ASEN 5044, Fall 2024

Statistical Estimation for Dynamical Systems

Lecture 18:
More on Stochastic Processes; White Noise Processes; Stochastic Linear CT and DT Models

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Tuesday 10/15/2024

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Announcements

- **Homework 5: due this Fri 10/18 on Gradescope**

- **HW 6 out this Thurs 10/17 (will be due in 2 weeks)**
- **Quiz 6 out this Fri 10/18, due Tues 10/22**

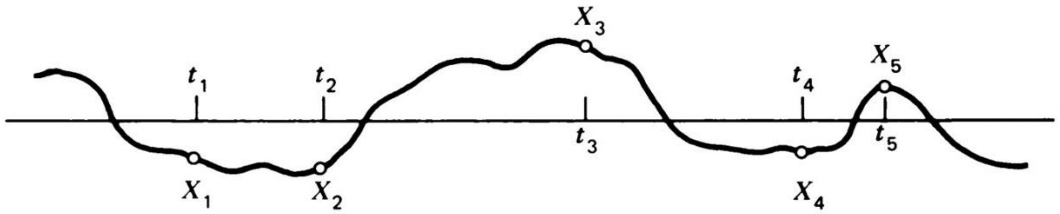
- **Midterm 1 being graded…**

Overview

<u>Last Time:</u>

- Functional transformations of random variables
- Derivation/proof that linear transformations y=f(x) of Gaussian random vectors x → Gaussian random vectors y

- Intro to stochastic processes
  - o Mean and autocorrelation functions



$$\bar{X}(t_i) = \bar{X}_i = E[X(t_i)] = \int_{-\infty}^{\infty} X(t_i) p(X(t_i)) dX(t_i)$$

$$R_X(t_i, t_j) = E[X(t_i)X(t_j)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t_i)X(t_j)p(X(t_i), X(t_j)) dX(t_i) dX(t_j)$$

$$X(t_1) \triangleq X_1$$

$$X(t_2) \triangleq X_2$$

$$\vdots$$

$$X(t_n) \triangleq X_n$$

Today…

More on stochastic processes

- Power spectral density (PSD) and white noise intensity
- Additive white noise: continuous/discrete time (CT/DT) versions
- DT Stochastic Linear State Space Systems

Example: Autocorrelations of Random Signals

- Let's look at ***sample values of*** $R_x(\tau)$ [time shift/lag $\tau = t_i - t_j$] for realizations of:

$$X_0(t) = 10\sin(2\pi t)$$

$$X_1(t) = 10\sin(2\pi t + \theta),\ \theta \sim \mathcal{U}[0, 2\pi]$$

$$X_2(t) = \text{jump to A, every 1.5 secs},\ A \sim \mathcal{U}[-0.5, 0.5]$$

$$X_3(t) = Q + \sin(0.2\pi t + \theta),\ Q \sim \mathcal{U}[-1, 1],\ \theta \sim \mathcal{U}[0, \pi/2]$$

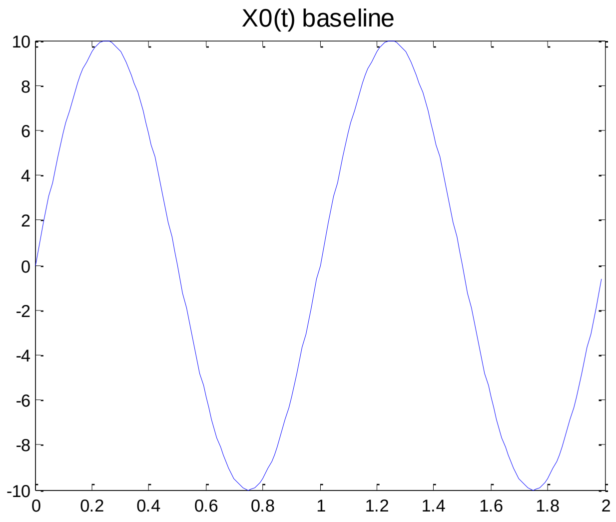
$$X_4(t) = A\sin(\theta),\ A \sim \mathcal{U}[0, 1],\ \theta \sim \mathcal{U}[0, 2\pi]$$

