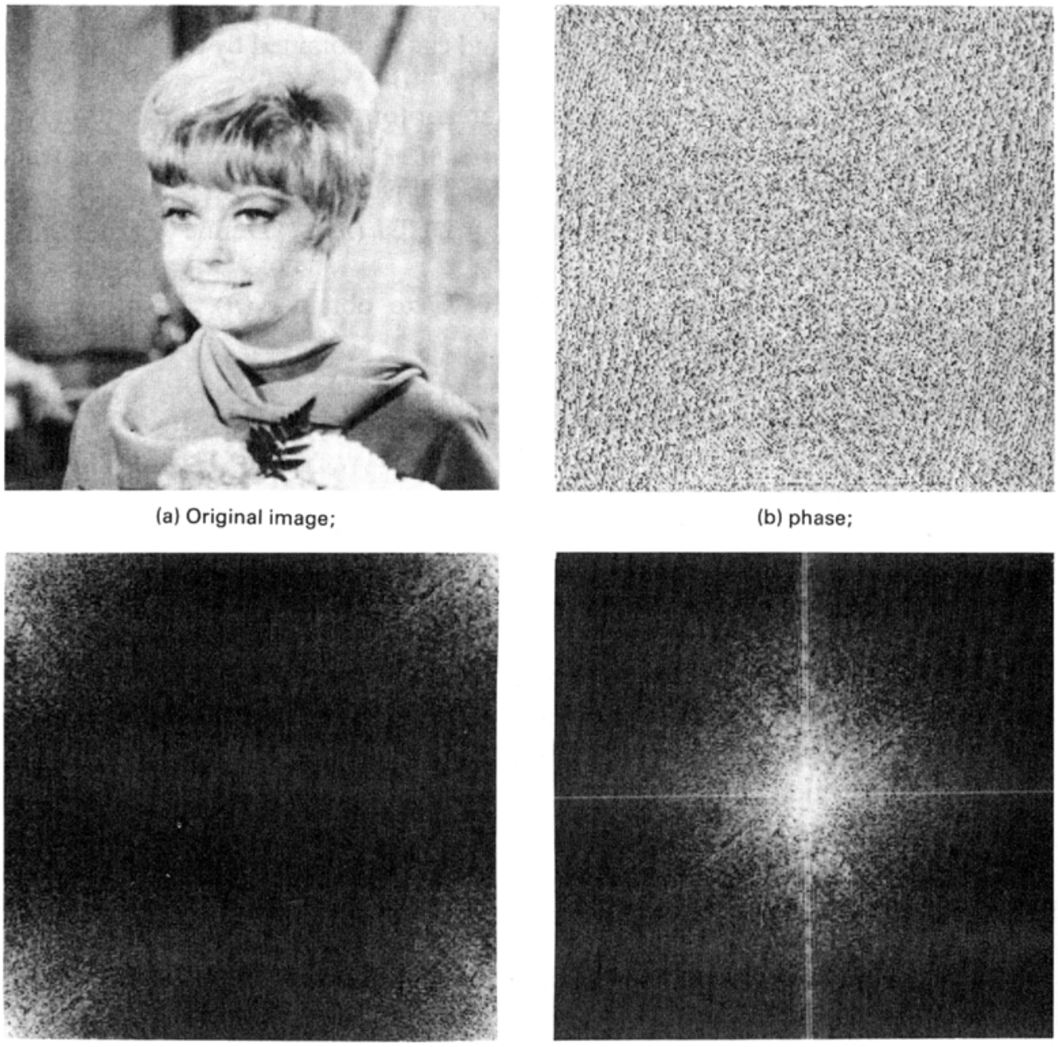
☞ 排序與重組

☞ SDM 複數運算

☞ 除以常數 N

```fortran
      SUBROUTINE FFT(F,LN)
      COMPLEX F(1024),U,W,T,CMPLX
      PI=3.141593
      N=2**LN
      NV2=N/2
      NM1=N-1
      J=1
      DO 3 I=1,NM1
          IF(I.GE.J) GO TO 1
          T=F(J)
          F(J)=F(I)
          F(I)=T
1         K=NV2
2         IF(K.GE.J) GO TO 3
          J=J-K
          K=K/2
          GO TO 2
3     J=J+K
      DO 5 L=1,LN
          LE=2**L
          LE1=LE/2
          U=(1.0,0.0)
          W=CMPLX(COS(PI/LE1),-SIN(PI/LE1))
          DO 5 J=1,LE1
              DO 4 I=J,N,LE
                  IP=I+LE1
                  T=F(IP)*U
                  F(IP)=F(I)-T
4             F(I)=F(I)+T
5     U=U*W
      DO 6 I=1,N
6     F(I)=F(I)/FLOAT(N)
      RETURN
      END
```

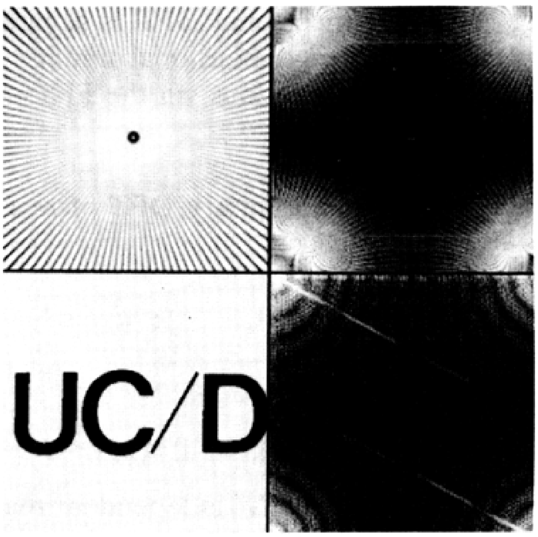
***Figure 3.23** A FORTRAN implementation of the successive-doubling FFT algorithm. (Adapted from Cooley* et al. *[1969].)*



(a) Original image; (b) phase;

(c) magnitude; (d) magnitude centered.

