ASEN 6020
Optimal Trajectories
Exam
Spring 2025

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Total points = 100
Hand in this copy of the test in addition to any work sheets that you use.

The exam is designed to be taken in a single 90 minute period, and you are allowed up to 180 minutes to complete, scan and upload the exam.
The test is open book and notes, but no internet access except to the course Gradescope and Canvas pages.

Write your answers clearly and neatly in the space provided.

If you have any questions as you take the test, feel free to send me a text message question at: 1-720-544-1260. Please only do this between the hours of 8AM and 11PM MT. I will respond as soon as I can.

Scan your exam and any worksheets you used during the exam and upload it to Gradescope as soon as you are done.

Please sign below to confirm that you have not collaborated with anyone else in taking this test and that you have worked on this exam for, at most, a total of 180 minutes. I will not grade this exam unless it is signed. E-signatures are OK.

Signature: JB

Starting Time and Date: 9:25 am 04/27/25

Ending Time and Date: 12:20 pm 04/27/25

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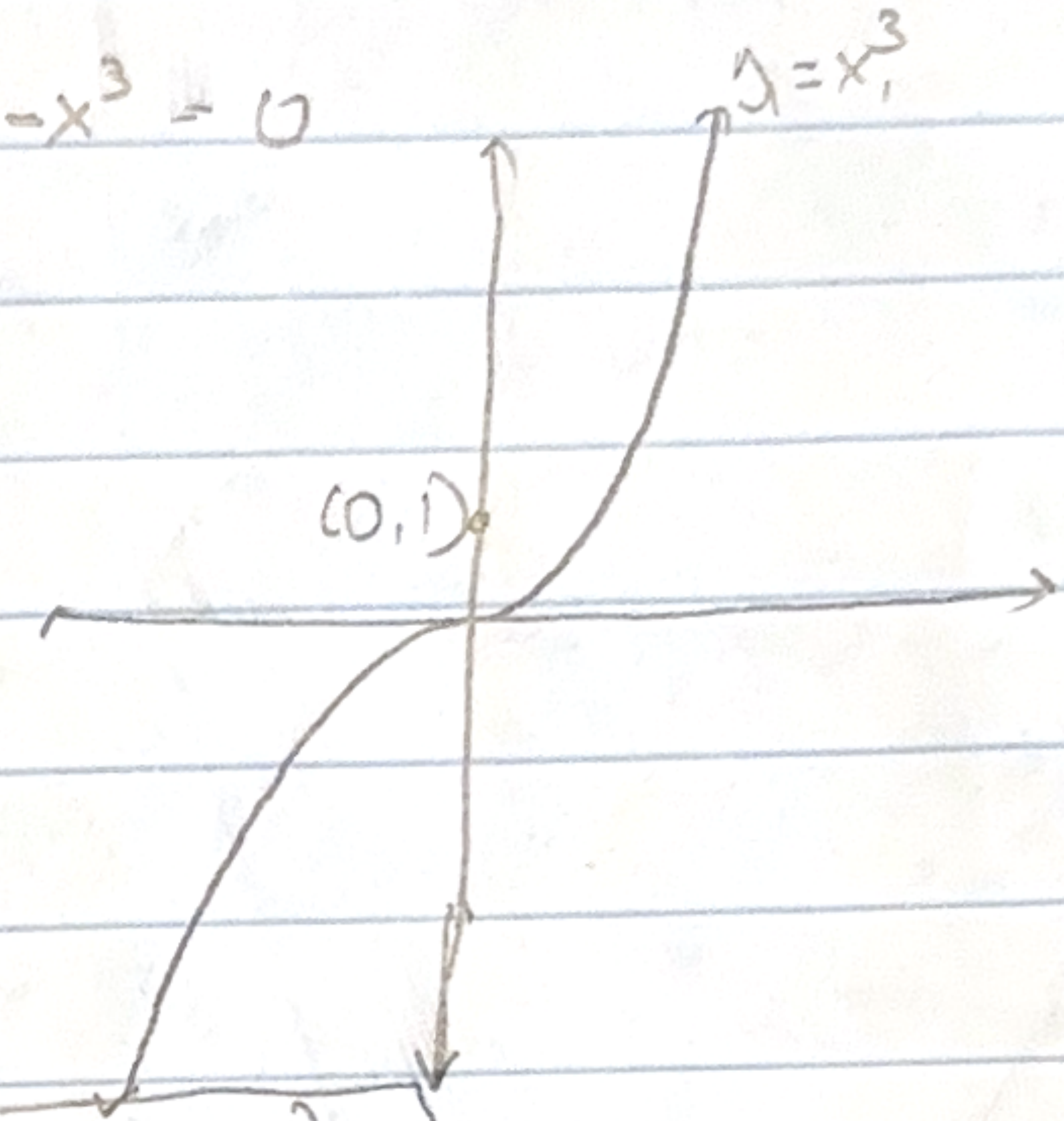
Jash Bhalavat