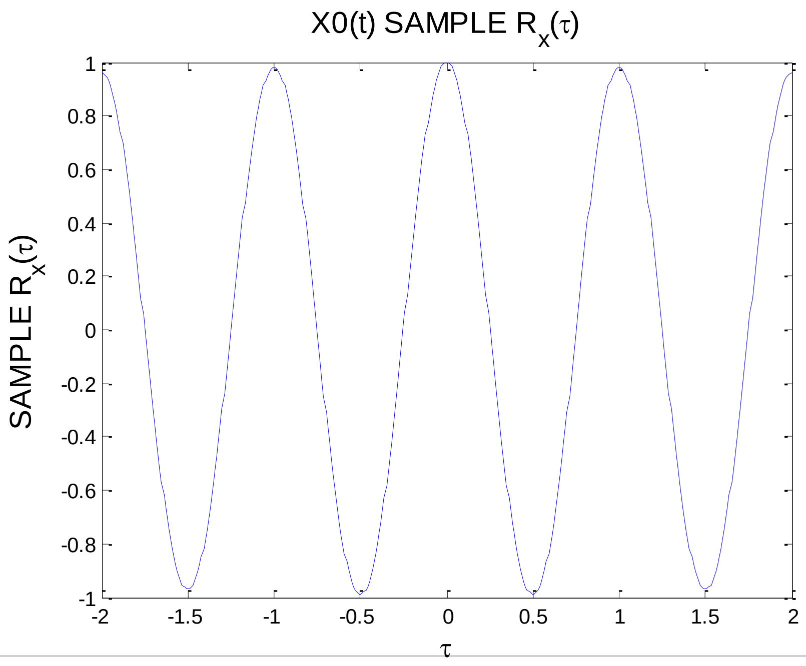
- Use the "**autocorr.m**" on sample realizations in Matlab to approximate $R_x(\tau)$ in CT with DT version (see Matlab doc for details on fxn)

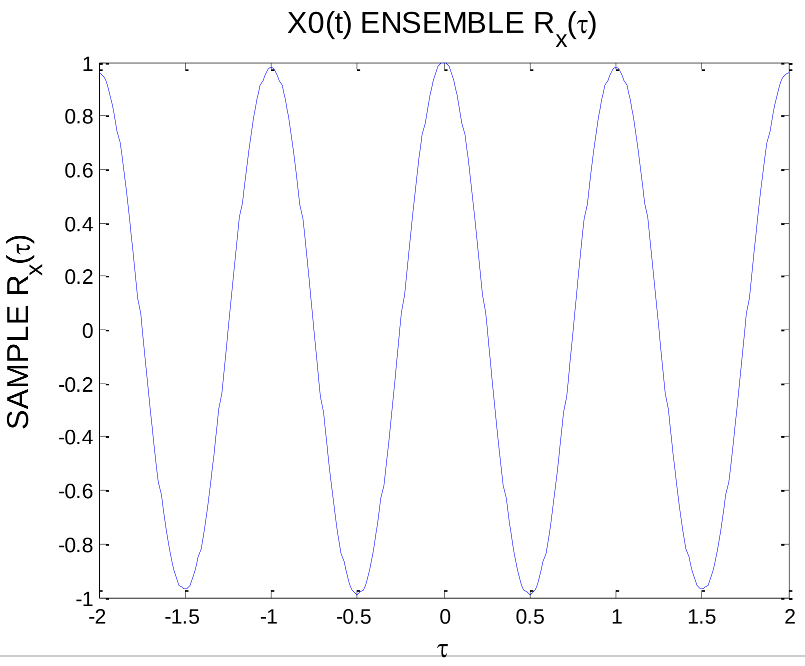
Example: Autocorrelations of Random Signals

- ***sample autocorrelation function*** for realizations of:

