ASEN 6060
ADVANCED ASTRODYNAMICS
Introduction to Periodic Orbits

Objectives:

- Introduce periodic orbit families, terminology for significant families in the CR3BP
- Generate initial guesses for Lyapunov orbits emanating from collinear equilibrium points

Periodic Orbits

A periodic orbit is a nonconstant trajectory that repeats in the rotating frame after a minimal period *T*: $\bar{x}_{PO}(t) = \bar{x}_{PO}(t+T)$

Periodic orbits can exist in dynamical systems that are autonomous (e.g., CR3BP) or periodic (e.g., ER3BP, orbit period constrained)

In the autonomous CR3BP:

- a periodic orbit is completely specified by the period *T* and a (nonunique) state, $\bar{x}_{PO}$
- Periodicity is defined in the rotating frame
- Calculate numerically using corrections algorithm
- Periodic orbits exist in continuous families (sets of solutions with a continuously varying set of initial conditions, orbit period, and other parameters)

Significance of Periodic Orbits

Infinite variety of families of periodic orbits

Contribute to the underlying dynamical structure in the CR3BP:

- May produce stable/unstable manifolds that govern natural transport through the system
- May produce nearby bounded motions, i.e., quasi-periodic trajectories
- Give rise to additional families of periodic orbits via bifurcations

→ Motivates study of orbital stability (soon!)

Know trajectory properties along periodic orbit for all time, beyond the integration time interval

Useful Definition

Direction of motion: defined in rotating frame using specific orbital angular momentum vector, calculated using state $\bar{x}(t)$ and measured relative to a reference point $\bar{x}_{ref}$, commonly a primary:

$$\bar{h} = \bar{r} \times \bar{v}$$

where $\bar{r} = (x - x_{ref})\hat{x} + (y - y_{ref})\hat{y} + (z - z_{ref})\hat{z}$

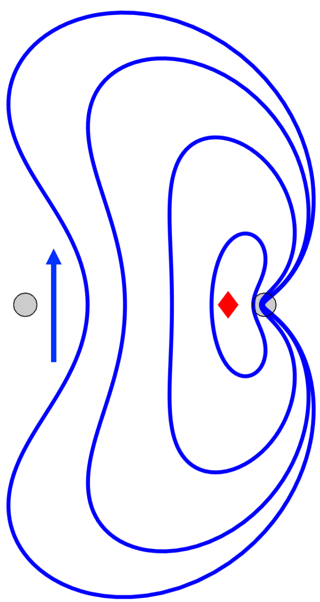
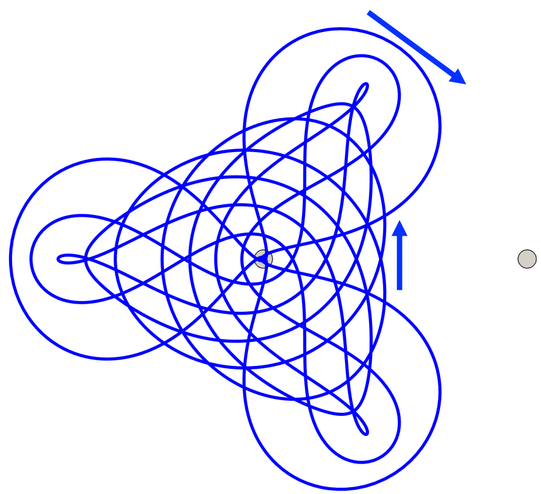
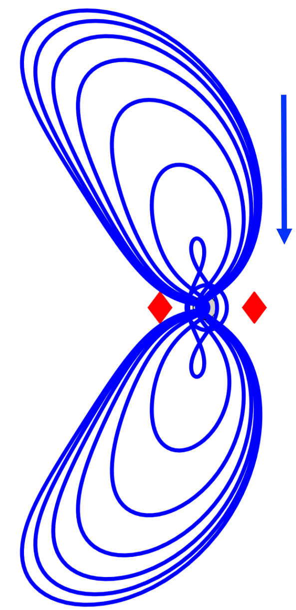
and only applies at a specific instant of time, *t*

- Prograde: $\bar{h} \cdot \hat{z} > 0$
- Retrograde: $\bar{h} \cdot \hat{z} < 0$

