ASEN 5044, Fall 2024

Statistical Estimation for Dynamical Systems

Lecture 12: Sums of Random Variables; Gaussian Distributions

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Tuesday 10/01/2024

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Announcements

- **HW 4 Due Thurs 10/03**
  - Submit to Gradescope

- **Office hours: usual times/places for THIS WEEK**

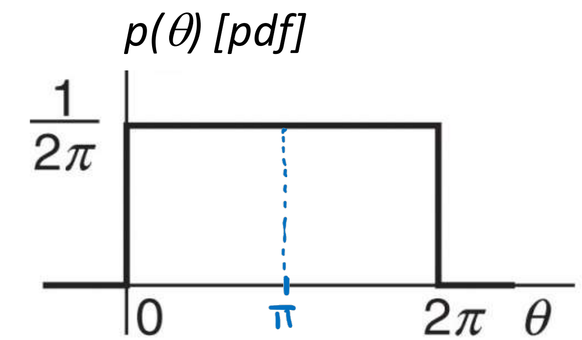
- **<u>Midterm 1: out this Thursday 10/01</u>**
  <u>[coverage: HWs 1-4 + Quizzes 1-4 + associated lectures]</u>
  - One week long take home exam posted to Canvas/Gradescope
  - Due Thurs 10/10/2024 on Gradescope, 11:59 pm
  - Open book/notes – must complete by yourself (honor code applies)
  - Quiz + HW solutions to be available (part of the notes)
  - **No quiz or HW this week (just midterm)**
  - **No office hours NEXT WEEK**

Last Time…

- Random Variables;
- Distributions for Discrete RVs;
- PDFs for Continuous RVs;
- Expected values and Expectation operator

Example: Moments of Uniform PDF

- Find mean, 2nd moment and variance of θ for spinning wheel problem



$$p(\theta) = \mathcal{U}_\theta[0, 2\pi] = \begin{cases} \frac{1}{2\pi}, & \text{if } \theta \in [0, 2\pi] \\ 0, & \text{o'wise} \end{cases}$$

$\rightarrow$ mean (1st moment) $= \bar{\theta} = E[\theta] = \int_{-\infty}^{\infty} \theta \cdot p(\theta)\, d\theta = \int_{-\infty}^{\infty} \theta \cdot \mathcal{U}_\theta[0, 2\pi]\, d\theta$

$\rightarrow$ Now use definition of $\mathcal{U}_\theta[0, 2\pi]$:

$$\bar{\theta} = \int_0^{2\pi} \theta \cdot \frac{1}{2\pi}\, d\theta \; \cancel{\left(+ \int_{-\infty}^{0} \theta \cdot 0\, d\theta + \int_{2\pi}^{\infty} \theta \cdot 0\, d\theta \right)}^{\,0} = \frac{1}{2\pi}\left[\frac{\theta^2}{2}\right]_0^{2\pi} = \boxed{\pi = \bar{\theta}}$$

$\rightarrow$ Now find 2nd moment: $E[\theta^2] = \int_{-\infty}^{\infty} \theta^2 \cdot p(\theta)\, d\theta = \int_{-\infty}^{\infty} \theta^2 \cdot \mathcal{U}_\theta[0, 2\pi]\, d\theta \Rightarrow \int_0^{2\pi} \theta^2 \cdot \frac{1}{2\pi}\, d\theta$

$\rightarrow$ following similar logic: $E[\theta^2] = \dfrac{(2\pi)^2}{3}$

$\rightarrow$ Then find $\text{var}(\theta) = E[(\theta - \bar{\theta})^2] \left(= \int_{-\infty}^{\infty} (\theta - \bar{\theta})^2 p(\theta)\, d\theta \right)$

(Fact #2 from end of previous lecture)

$$= E[\theta^2] - \bar{\theta}^2 = \frac{(2\pi)^2}{3} - (\pi^2) = \boxed{\frac{\pi^2}{3} = \text{var}(\theta)}$$