Figure 5.6 Two-dimensional unitary DFT of a 256 × 256 image.



a	*b*
c	*d*

Figure 5.7 Unitary DFT of images

(a) Resolution chart;
(b) its DFT;
(c) binary image;
(d) its DFT. The two parallel lines are due to the '/' sign in the binary image.

伍、其他可分性轉換

一、一般特性

１．Forward and Inverse Transformation Kernel：

$$T(u)=\sum_{x=0}^{N-1} f(x)\cdot g(x,u)$$

$$f(x)=\sum_{x=0}^{N-1} T(u)\cdot h(x,u)$$

二維轉換之正反轉換核心(Kernel)形式亦同。

２．可分性(Separability)

$$g(x,y,u,v) = g_1(x,u)\ g_2(y,v)$$

３．對稱性(Symmetry)

$$g(x,y,u,v) = g_1(x,u)\ g_1(y,v)$$

４．運算步驟：

$$T(x,v)=\sum_{y=0}^{N-1} f(x,y)\cdot g_2(y,v)$$

$$T(u,v)=\sum_{x=0}^{N-1} T(x,v)\cdot g_1(x,u)$$

５．矩陣表示式：

Ｔ　＝　ＡＦＡ，

Ｆ　＝　ＢＴＢ，　If　Ｂ　＝　$Ａ^{-1}$

近似式爲：　Ｆ　＝　ＢＡＦＡＢ

二、華許轉換(Walsh Transform)

1．定義：

$$g(x,u)=\frac{1}{N}\prod_{i=0}^{n-1}(-1)^{b_i(x)b_{n-1-i}(u)}$$

$$W(u)=\frac{1}{N}\sum_{x=0}^{N-1}f(x)\prod_{i=0}^{n-1}(-1)^{b_i(x)b_{n-1-i}(u)}$$

其中 $b_k(z)$是 z 以二進位表示時之第 k 個位元。

Table 3.3 Values of the 1-D Walsh Transformation Kernel for $N = 8$

u \ x	0	1	2	3	4	5	6	7
0	+	+	+	+	+	+	+	+
1	+	+	+	+	−	−	−	−
2	+	+	−	−	+	+	−	−
3	+	+	−	−	−	−	+	+
4	+	−	+	−	+	−	+	−
5	+	−	+	−	−	+	−	+
6	+	−	−	+	+	−	−	+
7	+	−	−	+	−	+	+	−

2．反轉換：

$$f(x)=\sum_{u=0}^{N-1}W(u)\prod_{i=0}^{n-1}(-1)^{b_i(x)b_{n-1-i}(u)}$$

3．二維轉換：

$$W(u,v)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)\prod_{i=0}^{n-1}(-1)^{[b_i(x)b_{n-1-i}(u)+b_i(y)b_{n-1-i}(v)]}$$

$$f(x,y)=\frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1}W(u,v)\prod_{i=0}^{n-1}(-1)^{[b_i(x)b_{n-1-i}(u)+b_i(y)b_{n-1-i}(v)]}$$

４．可分性與對稱性：

$$g(x,y,u,v)=g_1(x,u)g_1(y,v)$$
$$=h_1(x,u)h_1(y,v)$$
$$=[\frac{1}{\sqrt{N}}\prod_{i=0}^{n-1}(-1)^{b_i(x)b_{n-1-i}(u)}][\frac{1}{\sqrt{N}}\prod_{i=0}^{n-1}(-1)^{b_i(y)b_{n-1-i}(v)}]$$

５．快速演算法：

$$W(u)=\tfrac{1}{2}\{W_{even}(u)+W_{odd}(u)$$
$$W(u+M)=\tfrac{1}{2}\{W_{even}(u)-W_{odd}(u)$$

６．演算範例：

$$W(0)=\frac{1}{4}\sum_{x=0}^{3}[f(x)\prod_{i=0}^{1}(-1)^{b_i(x)b_{n-1-i}(0)}]=\frac{1}{4}[f(0)+f(1)+f(2)+f(3)]$$
$$W(1)=\frac{1}{4}\sum_{x=0}^{3}[f(x)\prod_{i=0}^{1}(-1)^{b_i(x)b_{n-1-i}(1)}]=\frac{1}{4}[f(0)+f(1)-f(2)-f(3)]$$
$$W(2)=\frac{1}{4}\sum_{x=0}^{3}[f(x)\prod_{i=0}^{1}(-1)^{b_i(x)b_{n-1-i}(2)}]=\frac{1}{4}[f(0)-f(1)+f(2)-f(3)]$$
$$W(3)=\frac{1}{4}\sum_{x=0}^{3}[f(x)\prod_{i=0}^{1}(-1)^{b_i(x)b_{n-1-i}(3)}]=\frac{1}{4}[f(0)-f(1)-f(2)+f(3)]$$

以快速演算法驗證：

$$W_{even}(0)=\tfrac{1}{2}[f(0)+f(2)] \qquad W_{odd}(0)=\tfrac{1}{2}[f(1)+f(3)]$$
$$W_{even}(1)=\tfrac{1}{2}[f(0)-f(2)] \qquad W_{odd}(1)=\tfrac{1}{2}[f(1)-f(3)]$$
$$W(0)=\frac{1}{2}[W_{even}(0)+W_{odd}(0)]=\frac{1}{4}[f(0)+f(1)+f(2)+f(3)]$$
$$W(1)=\frac{1}{2}[W_{even}(1)+W_{odd}(1)]=\frac{1}{4}[f(0)+f(1)-f(2)-f(3)]$$
$$W(2)=\frac{1}{2}[W_{even}(0)-W_{odd}(0)]=\frac{1}{4}[f(0)-f(1)+f(2)-f(3)]$$
$$W(3)=\frac{1}{2}[W_{even}(1)-W_{odd}(1)]=\frac{1}{4}[f(0)-f(1)-f(2)+f(3)]$$

