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Brief paper

Determination of optimal feedback terminal controllers for general boundary conditions using generating functions ☆

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Abstract

Given a nonlinear system and a performance index to be minimized, we present a general approach to expressing the finite time optimal feedback control law applicable to different types of boundary conditions. Starting from the necessary conditions for optimality represented by a Hamiltonian system, we solve the Hamilton–Jacobi equation for a generating function for a specific canonical transformation. This enables us to obtain the optimal feedback control for fundamentally different sets of boundary conditions only using a series of algebraic manipulations and partial differentiations. Furthermore, the proposed approach reveals an insight that the optimal cost functions for a given dynamical system can be decomposed into a single generating function that is only a function of the dynamics plus a term representing the boundary conditions. This result is formalized as a theorem. The whole procedure provides an advantage over methods rooted in dynamic programming, which require one to solve the Hamilton–Jacobi–Bellman equation repetitively for each type of boundary condition. The cost of this favorable versatility is doubling the dimension of the partial differential equation to be solved.

Keywords: Optimal control; Optimality; Optimal trajectory; Feedback control; Nonlinear control

1. Introduction

In a general optimal terminal control problem where a performance index is optimized subject to a nonlinear system for a finite time, the optimal state feedback control law can be derived from the solution to the Hamilton–Jacobi–Bellman equation (HJBE). As it usually becomes a nonlinear partial differential equation (PDE) even for a relatively simple formulation, it does not have a closed-form solution in general, or is extremely difficult to find if any. Thus, in addition to solving the HJBE directly, there have been a myriad of alternative techniques for obtaining optimal feedback control (Fax & Murray, 2000; Jamshidi, 1974; Krikelis & Kiriakidis, 1992; Longmuir & Bohn, 1967; Matuszewski, 1973; Spencer, Timlin, Sain, & Dyke, 1996; Yoshida & Loparo, 1989).

Except for Fax and Murray (2000), these contributions tend to consider only a special type of boundary condition (soft constraint problem). Though Fax and Murray, under very restrictive assumptions, consider another representative type of boundary condition (hard constraint problem), they fail to provide a general solution technique for the problem they considered. This is due, to a considerable degree, to the singularity of the optimal cost function at the terminal time, which is inherent in the hard constraint problem.

Recently, Park and Scheeres (2003, 2004) studied the optimal feedback control problem in the context of necessary conditions for optimality represented by a Hamiltonian system. They observed that the optimal cost function can be derived from a generating function for a class of canonical transformations, which allowed them to devise a systematic methodology to evaluate the optimal feedback control law for general boundary conditions; the generating function can be algebraically adapted to provide optimal feedback control laws for different types of boundary conditions. The cost of this favorable property is to solve the difficult Hamilton–Jacobi equation (HJE) to find a generating function. For a given problem, the HJE is a PDE

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for twice as many variables as those for the HJBE; the computational load grows more significantly as the dimension increases. Later Park, Guibout, and Scheeres (2005) applied this approach to a nonlinear optimal rendezvous problem in a central gravity field and were able to demonstrate the effectiveness of their method.

The current work is a synthesis and extension of these results. We show that our method can be applied to the optimal control of a given system with a variety of different types of boundary conditions. Furthermore, by exploiting algebraic links between generating functions, we present an algorithm for evaluating optimal feedback controls for different types of boundary conditions without having to solve the HJE or the HJBE repetitively.

This paper is constructed as follows. We first formulate the optimal feedback control problem for two representative types of boundary conditions (Section 2). The core discussion of how to use generating functions to obtain optimal feedback controls for these boundary conditions follows (Section 3). Then, the relation between the optimal cost function and a generating function is established and formalized as a theorem (Section 4). Justifying our method, we propose a systematic numerical implementation based on series expansion, and apply it to a spacecraft optimal maneuver problem as an example (Section 5).

2. Problem formulation

Consider minimization of a general performance index for an arbitrary initial point (x, t)

$$J(x,t) = \phi(x(t_f), t_f) + \int_t^{t_f} L(x(\tau), u(\tau), \tau)\,\mathrm{d}\tau$$

subject to the following system with final time constraints:

$$\dot{x}(t) = F(x(t), u(t), t), \quad \psi(x(t_f), t_f) = 0.$$

Here, $x \in \Re^n$, $t \in \Re$, $u \in \Re^m$, and $\psi \in \Re^{p \leqslant n}$. We assume that there exists no constraints on the state and control trajectories. Our objective is to evaluate the optimal trajectory satisfying the final time constraints and to find the optimal feedback control for an arbitrary initial point $(x, t) \in \Re^{n+1}$.

We consider two representative problem formulations, which are characterized by the types of terminal boundary conditions[1]:

- *Hard constraint problem* (*HCP*): Terminal boundary condition for states is pre-specified to a fixed point in $\Re^n$.
- *Soft constraint problem* (*SCP*): Terminal boundary condition for states is not pre-specified, but is indirectly affected by minimizing the final time performance index $\phi(x, t)$.