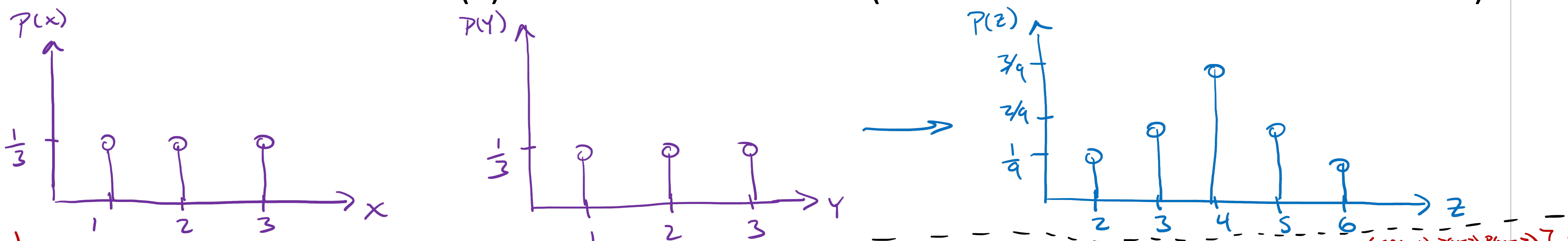
Today…

- Sums of independent random variables
  - Central limit theorem
- Gaussian (Normal) random variables and PDFs/distributions

READ SIMON BOOK, CHAPTER 2.6

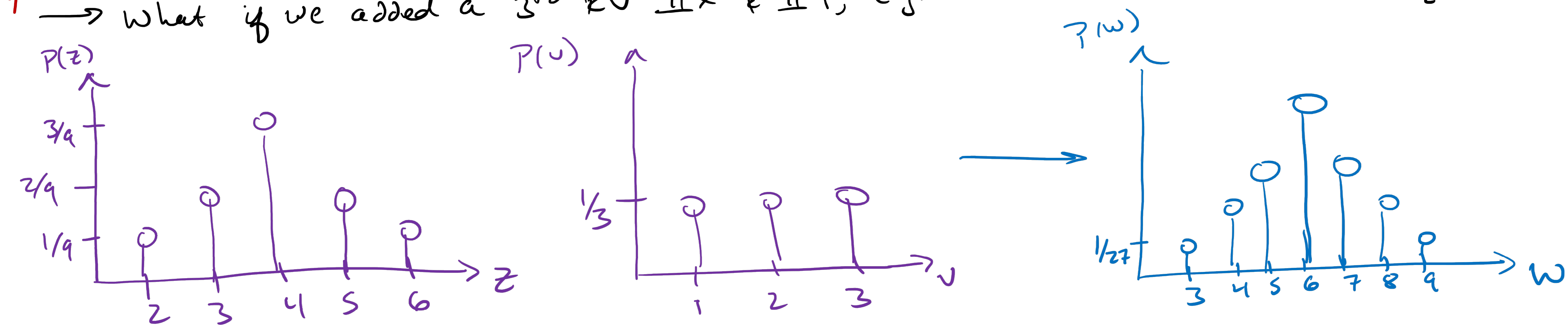
Sums of Independent Random Variables

- Suppose we have independent integer random variables X and Y with P(X) and P(Y)
- What's the distribution P(Z) of the sum Z = X + Y? (note: Z is another random variable!)



Resulting sum of discrete RVs has distribution given by convolution of starting RVs: $P(Z) = P(X) * P(Y) = \text{conv}([P(X=1), P(X=2), P(X=3)], [P(Y=1), P(Y=2), P(Y=3)])$

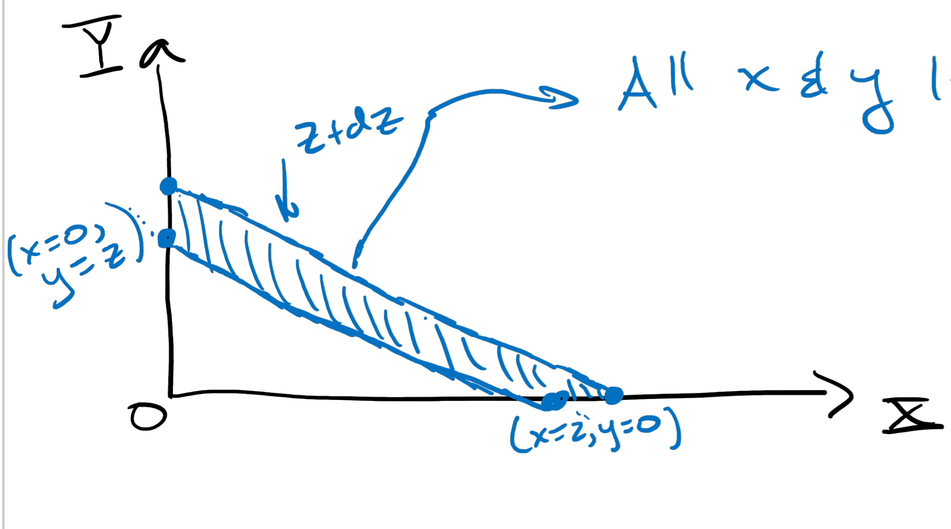
→ what if we added a 3rd RV $\perp\!\!\!\perp X$ & $\perp\!\!\!\perp Y$, e.g. $V \sim P(V)$? [so, get $W = \underbrace{X+Y}_{Z} + V = Z + V$]