● FWT FORTRAN 程式範例

```
      SUBROUTINE FWT(F,LN)
      REAL F(1024),T
      N=2**LN
      NV2=N/2
      NM1=N-1
      J=1
      DO 3 I=1,NM1
         IF(I.GE.J) GO TO 1
         T=F(J)
         F(J)=F(I)
         F(I)=T
1        K=NV2
2        IF(K.GE.J) GO TO 3
         J=J-K
         K=K/2
         GO TO 2
3     J=J+K
      DO 5 L=1,LN
         LE=2**L
         LE1=LE/2
         DO 5 J=1,LE1
            DO 4 I=J,N,LE
               IP=I+LE1
               T=F(IP)
               F(IP)=F(I)-T
4           F(I)=F(I)+T
5     CONTINUE
      DO 6 I=1,N
6     F(I)=F(I)/FLOAT(N)
      RETURN
      END
```

Figure 3.24 *Modification of the successive doubling FFT algorithm for computing the fast Walsh transform.*

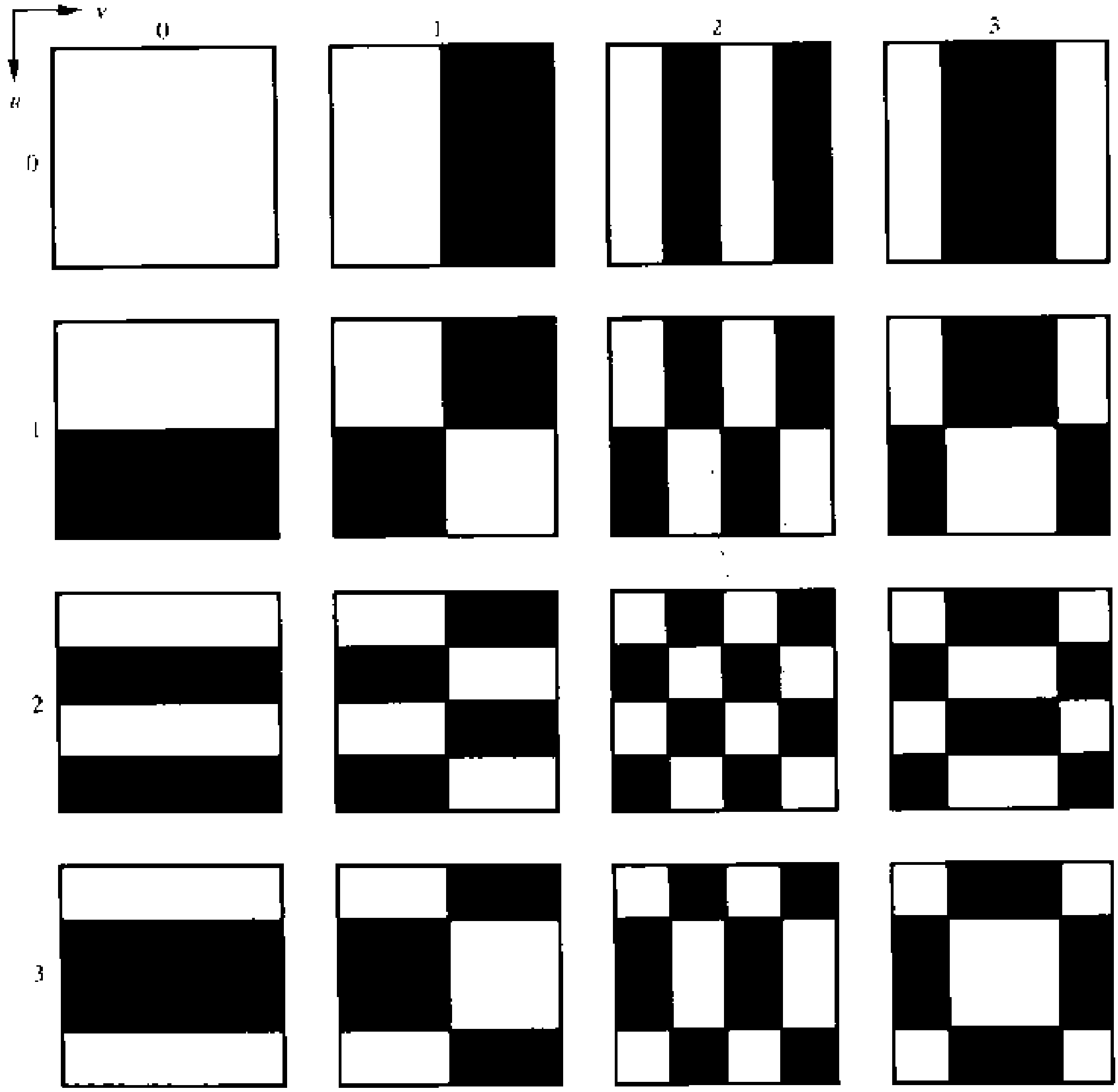


***Figure 3.25** Walsh basis functions for $N = 4$. Each block consists of 4×4 elements, corresponding to x and y varying from 0 to 3. The origin of each block is at its top left. White and black denote $+1$ and -1, respectively.*

三、哈達瑪轉換(Hadamard Transform)

