# ASEN 6060
# ADVANCED ASTRODYNAMICS
# Poincaré Mapping

Objectives:

- Define Poincaré mapping and procedure for construction
- Summarize interpretation and use of Poincaré maps

# *Poincaré Mapping*

Consider the system of autonomous differential equations:

$$\dot{\bar{x}} = \bar{f}(\bar{x})$$

Integrating a trajectory from initial conditions

$$\bar{x}_0 = [x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0]^T$$

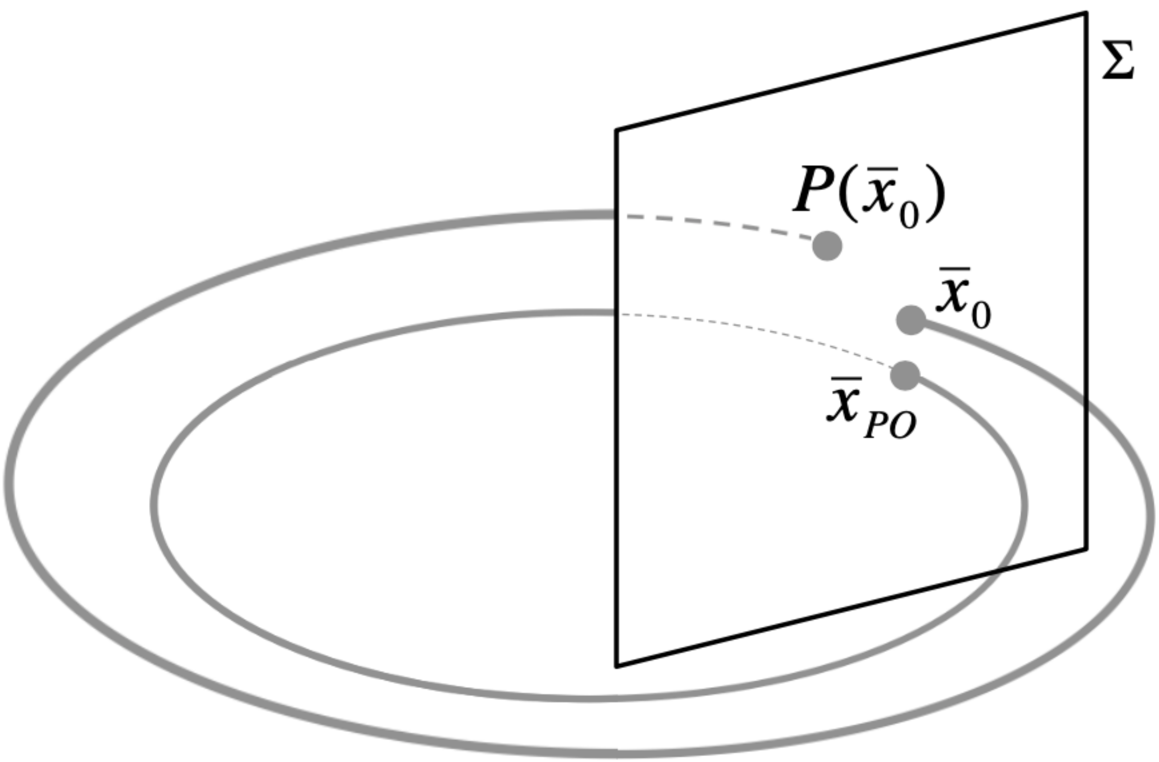
produces a continuous trajectory $\bar{x}(\bar{x}_0, t)$

A Poincaré map constructs a lower-dimensional representation of this continuous trajectory as a sequence of discrete states

# Surface of Section

Sample continuous trajectories using their intersections with a surface of section, Σ, that must be defined transverse to the flow

The $i$-th intersection with Σ labeled the $i$-th return to the map, $P^i(\bar{x}_0)$

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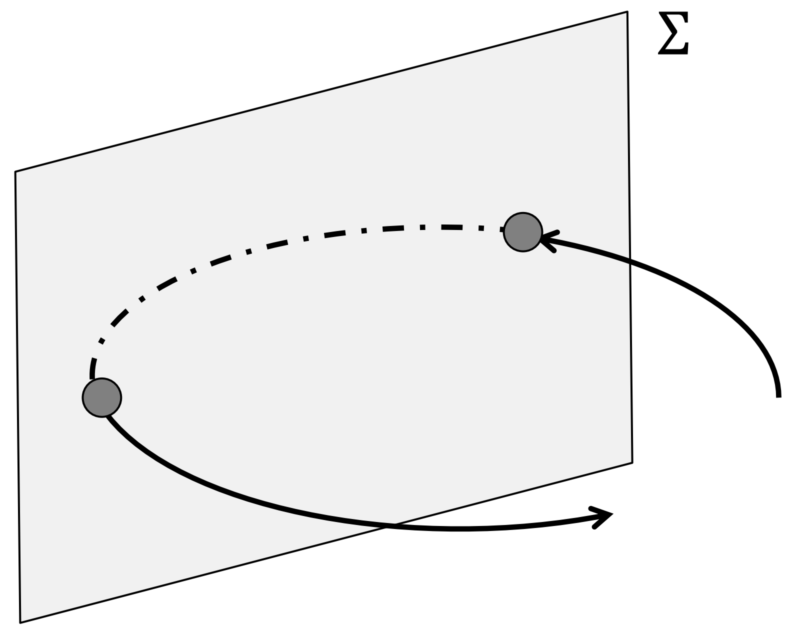
# *Terminology*

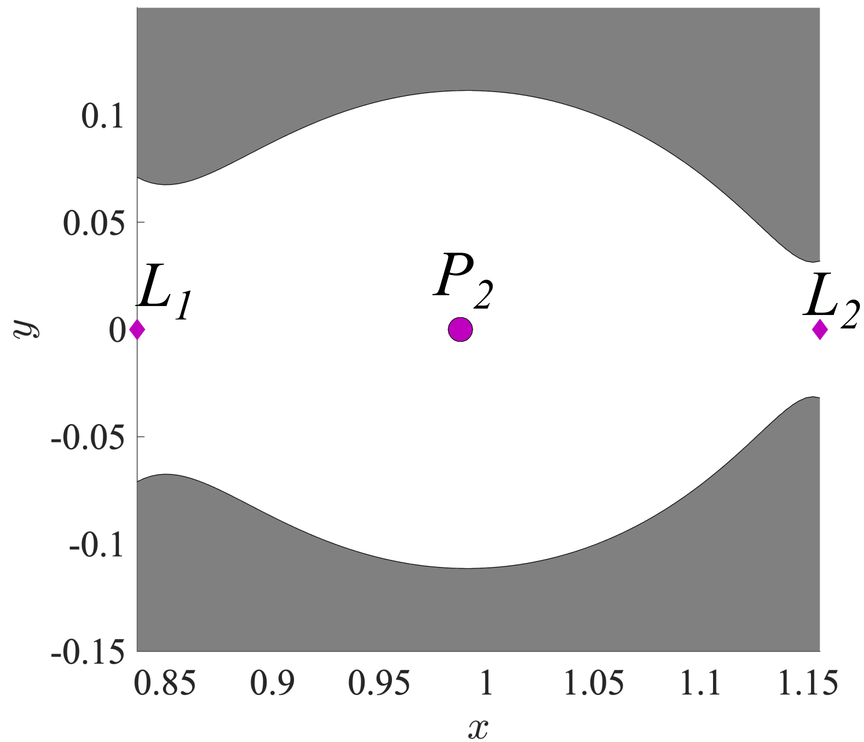
One-sided map:

Two-sided map:



# *Constructing a Poincaré Map: Example*

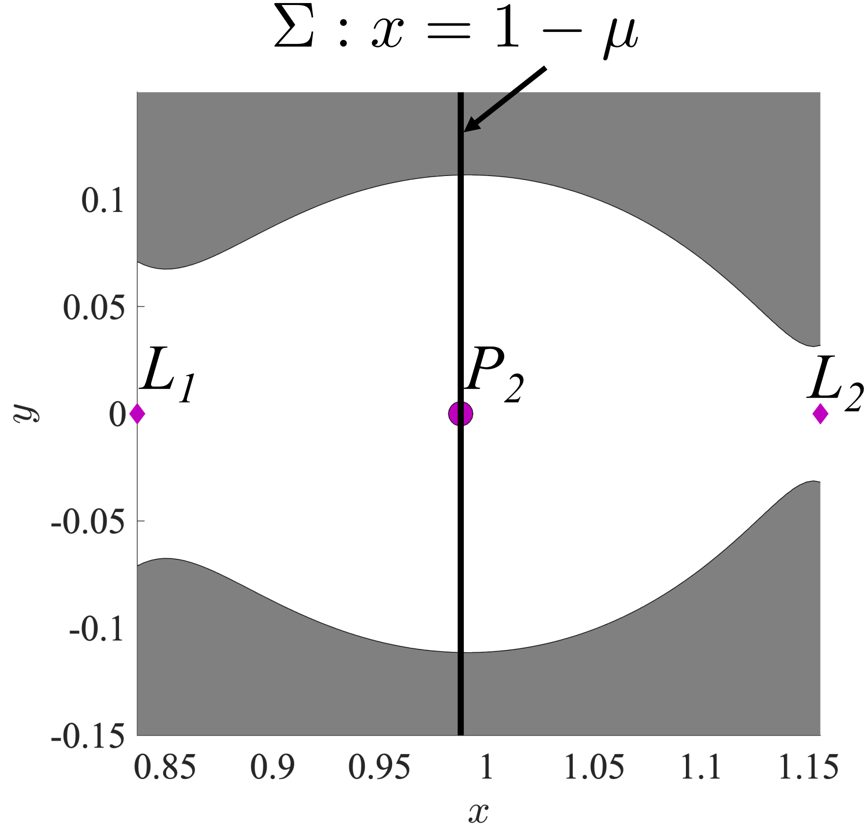
Construct a Poincaré map capturing general planar trajectories near $P_2$ at a specific Jacobi constant, $C_{des} = 3.17$, in the EM CR3BP



# *Constructing a Poincaré Map: Example*

Step 1: Define a surface of section as $\Sigma : x = 1 - \mu$

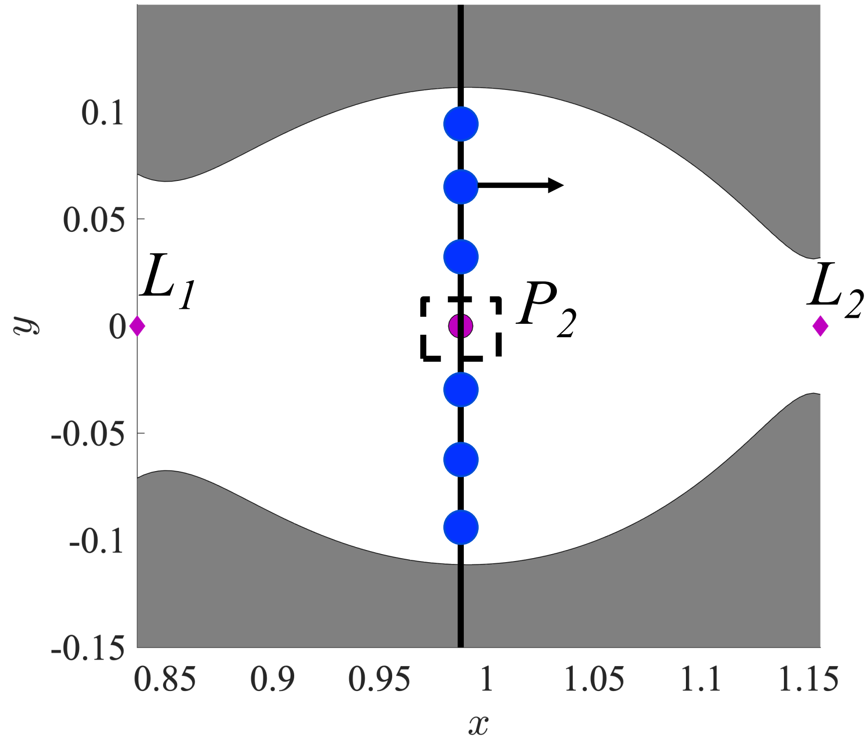
Define one-sided map to record only positive crossings, i.e., $\dot{x} > 0$



# *Constructing a Poincaré Map: Example*

Step 2: Seed initial conditions to produce desired trajectory set

- Define initial conditions (what kind of trajectories are of interest?)
- Select number of initial conditions, $N_{IC}$



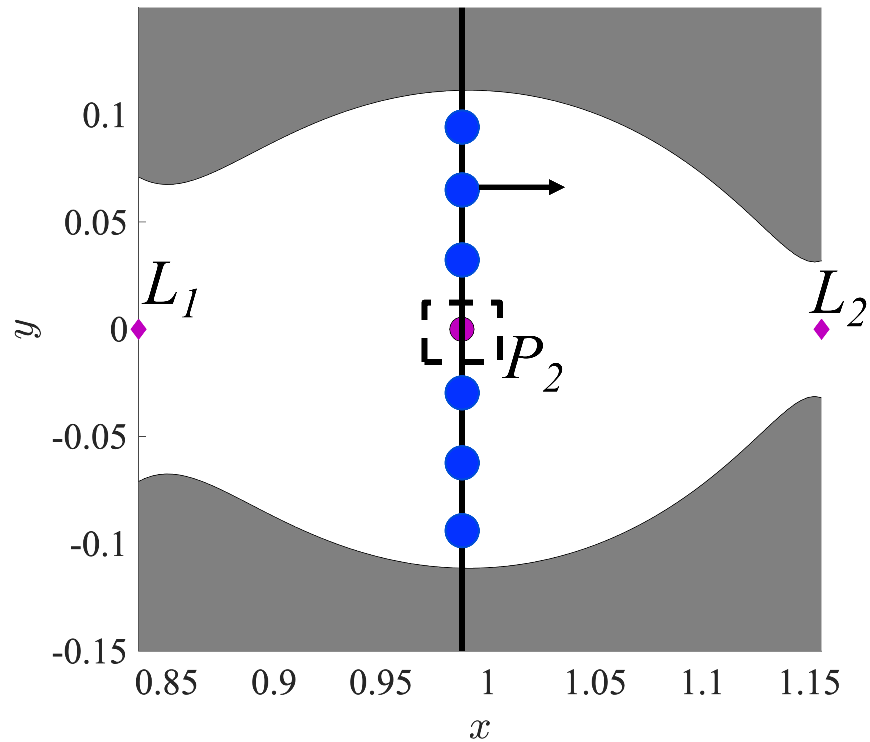
# *Constructing a Poincaré Map: Example*

Step 2: Seed initial conditions to produce desired trajectory set

$$\bar{x}_{i,0} = [x_{i,0}, y_{i,0}, z_{i,0}, \dot{x}_{i,0}, \dot{y}_{i,0}, \dot{z}_{i,0}]^T$$

- Select position components

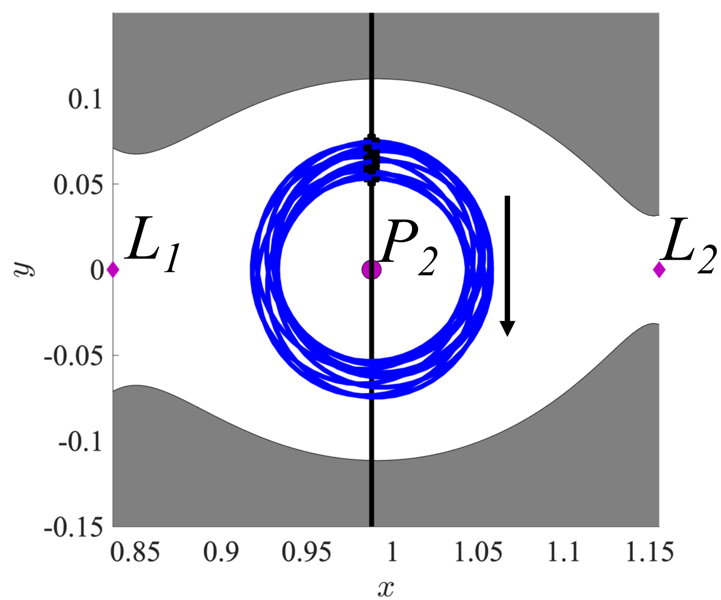
- Select velocity components



# Constructing a Poincaré Map: Example

Step 3: Propagate initial conditions and record crossings of surface of section until either:

E.g., 20 positive crossings of $\Sigma : x = 1 - \mu$ for one trajectory



# *Constructing a Poincaré Map: Example*

Step 4: Represent crossings of surface of section on a Poincaré map

- Each state along planar trajectory is described by state in form

$$\bar{x} = [x, y, 0, \dot{x}, \dot{y}, 0]^T$$

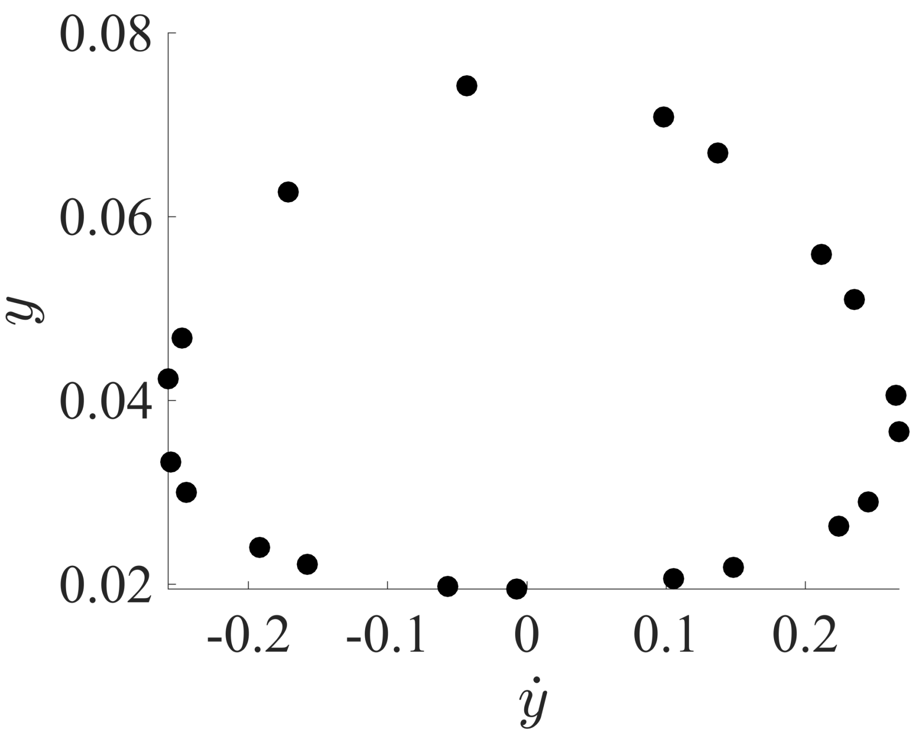
- At the crossing of the surface of section, $x = 1 - \mu$ , meaning all states take form

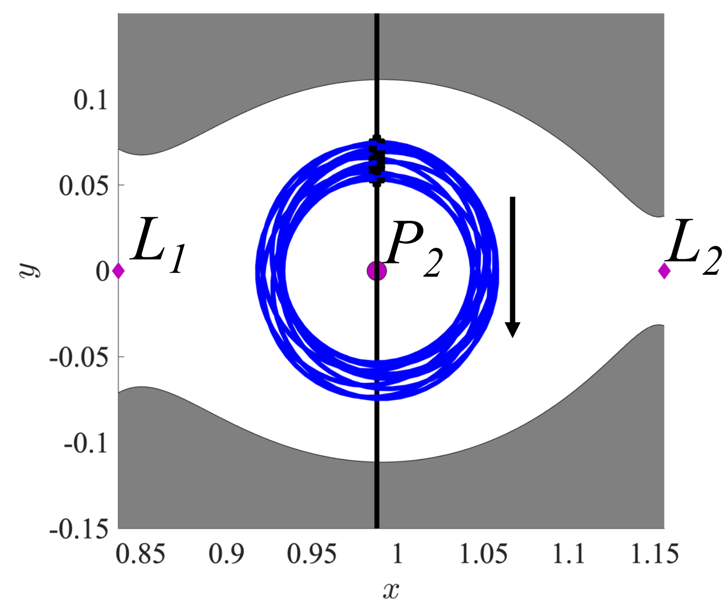
$$\bar{x} = [1 - \mu, y, 0, \dot{x}, \dot{y}, 0]^T$$

# Constructing a Poincaré Map: Example

Step 4: Represent crossings of surface of section on a Poincaré map

- One option: Construct 2-dimensional Poincaré map with horizontal and vertical axes representing $(\dot{y}, y)$ for each crossing
- For one trajectory: for $N_{ret} = 20$

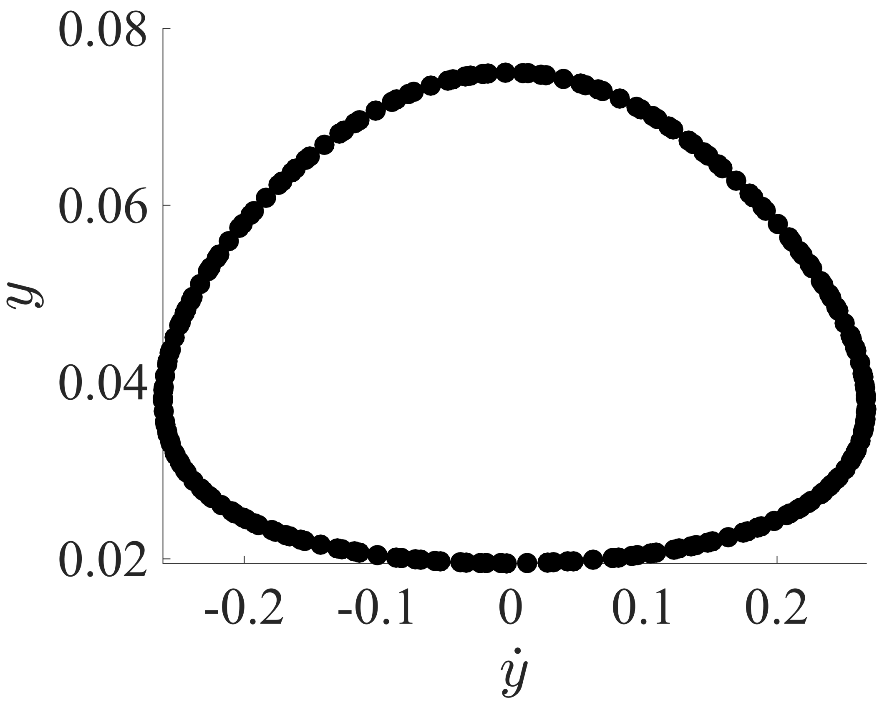


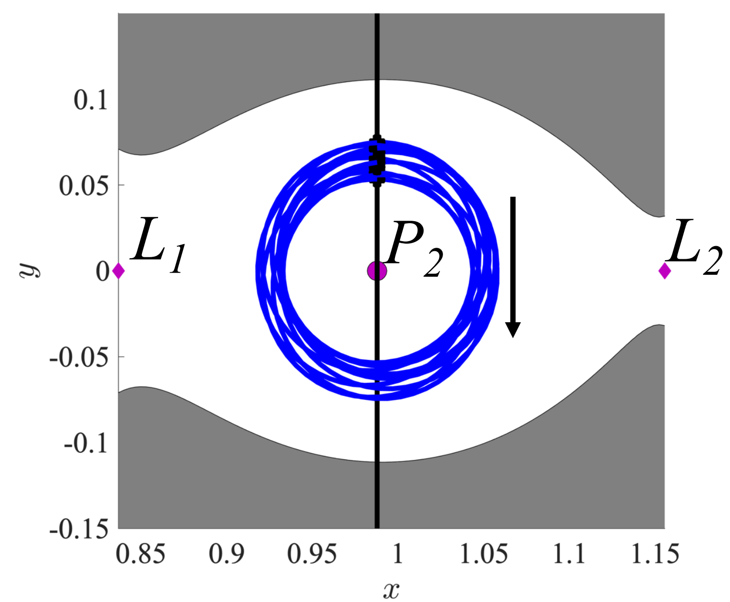



# *Constructing a Poincaré Map: Example*

Step 4: Represent crossings of surface of section on a Poincaré map

- One option: Construct 2-dimensional Poincaré map with horizontal and vertical axes representing $(y, \dot{y})$ for each crossing
- For one trajectory: increasing $N_{ret}$ to 200

