Nonlinear Spacecraft Control

ASEN 5010

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Outline

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- Lyapunov Functions
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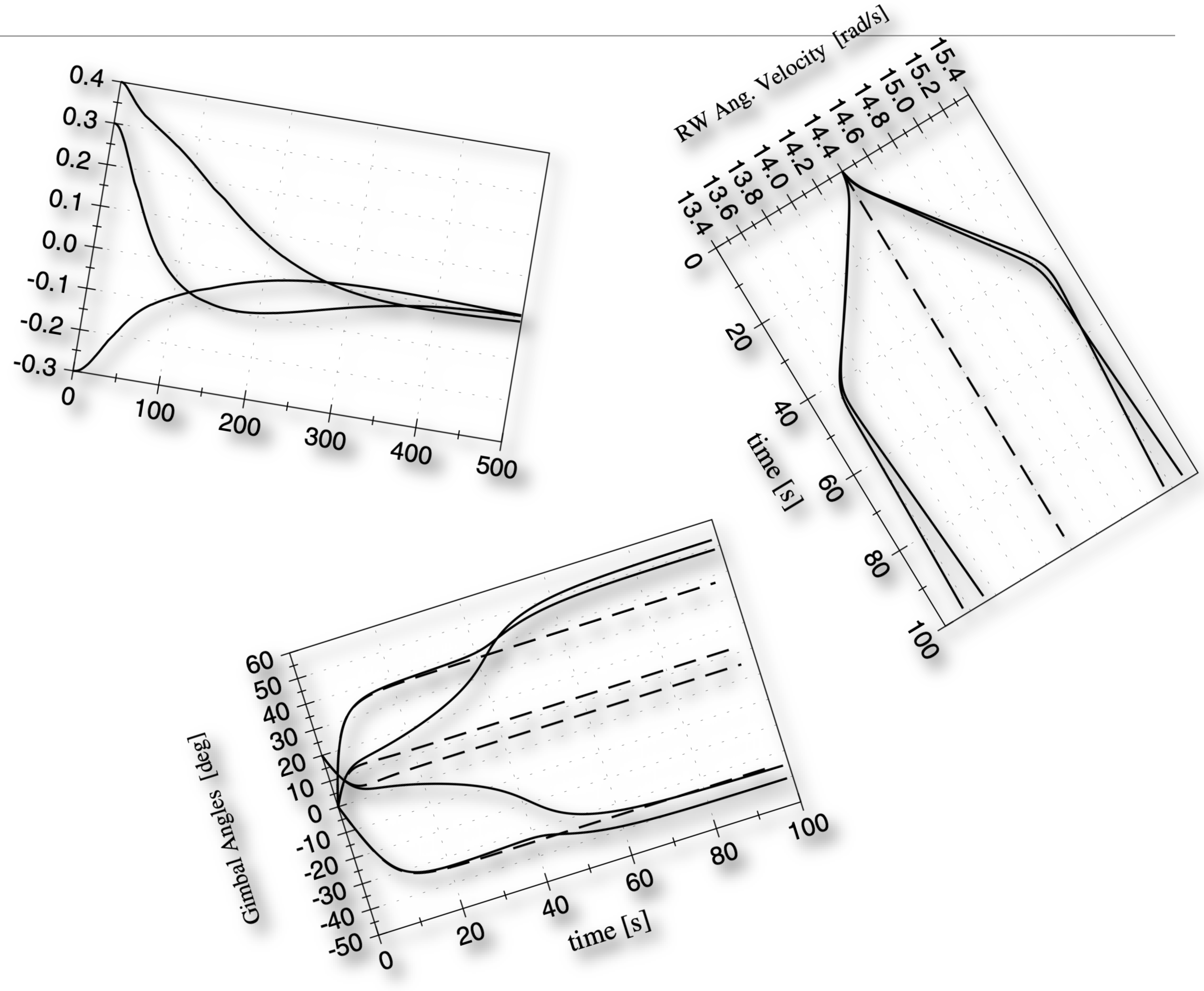


Stability Definitions

Why isn't stable just stable?

Definitions

State Vector: $\boldsymbol{x} = (x_1 \cdots \; x_N)^T$

EOM:

$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$ — Non-Autonomous System

$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ — Autonomous System

Control Vector: $\boldsymbol{u} = \boldsymbol{g}(\boldsymbol{x})$

Closed-Loop System: $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t)$

Equilibrium State: A state vector point $\boldsymbol{x}_e$ is said to be an equilibrium state (or equilibrium point) of a dynamical system described by $\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x},t)$ at time t_0 if

$$\boldsymbol{f}(\boldsymbol{x}_e, t) = 0 \qquad \forall \; t > t_0$$

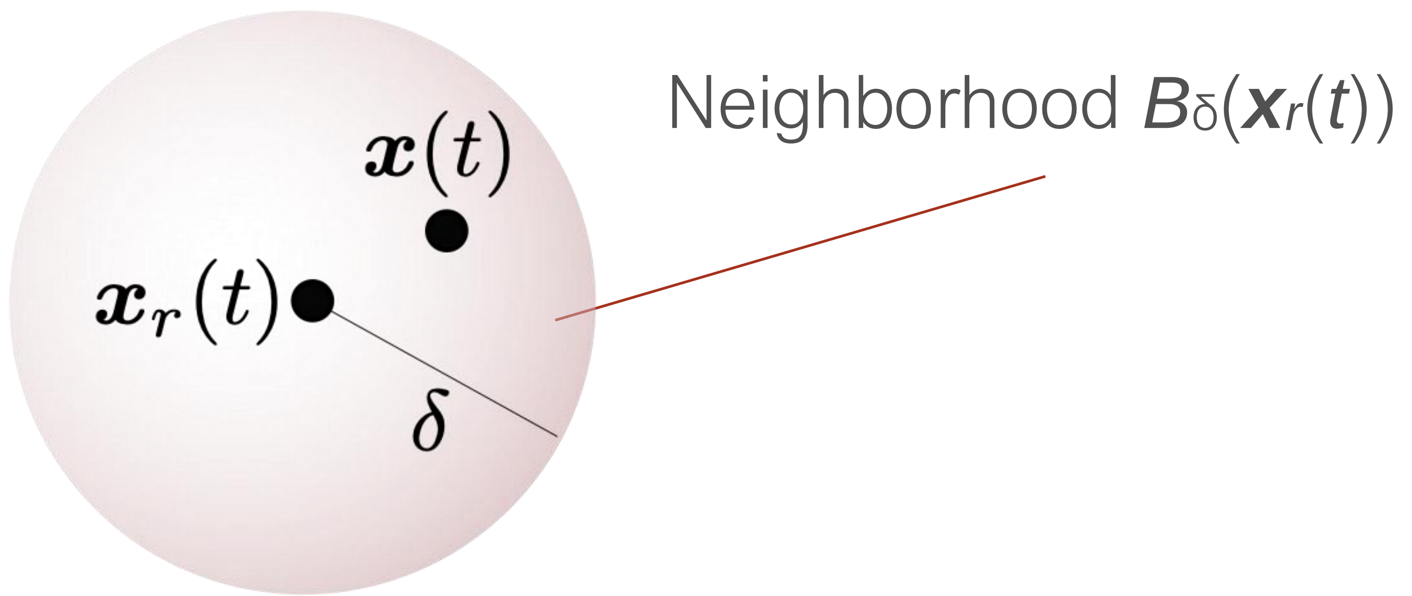
$$\dot{\boldsymbol{x}}_e = 0 \qquad \boldsymbol{x}_e = \text{constant}$$

Neighborhood: Given δ>0, a state vector ***x***(*t*) is said to be in the neighborhood $B_\delta(\boldsymbol{x}_r(t))$ of the state $\boldsymbol{x}_r(t)$ if

$$||\boldsymbol{x}(t) - \boldsymbol{x}_r(t)|| \; < \; \delta$$

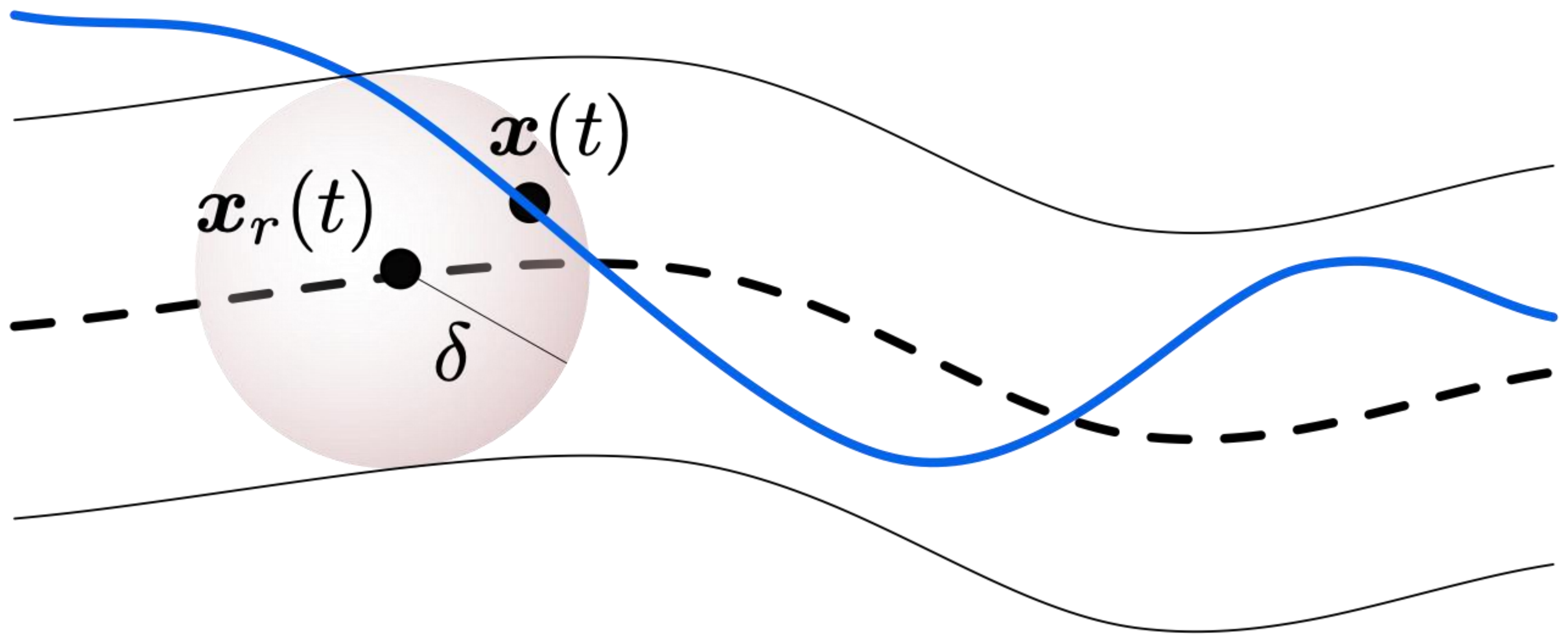
then

$$\boldsymbol{x}(t) \in B_\delta(\boldsymbol{x}_r(t))$$



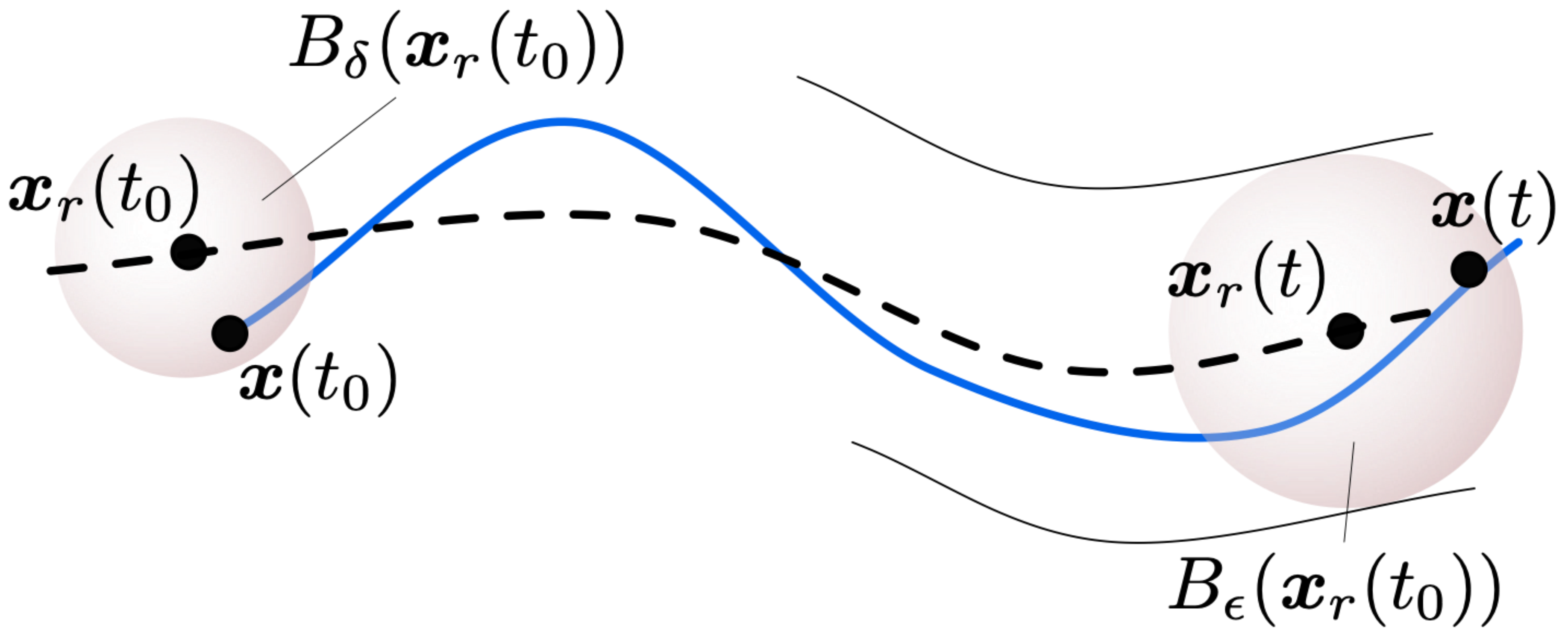
Lagrange Stability: The motion ***x***(*t*) is said to be Lagrange stable (or bounded) relative to $\boldsymbol{x}_r(t)$ if there exists a δ>0 such that

$$\boldsymbol{x}(t) \in B_\delta(\boldsymbol{x}_r(t)) \qquad \forall \; t > t_0$$



Lyapunov Stability: The motion ***x***(*t*) is said to be Lyapunov stable (or stable) relative to $\boldsymbol{x}_r$(t) if for each ε>0 there exists a δ(ε)>0 such that

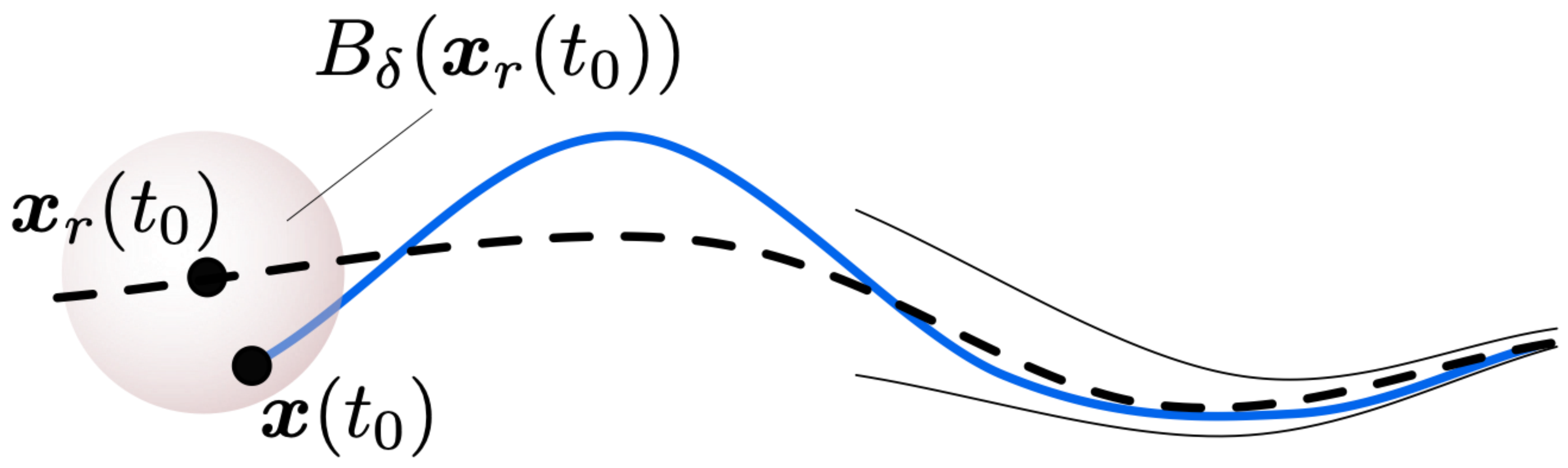
$$\boldsymbol{x}(t_0) \in B_\delta(\boldsymbol{x}_r(t_0)) \Longrightarrow \boldsymbol{x}(t) \in B_\epsilon(\boldsymbol{x}_r(t))$$
$$\forall \; t > t_0$$