１．定義：

$$g(x,u)=\frac{1}{N}(-1)^{\sum_{i=0}^{n-1}b_i(x)b_i(u)}$$

$$H(u)=\frac{1}{N}\sum_{x=0}^{N-1}f(x)(-1)^{\sum_{i=0}^{n-1}b_i(x)b_i(u)}$$

２．反轉換：

$$h(x,u)=(-1)^{\sum_{i=0}^{n-1}b_i(x)b_i(u)}=N\cdot g(x,u)$$

$$f(x)=\frac{1}{N}\sum_{u=0}^{N-1}H(u)(-1)^{\sum_{i=0}^{n-1}b_i(x)b_i(u)}$$

３．二維轉換：

$$H(u,v)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)(-1)^{\sum_{i=0}^{n-1}[b_i(x)b_i(u)+b_i(y)b_i(v)]}$$

$$f(x,y)=\frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1}H(u,v)(-1)^{\sum_{i=0}^{n-1}[b_i(x)b_i(u)+b_i(y)b_i(v)]}$$

４．對稱性與可分性：

$$\begin{aligned}g(x,y,u,v)&=g_1(x,u)g_1(y,v)\\&=h_1(x,u)h_1(y,v)\\&=[\frac{1}{\sqrt{N}}(-1)^{\sum_{i=0}^{n-1}b_i(x)b_i(u)}][\frac{1}{\sqrt{N}}(-1)^{\sum_{i=0}^{n-1}b_i(y)b_i(v)}]\end{aligned}$$

５．Hadamard 轉換與 Walsh 轉換之異同：

(1).次序性。

(2).N = 2^n 之條件。

(3).演算法之考慮(FWT & Recursive)。

Table 3.4 Values of the 1-D Hadamard Transformation Kernel for $N = 8$

u \ x	0	1	2	3	4	5	6	7
0	+	+	+	+	+	+	+	+
1	+	−	+	−	+	−	+	−
2	+	+	−	−	+	+	−	−
3	+	−	−	+	+	−	−	+
4	+	+	+	+	−	−	−	−
5	+	−	+	−	−	+	−	+
6	+	+	−	−	−	−	+	+
7	+	−	−	+	−	+	+	−

６．Hadamard 轉換之矩陣遞迴性質：

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H_{2N} = \begin{bmatrix} H_N & H_N \\ H_N & -H_N \end{bmatrix}$$

$$A = \frac{1}{\sqrt{N}} H_N$$

● 範例：

$$H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix} \quad H_8 = \begin{bmatrix} H_4 & H_4 \\ H_4 & -H_4 \end{bmatrix} = \begin{bmatrix} + & + & + & + & + & + & + & + \\ + & - & + & - & + & - & + & - \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ + & + & + & + & - & - & - & - \\ + & - & + & - & - & + & - & + \\ + & + & - & - & - & - & + & + \\ + & - & - & + & - & + & + & - \end{bmatrix}$$

７．序列(Sequency)與重組：

(1).序列爲矩陣中縱列變號之頻率。

(2).順序序列之重組(Ordered Sequency)。

$$g(x,u)=\frac{1}{N}(-1)^{\sum_{i=0}^{n-1} b_i(x)p_i(u)}$$

$$p_0(u)=b_{n-1}(u),$$

$$p_1(u)=b_{n-1}(u)+b_{n-2}(u),$$

$$\vdots$$

$$p_{n-1}(u)=b_1(u)+b_0(u).$$

Table 3.5 Values of the 1-D Ordered Hadamard Kernel for $N = 8$

$u \backslash x$	0	1	2	3	4	5	6	7
0	+	+	+	+	+	+	+	+
1	+	+	+	+	−	−	−	−
2	+	+	−	−	−	−	+	+
3	+	+	−	−	+	+	−	−
4	+	−	−	+	+	−	−	+
5	+	−	−	+	−	+	+	−
6	+	−	+	−	−	+	−	+
7	+	−	+	−	+	−	+	−