Direction of motion may change over time along any trajectory

Periodic Orbit Families in the CR3BP

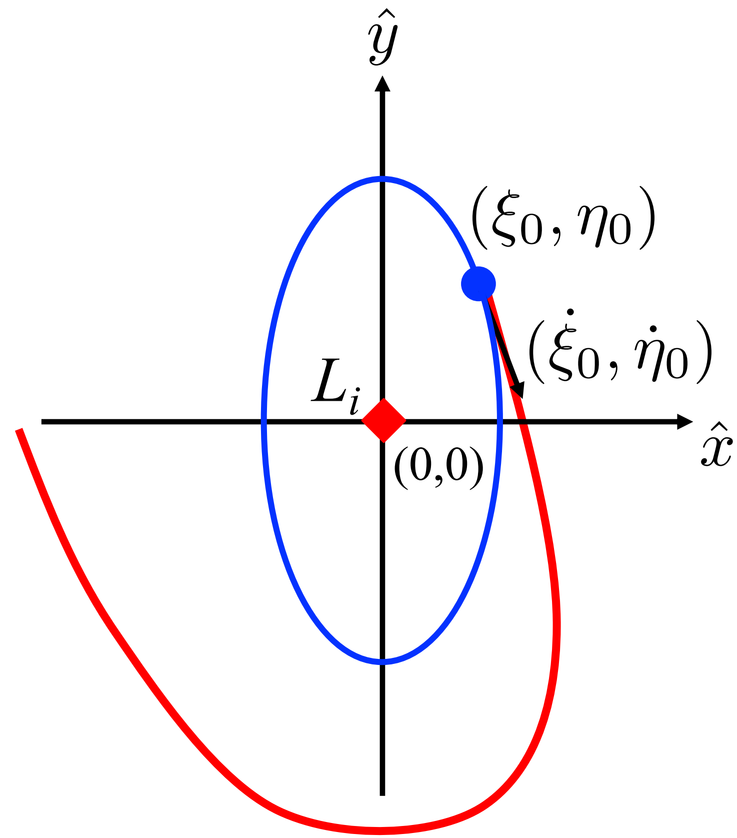
Some prominent periodic orbit families include:

Libration point orbits	Resonant orbits	Primary-centered orbits
		

Initial Guess for Libration Point Orbits

Use linearized dynamics near libration points to generate initial guesses for nearby periodic orbits

- Exploit oscillatory mode in linear model by carefully selecting the initial values of the variations
  - Selecting an initial condition that lies in the center eigenspace of the equilibrium point
- Integrate in nonlinear model to produce nonperiodic trajectory
- Use corrections to produce periodic trajectory in nonlinear model (to be covered in 2 weeks!)



For oscillatory eigenspace of real [A]

$$\bar{v}_{1,2} = \bar{v}_R \pm i\bar{v}_I$$

$$\bar{v}_R, \bar{v}_I$$

Initial Guess for Libration Point Orbits

Let's demonstrate one analytical approach for Lyapunov orbits near collinear L_i

Recall variational equations near L_i for planar motion:

$$\ddot{\xi} - 2\dot{\eta} = U^*_{xx}\Big|_{\bar{x}_{eq}}\xi + U^*_{xy}\Big|_{\bar{x}_{eq}}\eta$$

$$\ddot{\eta} + 2\dot{\xi} = U^*_{yx}\Big|_{\bar{x}_{eq}}\xi + U^*_{yy}\Big|_{\bar{x}_{eq}}\eta$$

Solutions for position components possess the following form:

$$\xi = \sum_{i=1}^{4} A_i e^{\lambda_i t} \qquad \eta = \sum_{i=1}^{4} B_i e^{\lambda_i t}$$

These scalar coefficients, A_i and B_i, are related due to coupling between the two equations.

Note: following Szebehely, 1967, 'Theory of Orbits', Section 5.3

Planar Solutions to Linear System

Plug solutions into variational equations to write A_i in terms of B_i

$$\ddot{\xi} - 2\dot{\eta} = U^*_{xx}\Big|_{\bar{x}_{eq}}\xi + U^*_{xy}\Big|_{\bar{x}_{eq}}\eta$$

$$\ddot{\eta} + 2\dot{\xi} = U^*_{yx}\Big|_{\bar{x}_{eq}}\xi + U^*_{yy}\Big|_{\bar{x}_{eq}}\eta$$

$$\xi = \sum_{i=1}^{4} A_i e^{\lambda_i t} \qquad \eta = \sum_{i=1}^{4} B_i e^{\lambda_i t}$$

At the initial time, $t_0=0$, these solutions are the initial conditions:

$$\xi_0 = \sum_{i=1}^{4} A_i e^{\lambda_i 0} \qquad \eta_0 = \sum_{i=1}^{4} B_i e^{\lambda_i 0}$$

$$\dot{\xi}_0 = \sum_{i=1}^{4} \lambda_i A_i e^{\lambda_i 0} \qquad \dot{\eta}_0 = \sum_{i=1}^{4} \lambda_i B_i e^{\lambda_i 0}$$

Planar Solutions to Linear System

Plug solutions into variational equations to write A_i in terms of B_i

$$\ddot{\xi} - 2\dot{\eta} = U^*_{xx}\big|_{\bar{x}_{eq}}\xi + U^*_{xy}\big|_{\bar{x}_{eq}}\eta$$

$$\ddot{\eta} + 2\dot{\xi} = U^*_{yx}\big|_{\bar{x}_{eq}}\xi + U^*_{yy}\big|_{\bar{x}_{eq}}\eta$$

