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ASEN 5050
Fall 2024

HW 8

Problem 1 → Given: $r_p = 7{,}500$ km, $r_a = 8{,}500$ km, $i = 105°$, $P = 110$ min, $R_{Planet} = 6500$ km,
$a_{Planet} = 2.25$ AU, S/C → SSO

Assumptions → 2BP, $\mu = G(m_{Planet} + m_{S/C}) \rightarrow \because m_{S/C} << m_{Planet} \rightarrow \mu = G m_{Planet}$

AU = 149597870.7 km, $G = 6.67 \times 10^{-20} \frac{km^3}{kg s^2}$

Planet - Sun → 2BP → $\mu = G(m_{Planet} + m_{Sun}) \rightarrow m_{Planet} << m_{Sun} \rightarrow \mu = G m_{Sun}$

Planet-Sun 2BP → $P_{Planet} = 2\pi \sqrt{\frac{a_{Planet}}{\mu}} = 1.0651 \times 10^{8}$ s

$\dot{\Omega} = \left(\frac{360° \cdot \pi}{180}\right)\left(\frac{1}{P_{Planet}}\right) = 5.8992 \times 10^{-8} \frac{rad}{s}$

S/C - Planet 2BP → $a_{S/C} = \frac{1}{2}(r_a + r_p) = 8000$ km, $e_{S/C} = \frac{r_a - r_p}{r_a + r_p} = 0.0625$

$P_{S/C} = 2\pi \sqrt{\frac{a_{S/C}^3}{\mu}} = 2\pi \sqrt{\frac{a_{S/C}^3}{G m_{Planet}}} \rightarrow \frac{a_{S/C}^3}{G m_{Planet}} = \left(\frac{P_{S/C}}{2\pi}\right)^2 \rightarrow \boxed{m_{Planet} = 6.9569 \times 10^{24} \text{ kg}}$

$\dot{\Omega} = -\frac{3}{2} \frac{\sqrt{\mu} J_2 R_{Planet}^2}{(1-e^2)^2 a^{7/2}} \cos(i) \rightarrow J_2 = \frac{-2\dot{\Omega}(1-e^2)^2 a^{7/2}}{3\sqrt{\mu} R_{Planet}^2 \cos(i)} = \boxed{2.3990 \times 10^{-4} = J_2}$

Additional Assumptions → Planet orbit is circular, J_2 is the only perturbation, S/C orbit maintains a, i, e throughout the mission, no third body effects, no drag/SRP

Problem 2 → Given: $\vec{R}_0 = 2489.63813\hat{x} - 3916.07418\hat{y} - 5679.05524\hat{z}$ km
$\vec{V}_0 = 9.13452\hat{x} - 1.91212\hat{y} + 2.57306\hat{z}$ km/s] @t_0 in GCRF, $\vec{X}_0 = [\vec{R}_0, \vec{V}_0]^T$

Assumption: 2BP, $\mu = G(m_{earth} + m_{S/C}) \rightarrow \because m_{earth} >> m_{S/C} \rightarrow \mu \approx G m_{earth} = 3.986004415 \times 10^5$ km³/s²

a) $R_0 = |\vec{R}_0| = 7.3339 \times 10^3$ km, $V_0 = |\vec{V}_0| = 9.6807$ km/s

$\mathcal{E}_0 = V_0^2/2 - \mu/R_0 = \boxed{-7.4925 \text{ km}^2/\text{s}^2 = \mathcal{E}_0}$

$h_0 = |\vec{R}_0 \times \vec{V}_0| = \boxed{6.9258 \times 10^4 \text{ km}^2/\text{s} = h_0}$

b) $\theta_1^* = 180°$, $p_0 = h^2/\mu = 1.2034 \times 10^4$ km $= p$

$\vec{e} = \frac{1}{\mu}\left[\left(V_0^2 - \frac{\mu}{R_0}\right)\vec{R}_0 - (\vec{R}_0 \cdot \vec{V}_0)\vec{V}_0\right] = -0.1120\hat{x} - 0.3118\hat{y} - 0.6617\hat{z}$

$e = |\vec{e}| = 0.74$, $\theta_0^* = \pm \cos^{-1}\left(\frac{\vec{e} \cdot \vec{R}_0}{e R_0}\right) = \pm 29.9999°$

$$\vec{R}_0 \cdot \vec{V}_0 = 1.5617 \times 10^4 \ \frac{km^2}{s} \rightarrow \because \vec{R}_0 \cdot \vec{V}_0 \geq 0 \rightarrow \theta_0^* \geq 0 \rightarrow \therefore \theta_0^* = +29.9999^\circ$$

$$\Delta\theta^* = \theta_1^* - \theta_0^* = 150.0001^\circ$$

$$R_1 = \frac{p}{1 + e\cos(\theta_1^*)} = 4.6284 \times 10^4 \ km$$

$$f = 1 - \frac{R_1}{p}(1 - \cos(\Delta\theta^*)) = -6.1770$$

$$g = \frac{R_0 R_1}{\sqrt{\mu p}}(\sin(\Delta\theta^*)) = 2.4505 \times 10^3 \ s$$

$$\dot{f} = \sqrt{\frac{\mu}{p}} \tan\left(\frac{\Delta\theta^*}{2}\right)\left(\frac{1 - \cos(\Delta\theta^*)}{p} - \frac{1}{R_1} - \frac{1}{R_0}\right) = -6.2474 \times 10^{-5} \ \frac{1}{s}$$

$$\dot{g} = 1 - \frac{R_0}{p}(1 - \cos(\Delta\theta^*)) = -0.1372$$

$\vec{R}_1 = f\vec{R}_0 + g\vec{V}_0$

$\vec{V}_1 = \dot{f}\vec{R}_0 + \dot{g}\vec{V}_0$

$$\boxed{\begin{aligned} \vec{R}_1 &= 7.006 \times 10^3 \hat{x} + 1.9504 \times 10^4 \hat{y} + 4.1385 \times 10^4 \hat{z} \ km \\ \vec{V}_1 &= -1.4083 \hat{x} + 0.5059 \hat{y} - 1.3557 \times 10^{-6} \hat{z} \ km/s \end{aligned}} \ @ t_1 \rightarrow \vec{X}_{ref} = [\vec{R}_1, \vec{V}_1]^T$$

No quadrant check req'd

$$E_0 = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}} \tan\left(\frac{\theta_0^*}{2}\right)\right) = 0.2064 \ rad$$

$$E_1 = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}} \tan\left(\frac{\theta_1^*}{2}\right)\right) = \pi \ rad$$

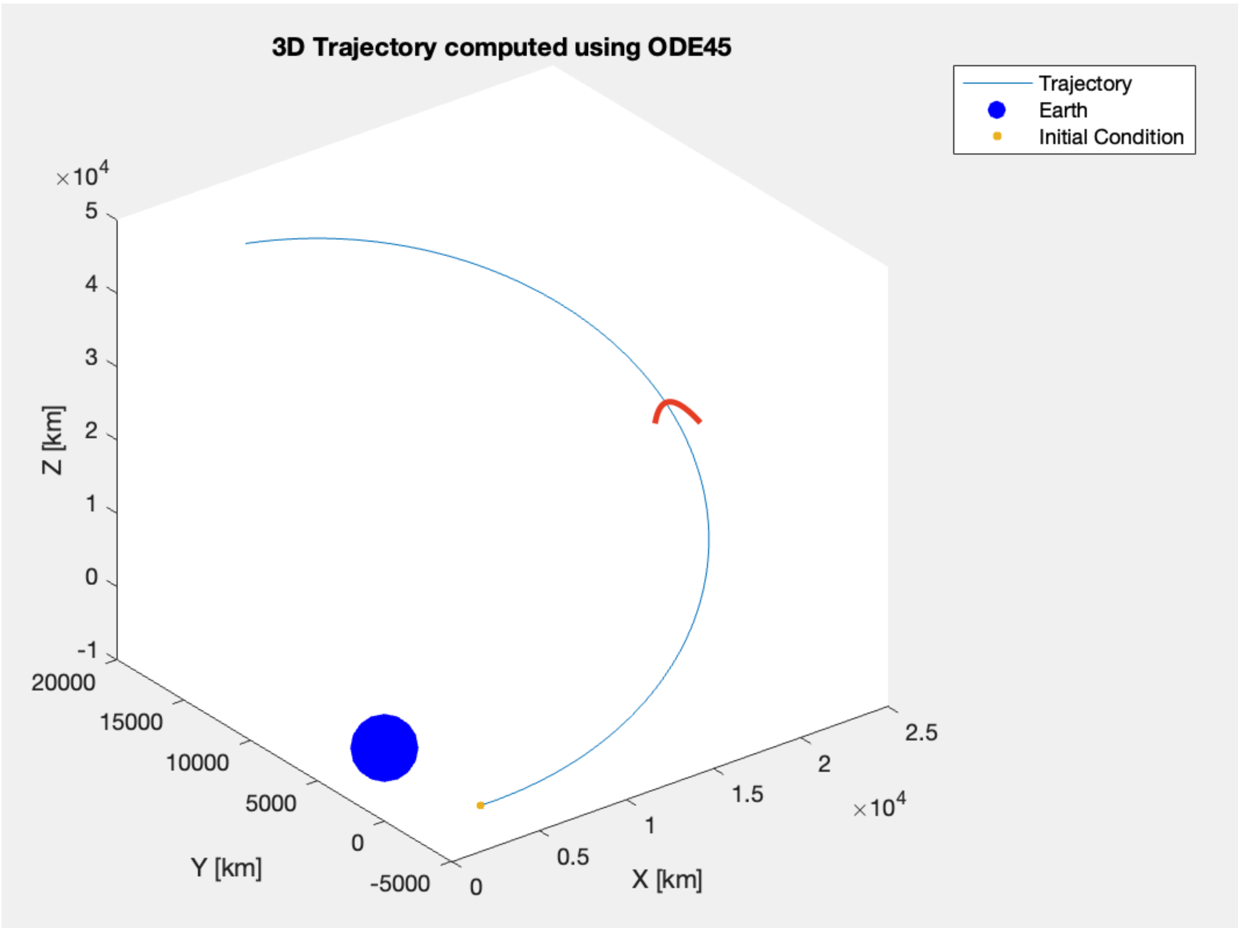
$$\rightarrow t_{01} = \frac{\mathbb{P}}{2\pi}\left[(E_1 - e\sin E_1) - (E_0 - e\sin E_0)\right] = t_1 - t_0$$