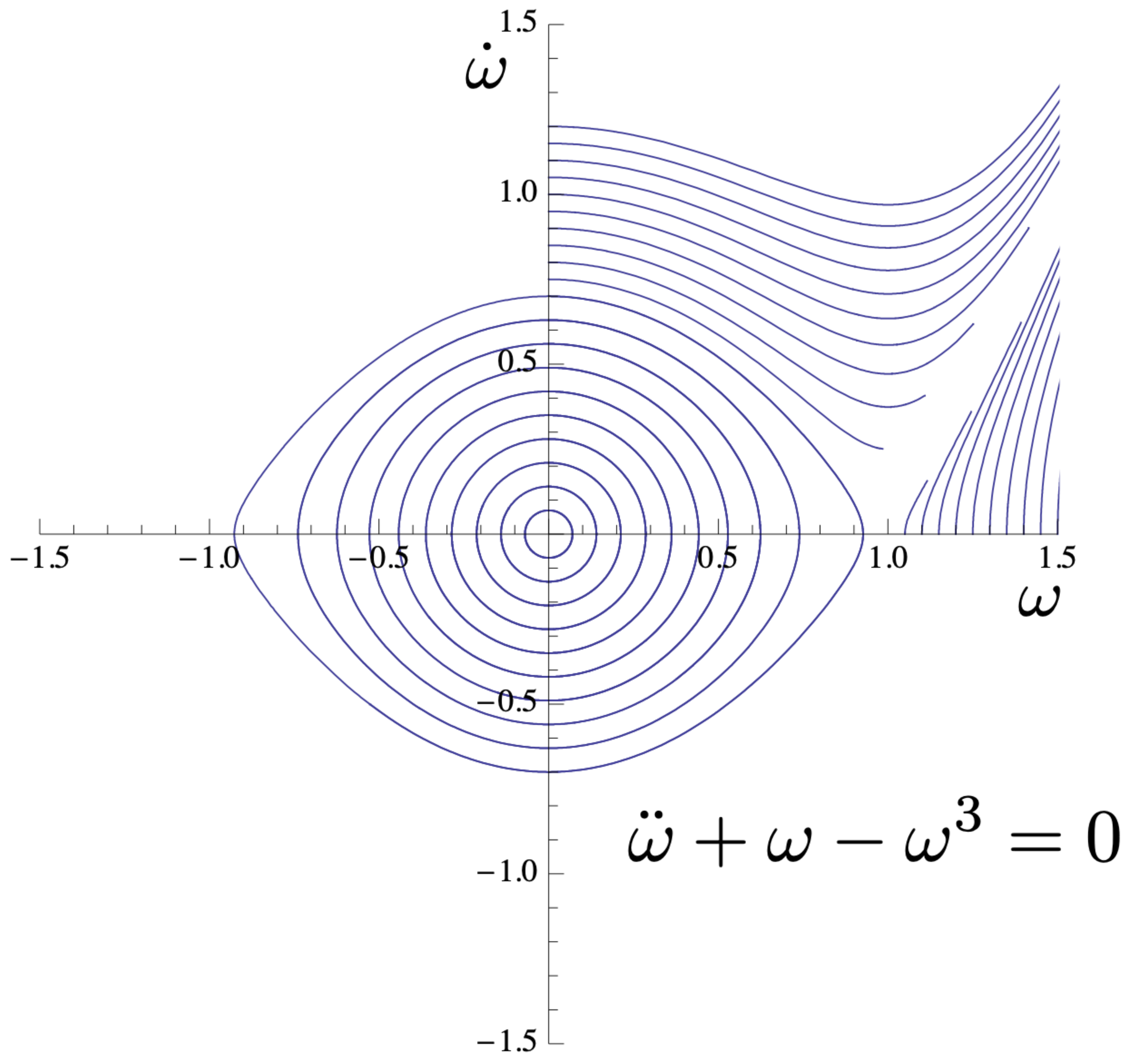


Asymptotic Stability: The motion $\boldsymbol{x}(t)$ is asymptotically stable relative to $\boldsymbol{x}_r(t)$ if $\boldsymbol{x}(t)$ is Lyapunov stable and there exists a δ>0 such that

$$\boldsymbol{x}(t_0) \in B_\delta(\boldsymbol{x}_r(t_0)) \Longrightarrow \lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{x}_r(t)$$

Global Stability: The motion $\boldsymbol{x}(t)$ is globally stable relative to $\boldsymbol{x}_r(t)$ if $\boldsymbol{x}(t)$ is stable for any initial state vector $\boldsymbol{x}(t_0)$.





(Show Mathematica Example)

Linearization of Dynamical System

Reference motion given by: $\dot{\boldsymbol{x}}_r = \boldsymbol{f}(\boldsymbol{x}_r, \boldsymbol{u}_r)$

Feedforward control

Nonlinear EOM: $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$

Feedback control: $\delta \boldsymbol{u} = \boldsymbol{u} - \boldsymbol{u}_r$

Departure motion: $\delta \boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_r$

Performing a Taylor Series expansion of ***x*** about ($\boldsymbol{x}_r$,$\boldsymbol{u}_r$) we obtain

$$\delta \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}_r, \boldsymbol{u}_r) + \frac{\partial \boldsymbol{f}(\boldsymbol{x}_r, \boldsymbol{u}_r)}{\partial \boldsymbol{x}} \delta \boldsymbol{x} + \frac{\partial \boldsymbol{f}(\boldsymbol{x}_r, \boldsymbol{u}_r)}{\partial \boldsymbol{u}} \delta \boldsymbol{u} + H.O.T - \boldsymbol{f}(\boldsymbol{x}_r, \boldsymbol{u}_r)$$

$$\delta \dot{\boldsymbol{x}} \simeq \frac{\partial \boldsymbol{f}(\boldsymbol{x}_r, \boldsymbol{u}_r)}{\partial \boldsymbol{x}} \delta \boldsymbol{x} + \frac{\partial \boldsymbol{f}(\boldsymbol{x}_r, \boldsymbol{u}_r)}{\partial \boldsymbol{u}} \delta \boldsymbol{u}$$

Let us define:

$$[A] = \frac{\partial \boldsymbol{f}(\boldsymbol{x}_r, \boldsymbol{u}_r)}{\partial \boldsymbol{x}}$$

$$[B] = \frac{\partial \boldsymbol{f}(\boldsymbol{x}_r, \boldsymbol{u}_r)}{\partial \boldsymbol{u}}$$

The linearized system is then written in standard form as

$$\boxed{\delta \dot{\boldsymbol{x}} \simeq [A] \delta \boldsymbol{x} + [B] \delta \boldsymbol{u}}$$

If the nominal reference motion is an equilibrium state $\boldsymbol{x}_e$, then the linearized EOM simplify to:

$$\dot{\boldsymbol{x}} \simeq [A] \boldsymbol{x} + [B] \boldsymbol{u}$$

Lyapunov’s Direct Method

Powerful method to prove nonlinear stability using energy methods...

More Definitions...

Positive (Negative) Definite Function: A scalar continuous function *V*(***x***) is said to be locally positive (negative) definite about $\boldsymbol{x}_r$ if

$$\boldsymbol{x} = \boldsymbol{x}_r \implies V(\boldsymbol{x}) = 0$$

and there exists a δ>0 such that

$$\forall\ \boldsymbol{x} \in B_\delta(\boldsymbol{x}_r) \implies V(\boldsymbol{x}) > 0 \quad (V(\boldsymbol{x}) < 0)$$

Positive (Negative) Semi-Definite Function: A scalar continuous function *V*(***x***) is said to be locally positive (negative) semi-definite about $\boldsymbol{x}_r$ if

$$\boldsymbol{x} = \boldsymbol{x}_r \implies V(\boldsymbol{x}) = 0$$

and there exists a δ>0 such that

$$\forall\ \boldsymbol{x} \in B_\delta(\boldsymbol{x}_r) \implies V(\boldsymbol{x}) \geq 0 \quad (V(\boldsymbol{x}) \leq 0)$$

Examples:

$$V(x, \dot{x}) = \frac{1}{2}x^2 + \frac{1}{2}\dot{x}^2$$

A matrix [*K*] is said to be positive or negative (semi-) definite if for every state vector ***x***:

$$\boldsymbol{x}^T[K]\boldsymbol{x} \begin{cases} > 0 & \Rightarrow \text{positive definite} \\ \geq 0 & \Rightarrow \text{positive semi-definite} \\ < 0 & \Rightarrow \text{negative definite} \\ \leq 0 & \Rightarrow \text{negative semi-definite} \end{cases}$$

Lyapunov Function

Lyapunov Function: The scalar function $V(\boldsymbol{x})$ is a Lyapunov function for the dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ if it is continuous and there exists a $\delta>0$ such that for any $\boldsymbol{x} \in B_\delta(\boldsymbol{x}_r)$

1) $V(\boldsymbol{x})$ is a positive definite function about $\boldsymbol{x}_r$
2) $V(\boldsymbol{x})$ has continuous partial derivatives
3) $\dot{V}(\boldsymbol{x})$ is negative semi-definite

Example: Consider the spring-mass system

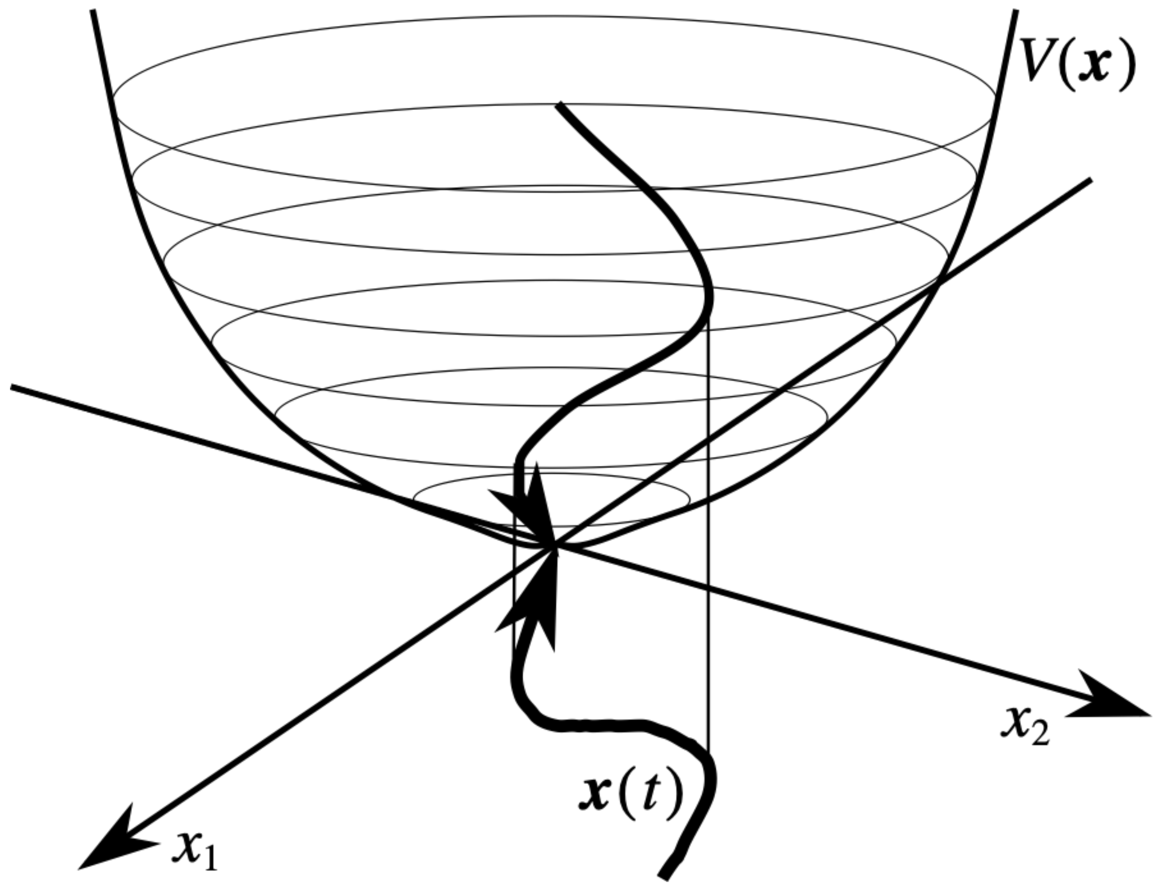
$$m\ddot{x} + kx = 0$$

Let us use the total system energy as a candidate Lyapunov function.

$$V(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

The Lyapunov rate is then expressed as

$$\dot{V}(x, \dot{x}) = (m\ddot{x} + kx)\,\dot{x} = 0 \leq 0$$



$$\dot{V} = \frac{\partial V}{\partial \boldsymbol{x}}^T \dot{\boldsymbol{x}} = \frac{\partial V}{\partial \boldsymbol{x}}^T \boldsymbol{f}(\boldsymbol{x}) \leq 0$$

All projections of the dynamical motion on to the Lyapunov function surface must point toward the reference state $\boldsymbol{x}_r$.

Lyapunov Stability: If a Lyapunov function $V(\boldsymbol{x})$ exists for the dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, then this system is stable about the origin.

Asymptotic Stability: Assume $V(\boldsymbol{x})$ is a Lyapunov function about $\boldsymbol{x}_r(t)$ for the dynamical system
$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$; then the system is asymptotically stable if
1) the system is stable about $\boldsymbol{x}_r$
2) $\dot{V}(\boldsymbol{x})$ is negative definite about $\boldsymbol{x}_r$

Example: Consider the spring-mass-damper system:

$$m\ddot{x} + c\dot{x} + kx = 0$$

with the Lyapunov function

$$V(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

Taking the derivative we only determine stability.

$$\dot{V}(x,\dot{x}) = (m\ddot{x} + kx)\,\dot{x} = -c\dot{x}^2 \leq 0$$

Evaluating the higher derivatives on the set $\dot{x} = 0$ yields:

$$\ddot{V}(\dot{x}=0) = -2c\ddot{x}\dot{x} = 2\frac{c}{m}\,(c\dot{x} + kx)\,\dot{x} = 0$$

$$\dddot{V} = -2\frac{c}{m^2}\left((c\dot{x}+kx)^2 + c^2\dot{x}^2 + ckx\dot{x} - k\dot{x}^2\right) \Rightarrow \dddot{V}(\dot{x}=0) = -2\frac{ck^2}{m^2}x^2 < 0$$

Theorem:* Assume there exists a Lyapunov function $V(\boldsymbol{x})$ of the dynamical system $\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x})$. Let Ω be the non-empty set of state vectors such that

$$\boldsymbol{x} \in \Omega \implies \dot{V}(\boldsymbol{x}) = 0$$

If the first k-1 derivatives of $V(\boldsymbol{x})$, evaluated on the set Ω, are zero

$$\frac{\mathrm{d}^i V(\boldsymbol{x})}{\mathrm{d}t^i} = 0 \quad \forall\, \boldsymbol{x} \in \Omega \qquad i = 1, 2, \ldots, k-1$$

and the k^{th} derivative is negative definite on the set Ω

$$\frac{\mathrm{d}^k V(\boldsymbol{x})}{\mathrm{d}t^k} < 0 \quad \forall\, \boldsymbol{x} \in \Omega$$

then the system $\boldsymbol{x}(t)$ is asymptotically stable if k is an odd number.

*R. Mukherjee and D. Chen, "Asymptotic Stability Theorem for Autonomous Systems," *Journal of Guidance, Control and Dynamics*, Vol. 16, Sept. –Oct. 1993, pp. 961–963.

Lyapunov Stability of Linear System

- Assume that the dynamical system is of the linear form:

$$\dot{\boldsymbol{x}} = [A]\boldsymbol{x}$$

- Let [*P*]>0 be a symmetric, p.d. matrix, then we define

$$V(\boldsymbol{x}) = \boldsymbol{x}^T[P]\boldsymbol{x}$$

$$\Downarrow$$

$$\dot{V} = \dot{\boldsymbol{x}}^T[P]\boldsymbol{x} + \boldsymbol{x}^T[P]\dot{\boldsymbol{x}}$$

$$\Downarrow$$

$$\dot{V} = \boldsymbol{x}^T\left([A]^T[P] + [P][A]\right)\boldsymbol{x} < 0 \quad \checkmark$$

is this negative definite?