Note: dropping subscript $\bar{x}_{eq}$, but still evaluating second partial derivatives at equilibrium point of interest

$$\lambda_i^2 A_i - 2\lambda_i B_i = U^*_{xx} A_i + \cancel{U^*_{xy} B_i}$$

$$\lambda_i^2 B_i + 2\lambda_i A_i = \cancel{U^*_{yx} A_i} + U^*_{yy} B_i$$

Note: $U^*_{xy} = U^*_{yx} = 0$ for a collinear equilibrium point

One expression relating A_i to B_i:

$$B_i = \frac{(\lambda_i^2 - U^*_{xx})}{2\lambda_i} A_i = \alpha_i A_i$$

Planar Solutions to Linear System

Plugging in $B_i(A_i)$, the initial conditions are a function of the coefficients A_1, A_2, A_3, A_4

$$\xi_0 = \sum_{i=1}^{4} A_i e^{\lambda_i 0} \qquad \eta_0 = \sum_{i=1}^{4} \alpha_i A_i e^{\lambda_i 0}$$

$$\dot{\xi}_0 = \sum_{i=1}^{4} \lambda_i A_i e^{\lambda_i 0} \qquad \dot{\eta}_0 = \sum_{i=1}^{4} \alpha_i A_i \lambda_i e^{\lambda_i 0}$$

Rewrite the initial conditions as a function of the coefficients

$$\begin{bmatrix} \xi_0 \\ \dot{\xi}_0 \\ \eta_0 \\ \dot{\eta}_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_1\lambda_1 & \alpha_2\lambda_2 & \alpha_3\lambda_3 & \alpha_4\lambda_4 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}$$

Planar Solutions to Linear System

However, the eigenvalues associated with the equilibrium points occur in pairs:

$$\lambda_1 = -\lambda_2 \qquad\qquad \lambda_3 = -\lambda_4$$

And if $$B_i = \frac{(\lambda_i^2 - U_{xx}^*)}{2\lambda_i} A_i = \alpha_i A_i$$

Then: $$\alpha_1 = -\alpha_2 \qquad\qquad \alpha_3 = -\alpha_4$$

The relationship between the initial conditions and coefficients then simplifies to:

$$\begin{bmatrix} \xi_0 \\ \dot{\xi}_0 \\ \eta_0 \\ \dot{\eta}_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & -\lambda_1 & \lambda_3 & -\lambda_3 \\ \alpha_1 & -\alpha_1 & \alpha_3 & -\alpha_3 \\ \alpha_1\lambda_1 & \alpha_1\lambda_1 & \alpha_3\lambda_3 & \alpha_3\lambda_3 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}$$

Planar Solutions to Linear System

This matrix system is then inverted to produce expressions for the coefficients A_i in terms of the initial conditions, λ_i and α_i

→ You will work through this step on your own in the homework!

Note: may find various versions of these expressions based on matrix inversion and simplification steps

One form appears in Ch 5.3 of Szebehely, 1967, ‘Theory of Orbits’

Planar Solutions to Linear System

We can select initial conditions to excite specific modes and produce desired solutions in the linear system

$$\xi = \sum_{i=1}^{4} A_i e^{\lambda_i t} \qquad \xi_0 = \sum_{i=1}^{4} A_i e^{\lambda_i 0} \qquad \dot{\xi}_0 = \sum_{i=1}^{4} \lambda_i A_i e^{\lambda_i 0}$$

$$\eta = \sum_{i=1}^{4} B_i e^{\lambda_i t} \qquad \eta_0 = \sum_{i=1}^{4} \alpha_i A_i e^{\lambda_i 0} \qquad \dot{\eta}_0 = \sum_{i=1}^{4} \alpha_i A_i \lambda_i e^{\lambda_i 0}$$

For a collinear equilibrium point, let's label the eigenvalues and associated coefficients as follows:

λ_1, λ_2 : real eigenvalues, stable/unstable modes associated with coefficients A_1, A_2

λ_3, λ_4 : imaginary eigenvalues, oscillatory modes associated with coefficients A_3, A_4

Planar Periodic Solutions to Linear System

To excite the oscillatory modes only, we can find the initial conditions that set $A_1 = A_2 = 0$

→Set the expressions for A_1, A_2 equal to 0 and solve for the associated initial velocity components that excite oscillatory modes for a given combination of variations in the position coordinates

$$\begin{bmatrix} \xi_0 \\ \dot{\xi}_0 \\ \eta_0 \\ \dot{\eta}_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & -\lambda_1 & \lambda_3 & -\lambda_3 \\ \alpha_1 & -\alpha_1 & \alpha_3 & -\alpha_3 \\ \alpha_1\lambda_1 & \alpha_1\lambda_1 & \alpha_3\lambda_3 & \alpha_3\lambda_3 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}$$

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \left[\mathbf{B}\right]^{-1} \begin{bmatrix} \xi_0 \\ \dot{\xi}_0 \\ \eta_0 \\ \dot{\eta}_0 \end{bmatrix}$$