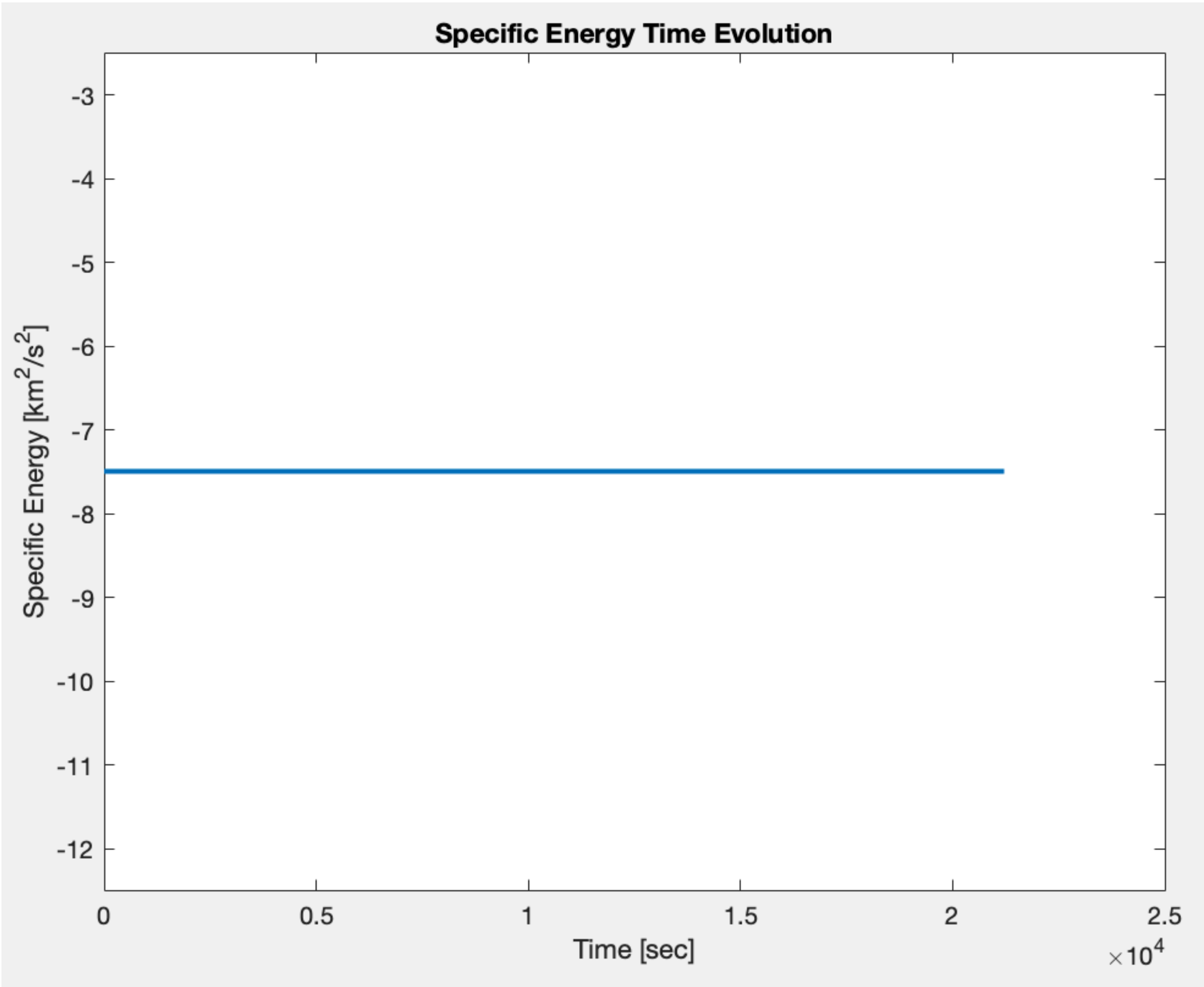
$$a = -\frac{\mu}{2\varepsilon_0} = 2.66 \times 10^4 \ km \rightarrow \mathbb{P} = 2\pi\sqrt{\frac{a^3}{\mu}} = 4.3175 \times 10^4 \ s$$

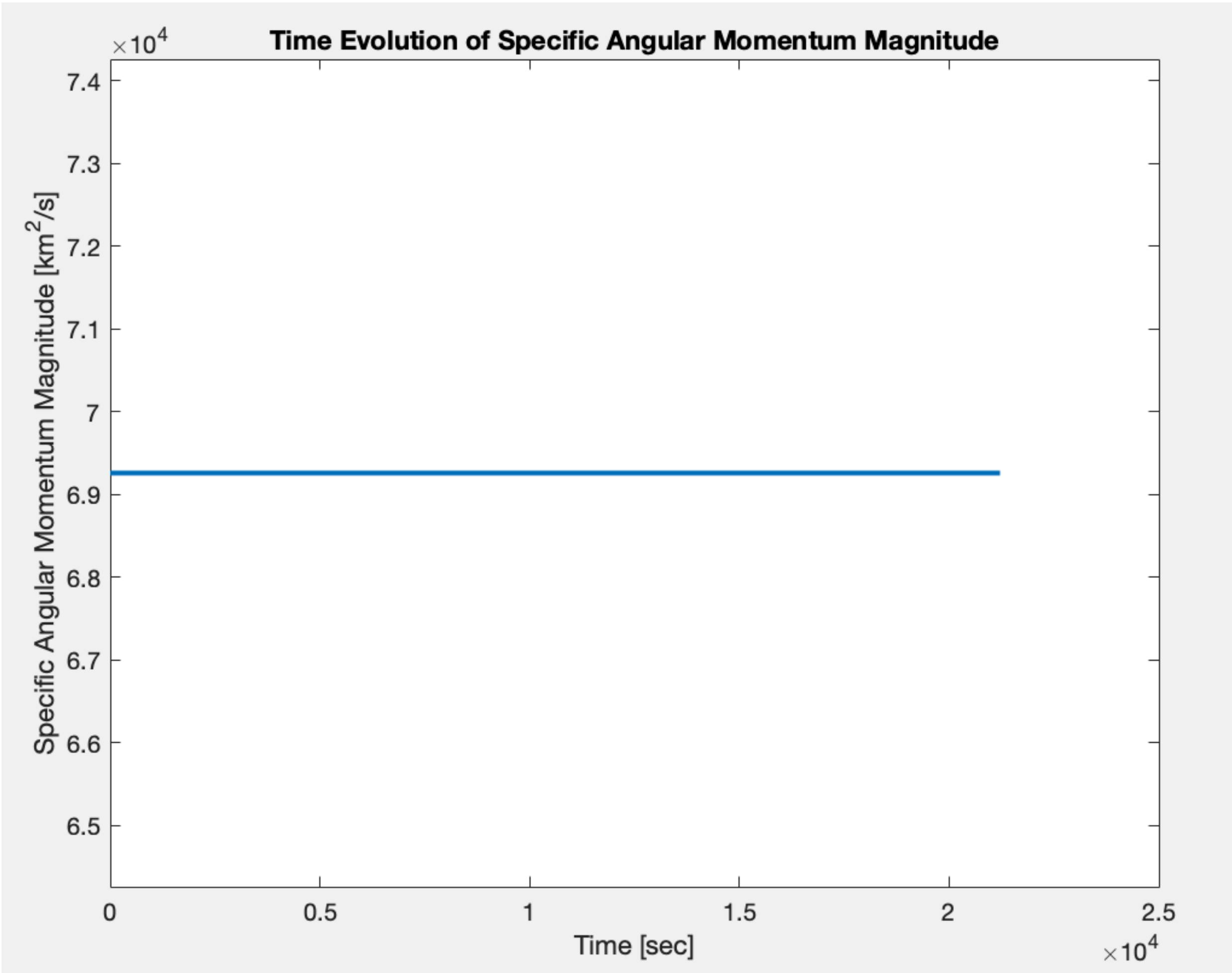
$\boxed{t_1 - t_0 = 2.1211 \times 10^4 \ s}$ → time from t_0 to t_1

Problem 2

Part c




Specific Energy Time Evolution
-3
-4
-5
-6
-7
-8
-9
-10
-11
-12
Specific Energy [km^2/s^2]
0
0.5
1
1.5
2
2.5
Time [sec]
×10^4



The code for problem 2 part c is attached below. The helper functions code is attached after Problem 3.

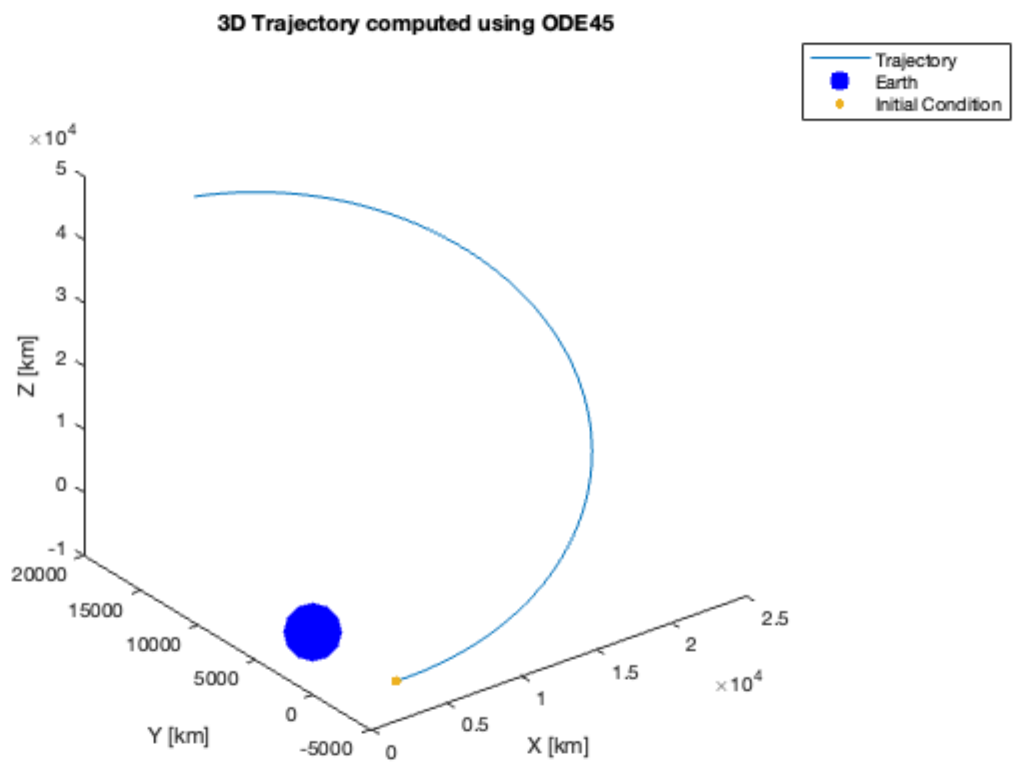
```
P = 2*pi*sqrt(a^3/mu_earth);
t0_to_t1 = P/(2*pi) * ((E_1 - e_1*sin(E_1)) - (E_0 - e_1*sin(E_0)));
```