(3).順序 Hadamard 轉換：

$$H(u)=\frac{1}{N}\sum_{x=0}^{N-1} f(x)(-1)^{\sum_{i=0}^{n-1} b_i(x)p_i(u)}$$

$$f(x)=\sum_{u=0}^{N-1} H(u)(-1)^{\sum_{i=0}^{n-1} b_i(x)p_i(u)}$$

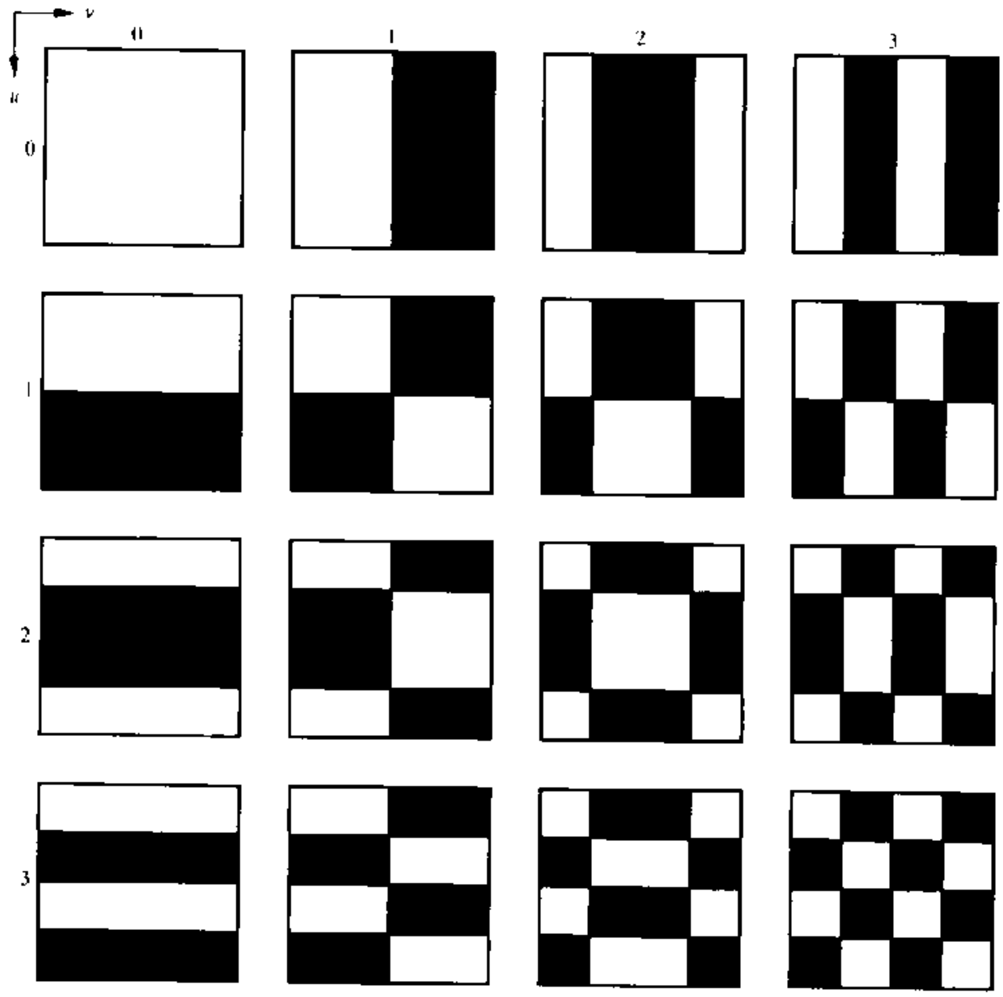


Figure 3.26 *Ordered Hadamard basis functions for N = 4. Each block consists of 4 × 4 elements, corresponding to x and y varying from 0 to 3. The origin of each block is at its top left. White and black denote +1 and −1, respectively.*

(a) (b)

Figure 3.27 *A simple image and the log magnitude of its Hadamard transform.*

四、離散餘弦轉換(Discrete Cosine Transform)

１．定義：

$$g(x,0)=\frac{1}{\sqrt{N}} \qquad g(x,u)=\sqrt{\frac{2}{N}}\cos\frac{(2\mathrm{x}+1)\mathrm{u}\pi}{2\mathrm{N}}$$

$$C(0)=\frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}f(x)$$

$$C(u)=\sqrt{\frac{2}{N}}\sum_{x=0}^{N-1}f(x)\cos\frac{(2\mathrm{x}+1)\mathrm{u}\pi}{2\mathrm{N}}$$

２．反轉換：

$$f(x)=\frac{1}{\sqrt{N}}C(0)+\sqrt{\frac{2}{N}}\sum_{u=1}^{N-1}C(u)\cos\frac{(2\mathrm{x}+1)\mathrm{u}\pi}{2\mathrm{N}}$$

３．二維轉換：

$$C(0,0)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)$$

$$C(u,v)=\frac{1}{2N^3}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)[\cos(2x+1)u\pi][\cos(2y+1)v\pi]$$

$$f(x,y)=\frac{1}{N}C(0,0)+\frac{1}{2N^3}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1}C(u,v)[\cos(2x+1)u\pi][\cos(2y+1)v\pi]$$

４．以 FFT 演算法求 DCT：

$$C(0)=\frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}f(x)$$

$$C(u)=\sqrt{\frac{2}{N}}\,\mathrm{Re}\{[\exp(\frac{-j\pi u}{2N})]\sum_{x=0}^{2N-1}f(x)\exp(\frac{-j\pi ux}{N})\}$$

$$\text{where u}=1,2,\ldots,\mathrm{N}-1;\ \mathrm{f(x)}=0\ \text{for x}=\mathrm{N},\mathrm{N}+1,\ldots,2\mathrm{N}-1$$