Exciting Oscillatory Modes

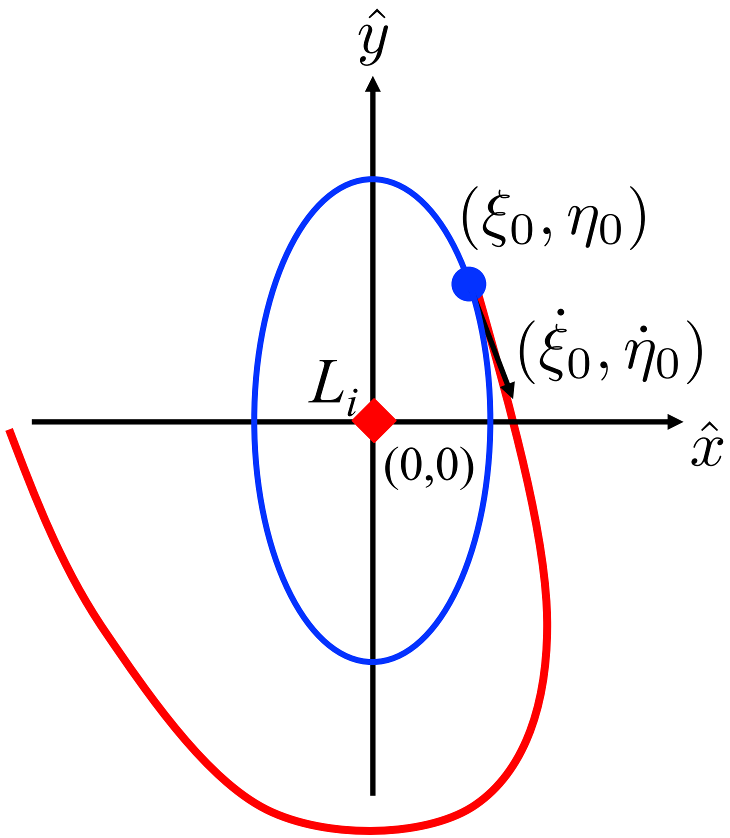
Following this process, initial conditions take the form:

$$\xi_0 = \xi_{0,des}$$

$$\eta_0 = \eta_{0,des}$$

$$\dot{\xi}_0 = \frac{\lambda_3 \eta_0}{\alpha_3}$$

$$\dot{\eta}_0 = \alpha_3 \lambda_3 \xi_0$$



Solutions in the linear system take form:

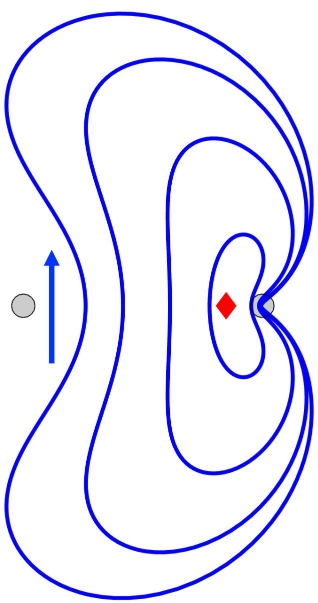
$$\xi(t) = A_3 e^{\lambda_3 t} + A_4 e^{-\lambda_3 t}$$

$$\eta(t) = A_3 \alpha_3 e^{\lambda_3 t} - A_4 \alpha_3 e^{-\lambda_3 t}$$

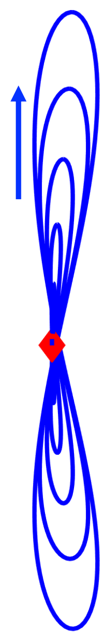
Selected Libration Point Orbits

- Lyapunov and vertical orbits emanate from L_1, L_2, L_3
- Halo and axial orbits emanate from bifurcations along Lyapunov and/or vertical orbit families (i.e., they intersect them)