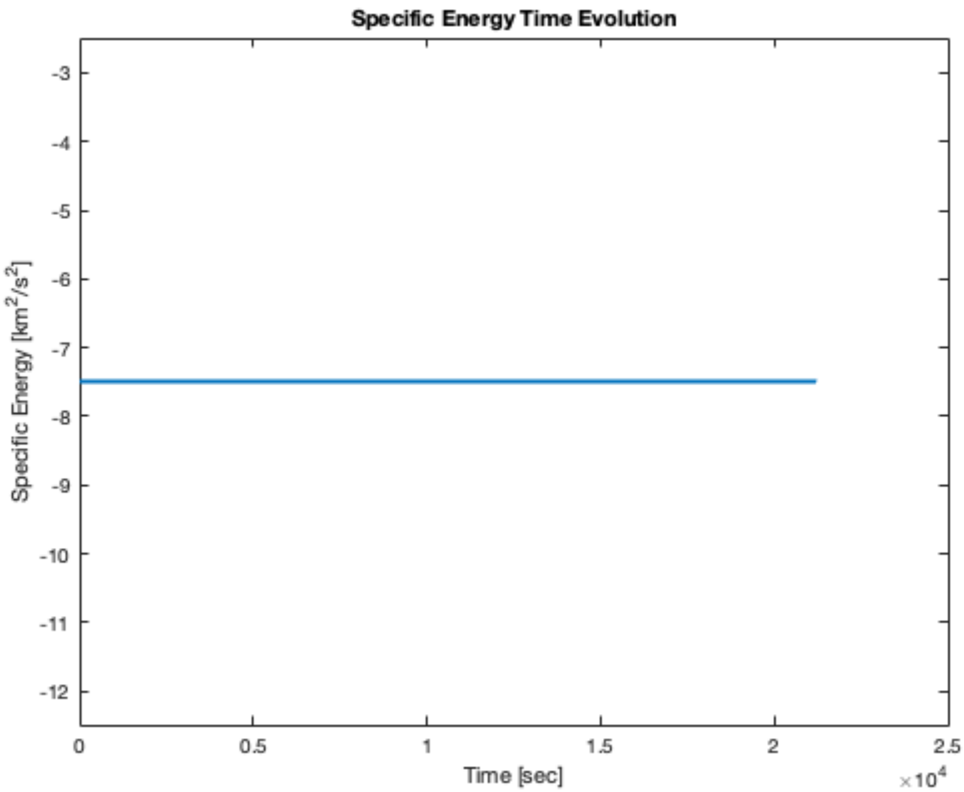
Part c

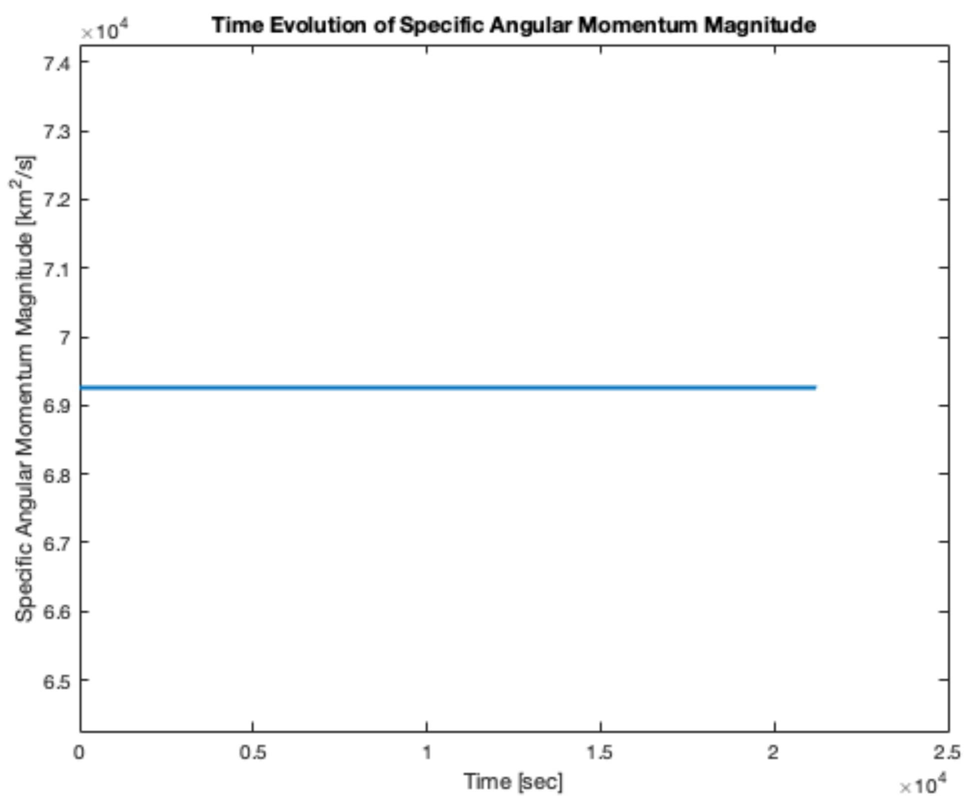
Initial State

```
state0 = [R0'; V0'];

% Set options for ODE45
options = odeset('RelTol', 1e-8, 'AbsTol', 1e-8);

% Call ODE45 function
[tout, xout] = ode45(@(t, state)x_2bp(t, state, mu_earth), [0 t0_to_t1],
state0, options);

% Calculate eps and h for all points
for i = 1:length(tout)
    [eps(i), h(i)] = rv2eps_h(xout(i, 1:3), xout(i, 4:6), mu_earth);
end

% Plot 3D trajectory
figure()
plot3(xout(:,1), xout(:,2), xout(:,3))
hold on
plot3(0, 0, 0, '.', 'MarkerSize',100, 'Color', 'blue')
plot3(R0(1), R0(2), R0(3), '.', 'MarkerSize', 10)
hold off
legend("Trajectory", "Earth", "Initial Condition")
title("3D Trajectory computed using ODE45")
xlabel("X [km]")
ylabel("Y [km]")
zlabel("Z [km]")

% Plot Specific Energy and Angular Momentum Magnitude
figure()
plot(tout, eps, 'LineWidth', 2)
ylim([eps_0-5, eps_0+5])
title("Specific Energy Time Evolution")
xlabel("Time [sec]")
ylabel("Specific Energy [km^2/s^2]")

figure()
plot(tout, h, 'LineWidth', 2)
ylim([h0-5000, h0+5000])
title("Time Evolution of Specific Angular Momentum Magnitude")
xlabel("Time [sec]")
ylabel("Specific Angular Momentum Magnitude [km^2/s]")

function statedot = x_2bp(t, state, mu_earth)
    % 2 Body Problem Function to pass to ODE45
    x = state(1);
    y = state(2);
```

```
    z = state(3);
    xdot = state(4);
    ydot = state(5);
    zdot = state(6);

    % r is magnitude of r_vec
    r = sqrt(x^2 + y^2 + z^2);

    % Derivative output is 2 body problem without any perturbations
    statedot(1, 1) = xdot;
    statedot(2, 1) = ydot;
    statedot(3, 1) = zdot;
    statedot(4, 1) = -mu_earth/r^3 * x;
    statedot(5, 1) = -mu_earth/r^3 * y;
    statedot(6, 1) = -mu_earth/r^3 * z;
end
```




Specific Energy Time Evolution
Specific Energy [km^2/s^2]
-3
-4
-5
-6
-7
-8
-9
-10
-11
-12
0
0.5
1
1.5
2
2.5
Time [sec]
×10^4


Time Evolution of Specific Angular Momentum Magnitude
×10^4
Specific Angular Momentum Magnitude [km^2/s]
7.4
7.3
7.2
7.1
7
6.9
6.8
6.7
6.6
6.5
0
0.5
1
1.5
2
2.5
Time [sec]
×10^4

Part d

Absolute & Relative Tolerance	ΔR (km)	ΔV (km/s)	Δε (km^2/s^2)	Δh (km^2/s)	Computational Time (sec)
1e-4	7.0296	1.655e-4	0.0011	2.8515	3.8888e-4
1e-6	0.0593	1.3225e-6	9.0529e-6	0.0275	3.1292e-4
1e-8	3.2370e-4	7.1756e-9	4.9494e-8	1.5226e-4	5.5646e-4
1e-10	1.6745e-6	3.4398e-11	2.6010e-10	9.1374e-7	0.0013
1e-12	9.5824e-9	1.6964e-13	1.5286e-12	6.5775e-9	0.0030

Part e

- As the tolerance decreases (going from 1e-4 to 1e-12), the final state vector becomes more accurate i.e. it gets closer to the reference state vector.
- As the tolerance decreases (going from 1e-4 to 1e-12), the constant values (specific energy and specific angular momentum magnitude get closer to the initial values.
- As the tolerance decreases (going from 1e-4 to 1e-12), the computational time increases.
- I would select 1e-8 as the tolerance because it strikes a good balance between computational time and accuracy of various parameters. After 1e-8 the computational time increases rapidly and for longer simulations, that can really add up. The accuracy in R, V, ε, and h are not noticeable when those values are in the thousands.

Part f

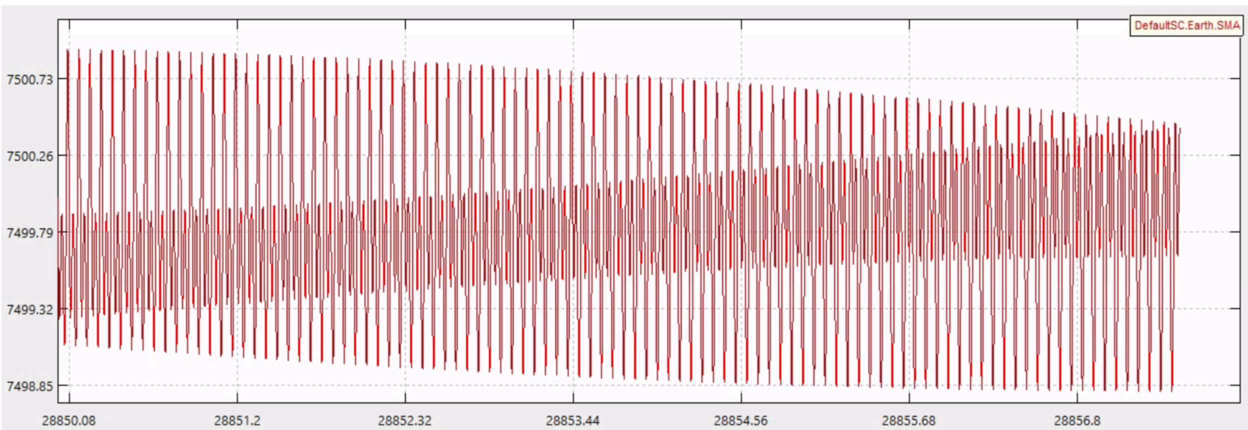
- If the dynamical model incorporated the J2 perturbation, the periapsis would be stationary because the inclination is 63.4°.
- Additionally, the orbit is prograde and since the inclination is 63.4°, the change in the line of nodes is westward i.e. change in the right ascension of the ascending node is less than 0.
- The above mentioned effects are secular. The rest of the orbital elements may face shorter-period oscillatory effects that are not modeled analytically.

