Question 1 - attempt 2

Wednesday, February 16, 2022 15:40

CONSIDER THE COMPARISON BETWEEN THE COST OF A HOHMANN TRANSFER, J_H, AND A BI-PARABOLIC TRANSFER, J_{BP}, BETWEEN TO CIRCULAR ORBITS WITH RADIUS RATIO $r = r_2/r_1$.

$$J_H = \sqrt{\frac{2r}{1+r}} - 1 + \frac{1}{\sqrt{r}} - \sqrt{\frac{2}{r(1+r)}}$$

$$J_{BP} = (\sqrt{2}-1)\left[1 + \frac{1}{\sqrt{r}}\right]$$

SHOW THAT THE FOLLOWING POLYNOMIAL EQUATION CAN BE DERIVED, STARTING FROM THE CONDITION $J_{BP} \le J_H$ AND ASSUMING $r > 1$.

$$r^3 - (7+4\sqrt{2})r^2 + (3+4\sqrt{2})r - 1 \geq 0$$

$$(\sqrt{2}-1)\left[1+\frac{1}{\sqrt{r}}\right] \le \sqrt{\frac{2r}{1+r}} - 1 + \frac{1}{\sqrt{r}} - \sqrt{\frac{2}{r(1+r)}}$$

$$(\sqrt{2}-1)\left[\frac{\sqrt{r}}{\sqrt{r}} + \frac{1}{\sqrt{r}}\right] \le \frac{\sqrt{2r} - \sqrt{r(1+r)} + \sqrt{1+r} - \sqrt{2}}{\sqrt{r}\sqrt{1+r}}$$

$$\sqrt{1+r}\,(\sqrt{2}-1)(1+\sqrt{r}) \le \sqrt{2}(r-1) + \sqrt{1+r}\,(1-\sqrt{r})$$

$$\sqrt{2}\sqrt{1+r} + \sqrt{2}\sqrt{r}\sqrt{1+r} - \sqrt{1+r} - \cancel{\sqrt{r}\sqrt{1+r}} \le \sqrt{2}r - \sqrt{2} + \sqrt{1+r} - \cancel{\sqrt{r}\sqrt{1+r}}$$

$$\sqrt{2}\sqrt{1+r} - 2\sqrt{1+r} \le \sqrt{2}r - \sqrt{2} - \sqrt{2}\sqrt{r}\sqrt{1+r}$$

$$(2-\sqrt{2})(\sqrt{1+r}) \le \sqrt{2}(r - \sqrt{r}\sqrt{1+r} - 1)$$

SQUARE BOTH SIDES

$$(2-\sqrt{2})^2(1+r) \le 2(r - \sqrt{r}\sqrt{1+r} - 1)^2$$

$$(4-4\sqrt{2}+2)(1+r) \le 2(2r^2 - r - 2r\sqrt{r}\sqrt{1+r} + 2\sqrt{r}\sqrt{1+r} + 1)$$

$$(6-4\sqrt{2})(1+r)$$

$$\cancel{2}(3-2\sqrt{2})(1+r) \le \cancel{2}(2r^2 - r - 2r\sqrt{r}\sqrt{1+r} + 2\sqrt{r}\sqrt{1+r} + 1)$$

$$2\sqrt{r}\sqrt{1+r}\,(r-1) \le 2r^2 - r + 1 - (3-2\sqrt{2})(1+r)$$

$$2r^2 - r - (3-2\sqrt{2})r - (3-2\sqrt{2}) + 1$$

$$\cancel{2}\sqrt{r}\sqrt{1+r}\,(r-1) \le \cancel{2}r^2 - (\cancel{4} - \cancel{2}\sqrt{2})r - \cancel{2} + \cancel{2}\sqrt{2}$$

$$\sqrt{r}\sqrt{1+r}\,(r-1) \le r^2 - (2-\sqrt{2})r - 1 + \sqrt{2}$$

SQUARE BOTH SIDES

$$(r)(1+r)(r-1)^2 \le (r^2 - \underbrace{(2-\sqrt{2})}_{a}r - \underbrace{(1-\sqrt{2})}_{b})^2$$

$$r(1+r)(r^2-2r+1)$$

$$(r^2+r)(r^2-2r+1)$$

$$\cancel{r^4} - r^3 - r^2 + r$$

$$(r^2-ar-b)(r^2-ar-b)$$

$$r^4 - 2ar^3 - 2br^2 + a^2r^2 + 2abr + b^2$$

$$\cancel{r^4} - 2ar^3 + (a^2-2b)r^2 + 2abr + b^2$$

$$0 \le (1-2a)r^3 + (1+a^2-2b)r^2 + (2ab-1)r + b^2$$

$$0 \le r^3 + \frac{(1+a^2-2b)}{1-2a}r^2 + \frac{(2ab-1)}{(1-2a)} + \frac{b^2}{1-2a}$$

$a = 2-\sqrt{2}$; $b = 1-\sqrt{2}$

$$a^2 = (2-\sqrt{2})(2-\sqrt{2}) = 4-4\sqrt{2}+2$$

$$a^2 = 6-4\sqrt{2}$$

$$ab = (2-\sqrt{2})(1-\sqrt{2}) = 2-\sqrt{2}-2\sqrt{2}+2$$

$$ab = 4-3\sqrt{2}$$

$$b^2 = (1-\sqrt{2})(1-\sqrt{2})$$

$$= 1-2\sqrt{2}+2$$

$$b^2 = 3-2\sqrt{2}$$

$$1-2a = 1-2(2-\sqrt{2})$$

$$= 1-4+2\sqrt{2}$$

$$1-2a = -3+2\sqrt{2}$$

$$\frac{1+a^2-2b}{1-2a} = \frac{1+6-4\sqrt{2}-2(1-\sqrt{2})}{-3+2\sqrt{2}}$$

$$= \frac{7-4\sqrt{2}-2+2\sqrt{2}}{-3+2\sqrt{2}}$$

$$= \left(\frac{5-2\sqrt{2}}{-3+2\sqrt{2}}\right)\left(\frac{-3-2\sqrt{2}}{-3-2\sqrt{2}}\right)$$

$$= \frac{-15+6\sqrt{2}-10\sqrt{2}+8}{9-8}$$

$$= (-15+8) - 4\sqrt{2}$$

$$-7-4\sqrt{2}$$

$$= -(7+4\sqrt{2})$$ COEFFICIENT TO r^2 TERM

$$\frac{2ab-1}{1-2a} = \frac{2(4-3\sqrt{2})-1}{-3+2\sqrt{2}} = \left(\frac{7-6\sqrt{2}}{-3+2\sqrt{2}}\right)\left(\frac{-3-2\sqrt{2}}{-3-2\sqrt{2}}\right)$$

$$= -21+18\sqrt{2}-14\sqrt{2}+24$$

$$= 3+4\sqrt{2}$$ COEFFICIENT TO r TERM

$$\frac{b^2}{1-2a} = \frac{3-2\sqrt{2}}{-3+2\sqrt{2}} = -1$$ CONSTANT TERM

$$\boxed{0 \le r^3 - (7+4\sqrt{2})r^2 + (3+4\sqrt{2})r - 1}$$

ROOTS: 0.14655, 0.57153, 11.9388

→ The roots < 1 have no physical significance, since the polynomial is derived under the assumption that $r > 1$.

$1 < r < 11.94$ $J_H \le J_{BP}$ (Hohmann optimal)

$r > 11.94$ $J_{BP} \le J_H$ (Bi-Parabolic optimal)

IF $r < 1$, THEN THE HOHMANN COST FUNCTION BECOMES

$$J_H = 1 - \sqrt{\frac{2r}{1+r}} + \sqrt{\frac{2}{r(1+r)}} - \frac{1}{\sqrt{r}}$$

BI-PARABOLIC COST FUNCTION BECOMES

$$J_{BP} = (\sqrt{2}-1)\left[1+\frac{1}{\sqrt{r}}\right]$$

$$J_{BP} < J_H$$

$$(\sqrt{2}-1)\left[1+\frac{1}{\sqrt{r}}\right] \le 1 - \sqrt{\frac{2r}{1+r}} + \sqrt{\frac{2}{r(1+r)}} - \frac{1}{\sqrt{r}}$$

$$\sqrt{2} + \frac{\sqrt{2}}{\sqrt{r}} - 1 - \cancel{\frac{1}{\sqrt{r}}} \le 1 - \sqrt{\frac{2r}{1+r}} + \sqrt{\frac{2}{r(1+r)}} - \cancel{\frac{1}{\sqrt{r}}}$$

$$\sqrt{2} - 2 + \frac{\sqrt{2}}{\sqrt{r}} \le \sqrt{\frac{2}{1+r}}\left(-\sqrt{r} + \frac{1}{\sqrt{r}}\right)$$

$$\sqrt{r}\left[1 - \sqrt{2} + \frac{1}{\sqrt{r}}\right] \le \sqrt{\frac{1}{1+r}}\left(\frac{1-r}{\sqrt{r}}\right)\sqrt{r}$$ MULTIPLY BOTH SIDES BY $\sqrt{r}$

