- Pseudo Spectral Methods
- Time optimization / Reachable Sets
- Singular Controls
- Links between Ham. Dynamics & Opt. Control
- Dynamic Programming

⋮

Exam Next Week

Pseudo-Spectral Methods:

- DIDO
- GPOPS

Given

$$J = K(\vec{x}_0, t_0, \vec{x}_f, t_f) + \int_{t_0}^{t_f} L(\vec{x}, \vec{u}, \tau)\, d\tau$$

$$\dot{\vec{x}} = \vec{F}(\vec{x}, \vec{u}, t)$$

$$\vec{g}(\vec{x}_0, t_0, \vec{x}_f, t_f) = 0$$

Minimize J subject to $\dot{\vec{x}} = \vec{F}$ & $\vec{g} = 0$.

Main Idea: Replace $\vec{x}(t)$, $\vec{u}(t)$, "arbitrary functions of time" with an explicit decomposition:

$$\left.\begin{aligned} \vec{x} &\simeq X(t) = \sum_{i=0}^{N} X_i \,\phi_i(t) \\ \vec{u} &\simeq U(t) = \sum_{i=0}^{N} U_i \,\phi_i(t) \end{aligned}\right\}$$

what choices exist for $\phi_i(t)$?

Many options for $\phi_i(t)$ exist ..... include Lagrange Interpolating Polynomials & Legendre functions, and others .....

Lagrange Interpolating Polynomials

Classical result: Given a function specified at N points, (x_i, t_i), we can form a polynomial in time that explicitly satisfies these points & interpolates between them.

$$\boxed{L_i(t) = \prod_{\substack{j=0 \\ j \neq i}}^{N} \frac{(t - t_j)}{(t_i - t_j)}}$$

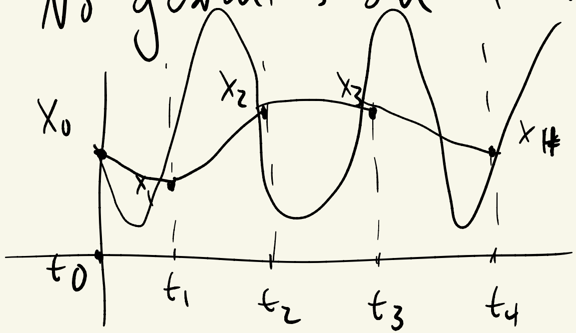
using this,

$$\boxed{X(t) = \sum_{i=0}^{N} L_i(t) X_i}$$

Note: $L_i(t_k) = \begin{cases} 1 & t_k = t_i \\ 0 & t_k = t_j \neq t_i \end{cases}$

Thus $X(t_k) = \sum_{i=0}^{N} \underbrace{L_i(t_k)}_{\delta_{ik}} X_i = X_k$

No gurantees on "Fitness" of the Function for $t \neq t_k$.



Have to deal with

$$\frac{d\vec{X}}{dt} \simeq \frac{dX}{dt} = \frac{d}{dt}\sum_{i=0}^{N} L_i(t)\, X_i = \sum_{i=0}^{N} \dot{L}_i(t)\, X_i$$

$$\dot{L}_i(t) = \frac{d}{dt}\prod_{\substack{j=0\\ j\neq i}}^{N} \frac{(t-t_j)}{(t_i-t_j)} = \frac{1}{\prod_{\substack{j=0\\ j\neq i}}^{N}(t_i-t_j)} \sum_{k=0}^{N} \prod_{\substack{j=0\\ j\neq i,k}}^{N} (t-t_j)$$

What is

$$\boxed{\dot{L}_i(t_\ell) = D_{\ell i}} \Rightarrow \text{Use this in the Dynamics}$$

$$\dot{\vec{x}} \approx \dot{X} = \sum_{l=0}^{N} \dot{L}_l(t)\, X_l = \vec{F}\left(X(t), \underline{\underline{U}}(t), t\right)$$

Evaluate at the time nodes, t_k,

$$\dot{X}_k = \boxed{\sum_{l=0}^{N} D_{kl}\, X_l = \vec{F}\left(X_k, U_k, t_k\right)} \Rightarrow$$

An explicit constraint from the dynamics that involves the states X_k.