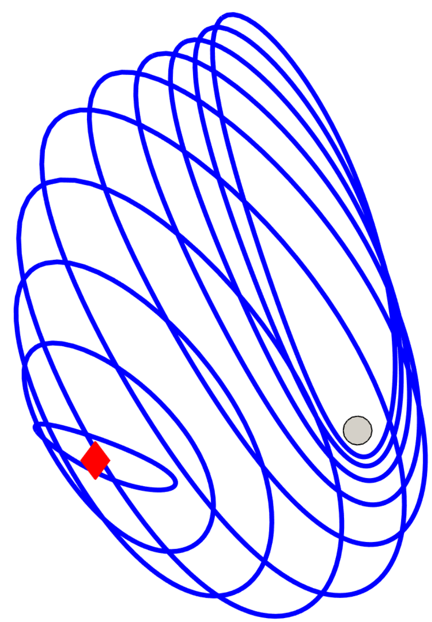
L_1 Lyapunov orbits



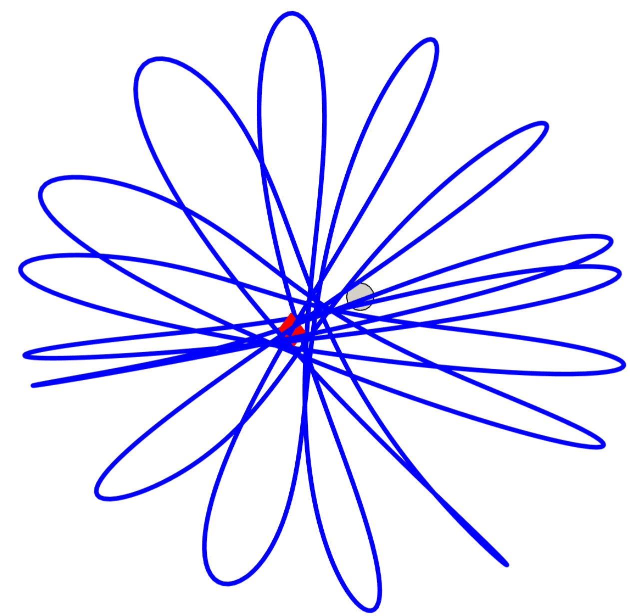
L_1 vertical orbits



L_1 halo orbits



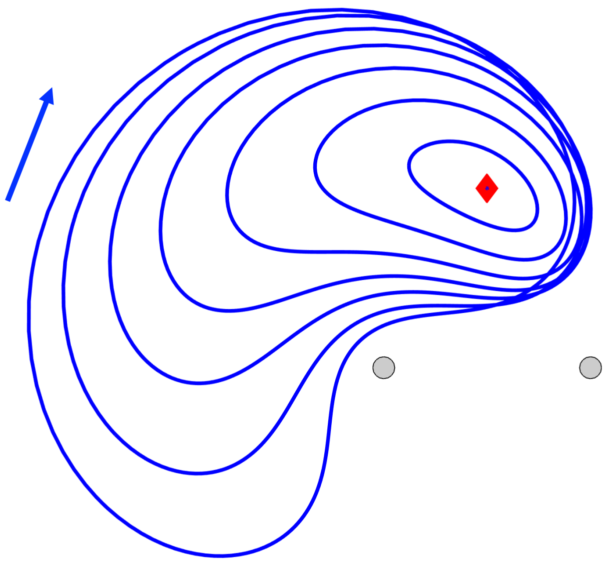
L_1 axial orbits



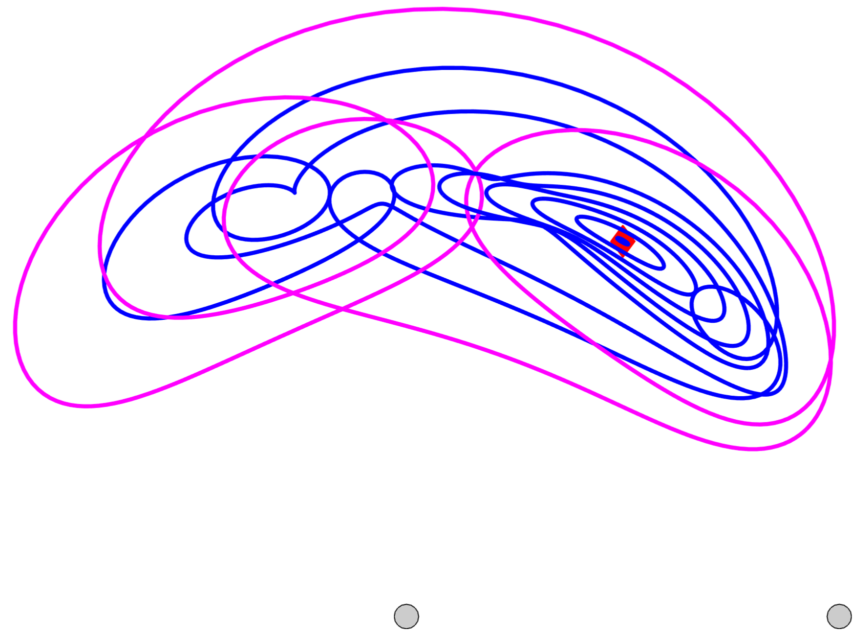
Selected Libration Point Orbits

Similar families about L_5 exist and are symmetric about the *x*-axis

L_4 short period orbits



L_4 long period orbits



Selected Resonant Orbits

- Consider *orbital resonances* (*mean motion resonances*)
- Definition is derived from the two-body problem (2BP):
  - For a *p:q* orbital resonance, P_3 (s/c) completes *p* revolutions about P_1 (Earth) in the inertial frame in the same time interval that P_2 (Moon) completes *q* revolutions in its orbit relative to P_1 (Earth)

$$\frac{p}{q} = \frac{\mathcal{P}_2}{\mathcal{P}_3} = \frac{n_3}{n_2}$$

where $\mathcal{P}_i$ = period of body *i* in inertial frame in Keplerian orbit

n_i = mean motion of body *i* in Keplerian orbit

- Interior resonance: $p > q$
- Exterior resonance: $p < q$, orbit in exterior region, sometimes 1+ close passes to primaries or equilibrium points

Resonant Orbits in the 2BP

Useful image examples from Murray and Dermott, 2000, Solar System Dynamics, Cambridge University Press, pp. 322-324