$$\sqrt{r} - \sqrt{2}\sqrt{r} + 1 \le \sqrt{\frac{1}{1+r}}(1-r)$$

$$\sqrt{r}(1-\sqrt{2}) + 1 \le \sqrt{\frac{1}{1+r}}(1-r)$$ SQUARE BOTH SIDES

$$(\sqrt{r}(1-\sqrt{2})+1)(\sqrt{r}(1-\sqrt{2})+1) \le \frac{1}{1+r}(1-r)^2$$

$$r(1-\sqrt{2})^2 + 2\sqrt{r}(1-\sqrt{2}) + 1 \le \frac{1}{1+r}(1-r)^2$$

$$r(r+1)(1-\sqrt{2})^2 + 2\sqrt{r}(1-\sqrt{2})(1+r) + r + 1 - (1-r)^2 \le 0$$

$$(r^2+r)(1-2\sqrt{2}+2) + r + 1 - (1-2r+r^2) \le 2\sqrt{r}(1-\sqrt{2})(1+r)$$

$$\cancel{r^2} - 2\sqrt{2}r^2 + 2r^2 + r - 2\sqrt{2}r + 2r + r \cancel{+1} \cancel{-1} + 2r - \cancel{r^2} \le$$

$$r^2(2-2\sqrt{2}) + (6-2\sqrt{2})r \le 2\sqrt{r}(1-\sqrt{2})(1+r)$$

SQUARE BOTH SIDES AGAIN

$$[(2-2\sqrt{2})r^2 + (6-2\sqrt{2})r][(2-2\sqrt{2})r^2 + (6-2\sqrt{2})r] \le 4r(1-\sqrt{2})^2(1+r)^2$$

$$4(1-\sqrt{2})^2r^4 + 4(1-\sqrt{2})(6-2\sqrt{2})r^3 + (6-2\sqrt{2})^2r^2 \le 4r(1-\sqrt{2})^2(1+2r+r^2)$$

$$4(1-\sqrt{2})^2(r^3+2r^2+r)$$

$$4(1-\sqrt{2})^2r^3 + 4(1-\sqrt{2})(6-2\sqrt{2})r^2 + (6-2\sqrt{2})^2r \le 4(1-\sqrt{2})^2(r^2+2r+1)$$

$$r^3 + \frac{6-2\sqrt{2}}{1-\sqrt{2}}r^2 + \frac{(6-2\sqrt{2})^2}{4(1-\sqrt{2})^2}r \le r^2+2r+1$$

$$r^3 + \underbrace{\left(\frac{6-2\sqrt{2}}{1-\sqrt{2}} - 1\right)}_{-(3+4\sqrt{2})}r^2 + \underbrace{\left(\frac{(6-2\sqrt{2})}{4(1-\sqrt{2})^2} - 2\right)}_{=7+4\sqrt{2}}r - 1 \le 0$$

$$\boxed{r^3 - (3+4\sqrt{2})r^2 + (7+4\sqrt{2})r - 1 \le 0}$$

Roots are 6.8234, 1.7497, 0.0838

→ Only roots $r<1$ are physical, since we defined the cost function for $r<1$

→ $r \le 0.0838$ parabolic transfer is optimal

★ The same results can be derived by substituting $r = 1/r'$ ★ into the original polynomial & simplifying

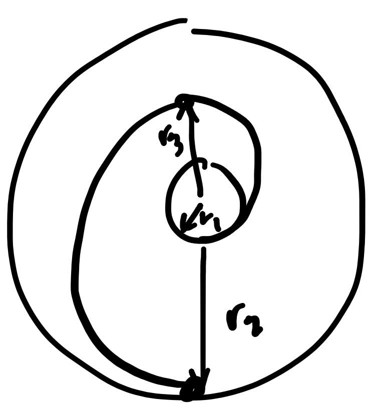
Question 2

Wednesday, February 16, 2022 12:45

WHAT IS THE COST OF A BI-ELLIPTIC TRANSFER, AS COMPARED TO A HOHMANN TRANSFER, WHEN THE INTERMEDIATE ORBIT APOAPSIS ℓ FALLS IN THE INTERVAL $1 < \ell < r$? DISCUSS.

$$r = \frac{r_2}{r_1} \qquad \ell = \frac{r_3}{r_1}$$

$$1 < \frac{r_3}{r_1} < \frac{r_2}{r_1} \Rightarrow r_1 < r_3 < r_2$$



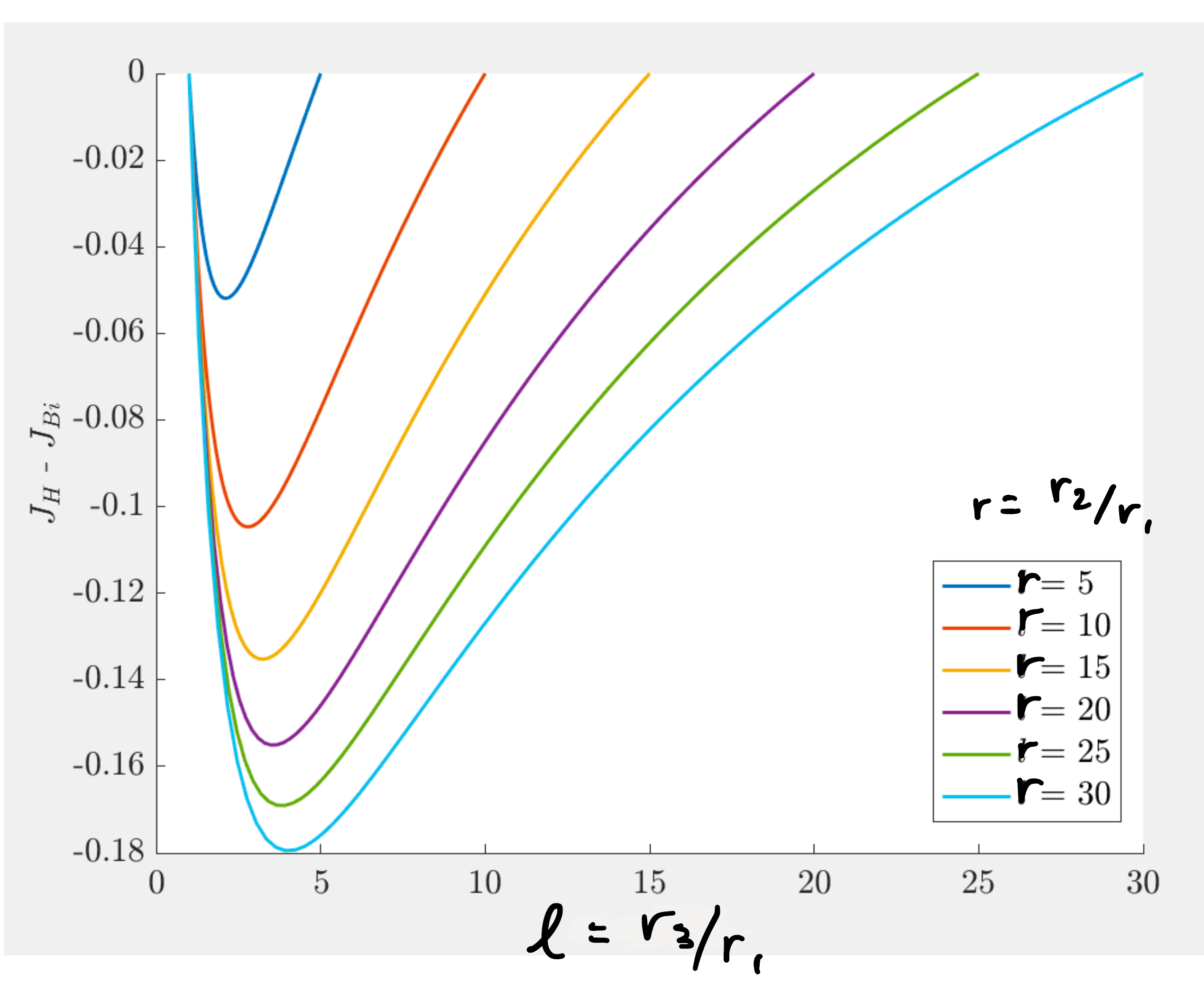
COST OF BI-ELLIPTIC

$$\tilde{J}_{Bi}(r,\ell) = \sqrt{\frac{2\ell}{1+\ell}} - 1 + \sqrt{\frac{2r}{\ell(\ell+r)}} - \sqrt{\frac{2}{\ell(1+\ell)}} - \sqrt{\frac{2\ell}{r(\ell+r)}} + \frac{1}{\sqrt{r}}$$

COST OF HOHMANN

$$\tilde{J}_H(r) = \sqrt{\frac{2r}{1+r}} - 1 + \frac{1}{\sqrt{r}} - \sqrt{\frac{2}{r(1+r)}}$$

→ DO A DIRECT NUMERICAL COMPARISON & PLOT $J_H - J_{Bi}$



ALL VALUES SHOWN $J_H - J_{Bi} < 0 \Rightarrow$ THERE ARE NO INSTANCES WHERE BI-ELLIPTIC TRANSFER COSTS LESS THAN HOHMANN TRANSFER

☆ These results could also have been shown analytically ☆

Question 3

Monday, February 7, 2022 11:41

RECALL THE COMPARISON BETWEEN THE COST OF A HOHMANN TRANSFER AND AN AEROBRAKING ELLIPTIC TRANSFER, $g_I(\alpha_1,\alpha_2) = J_H - J_{AE}$ FOR $\alpha_2 < \alpha_1$:

$$g_I(\alpha_1,\alpha_2) = \sqrt{\frac{2}{\alpha_1(1+\alpha_1)}} - \sqrt{\frac{2\alpha_2}{\alpha_1(\alpha_2+\alpha_1)}} + \sqrt{\frac{2\alpha_1}{\alpha_2(\alpha_2+\alpha_1)}} + \sqrt{\frac{2}{\alpha_2(1+\alpha_2)}} - \frac{2}{\sqrt{\alpha_2}}$$