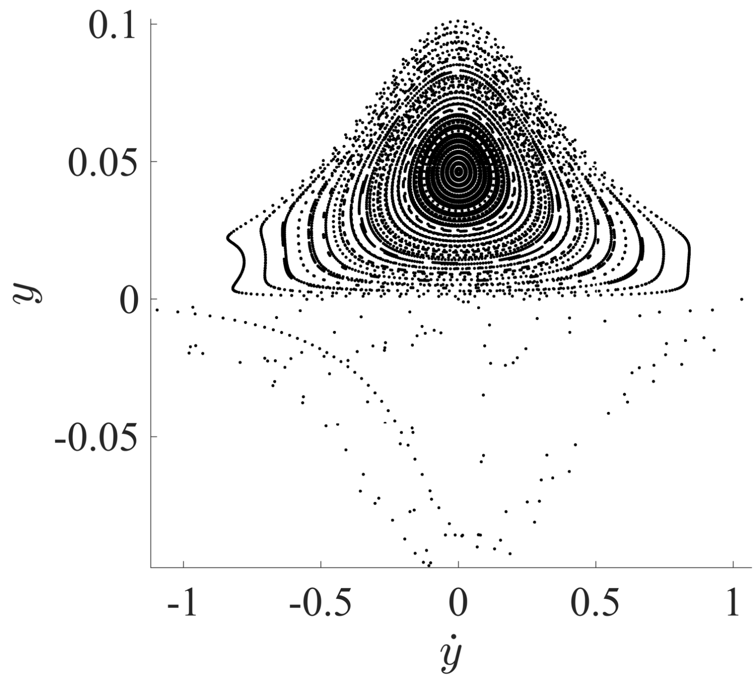


# *Constructing a Poincaré Map: Example*

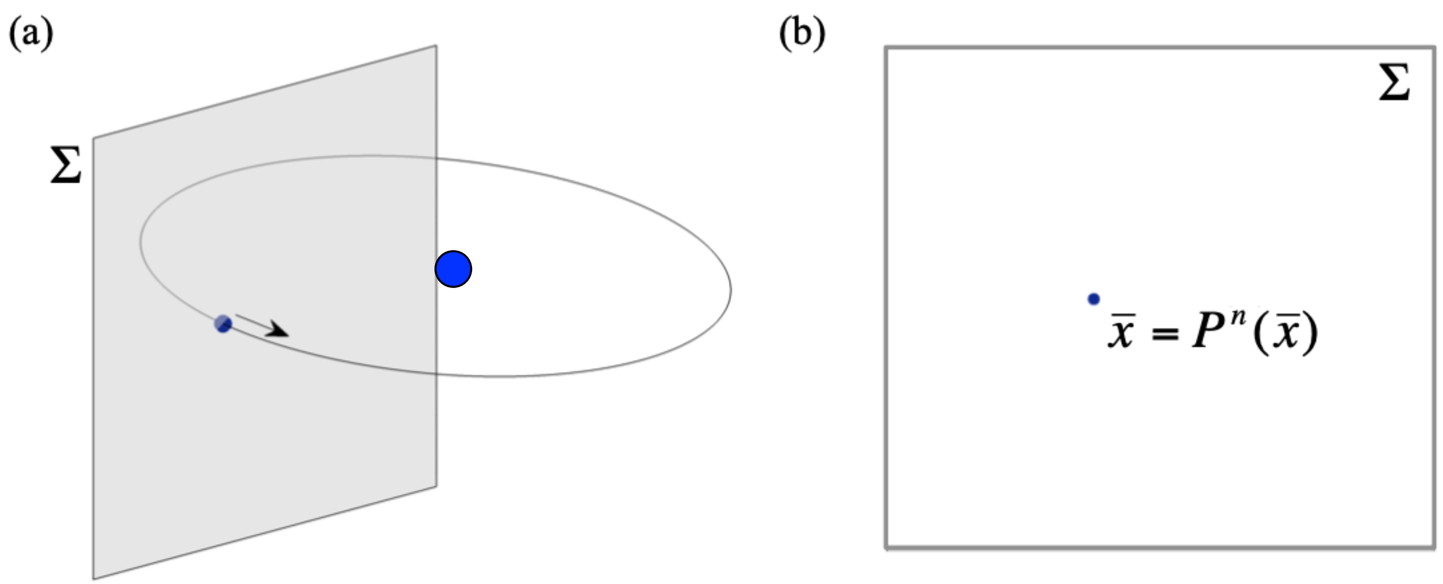
Step 4: Represent crossings of surface of section on a Poincaré map

- One option: Construct 2-dimensional Poincaré map with horizontal and vertical axes representing $(y, \dot{y})$ for each crossing
- For 1000 trajectories with $\dot{x} > 0$



# *Identifying Periodic Orbits on a Map*

In an appropriately constructed one-sided Poincaré map, a periodic orbit with one intersection with the surface of section over an orbit period appears as a

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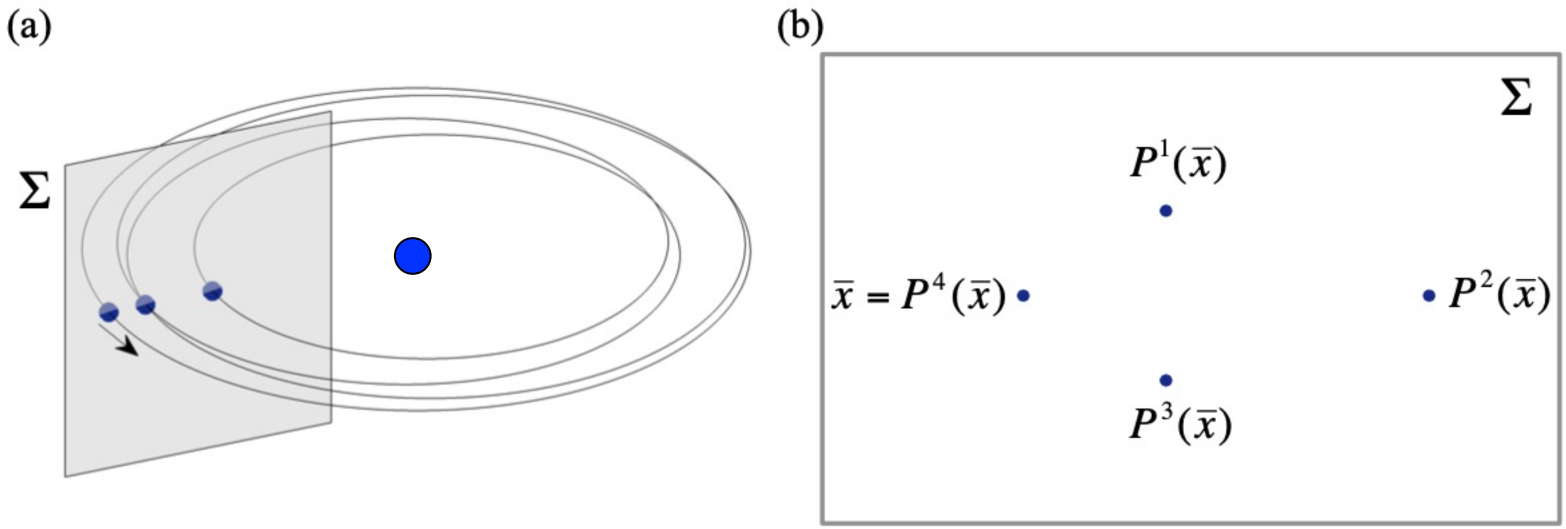
# Identifying Periodic Orbits on a Map