$$X_0(t) = 10\sin(2\pi t)$$



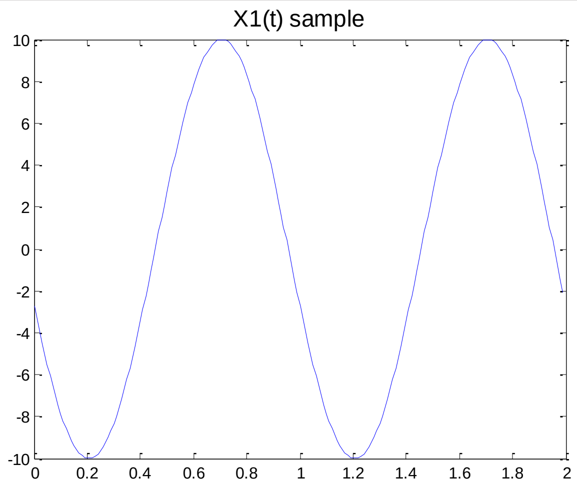


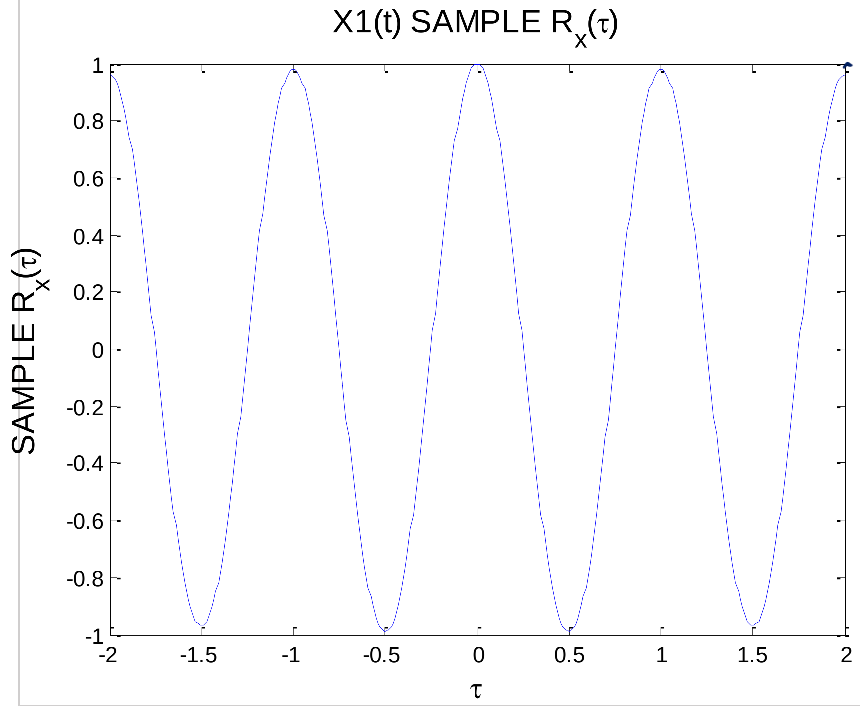


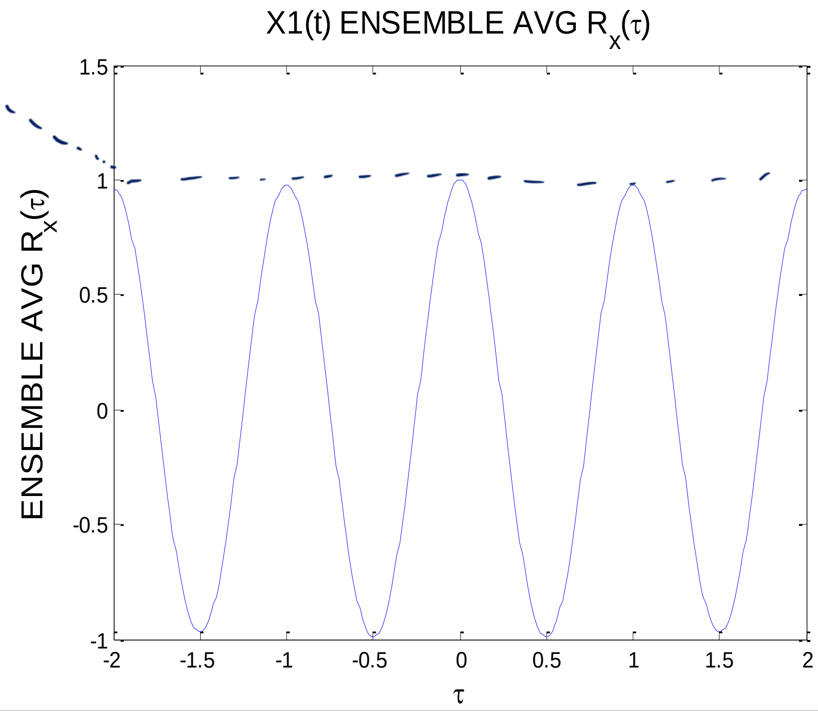
Example: Autocorrelations of Random Signals

- ***sample autocorrelation function*** for realizations of:

$$X_1(t) = 10\sin(2\pi t + \theta),\ \ \theta \sim \mathcal{U}[0, 2\pi]$$



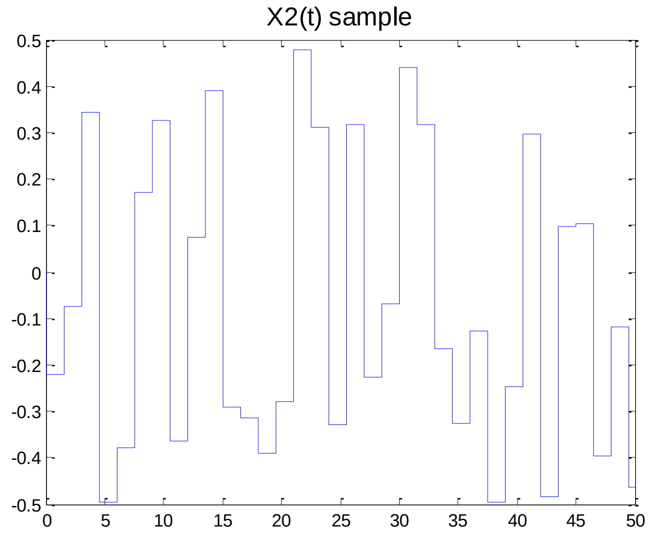


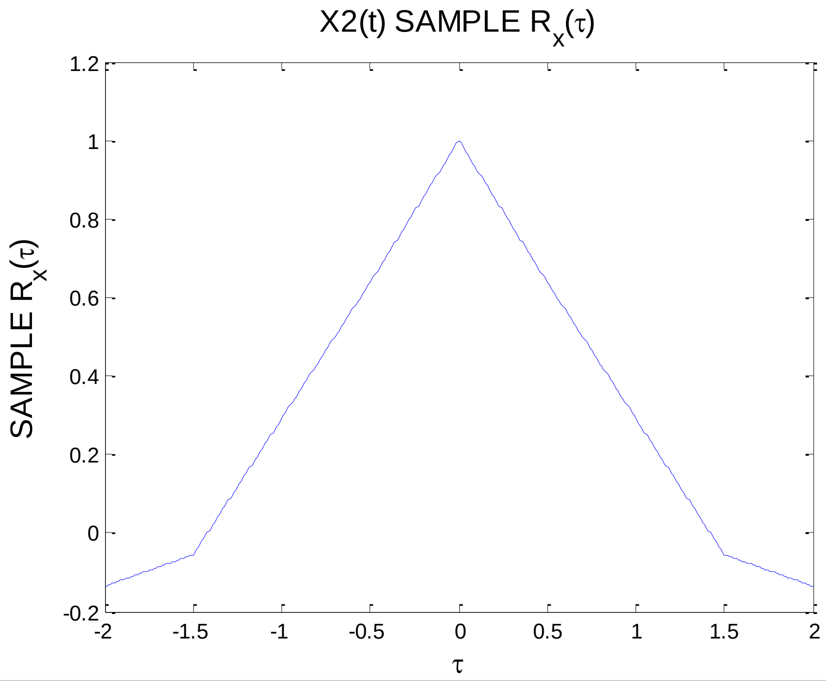


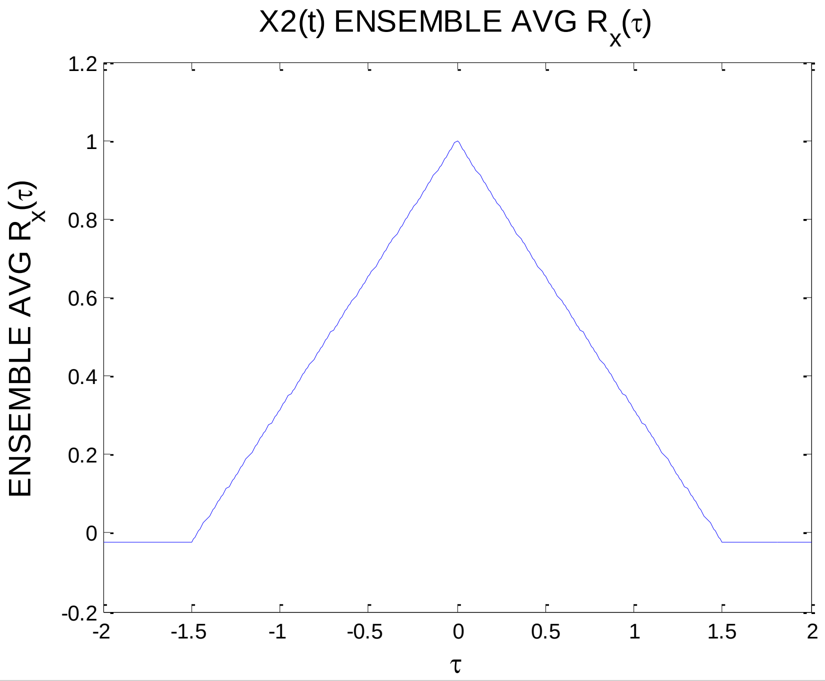
Example: Autocorrelations of Random Signals