$\boxed{\text{Mid-Term}}$

Start Time: 9:25 AM, 04/27/2025

End Time: , 04/27/2025

Problem 1 → $x_0=0,\ y_0=1,\ y_1=x_1^3,\ J=x^2+(y-1)^2,\ g(x,y)=y-x^3=0$



a) KKT necessary conditions:

$L(x,y,\lambda_0,\lambda_1)=\lambda_0(x^2+(y-1)^2)+\lambda_1(y-x^3)$

$\vec{x}\in\mathbb{R}^2,\ \lambda_0\in\mathbb{R},\ \lambda_1\in\mathbb{R},\ g\in\mathbb{R}$

$L=\lambda_0x^2+\lambda_0(y-1)^2+\lambda_1y-\lambda_1x^3$

$\frac{\partial L}{\partial x}=2\lambda_0x-3\lambda_1x^2=0$

$\frac{\partial L}{\partial y}=2\lambda_0(y-1)+\lambda_1=0$

$2\lambda_0x=3\lambda_1x^2,\quad 2\lambda_0y-2\lambda_0+\lambda_1=0\to 2\lambda_0y=2\lambda_0-\lambda_1\to\boxed{y=1-\frac{\lambda_1}{2\lambda_0}}$

∵ No inequality conditions, $\lambda_0=1$, conditions become KKT conditions.

$\boxed{y=1-\frac{\lambda_1}{2}},\quad 2x=3\lambda_1x^2\to\boxed{x=0}\to\boxed{x=\frac{2}{3\lambda_1}}$

Plug into g → $g(x,y)=y-x^3=1-\frac{\lambda_1}{2}-\left(\frac{2}{3\lambda_1}\right)^3=1-\frac{\lambda_1}{2}-\frac{8}{27\lambda_1^3}=0$

$\frac{\lambda_1}{2}+\frac{8}{27\lambda_1^3}=1\to\frac{27\lambda_1^4+16}{54\lambda_1^3}=1\to 27\lambda_1^4+16=54\lambda_1^3\to\boxed{27\lambda_1^4-54\lambda_1^3+16=0}$

Quad eqn. $\boxed{\lambda_1^2=\frac{54\pm\sqrt{(54)^2-4(27)(16)}}{2(27)}}\to\boxed{\lambda_1=\sqrt{\frac{54\pm\sqrt{54^2-4(27)(16)}}{54}}}$

b) $x=0\to g(x,y)=y-x^3=0\to y=0$

$\frac{\partial L}{\partial x}=0,\quad\frac{\partial L}{\partial y}=-2+\lambda_1\to\lambda_1=2\to y=0$

Local opt. cond → $\left[\frac{\partial^2 f}{\partial\vec{x}^2}+\frac{\partial^2(\vec{\lambda}\cdot\vec{g})}{\partial\vec{x}^2}\right]\Bigg|_{\substack{x=0\\y=0}}=$

$\frac{\partial^2 f}{\partial\vec{x}^2}=\begin{bmatrix}0\end{bmatrix}$

c) ∵ J is 1 @x=0 & x=1 → x=0 is the local minimum because for x to be a global minimum $J(x^*+\delta x^*)>J(x^*)$ on the whole set of X