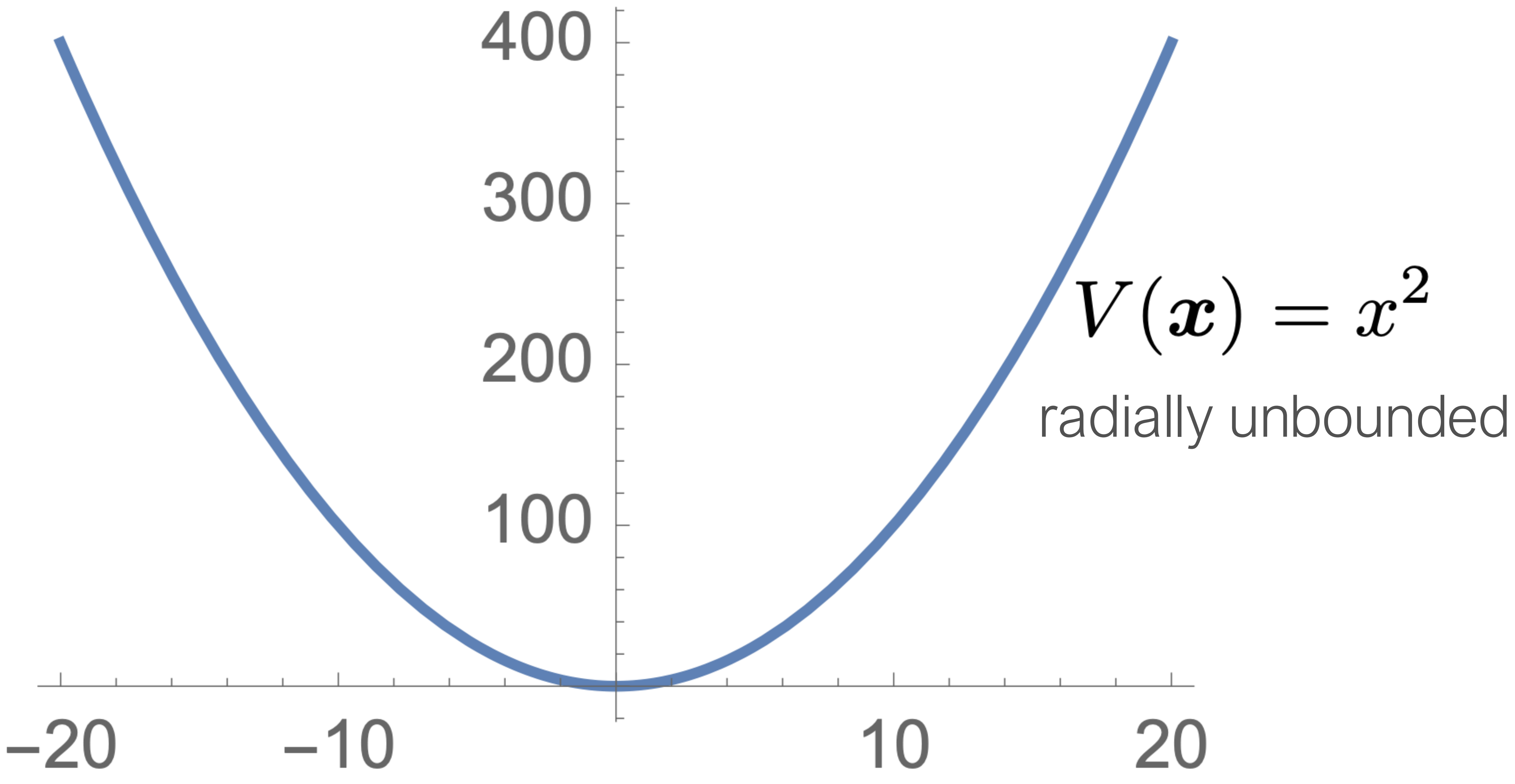
Theorem: An autonomous linear system $\dot{\boldsymbol{x}} = [A]\boldsymbol{x}$ is stable if and only if for any symmetric, positive definite [*R*] there exists a corresponding symmetric, positive definite [*P*] such that

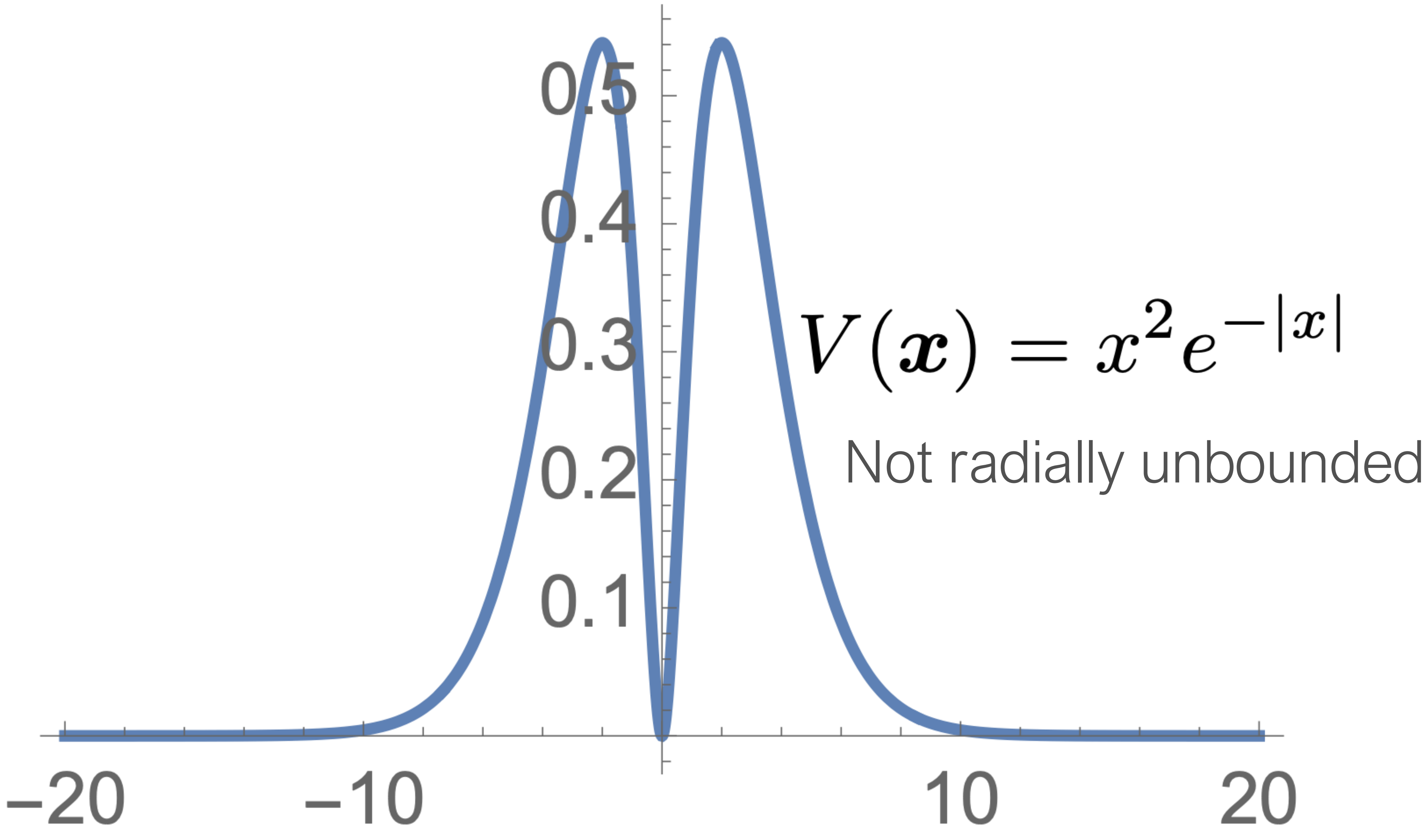
$$[A]^T[P] + [P][A] = -[R]$$

algebraic Lyapunov equation

Global Stability

- The stability argument holds for any initial conditions
- The V(**x**) function is radially unbounded $\lim_{|\boldsymbol{x}|\to\infty} V(\boldsymbol{x}) = \infty$





Lyapunov Functions

Elegant energy functions to make the control design/analysis simpler...

Elemental Velocity-Based Lyapunov Functions

Finding proper Lyapunov functions can be a difficult task for many systems.

We will break up this search into rate and position based Lyapunov functions.

Goal: drive only the state rates to zero

$$\dot{\boldsymbol{q}} \rightarrow 0$$

General Mechanical System

State Vector: $(\boldsymbol{q}, \dot{\boldsymbol{q}})$

Goal: $\dot{\boldsymbol{q}} \rightarrow 0$

Kinetic Energy: $\boxed{T = \frac{1}{2}\dot{\boldsymbol{q}}^T[M]\dot{\boldsymbol{q}}}$ True for any natural mechanical system

with $[M] = [M]^T > 0$

EOM: $\boxed{[M(\boldsymbol{q})]\ddot{\boldsymbol{q}}} = -[\dot{M}(\boldsymbol{q}, \dot{\boldsymbol{q}})]\dot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T[M_q(\boldsymbol{q})]\dot{\boldsymbol{q}} + \boxed{\boldsymbol{Q}}$

Generalized Force Vector

with $\dot{\boldsymbol{q}}^T[M_{\boldsymbol{q}}(\boldsymbol{q})]\dot{\boldsymbol{q}} \equiv \begin{pmatrix} \dot{\boldsymbol{q}}^T \left[\frac{\partial M}{\partial q_1}\right] \dot{\boldsymbol{q}} \\ \vdots \\ \dot{\boldsymbol{q}}^T \left[\frac{\partial M}{\partial q_N}\right] \dot{\boldsymbol{q}} \end{pmatrix}$

Lyapunov Function: $\boxed{V(\dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T[M(\boldsymbol{q})]\dot{\boldsymbol{q}}}$ Let’s chose to use the kinetic energy function as our Lyapunov function!

Lyapunov Rate: $\dot{V} = \dot{\boldsymbol{q}}^T\boxed{[M]\ddot{\boldsymbol{q}}} + \frac{1}{2}\dot{\boldsymbol{q}}^T[\dot{M}]\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^T\left(-\frac{1}{2}[\dot{M}]\dot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T[M_{\boldsymbol{q}}]\dot{\boldsymbol{q}} + \boldsymbol{Q}\right)$

Note that $\dot{\boldsymbol{q}}^T\left(\dot{\boldsymbol{q}}^T[M_{\boldsymbol{q}}]\dot{\boldsymbol{q}}\right) = \sum_{i=1}^{N} \dot{q}_i\left(\dot{\boldsymbol{q}}^T[M_{q_i}]\dot{\boldsymbol{q}}\right) = \dot{\boldsymbol{q}}^T[\dot{M}]\dot{\boldsymbol{q}}$

$\boxed{\dot{V} = \dot{\boldsymbol{q}}^T\boldsymbol{Q}}$ This is the generalized work/energy equation! $\Rightarrow$ $\boldsymbol{Q} = -[P]\dot{\boldsymbol{q}}$ $\Rightarrow$ $\dot{V} = -\dot{\boldsymbol{q}}^T[P]\dot{\boldsymbol{q}} < 0$ Globally asymptotically stabilizing

- Next, let us consider the tracking problem of a generalized mechanical system:

Reference States: $(\boldsymbol{q}_r, \dot{\boldsymbol{q}}_r)$

State Vector: $(\boldsymbol{q}, \dot{\boldsymbol{q}})$

Tracking Error: $\delta\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_r$

Goal: $\delta\dot{\boldsymbol{q}} \to 0$

Lyapunov Function: $$V(\dot{\boldsymbol{q}}) = \frac{1}{2}\delta\dot{\boldsymbol{q}}^T[M(\boldsymbol{q})]\delta\dot{\boldsymbol{q}}$$

Energy-function-like positive definite measure of tracking error.

Lyapunov Rate: $$\dot{V} = \delta\dot{\boldsymbol{q}}^T\left(-\frac{1}{2}[\dot{M}]\left(\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}_r\right) + \frac{1}{2}\dot{\boldsymbol{q}}^T[M_q]\dot{\boldsymbol{q}} - [M]\ddot{\boldsymbol{q}}_r + \boldsymbol{Q}\right)$$

Note that the work/energy principle doesn't hold with these non-mechanical energy function, and the Lyapunov rate is no longer the simply the power equation.

Proposed Control: $$\boldsymbol{Q} = \frac{1}{2}[\dot{M}]\left(\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}_r\right) - \frac{1}{2}\dot{\boldsymbol{q}}^T[M_q]\dot{\boldsymbol{q}} + [M]\ddot{\boldsymbol{q}}_r - [P]\delta\dot{\boldsymbol{q}}$$

Feedback linearization

Feedforward compensation

Proportional Feedback

$$\Rightarrow \quad \dot{V}(\delta\dot{\boldsymbol{q}}) = -\delta\dot{\boldsymbol{q}}^T[P]\delta\dot{\boldsymbol{q}} < 0$$

Globally asymptotically stabilizing

- Example of Mechanical System Stabilization: (Ex: 8.8 in S&J)

State vector: $\boldsymbol{q} = (\theta_1, \theta_2, \theta_3)^T$

Goal: $\dot{\boldsymbol{q}} \to 0$

EOM:

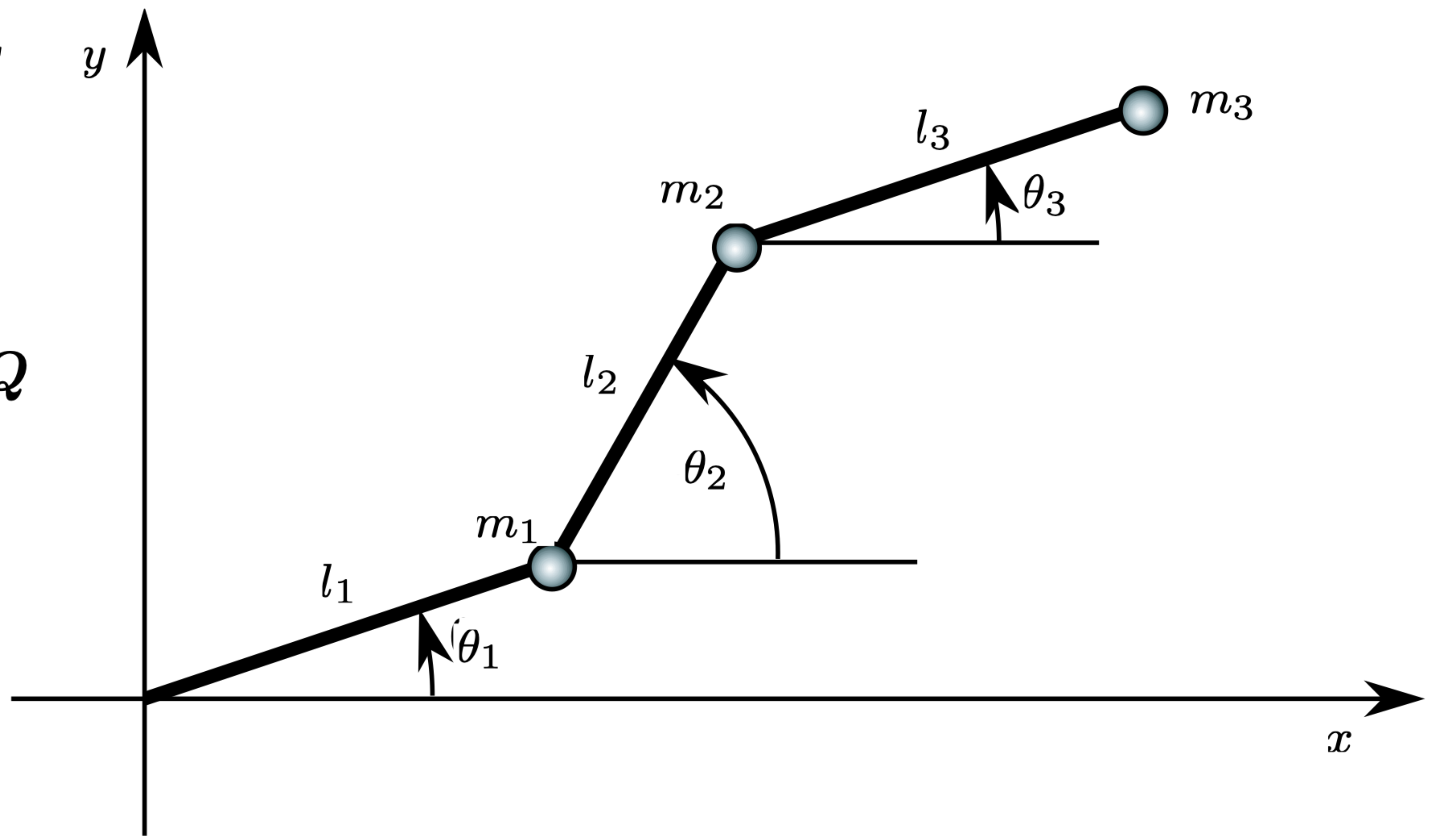
$$[M]\ddot{\boldsymbol{q}} + [\dot{M}]\dot{\boldsymbol{q}} - \frac{1}{2}\dot{\boldsymbol{q}}^T[M_{\boldsymbol{q}}]\dot{\boldsymbol{q}} = \boldsymbol{Q}$$

Lyapunov Function:

$$V(\dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T[M(\boldsymbol{q})]\dot{\boldsymbol{q}}$$

Lyapunov Rate:

$$\dot{V} = \dot{\boldsymbol{q}}^T\boldsymbol{Q}$$



Control:

$$\boxed{\boldsymbol{Q}_1 = -P_1\dot{\boldsymbol{q}}}$$ Rate Feedback

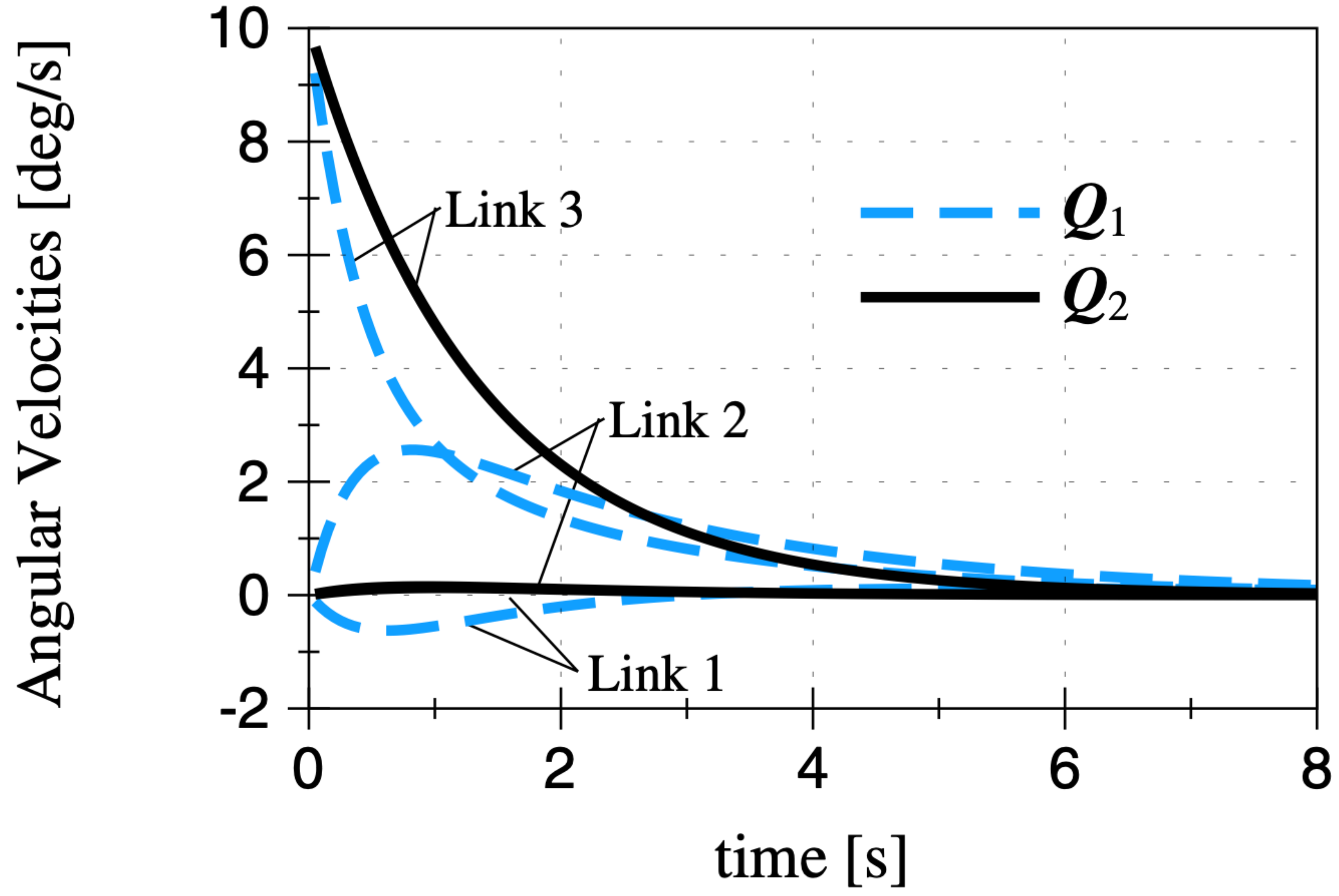
$$\boxed{\boldsymbol{Q}_2 = -P_2[M(\boldsymbol{q})]\dot{\boldsymbol{q}}}$$ “Momentum” Feedback