- ***sample autocorrelation function*** for realizations of:

$$X_2(t) = \text{jump to A, every 1.5 secs, } A \sim \mathcal{U}[-0.5, 0.5]$$



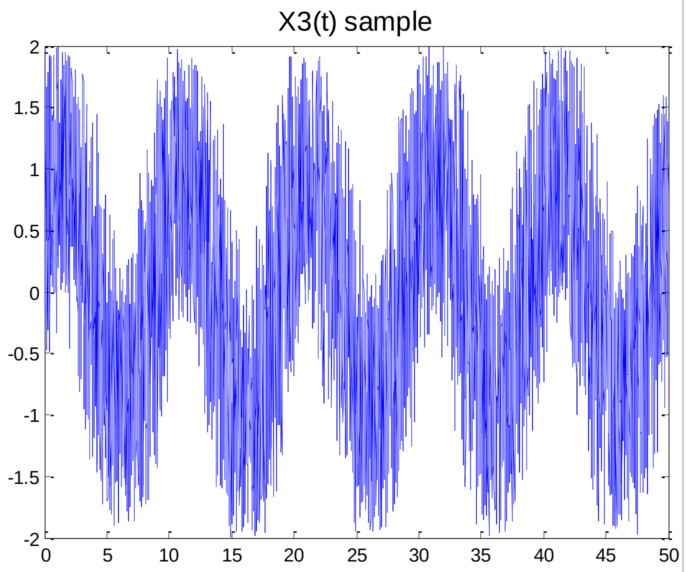


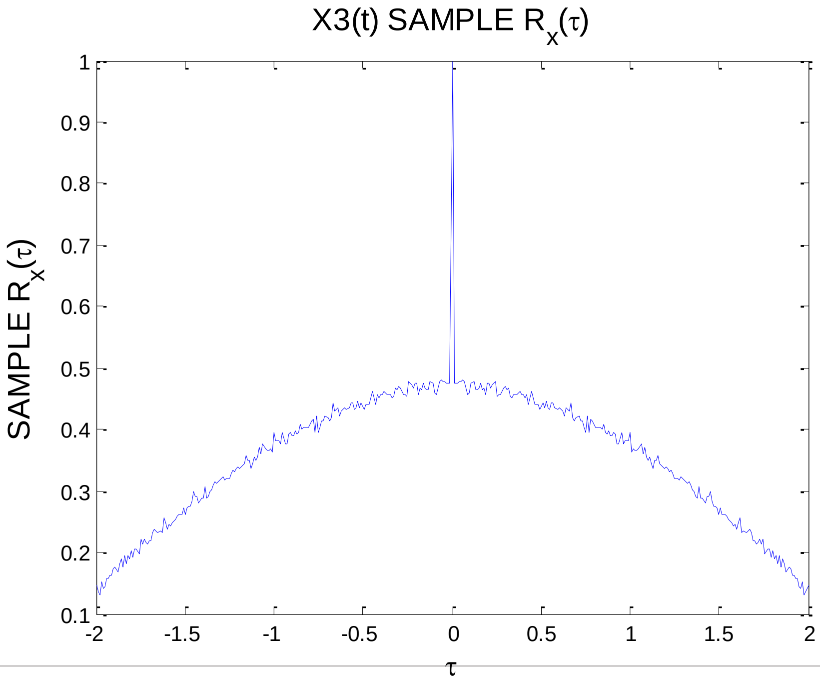


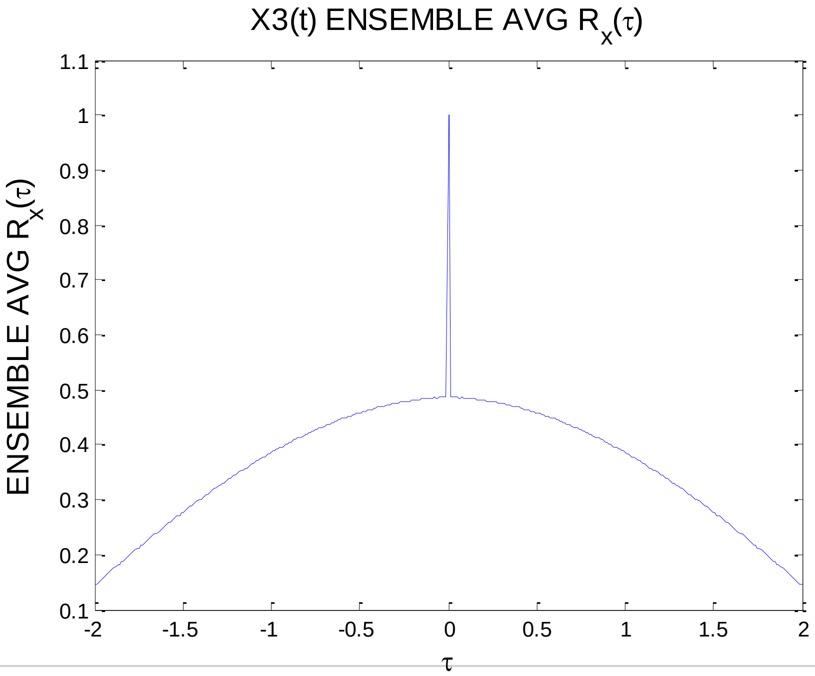
Example: Autocorrelations of Random Signals