Asteroid starts at perihelion in 2:1 resonance

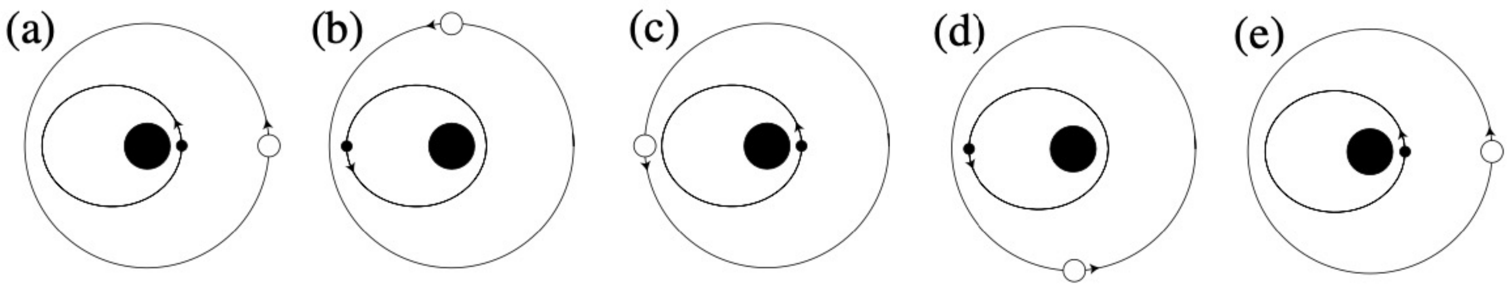


Fig. 8.1. The relative positions of Jupiter (white circle) and an asteroid (small filled circle) for the stable configuration when their orbital periods are in a ratio of 2:1. If T_J is the period of Jupiter's orbit then the diagrams illustrate the configurations at times (a) $t = 0$, (b) $t = \frac{1}{4}T_J$, (c) $t = \frac{1}{2}T_J$, (d) $t = \frac{3}{4}T_J$, and (e) $t = T_J$.

Resonant Orbits in the 2BP

Useful image examples from Murray and Dermott, 2000, Solar System Dynamics, Cambridge University Press, pp. 324

Resonant orbits in 2BP in inertial vs P_1-P_2 rotating frames

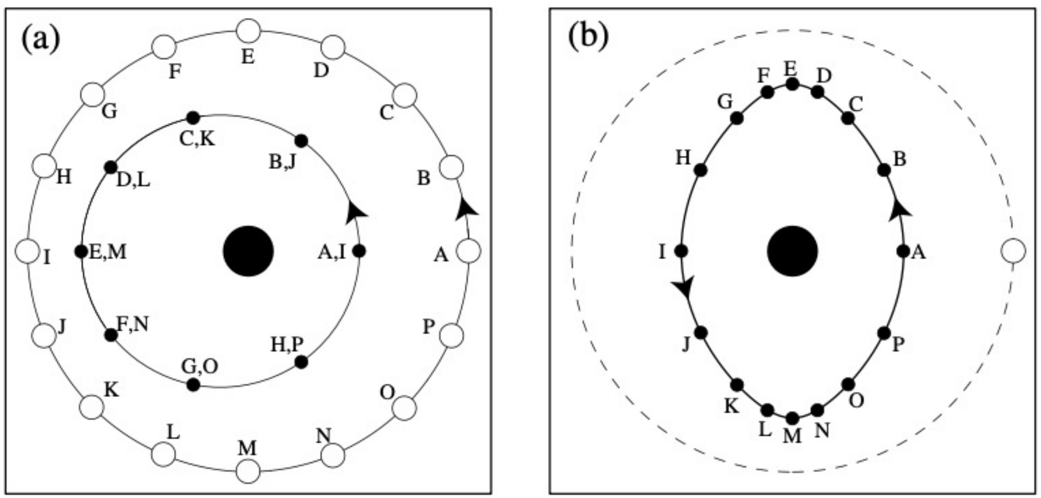


Fig. 8.3. (a) Points along the orbit of an asteroid (small black circles) and Jupiter (white circles) at fixed time intervals of 1/16 Jupiter period in a nonrotating reference frame for the 2:1 resonance. The letters at each point on one orbit match up with an equivalent point on the other orbit. (b) The path of the asteroid in a rotating frame moving with the mean motion of Jupiter. The letters denote the same points as shown in Fig. 8.3a. The eccentricity of the asteroid's orbit is 0.2.

Resonant Orbits in the 2BP

Useful image examples from Murray and Dermott, 2000, Solar System Dynamics, Cambridge University Press, pp. 325