Lagrange Polys have convergence problems.....

A Better Formulation will use Legendre Polynomials as the interpolating functions, due to their convergence properties. These are orthogonal functions, thus each term picks up a unique aspect of the discrete trajectory.

Legendre PseudoSpectral Approx.

$(-1 \le t \le 1)$

$$\vec{x} \simeq X(t) = \sum_{l=0}^{N} X_l \,\phi_l(t)$$

$\mathcal{L}_l(t) = l^{th}$ order Legendre Polynomial

$$\phi_l(t) = \frac{1}{N(N+1)\mathcal{L}_l(t_l)} \frac{(t^2-1)}{(t-t_l)} \mathcal{L}_l(t)$$

$\mathcal{L}_1(t) = t$

$$\phi_l(t_k) = \begin{cases} 1 & l=k \\ 0 & l \neq k \end{cases}$$

$$D_{kl} = \dot{\phi}_l(t_k) = \begin{cases} \dfrac{\mathcal{L}_k(t_k)}{\mathcal{L}_l(t_l)} \dfrac{1}{(t_k-t_l)} & k \neq l \\ -\dfrac{N(N+1)}{4} & k=l=0 \\ N(N+1)/4 & k=l=N \end{cases}$$

0 else

Consider the cost function: ($t_f = t_N$)

$$J = K(x_0, t_0, x_N, t_N) + \underbrace{\int_{t_0}^{t_N} L(x(t), u(t), t)\, dt}_{\text{Gauss-Lobatto Integration Formula}}$$

$$= \left(\sum_{k=0}^{N} L(x_k, u_k, t_k)\, w_k \right)$$

$$\boxed{w_k = \frac{2}{N(N+1)} \frac{1}{\mathcal{L}_k^2(t_k)}}$$

Further relationships:

$$w_i D_{ik} + w_k D_{ki} = 0 \qquad i, k = 1, 2, \ldots, N-1$$

$$2 w_0 D_{00} = -1, \quad 2 w_N D_{NN} = 1, \quad \sum_{i=0}^{N} w_i = 2$$

Technical Issue: Interp. polys are defined over $[-1, 1]$, but our time goes from $t_0 \to t_f$.

Define a new time parameter $\tau \in [-1, 1]$, such that

$$t = \frac{1}{2}\left[(t_f - t_0)\tau + (t_f + t_0)\right] \Rightarrow \boxed{dt = \frac{1}{2}(t_f - t_0)\, d\tau}$$

Problem statement becomes

$$J = K(X(-1), t_0, X(1), t_f) + \frac{1}{2}(t_f - t_0)\int_{-1}^{1} L(X(\tau), U(\tau), \tau)\, d\tau$$

$$\frac{1}{2}(t_f - t_0)\frac{dX}{dt} = F(X, U)$$

$$g(X(-1), t_0, X(1), t_f) = 0$$

$\Rightarrow$

Discritize the Direct Problem

$$J = K(X_0, X_N) + \sum_{k=0}^{N} L(X_k, U_k)\, w_k$$

subject to

$$F(X_k, U_k) - \sum_{l=0}^{N} D_{kl} X_l = 0 \qquad k = 0, 1, \dots, N$$

$$\vec{g}(X_0, X_N) = 0$$

KKT conditions

$$\Lambda = K(X_0, X_N) + \sum_{k=0}^{N} L(X_k, U_k)\, w_k + \tilde{\nu} \cdot \vec{g}(X_0, X_N) + \sum_{k=0}^{N} \tilde{\lambda}_k \cdot \left[F(X_k, U_k) - \sum_{l=0}^{N} D_{kl} X_l \right]$$

Nec. Conds are:

$$\frac{\partial \Lambda}{\partial x_k} = 0 \quad k = 0, 1, \ldots, N \qquad \frac{\partial \Lambda}{\partial \tilde{\nu}} = 0$$

$$\frac{\partial \Lambda}{\partial u_k} = 0 \quad k = 0, 1, \ldots, N \qquad \frac{\partial \Lambda}{\partial \tilde{\lambda}_k} = 0 \quad k = 0, 1, \ldots, N$$

Can form the Hamiltonian Nec. Conditions first & then discritize.