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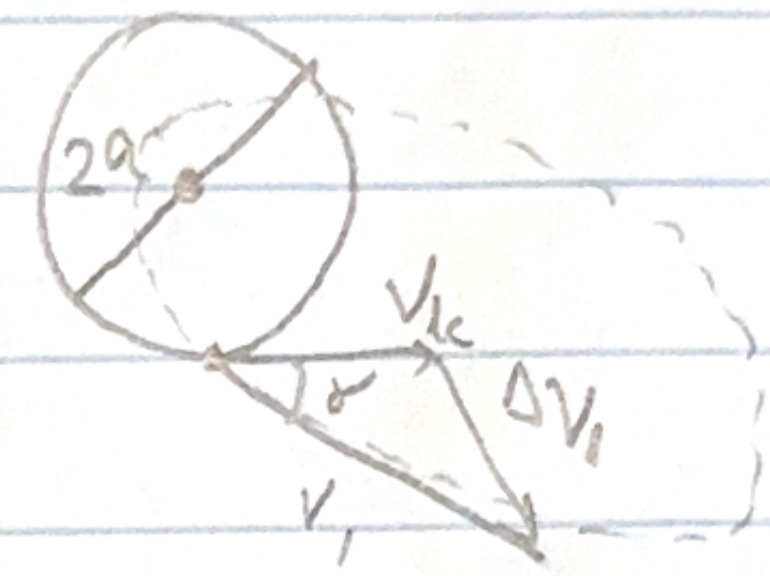
Problem 2 →

a) Circular orbit, velocity at any point



$$V_c = \sqrt{\frac{\mu}{a}}$$

$$\Delta V_1^2 = V_1^2 + V_c^2 - 2V_1V_c\cos\gamma$$

θ - True anomaly

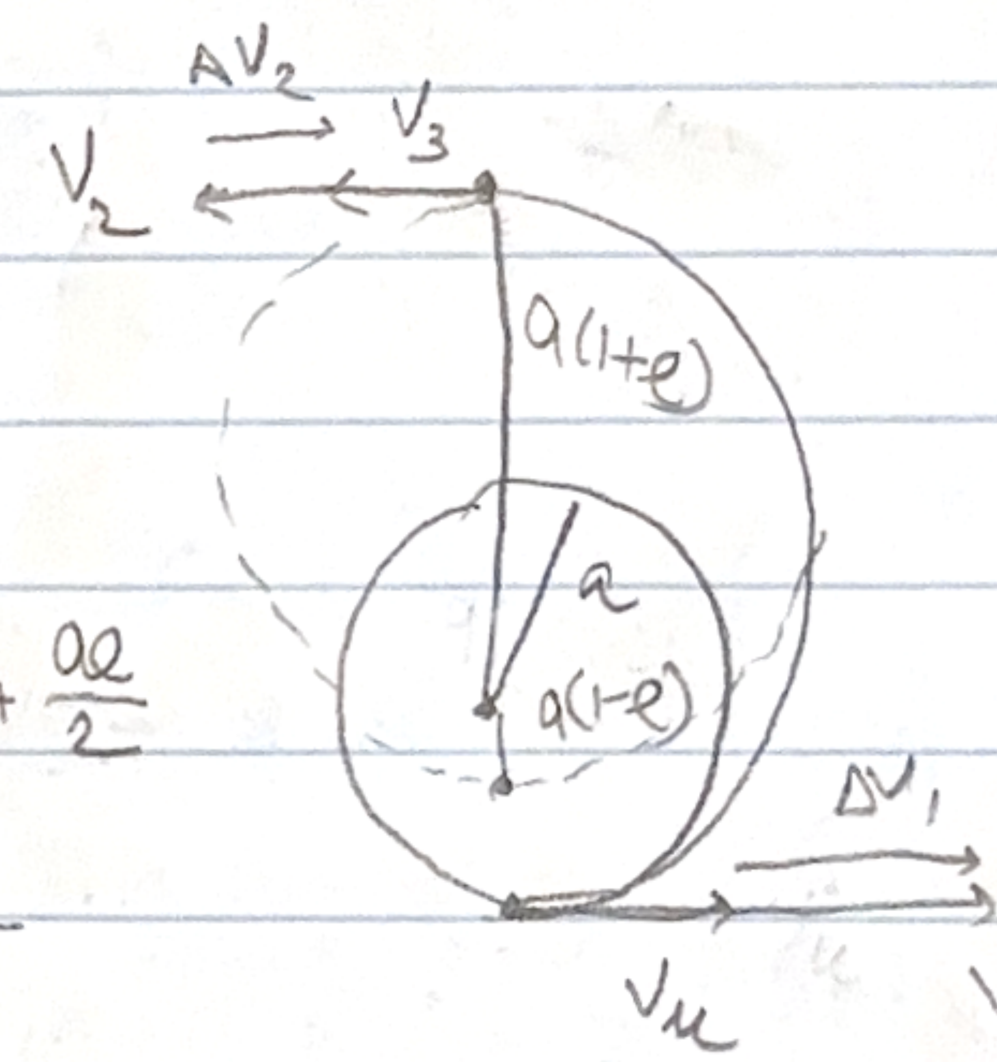
$$V_1 = \sqrt{\frac{\mu}{p}(1+2e\cos\theta+e^2)}, \quad p = a(1-e^2)$$