Sums of Independent Random Variables

- Frequently need to add *independent continuous* random variables X and Y
- If we know p(X) and p(Y), what does the pdf p(Z) for Z = X+Y look like?

Suppose $Z = z$ (some fixed #): what are <u>all</u> the possible ways to get z from X & Y?
" " " " " " " " z+dz from x+dx & y+dy?



→ All x & y lying in differential strip map to points between upper & lower band lines z & z+dz in z-space

$$\rightarrow P(z \le Z \le z+dz) = P(X \,\&\, Y \text{ lie inside the differential strip})$$

$$= \iint_{\text{Diff. strip}} p_x(x) \cdot p_y(y)\, dx\, dy \quad [\text{ b/c } X \perp\!\!\!\perp Y]$$

→ But we also know that since $x+y=z$ = some known # → $y = z-x$ → $dy = dz$ (for some fixed x in integration)

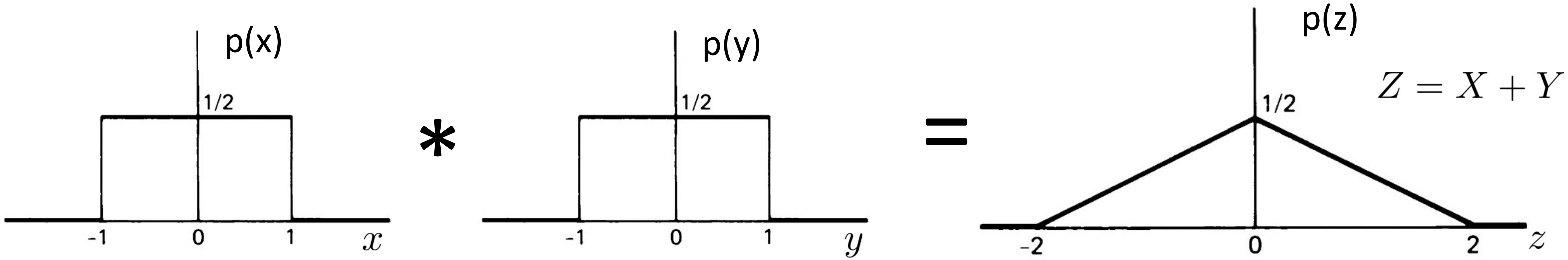
$$\rightarrow P(z \le Z \le z+dz) = \iint_{\text{Diff. strip}} p_x(x)\cdot p_y(z-x)\, dx\, dz \xrightarrow[\text{x \& z values}]{\text{Integrate over all possible}} P(\eta \le Z \le \xi) = \int_{\eta}^{\xi} \underbrace{\left[\int_{-\infty}^{\infty} p_x(x)\, p_y(z-x)\, dx\right]}_{p(z)} dz$$

$$(*) \rightarrow \boxed{\therefore\ p(z) = \int_{-\infty}^{\infty} p_x(x)\, p_y(z-x)\, dx = p_x(x) * p_y(y) = \text{convolution (integral) of functions } p(x) \,\&\, p(y)}$$