The mathematical expressions for the boundary conditions become distinct from each other:

- HCP: $\phi(x(t_f), t_f) \equiv 0$, $\psi(x(t_f), t_f) = x(t_f) - x_f$. The pair $(x(t_f), t_f)$ is given a priori.
- SCP: $\psi(x(t_f), t_f)$ does not exist and $\phi(x(t_f), t_f) \in \Re$. The pair $(x(t_f), t_f)$, in contrast to the HCP, is not given a priori.

Now define the pre-Hamiltonian $\bar{H}$ as

$$\bar{H}(x, \lambda, u, t) = L(x, u, t) + \lambda^T F(x, u, t), \tag{1}$$

where λ represents the costates. Then, Pontryagin's principle provides the first order necessary conditions for optimality (Bryson & Ho, 1975):

$$\dot{x} = \bar{H}_\lambda(x, \lambda, u, t), \tag{2}$$

$$\dot{\lambda} = -\bar{H}_x(x, \lambda, u, t), \tag{3}$$

$$u = \arg\min_{\bar{u}} \bar{H}(x, \lambda, \bar{u}, t). \tag{4}$$

Substituting (4) into (1)–(3) defines a standard Hamiltonian system for states and costates only:

$$H(x, \lambda, t) = \bar{H}(x, \lambda, \arg\min_{\bar{u}} \bar{H}(x, \lambda, \bar{u}, t), t), \tag{5}$$

$$\dot{x} = H_\lambda(x, \lambda, t), \tag{6}$$

$$\dot{\lambda} = -H_x(x, \lambda, t). \tag{7}$$

Evaluating the optimal trajectory corresponds to solving this system of ordinary differential equations (ODEs) satisfying the given boundary conditions. For the HCP, the initial states x_0 and terminal states x_f are given and the initial costates λ_0 and terminal costates λ_f are to be determined. For SCP, the initial states are given, whereas terminal states, initial costates, and terminal costates are to be determined. Note, however, that the well-known transversality condition applies for this case (Bryson & Ho, 1975)

$$\lambda(t_f) = \frac{\partial \phi(x(t_f), t_f)}{\partial x(t_f)}, \tag{8}$$

which relates the terminal states and costates and provides an additional boundary condition for the SCP.[2] For both cases, we need to solve a system of ODEs with split boundary conditions, thus the optimal control problem is again reduced to a two-point boundary value problem (TPBVP).

3. Optimal solution by generating function

Given the Hamiltonian system (5)–(7), we treat the trajectory $(x(t), \lambda(t))$ as a transformation between (moving) terminal coordinates (x, λ, t) and (fixed) initial coordinates (x_0, λ_0, t_0),

[1] This classification is just for simplicity of argument. It does not imply that the applicability of our approach is confined to these two kinds of boundary conditions. There exist other types of boundary conditions between these two extreme cases, which are specified by a terminal hyperplane. Later, we will briefly discuss how to manipulate such situations.

[2] Other than the HCP and SCP, if there exists a terminal hyperplane $\psi(x(t_f), t_f) \in \Re^{p<n}$, then we have a more general type of terminal boundary conditions, where the terminal states are partially determined and the transversality condition relates the undetermined terminal states with terminal costates (Bryson & Ho, 1975).

which is by definition a canonical transformation.[3] Thus, there exist generating functions for this transformation which can have the following four classical forms:

$$F_1(x, x_0, t, t_0), \quad F_2(x, \lambda_0, t, t_0),$$
$$F_3(\lambda, x_0, t, t_0), \quad F_4(\lambda, \lambda_0, t, t_0).$$

These functions can *generate* the given canonical transformation by the following relations:

$$\lambda = \frac{\partial F_1(x, x_0, t, t_0)}{\partial x}, \quad \lambda_0 = -\frac{\partial F_1(x, x_0, t, t_0)}{\partial x_0}, \tag{9}$$

$$\lambda = \frac{\partial F_2(x, \lambda_0, t, t_0)}{\partial x}, \quad x_0 = \frac{\partial F_2(x, \lambda_0, t, t_0)}{\partial \lambda_0}, \tag{10}$$

$$x = -\frac{\partial F_3(\lambda, x_0, t, t_0)}{\partial \lambda}, \quad \lambda_0 = -\frac{\partial F_3(\lambda, x_0, t, t_0)}{\partial x_0}, \tag{11}$$

$$x = \frac{\partial F_4(\lambda, \lambda_0, t, t_0)}{\partial \lambda}, \quad x_0 = -\frac{\partial F_4(\lambda, \lambda_0, t, t_0)}{\partial \lambda_0}. \tag{12}$$