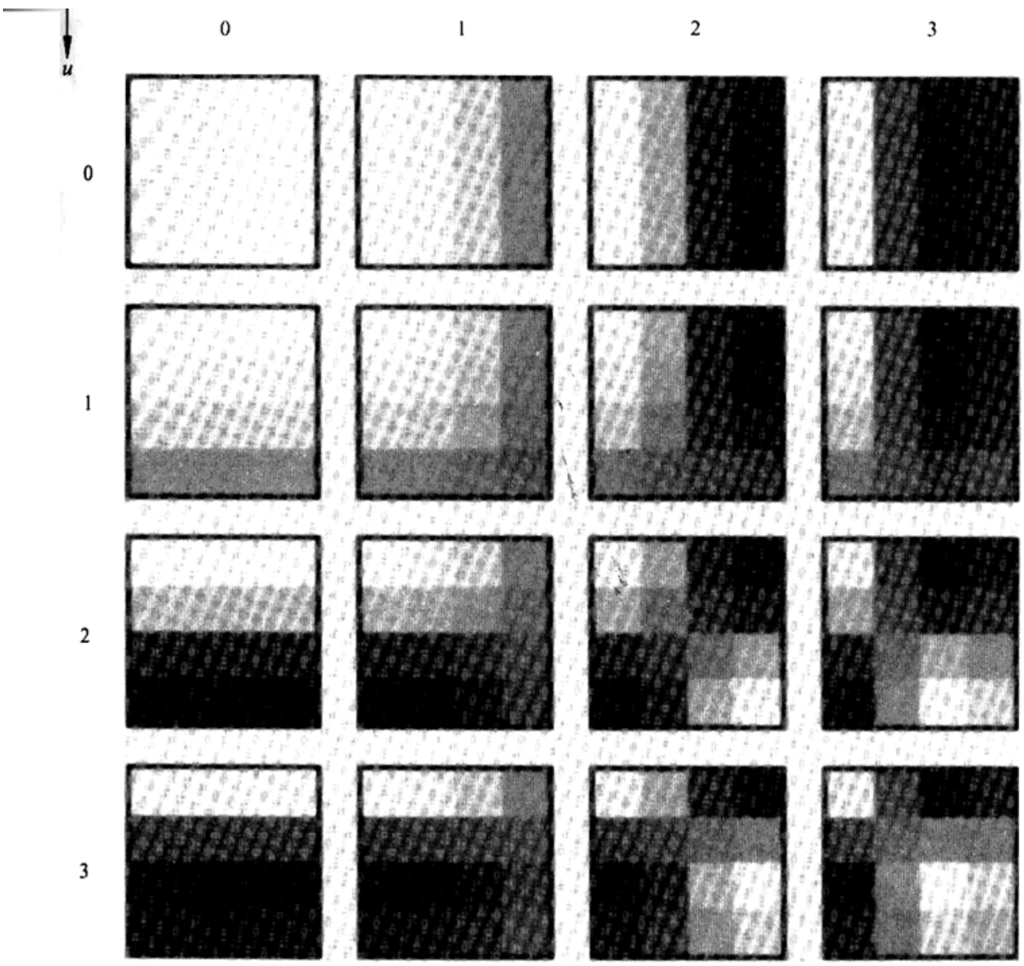


Figure 3.28 Discrete cosine transform basis functions for $N = 4$. Each block consists of 4×4 elements, corresponding to x and y varying from 0 to 3. The origin of each block is at its top left. The highest value is shown in white. Other values are shown in grays, with darker meaning smaller.

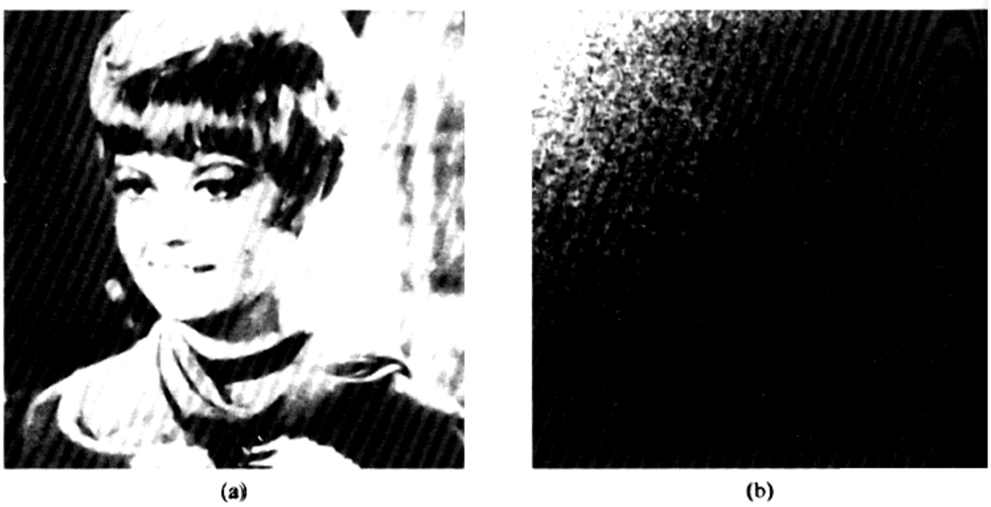
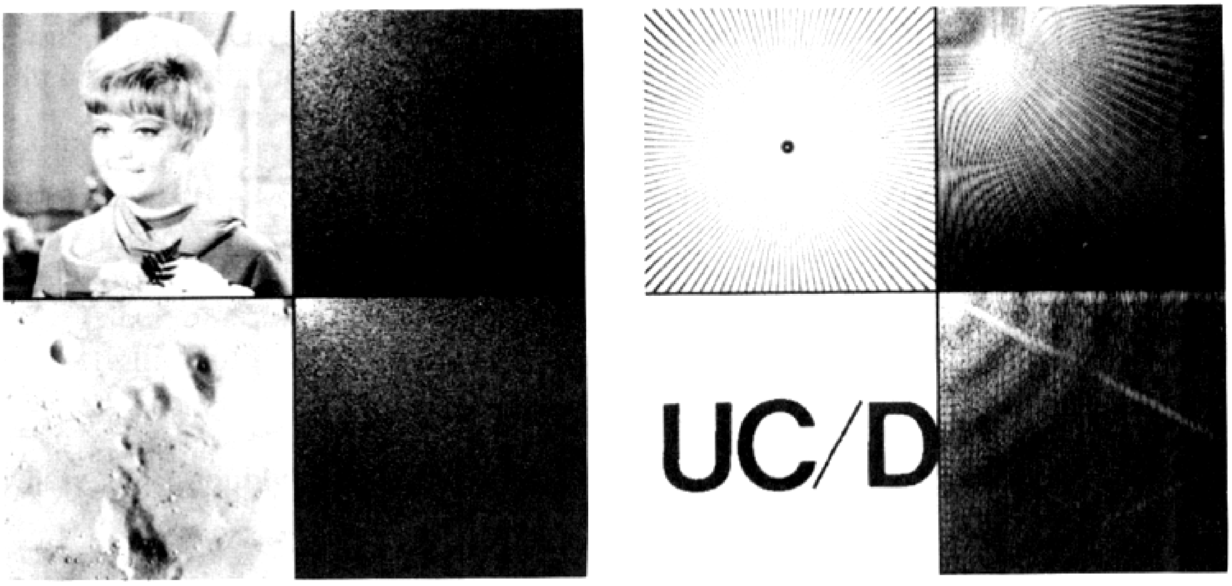