DERIVE AND VERIFY THIS RELATIONSHIP. THEN, SOLVE THE EQUATION IN THE VICINITY OF THE POINT $\alpha_1 = \alpha_2 = 1$

$$g_I(1+a\epsilon, 1+\epsilon) = 0$$

TO DO THIS ASSUME THAT $\epsilon \ll 1$ AND PERFORM THE TAYLOR SERIES EXPANSION. TO SATISFY THIS RELATIONSHIP YOU SHOULD FIND THAT $a = 3$. THUS, SHOW THAT IN THE VICINITY OF THE (1,1) ORIGIN ON THE (α_1,α_2) CHART, THE LINE BETWEEN OPTIMAL HOHMANN & OPTIMAL ELLIPTIC AERDASSIST IS DEFINED AS:

$$\alpha_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}$$

HOHMANN TRANSFER COST FUNCTION

$r_2 < r_1$

$$J_H = -\sqrt{\frac{2\mu}{r_1+r_2}\frac{r_2}{r_1}} + \sqrt{\frac{\mu}{r_1}} - \sqrt{\frac{\mu}{r_2}} + \sqrt{\frac{2\mu}{r_1+r_2}\frac{r_1}{r_2}} \qquad \alpha_1 = \frac{r_1}{R} \quad \alpha_2 = \frac{r_2}{R}$$

$$= -\sqrt{\frac{2\mu}{R\alpha_1 + R\alpha_2}\frac{\cancel{R}\alpha_2}{\cancel{R}\alpha_1}} + \sqrt{\frac{\mu}{\alpha_1 R}} - \sqrt{\frac{\mu}{\alpha_2 R}} + \sqrt{\frac{2\mu}{\alpha_1 R + \alpha_2 R}\frac{\cancel{R}\alpha_1}{\cancel{R}\alpha_2}}$$

$$= -\sqrt{\frac{\mu}{R}}\sqrt{\frac{2\alpha_2}{\alpha_1(\alpha_1+\alpha_2)}} + \sqrt{\frac{\mu}{R}}\sqrt{\frac{1}{\alpha_1}} - \sqrt{\frac{\mu}{R}}\sqrt{\frac{1}{\alpha_2}} + \sqrt{\frac{\mu}{R}}\sqrt{\frac{2\alpha_1}{\alpha_2(\alpha_1+\alpha_2)}}$$

→ NORMALIZE BY $\sqrt{\frac{\mu}{R}}$

$$\boxed{J_H(\alpha_1,\alpha_2) = \frac{1}{\sqrt{\alpha_1}} - \sqrt{\frac{2\alpha_2}{(\alpha_1+\alpha_2)\alpha_1}} + \sqrt{\frac{2\alpha_1}{(\alpha_1+\alpha_2)\alpha_2}} - \frac{1}{\sqrt{\alpha_2}}}$$

AERO-ASSIST COST FUNCTION

$$\Delta V_{AE} = \sqrt{\frac{\mu}{r_1}} - \sqrt{\frac{2\mu}{(r_1+R)}\frac{R}{r_1}} + \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu}{(r_2+R)}\frac{R}{r_2}} \qquad \alpha_1 = \frac{r_1}{R} \quad \alpha_2 = \frac{r_2}{R}$$

$$= \sqrt{\frac{\mu}{\alpha_1 R}} - \sqrt{\frac{2\mu}{(\alpha_1 R + R)}\frac{\cancel{R}}{\cancel{R}\alpha_1}} + \sqrt{\frac{\mu}{\alpha_2 R}} - \sqrt{\frac{2\mu}{(\alpha_2 R + R)}\frac{\cancel{R}}{\alpha_2\cancel{R}}}$$

$$\Delta V_{AE} = \sqrt{\frac{\mu}{R}}\sqrt{\frac{1}{\alpha_1}} - \sqrt{\frac{\mu}{R}}\sqrt{\frac{2}{\alpha_1(\alpha_1+1)}} + \sqrt{\frac{\mu}{R}}\sqrt{\frac{1}{\alpha_2}} - \sqrt{\frac{\mu}{R}}\sqrt{\frac{2}{\alpha_2(\alpha_2+1)}}$$

→ NORMALIZE BY $\sqrt{\frac{\mu}{R}}$

$$\boxed{J_{AE}(\alpha_1,\alpha_2) = \frac{1}{\sqrt{\alpha_1}} - \sqrt{\frac{2}{\alpha_1(1+\alpha_1)}} + \frac{1}{\sqrt{\alpha_2}} + \sqrt{\frac{2}{\alpha_2(1+\alpha_2)}}}$$

$$g_I(\alpha_1,\alpha_2) = J_H - J_{AE}$$

$$= \cancel{\frac{1}{\sqrt{\alpha_1}}} - \sqrt{\frac{2\alpha_2}{(\alpha_1+\alpha_2)\alpha_1}} + \sqrt{\frac{2\alpha_1}{(\alpha_1+\alpha_2)\alpha_2}} - \frac{1}{\sqrt{\alpha_2}}$$

$$- \left(\cancel{\frac{1}{\sqrt{\alpha_1}}} - \sqrt{\frac{2}{\alpha_1(1+\alpha_1)}} + \frac{1}{\sqrt{\alpha_2}} + \sqrt{\frac{2}{\alpha_2(1+\alpha_2)}} \right)$$

$$\boxed{g_I(\alpha_1,\alpha_2) = \sqrt{\frac{2}{\alpha_1(1+\alpha_1)}} - \sqrt{\frac{2\alpha_2}{\alpha_1(\alpha_1+\alpha_2)}} + \sqrt{\frac{2\alpha_1}{\alpha_2(\alpha_1+\alpha_2)}} + \sqrt{\frac{2}{\alpha_2(1+\alpha_2)}} - \frac{2}{\sqrt{\alpha_2}}}$$

TAYLOR SERIES EXPANSION

$$g_I(1+a\epsilon, 1+\epsilon) = 0$$

$$g_I \simeq g_I(1+a\epsilon, 1+\epsilon) - \left.\frac{\partial g_I}{\partial \alpha_1}\right|_{(1,1)}(a\epsilon) - \left.\frac{\partial g_I}{\partial \alpha_2}\right|_{(1,1)}(\epsilon) = 0$$

$$g_I(1+a\epsilon, 1+\epsilon) = \left.\frac{\partial g_I}{\partial \alpha_1}\right|_{(1,1)}(a\epsilon) + \left.\frac{\partial g_I}{\partial \alpha_2}\right|_{(1,1)}(\epsilon)$$

$$\frac{\partial g_I}{\partial \alpha_1} = -\frac{1+2\alpha_1}{\sqrt{2}\,\alpha_1^{3/2}(\alpha_1+1)^{3/2}} + \frac{\sqrt{\alpha_2}(\alpha_2+2\alpha_1)}{\sqrt{2}\,\alpha_1^{3/2}(\alpha_1+\alpha_2)^{3/2}} + \frac{\sqrt{\alpha_2}}{\sqrt{2\alpha_1}(\alpha_1+\alpha_2)^{3/2}}$$

$$\left.\frac{\partial g_I}{\partial \alpha_1}\right|_{(1,1)} = -\frac{1+2}{2^{1/2}2^{3/2}} + \frac{3}{2^{1/2}2^{3/2}} + \frac{1}{2^{1/2}2^{3/2}} \qquad 2^{1/2}\cdot 2^{3/2} = 2^{4/2} = 2^2 = 4$$

$$= -\frac{3}{4} + \frac{3}{4} + \frac{1}{4} = 1/4$$

$$\frac{\partial g_I}{\partial \alpha_2} = -\frac{\sqrt{\alpha_1}}{\sqrt{2}\sqrt{\alpha_2}(\alpha_2+\alpha_1)^{3/2}} - \frac{\sqrt{\alpha_1}(2\alpha_2+\alpha_1)}{\sqrt{2}\,\alpha_2^{3/2}(\alpha_2+\alpha_1)^{3/2}} - \frac{1+2\alpha_2}{\sqrt{2}\,\alpha_2^{3/2}(\alpha_2+1)^{3/2}} + \frac{1}{\alpha_2^{3/2}}$$

$$\left.\frac{\partial g_I}{\partial \alpha_2}\right|_{(1,1)} = -\frac{1}{2^{1/2}2^{3/2}} - \frac{3}{2^{1/2}2^{3/2}} - \frac{3}{2^{1/2}2^{3/2}} + \frac{1}{1}$$

$$= -1/4 - 3/4 - 3/4 + 1 = -3/4$$

$$g_I(\underbrace{1+a\epsilon}_{\alpha_1}, \underbrace{1+\epsilon}_{\alpha_2}) = 0 = 1/4(a\epsilon) - 3/4\,\epsilon$$

$$1/4(a\epsilon) = 3/4\,\epsilon$$

$$a\epsilon = 3\epsilon$$

$$\Rightarrow \boxed{a = 3}$$

$$\alpha_1 = 1 + 3\epsilon \qquad \alpha_1 = 1 + 3\epsilon$$

$$\alpha_2 = 1 + \epsilon \quad \rightarrow \quad -\;3\alpha_2 = 3 + 3\epsilon$$

$$\alpha_1 - 3\alpha_2 = -2$$

$$3\alpha_2 = \alpha_1 + 2$$

$$\Rightarrow \boxed{\alpha_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}}$$