In an appropriately constructed one-sided Poincaré map, a periodic orbit with multiple intersections with the surface of section over an orbit period produces

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# Quasi-Periodic Trajectories

Quasi-periodic trajectory / quasi-periodic orbit (QPO):

- Nonperiodic, bounded motion, lies on surface of an invariant torus
- Exists near a periodic orbit with oscillatory modes
- Existence and persistence covered by Kolmogorov-Arnold-Moser (KAM) theory

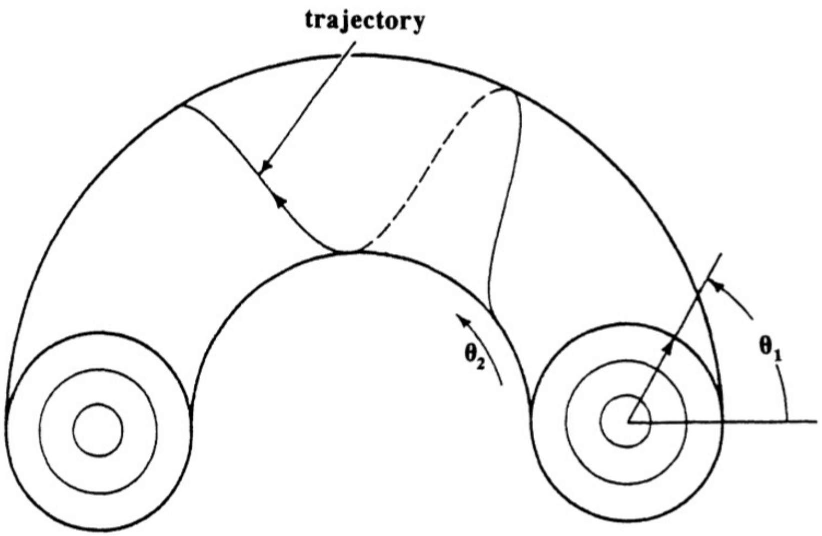
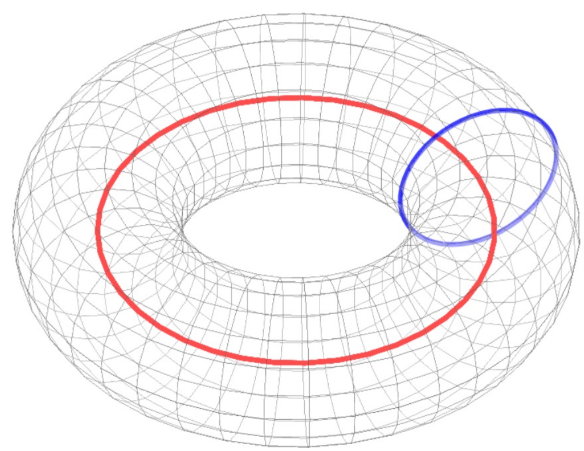



Image credit: Contopoulos, 2004, "Order and Chaos in Dynamical Astronomy"



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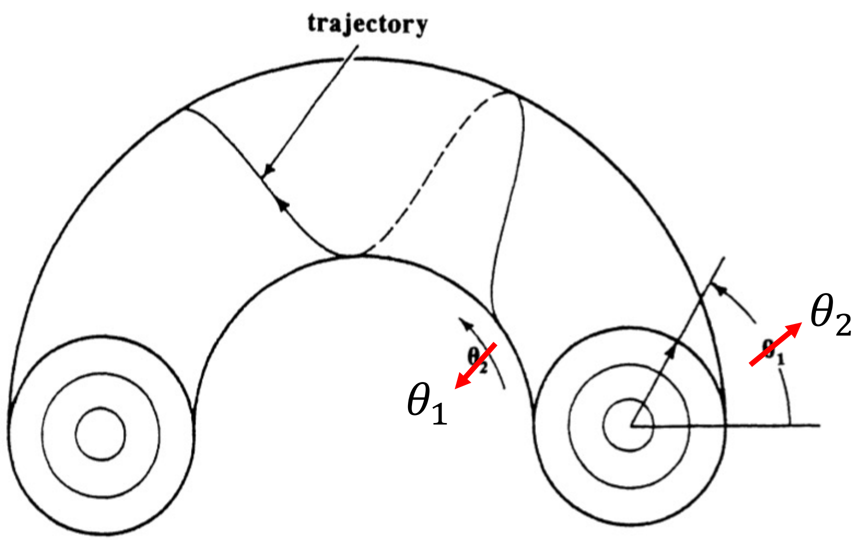
# *Quasi-Periodic Trajectories*

Quasi-periodic trajectory on the surface of a two-dimensional torus defined by two incommensurate, fundamental frequencies: $\omega_1, \omega_2$

- $\omega_1$ = Central frequency
- $\omega_2$ = Transverse frequency

(Note: notation opposite to picture below!)

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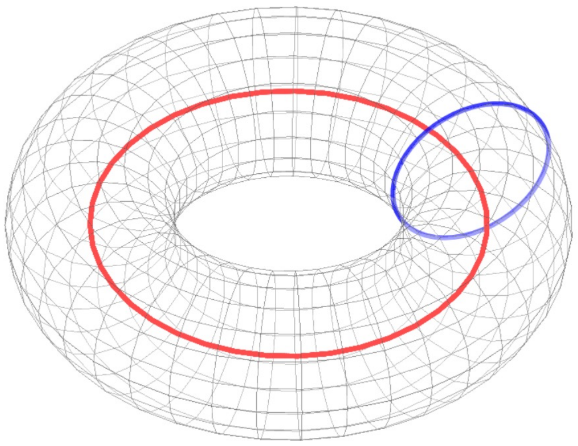

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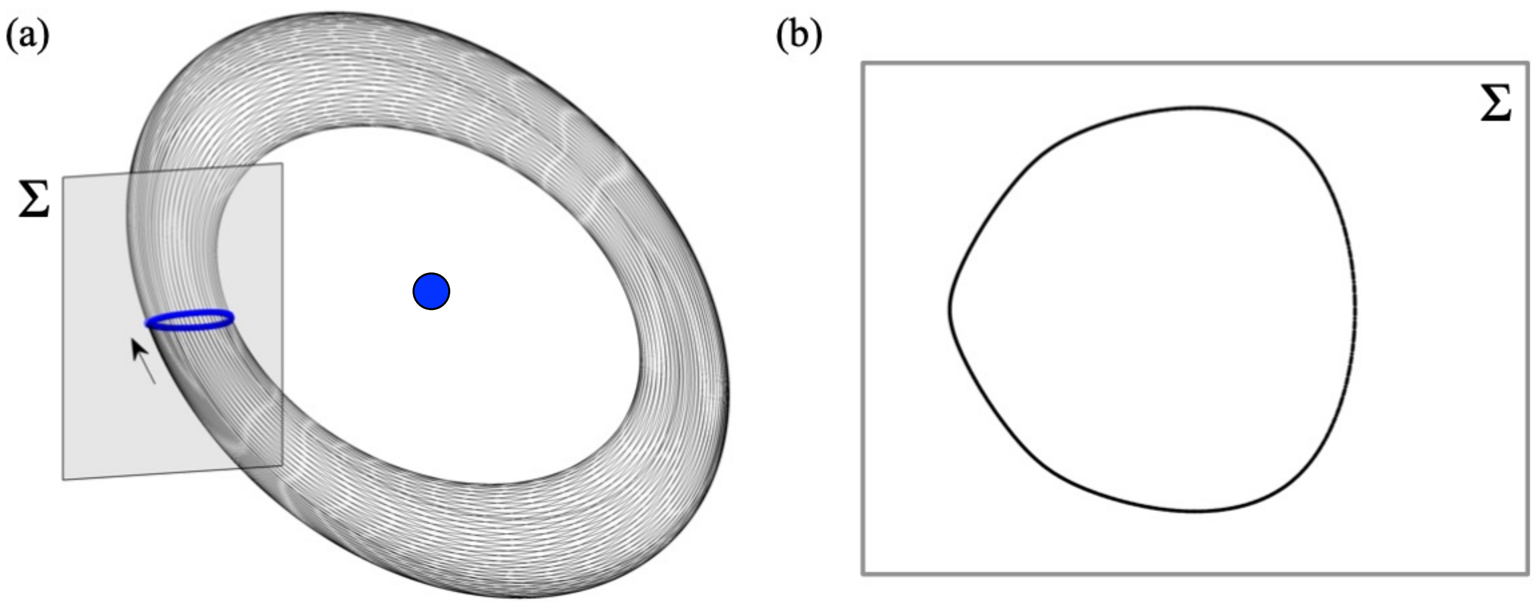
# Identifying Quasi-Periodic Orbits on a Map