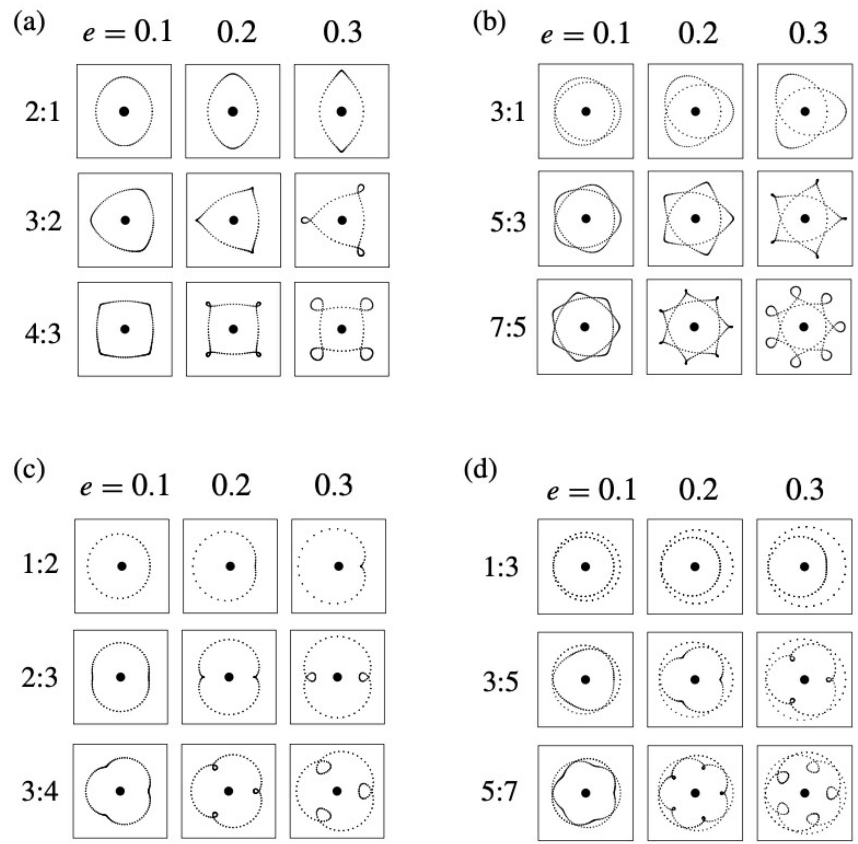


Fig. 8.4. Paths in the rotating frame for a test particle at (a) the 2:1, 3:2, and 4:3 first-order, interior resonances, (b) the 3:1, 5:3, and 7:5 second-order, interior resonances, (c) the 1:2, 2:3, and 3:4 first-order, exterior resonances, and (d) the 1:3, 3:5, and 5:7 second-order, exterior resonances for values of the eccentricity $e = 0.1$, 0.2, and 0.3. The positions of the particle along each path are drawn at equal time intervals.

Resonant Orbits in the CR3BP

- Due to gravity of third-body, common to update the definition: for a *p:q* orbital resonance, P_3 (s/c) completes *p* revolutions about P_1 (Earth) in the inertial frame in approximately the time that P_2 (Moon) completes *q* revolutions in its orbit relative to P_1 (Earth)

- How to compute? Use a *p:q* resonant orbit from 2BP (transformed to P_1-P_2 rotating frame) to produce an initial guess and correct in the CR3BP to produce a periodic orbit.

- Associated continuous family of periodic orbits in CR3BP: *p:q* resonant orbit family

Initial Guess for Resonant Orbit

One approach:

1. Calculate orbit period for *p:q* resonance, T_{sc}, in inertial frame
2. Define IC at prograde/retrograde periapsis/apoapsis in inertial frame for conic with period T_{sc} using 2BP (vary e)

$$a = \left(\mu_{2BP} \left(\frac{T_{sc}}{2\pi} \right)^2 \right)^{1/3} \qquad r = a(1-e^2)/(1+e\cos(\theta))$$

$$v = \sqrt{(2\mu_{2BP}/r - \mu_{2BP}/a)}$$

3. Nondimensionalize state vector and convert to rotating frame
4. Integrate for orbit period $T_{sc,rot} = p\, T_{sc}$ in CR3BP
5. Select value of eccentricity that produces good initial guess for a periodic orbit
6. Use corrections scheme to compute a periodic orbit in CR3BP

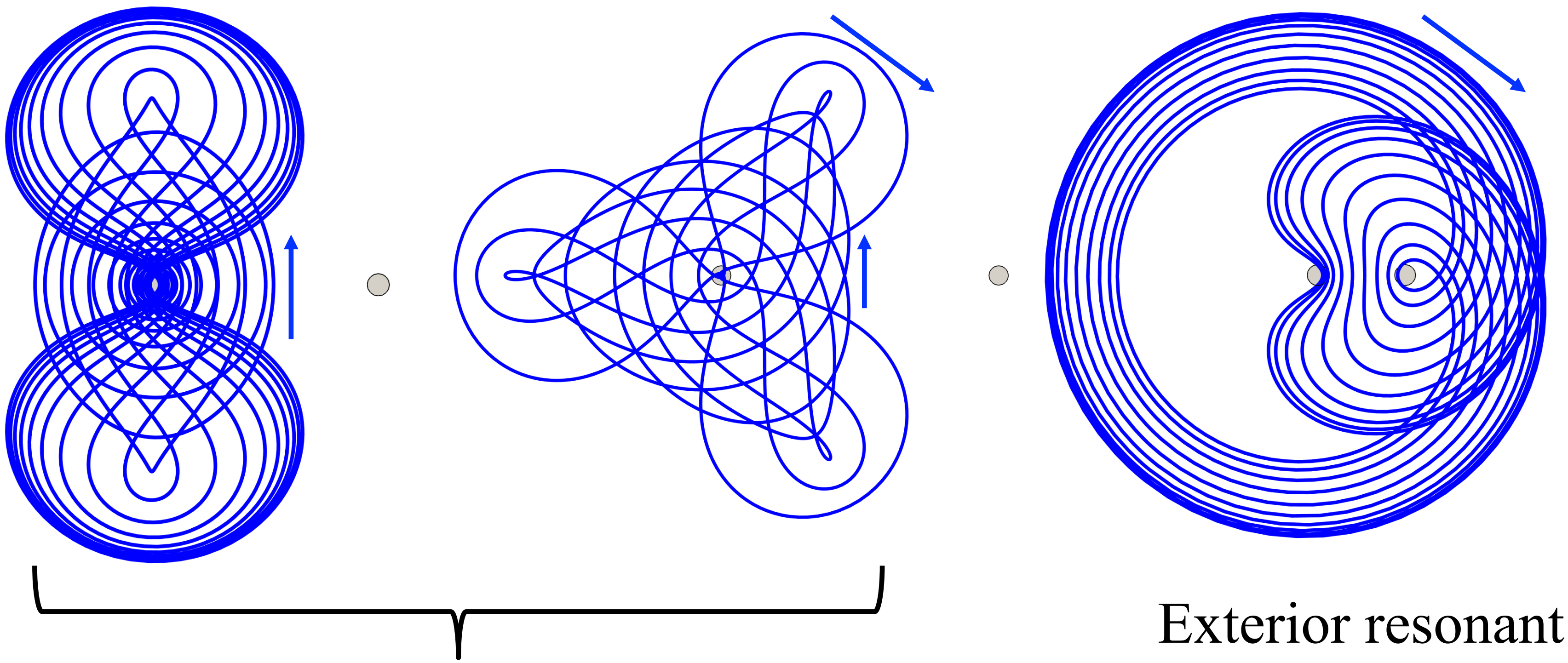
For more details, see Vaquero, M., 2013, "Spacecraft Transfer Trajectory Design Exploiting Resonant Orbits In Multi-body Environments ", PhD Dissertation, Purdue University