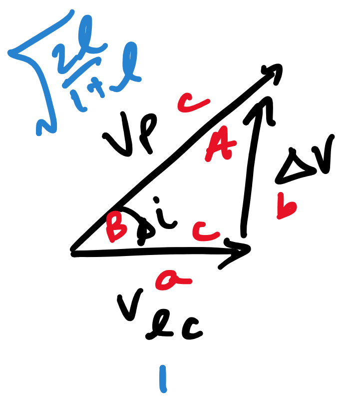
Question 4

Wednesday, February 16, 2022 12:48

CONSIDER A BI-ELLIPTIC PLANE CHANGE MANEUVER. THE DESIRED PLANE CHANGE IS $\Delta i = 90°$, BUT THE LIMIT ON THE ELLIPTIC RADIUS IS $\ell = 10$. COMPARE THE COST OF PERFORMING THE ENTIRE PLANE CHANGE AT APOAPSIS OF THE ELLIPSE WITH DIVIDING THE COST BETWEEN PLANE CHANGES AT THE FIRST & THIRD BURNS, Δi_1 AND Δi_3, RESPECTIVELY, WITH THE REMAINDER OCCURING AT THE SECOND BURN AT APOAPSIS. COMPUTE AND PRESENT A CONTOUR PLOT SHOWING THE VALUES OF THE TOTAL COST OF THE TRANSFER PLOTTED AGAINST Δi_1 AND Δi_3. FIND THE OPTIMAL COMBINATION FOR THIS PROBLEM.

WHEN ALL PLANE CHANGE OCCURS AT APOAPSIS

$$\Delta \tilde{v}_{Bi} = 2\left[\sqrt{\frac{2\ell}{1+\ell}} - 1\right] + 2\sqrt{\frac{2}{\ell(1+\ell)}}\,\sin(\Delta i/2)$$



USE LAW OF COSINES TO SOLVE FOR Δv

$$b^2 = a^2 + c^2 - 2ac\cos B$$

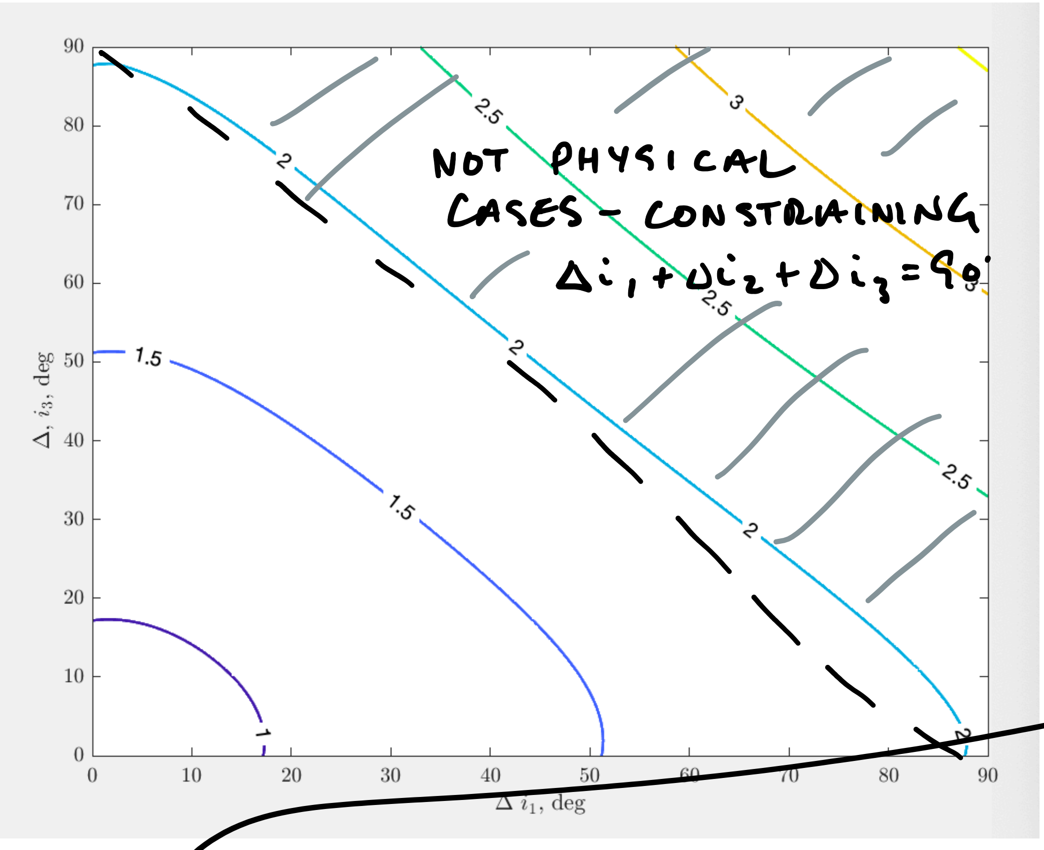
$$\Delta v^2 = 1^2 + \frac{2\ell}{1+\ell} - 2(1)\left(\sqrt{\frac{2\ell}{1+\ell}}\right)\cos(\Delta i)$$

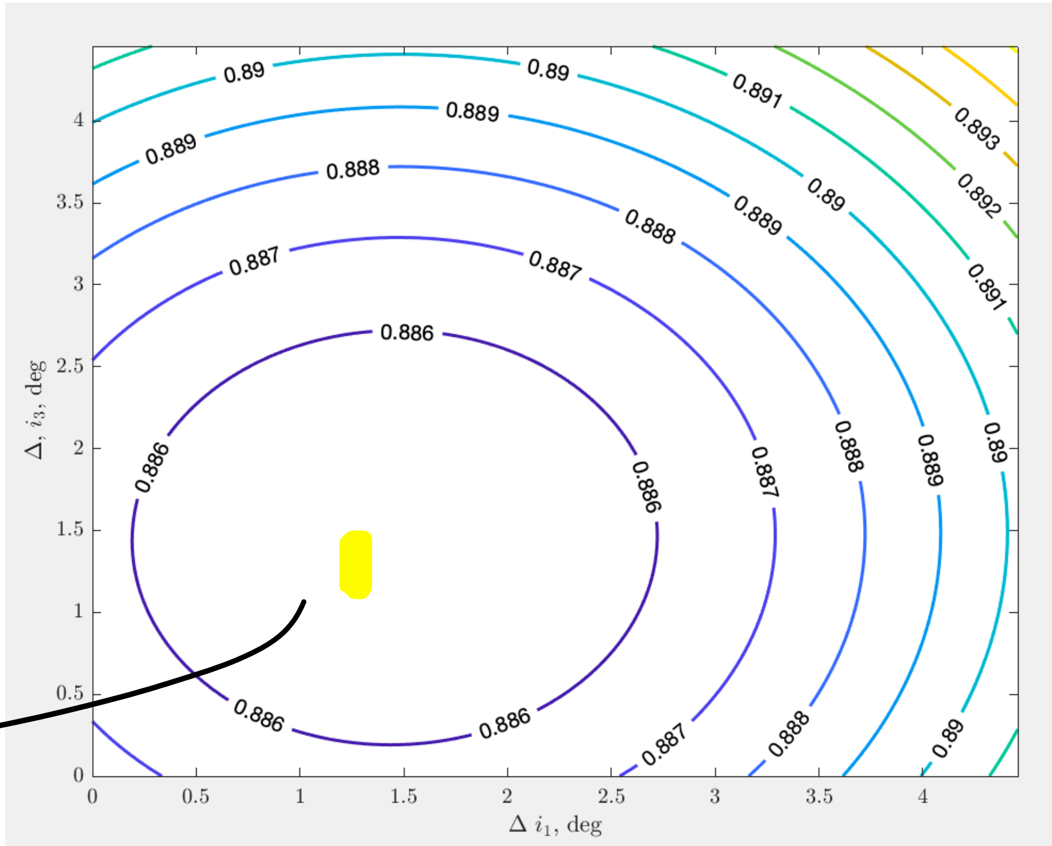
$$\Delta v = \left[1 + \frac{2\ell}{1+\ell} - 2\sqrt{\frac{2\ell}{1+\ell}}\cos(\Delta i)\right]^{1/2}$$

→ THIS IS Δv_1 & Δv_3

COST WHEN SPLIT UP $\qquad \Delta i_1 + \Delta i_2 + \Delta i_3 = 90°$

$$\begin{aligned}\Delta \tilde{v} = &\left[1 + \frac{2\ell}{1+\ell} - 2\sqrt{\frac{2\ell}{1+\ell}}\cos\Delta i_1\right]^{1/2} && \text{maneuver 1}\\ + &\left[1 + \frac{2\ell}{1+\ell} - 2\sqrt{\frac{2\ell}{1+\ell}}\cos\Delta i_3\right]^{1/2} && \text{maneuver 3}\\ + &\, 2\left[\cancel{\sqrt{\frac{2\ell}{1+\ell}} - 1}\right] + 2\sqrt{\frac{2}{\ell(1+\ell)}}\sin(\Delta i_2/2) && \text{remainder @ apoapsis \& recircularizing}\end{aligned}$$