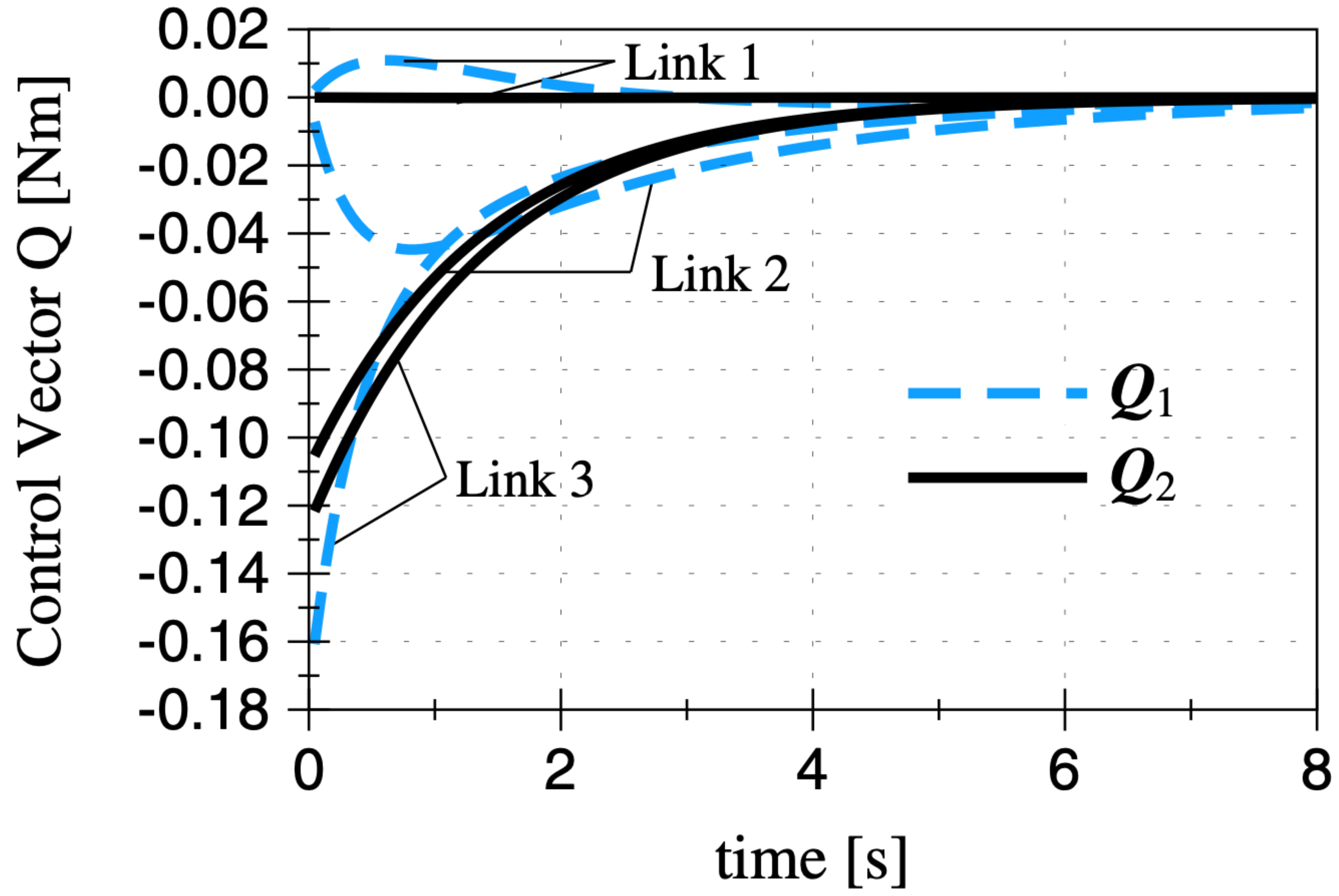
Symmetric, positive definite Mass Matrix:

$$M(\boldsymbol{q}) = \begin{bmatrix} (m_1+m_2+m_3)l_1^2 & (m_2+m_3)l_1l_2\cos(\theta_2-\theta_1) & m_3l_1l_3\cos(\theta_3-\theta_1) \\ (m_2+m_3)l_1l_2\cos(\theta_2-\theta_1) & (m_2+m_3)l_2^2 & m_3l_1l_3\cos(\theta_3-\theta_2) \\ m_3l_1l_3\cos(\theta_3-\theta_1) & m_3l_2l_3\cos(\theta_3-\theta_2) & m_3l_3^2 \end{bmatrix}$$

Simulation Parameters

Parameter	Value	Units
l_i	1	m
m_i	1.0	kg
P_1	1.0	kg-m^2/sec
P_2	0.72	kg-m^2/sec
$\boldsymbol{x}(t_0)$	$[-90\ 30\ 0]$	deg
$\dot{\boldsymbol{x}}(t_0)$	$[0.0\ 0.0\ 10]$	deg/sec





While both controls are asymptotically stabilizing, the 2nd control solution is actually exponentially stabilizing, and successfully isolates the motion of the third link from the first link.

Rigid Body Detumbling

State Vector: $\boldsymbol{\omega}$ Goal: $\boldsymbol{\omega} \rightarrow 0$

EOM: $$[I]\dot{\boldsymbol{\omega}} = -[\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} + \boldsymbol{Q}$$

External control torque

Lyapunov Function:

$$V(\boldsymbol{\omega}) = T = \frac{1}{2}\boldsymbol{\omega}^T[I]\boldsymbol{\omega}$$

Constant in Body frame components

Lyapunov Rate:

$$\dot{V} = \boldsymbol{\omega}^T([I]\dot{\boldsymbol{\omega}}) = \boldsymbol{\omega}^T(-[\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} + \boldsymbol{Q})$$

$$= \boldsymbol{\omega}^T\boldsymbol{Q}$$

Power form of work/energy equation

Control: $\boxed{\boldsymbol{Q} = -[P]\boldsymbol{\omega}}$ with $[P] = [P]^T > 0$

$$\dot{V}(\boldsymbol{\omega}) = -\boldsymbol{\omega}^T[P]\boldsymbol{\omega} < 0$$

Globally asymptotically stabilizing

Note: This control result does not require any knowledge of the inertia matrix! It is very robust to inertia modeling errors.

Reference: $\boldsymbol{\omega}_r$ Goal: $\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_r \rightarrow 0$

Note: $${}^{\mathcal{B}}\delta\boldsymbol{\omega} = {}^{\mathcal{B}}\boldsymbol{\omega} - [BR]\,{}^{\mathcal{R}}\boldsymbol{\omega}_r$$

Lyapunov Function: $$V(\delta\boldsymbol{\omega}) = \frac{1}{2}\delta\boldsymbol{\omega}^T[I]\delta\boldsymbol{\omega}$$

Lyapunov Rate: $$\dot{V} = \delta\boldsymbol{\omega}^T[I]\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{\omega})$$

Note: $$\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{\omega}) = \dot{\boldsymbol{\omega}} - \dot{\boldsymbol{\omega}}_r + \boldsymbol{\omega} \times \boldsymbol{\omega}_r$$

$$\dot{V} = \delta\boldsymbol{\omega}^T\left(-[\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} + [I]\boldsymbol{\omega} \times \boldsymbol{\omega}_r - [I]\dot{\boldsymbol{\omega}}_r + \boldsymbol{Q}\right)$$

Control:

$$\boxed{\boldsymbol{Q} = [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} - [I][\tilde{\boldsymbol{\omega}}]\boldsymbol{\omega}_r + [I]\dot{\boldsymbol{\omega}}_r - [P]\delta\omega}$$

$$\dot{V}(\delta\boldsymbol{\omega}) = -\delta\boldsymbol{\omega}^T[P]\delta\boldsymbol{\omega} < 0$$

Globally asymptotically stabilizing

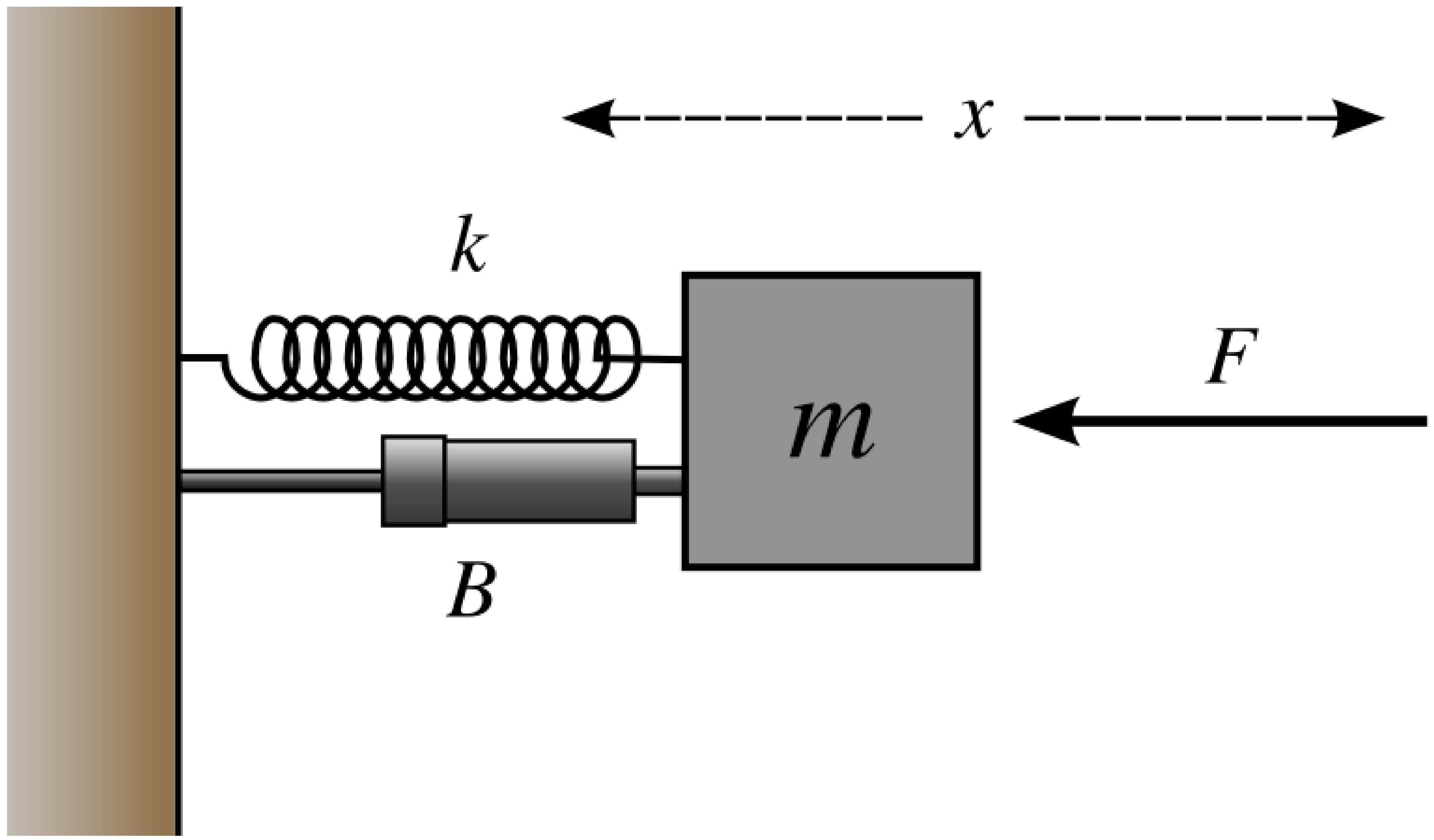
Elemental Position-Based Lyapunov Functions

Physical Motivation is the linear spring energy with stiffness k:

$$T = \frac{1}{2}kx^2$$

We will seek similar energy-like functions which provide positive definite error measures of the position-errors.

The state rate $\dot{x}$ is treated as a control variable in this discussion. This is typically the case in robotic control where a lower-level servo loop implements the required x.



Euler Angle Potential Functions

State Vector: $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^T$

Lyapunov Function: $V(\boldsymbol{\theta}) = \frac{1}{2}\boldsymbol{\theta}^T[K]\boldsymbol{\theta}$ with $[K] = [K]^T > 0$

Kinematic Differential Equations: $\dot{\boldsymbol{\theta}} = [B(\boldsymbol{\theta})]\boldsymbol{\omega}$

Lyapunov Rate: $\dot{V} = \boldsymbol{\omega}^T \left([B(\boldsymbol{\theta})]^T[K]\boldsymbol{\theta}\right)$

This function can later on be used in Lyapunov position and rate feedback law developments.

Attitude Tracking Error between *B* and reference frame *R*: $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^T$

Kinematic Differential Equations: $\dot{\boldsymbol{\theta}} = [B(\boldsymbol{\theta})]\delta\boldsymbol{\omega}$ with $\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_r$

Lyapunov Function: $V(\boldsymbol{\theta}) = \frac{1}{2}\boldsymbol{\theta}^T[K]\boldsymbol{\theta}$

There is no algebraic distinction between position based regulator and tracking Lyapunov functions, and we won’t distinguish the two from remainder of the position-based Lyapunov function discussion.

Classical Rodrigues Parameters

A brute force approach would define a candidate Lyapunov function as spring-mass energy-like form:

$$V(\boldsymbol{q}) = \boldsymbol{q}^T[K]\boldsymbol{q}$$

Taking the derivative we find

$$\dot{V} = \boldsymbol{\omega}^T\left(\left(I - [\tilde{\boldsymbol{q}}] + \boldsymbol{q}\boldsymbol{q}^T\right)[K]\boldsymbol{q}\right)$$

which gets reduced to

$$\dot{V} = \boldsymbol{\omega}^T\left(K\left(1 + q^2\right)\boldsymbol{q}\right)$$

if the gain matrix [*K*] is a scalar *K* and $q^2 = \boldsymbol{q}^T\boldsymbol{q}$

This term may lead to nonlinear feedback laws.

A more elegant Gibbs-vector Lyapunov function is given by:

$$V(\boldsymbol{q}) = K\ln\left(1 + \boldsymbol{q}^T\boldsymbol{q}\right)$$

Taking the derivative, and substituting the differential kinematic equations, a surprisingly simple form is found:

$$\boxed{\dot{V} = \boldsymbol{\omega}^T\left(K\boldsymbol{q}\right)}$$

leads to linear attitude feedback!

Let's develop an attitude servo law, we define

$$V(\boldsymbol{q}) = \ln\left(1 + \boldsymbol{q}^T\boldsymbol{q}\right) \quad \text{with} \quad \dot{V} = \boldsymbol{\omega}^T\boldsymbol{q}$$

The body rate control vector is then defined as

$$\boldsymbol{\omega} = -[K]\boldsymbol{q} \Rightarrow \dot{V}(\boldsymbol{q}) = -\boldsymbol{q}^T[K]\boldsymbol{q} < 0$$

Other Parameters

Modified Rodrigues Parameters:

Lyapunov Function:

$$V(\boldsymbol{\sigma}) = 2K \ln\left(1 + \boldsymbol{\sigma}^T \boldsymbol{\sigma}\right)$$

If you switch to the shadow MRP set on the $\sigma^2=1$ surface, then this Lyapunov function is continuous.

Lyapunov Rate:

$$\dot{V} = \boldsymbol{\omega}^T \left(K\boldsymbol{\sigma}\right)$$

This leads to elegant linear attitude feedback laws which are globally stabilizing by switching between the original and shadow MRP set.

Euler Parameters:

Ideal Attitude: $\hat{\boldsymbol{\beta}} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^T$

Lyapunov Function:

$$V(\boldsymbol{\beta}) = K\left(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}\right)^T \left(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}\right)$$

Lyapunov Rate:

$$\dot{V} = K\boldsymbol{\omega}^T [B(\boldsymbol{\beta})]^T \left(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}\right)$$

recall that $[B(\boldsymbol{\beta})]^T \boldsymbol{\beta} = 0$

This leads to

$$\dot{V} = K\boldsymbol{\omega}^T \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \boldsymbol{\omega}^T \left(K\boldsymbol{\epsilon}\right)$$

Note that will will stabilize the attitude to $\beta_0 = \pm 1$, which is the same attitude. However, no guarantee is made if the long or short rotational path is used.

Nonlinear Feedback

Finally, we look at the complete 3-axis control of spacecraft attitude...

Unconstrained Control

- First let us assume that the external control (thrusters) is unconstrained in magnitude, and that the thruster can point in any direction.