$$V_1 = \sqrt{\frac{\mu(1+2e\cos\theta+e^2)}{a(1-e^2)}}$$

$$\Delta V_1^2 = \frac{\mu}{a(1-e^2)}(1+2e\cos\theta+e^2) + \frac{\mu}{a} - 2\frac{\mu}{a}\sqrt{\frac{(1+2e\cos\theta+e^2)}{(1-e^2)}}\cos(\gamma)$$

$$\boxed{\Delta V_1 = \sqrt{\frac{\mu(1+2e\cos\theta+e^2)}{a(1-e^2)} + \frac{\mu}{a} - \frac{2\mu\cos\gamma}{a}\sqrt{\frac{(1+2e\cos\theta+e^2)}{(1-e^2)}}}}$$

$\geq 0 \because \Delta V_1$ is a magnitude



b) $V_c = \sqrt{\frac{\mu}{a}}$,

@ periapsis → Transfer ellipse → $a_T = \frac{1}{2}(a(1+e)+a) = \frac{1}{2}(2a+ae) = a + \frac{ae}{2}$

$r_a = a(1+e), \; r_p = a$

$$V_p = V_1 = \sqrt{\frac{\mu}{a_T}\frac{r_a}{r_p}} = \sqrt{\frac{\mu}{a+\frac{ae}{2}}\frac{a(1+e)}{a}} = \sqrt{\frac{\mu(1+e)}{a+ae/2}}$$

$$\Delta V_1 = V_1 - V_c = \sqrt{\frac{\mu(1+e)}{a+ae/2}} - \sqrt{\frac{\mu}{a}}$$

@ apoapsis → $V_2 = V_a = \sqrt{\frac{\mu}{a_T}\frac{r_p}{r_a}} = \sqrt{\frac{\mu}{a+ae/2}\cdot\frac{\cancel{a}}{\cancel{a}(1+e)}} = \sqrt{\frac{\mu}{(a+\frac{ae}{2})(1+e)}}$

$$V_3 = V_a = \sqrt{\frac{\mu}{a}\frac{\cancel{a}(1-e)}{\cancel{a}(1+e)}} = \sqrt{\frac{\mu(1-e)}{a(1+e)}}$$

$$\Delta V_2 = \sqrt{\frac{\mu}{(a+\frac{ae}{2})(1+e)}} - \sqrt{\frac{\mu(1-e)}{a(1+e)}}$$

$$\Delta V_{total} = \Delta V_1 + \Delta V_2 = \boxed{\sqrt{\frac{\mu(1+e)}{a+ae/2}} - \sqrt{\frac{\mu}{a}} + \sqrt{\frac{\mu}{(a+\frac{ae}{2})(1+e)}} - \sqrt{\frac{\mu(1-e)}{a(1+e)}}}$$