Optimal combination $\Delta i_1 = \Delta i_3 = 1.45° \Rightarrow \Delta i_2 = 87.1°$

which gives a cost $\tilde{J} = 0.885$

Question 5

Wednesday, February 16, 2022 18:02

EXPLORE THE OPTIMALITY OF "DOG-LEG" PLANE CHANGE MANEUVERS IN MORE DETAIL

IMPLEMENT THE OPTIMIZATION PROCESS DESCRIBED IN SECTION II.B OF THE PAPER, "Three Dimensional Transfers" AND SUMMARIZED WITH Eqs. (13-15), USING THE FOLLOWING CONDITIONS WHICH SIMPLIFY THE PROBLEM.

$$v_0 = \sqrt{\frac{\mu}{r_1}}$$

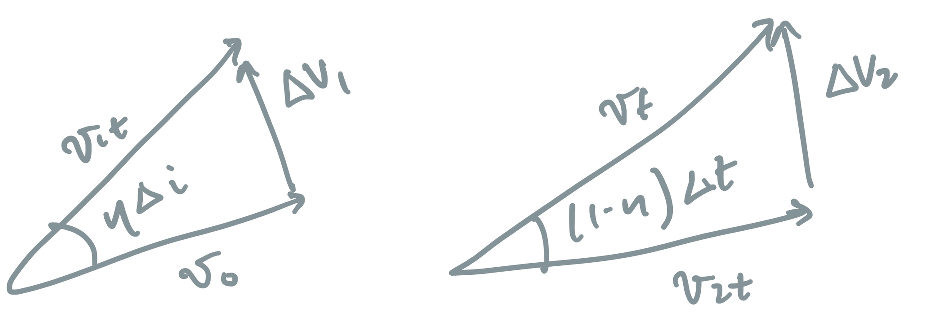
$$v_{1t} = \sqrt{\frac{2\mu}{r_1+r_2}\frac{r_2}{r_1}}$$

$$v_f = v_{2t} = \sqrt{\frac{2\mu}{r_1+r_2}\frac{r_1}{r_2}}$$

NOTE THAT $r_1 \le r_2$

- DERIVE Eqs. (13-15) WITH OUR SIMPLIFICATIONS. DIVIDE THE COST FUNCTION BY INITIAL LOCAL CIRCULAR SPEED, v_0. THIS ELIMINATES THE GRAVITATIONAL PARAMETER μ & REDUCES THE FREE VARIABLES TO $l = r_2/r_1 \ge 1$, η & Δi

$$J(\quad) = \Delta V_{total} = \Delta v_1 + \Delta v_2$$



APPLY LAW OF COSINES TO FIND:

$$= \underbrace{\sqrt{v_{1t}^2 + v_0^2 - 2v_{1t}v_0\cos(\eta\Delta i)}}_{1} + \underbrace{\sqrt{v_f^2 + v_{2t}^2 - 2v_{2t}v_f\cos((1-\eta)\Delta i)}}_{2}$$

JUST LOOK AT 1

$$\left[\left(\sqrt{\frac{2\mu}{r_1+r_2}\frac{r_2}{r_1}}\right)^2 + \left(\sqrt{\frac{\mu}{r_1}}\right)^2 - 2\sqrt{\frac{2\mu}{r_1+r_2}\frac{r_2}{r_1}}\sqrt{\frac{\mu}{r_1}}\cos(\eta\Delta i)\right]^{1/2} \qquad \frac{2\mu^2 r_2}{(r_1+r_2)r_1^2}$$

$$\frac{2r_2}{r_1+r_2}$$

$$= \sqrt{\frac{2\mu}{r_1+r_2}\frac{r_2}{r_1} + \frac{\mu}{r_1} - 2\sqrt{\frac{2\mu}{r_1+r_2}\frac{r_2}{r_1}}\sqrt{\frac{\mu}{r_1}}\cos(\eta\Delta i)}$$

$$= \sqrt{\frac{\mu}{r_1}\left(\frac{2r_2}{r_1+r_2}\right) + \frac{\mu}{r_1} - 2\sqrt{\frac{2\mu^2 r_2}{(r_1+r_2)r_1^2}}\cos(\eta\Delta i)}$$

$$= \sqrt{\frac{\mu}{r_1}\left(\frac{2r_2}{r_1+r_2}\right) + \frac{\mu}{r_1} - 2\frac{\mu}{r_1}\sqrt{\frac{2r_2}{(r_1+r_2)}}\cos(\eta\Delta i)}$$

$$= \underbrace{\sqrt{\frac{\mu}{r_1}}}_{v_0}\sqrt{\frac{2r_2}{r_1+r_2} + 1 - 2\sqrt{\frac{2r_2}{(r_1+r_2)}}\cos(\eta\Delta i)}$$

→ NORMALIZE BY v_0 $\quad l = r_2/r_1 \quad r_1 = \frac{r_2}{l} \quad r_2 = r_1 l$

$$= \sqrt{\frac{2r_2}{r_1(1+\frac{r_2}{r_1})} + 1 - 2\sqrt{\frac{2r_2}{r_1(\frac{r_2}{r_1}+1)}}\cos(\eta\Delta i)}$$

$$1 \quad = \sqrt{\frac{2l}{1+l} + 1 - 2\sqrt{\frac{2l}{(1+l)}}\cos(\eta\Delta i)}$$

JUST LOOK AT TERM 2

$$\sqrt{v_f^2 + v_{2t}^2 - 2v_{2t}v_f\cos((1-\eta)\Delta i)}$$

$$= \sqrt{2\left[\frac{2\mu}{r_1+r_2}\frac{r_1}{r_2}\right] - 2\left[\frac{2\mu}{r_1+r_2}\frac{r_1}{r_2}\right]\cos((1-\eta)\Delta i)}$$

$$\sqrt{2\left(\frac{\mu}{r_1}\right)\left[\frac{2r_1^2}{(r_1+r_2)r_2}\right] - 2\left(\frac{\mu}{r_1}\right)\left(\frac{2r_1^2}{(r_1+r_2)(r_2)}\right)\cos((1-\eta)\Delta i)}$$

$$\sqrt{\frac{\mu}{r_1}}\sqrt{\frac{4r_1^2}{(\frac{r_1}{r_2}+1)r_2^2} - \frac{4r_1^2}{(\frac{r_1}{r_2}+1)r_2^2}\cos((1-\eta)\Delta i)}$$

NORMALIZE BY v_0

$$\sqrt{\frac{4}{l(1+l)} - \frac{4}{l(1+l)}\cos((1-\eta)\Delta i)}$$

$$2 \quad = 2\sqrt{\frac{1}{l(1+l)}\left[1 - \cos((1-\eta)\Delta i)\right]}$$

COMBINE TERMS

$$\boxed{J = \sqrt{\frac{2l}{1+l} + 1 - 2\sqrt{\frac{2l}{1+l}}\cos(\eta\Delta i)} + 2\sqrt{\frac{1}{l(1+l)}\left[1 - \cos((1-\eta)\Delta i)\right]}}$$