Selected Resonant Orbits

2:1 resonant orbit family

3:1 resonant orbit family

1:2 resonant orbit family

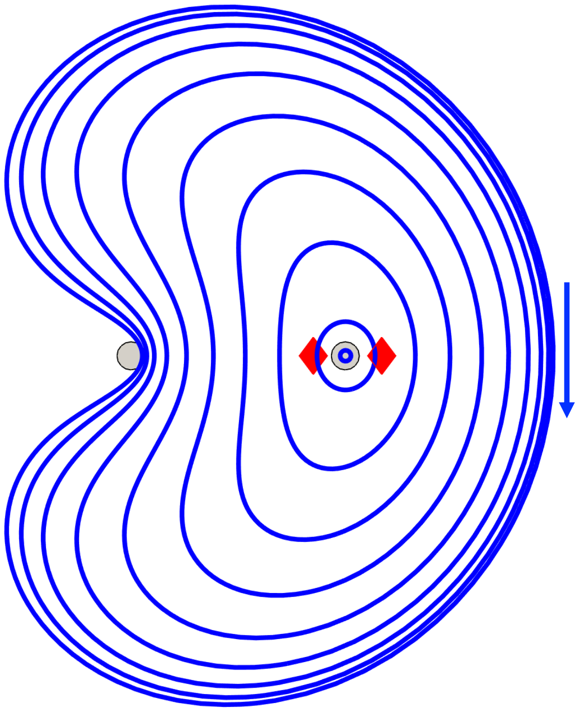


Interior resonant orbit families

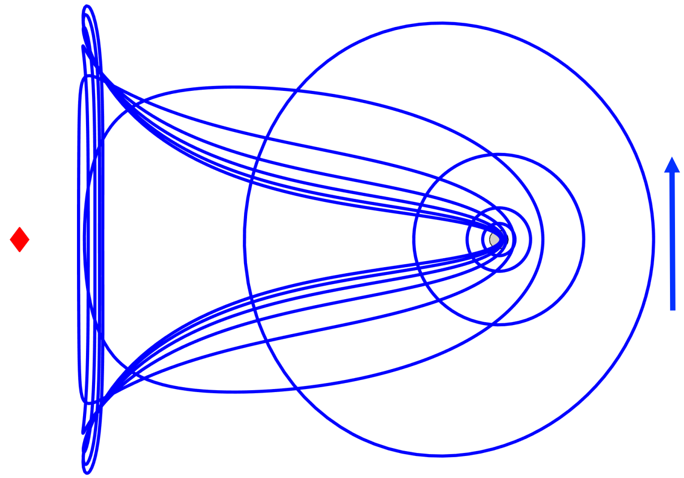
Exterior resonant orbit family

Selected Primary-Centered Orbits

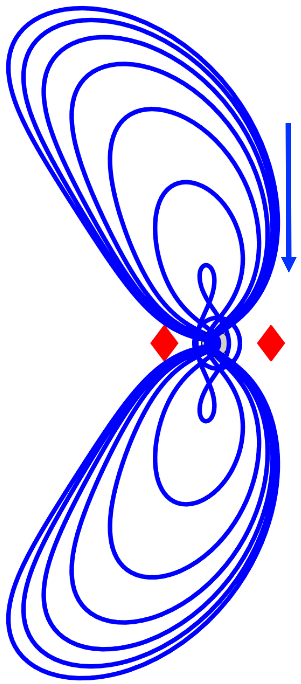
Distant retrograde orbits



Low prograde orbits



Distant prograde orbits



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