Furthermore, they satisfy their own version of the HJE:

$$\frac{\partial F_1(x, x_0, t, t_0)}{\partial t} + H\left(x, \frac{\partial F_1(x, x_0, t, t_0)}{\partial x}, t\right) = 0, \tag{13}$$

$$\frac{\partial F_2(x, \lambda_0, t, t_0)}{\partial t} + H\left(x, \frac{\partial F_2(x, \lambda_0, t, t_0)}{\partial x}, t\right) = 0, \tag{14}$$

$$\frac{\partial F_3(\lambda, x_0, t, t_0)}{\partial t} + H\left(-\frac{\partial F_3(\lambda, x_0, t, t_0)}{\partial \lambda}, \lambda, t\right) = 0, \tag{15}$$

$$\frac{\partial F_4(\lambda, \lambda_0, t, t_0)}{\partial t} + H\left(-\frac{\partial F_4(\lambda, \lambda_0, t, t_0)}{\partial \lambda}, \lambda, t\right) = 0. \tag{16}$$

Alternatively, we can also derive a similar result between a set of *fixed* terminal coordinates (x_f, λ_f, t_f) and the *moving* initial coordinates (x, λ, t) with $t \leqslant t_f$. For example, for F_1 the following relation holds for this set of coordinates:

$$\begin{aligned} &-\frac{\partial F_1(x_f, x, t_f, t)}{\partial t} + H\left(x, -\frac{\partial F_1(x_f, x, t_f, t)}{\partial x}, t\right) = 0, \\ &\lambda = -\frac{\partial F_1(x_f, x, t_f, t)}{\partial x}, \quad \lambda_f = \frac{\partial F_1(x_f, x, t_f, t)}{\partial x_f}. \end{aligned} \tag{17}$$

Instead of solving the TPBVP numerically and repetitively for varying boundary conditions, our approach solves for these generating functions. The choice of the appropriate generating function depends on the type of boundary condition. For the HCP, $F_1(x, x_0, t, t_0)$ is the most appropriate as we know the initial and terminal states. Indeed if we can find F_1, we can directly evaluate the initial and final costates from (9)

$$\lambda_f = \left.\frac{\partial F_1}{\partial x}\right|_{t=t_f, x=x_f} = \frac{\partial F_1(x_f, x_0, t_f, t_0)}{\partial x_f}, \tag{18}$$

$$\lambda_0 = -\left.\frac{\partial F_1}{\partial x_0}\right|_{t=t_f, x=x_f} = -\frac{\partial F_1(x_f, x_0, t_f, t_0)}{\partial x_0}. \tag{19}$$

[3] Refer to Goldstein (1965), Greenwood (1977), and Guibout and Scheeres (2003) for a review of canonical transformations and generating functions.

Furthermore, since any time $t \leqslant t_f$ can be the initial time, (19) should hold for arbitrary initial conditions $x = x(t)$ and $\lambda = \lambda(t)$:

$$\lambda = -\frac{\partial F_1(x_f, x, t_f, t)}{\partial x}. \tag{20}$$

Substitution of (20) into (4) yields the optimal feedback control for the HCP:

$$u = \arg\min_{\bar{u}} \bar{H}\left(x, -\frac{\partial F_1(x_f, x, t_f, t)}{\partial x}, \bar{u}, t\right). \tag{21}$$

For the SCP, it is not immediately apparent which generating function is the most appropriate since we have $3n$ unknown boundary conditions $(\lambda_0, x_f, \lambda_f)$. Whatever generating function we may choose, we must solve a set of implicit equations. However, since we are interested in both HCP and SCP, we choose F_1 to avoid evaluating an additional generating function.

Consider again the $2n$ relations (18)–(19) along with the n transversality conditions (8). Taking the initial states x_0 as independent parameters and solving these $3n$ implicit equations for $(x_f, \lambda_0, \lambda_f)$ results in

$$x_f = x_f(x_0, t_f, t_0), \tag{22}$$

$$\lambda_0 = \lambda_0(x_0, t_f, t_0), \tag{23}$$

$$\lambda_f = \lambda_f(x_0, t_f, t_0). \tag{24}$$

Finally, a similar procedure leads to the optimal feedback control for the SCP[4] :

$$u = \arg\min_{\bar{u}} \bar{H}\left(x, -\left.\frac{\partial F_1(x_f, x, t_f, t)}{\partial x}\right|_{x_f = x_f(x, t_f, t)}, \bar{u}, t\right). \tag{25}$$

As is shown, once the F_1 generating function has been found, the unknown boundary conditions are simply evaluated by a series of partial differentiations and algebraic manipulations without solving a differential equation. Furthermore, the evaluation of the initial costates λ_0 enables us to develop the optimal trajectory by simple forward integration.