↳ DERIVED FROM (13) $\qquad = \Delta V_{total}$

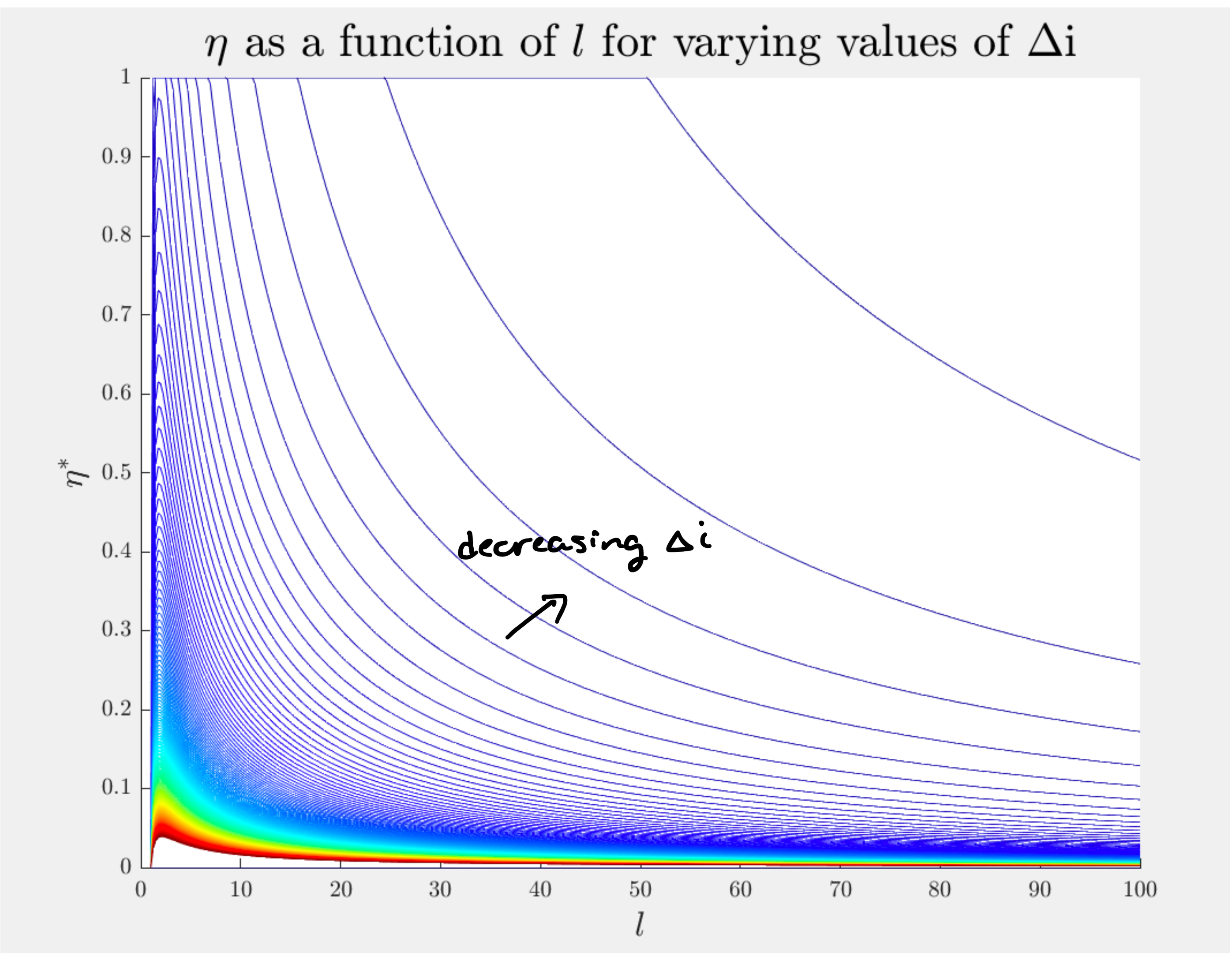
(14) $\quad \Delta i = |i_f - i_0|$

(15) $\quad F = \dfrac{\partial \Delta V_{TOTAL}}{\partial \eta}$

$$= \frac{\partial}{\partial \eta}\left(\sqrt{\frac{2l}{1+l} + 1 - 2\sqrt{\frac{2l}{l(1+l)}}\cos(\eta\Delta i)} + 2\sqrt{\frac{1}{l(1+l)}\left[1 - \cos((1-\eta)\Delta i)\right]}\right)$$

$$\boxed{\begin{aligned} F &= \frac{\Delta i}{2}\left(\frac{2l}{1+l} + 1 - 2\sqrt{\frac{2l}{(1+l)}}\cos(\eta\Delta i)\right)^{-1/2}\left(2\sqrt{\frac{2l}{(1+l)}}\sin(\eta\Delta i)\right) \\ &\quad - \Delta i\left(\frac{1}{l(1+l)}\left[1 - \cos((1-\eta)\Delta i)\right]\right)^{-1/2}\left(\frac{1}{l(1+l)}\sin((1-\eta)\Delta i)\right) \end{aligned}}$$

- WRITE A SCRIPT THAT WILL SOLVE FOR THE OPTIMAL VALUE OF η^* FOR A GIVEN Δi AND l.
- USING THIS PROGRAM, EXPLORE HOW THE OPTIMAL VALUE OF η^* CHANGES AS A FUNCTION OF Δi & l. FOR EXAMPLE, FIX A VALUE OF Δi AND SOLVE FOR η^* FOR A RANGE OF l VALUES FROM $l=1$ to $l=100$. THEN REPEAT FOR A DIFFERENT VALUE OF Δi. PLOT EACH OF THESE CURVES ON A SINGLE CHART OF η^* vs l



Question 6

Monday, February 24, 2025 16:15

ASSUME A SPACECRAFT ORBIT ABOUT A PLANET WITH GRAVITATIONAL PARAMETER μ. THE ORBIT HAS A SEMI-MAJOR AXIS a, AN ECCENTRICITY e, AND AN ARGUMENT OF PERIAPSIS $\omega = 0$. WE WANT TO ROTATE THE ARGUMENT OF PERIAPSIS BY 180°. THERE ARE MANY WAYS TO DO THIS, WE WANT TO COMPARE 3: A ONE-IMPULSE ROTATION AND TWO TWO-IMPULSE ROTATIONS. ANSWER THE FOLLOWING QUESTIONS REGARDING THESE. THE FOLLOWING RESULTS FROM ORBIT THEORY MAY BE USEFUL:

$$r(\nu) = \frac{p}{1 + e\cos\nu}$$

$$V(\nu) = \sqrt{\frac{\mu}{p}}\sqrt{1 + 2e\cos\nu + e^2}$$

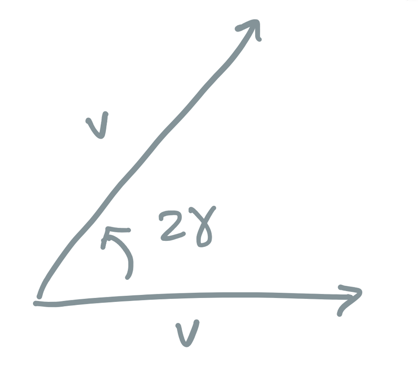
$$\tan\gamma = \frac{e\sin\nu}{1 + e\cos\nu}$$

$$p = a(1 - e^2)$$

r: radius
V: speed
γ: flight path angle
ν: true anomaly
p: orbit parameter

(a) ONE-IMPULSE: THIS MANEUVER IS PERFORMED WHEN $\nu = \pm 90°$ AND ROTATES THE VELOCITY VECTOR THROUGH AN ANGLE 2γ. SHOW THAT THE COST OF THIS MANEUVER EQUALS:

$$\Delta V_a = 2\sqrt{\frac{\mu}{p}}\, e$$



$$\Delta V_a = \sqrt{2V^2 - 2V^2\cos(2\gamma)}$$
$$= \sqrt{2}\, V \sqrt{(1 - \cos 2\gamma)}$$

$$\tan(\gamma) = \frac{e\sin(\pm 90°)}{1 + e\cos(\pm 90°)} = \pm e$$

$$V = \sqrt{\frac{\mu}{p}}\sqrt{1 + 2e\cos\nu + e^2}$$
$$\sqrt{\frac{\mu}{p}}\sqrt{1 + 2e\cos(\pm 90°) + e^2}$$
$$V = \sqrt{\frac{\mu}{p}}\sqrt{1+e^2}$$

$$\tan(2\gamma) = \frac{2\tan\gamma}{1 - \tan^2\gamma} = \frac{\pm 2e}{1 - e^2}$$

$$\tan^2(2\gamma) + 1 = \frac{1}{\cos^2(2\gamma)}$$

$$\frac{4e^2}{(1-e^2)^2} + \frac{(1-e^2)^2}{(1-e^2)^2} = \frac{1}{\cos^2(2\gamma)}$$

$$\cos^2(2\gamma) = \frac{(1-e^2)^2}{4e^2 + (1-e^2)^2}$$

$$= \frac{(1-e^2)}{\sqrt{\underbrace{4e^2 + 1 + e^4 - 2e^2}_{(1+e^2)^2}}} = \frac{1-e^2}{1+e^2}$$

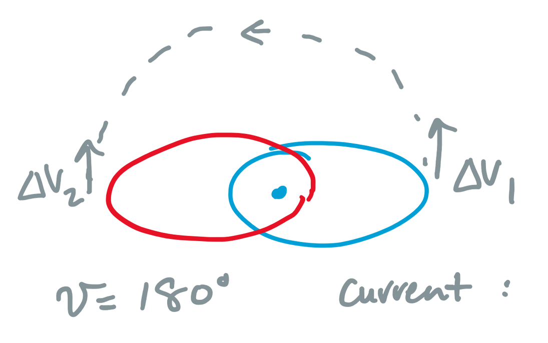
$$\cos 2\gamma = \frac{1-e^2}{1+e^2}$$

$$\Delta V_a = \sqrt{2}\sqrt{\frac{\mu}{p}}\sqrt{1+e^2}\sqrt{1 - \frac{1-e^2}{1+e^2}}$$
$$= \sqrt{2}\sqrt{\frac{\mu}{p}}\sqrt{1+e^2}\sqrt{\frac{1+e^2-1+e^2}{1+e^2}}$$
$$= \sqrt{2}\sqrt{\frac{\mu}{p}}\cancel{\sqrt{1+e^2}}\sqrt{\frac{2e^2}{\cancel{1+e^2}}}$$

$$\boxed{\Delta V_a = 2\sqrt{\frac{\mu}{p}}\, e}$$

(b) TWO-IMPULSE APOAPSIS: THE FIRST MANEUVER IS PERFORMED AT APOAPSIS AND PLACES THE S/C ON A CIRCULAR ORBIT. AFTER A HALF-ORBIT THE S/C DROPS PERIAPSIS BACK TO THE ORIGINAL VALUE. SHOW THAT THE COST OF THIS MANEUVER EQUALS:

$$\Delta V_b = 2\sqrt{\frac{\mu}{p}}\left[\sqrt{1-e} - (1-e)\right]$$



$\nu = 180°$

$$\Delta V = \Delta V_1 + \Delta V_2 = 2(\Delta V_1) \quad \text{symmetric}$$

current: $V_{apo} = \sqrt{\frac{\mu}{p}}\sqrt{1 + e^2 - 2e} = \sqrt{\frac{\mu}{p}}(1-e)$

desired: $V_{circ} = \sqrt{\frac{\mu}{r_{apo}}}$ $\qquad r_{apo} = \frac{p}{1-e}$

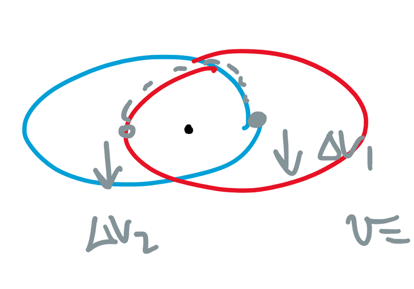
$$= \sqrt{\frac{\mu(1-e)}{p}} = \sqrt{\frac{\mu}{p}}\sqrt{1-e}$$

$$\Delta V_1 = \text{desired} - \text{current} = \sqrt{\frac{\mu}{p}}\left(\sqrt{1-e} - (1-e)\right)$$

$$\boxed{\Delta V_b = 2\Delta V_1 = 2\sqrt{\frac{\mu}{p}}\left[\sqrt{1-e} - (1-e)\right]}$$