convolution operator

Example: Sum of Two Uniform RVs

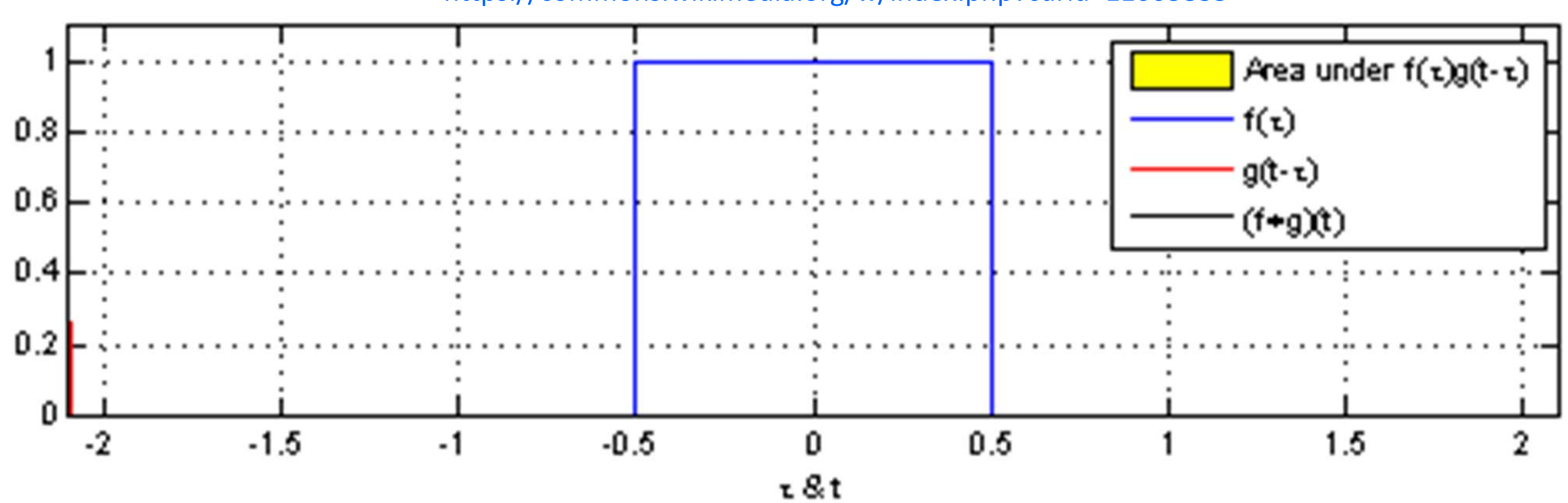
- Suppose p(X) = U(-1,1) and p(Y)=U(-1,1)



$$p(z) = \int_{-\infty}^{\infty} p_x(x)\, p_y(z-x)\, dx = p_x(x) * p_y(y)$$

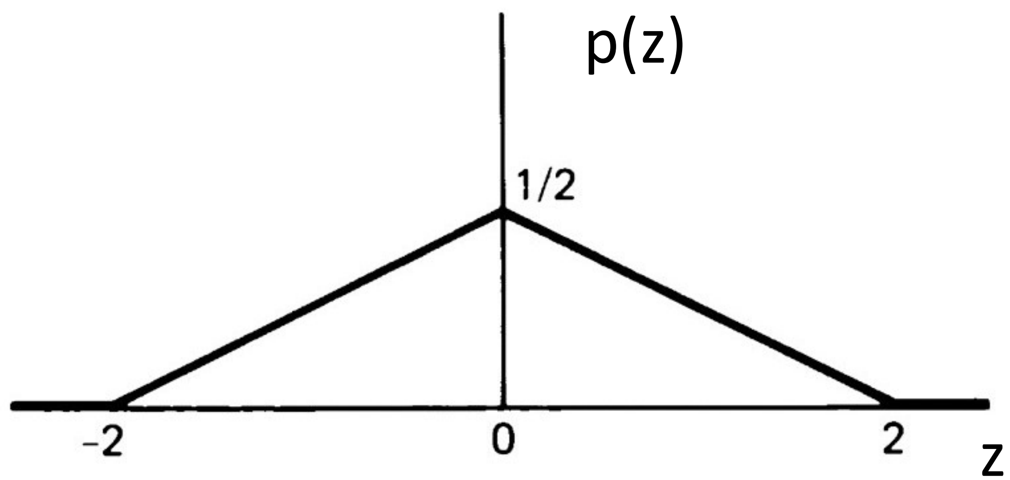
Animated Convolution Example (not to same scale as pdf figures)