External torque
(atmospheric torque)

EOM: $$[I]\dot{\boldsymbol{\omega}} = -[\tilde{\omega}][I]\boldsymbol{\omega} + \boldsymbol{u} + \boldsymbol{L}$$

control vector
(thrusters)

Goal: $$\delta\boldsymbol{\omega} = \boldsymbol{\omega}_{B/R} = \boldsymbol{\omega} - \boldsymbol{\omega}_r \to 0$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{B/R} \to 0$$

angular velocity
error

body angular
velocity

reference
angular velocity

Attitude error between body
frame *B* and reference frame *R*
using MRPs

Exact attitude tracking error kinematic differential equations:

$$\dot{\boldsymbol{\sigma}} = \frac{1}{4}\left[(1-\sigma^2)I + 2[\tilde{\boldsymbol{\sigma}}] + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right]\delta\boldsymbol{\omega}$$

Lyapunov function definition:

$$V(\delta\boldsymbol{\omega}, \boldsymbol{\sigma}) = \frac{1}{2}\delta\boldsymbol{\omega}^T[I]\delta\boldsymbol{\omega} + 2K\ln\left(1+\boldsymbol{\sigma}^T\boldsymbol{\sigma}\right)$$

kinetic-energy-like p.d. function   p.d. MRP attitude error function

Note that the angular rate and inertia components are taken with respect to the body frame.

$$\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}([I]) = 0 \qquad \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{\omega})$$

To guarantee stability, we force $\dot{V}$ to be **negative semi-definite** by setting it equal to

$$\dot{V} = -\delta\boldsymbol{\omega}^T[P]\delta\boldsymbol{\omega}$$

$$[P] = [P]^T > 0$$

Differentiating V we find:

$$\dot{V} = \delta\boldsymbol{\omega}^T\left([I]\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{\omega}) + K\boldsymbol{\sigma}\right) = -\delta\boldsymbol{\omega}^T[P]\delta\boldsymbol{\omega}$$

$$\boxed{[I]\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{\omega}) + [P]\delta\boldsymbol{\omega} + K\boldsymbol{\sigma} = 0}$$

closed-loop dynamics

Using $\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{\omega}) = \dot{\boldsymbol{\omega}} - \dot{\boldsymbol{\omega}}_r + \boldsymbol{\omega}\times\boldsymbol{\omega}_r$ yields

$$\boxed{[I](\dot{\boldsymbol{\omega}}} - \dot{\boldsymbol{\omega}}_r + \boldsymbol{\omega}\times\boldsymbol{\omega}_r) + [P](\boldsymbol{\omega}-\boldsymbol{\omega}_r) + K\boldsymbol{\sigma} = 0$$

Substitute EOM: $\boxed{[I]\dot{\boldsymbol{\omega}}} = -[\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} + \boldsymbol{u} + \boldsymbol{L}$

$$\boldsymbol{u} = -K\boldsymbol{\sigma} - [P]\delta\boldsymbol{\omega} + [I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\boldsymbol{\omega}}]\boldsymbol{\omega}_r\right) + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} - \boldsymbol{L}$$

$${}^{\mathcal{B}}\boldsymbol{\omega}_r = [BR]\,{}^{\mathcal{R}}\boldsymbol{\omega}_r \qquad {}^{\mathcal{B}}\dot{\boldsymbol{\omega}}_r = [BR]\,{}^{\mathcal{R}}\dot{\boldsymbol{\omega}}_r$$

Global Stability?

The feedback law found is of the form

$$\boldsymbol{u} = -K\boldsymbol{\sigma} - [P]\delta\boldsymbol{\omega} + [I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\boldsymbol{\omega}}]\boldsymbol{\omega}_r\right) + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} - \boldsymbol{L}$$

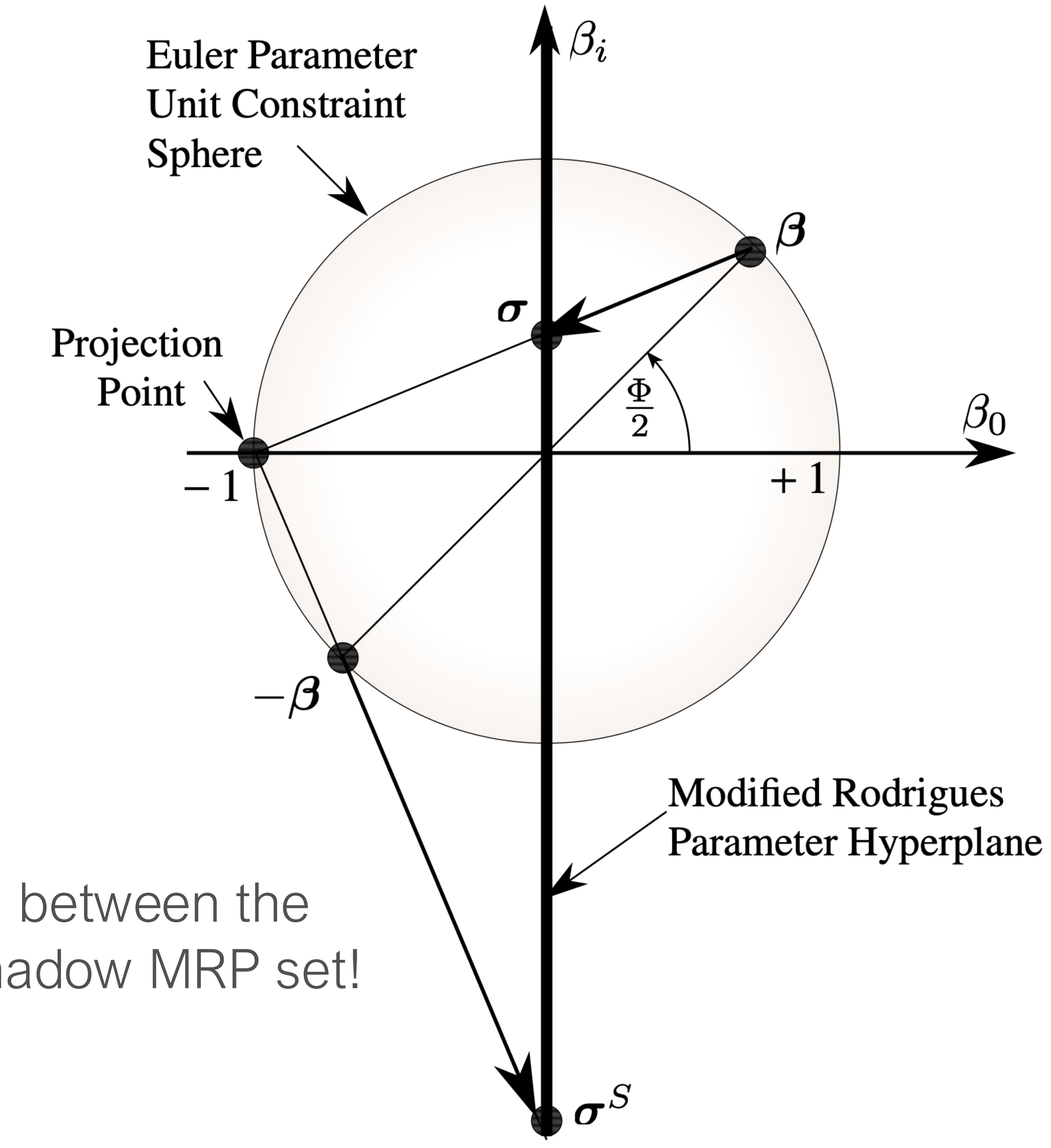
With the associated Lyapunov function being defined as:

$$V(\boldsymbol{\omega}, \boldsymbol{\sigma}) = \frac{1}{2}\delta\boldsymbol{\omega}^T[I]\delta\boldsymbol{\omega} + 2K\ln\left(1 + \boldsymbol{\sigma}^T\boldsymbol{\sigma}\right)$$

$$V \to \infty \quad \text{as} \quad \delta\boldsymbol{\omega}, \boldsymbol{\sigma} \to \infty$$

Globally Stabilizing

However, the MRP attitude can go singular? What if the body is tumbling and we make a 360º revolution?



We can switch between the original and shadow MRP set!

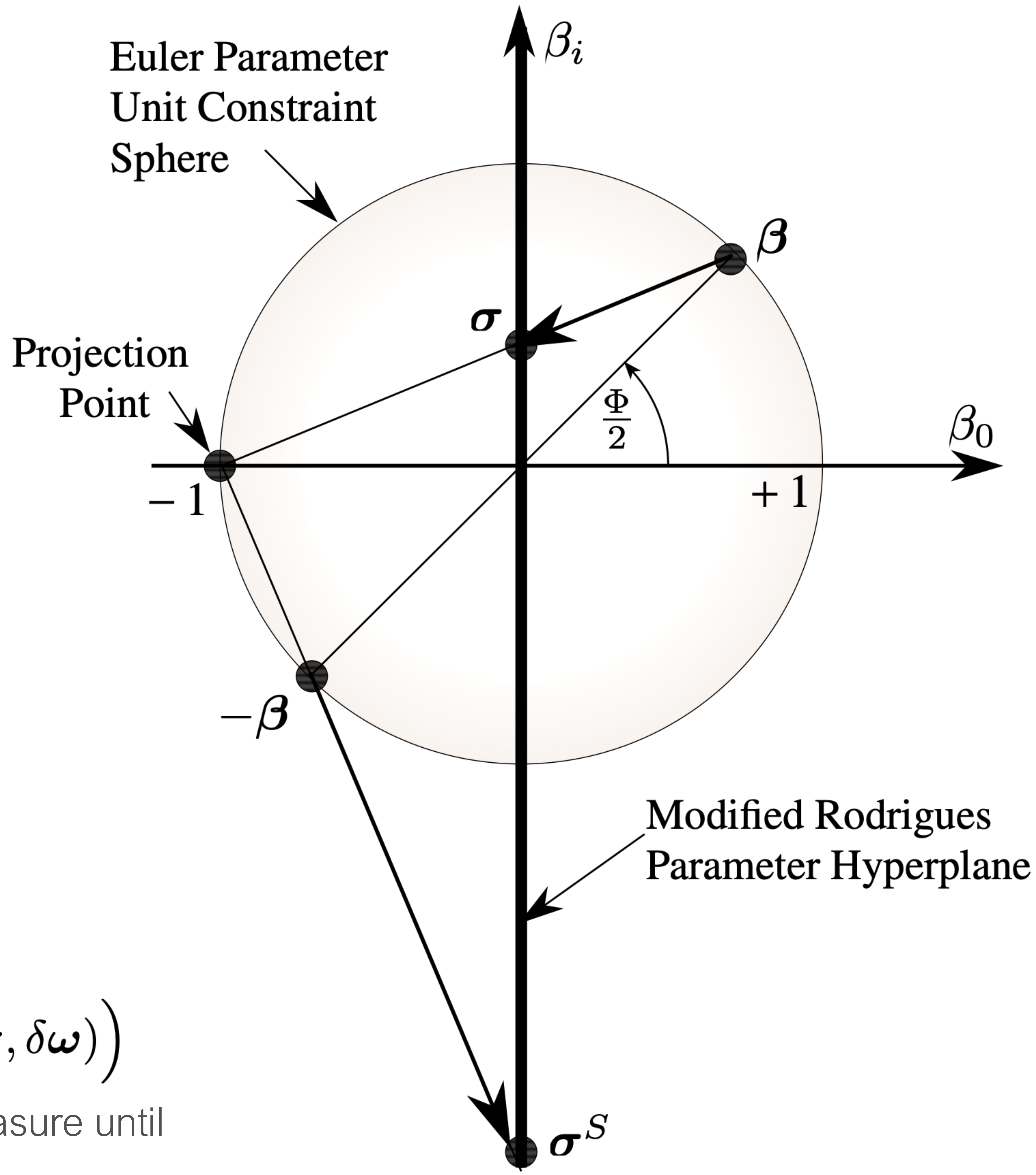
A convenient MRP switching surface is

$$\boldsymbol{\sigma}^T\boldsymbol{\sigma} = \sigma^2 = 1$$

where the body is “upside-down” relative to the reference attitude. The mapping to the shadow set is simply

$$\boldsymbol{\sigma}^S = -\boldsymbol{\sigma}$$

Note that the Lyapunov function *V* is *continuous* during this MRP switching with this switching surface!



Assume that V_1 is the Lyapunov tracking how the original state errors are being reduced.

$V_1(\boldsymbol{\sigma}, \delta\boldsymbol{\omega})$ is reduced until $|\boldsymbol{\sigma}| = 1$

MRP mapping: $\boldsymbol{\sigma}^S = -\boldsymbol{\sigma}$

New Lyapunov function: $V_2(\boldsymbol{\sigma}^S, \delta\boldsymbol{\omega}) \quad \left(= V_1(\boldsymbol{\sigma}, \delta\boldsymbol{\omega})\right)$

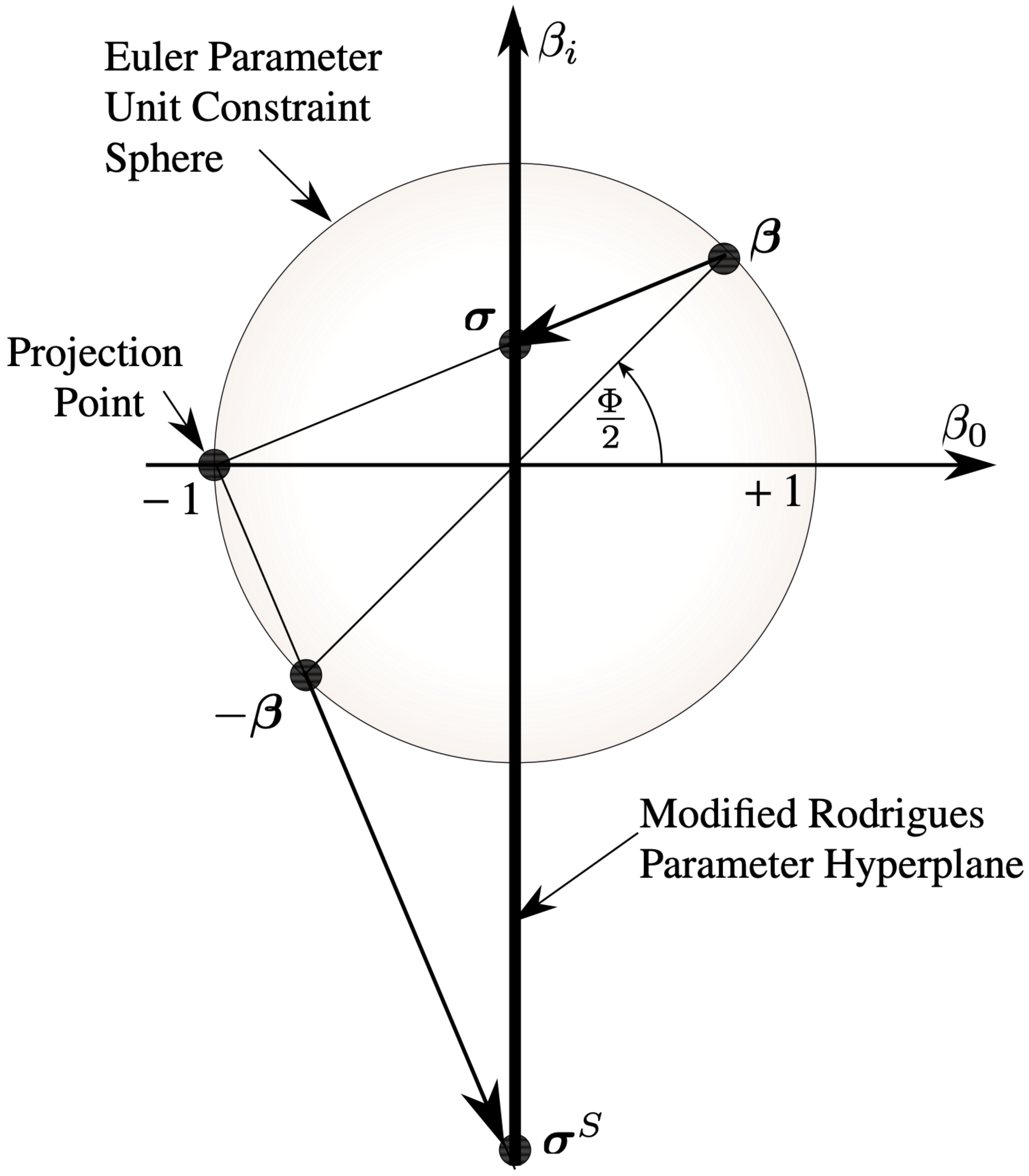
We can reset the stability analysis now to track this new error measure until either $|\boldsymbol{\sigma}^S|$ or $= 1$ $\quad |\boldsymbol{\sigma}^S| \rightarrow = 0$

Comment:

During the switching the *V* partial derivatives are well defined, but not continuous. Thus, we cannot use standard Lyapunov theory to argue global stability. However, by breaking up the problem into a series of every decreasing Lyapunov functions, we can argue that global stability will be achieved.

Application:

This MRP switching provides very elegant attitude feedback laws which are linear in MRP and will automatically de-tumble a body by always rotating it back to the reference attitude using the shortest rotation distance. Attitude error wind-up is avoided.