- ***sample autocorrelation function*** for realizations of:

$$X_3(t) = Q + \sin(0.2\pi t + \theta),\ \ Q \sim \mathcal{U}[-1,1],\ \ \theta \sim \mathcal{U}[0,\pi/2]$$



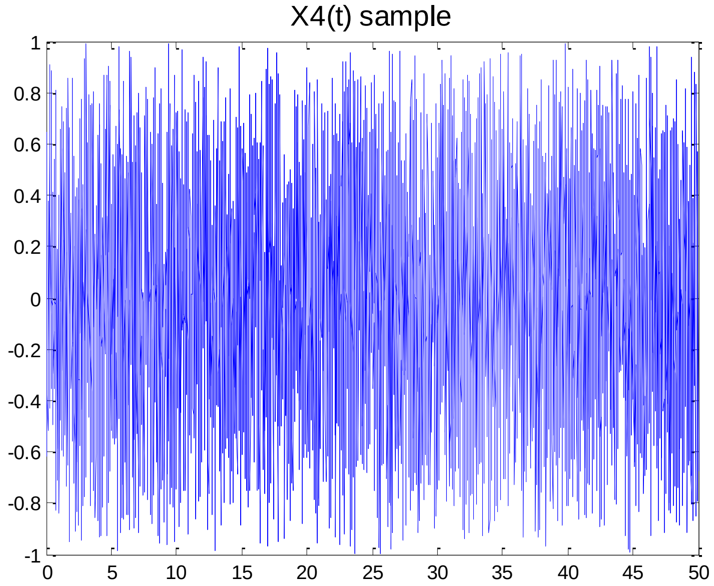


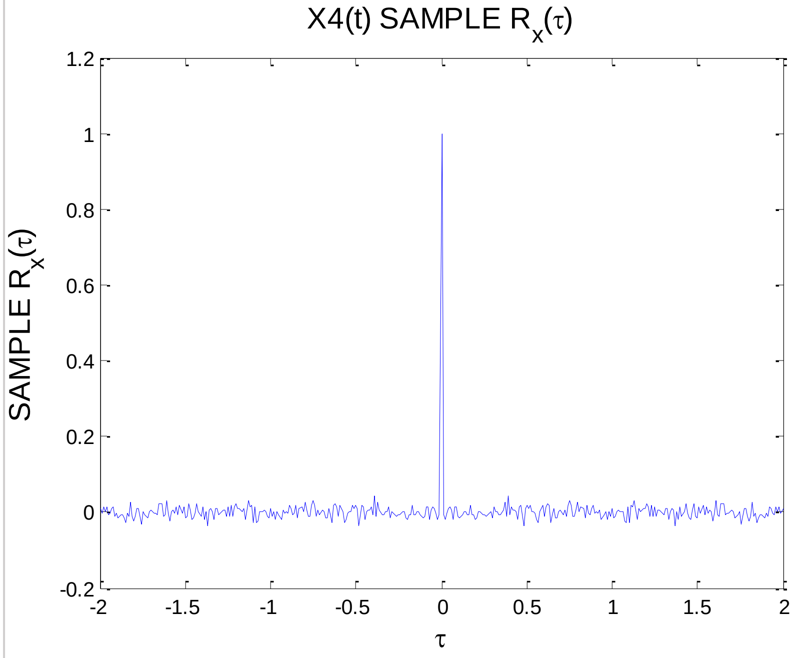


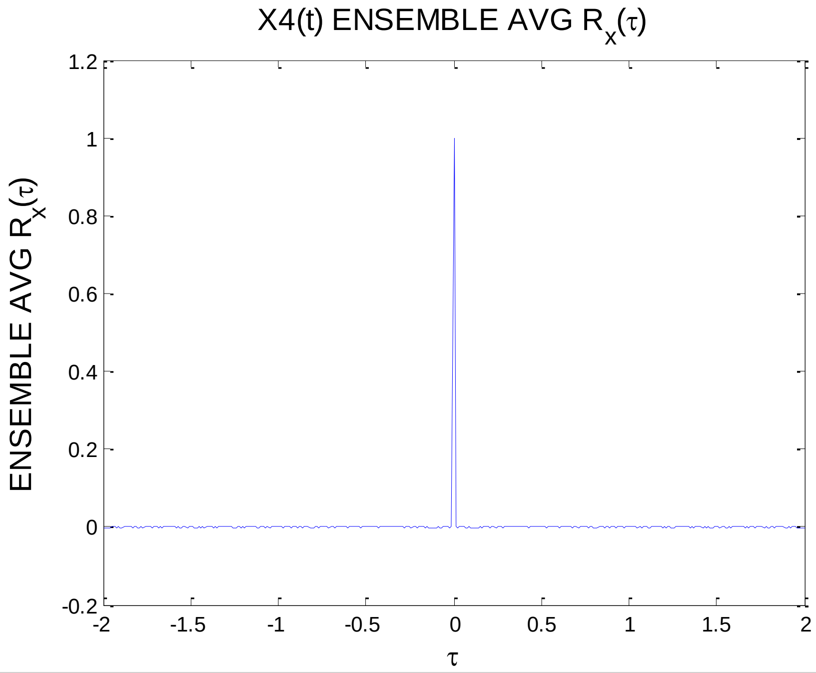
Example: Autocorrelation of Random Signals