(c) TWO-IMPULSE PERIAPSIS: THE FIRST MANEUVER IS PERFORMED AT PERIAPSIS AND PLACES THE S/C ON A CIRCULAR ORBIT. AFTER A HALF-ORBIT THE S/C BOOST THE APOAPSIS BACK TO THE ORIGINAL VALUE. SHOW THAT THE COST OF THE MANEUVER EQUALS:

$$\Delta V_c = 2\sqrt{\frac{\mu}{p}}\left[(1+e) - \sqrt{1+e}\right]$$



$\nu = 0°$

$$\Delta V = \Delta V_1 + \Delta V_2 = 2\Delta V_1$$

current: $V_{periapsis} = \sqrt{\frac{\mu}{p}}\sqrt{\underbrace{1 + 2e + e^2}_{(e+1)^2}}$

$$= \sqrt{\frac{\mu}{p}}(1+e)$$

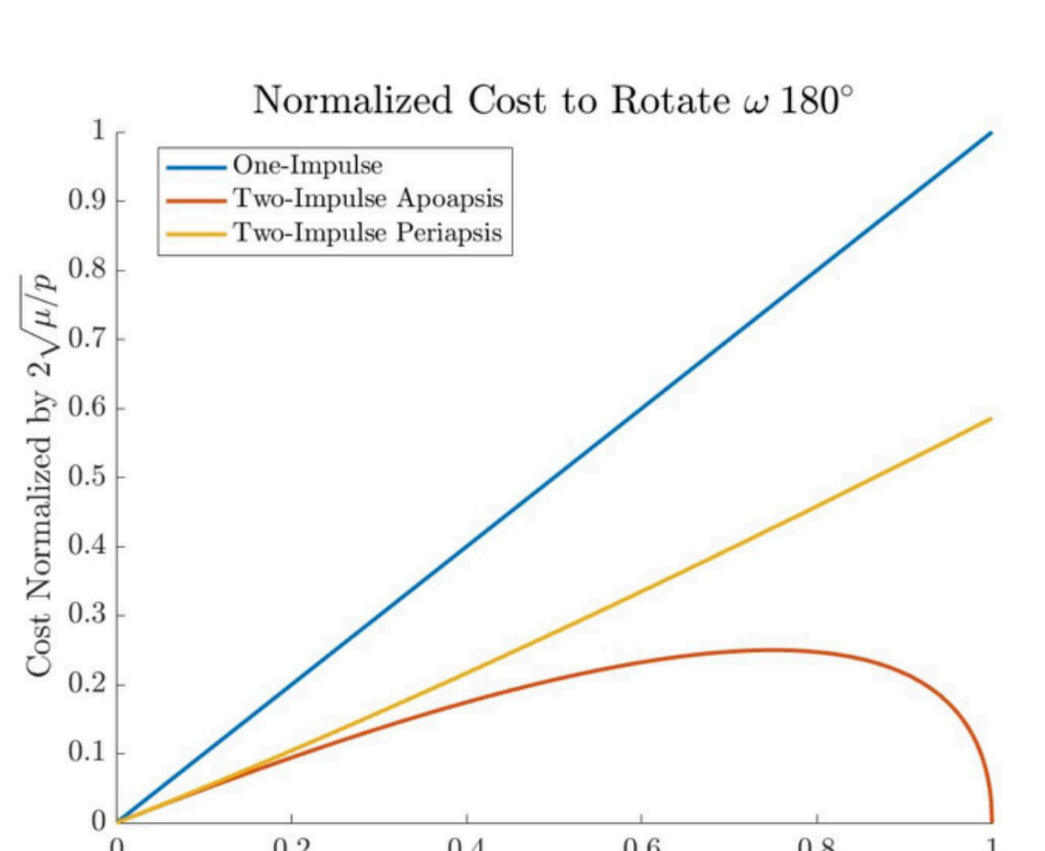
desired: $V_{circ} = \sqrt{\frac{\mu}{r_{per}}}$ $\qquad r_{per} = \frac{p}{1+e}$

$$\sqrt{\frac{\mu}{p}}\sqrt{1+e}$$

$$\Delta V_1 = \text{desired} - \text{current} = V_p - V_{circ}$$
$$= \sqrt{\frac{\mu}{p}}\left[(1+e) - \sqrt{1+e}\right]$$

$$\boxed{\Delta V_c = 2\Delta V_1 = 2\sqrt{\frac{\mu}{p}}\left[(1+e) - \sqrt{1+e}\right]}$$

(d) COMPARE THESE MANEUVERS AND DETERMINE WHICH ONE IS OPTIMAL FOR ALL VALUES OF ECCENTRICITY, $0 \le e \le 1$



The cost analysis shows that the two-impulse apoapsis maneuver is optimal for all values of eccentricity between 0 and 1. At zero eccentricity, the argument of periapsis is not well-defined, since the orbit is circular, and the cost of all maneuvers is zero.

Question 7 (2025)

Wednesday, February 23, 2022 15:04

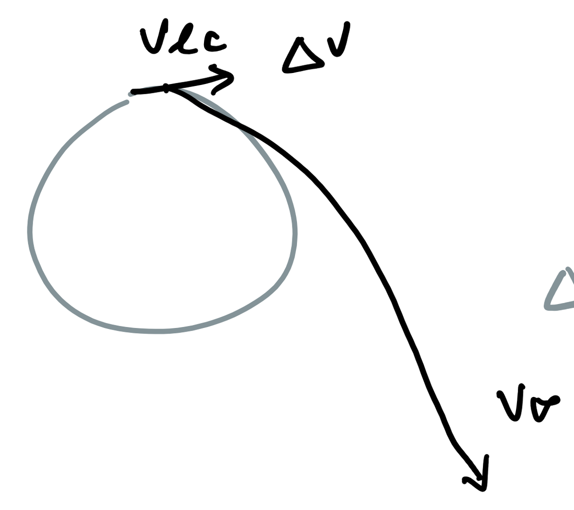
PERFORM A COMPLETE OPTIMAL ANALYSIS OF 1 & 2-IMPULSE ESCAPE MANEUVERS.

CONSIDER A S/C IN AN INITIALLY CIRCULAR ORBIT AT A RADIUS r ABOUT A CENTRAL PLANET W/ MASS PARAMETER μ. THE S/C WILL BE PLACED ON AN ESCAPE TRAJECTORY W/ A SPECIFIED ESCAPE ENERGY PARAMETRIZED AS

$$2E = V_\infty^2$$

WHERE V_∞ IS THE HYPERBOLIC ESCAPE SPEED OF THE S/C.

DERIVE THE COST FUNCTION FOR A 1-IMPULSE ESCAPE



$$\Delta V = V - V_{lc}$$

$$V_e = \sqrt{\frac{2\mu}{r}}$$

$$V^2 = V_\infty^2 + V_e^2$$

$$V = \sqrt{V_\infty^2 + \frac{2\mu}{r}}$$

$$\Delta V = \sqrt{V_\infty^2 + \underbrace{V_e^2}_{\frac{2\mu}{r}}} - \underbrace{V_c}_{\sqrt{\frac{\mu}{r}}}$$

$$\boxed{J_{1\text{-}Imp} = \Delta V = \sqrt{V_\infty^2 + \frac{2\mu}{r}} - \sqrt{\frac{\mu}{r}}}$$

NORMALIZE BY $\sqrt{\frac{\mu}{r}}$

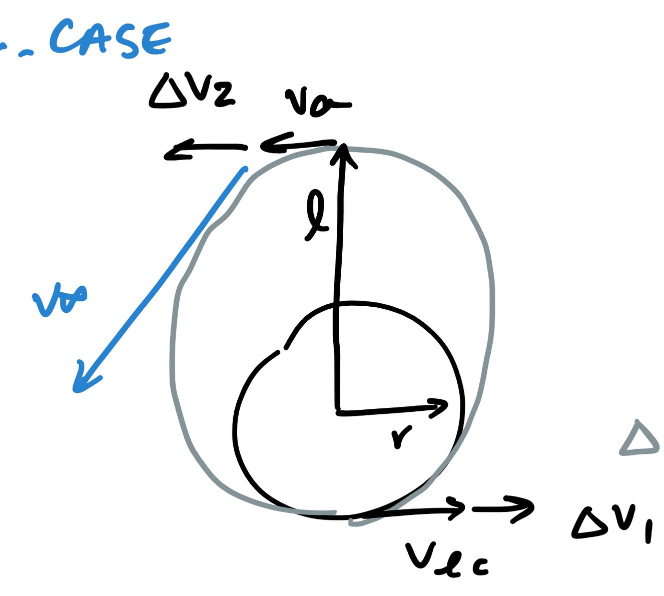
$$= \sqrt{\frac{\mu}{r}\left(\frac{V_\infty^2}{V_{lc}^2} + 2\right)} - \sqrt{\frac{\mu}{r}}$$

$$= \sqrt{\frac{V_\infty^2}{V_{lc}^2} + 2} - 1$$

$$J_{1\text{-}imp} = \sqrt{\frac{V_\infty^2}{V_{lc}^2} + 2} - 1$$

NOW CONSIDER A 2-IMPULSE ESCAPE WHERE THE S/C IS FIRST PLACED ON AN ELLIPTIC ORBIT AND THEN PERFORMS ITS ESCAPE BURN AT THE FIRST APSE PASSAGE. DERIVE THE COST FUNCTION FOR THE CASES WHERE THE ORBIT IS INCREASED (OCCURS @ APOAPSIS) AND FOR ORBIT IS DECREASED (OCCURS @ PERIAPSIS)