Tumbling Body Example:

Single-axis rotation with

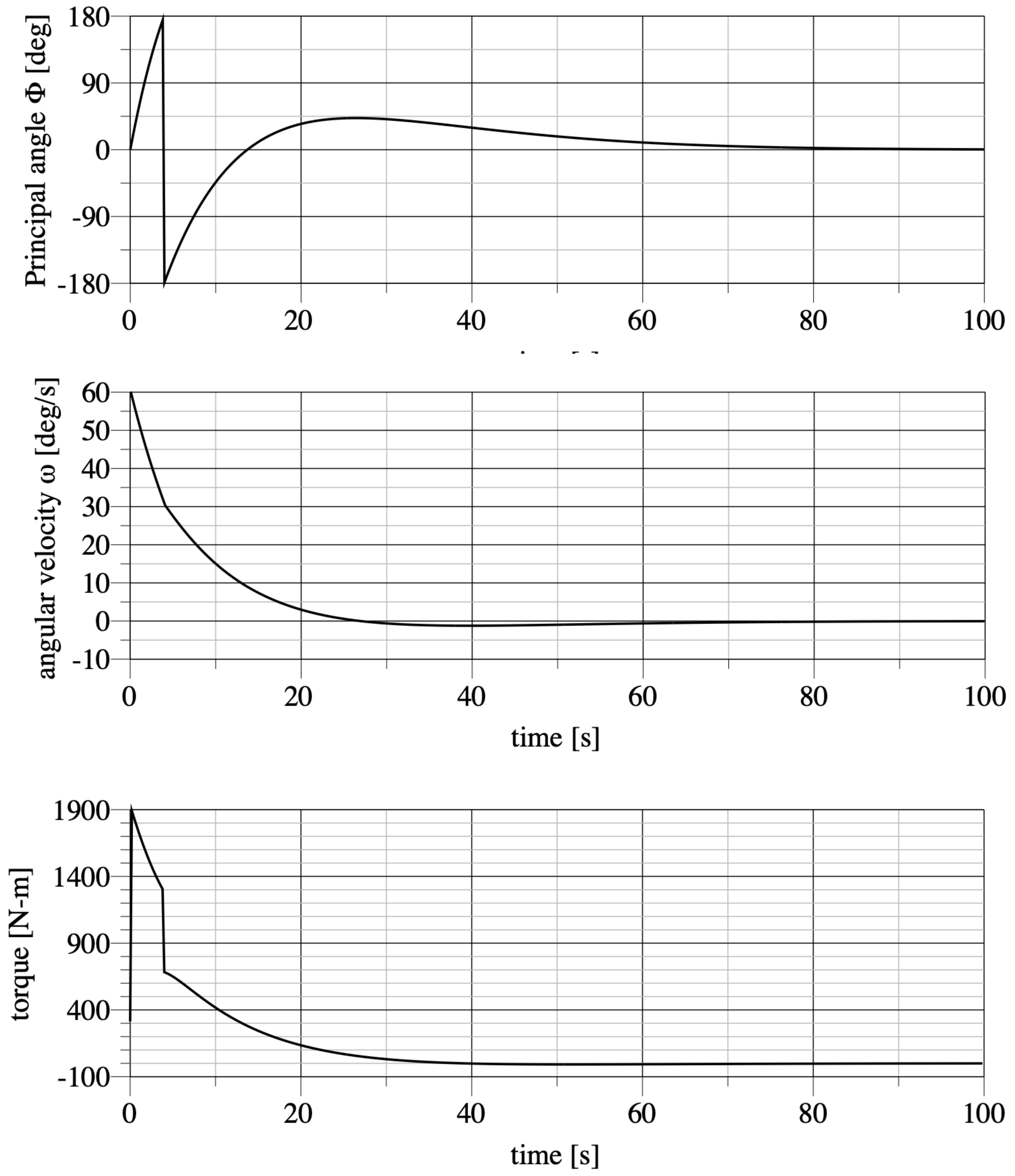
$$J = 12000 \text{ kg m}^2$$

$$K = 300$$

$$P = 1800$$

$$\omega = 60^{\mathrm{o}}/\mathrm{s}$$

Goal: Bring the body to rest at the zero attitude. (regulator problem)



Asymptotic Convergence

Note that the previous Lyapunov function only had a negative semi-definite derivative.

$$\dot{V}(\boldsymbol{\sigma}, \delta\boldsymbol{\omega}) = -\delta\boldsymbol{\omega}^T [P] \delta\boldsymbol{\omega}$$

$\Rightarrow$ globally stabilizing

Let us analyze this control to see when it is asymptotically stabilizing. We do so by investigating higher derivatives of *V*.

Note $\dot{V} = 0 \quad \Rightarrow \quad \Omega = \{\delta\boldsymbol{\omega} = 0\}$

2nd: $$\ddot{V} = -2\delta\boldsymbol{\omega}^T [P] \delta\boldsymbol{\omega}'$$

$$\ddot{V}(\boldsymbol{\sigma}, \delta\boldsymbol{\omega} = 0) = 0$$

3rd: $$\dddot{V} = 2\delta\boldsymbol{\omega}^T [P] \delta\boldsymbol{\omega}'' - 2\delta\boldsymbol{\omega}'^T [P] \delta\boldsymbol{\omega}'$$

Recall $$[I]\delta\boldsymbol{\omega}' + [P]\delta\boldsymbol{\omega} + K\boldsymbol{\sigma} = 0$$

closed-loop dynamics

$$\delta\boldsymbol{\omega}' = -[I]^{-1} \left([P]\delta\boldsymbol{\omega} + K\boldsymbol{\sigma}\right)$$

$$\delta\boldsymbol{\omega}' = -[I]^{-1} \, K\boldsymbol{\sigma} \qquad \text{if} \qquad \delta\boldsymbol{\omega} = 0$$

$\Downarrow$

$$\dddot{V}(\boldsymbol{\sigma}, \delta\boldsymbol{\omega} = 0) = -K^2 \boldsymbol{\sigma}^T \left([I]^{-1}\right) [P][I]^{-1} \boldsymbol{\sigma}$$

This 3rd derivative is negative definite in MRPs, and thus the control is asymptotically stabilizing.

External Torque Model Error

If some *un-modeled* external torque $\Delta \boldsymbol{L}$ is present, then the EOM are written as:

$$[I]\dot{\boldsymbol{\omega}} = -[\tilde{\omega}][I]\boldsymbol{\omega} + \boldsymbol{u} + \boldsymbol{L} + \Delta \boldsymbol{L}$$

The Lyapunov rate is now written as

$$\dot{V} = -\delta\boldsymbol{\omega}^T[P]\delta\boldsymbol{\omega} + \delta\boldsymbol{\omega}^T\Delta\boldsymbol{L}$$

This is no longer negative semi-definite! However, assume that the un-modeled torque vector is constant and bounded (as seen by body frame *B*).

However, this $\dot{V}$ does show that the $\delta\boldsymbol{\omega}$ errors will be bounded by the control. For large $\delta\boldsymbol{\omega}$ the quadratic term will dominate.

The new closed-loop EOM are:

$$[I]\delta\boldsymbol{\omega}' + [P]\delta\boldsymbol{\omega} + K\boldsymbol{\sigma} = \Delta\boldsymbol{L}$$

Differentiating using $\dot{\boldsymbol{\sigma}} = \frac{1}{4}[B(\boldsymbol{\sigma})]\delta\boldsymbol{\omega}$ yields:

$$[I]\delta\boldsymbol{\omega}'' + [P]\delta\boldsymbol{\omega}' + \frac{K}{4}[B(\boldsymbol{\sigma})]\delta\boldsymbol{\omega} = \Delta\boldsymbol{L}' \approx 0$$

This is a spring-mass-damper system with a nonlinear spring. To show that the stiffness matrix here is positive definite note that

$$\boldsymbol{\omega}^T[B(\boldsymbol{\sigma})]\boldsymbol{\omega} = \boldsymbol{\omega}^T\left[(1-\sigma^2)I + 2[\tilde{\boldsymbol{\sigma}}] + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right]\boldsymbol{\omega}$$

$$= (1-\sigma^2)\boldsymbol{\omega}^T\boldsymbol{\omega} + 2\left(\boldsymbol{\omega}^T\boldsymbol{\sigma}\right)^2 > 0$$

Note that we assume that the MRP vector is maintained to have a magnitude less than 1!

$$[I]\delta\boldsymbol{\omega}'' + [P]\delta\boldsymbol{\omega}' + \frac{K}{4}[B(\boldsymbol{\sigma})]\delta\boldsymbol{\omega} = 0$$

Because the closed-loop dynamics above are stable, the angular velocity tracking errors will reach a steady state value.

$$K[B(\boldsymbol{\sigma}_{ss})]\delta\boldsymbol{\omega}_{ss} = 0$$

However, the matrix $[B(\boldsymbol{\sigma}_{ss})]$ has been shown to be near-orthogonal and is always full-rank. This leads to

$$\delta\boldsymbol{\omega}_{ss} = 0$$

Using this result in the closed-loop dynamics:

$$[I]\delta\boldsymbol{\omega}' + [P]\delta\boldsymbol{\omega} + K\boldsymbol{\sigma} = \Delta\boldsymbol{L}$$

the attitude steady-state error is found to be

$$\boldsymbol{\sigma}_{ss} = \lim_{t\to\infty}\boldsymbol{\sigma} = \frac{1}{K}\Delta\boldsymbol{L}$$

The larger the attitude feedback gain *K* is, the smaller the steady-state attitude error will be.

Example:

A symmetric rigid body with all three principal inertias being 10 kg m^2 is studied. An unmodeled external torque is present.

$$\Delta \boldsymbol{L} = (0.05, 0, 10, -0.10)^T \text{ Nm}$$

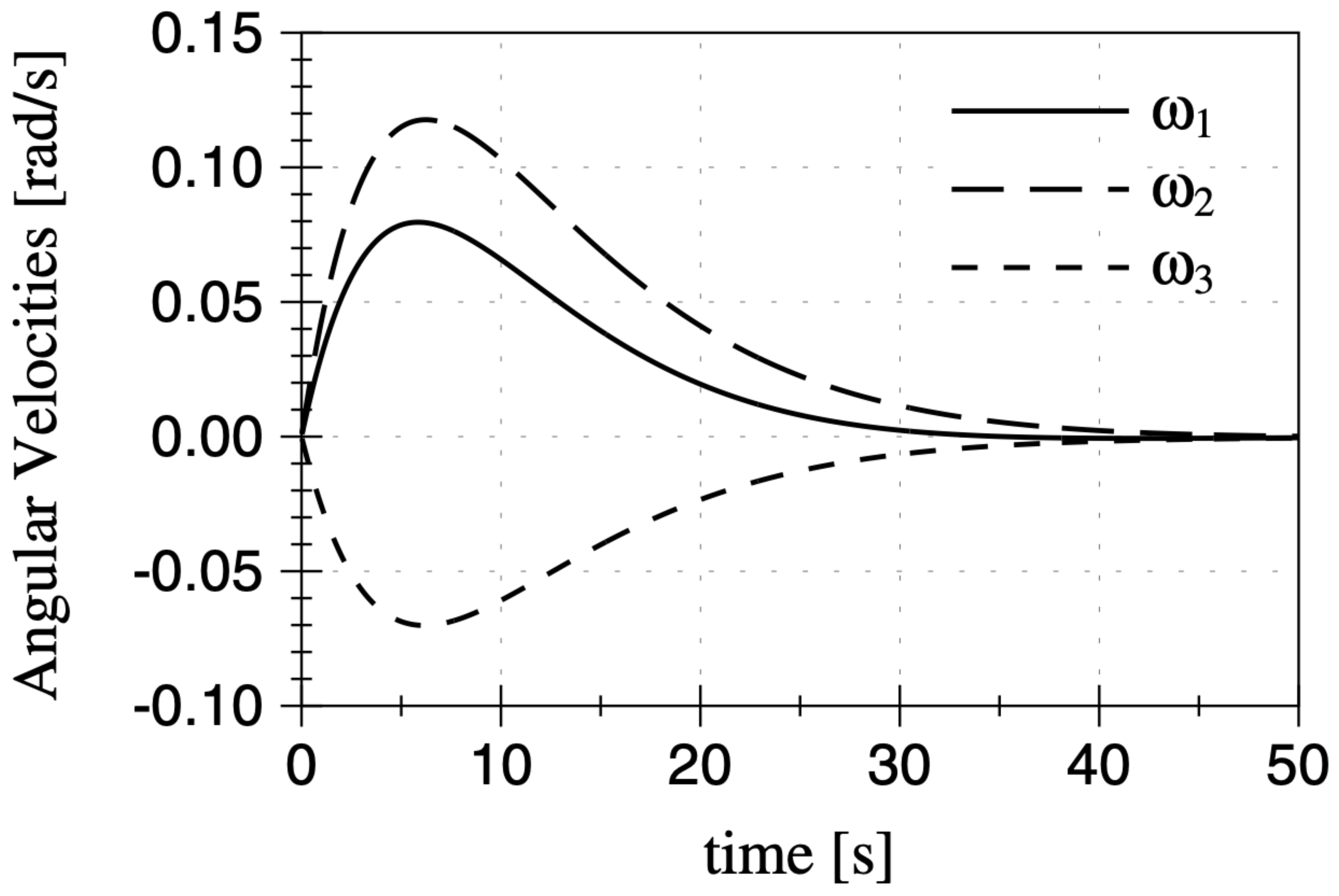
Goal: $\boldsymbol{\omega} \to 0 \qquad \boldsymbol{\sigma} \to 0$

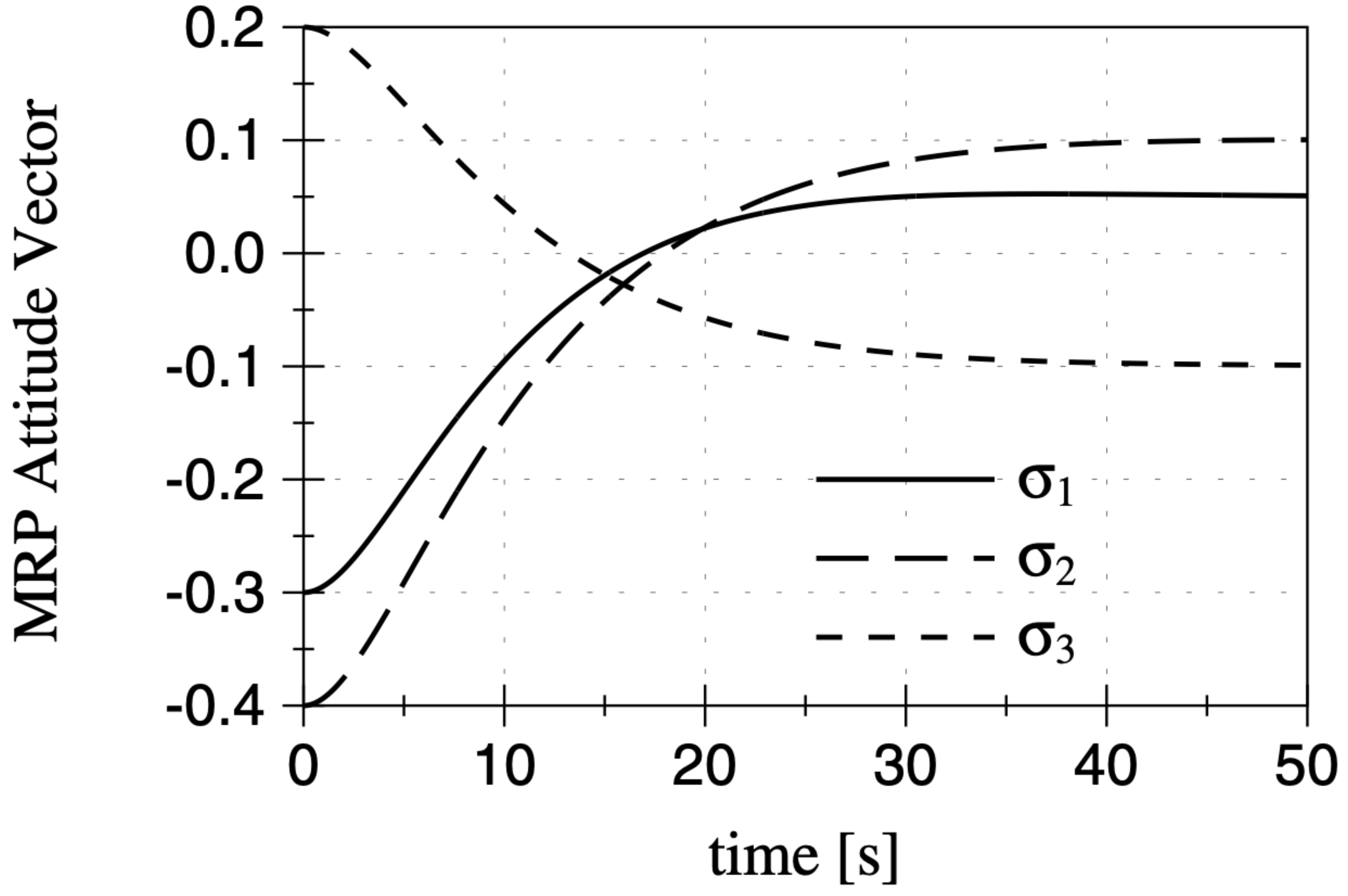
Initial Error:
$$\boldsymbol{\sigma}(t_0) = (-0.3, -0.4, 0.2)^T$$
$$\boldsymbol{\omega}(t_0) = (0.0, 0.0, 0.0)^T$$

Gains:
$$K = 1 \text{ kg m}^2/\text{s}^2$$
$$P = 3 \text{ kg m}^2/\text{s}$$

Predicted steady state errors:

$$\delta\boldsymbol{\omega}_{ss} = 0$$
$$\boldsymbol{\sigma}_{ss} = \frac{1}{K}\Delta\boldsymbol{L} = \begin{pmatrix} 0.05 \\ 0.10 \\ -0.10 \end{pmatrix}$$





Integral Feedback

- Next, let us investigate adding an integral feedback term to make the attitude control more robust to un-modeled external torques.

Let us introduce the new state vector $\boldsymbol{z}$:

$$\boldsymbol{z}(t) = \int_0^t \left(K\boldsymbol{\sigma} + [I]\delta\boldsymbol{\omega}'\right) \mathrm{d}t$$

Note that $\boldsymbol{z}$ will grow unbounded if there is any finite steady state attitude errors!

Thus, we want a new control law that will force $\boldsymbol{z}$ to go to zero, and thus drive any steady-state attitude errors to zero as well.