- ***sample autocorrelation function*** for realizations of:

$$X_4(t) = A\sin(\theta),\ A \sim \mathcal{U}[0,1],\ \theta \sim \mathcal{U}[0,2\pi]$$







White Noise

- Let’s take the result of last example to the extreme: what if X(t) is COMPLETELY uncorrelated with itself for ALL τ EXCEPT at $\tau = 0$?

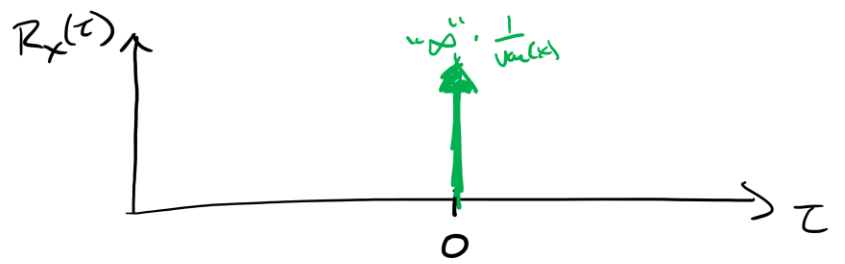
$$R_x(\tau) = \frac{1}{Var(x(t))} \cdot \delta(t) = \begin{cases} \infty, & \text{if } \tau = 0 \\ 0, & \text{if } \tau \neq 0 \end{cases}$$

(normalized)

where Dirac Delta fxn $\delta(t)$ has properties:

(i) $\int_{-\infty}^{\infty} \delta(t)\,dt \triangleq 1$

(ii) "sifting property": $\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = f(a)$

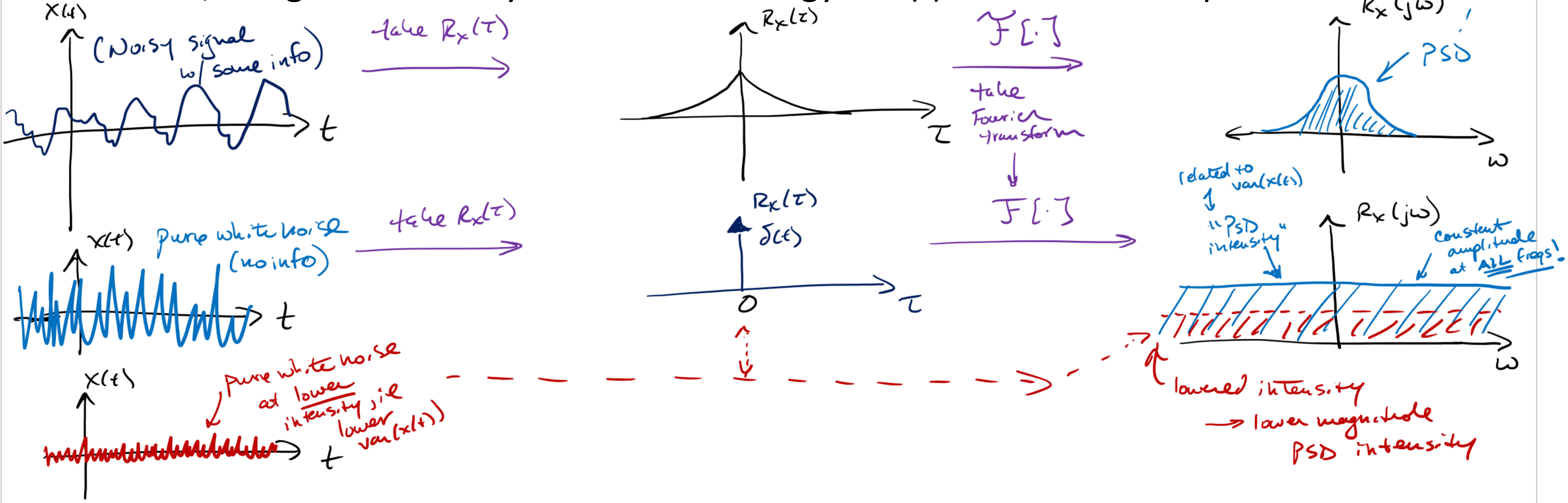


- In other words, such a signal has absolutely no memory of itself from one infinitesimal instant to the next! (i.e. contains no predictable information)

- If $R_x(\tau)$ = Dirac delta, then X(t) is said to be **white noise**

Why “White” Noise?