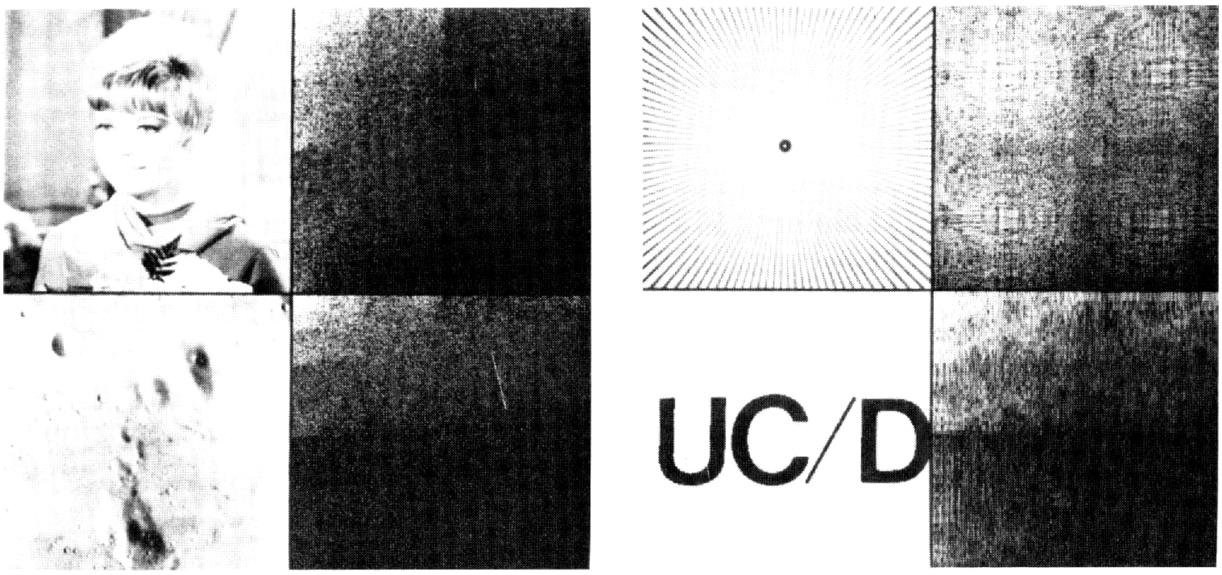
Figure 3.29 A simple image and the log magnitude of its discrete cosine transform.



(a) Cosine transform examples of monochrome images;

(b) Cosine transform examples of binary images.

Figure 5.11



(a) Hadamard transforms of monochrome images.

(b) Hadamard transforms of binary images.

Figure 5.13 Examples of Hadamard transforms.

陸、Hotelling 轉換

一、定義：

向量　x　= $[x_1, x_2, \ldots, x_n]^T$

其平均向量為：

$m_x = E\{x\}$　即 x 之期望值

變異數矩陣為：

$C_x = E\{(x - m_x)(x - m_x)^T\}$

對於 M 個取樣自隨機族群之向量，其平均向量為：

$$m_x = \frac{1}{M}\sum_{k=1}^{M} x_k$$

變異數矩陣為：

$$C_x = \frac{1}{M}\sum_{k=1}^{M} x_k x_k^T - m_x m_x^T$$

C_x 為實數對稱矩陣，故可由其找出一組正交特徵值向量 e_i，其對應之特徵值分別為 λ_i, $i = 1, 2, \ldots, n$。($\lambda_j \geqq \lambda_{j+1}$)。

令 A 為 n 個特徵向量所組成之矩陣，則將下面將 x 轉換為 y 向量之式子稱為 Hotelling 轉換：

$y = A(x - m_x)$

二、性質：

1．$m_y = 0$

2．$C_y = A C_x A^T$

3．$$C_y = \begin{bmatrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & \bullet & & \\ & & & \bullet & \\ 0 & & & & \lambda_n \end{bmatrix}$$

4．C_x與C_y具有相同之特徵值。

5．$A^{-1} = A^T$

三、影像旋轉之應用：

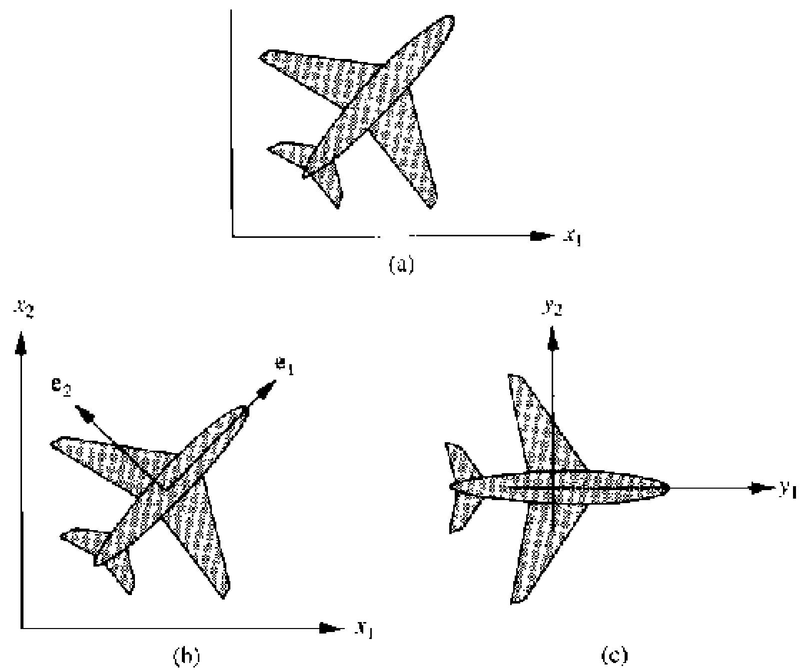


Figure 3.30 *(a) A binary object; (b) its principal axes (eigenvectors); (c) object rotated by using Eq. (3.6-6).*