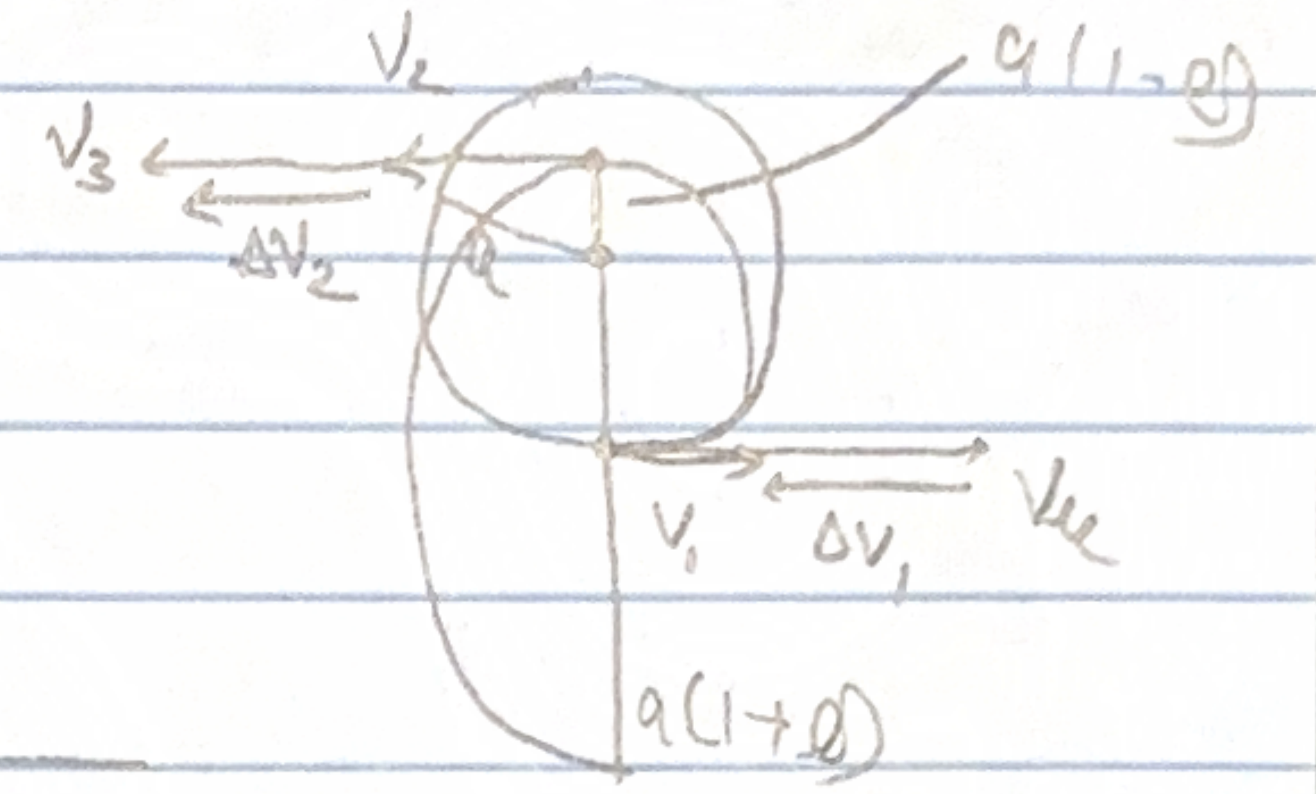
$$= \sqrt{\frac{\mu}{a}}\left(\sqrt{\frac{1+e}{1+e/2}} - 1 + \sqrt{\frac{1}{(1+e/2)(1+e)}} - \sqrt{\frac{1-e}{1+e}}\right)$$

$\sqrt{1+\ell} = 1 + \frac{\ell}{2}$

$\frac{1}{\sqrt{1+\ell/2}} = 1 - \frac{\ell}{4}$

$\ell \ll 1 \rightarrow$ expansion $\rightarrow \Delta V \approx \sqrt{\frac{4}{a}} \left(\left(1+\frac{\ell}{2}\right)\left(1-\frac{\ell}{4}\right) - 1 + \left(1-\frac{\ell}{4}\right)\left(1-\frac{\ell}{2}\right) - \left(1-\frac{\ell}{2}\right)\left(1-\frac{\ell}{2}\right)\right)$

c) $V_c = \sqrt{\frac{\mu}{a}}$



1) apoapsis

Transfer ellipse $\rightarrow r_p = a(1-e)$, $r_a = a$

$a_T = \frac{1}{2}(r_a + r_p) = \frac{1}{2}(a + a - ae) = a - \frac{ae}{2}$

$V_a = V_1 = \sqrt{\frac{\mu}{a_T} \cdot \frac{r_p}{r_a}} = \sqrt{\frac{\mu}{a - \frac{ae}{2}} \cdot \frac{a(1-e)}{a}} = \sqrt{\frac{\mu(1-e)}{a - ae/2}}$

$\Delta V_1 = \sqrt{\frac{\mu}{a}} - \sqrt{\frac{\mu(1-e)}{a - ae/2}}$

2) Periapsis

$V_p = V_2 = \sqrt{\frac{\mu}{a_T} \cdot \frac{r_a}{r_p}} = \sqrt{\frac{\mu}{a - \frac{ae}{2}} \cdot \frac{a}{a(1-e)}} = \sqrt{\frac{\mu}{(a - \frac{ae}{2})(1-e)}}$