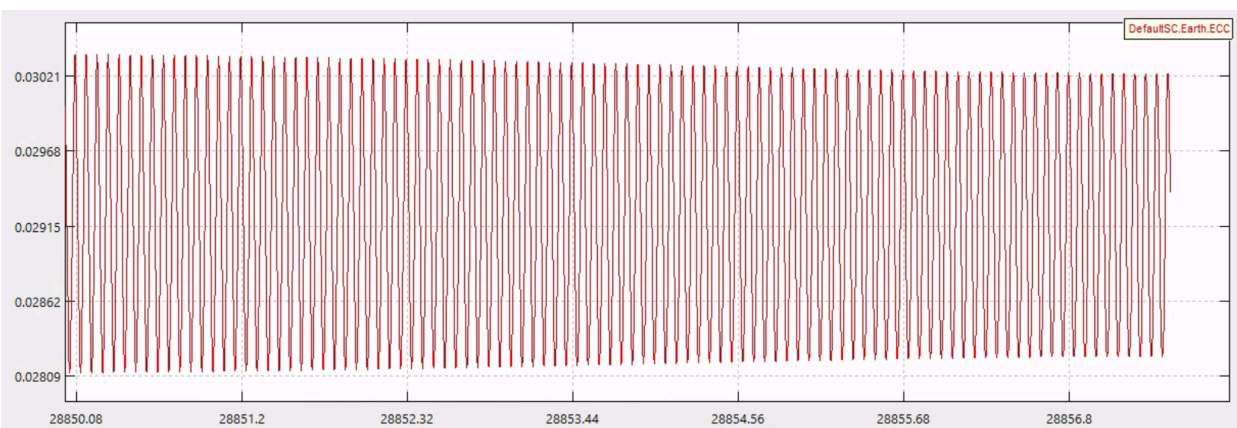
Problem 3

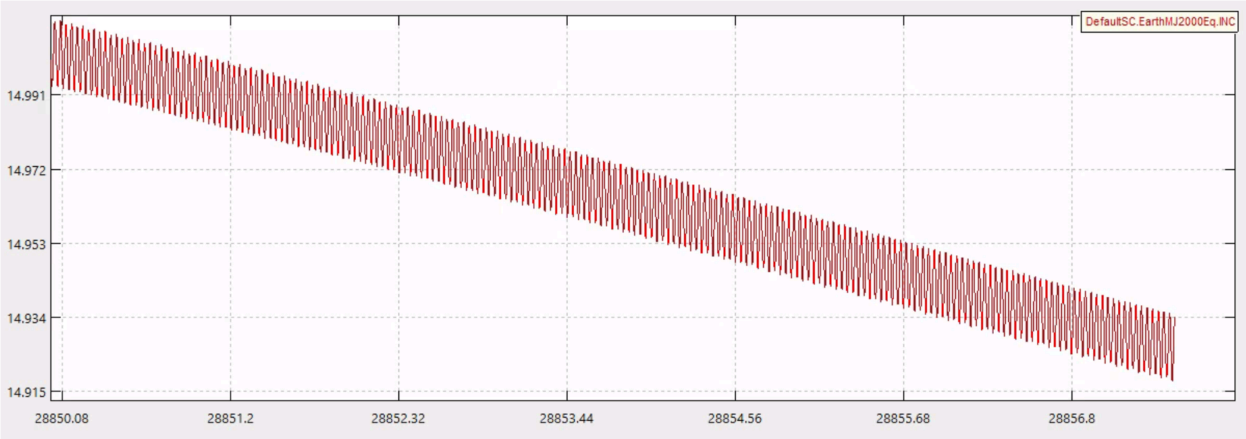
Part a

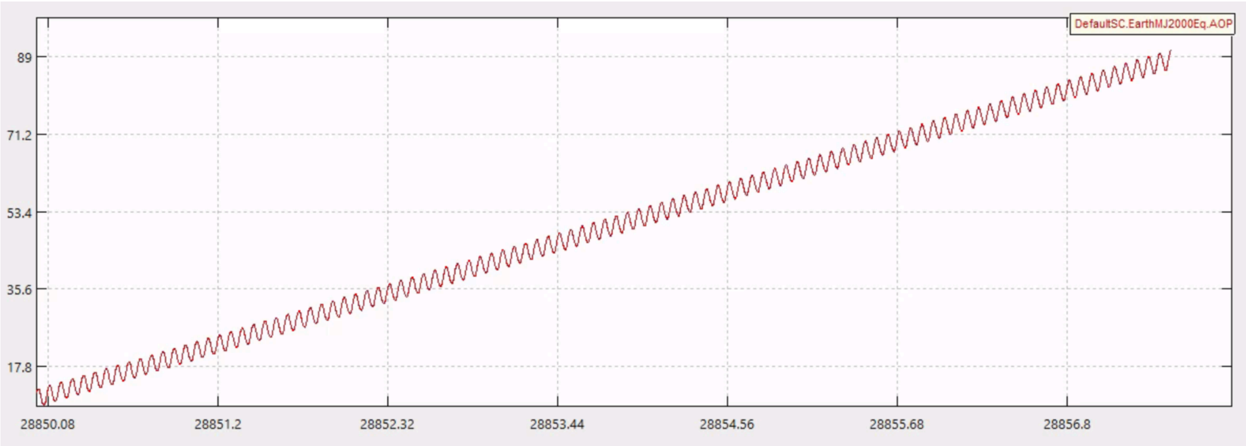
- The integrator used in the “EarthPM” propagator is a RungeKutta89
- The accuracy used is 9.9999e-12
- The semi-major axis from the report is 7500.000001147 km and eccentricity is 0.02999999999 and those values are slightly different from 7500 km and 0.03 semi-major axis and eccentricity respectively. We can use these values because they’re supposed to stay consistent during the numerical integration. The difference between expected and actual is 1e-6 km in the SMA and 1e-11. The SMA difference is higher than expected, but eccentricity is exactly what’s expected.