四、影像資料壓縮之應用：

由 Hotelling 轉換式可反求得：

$$x = A^T y + m_x$$

若由 y 重建 x 時僅用對應於較大特徵值之 K 個特徵向量，則所求得重建之近似 x 爲：

$$\hat{x} = A_K^T y + m_x$$

此近似向量與原向量 x 之誤差爲：

$$e_{ms} = \sum_{j=1}^{n} \lambda_j - \sum_{j=1}^{K} \lambda_j = \sum_{j=K+1}^{n} \lambda_j$$

Table 3.6 Channel Numbers and Wavelengths

Channel	*Wavelength Band* (μm)
1	0.40–0.44
2	0.62–0.66
3	0.66–0.72
4	0.80–1.00
5	1.00–1.40
6	2.00–2.60

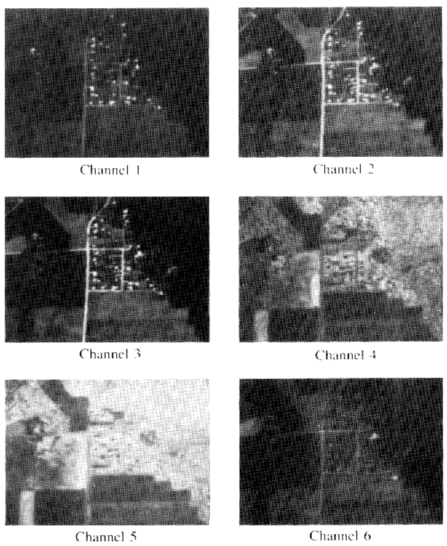


Figure 3.31 *Six spectral images from an airborne scanner. (Courtesy of the Laboratory for Applications of Remote Sensing, Purdue University.)*

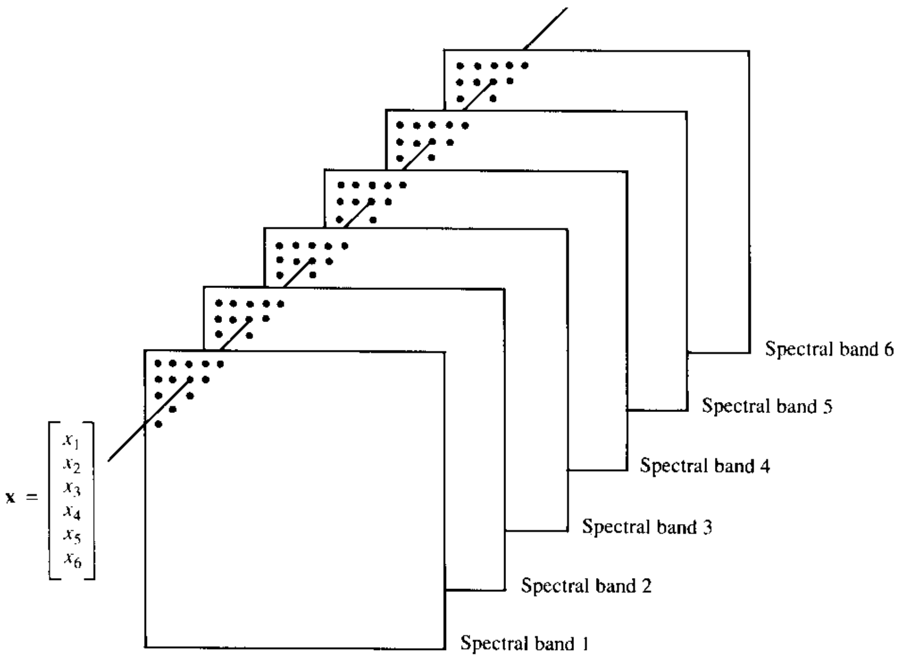


Figure 3.32 *Formation of a vector from corresponding pixels in six images.*

Table 3.7 Eigenvalues of the Covariance Matrix of the Images Shown in Fig. 3.31

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
3210	931.4	118.5	83.88	64.00	13.40

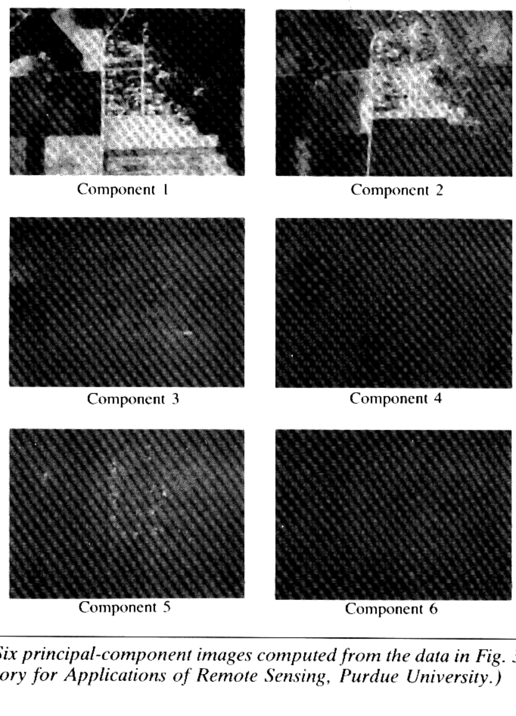


Figure 3.33 *Six principal-component images computed from the data in Fig. 3.31. (Courtesy of the Laboratory for Applications of Remote Sensing, Purdue University.)*