In an appropriately constructed one-sided Poincaré map, a quasi-periodic trajectory appears as a

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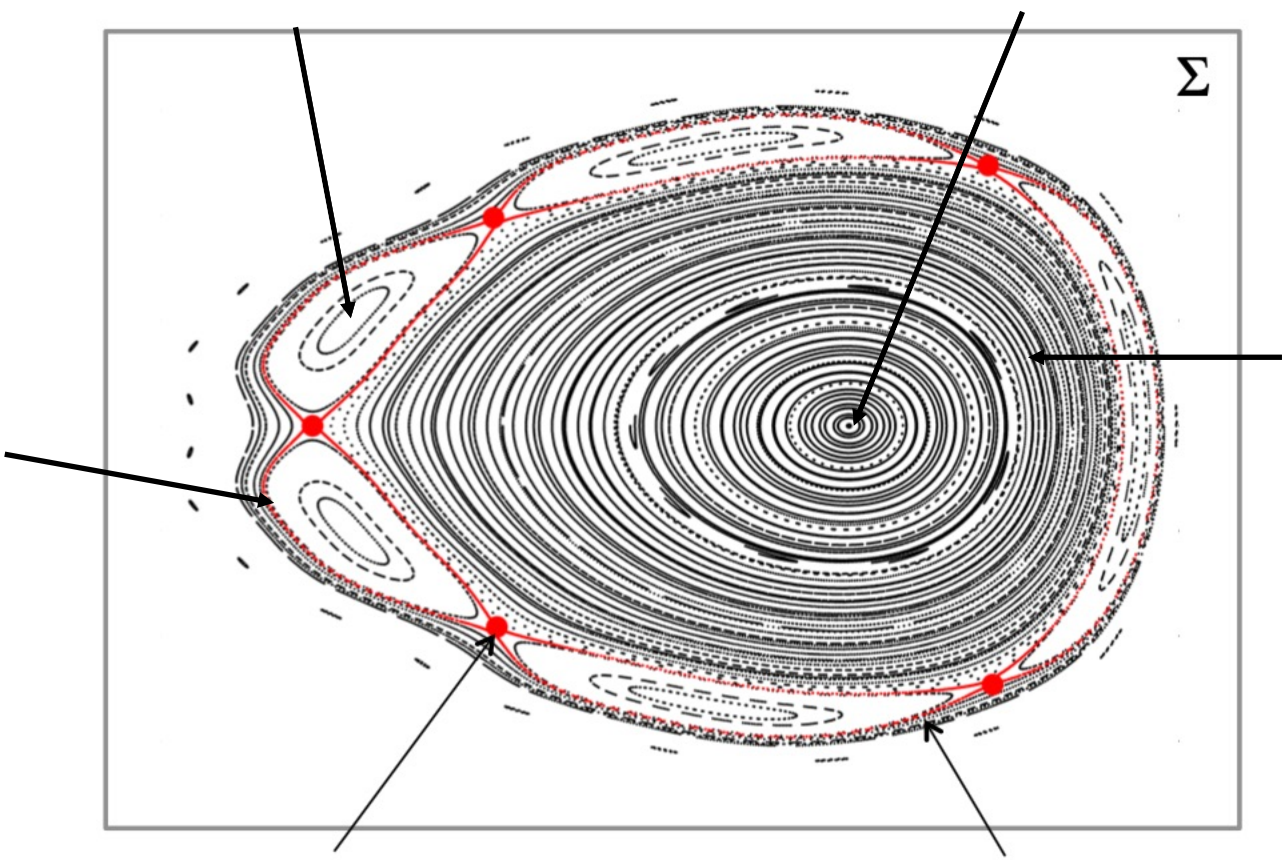
# *Complex Islands and Island Chains*

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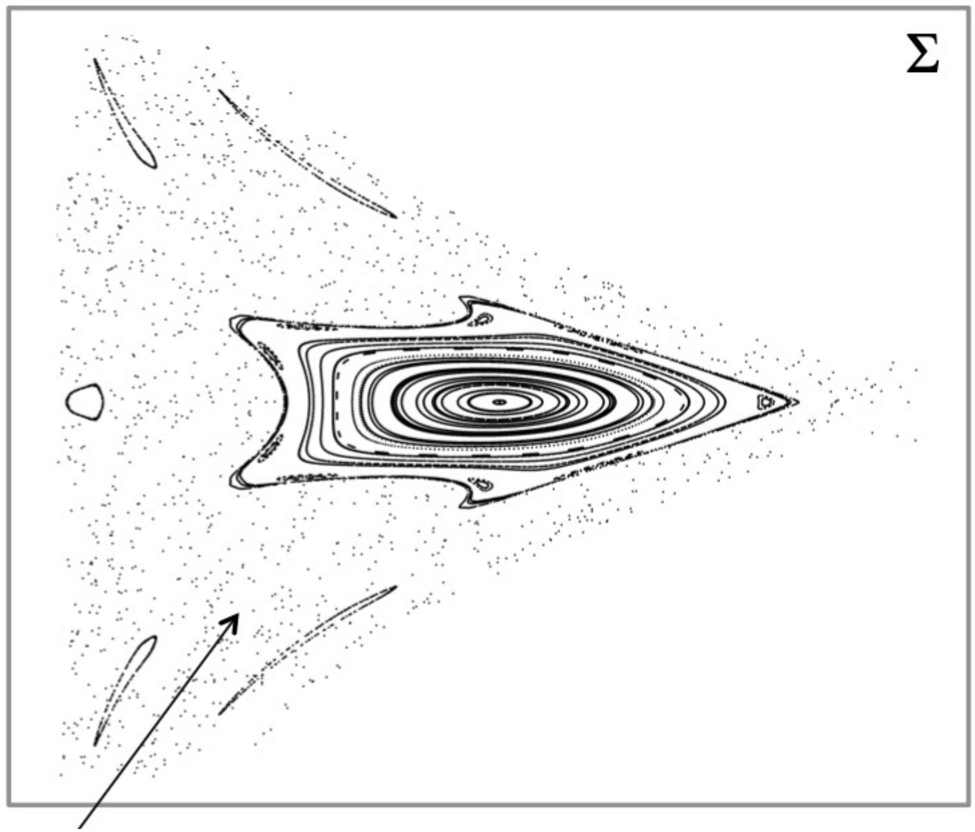
# Chaotic Regions