$V_a' = V_3 = \sqrt{\frac{\mu}{a} \cdot \frac{r_p}{r_a}} = \sqrt{\frac{\mu}{a} \cdot \frac{a(1-e)}{a(1+e)}} = \sqrt{\frac{\mu(1-e)}{a(1+e)}}$

$\Delta V_2 = V_3 - V_2 = \sqrt{\frac{\mu(1-e)}{a(1+e)}} - \sqrt{\frac{\mu}{(a - \frac{ae}{2})(1-e)}}$

$\Delta V_{tot} = \Delta V_1 + \Delta V_2 = \sqrt{\frac{\mu}{a}} - \sqrt{\frac{\mu(1-e)}{a - ae/2}} + \sqrt{\frac{\mu(1-e)}{a(1+e)}} - \sqrt{\frac{\mu}{(a - \frac{ae}{2})(1-e)}}$

$= \sqrt{\frac{\mu}{a}}\left(1 - \sqrt{\frac{1-e}{1-e/2}} + \sqrt{\frac{1-e}{1+e}} - \sqrt{\frac{1}{(1-\frac{e}{2})(1-e)}}\right)$

$e << 1 \rightarrow$ expansion $\rightarrow \Delta V \approx \sqrt{\frac{\mu}{a}}\left(1 - (1-\frac{e}{2})(1+\frac{e}{4}) + (1-\frac{e}{2})(1-\frac{e}{2}) - (1+\frac{e}{4})(1+\frac{e}{2})\right)$

$=$

d)

e) When e is not small $\rightarrow \Delta V_3$ must be optimal because it is favorable to perform ΔV_s close to the body when possible.

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Problem 3 → $\vec{x} = \begin{bmatrix} r \\ v \end{bmatrix}$, $\dot{\vec{x}} = \begin{bmatrix} \dot{r} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ u \end{bmatrix}$, $|u| \le 1$, $J = \int_0^{t_f} |u| \, d\tau$, $r(t_0) = r_0$, $v(t_0) = v_0$, $t_0 = 0$

a) $H = L + \vec{p} \cdot \vec{f}$, $L = |u|$, $\vec{p} = \begin{bmatrix} p_r \\ p_v \end{bmatrix}$, $\dot{\vec{x}} = \vec{f} = \begin{bmatrix} v \\ u \end{bmatrix}$

$\boxed{H = |u| + p_r v + p_v u}$ $\quad |u| = u \cdot \text{sign}(u)$

$= u\,\text{sgn}(u) + p_v u + p_r v = \boxed{p_r v + u(\text{sgn}(u) + p_v) = H}$

b) $u^* = \underset{u}{\text{argmin}}\, H = \underset{u}{\text{argmin}} \left[\cancel{p_r v} + u(\text{sgn}(u) + p_v) \right]$ (Not influenced by u, so remove $p_r v$)

Test cases → $p_v = 1 - \epsilon \rightarrow (1 - \epsilon) u + u\,\text{sign}(u)$ (ϵ is very small)

If $u = +1 \rightarrow 1 - \epsilon + 1 \rightarrow 2 - \epsilon \rightarrow +ve$

If $u = -1 \rightarrow \epsilon - 1 + 1 \rightarrow +ve$

If $u = 0 \rightarrow 0 \rightarrow$ $\therefore p_v \le 1 \rightarrow u^* = 0$

$p_v = 1 + \epsilon \rightarrow (1 + \epsilon) u + u\,\text{sgn}(u)$

If $u = +1 \rightarrow 1 + \epsilon + 1 \rightarrow 2 + \epsilon \rightarrow +ve$

If $u = -1 \rightarrow -\epsilon - 1 + 1 \rightarrow -\epsilon \rightarrow -ve$

If $u = 0 \rightarrow 0 \rightarrow$ $\therefore p_v > 1 \rightarrow u^* = -u_{max}$

$p_v = -\frac{1}{2} \rightarrow -\frac{u}{2} + u\,\text{sign}(u)$

If $u = 0 \rightarrow 0$

If $u = -x \rightarrow +\frac{x}{2} + x = \frac{3x}{2} \rightarrow +ve$

If $u = x \rightarrow -\frac{x}{2} + x \rightarrow \frac{x}{2} \rightarrow +ve \rightarrow \therefore p_v = -\frac{1}{2} \rightarrow u^* = 0$

$p_v = \frac{1}{2} \rightarrow \frac{u}{2} + \frac{u}{2}\text{sgn}(u)$

If $u = 0 \rightarrow 0$

If $u = -x \rightarrow -\frac{x}{2} + \frac{x}{2} \rightarrow \frac{x}{2} \rightarrow +ve$