INC. CASE



$$\Delta V_1 = \underbrace{\sqrt{\mu\left(\frac{2}{r} - \frac{2}{(r+l)}\right)}}_{=\sqrt{\frac{2\mu l}{r(r+l)}}} - \sqrt{\frac{\mu}{r}}$$

$$\Delta V_2 = \sqrt{V_\infty^2 + \underbrace{V_e^2}_{\frac{2\mu}{r}}} - \underbrace{\sqrt{\mu\left(\frac{2}{l} - \frac{2}{r+l}\right)}}_{=\sqrt{\frac{2\mu r}{l(l+r)}}}$$

$$\Delta V_2 = \sqrt{V_\infty^2 + \frac{2\mu}{r}} - \sqrt{\mu\left(\frac{2}{l} - \frac{2}{r+l}\right)}$$

$(l > r)$

$$\boxed{J = \Delta V_{TOTAL} = \underbrace{\sqrt{\frac{2\mu l}{r(r+l)}} - \sqrt{\frac{\mu}{r}}}_{\Delta V_1} + \underbrace{\sqrt{V_\infty^2 + \frac{2\mu}{l}} - \sqrt{\frac{2\mu r}{l(l+r)}}}_{\Delta V_2}}$$

NORMALIZE BY $V_{lc} = \sqrt{\frac{\mu}{r}}$

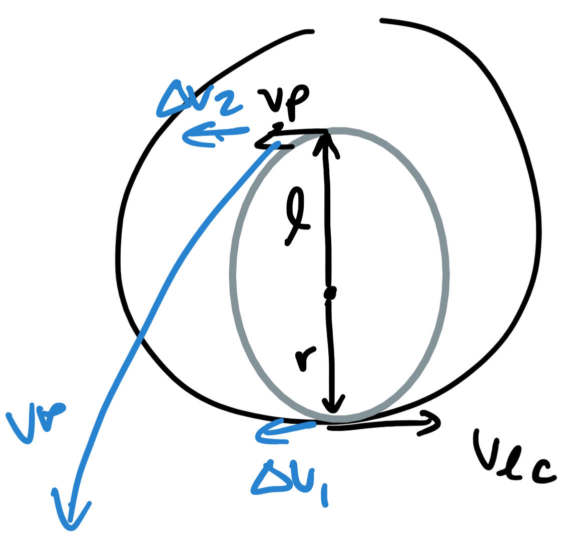
$$\sqrt{\left(\frac{2l}{r+l}\right)} - 1 + \sqrt{\frac{V_\infty^2 r}{\mu} + \frac{2r}{l}} - \sqrt{\frac{2r^2}{l(l+r)}}$$

$\pi = \frac{l}{r}$

$$\sqrt{\frac{2\pi}{1+\pi}} - 1 + \sqrt{\frac{V_\infty^2}{V_{lc}^2} + \frac{2}{\pi}} - \sqrt{\frac{2}{\pi(1+\pi)}}$$

DEC. CASE

$V_a < V_{lc} \rightarrow$ negative



$$\Delta V_1 = \sqrt{\mu\left(\frac{2}{r} - \frac{2}{l+r}\right)} - \sqrt{\frac{\mu}{r}}$$

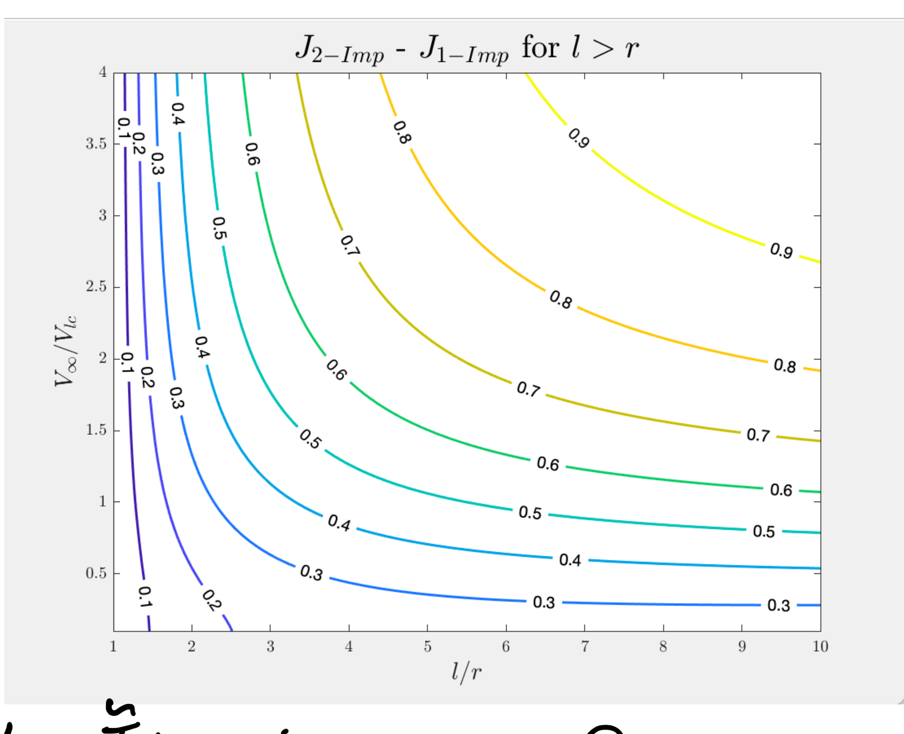
$$\Delta V_2 = \sqrt{V_\infty^2 + \frac{2\mu}{r}} - \sqrt{\mu\left(\frac{2}{l} - \frac{2}{l+r}\right)}$$

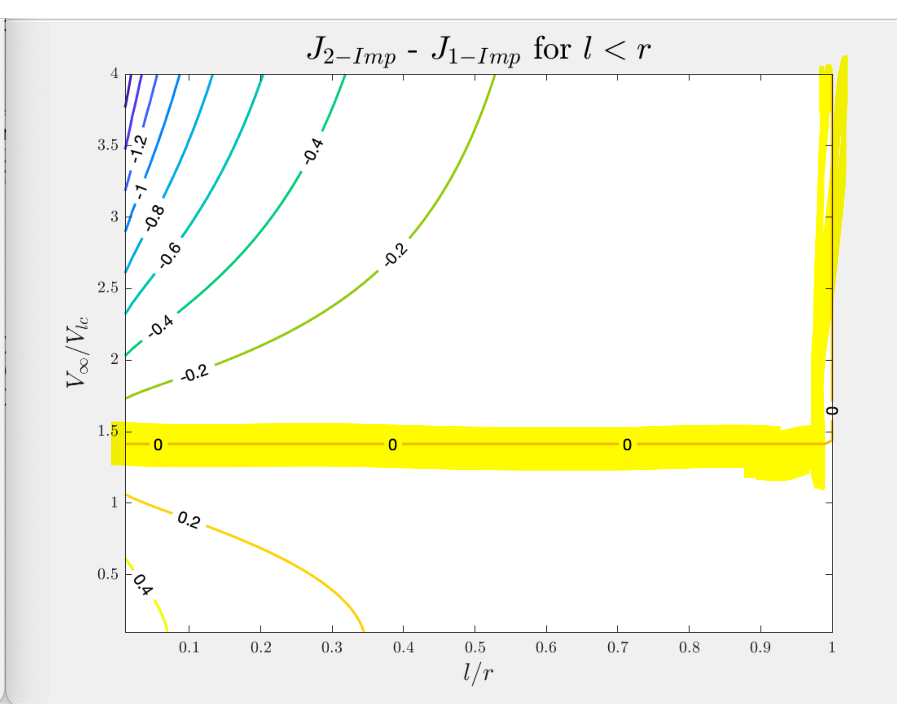
$$J = \Delta V_{TOTAL} = \overset{V_{lc} - V_a}{|\Delta V_1|} + \Delta V_2 \qquad (l < r)$$

$$\boxed{J = \sqrt{\frac{2\mu l}{r(r+l)}} - \sqrt{\frac{\mu}{r}} + \sqrt{V_\infty^2 + \frac{2\mu}{l}} - \sqrt{\frac{2\mu r}{l(l+r)}}}$$

BOTH 2-IMPULSE CASES CAN BE WRITTEN AS

$$\boxed{J = \Delta V_{TOTAL} = \underbrace{\left|\sqrt{\frac{2\mu l}{r(r+l)}} - \sqrt{\frac{\mu}{r}}\right|}_{|\Delta V_1|} + \underbrace{\sqrt{V_\infty^2 + \frac{2\mu}{l}} - \sqrt{\frac{2\mu r}{l(l+r)}}}_{\Delta V_2}}$$





↳ $\tilde{J} > 0 \Rightarrow$ change @ apoapsis is never optimal

Change in sign of cost function at $\sqrt{2}\frac{V_\infty}{V_{lc}} \rightarrow$ change @ periapsis is optimal for $\frac{V_\infty}{V_{lc}} > \sqrt{2}$

Optimality Chart