In order to find F_1, we need to solve the difficult HJE. A useful observation is that we do not necessarily have to solve for F_1. The Legendre transformations

$$F_2(x, \lambda_0, t, t_0) = F_1(x, x_0, t, t_0) + \lambda_0^T x_0, \tag{26}$$

$$F_3(\lambda, x_0, t, t_0) = F_1(x, x_0, t, t_0) - \lambda^T x, \tag{27}$$

$$F_4(\lambda, \lambda_0, t, t_0) = F_2(x, \lambda_0, t, t_0) - \lambda^T x \tag{28}$$

enable us to compute F_1 from any other generating functions simply by algebraic manipulations and partial differentiations. This observation is at the heart of our application and can provide a computationally significant advantage, as we can initially choose a generating function which may be easier to solve than others. Later we will show that for our applications solving

[4] For the terminal constraint given by a hyperplane $\psi(x(t_f), t_f) = 0$ in $\Re^{p<n}$, we will have mixed terminal conditions for both states and costates in general. In this case, a more generalized kind of generating function is required, which would mix all four kinds of variables (initial and terminal states and costates).

the HJE for F_2 or F_3 is far easier than for F_1 or F_4, and take full advantage of this property.

4. The optimal cost function and the F_1 generating function

We have shown that generating functions can be used to find optimal feedback control laws. This strongly suggests that there should be connections between the optimal cost function (which is the solution to the HJBE) and the generating functions (which are the solutions to the HJE). In order to explore this, we first present the sufficient condition for an optimal control. Then by establishing and proving a theorem, we show that the optimal cost function can be expressed as a combination of the F_1 generating function and the terminal cost function $\phi(x_f, t_f)$.

Theorem 1 (*Sufficient condition for optimality*). *Suppose the following two conditions hold*:

(1) *In the domain considered for* (x, t), *the pre-Hamiltonian* (1) *has a unique minimizer with respect to control such that*

$$u = \arg\min_{\bar{u}} \bar{H}\left(x, \frac{\partial J}{\partial x}, \bar{u}, t\right).$$

(2) $J(x, t)$ *is sufficiently smooth* (*or analytic*) *and satisfies the HJBE with the given boundary condition*

$$0 = \frac{\partial J}{\partial t}(x, t) + \min_{\bar{u}} \bar{H}\left(x, \frac{\partial J}{\partial x}, \bar{u}, t\right),$$
$$J(x(t_f), t_f) = \phi(x(t_f), t_f) \quad \text{on } \psi(x(t_f), t_f) = 0. \tag{29}$$

Then J is the optimal cost and u is the corresponding optimal feedback control law.

Proof. Refer to Bryson and Ho (1975) or Athans and Falb (1966, pp. 351–363). □

Theorem 2 (*Optimal cost derived from* F_1). *For both the HCP and SCP*, *the function*

$$V(x, t) = -F_1(x_f, x, t_f, t) + \phi(x_f, t_f)$$

satisfies the HJBE and the corresponding boundary condition (29), *thus it is the optimal cost function. Furthermore, the optimal feedback control can be expressed as*

$$u = \arg\min_{\bar{u}} \bar{H}\left(x, \frac{\partial V(x, t)}{\partial x}, \bar{u}, t\right).$$

Proof. We first consider the HCP. Note that $\phi(x_f, t_f) \equiv 0$ by definition. Since $-V = F_1$ satisfies the modified HJE and the associated relations (17), we have

$$0 = \frac{\partial V}{\partial t}(x, t) + H\left(x, \frac{\partial V}{\partial x}, t\right)$$
$$\Rightarrow 0 = \frac{\partial V}{\partial t}(x, t) + \min_{\bar{u}} \bar{H}\left(x, \frac{\partial V}{\partial x}, \bar{u}, t\right), \tag{30}$$

which is indeed the HJBE for V.

Now consider the boundary condition for V. For the HCP, $V(x = x_f, t = t_f) = 0$ is the boundary condition to be satisfied, as is easily seen from the above statement of the sufficient condition. First, recall that x and t are *moving* variables, representing initial state and time, respectively. As the (candidate) cost function V is represented by the F_1 generating function, we should define a functional form for F_1 which reflects this boundary condition at $t = t_f$. Noting that F_1 treats the trajectory of our dynamical system as a canonical transformation between initial and terminal coordinates, we see that the flow defined by F_1 should define the following simple identity:

$$x = x_f, \quad \lambda = \lambda_f \quad \text{at } t = t_f.$$

Here lies a difficulty, as F_1 cannot realize such an identity transformation at $t = t_f$; it becomes singular as $t \to t_f$ since its independent arguments, the initial and terminal positions, are equal and not independent at $t = t_f$.