- The “color of noise” has to do with **frequency content** of the signal
- Look at spectral power via **Fourier transform of $R_x(\tau)$**: **power spectral density (PSD)** *
- Math: Parseval’s theorem: energy in time domain = energy in frequency domain
- Therefore, integral of PSD says how much energy in X(t) at different frequencies



White Noise is Physically Non-Realizable!!

- White noise is pure mathematical fiction
  - Impossible to have a signal that can go infinitely far infinitely fast!
  - All "real world" noise has SOME physical memory/inertia

$$x(k+1) = Fx(k) + Gu(k) + w(k)$$

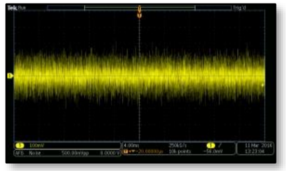
proc. noise

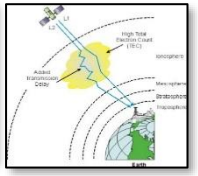
meas. noise

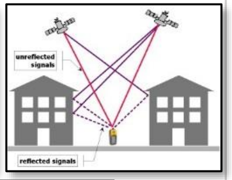
$$y(k+1) = Hx(k+1) + v(k+1)$$

- So why bother with white noise?
  - "worst case" scenario for noise: hit with everything possible!
  - Makes math analysis MUCH simpler
    (band limited noise models are nasty)
  - All "colored" or correlated noise (with finite spectral energy):
    can be modeled as white noise passing through a filter!

white noise → $\boxed{\text{Low Pass Filter "Prewhitening Filter"}}$ → correlated noise







Discrete Time Random Sequences (DT Processes)

Let $\{\cdots, x(k-2), x(k-1), x(k), x(k+1), \cdots\}$ be a set of random vectors ordered by time index k.

Define $X^k = \{\cdots, x(k-1), x(k)\}$ to be a set of all random vectors in the sequence up to time k.

- <u>Markov sequence:</u> if $p(\ x(k) \mid X^j = \{\cdots, x(j-1), x(j)\}\ ) = p(\ x(k) \mid x(j)\ )\ \forall k > j$

  then the sequence is said to be a **Markov sequence** (i.e. has the Markov property)

  $\Rightarrow$ means that all important info about past for purposes of (imperfectly) predicting the future $x(k)$ is completely summarized by $x(j)$

- <u>White noise sequence:</u> random sequence Σ of elements $w(k) \in \mathbb{R}^n$ such that

  $(i)\quad E[w(k)] = 0\ \forall k$

  $(ii)\quad \text{cov}(w(k), w(j)) = E[w(k)w^T(j)] = \delta_{kj} \cdot Q(k)\ \forall k$

  where $\delta_{kj} = \begin{cases} 1, & \text{if } k = j \\ 0, & \text{if } k \neq j \end{cases}$ (Kronecker delta)

  and $Q(k)$ = posdef covariance matrix (**white noise intensity covariance matrix**)

Describing DT Random Sequences

- Can also define mean and autocorrelation functions

$$\text{(mean:)}\ \ \bar{x}(k) = E[x(k)] = \int_{-\infty}^{\infty} x(k)\cdot p(\ x(k)\ )dx(k)$$

$$\text{(autocorr:)}\ \ R(k,j) = E[x(k)x^T(j)] = \int_{-\infty}^{\infty} x(k)x^T(j)\cdot p(\ x(k),\ x(j)\ )\,dx(k)\,dx(j)$$

- <u>**Stationarity:**</u> sequence $X = \{\cdots, x(k-1), x(k), x(k+1), \cdots\}$ is *wide sense stationary (WSS)* if:

$$\text{(i)}\ \ \bar{x}(k) = \text{constant indep. of time}$$

$$\text{(ii)}\ \ R(k,j) = R(k-j,0) = R(k-j)\ \ \text{(function of time shift/difference only)}$$

- <u>**Ergodicity:**</u> sequence X is ergodic if ensemble averages are same as time averages:

$$E[f(x(k))] = \int_{-\infty}^{\infty} f(x(k))\cdot p(\ x(k)\ )dx(k) = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{k=-N}^{N} f(x(k))$$

(where $f(x(k))$ is some arbitrary function of $x(k)$)