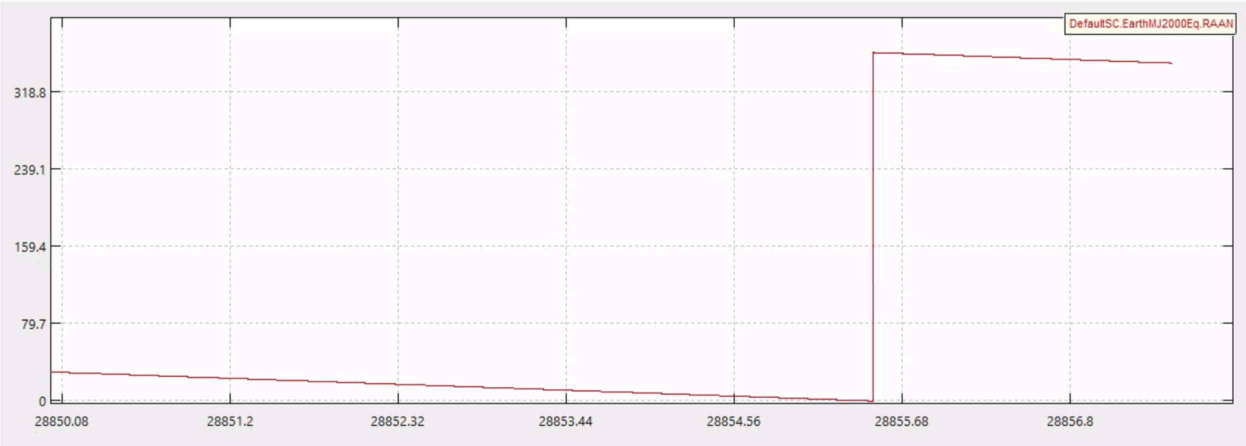
Part b





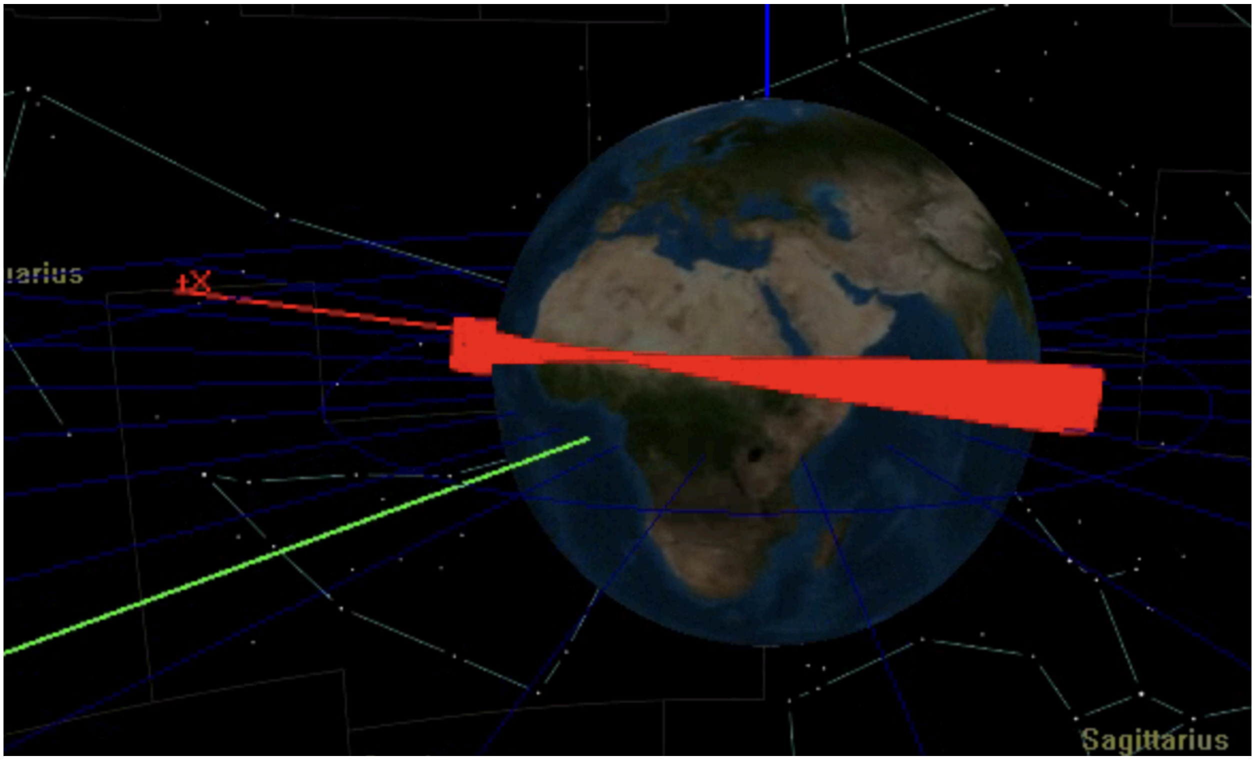






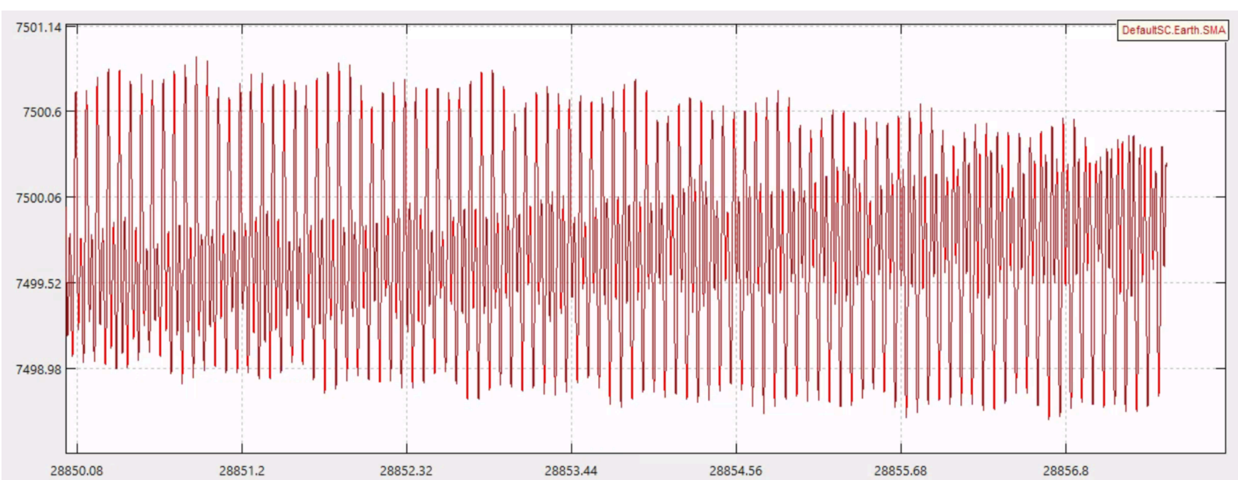
- The semi-major axis, eccentricity, and inclination values are highly oscillatory without any major secular effects. The inclination may seem to present a trend, but it's really minute over 100 orbits.
- The argument of periapsis has a very clear trend. It increases as time goes on.
- The right ascension of the ascending node, on the other hand, decreases as time goes on.
- This is consistent with the concept covered in lecture because the direction of the change in RAAN and AOP depends on the inclination and since the inclination is <63.4°, this behavior concurs with that.

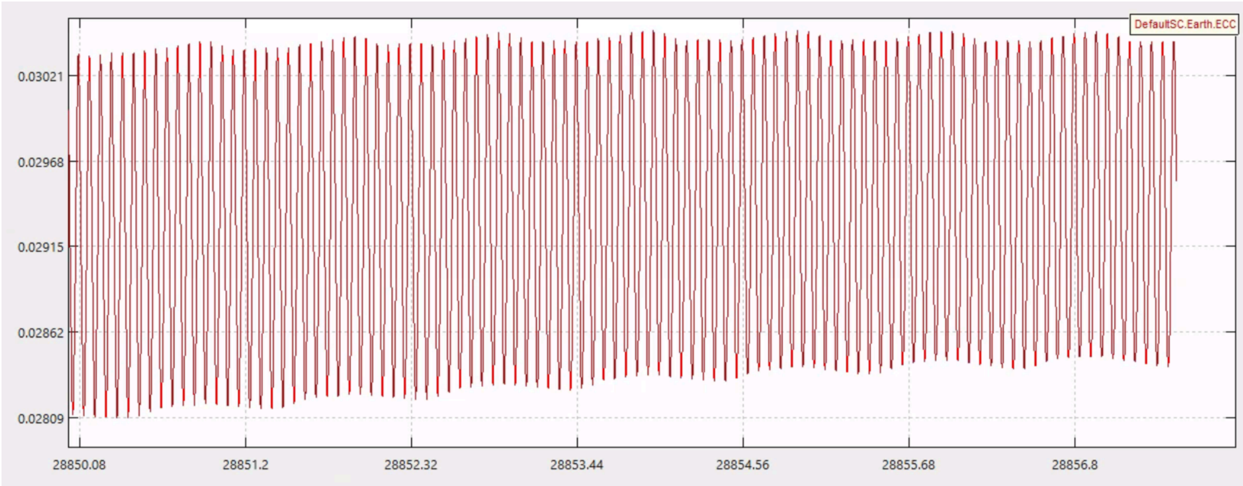
Part c

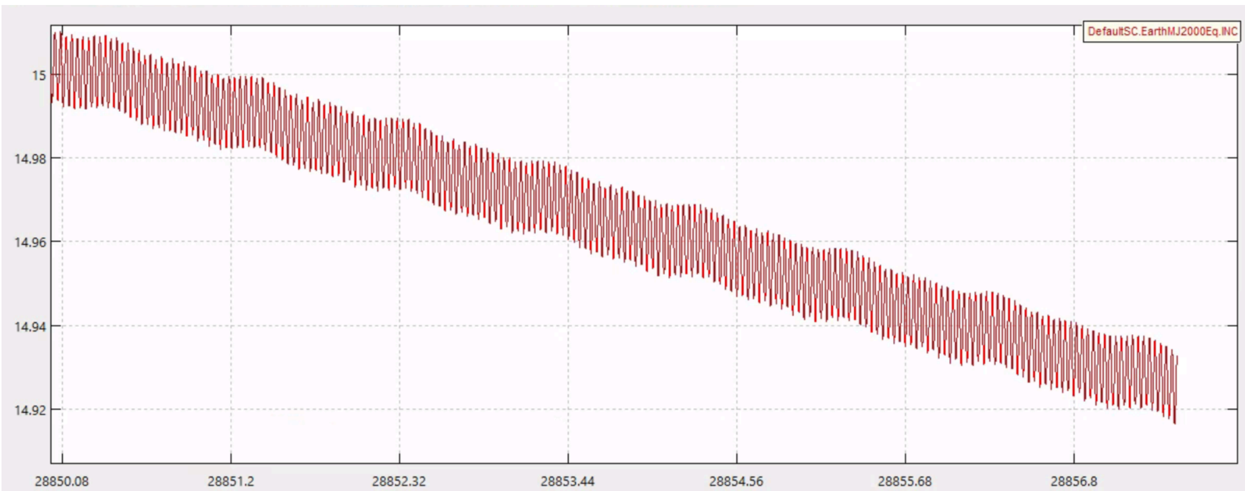


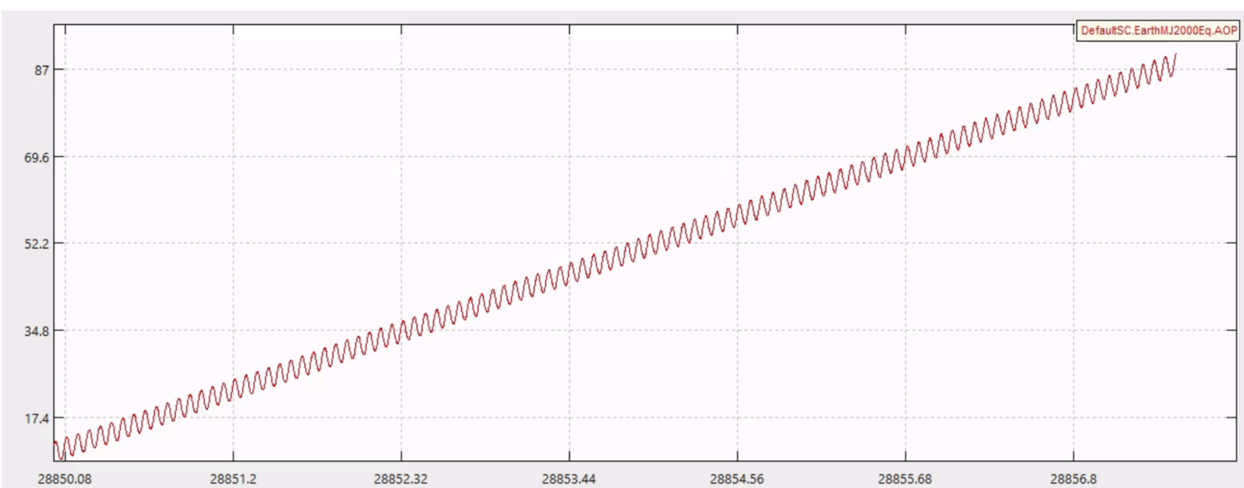
- As can be seen from the above image, the orbit perturbs as it passes through time.
- The perturbation can be seen as the orbit shifts while remaining constant at two points that are 180° apart. One of those points can be seen right over Africa in the above image.