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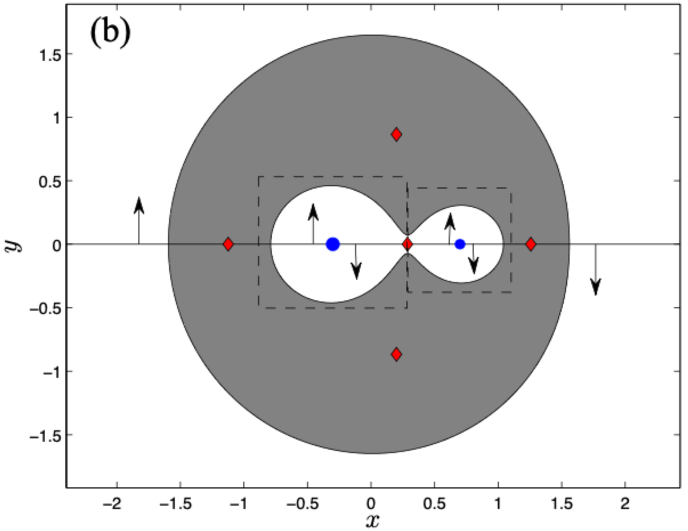
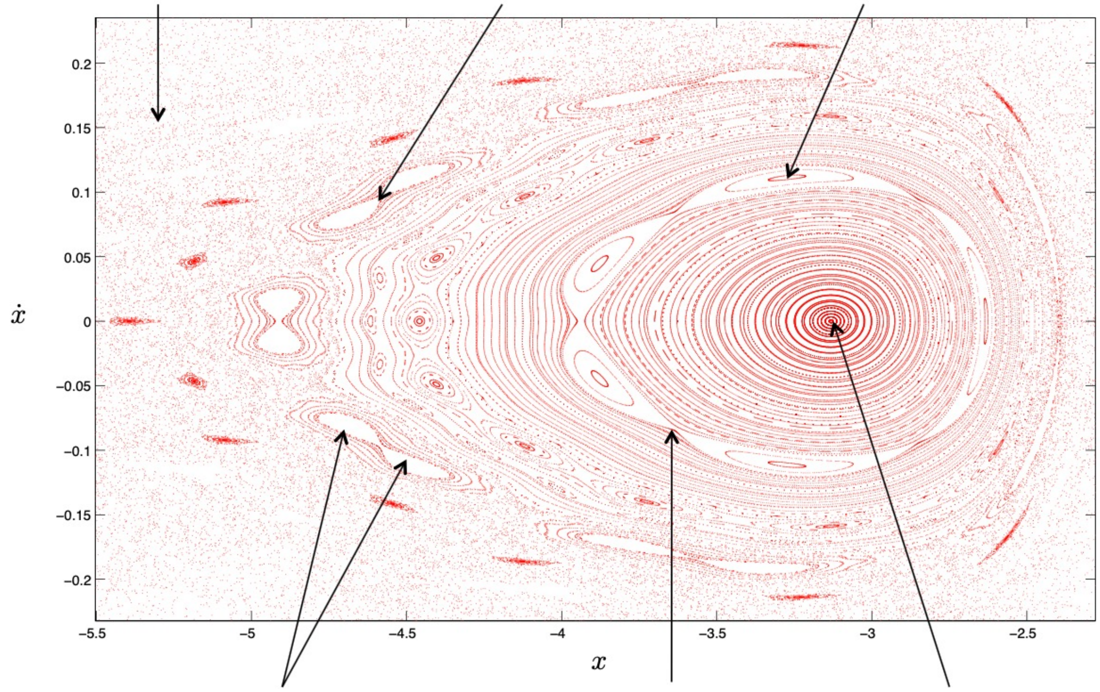
# *Identifying Structures on Poincaré Map*

Poincaré map in CR3BP with $\mu = 0.30$ for planar, initially retrograde trajectories at $C_{des} \approx 3.885$ with $\Sigma: y = 0, \dot{y} > 0$

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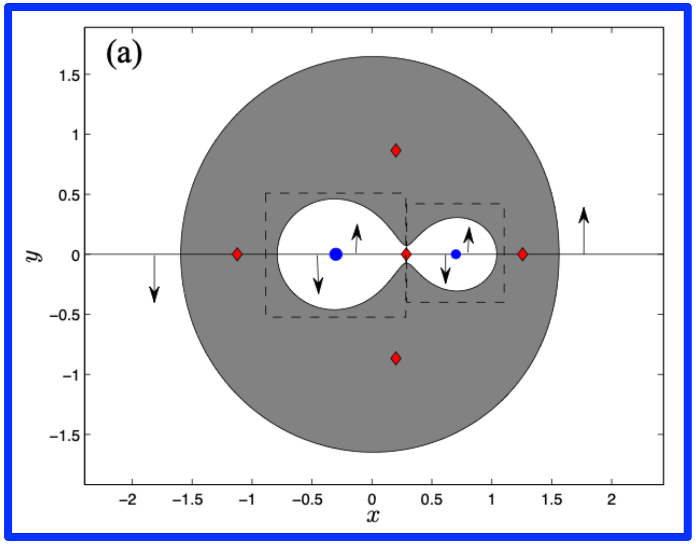
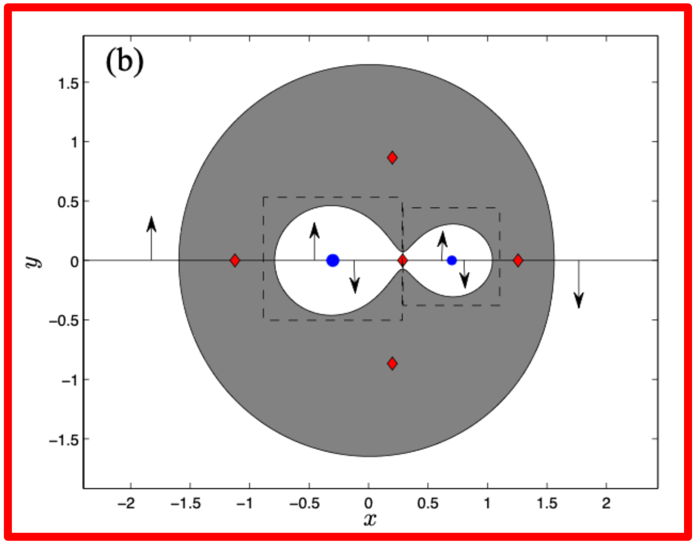
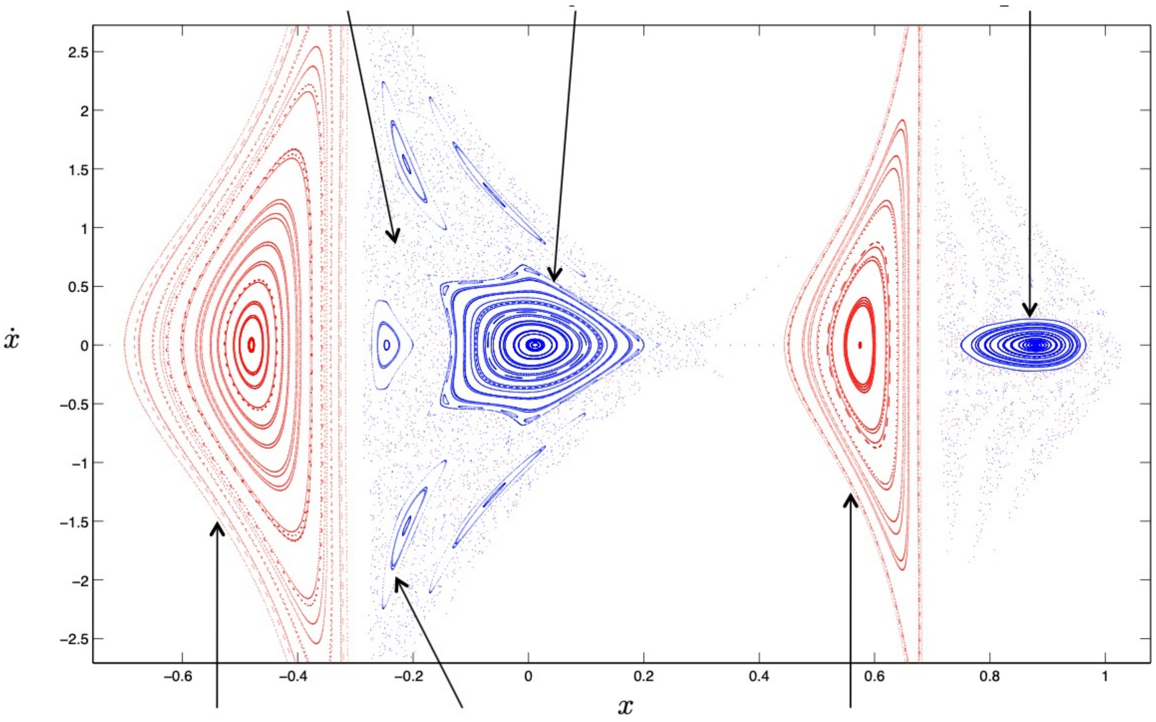
# Identifying Structures on Poincaré Map