New Lyapunov function:

$$V(\delta\boldsymbol{\omega},\boldsymbol{\sigma},\boldsymbol{z}) = \frac{1}{2}\delta\boldsymbol{\omega}^T[I]\delta\boldsymbol{\omega} + 2K\log\left(1+\boldsymbol{\sigma}^T\boldsymbol{\sigma}\right) + \frac{1}{2}\boldsymbol{z}^T\boxed{[K_I]}\boldsymbol{z}$$

s.p.d.

Assume at first that there is no un-modeled external torque. In this case we set the Lyapunov rate equal to

$$\dot{V} = -\left(\delta\boldsymbol{\omega} + [K_I]\boldsymbol{z}\right)^T[P]\left(\delta\boldsymbol{\omega} + [K_I]\boldsymbol{z}\right)$$

and solve for the control vector $\boldsymbol{u}$:

$$\boldsymbol{u} = -K\boldsymbol{\sigma} - [P]\delta\boldsymbol{\omega} - [P][K_I]\boldsymbol{z} + [I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\boldsymbol{\omega}}]\boldsymbol{\omega}_r\right) + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} - \boldsymbol{L}$$

Measuring $\boldsymbol{z}$ direction is not convenient because it required the derivative of $\delta\boldsymbol{\omega}$. Instead, note that we can write

$$\boldsymbol{z}(t) = K\int_0^t \boldsymbol{\sigma}\mathrm{d}t + [I]\left(\delta\boldsymbol{\omega} - \delta\boldsymbol{\omega}_0\right)$$

This allows us to re-write the feedback control law in the final form:

$$\boxed{\begin{aligned}\boldsymbol{u} = &- K\boldsymbol{\sigma} - ([P] + [P][K_I][I])\,\delta\boldsymbol{\omega} \\ &- K[P][K_I]\int_0^t \boldsymbol{\sigma}\mathrm{d}t + [P][K_I][I]\delta\boldsymbol{\omega}_0 \\ &+ [I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\omega}]\boldsymbol{\omega}_r\right) + [\tilde{\omega}][I]\boldsymbol{\omega} - \boldsymbol{L}\end{aligned}}$$

Next, let's analyze the stability of this control law. The Lyapunov rate is semi-negative definite

$$\dot{V} = -\left(\delta\boldsymbol{\omega} + [K_I]\boldsymbol{z}\right)^T [P]\left(\delta\boldsymbol{\omega} + [K_I]\boldsymbol{z}\right)$$

This guarantees that **ω**, **σ**, and ***z*** are stable, and that

$$\delta\boldsymbol{\omega} + [K_I]\boldsymbol{z} \rightarrow 0$$

To study asymptotic convergence, we investigate the higher order derivative of the Lyapunov function *V*.

The first non-zero higher derivative evaluated on the set where $\dot{V}$ is zero is

$$\dddot{V}(\boldsymbol{\sigma}, \delta\boldsymbol{\omega}+[K_I]\boldsymbol{z}=0) = -K^2\boldsymbol{\sigma}^T\left([I]^{-1}\right)[P][I]\boldsymbol{\sigma}$$

$$\Rightarrow \quad \boldsymbol{\sigma} \rightarrow 0 \quad \Rightarrow \quad \delta\boldsymbol{\omega} \rightarrow 0$$

Kinematic Relationship

$$\delta\boldsymbol{\omega} + [K_I]\boldsymbol{z} \rightarrow 0 \quad \Rightarrow \quad \boldsymbol{z} \rightarrow 0$$

If unmodeled external torques are included, then the Lyapunov rate is expressed as:

$$\dot{V} = -\left(\delta\boldsymbol{\omega}+[K_I]\boldsymbol{z}\right)^T\left([P](\delta\boldsymbol{\omega}+[K_I]\boldsymbol{z}) - \Delta\boldsymbol{L}\right)$$

This is no longer n.s.d. However, we can conclude for bounded Δ***L*** the states δ**ω** and ***z*** must remain bounded.

Recall $$\boldsymbol{z}(t) = \int_0^t \left(K\boldsymbol{\sigma} + [I]\delta\dot{\boldsymbol{\omega}}\right)\mathrm{d}t$$

$$\Rightarrow \quad \boldsymbol{\sigma} \rightarrow 0 \quad \Rightarrow \quad \delta\boldsymbol{\omega} \rightarrow 0$$

Next, let's study the state ***z*** as the Lyapunov rate approaches zero at steady-state. This requires that

$$\lim_{t\to\infty}\left([P]\left(\delta\boldsymbol{\omega}+[K_I]\boldsymbol{z}\right)-\Delta\boldsymbol{L}\right)=0$$

Because $\delta\boldsymbol{\omega}\rightarrow 0$, the steady-state value of ***z*** is expressed as:

$$\boxed{\lim_{t\to\infty}\boldsymbol{z}=[K_I]^{-1}[P]^{-1}\Delta\boldsymbol{L}}$$

Example:

A symmetric rigid body with all three principal inertias being 10 kg m^2 is studied. An unmodeled external torque is present.

$$\Delta \boldsymbol{L} = (0.05, 0, 10, -0.10)^T \text{ Nm}$$

Goal: $\boldsymbol{\omega} \rightarrow 0 \qquad \boldsymbol{\sigma} \rightarrow 0$

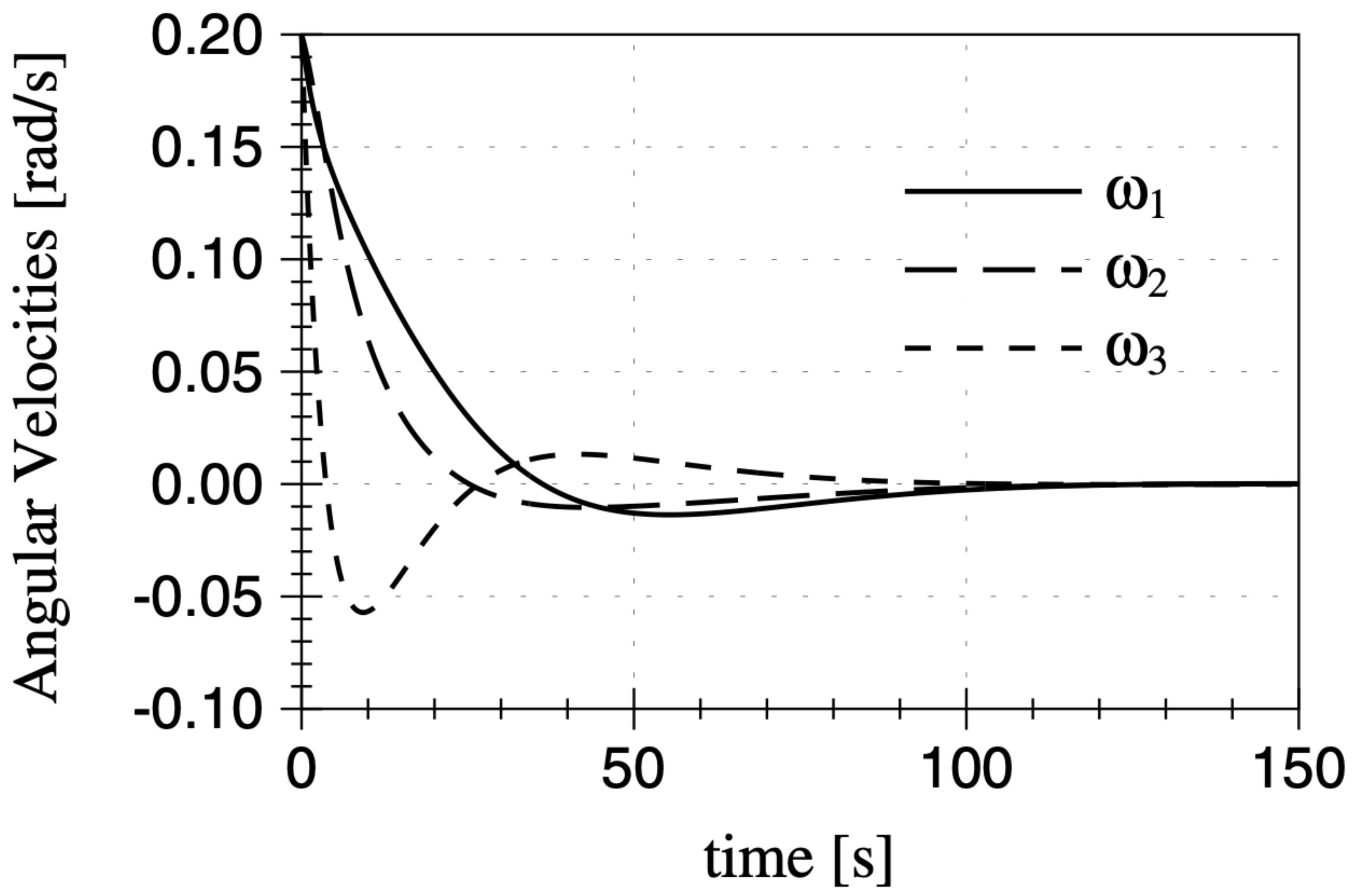
Initial Error:
$$\boldsymbol{\sigma}(t_0) = (-0.3, -0.4, 0.2)^T$$
$$\boldsymbol{\omega}(t_0) = (0.2, 0.2, 0.2)^T \text{ rad/s}$$

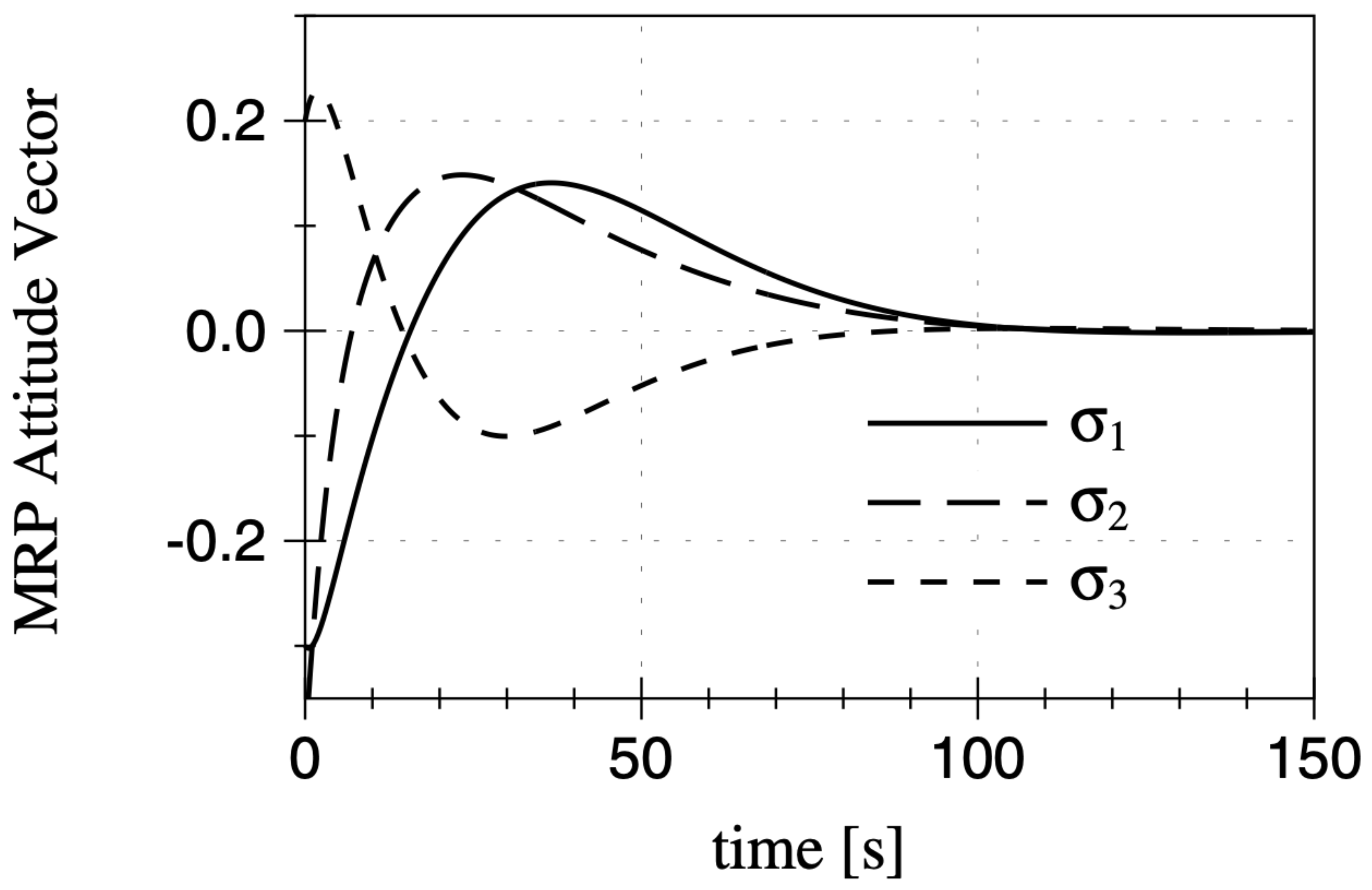
Gains:
$$K = 1 \text{ kg m}^2/\text{s}^2$$
$$P = 3 \text{ kg m}^2/\text{s}$$
$$K_I = 0.01 \text{ s}^{-1}$$

Predicted steady state errors:

$$\delta\boldsymbol{\omega}_{ss} = 0 \qquad \boldsymbol{\sigma}_{ss} = 0$$

$$\lim_{t\rightarrow\infty} \boldsymbol{z} = \frac{\Delta \boldsymbol{L}}{K_I P} = \begin{pmatrix} 1.66 \\ 3.33 \\ -3.33 \end{pmatrix} \text{kg-m}^2/\text{sec}$$





Example:

A symmetric rigid body with all three principal inertias being 10 kg m^2 is studied. An unmodeled external torque is present.

$$\Delta \boldsymbol{L} = (0.05, 0, 10, -0.10)^T \text{ Nm}$$

Goal: $\boldsymbol{\omega} \to 0$ $\quad\quad$ $\boldsymbol{\sigma} \to 0$

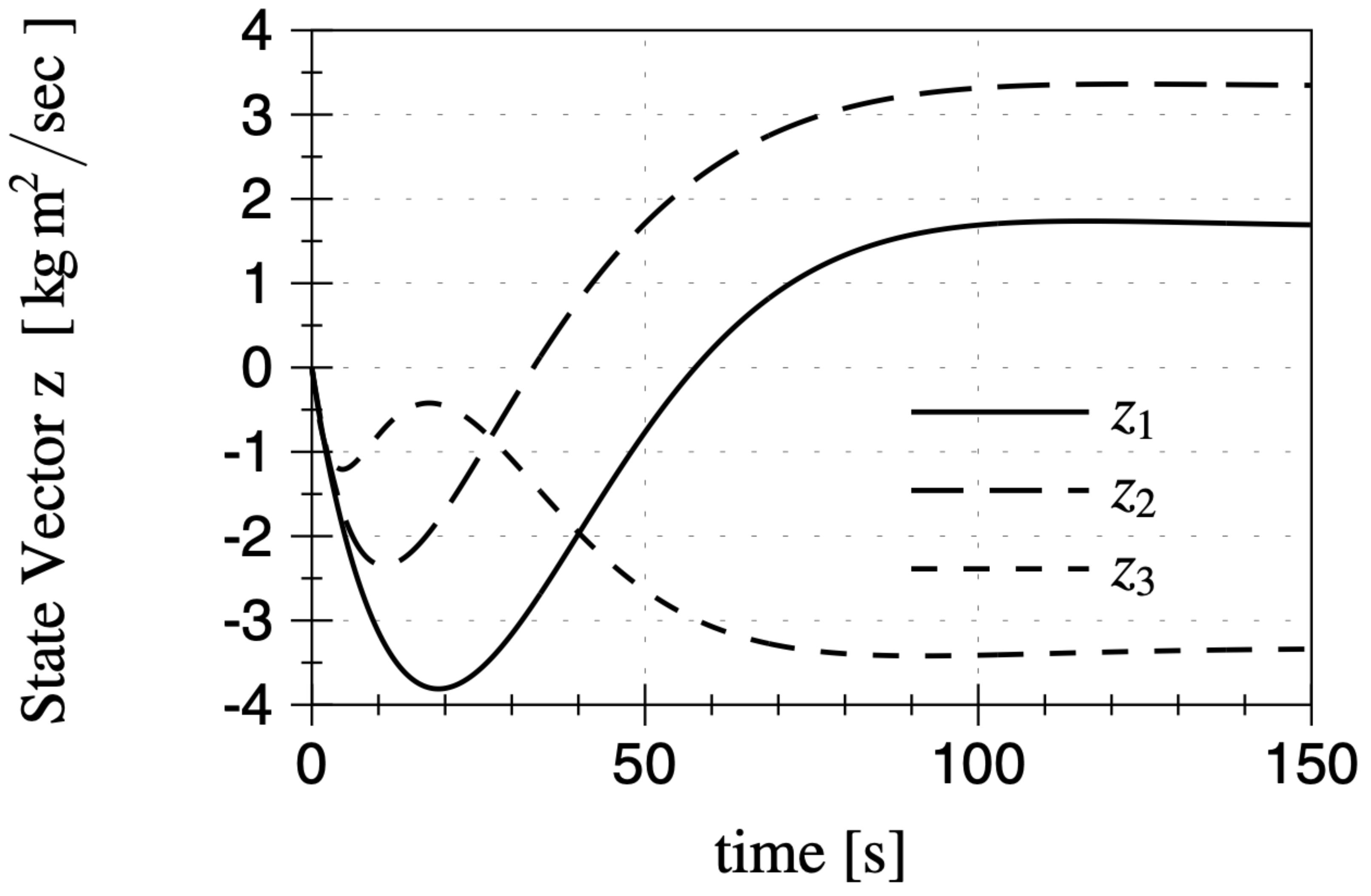
Initial Error:
$$\boldsymbol{\sigma}(t_0) = (-0.3, -0.4, 0.2)^T$$
$$\boldsymbol{\omega}(t_0) = (0.2, 0.2, 0.2)^T \text{ rad/s}$$