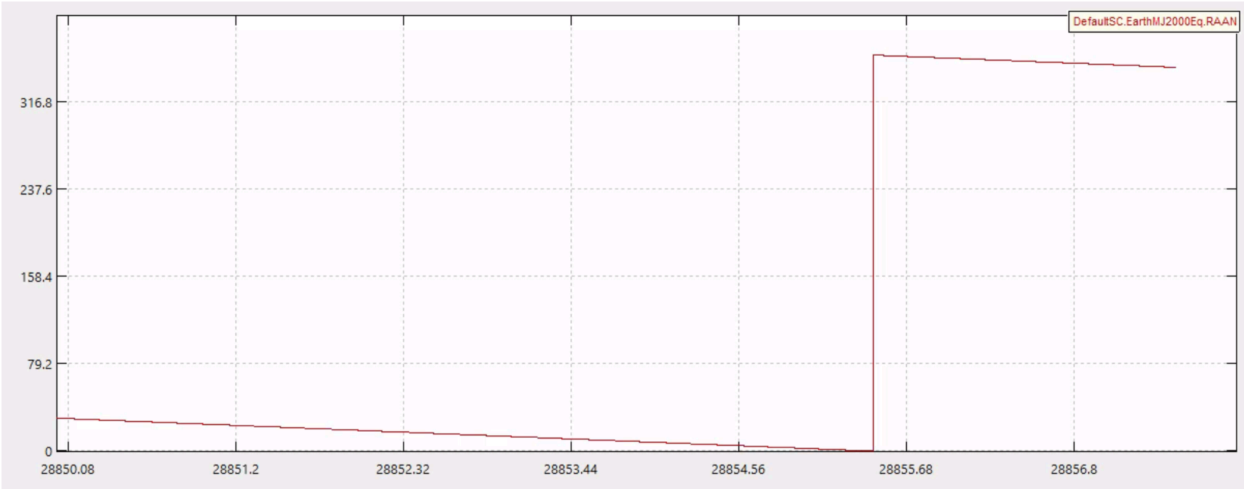
Part d




DefaultSC.Earth.ECC
0.03021
0.02968
0.02915
0.02862
0.02809
28850.08
28851.2
28852.32
28853.44
28854.56
28855.68
28856.8

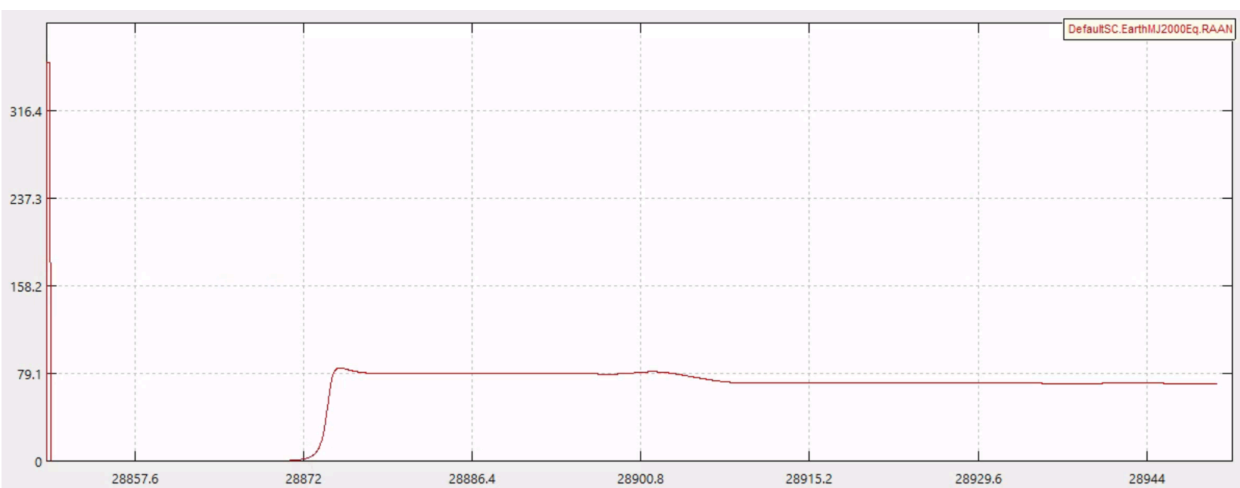

DefaultSC.EarthMJ2000Eq.INC
15
14.98
14.96
14.94
14.92
28850.08
28851.2
28852.32
28853.44
28854.56
28855.68
28856.8

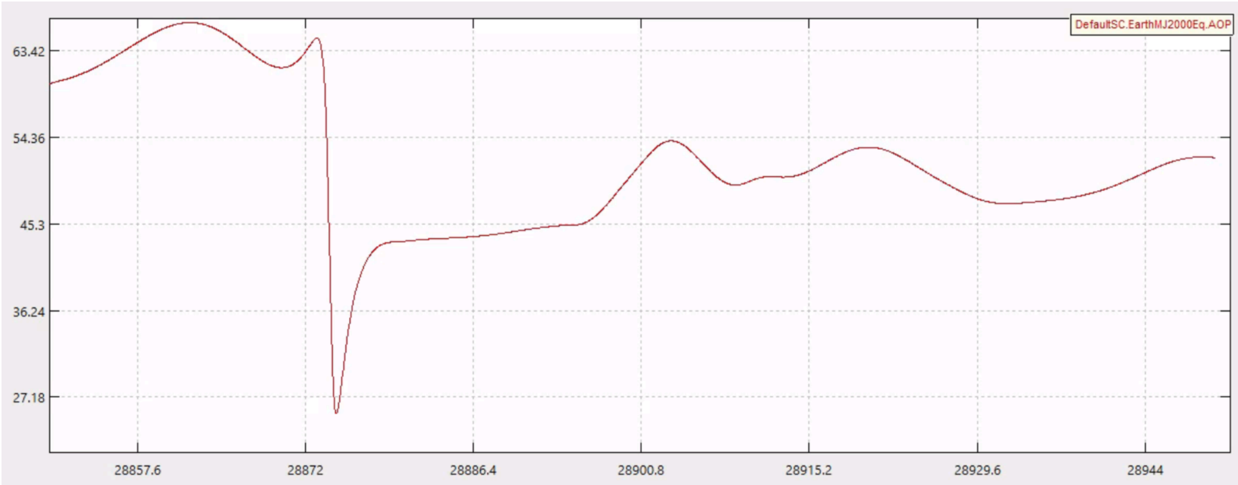

DefaultSC.EarthMJ2000Eq.AOP
87
69.6
52.2
34.8
17.4
28850.08
28851.2
28852.32
28853.44
28854.56
28855.68
28856.8



- The inclination, semi-major axis, and eccentricity plots seem to be affected the most when using this propagator.
    - There are some shorter period oscillations in the inclination and SMA plots.
    - The eccentricity magnitude seems to decrease over time.
- The RAAN and AOP seem more consistent with the previous plots.
- If planning an orbit, the 2-body problem is not enough because the J2 perturbations are highly influential. However, seems like J2 is sufficient for a preliminary analysis because the higher order terms don't influence the orbit enough.

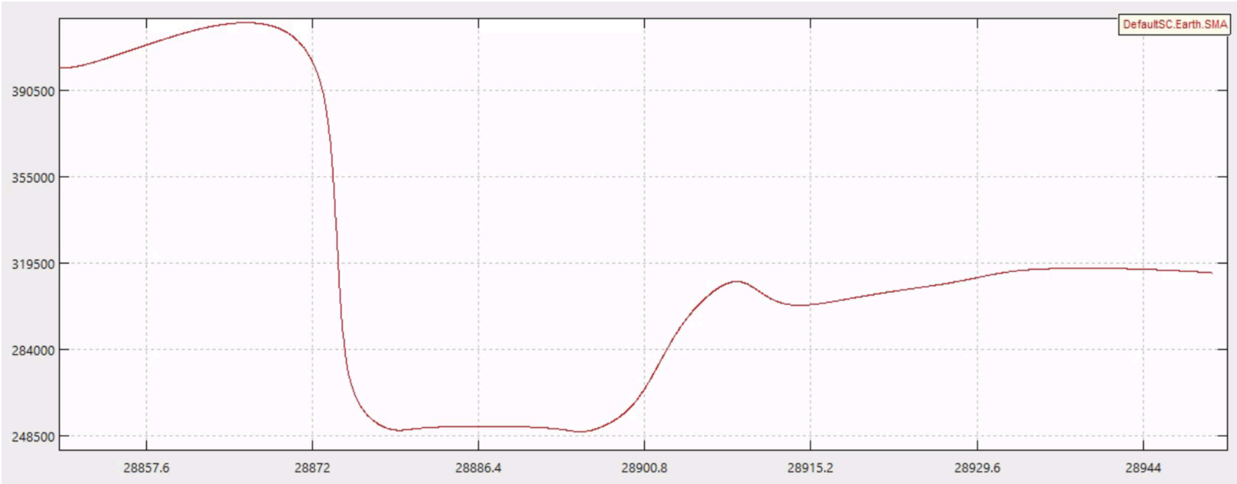
Part e

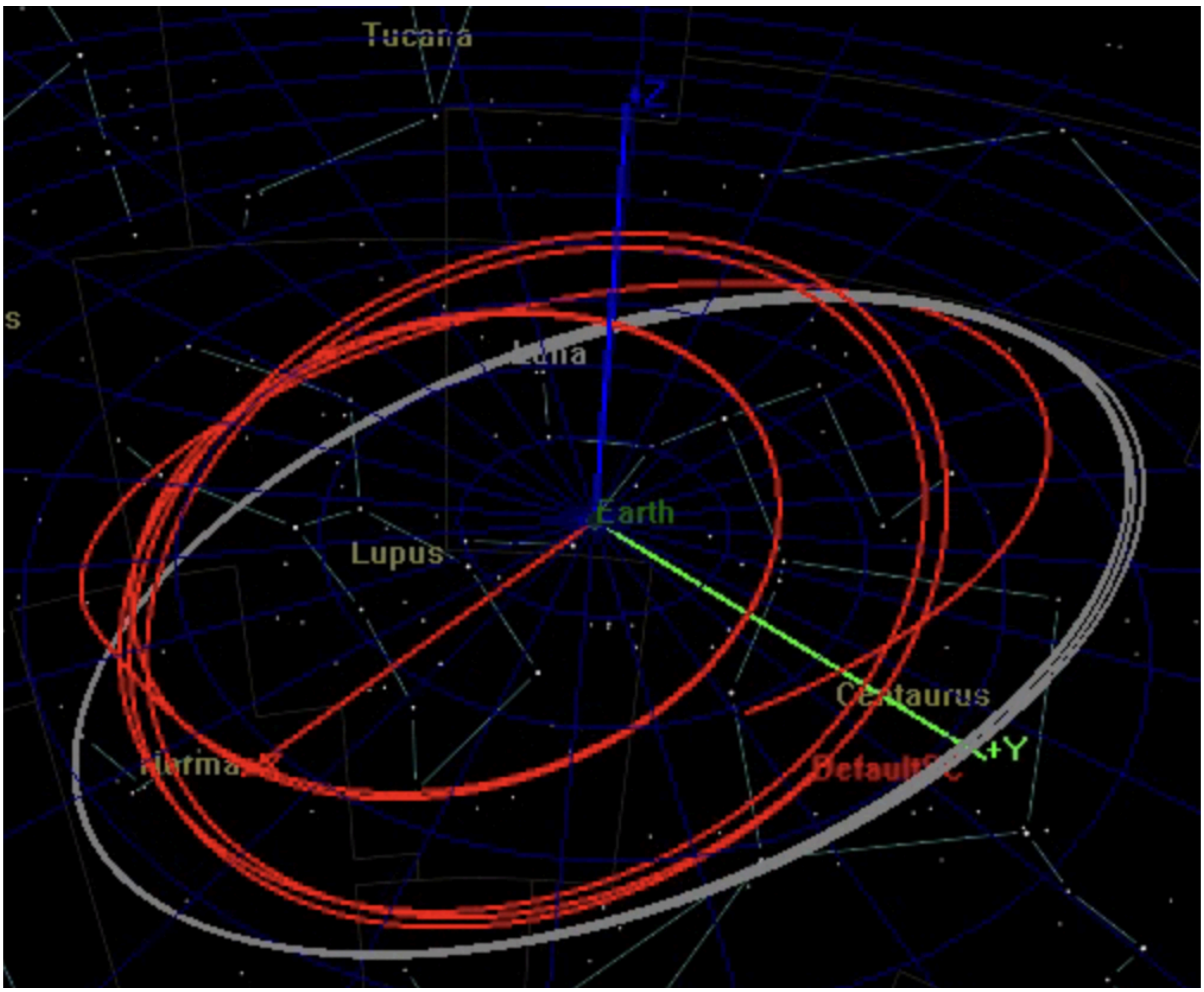



DefaultSC.EarthMJ2000Eq.AOP
63.42
54.36
45.3
36.24
27.18
28857.6
28872
28886.4
28900.8
28915.2
28929.6
28944