Instead, we can easily define the functional form of F_2 which realizes this identity; indeed the form $F_2 = x_f^T \lambda$ generates the identity transformation related to our solution flow, that is, from (10)

$$x = \frac{\partial F_2}{\partial \lambda} = x_f, \quad \lambda_f = \frac{\partial F_2}{\partial x_f} = \lambda.$$

Thus, we know the form of F_2 at the terminal time. Using this, we indirectly determine the value of F_1 at the terminal time from the Legendre transformation (26)[5]

$$F_1(x_f, x_f, t_f, t_f) = [F_2(x_f, \lambda, t_f, t) - \lambda^T x]|_{t=t_f} \equiv 0,$$

which finally yields

$$V(x_f, t_f) = -F_1(x_f, x_f, t_f, t_f) \equiv 0. \tag{31}$$

Combining (30) and (31), we see that $V = -F_1$ satisfies the HJBE and the associated boundary condition. The optimal control law has been obtained from (21), which completes the proof for the HCP.

Now consider the SCP. First from the HJE and associated relations (17) along with the transversality conditions (8), we have

$$\lambda_f = \frac{\partial F_1}{\partial x_f} = \frac{\partial \phi}{\partial x_f}. \tag{32}$$

Also recall from (22) that $x_f = x_f(x, t_f, t)$. Taking partial derivatives of $V = -F_1 + \phi$ with respect to t and simply applying the chain rule, we obtain

$$\frac{\partial V}{\partial t} = -\frac{\partial F_1}{\partial t} - \frac{\partial F_1}{\partial x_f}\left(\frac{\partial x_f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial x_f}{\partial t}\right)$$
$$+ \frac{\partial \phi}{\partial x_f}\left(\frac{\partial x_f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial x_f}{\partial t}\right).$$

[5] We see that F_1 is singular as it loses independence of its arguments. In fact, this is an equivalent statement that the optimal cost function becomes singular at the terminal time for the HCP.

Introduction of (32) into the above yields

$$\frac{\partial V}{\partial t} = -\frac{\partial F_1}{\partial t}.$$

Similarly, taking partial derivatives of V with respect to x gives

$$\frac{\partial V}{\partial x} = -\frac{\partial F_1}{\partial x} - \frac{\partial F_1}{\partial x_{\mathrm{f}}}\frac{\partial x_{\mathrm{f}}}{\partial x} + \frac{\partial \phi}{\partial x_{\mathrm{f}}}\frac{\partial x_{\mathrm{f}}}{\partial x} = -\frac{\partial F_1}{\partial x}.$$

Now substituting $-\partial F_1/\partial t = \partial V/\partial t$ and $-\partial F_1/\partial x = \partial V/\partial x$ into the HJE (17) finally yields

$$\begin{aligned} 0 &= \frac{\partial V}{\partial t}(x,t) + H\left(x, \frac{\partial V}{\partial x}, t\right) \\ \Rightarrow 0 &= \frac{\partial V}{\partial t}(x,t) + \min_{\bar{u}} \bar{H}\left(x, \frac{\partial V}{\partial x}, \bar{u}, t\right), \end{aligned}$$

which is indeed the HJBE for V. In order to consider the boundary condition for the SCP, note that the evaluation of expression (22) at the terminal time simply yields the identity:

$$x_{\mathrm{f}}(x, t_{\mathrm{f}}, t)|_{t=t_{\mathrm{f}}, x=x_{\mathrm{f}}} = x_{\mathrm{f}}.$$

Then similarly as in the HCP, consideration of the Legendre transformation at the terminal time yields

$$\begin{aligned} V(x,t)|_{x=x_{\mathrm{f}}, t=t_{\mathrm{f}}} &= [-F_1(x_{\mathrm{f}}(x, t_{\mathrm{f}}, t), x, t_{\mathrm{f}}, t) \\ &\quad + \phi(x_{\mathrm{f}}(x, t_{\mathrm{f}}, t), t_{\mathrm{f}})]_{x=x_{\mathrm{f}}, t=t_{\mathrm{f}}} \\ &= \phi(x_{\mathrm{f}}, t_{\mathrm{f}}), \end{aligned}$$

which satisfies the boundary condition for the SCP. Hence $V = -F_1 + \phi$ is the optimal cost function for the SCP. The optimal control law has been determined from (25), which completes the proof for the SCP. □

Note that the same F_1 function is employed for both the HCP and SCP. However, the ultimate cost function and the associated feedback control law are quite different from each other. F_1 is used directly for the HCP, whereas we introduce $x_{\mathrm{f}} = x_{\mathrm{f}}(x, t_{\mathrm{f}}, t)$ into F_1 to evaluate the optimal cost for the SCP. Also note that once F_1 is determined, all additional steps of finding the cost function for the SCP contain only algebraic manipulations and partial differentiations; we do not need to solve the difficult HJE again. Thus, we have identified a more fundamental function that lies behind the optimal cost function for the given dynamical system, which can be used for different sets of boundary conditions. The cost of this favorable versatility is that the HJE is a function of $2n$ variables as opposed to n variables for the HJBE. As the dimension of the problem increases, the computational capability of solving the HJE becomes more restrictive than that of HJBE due to the 'curse' of dimensionality.