If $u = x \rightarrow \frac{x}{2} + \frac{x}{2} \rightarrow x \rightarrow +ve \rightarrow \therefore p_v = \frac{1}{2} \rightarrow u^* = 0$

$p_v = -2 \rightarrow -2u + u\,\text{sign}(u)$

If $u = 0 \rightarrow 0$

If $u = -x \rightarrow +2x - x \rightarrow x \rightarrow +ve$

If $u = x \rightarrow -2x + x \rightarrow -x \rightarrow$ $\therefore p_v = -2 \rightarrow u^* = x$

$P_v = 2 \rightarrow \quad 2u + u\,sgn(u)$

If $u = 0 \rightarrow 0$

If $u = -x \rightarrow -2x + x \rightarrow -x \rightarrow -ve$

If $u = x \rightarrow 2x + x \rightarrow 3x \rightarrow +ve \rightarrow$ if $P_v = 2$, $u^* = -x$

$|P_v| \le 1 : u^* = 0$

$|P_v| > 1 : u^* = -u_{max}\, sgn(P)$

$$\boxed{u^* = \begin{cases} 0 & \text{if } |P_v| \le 1 \\ -u_{max}\, sgn(P_v) & \text{if } |P_v| > 1 \end{cases}}$$

$|u| \le 1 \rightarrow u_{max} = 1$

c) $H = P_r v + u(sgn(u) + P_v)$

$u^* = 0 \rightarrow H^* = P_r v$

$u^* = -sgn(P_v) \rightarrow H^* = P_r v + \underbrace{(sign(P_v)\, sign(u))}_{\text{goes to 1 as } sgn(u) = -sgn(P_v)} - \underbrace{(sign(P_v)\, P_v)}_{|P_v|}$

$= P_r v - sgn(P_v)\, sign(sign(P_v)) - |P_v|$

$= P_r v - |P_v| + 1$

$$\boxed{H^* = \begin{cases} P_r v & \text{if } |P_v| \le 1 \\ P_r v - |P_v| + 1 & \text{if } |P_v| > 1 \end{cases}}$$

$\dot{\vec{x}} = \frac{\partial H^*}{\partial \vec{p}} \rightarrow \boxed{\dot{r} = \frac{\partial H^*}{\partial P_r} = v}$

$$\boxed{\dot{v} = \frac{\partial H^*}{\partial P_v} = \begin{cases} 0 & \text{if } |P_v| \le 1 \\ -\frac{P_v}{|P_v|} & \text{if } |P_v| > 1 \end{cases}}$$

$\dot{\vec{p}} = \frac{\partial H^*}{\partial \vec{x}}$ $\boxed{\dot{P}_r = \frac{\partial H^*}{\partial r} = 0 \qquad \dot{P}_v = \frac{\partial H^*}{\partial v} = P_r}$

P_r is constant, $\boxed{P_{r_0} = P_{r_f}}$

$P_v = P_r t + P_{v_0}$

d) $\vec{P}_0 = -\cancel{\frac{\partial K}{\partial \vec{x}_0}}^{0} - \vec{\lambda}\frac{\partial g}{\partial \vec{x}_0}$, $g_1 = r(t_0) - r_0 = 0$, $g_2 = v(t_0) - v_0 = 0$; $g_3 = t(0) - t_0 = 0$

$\vec{\lambda} = [\lambda_{r_0}, \lambda_{v_0}, \lambda_{t_0}]^T$

$P_{r_0} = +\lambda_{r_0}$

$P_{v_0} = +\lambda_{v_0}$

$H_0 = \cancel{\frac{\partial K}{\partial t_0}} + \vec{\lambda} \cdot \frac{\partial \vec{g}}{\partial t_0} = -\lambda_{t_0}$

$P_{r_f} = \cancel{\frac{\partial K}{\partial r_f}}^{0} + \vec{\lambda}\frac{\partial \vec{g}}{\partial r_f} = 0$

$P_{v_f} = \frac{\partial K}{\partial v_f} + \vec{\lambda}\frac{\partial \vec{g}}{\partial v_f} = 0$, $H_f = -\cancel{\frac{\partial K}{\partial t_f}} - \vec{\lambda} \cdot \frac{\partial \vec{g}}{\partial t_f} = 0$