https://commons.wikimedia.org/w/index.php?curid=11003835



Example: Sum of Three Uniform RVs

- Suppose X~U(-1,1), Y~U(-1,1), and V~U(-1,1), such that W = X + Y + V



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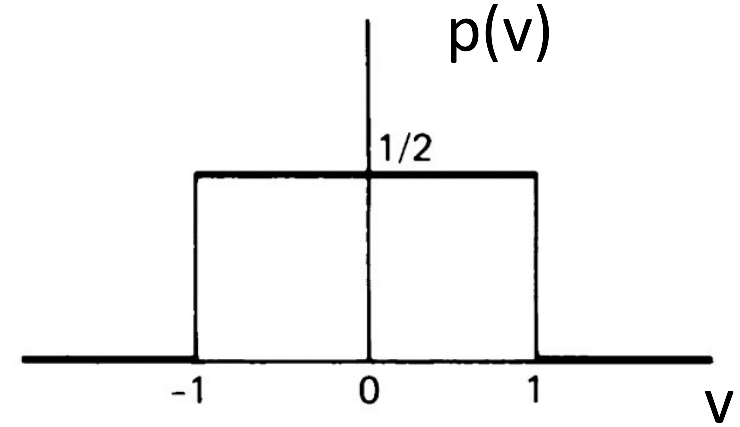
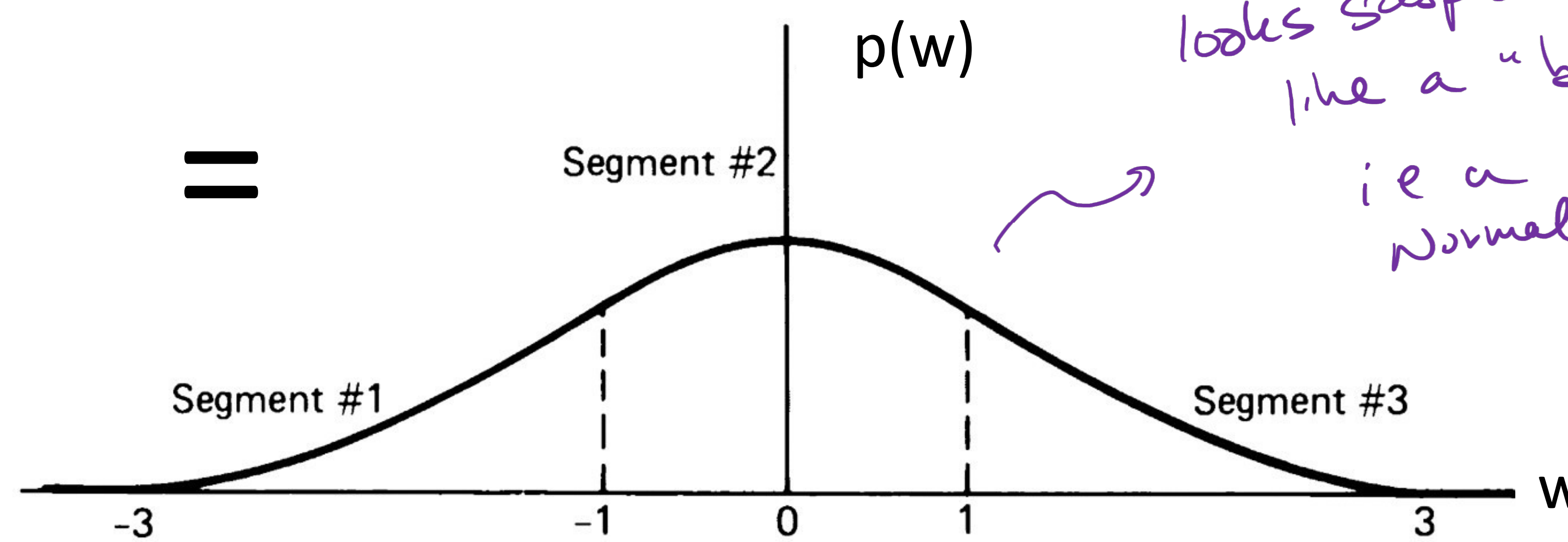


$$p_w(w) = \int_{-\infty}^{\infty} p_z(z)\, p_v(w-z)\, dz$$

$$= p_z(z) * p_v(v)$$

=



looks suspiciously like a "bell curve"