Summarized, all these results imply that the optimal feedback control problem can be considered as part of a more comprehensive field of canonical transformations for Hamiltonian systems.

5. Numerical implementation and an illustrative example

In the following, we briefly discuss a method of solving for the generating functions of a Hamiltonian system discussed in Guibout and Scheeres (2003), and then use this method for an illustrative example of Theorem 2.

5.1. A solution strategy for generating functions

We have shown that the F_1 generating function can be used to solve optimal feedback control problems for different types of boundary conditions. In order to find F_1, we need to solve the HJE at least (and at most) once for one kind of generating function. This requires, at the least, that the functional form of the generating function be specified at some epoch. Now recall that for our canonical transformation the old and new coordinates are equal when $t = t_0$, and thus the generating function (solution to the HJE) must define an identity transformation at $t = t_0$. As is discussed in the previous section, F_1 (also F_4) cannot generate such a transformation, on the other hand F_2 (also F_3) are well defined at $t = t_0$ and can generate the identity transformation.[6] Indeed, as given in the proof of the previous section, $F_2(x, \lambda_0, t = t_0, t_0) = x^{\mathrm{T}}\lambda_0$ generates the identity transformation. Also, similar arguments hold for $F_3(\lambda, x_0, t = t_0, t_0) = -x_0^{\mathrm{T}}\lambda$. Therefore, given the Hamiltonian of a system, we can solve the HJE for F_2 or F_3 from the initial time and then evaluate the other generating functions through the Legendre transformations at a later time.

For example, suppose we have computed $F_2(x, \lambda_0, t, t_0)$. In the second equation of (10), assuming the uniqueness of inversion for the initial costates λ_0,[7] we can express λ_0 as a function of the initial and terminal states (x_0, x):

$$\lambda_0 = \lambda_0(x_0, x, t, t_0).$$

Then, substituting into (26) yields F_1 as a function of the desired variables:

$$\begin{aligned} F_1(x, x_0, t, t_0) &= F_2(x, \lambda_0(x_0, x, t, t_0), t, t_0) \\ &\quad - x_0^{\mathrm{T}}\lambda_0(x_0, x, t, t_0). \end{aligned}$$

In Guibout and Scheeres (2003), this fact is used to derive a specific solution algorithm for a class of problems where

(1) the system $\dot{x} = F(x(t), u(t), t)$ is analytic and has an *unforced* zero equilibrium, i.e., $F(x = 0, u = 0, t) \equiv 0$,
(2) the integrand of the cost function $L(x(t), u(t), t)$ is analytic.

Following the procedure in Guibout and Scheeres (2003), we expand the Hamiltonian H and the generating functions F_2 as Taylor series in their spatial arguments about the zero condition, where these expansions can be made to arbitrarily high

[6] This identity transformation is invariant and does not depend on the boundary conditions of a given system.

[7] That is, there exists an open neighborhood in $\mathscr{R}^n$ such that $\partial^2 F_2(x, \lambda_0, t, t_0)/\partial\lambda_0^2 \neq 0$.

order. Then, using the HJE, we find a series of ODEs for the coefficients of the series expansion F_2. We solve these equations for F_2, using our initial boundary conditions to generate initial conditions for the ODEs. Then we use the Legendre transformations and inversion of series to compute the coefficients of F_1.

5.2. Example: nonlinear optimal maneuvers in a central gravity field

Consider minimizing

$$J = \frac{1}{2} x_{\mathrm{f}}^{\mathrm{T}} Q_{\mathrm{f}} x_{\mathrm{f}} + \frac{1}{2}\int_{t_0}^{t_{\mathrm{f}}} u^{\mathrm{T}}(t)u(t)\,\mathrm{d}t$$

subject to the system dynamics

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ 2x_4 - (1+x_1)\left(\dfrac{1}{r^3}-1\right) + u_1 \\ -2x_3 - x_2\left(\dfrac{1}{r^3}-1\right) + u_2 \end{bmatrix},$$

where $r = \sqrt{(x_1+1)^2 + x_2^2}$. This system represents the planar motion of a particle in a central gravity field, expressed in a rotating coordinate frame. The origin of this frame corresponds to a circular orbit, the coordinates (x_1, x_2, x_3, x_4) represent radial displacement, tangential displacement, radial velocity, and tangential velocity deviations from the circular orbit, and (u_1, u_2) represent the radial and tangential control input, respectively.[8]

Formulating this problem as a Hamiltonian system and using the well-known Lawden's primer vector theory (Lawden, 1963), we obtain the Hamiltonian as a function of states and costates only:

$$H = -\frac{1}{2}(\lambda_3^2 + \lambda_4^2) + \lambda_1 x_3 + \lambda_2 x_4 + \lambda_3\left[2x_4 \cdots - (1+x_1)\left(\frac{1}{r^3}-1\right)\right] - \lambda_4\left[2x_3 + x_2\left(\frac{1}{r^3}-1\right)\right]. \tag{33}$$

Now, we can apply our generating function approach to the Hamiltonian above. In our implementation, we expand H, F_1, and F_2 to the fourth order using Matlab®, then follow the same procedure as in Guibout and Scheeres (2003).