$$H = \sum_{k=0}^{N} H_k$$

$$\underbrace{H_k = L(x_k, U_k) + P_k \cdot \vec{F}(x_k, U_k)}$$

$k = 0, \dots, N$

Solve $\dfrac{\partial H_k}{\partial U_k} = 0 \Rightarrow$ optimal control U_k^* at a given x_k & P_k.

$$\left\{ \begin{aligned} &\vec{F}(x_k, U_k^*) - \sum_{l=0}^{N} D_{kl} X_l = 0 \\ &\left[-\frac{\partial H_k}{\partial X_k} - \sum_{l=0}^{N} D_{kl} P_l = 0 \right] \end{aligned} \right\}$$

$\underline{\text{Dynamics}} + \vec{g}(x_0, x_N) = \vec{0}$

$$P_0 = -\frac{\partial K}{\partial X_0} - \vec{\nu} \cdot \frac{\partial \vec{g}}{\partial x_0} ; \quad P_N = \frac{\partial K}{\partial x_f} + \vec{\nu} \cdot \frac{\partial \vec{g}}{\partial x_f} ; \quad H_0 = -\frac{\partial K}{\partial t_0}\bigg|_{-1} \pm \vec{\nu} \cdot \frac{\partial \vec{g}}{\partial t_0}\bigg|_{\tau=-1}$$

$$H_N = \frac{\partial K}{\partial t_f}\bigg|_{+1} + \vec{\nu} \frac{\partial \vec{g}}{\partial t_f}\bigg|_{+1}$$

Nonlinear Programming Problem Find

$(x_0, x_1, \dots, x_N, p_0, p_1, \dots, p_N, \vec{\nu})$ that satisfy the prev. conditions

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These 2 approaches appear to be different.

$$
\begin{tikzcd}
\text{Indirect Problem } B^{\lambda} \arrow[r, "\text{Discretization}"'] & B^{\lambda N} \arrow[d, leftrightarrow, "?"] \\
\text{Direct Problem } B \arrow[u] \arrow[r, "\text{Discretized}"'] & B^{N} \text{ (Direct Discretization)}
\end{tikzcd}
$$

$\boxed{C_N = 0}$

How are these related to each other?

KKT

$C_N - C_N = 0$

Start with the KKT conditions:

$$\frac{\partial \Lambda}{\partial X_k} = \frac{\partial L_k}{\partial X_k} w_k + \tilde{\lambda}_k \cdot \frac{\partial F_k}{\partial X_k} - \frac{\partial}{\partial X_k}\left(\sum_{m=0}^{N} \sum_{l=0}^{N} \tilde{\lambda}_m \cdot D_{ml} \cdot X_l \right)$$

$k \neq 0, N$

$$\frac{\partial}{\partial X_k}\left[\sum_{m=0}^{N} \sum_{l=0}^{N} \tilde{\lambda}_m \cdot D_{ml} \cdot X_l \right] = \sum_{m=0}^{N} D_{mk} \tilde{\lambda}_m$$

$$D_{mk} = \frac{-w_k}{w_m} D_{km} \qquad \boxed{w_m D_{mk} + w_k D_{km} = 0}$$

$$\frac{\partial \Lambda}{\partial X_k} = \frac{\partial L_k}{\partial X_k} w_k + \tilde{\lambda}_k \cdot \frac{\partial F_k}{\partial X_k} + \sum_{m=0}^{N} \frac{D_{km}}{w_m} \tilde{\lambda}_m w_k = 0$$

w_k

$$\frac{\partial L_k}{\partial X_k} + \left(\frac{\tilde{\lambda}_k}{w_k}\right) \cdot \frac{\partial F_k}{\partial X_k} + \sum_{m=0}^{N} D_{km}\left(\frac{\tilde{\lambda}_m}{w_m}\right) = 0 \qquad H_k = L_k + P_k F_k$$

$$\frac{\partial H_k}{\partial X_k} + \sum_{l=0}^{N} D_{kl} P_l = 0 =$$

$$\frac{\partial L_k}{\partial X_k} + \left(\vec{P}_k\right) \cdot \frac{\partial \vec{F}_k}{\partial X_k} + \sum_{l=0}^{N} D_{kl} P_l = 0$$