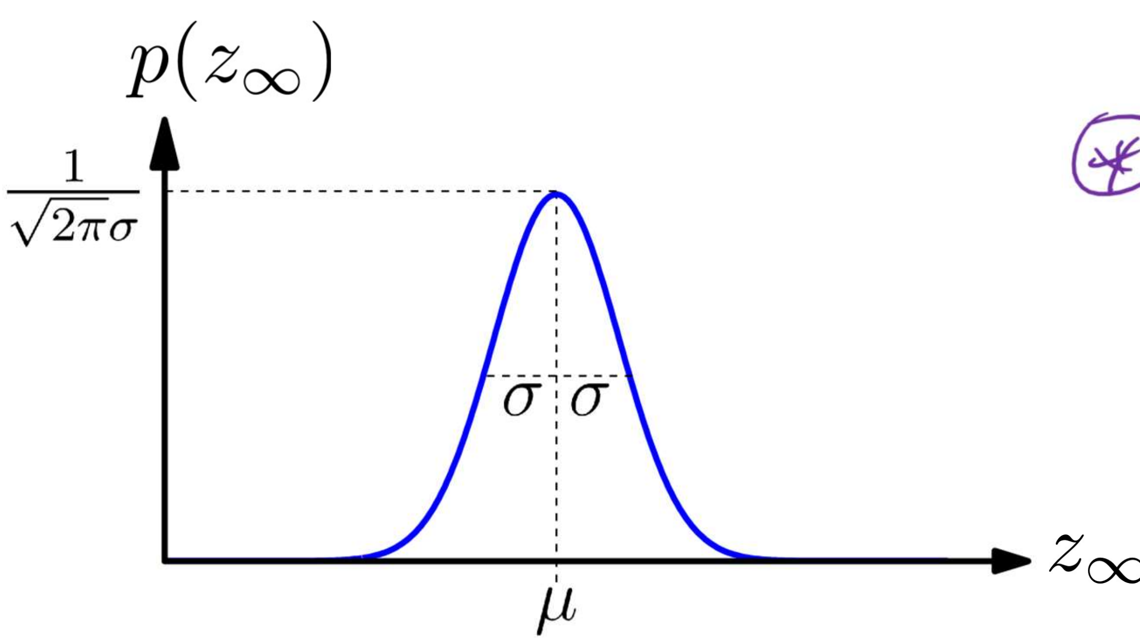
i.e. a Normal (Gaussian) pdf!!

The Central Limit Theorem

- If the sequence of RVs x_i , i=1,2,…,n,… consists of independent random variables, then (under some reasonably mild conditions) the pdf $p(z_n)$ of the sum