As an illustration of the optimal trajectory generated by the F_1 generating function, we choose a set of initial condition such that the initial positions are located along the circle of radius $r = 0.10$ and initial velocities are identically zero (that is, $x_0 = [0.10\cos\theta\, 0.10\sin\theta\, 0\, 0]$, with θ varying from 0 to 2π radians).

[8] Refer to Park et al. (2005) for more detailed description of the system dynamics as well as the formulation of Hamiltonian system and optimal control logic.

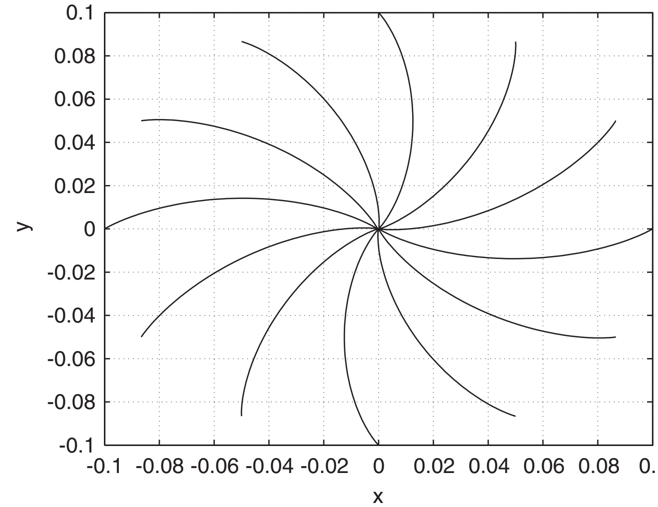

Fig. 1. Radial position x_1 vs. tangential position x_2, $r = 0.10, 0 \leqslant \theta \leqslant 360$ (optimal rendezvous HCP).

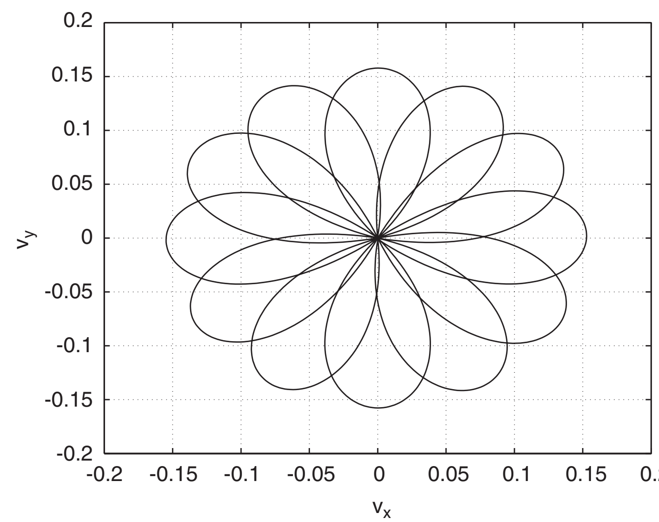

Fig. 2. Radial velocity x_3 vs. tangential velocity x_4, $r = 0.10, 0 \leqslant \theta \leqslant 360$ (optimal rendezvous HCP).

For the HCP, we locate the terminal boundary condition at the origin (that is, $x_{\mathrm{f}} = [0\ \ 0\ \ 0\ \ 0]$), Figs. 1–2 show the trajectories for position and velocity variables for the given set of initial condition. It is clear that our solutions satisfy the terminal boundary condition.

For the SCP where the terminal condition is not fixed by definition, we adjust the terminal weight function Q_{f} to indirectly affect the terminal condition. Fig. 3 shows the trajectories for position variables for $Q_{\mathrm{f}} = \alpha[10\ \ 10\ \ 2.5\ \ 2.5]$, $\alpha = 5, 50, 500$. It is seen that the heavier the terminal weight is, the closer the terminal condition is to the origin.