DefaultSC.EarthMJ2000Eq.INC
21.56
20.02
18.48
16.94
15.4
28857.6
28872
28886.4
28900.8
28915.2
28929.6
28944


DefaultSC.Earth.ECC
0.567
0.486
0.405
0.324
0.243
28857.6
28872
28886.4
28900.8
28915.2
28929.6
28944

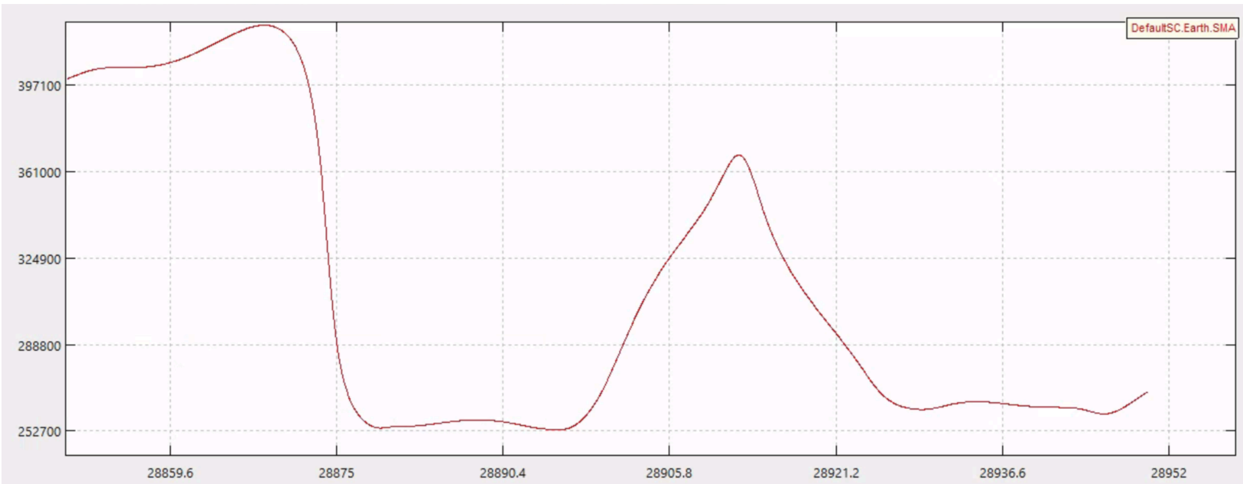


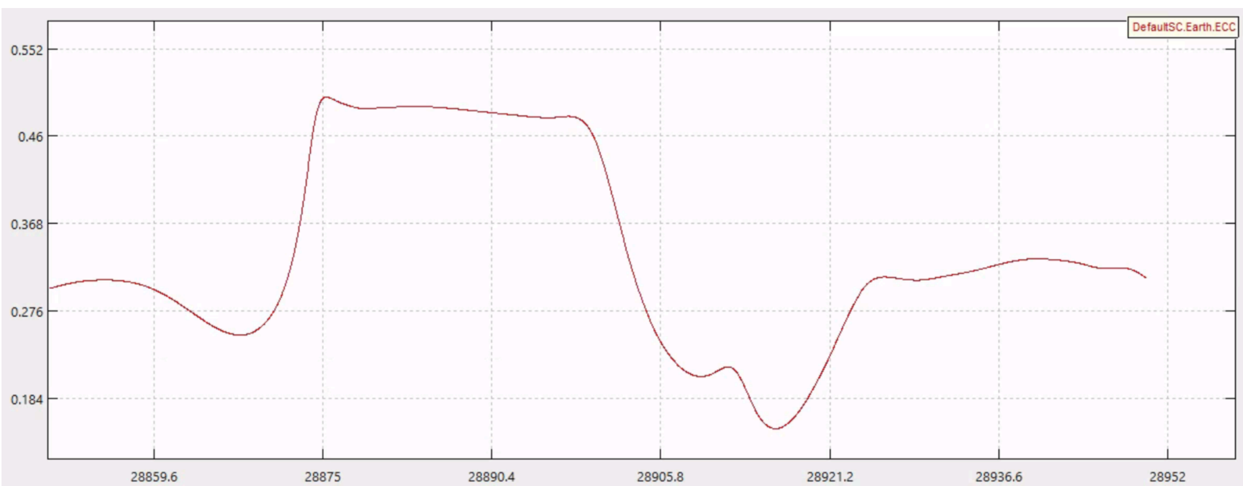


- The plots show a very erratic trajectory and it can be confirmed by seeing the 3D trajectory. When seen in GMAT with the lunar orbit, the moon's position influences the orbit. And, the spacecraft position with respect to the moon should also be taken into account here. Hence, it's not trivial to come up with a generalized equation for this

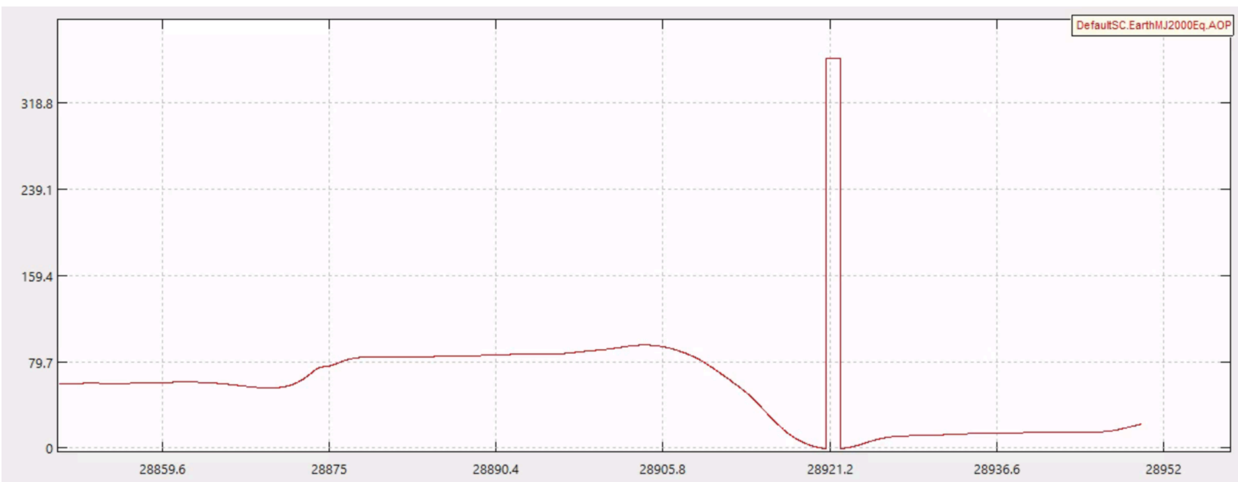
trajectory.
- The plots have spikes in the first half and all of those can be attributed to the moon passing close to the spacecraft.
- The J2 perturbations are not visible anymore.