$$z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i$$

will tend to a **Gaussian pdf** as $n \to \infty$



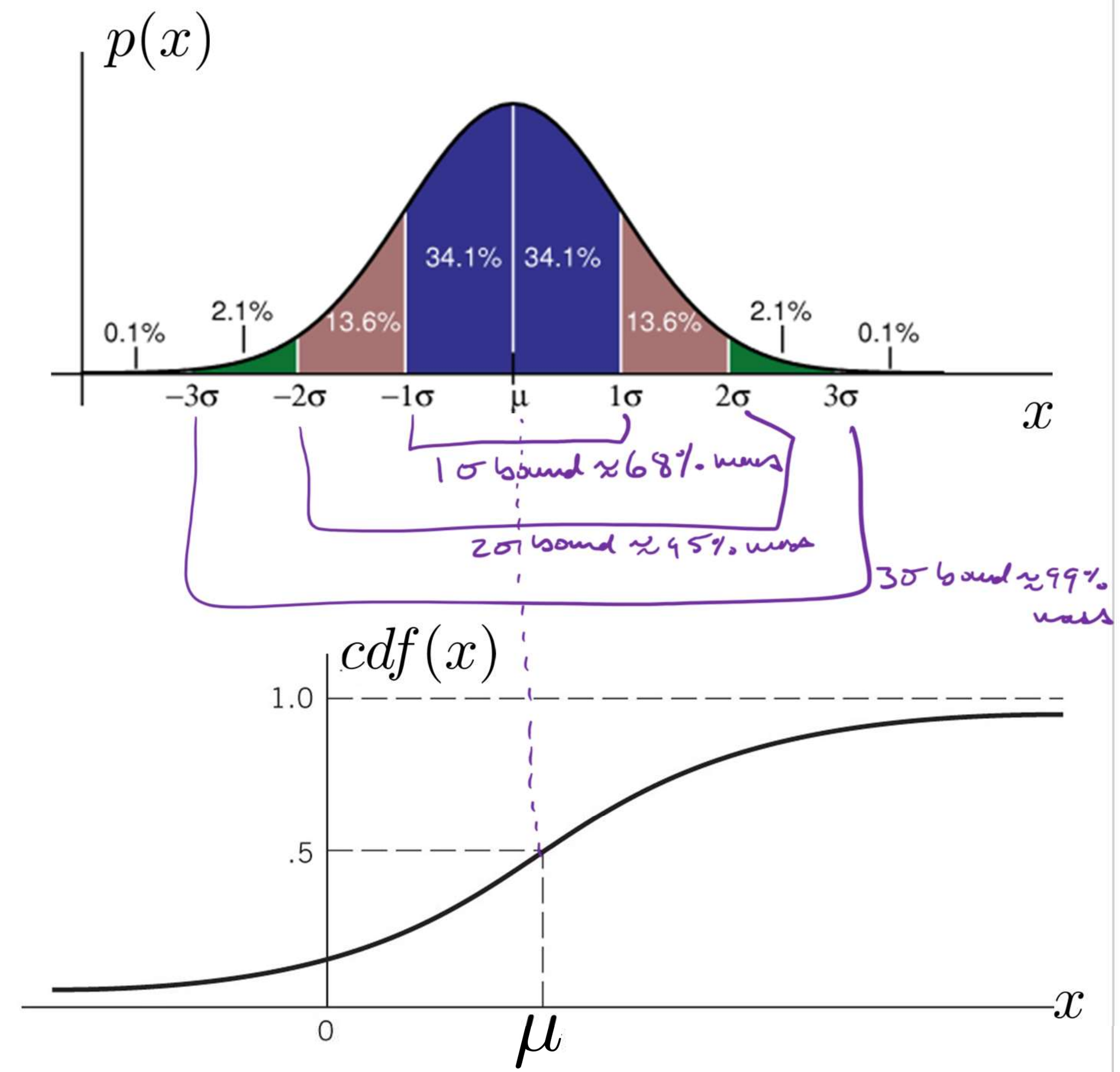
$$* \quad \boxed{\mathcal{N}_{z_\infty}(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\frac{1}{2\sigma^2}(z_\infty - \mu)^2\right]}$$

- Moreover, if the x_i are **independent and identically distributed (i.i.d.)** with zero mean and some finite variance σ^2, then $p(z_n) \to \mathcal{N}(0, \sigma^2) \quad \text{as } n \to \infty$

The Gaussian (or Normal) Distribution

- The continuous scalar random variable X with realizations x is normally distributed (has a Gaussian distribution) if its pdf is

$$\boxed{p(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right] = \mathcal{N}(\mu, \sigma^2)}$$



- This pdf is <u>completely</u> defined by:

$$\text{mean} = E[x] = \int_{-\infty}^{\infty} x p(x) dx = \mu$$

$$\text{var(x)} = E[(x-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 p(x) dx = \sigma^2$$

- Often we will write $x \sim \mathcal{N}(\mu, \sigma^2)$

Gaussians Are Your Friends!

- As we will see throughout course, Gaussians have many exceedingly useful properties
  - Completely described by mean and variance (1st two moments)
  - Single maximum peak at the mean (unimodal), symmetric
  - Show up naturally in physical systems (consequence of Central Limit Theorem)
  - Expectation operations generally easy (especially with linear functions/dynamics)
  - Marginalization of multivariate Gaussian pdfs gives univariate Gaussian pdfs
- Some useful Matlab functions
  - `normpdf`: to compute the pdf of N(μ,σ) at some value x
  - `normcdf`: to compute the cdf of N(μ,σ) at some value x
  - `randn`: to draw samples of standard normal random variables x ~ N(0,1)

To get random samples $s_i \sim \mathcal{N}(\mu, \sigma)$: *Step 1: draw $x_i \sim \mathcal{N}(0,1)$, i.e. $x_i =$ **randn**;

*Step 2: compute $s_i = \mu + (\sigma \cdot x_i)$;

$\rightarrow$ can easily check: $E[s_i] = E[\mu + \sigma \cdot x_i] = E[\mu] + E[\sigma \cdot x_i] = E[\mu] + \sigma \cdot E[x_i] = \mu + \sigma \cdot 0 = \mu$

$$\text{var}(s_i) = E[(s_i - \mu)^2] = (\text{...do it yourself!...}) = \sigma^2$$