Note that the initial conditions for all these results are obtained algebraically from the F_1 generating function which we only computed once. Given this solution, we can generate optimal trajectories for any initial and final conditions within the

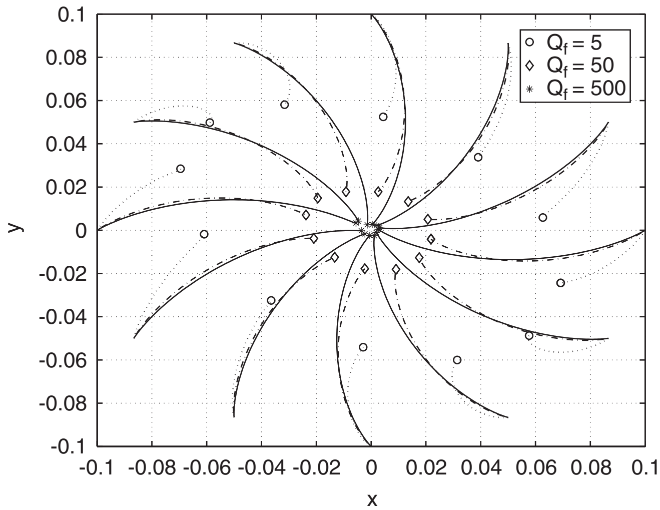

Fig. 3. Radial position x_1 vs. tangential position x_2, $r = 0.10, \alpha = 5, 50, 500, 0 \leqslant \theta \leqslant 360$ (optimal rendezvous SCP).

convergence region of our series expansion, for a variety of boundary conditions.

6. Conclusion

We have introduced a new expression of the optimal feedback control that is valid for changing types of boundary conditions. To exemplify this, we have applied our approach to two extreme cases of boundary conditions, the hard and soft constraint problem. Proving a theorem that decomposes the optimal cost function into a generating function of the relevant canonical transformation of the necessary conditions plus a boundary function, our method is advantageous over the classical dynamical programming approach in that we do not need to solve the HJBE repetitively for different types of boundary conditions. Furthermore, the final time singularity for the hard constraint problem can be circumvented to evaluate the optimal feedback control and cost function. These imply that the optimal feedback control problem can be treated in the context of canonical transformations for Hamiltonian systems. The cost of this approach is a doubling of the dimension of the HJE we must solve, compared with the HJBE.

Acknowledgments

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References

Athans, M., & Falb, P. L. (1966). *Optimal control: An introduction to the theory and its applications*. New York, NY: McGraw-Hill.

Bryson, A. E., & Ho, Y. (1975). *Applied optimal control*. London, England: Hemisphere Publishing Corporation.

Fax, J. A., & Murray, R. M. (2000). Finite-horizon optimal control and stabilization of time-scalable systems. *Proceedings of the 39th IEEE conference on decision and control* (pp. 748–753), Sydney, Australia.

Goldstein, H. (1965). *Classical mechanics*. Reading, MA: Addision-Wesley.

Greenwood, D. T. (1977). *Classical dynamics*. Englewood Cliffs, NJ: Prentice-Hall.

Guibout, V., & Scheeres, D. J. (2003). Solving relative two point boundary value problems: Applications to spacecraft formation flight transfers. *Journal of Guidance, Control, and Dynamics*, *27*(4), 693–704.

Jamshidi, M. (1974). 3-stage near-optimum design of nonlinear control processes. *Proceedings of the Institution of Electrical Engineers—London*, *121*(8), 886–892.

Krikelis, N. J., & Kiriakidis, K. I. (1992). Optimal feedback control of nonlinear systems. *International Journal of Systems Science*, *23*(12), 2141–2153.

Lawden, D. F. (1963). *Optimal trajectories for space navigation*. London, England: Butterworths.

Longmuir, A. G., & Bohn, E. V. (1967). The synthesis of suboptimal feedback control laws. *IEEE Transactions on Automatic Control*, *12*, 755–759.

Matuszewski, J. P. (1973). Suboptimal terminal feedback control of nonstationary nonlinear systems. *IEEE Transactions on Automatic Control*, *18*(3), 271–274.

Park, C., Guibout, V. M., & Scheeres, D. J. (2005). Solving optimal continuous thurst rendezvous problems with generating functions. *Journal of Guidance*, *Control*, *and Dynamics*, accepted for publication.

Park, C. & Scheeres, D. J. (2003). Solutions of optimal feedback control problem using Hamiltonian dynamics and generating functions. *Proceedings of the 43rd IEEE conference on decision and control* (pp. 1222–1227), Maui, Hawaii.

Park, C., & Scheeres, D. J. (2004). A generating function for optimal feedback control laws that satisfies the general boundary conditions of a system. *Proceedings of the American Control Conference* (pp. 679–684), Boston, MA.

Spencer, B. F., Timlin, T. L., Sain, M. K., & Dyke, S. J. (1996). Series solution of a class of nonlinear optimal regulators. *Journal of Optimization Theory and Applications*, *91*(2), 321–345.

Yoshida, T., & Loparo, K. A. (1989). Quadratic regulatory theory for analytic non-linear systems with additive controls. *Automatica*, *25*(4), 531–544.

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