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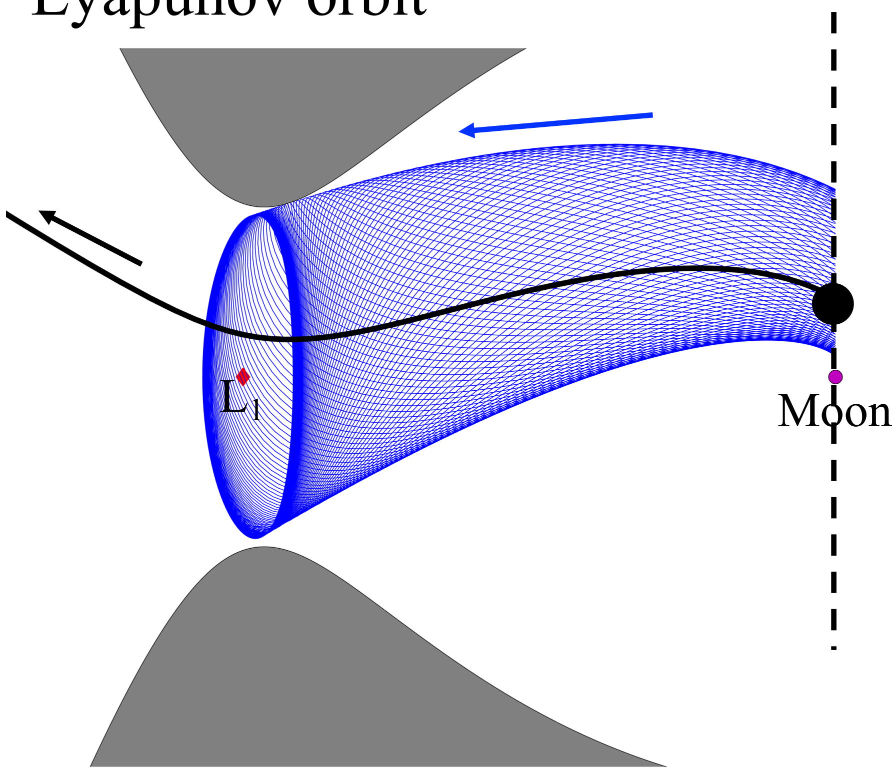
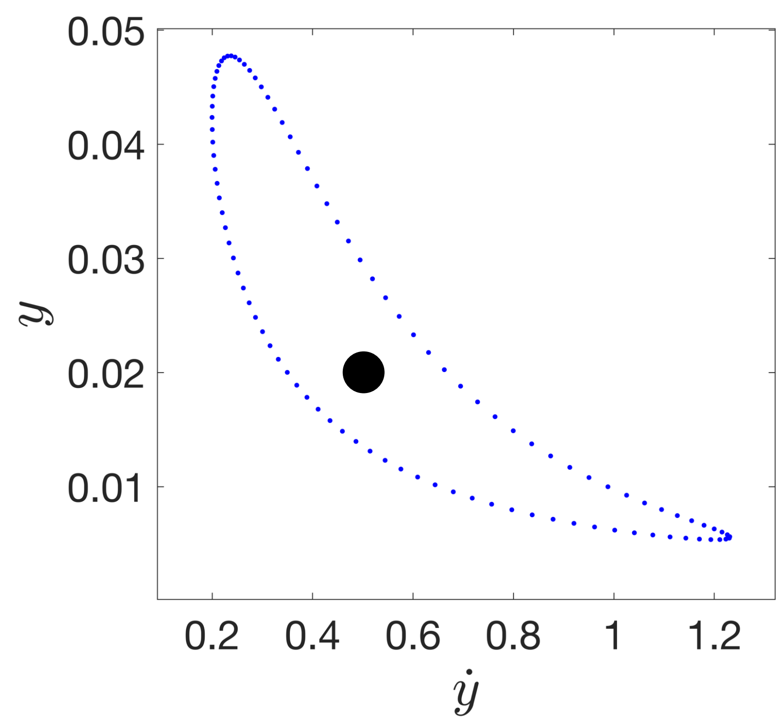
# *Summarizing Stable/Unstable Manifolds*

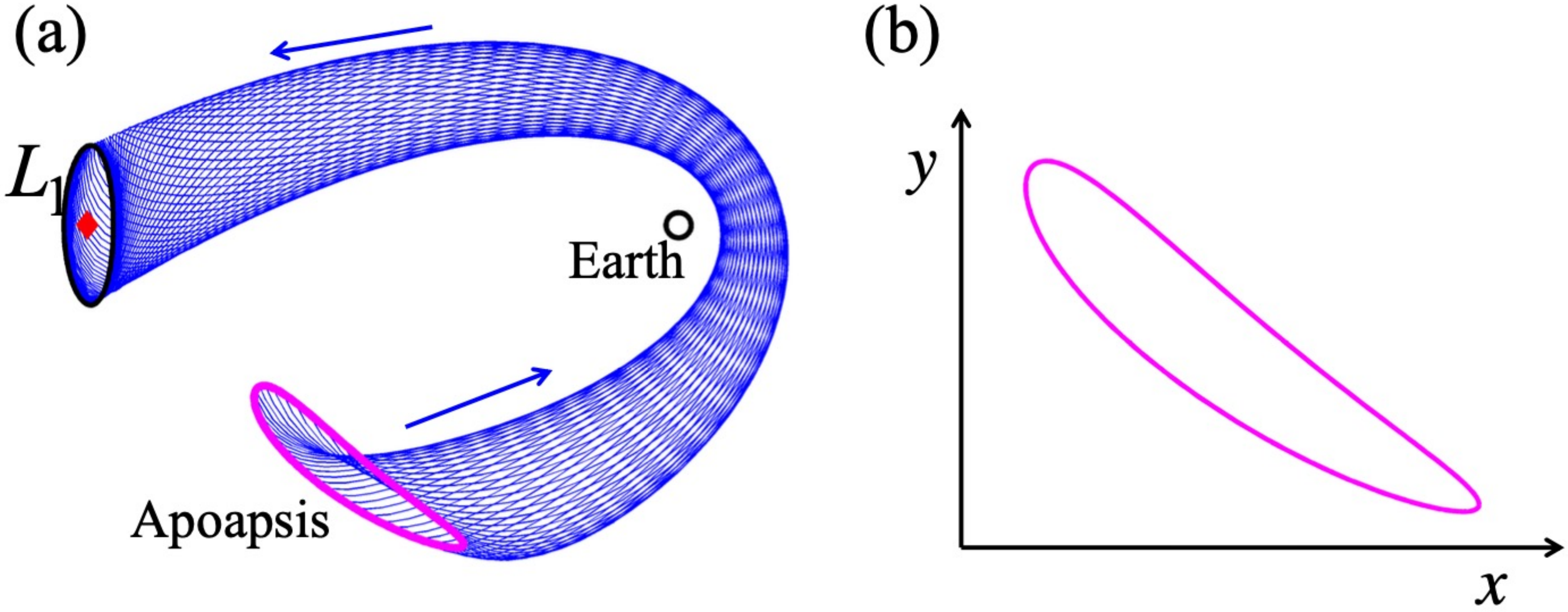
Hyperbolic invariant manifolds associated with periodic and quasi-periodic orbits near the collinear libration points govern trajectories passing through the gateways

E.g.: Planar trajectory starting within stable manifold of $L_1$ Lyapunov orbit



# *Summarizing Stable/Unstable Manifolds*

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