$\Rightarrow$ IF $P_k = \dfrac{\tilde{\lambda}_k}{w_k}$

Then equivalent.

$$\frac{\partial \Lambda}{\partial U_k} = \frac{\partial L_k}{\partial U_k} w_k + \tilde{\lambda}_k \cdot \frac{\partial F_k}{\partial U_k} = 0$$

$$\frac{\partial H}{\partial U_k} = \frac{\partial L_k}{\partial U_k} + P_k \cdot \frac{\partial F_k}{\partial U_k} = 0$$

Equivalent

$$\boxed{\vec{g}(X_0, X_N) = 0 \qquad F_k - \sum D_{k\ell} X_\ell = 0}$$ Identical for both

Transversality Conditions: $k = 0, N$

$$\frac{\partial \Lambda}{\partial X_N} = \frac{\partial K}{\partial X_N} + \frac{\partial L_N}{\partial X_N} w_N + \tilde{\nu} \cdot \frac{\partial \vec{g}}{\partial X_N} + \tilde{\lambda}_N \cdot \frac{\partial F_N}{\partial X_N} - \sum_{k=0}^{N} D_{kN} \frac{\tilde{\lambda}_k}{w_N} w_N = 0$$

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$$D_{iN} = -\frac{w_N}{w_i} D_{Ni} \quad ; \quad 2 D_{NN} = \frac{1}{w_N}$$

$$\sum_{k=0}^{N} D_{kN} \tilde{\lambda}_k = -\sum_{k=0}^{N-1} \frac{w_N}{w_k} D_{Nk} \tilde{\lambda}_k + D_{NN} \tilde{\lambda}_N \pm D_{NN} \tilde{\lambda}_N$$

$$= -\sum_{k=0}^{N} \frac{w_N}{w_k} D_{Nk} \tilde{\lambda}_k + \frac{\tilde{\lambda}_N}{w_N}$$

$$\frac{\partial K}{\partial X_N} + \frac{\partial L_N}{\partial X_N} w_N + \tilde{\nu} \cdot \frac{\partial g}{\partial X_N} + \tilde{\lambda}_N \cdot \frac{\partial F_N}{\partial X_N} + w_N \sum_{k=0}^{N} D_{Nk} \left(\frac{\tilde{\Lambda}_k}{w_k} \right) - \frac{\tilde{\Lambda}_N}{w_N} = 0$$

Trns. Cnditins for (N) tpj

KKT $\Rightarrow = 0.$

$$\boxed{P_N = \frac{\partial K}{\partial X_N} + \vec{\nu} \cdot \frac{\partial \vec{g}}{\partial X_N}} \qquad \frac{\tilde{\Lambda}_N}{w_N} = P_N \qquad \underbrace{P_0 = -\frac{\partial K}{\partial X_0} - \vec{\nu} \cdot \frac{\partial \vec{g}}{\partial X_0}}$$

$\Downarrow C_0$

$$\underbrace{\frac{\partial K}{\partial X_N} + \tilde{\nu} \cdot \frac{\partial g}{\partial X_N} - P_N}_{C_N = 0} + w_N \left[\frac{\partial L_N}{\partial X_N} + \underbrace{\left(\frac{\tilde{\Lambda}_N}{w_N} \right)}_{P_N} \cdot \frac{\partial F_N}{\partial X_N} + \sum_{k=0}^{N} D_{Nk} \underbrace{\left(\frac{\tilde{\Lambda}_k}{w_k} \right)}_{P_k} \right] = 0$$

$- C_N$

Trns. Cnds. hold.

Indirect N.C.

$$
\begin{tikzcd}
B^{\lambda} \arrow[r, "\text{Indirect Discretization}"] \arrow[d] & B^{\lambda N} \arrow[d, leftrightarrow] \\
B \arrow[r, "\text{Direct Discretization}"'] & B^{N}
\end{tikzcd}
$$

O.C. Prob.

KKT

If $\left.\begin{array}{l} C_N = 0 \\ C_0 = 0 \end{array}\right\}$ Closure Conditions

F.M. Ross Covector Mapping Theorem

Two approaches are equivalent if the Closure Conditions hold.