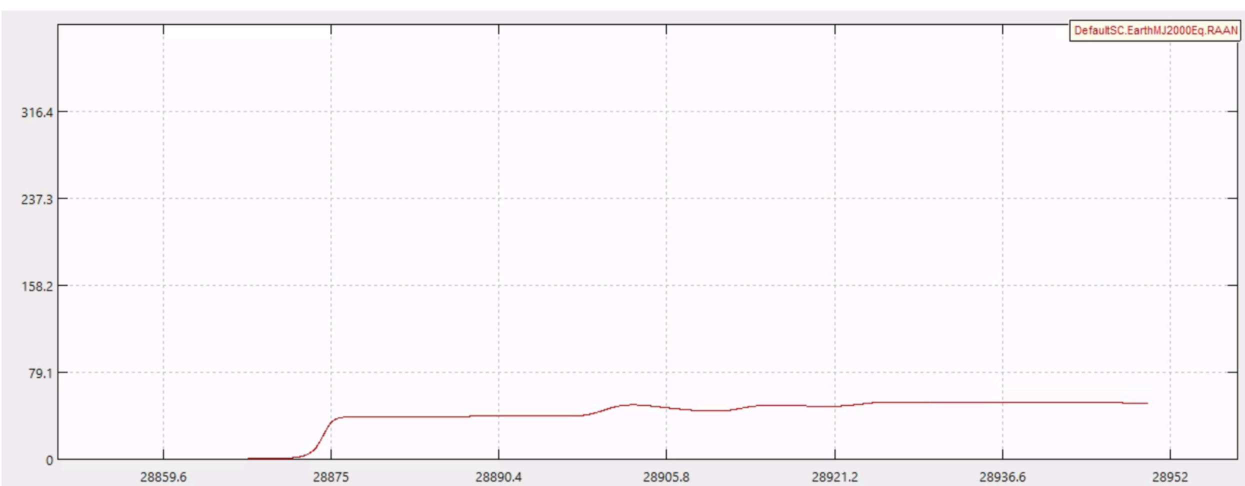
Part f

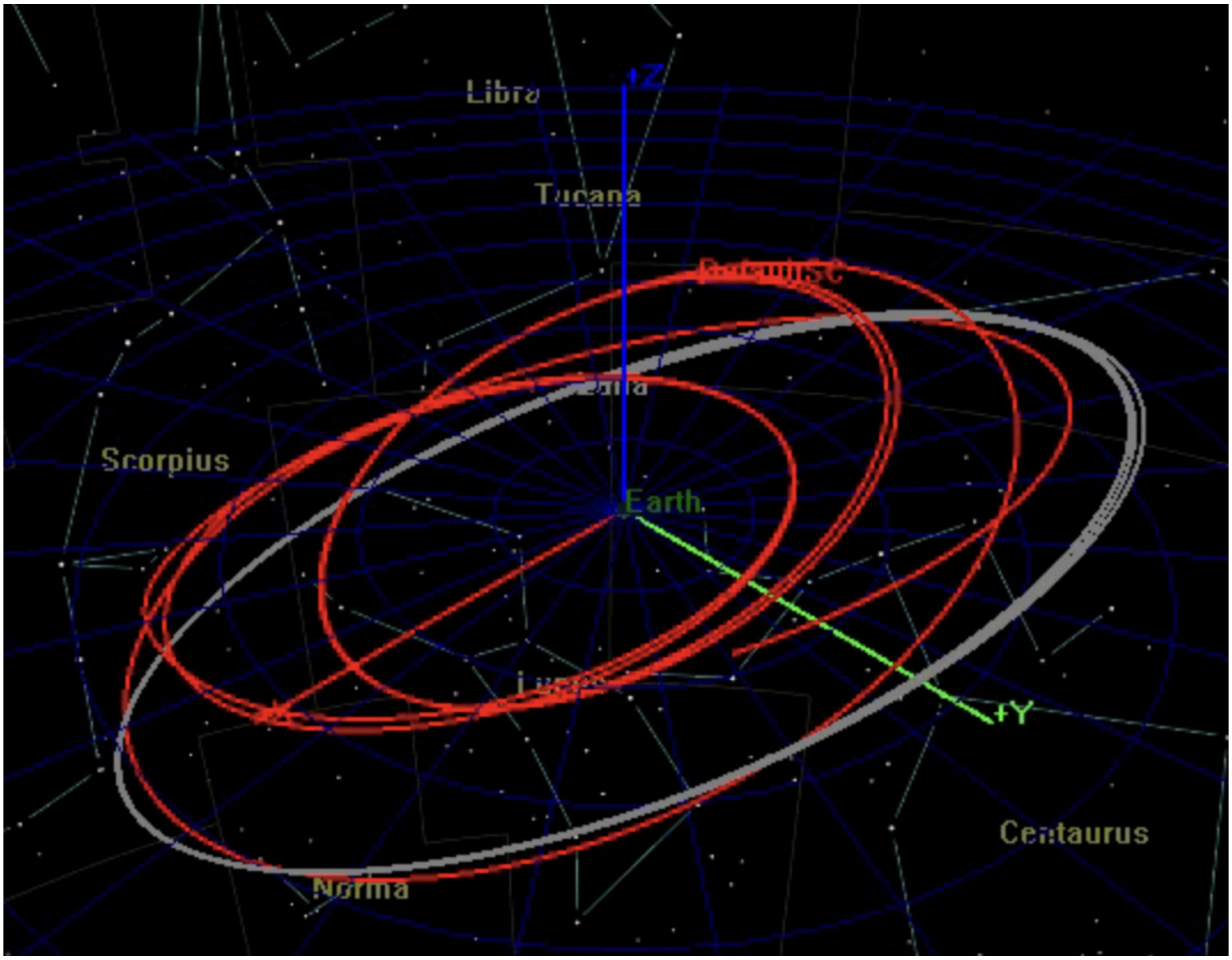





DefaultSC.EarthMJ2000Eq.INC
24.1
21.69
19.28
16.87
14.46
28859.6
28875
28890.4
28905.8
28921.2
28936.6
28952


DefaultSC.EarthMJ2000Eq.AOP
318.8
239.1
159.4
79.7
0
28859.6
28875
28890.4
28905.8
28921.2
28936.6
28952


DefaultSC.EarthMJ2000Eq.RAAN
316.4
237.3
158.2
79.1
0
28859.6
28875
28890.4
28905.8
28921.2
28936.6
28952



DefaultSC.UTCModJulian	DefaultSC.Earth.SMA	DefaultSC.Earth.ECC	DefaultSC.EarthMJ2000Eq.INC	DefaultSC.EarthMJ2000Eq.AOP	DefaultSC.EarthMJ2000Eq.RAAN
28950	269327.5071924043	0.3107165358049436	24.24048225435369	22.72603456405096	50.51249288908338

- As seen in the report above, the final orbital elements are no longer similar to the initial elements:
    - SMA - 400,000 km
    - e - 0.3
    - i - 20°
    - aop - 0°
    - raan - 60°
    - true anomaly - 0°
- Adding the sun and the moon have completely changed the orbit trajectory from a 2BP. Comparing to adding the moon, the effects on the orbit are significant but not generalizable.

Table of Contents


```
clear; clc; close all;

% ASEN 5050 - HW 8 - Problem 2
% Fall 2024
% Jash Bhalavat

% Given
mu_earth = 3.986004415e5; % km3/s2

% @ t0 in GCRF
R0 = [2489.63813, -3916.07418, -5679.05524]; % km
V0 = [9.13452, -1.91212, 2.57306]; % km/s
```

Part a

Calculate eps and h at t0

```
[eps_0, h0] = rv2eps_h(R0, V0, mu_earth);
```

Part b

```
theta_star_1 = pi;

% Get E0
coe_0 = rv2coe(R0, V0, mu_earth);
theta_star_0 = coe_0(6);
e_0 = coe_0(2);
E_0 = 2 * atan(sqrt((1-e_0)/(1+e_0)) * tan(theta_star_0/2));

% Calculate X_ref using fg f'ns
X_ref = fg(R0, V0, theta_star_1, mu_earth);
R1 = X_ref(1,:);
V1 = X_ref(2,:);

% Get E1
coe_1 = rv2coe(R1, V1, mu_earth);
e_1 = coe_1(2);
E_1 = 2 * atan(sqrt((1-e_1)/(1+e_1)) * tan(theta_star_1/2));

% Calculate time from t0 to t1
a = coe_0(1);
```

Part d

```
abs_rel_tols = [1e-4, 1e-6, 1e-8, 1e-10, 1e-12];

ref_R1_mag = norm(R1);
ref_V1_mag = norm(V1);

for i = 1:length(abs_rel_tols)
    options = odeset('RelTol', abs_rel_tols(i), 'AbsTol', abs_rel_tols(i));

    tic;
    [tout, xout] = ode45(@(t, state)x_2bp(t, state, mu_earth), [0 t0_to_t1],
state0, options);
    time_taken(i) = toc;

    ode_R1_mag = norm(xout(end,1:3));
    ode_V1_mag = norm(xout(end,4:6));

    [ode_eps_1, ode_h_1] = rv2eps_h(xout(end, 1:3), xout(end, 4:6), mu_earth);

    delta_R(i) = abs(ode_R1_mag - ref_R1_mag);
    delta_V(i) = abs(ode_V1_mag - ref_V1_mag);

    delta_eps(i) = abs(eps_0 - ode_eps_1);
    delta_h(i) = abs(h0 - ode_h_1);
end


% Part f

% Perturbations Overview, Slide 24
% Inclination is 63.4 deg, and AOP is 270 deg
% So, RAAN_dot is less than 0 and line of nodes shifts west, orbit is
% prograde
% Since inclination is 63.4 deg, the periapsis is stationary
```

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```
function coe = rv2coe(r, v, mu)
    % Convert pos/vel to classical orbital elements
    % Output vector is arranged as follows:
    % coe - [a, e, i, raan, aop, theta_star]

    r_norm = norm(r);
    r_hat = r/r_norm;

    v_norm = norm(v);
    v_hat = v/v_norm;

    h = cross(r, v);
    h_norm = norm(h);
    h_hat = h/h_norm;

    K = [0 0 1];
    n = cross(K, h);
    n_norm = norm(n);
    n_hat = n/n_norm;

    e = 1/mu * ((v_norm^2 - mu/r_norm)*r - dot(r,v)*v);
    e_norm = norm(e);
    e_hat = e/e_norm;

    eps = v_norm^2/2 - mu/r_norm;

    if e == 1.0
        p = h_norm^2/mu;
    else
        a = -mu/(2*eps);
        p = a * (1 - e_norm^2);
    end

    i = acos(h_hat(3));
    raan = sign(n_hat(2)) * abs(acos(n_hat(1)));
    aop = sign(e(3)) * abs(acos(dot(n, e)/(n_norm * e_norm)));
    theta_star = sign(dot(r, v)) * abs(acos(dot(e, r)/(e_norm * r_norm)));

    % Special cases
    aop_true = acos(e_hat(1));
    if e(2) < 0
        aop_true = 2*pi - aop_true;
    end

    coe = [a, e_norm, i, raan, aop, theta_star];
end
```

```
Not enough input arguments.

Error in rv2coe (line 6)
    r_norm = norm(r);
```

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```
function [eps, h_norm] = rv2eps_h(r, v, mu)
    % Convert pos/vel to eps and h
    % Output vector is arranged as follows:
    % [eps, h]

    r_norm = norm(r);
    r_hat = r/r_norm;

    v_norm = norm(v);
    v_hat = v/v_norm;

    h = cross(r, v);
    h_norm = norm(h);

    eps = v_norm^2/2 - mu/r_norm;

end
```

```
Not enough input arguments.

Error in rv2eps_h (line 6)
    r_norm = norm(r);
```

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```
function rv = fg(r0, v0, theta_star_1, mu)
    % Calculate rv at a different position giving initial r0, v0, final
theta_star, and
    % gravitational parameter of central body in a 2 BP

    coe = rv2coe(r0, v0, mu);
    h = norm(cross(r0, v0));
    p = h^2/mu;
    e = coe(2);
    theta_star_0 = coe(6);

    delta_theta_star = theta_star_1 - theta_star_0;

    r = p / (1 + e*cos(theta_star_1));

    r0_mag = norm(r0);

    f = 1 - r/p * (1 - cos(delta_theta_star));
    g = r0_mag*r/sqrt(mu*p) * sin(delta_theta_star);

    f_dot = sqrt(mu/p) * tan(delta_theta_star/2) * ((1 -
cos(delta_theta_star))/(p) - 1/r - 1/r0_mag);
    g_dot = 1 - (r0_mag/p) * (1 - cos(delta_theta_star));

    r_out = f*r0 + g*v0;
    v_out = f_dot*r0 + g_dot*v0;

    rv = [r_out; v_out];
end
```

```
Not enough input arguments.

Error in fg (line 5)
    coe = rv2coe(r0, v0, mu);
```

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