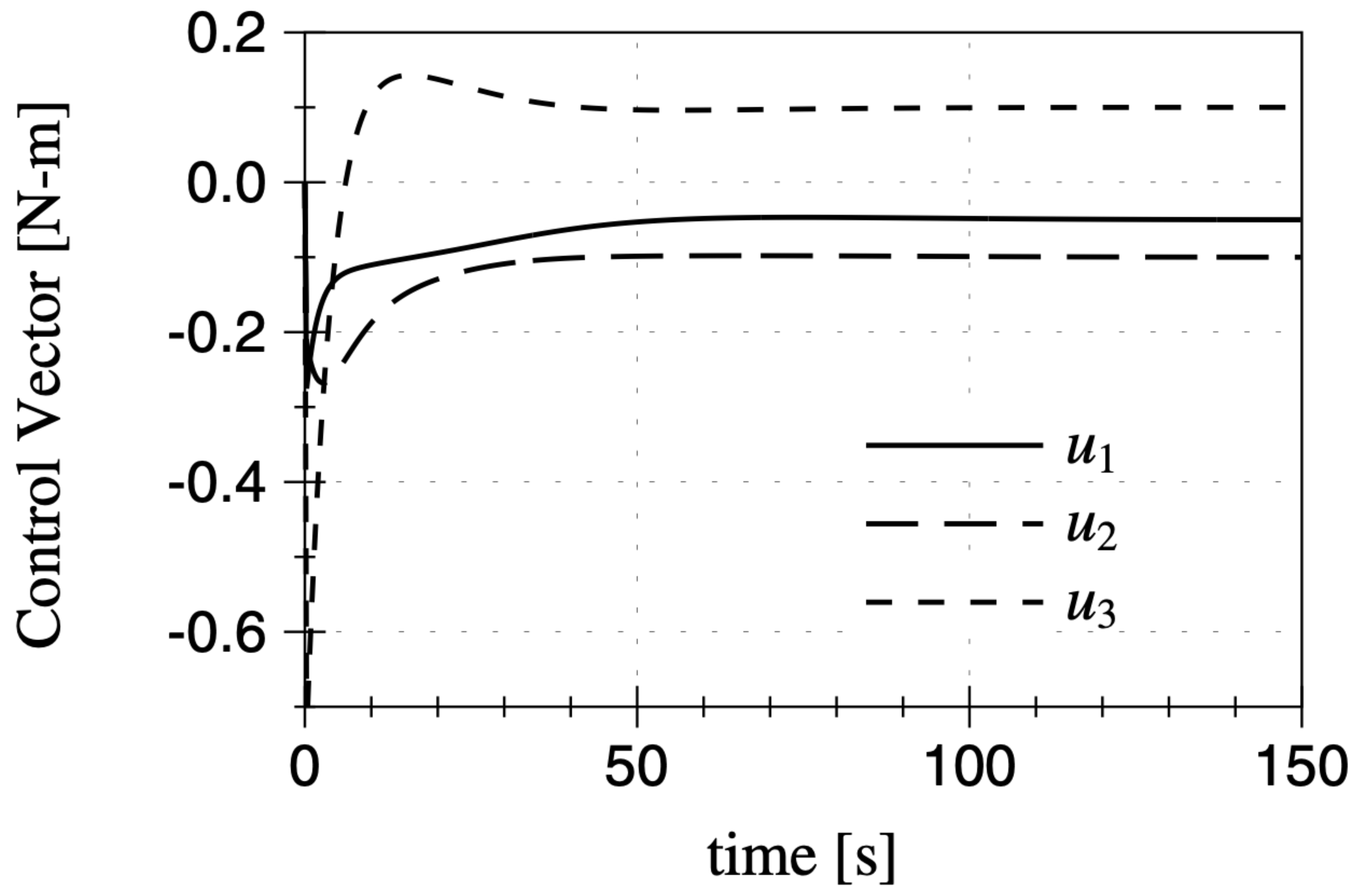
Gains:
$$K = 1 \text{ kg m}^2/\text{s}^2$$
$$P = 3 \text{ kg m}^2/\text{s}$$
$$K_I = 0.01 \text{ s}^{-1}$$

Predicted steady state errors:

$$\delta\boldsymbol{\omega}_{ss} = 0 \quad\quad \boldsymbol{\sigma}_{ss} = 0$$

$$\lim_{t\to\infty} \boldsymbol{z} = \frac{1}{K_I I}\Delta \boldsymbol{L} = \begin{pmatrix} 1.66 \\ 3.33 \\ -3.33 \end{pmatrix} \text{kg-m}^2/\text{sec}$$





Feedback Gain Selection

- Lyapunov theory is great to develop globally stabilizing nonlinear feedback control law. However, how does one select the feedback gains to get good performance?

Consider the “PD-like” nonlinear feedback control law:

$$\boldsymbol{u} = -K\boldsymbol{\sigma} - [P]\delta\boldsymbol{\omega} + [I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\boldsymbol{\omega}}]\boldsymbol{\omega}_r\right) + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} - \boldsymbol{L}$$

Without external torque modeling errors, the closed-loop dynamics are written as:

$$[I]\delta\boldsymbol{\omega}' + [P]\delta\boldsymbol{\omega} + K\boldsymbol{\sigma} = 0$$

Linear in $\boldsymbol{\sigma}$ thanks to the use of MRP

Differential kinematic eqn: $$\dot{\boldsymbol{\sigma}} = \frac{1}{4}[B(\boldsymbol{\sigma})]\delta\boldsymbol{\omega}$$

Let’s write the tracking error state vector $\boldsymbol{x}$ as:

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{\sigma} \\ \delta\boldsymbol{\omega} \end{pmatrix}$$

Nonlinear state-space formulation:

$$\dot{\boldsymbol{x}} = \begin{pmatrix} \dot{\boldsymbol{\sigma}} \\ \delta\boldsymbol{\omega}' \end{pmatrix} = \begin{bmatrix} 0 & \boxed{\frac{1}{4}B(\boldsymbol{\sigma})} \\ -K[I]^{-1} & -[I]^{-1}[P] \end{bmatrix} \begin{pmatrix} \boldsymbol{\sigma} \\ \delta\boldsymbol{\omega} \end{pmatrix}$$

Approximate differential kinematic equations: $$\dot{\boldsymbol{\sigma}} \simeq \frac{1}{4}\delta\boldsymbol{\omega}$$

$$\begin{pmatrix} \dot{\boldsymbol{\sigma}} \\ \delta\boldsymbol{\omega}' \end{pmatrix} = \begin{bmatrix} 0 & \frac{1}{4}I \\ -K[I]^{-1} & -[I]^{-1}[P] \end{bmatrix} \begin{pmatrix} \boldsymbol{\sigma} \\ \delta\boldsymbol{\omega} \end{pmatrix}$$

Linear state-space form

If a principal body coordinate frame is chosen, then the inertia matrix is diagonal and the linearized tracking error simplify to:

$$\begin{pmatrix} \dot{\sigma}_i \\ \delta\dot{\omega}_i \end{pmatrix} = \begin{bmatrix} 0 & \frac{1}{4} \\ -\frac{K}{I_i} & -\frac{P_i}{I_i} \end{bmatrix} \begin{pmatrix} \sigma_i \\ \delta\omega_i \end{pmatrix} \qquad i = 1, 2, 3$$

3 uncoupled differential equations

The roots of the corresponding characteristic equation are expressed as

$$\lambda_i = -\frac{1}{2I_i}\left(P_i \pm \sqrt{-KI_i + P_i{}^2}\right) \qquad i = 1, 2, 3$$

The feedback gains P_i and K can now be chosen such that the system is either under-damped (complex roots), critically damped (double real root), or over-damped (two unique real roots).

Let's consider an under-damped response.

$$\omega_{n_i} = \frac{\sqrt{KI_i}}{2I_i}$$ natural frequency

$$\xi_i = \frac{P_i}{\sqrt{KI_i}}$$ damping ratio

$$T_i = \frac{2I_i}{P_i}$$ Time decay constant

$$\omega_{d_i} = \frac{1}{2I_i}\sqrt{KI_i - P_i^2}$$ damped natural frequency

Feedback Gain Selection Example:

Parameter	Value	Units
I_1	140.0	kg-m^2
I_2	100.0	kg-m^2
I_3	80.0	kg-m^2
$\boldsymbol{\sigma}(t_0)$	[0.60 − 0.40 0.20]	
$\boldsymbol{\omega}(t_0)$	[0.70 0.20 − 0.15]	rad/sec
$[P]$	[18.67 2.67 10.67]	kg-m^2/sec
K	7.11	kg-m^2/sec^2

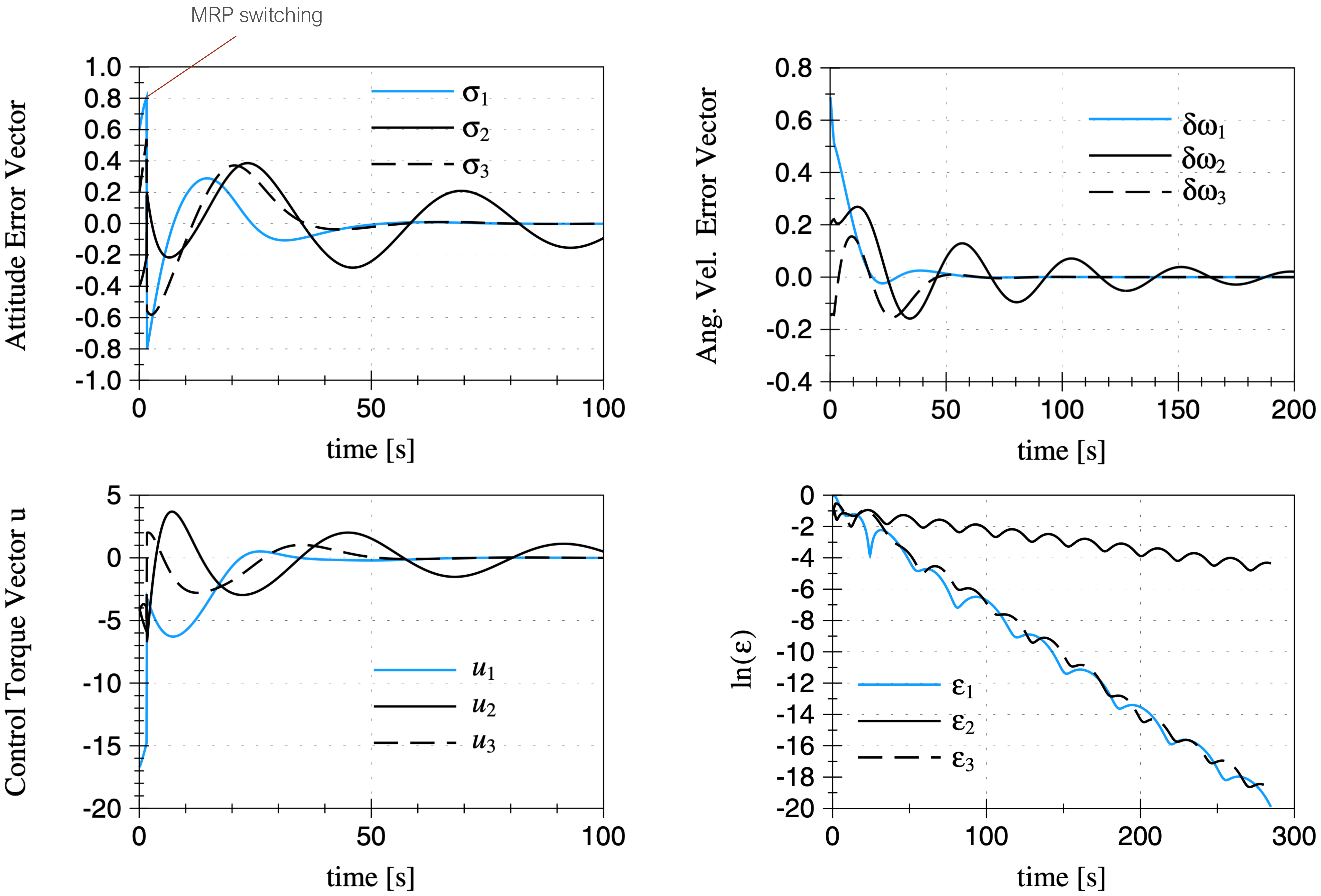
A principal body coordinate frame is chosen which diagonalizes the inertia matrix.

Large initial errors will cause the body to tumble "up-side-down".

Let us define the new state ε_i to track the 3 decoupled linearized tracking error dynamics.

$$\epsilon_i = \sqrt{\sigma_i^2 + \omega_i^2} \qquad i = 1, 2, 3$$

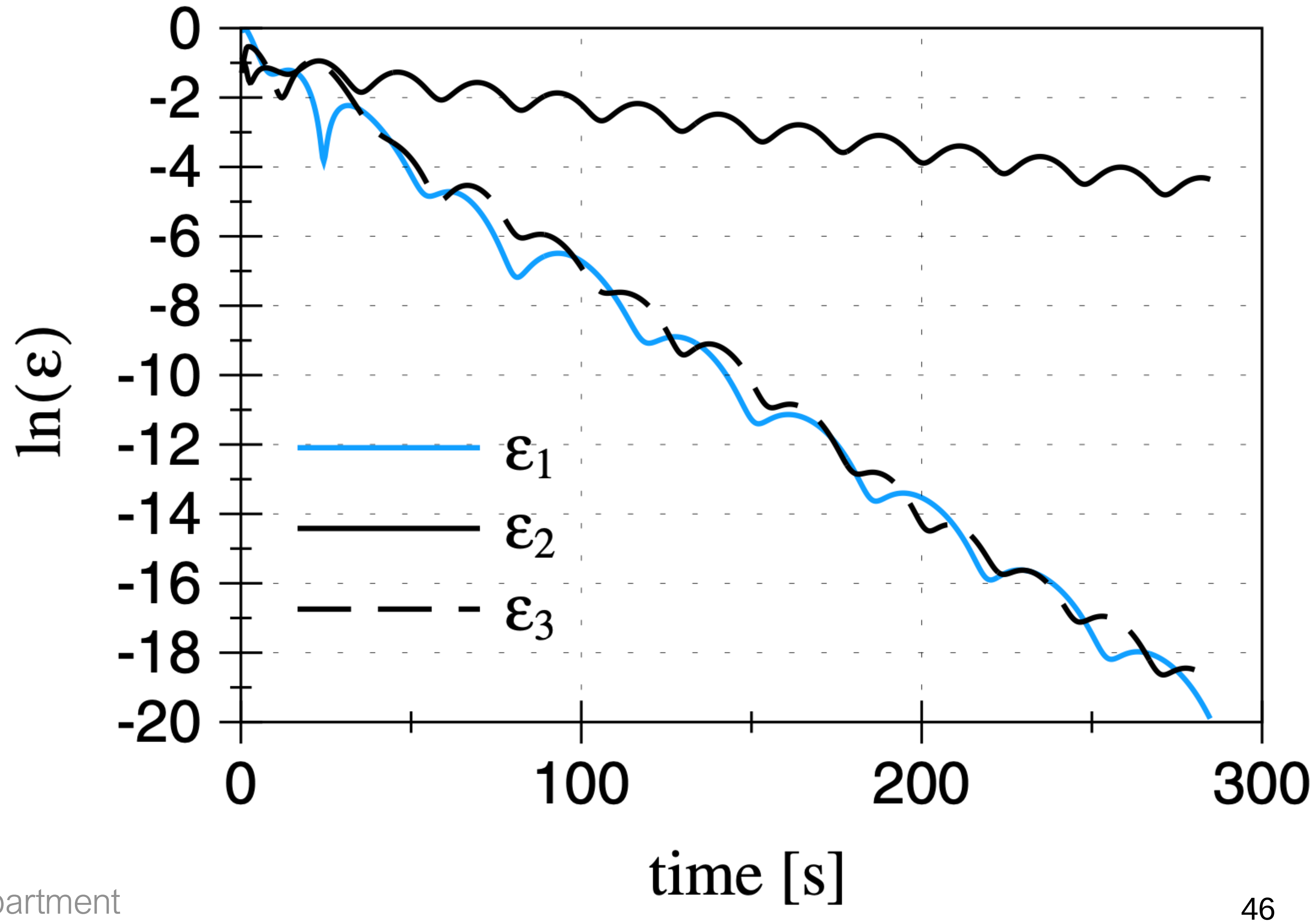
We can then evaluate this state in the nonlinear simulation, and compare to the predicted decay rate of the linearized analysis.


MRP switching
Attitude Error Vector
σ1
σ2
σ3
time [s]
Ang. Vel. Error Vector
δω1
δω2
δω3
time [s]
Control Torque Vector u
u1
u2
u3
time [s]
ln(ε)
ε1
ε2
ε3
time [s]

Parameter	Actual Average	Predicted Value	Percent Difference
T_1	14.71 s	15.00 s	1.97%
T_2	76.92 s	75.00 s	-2.50%
T_3	14.71 s	15.00 s	1.97%
ω_{d_1}	0.0938 rad/s	0.0909 rad/s	-3.12%
ω_{d_2}	0.1326 rad/s	0.1326 rad/s	0.08%
ω_{d_3}	0.1343 rad/s	0.1333 rad/s	-0.74%

This table compares the actual, nonlinear response to that of the linearized prediction of the gain selection method.

Because the MRP behave very linearly, the predicted tracking error dynamics matches the nonlinear motion very well, even though the is tumbling up-side-down!



Lyapunov Optimal Feedback

What if your thrusters are just too wimpy?

Stabilization of General System

Generalized Coordinates: $(q_i, \dot{q}_i)$

Goal: $\dot{q}_i \rightarrow 0$

EOM:

$$[M(\boldsymbol{q})]\ddot{\boldsymbol{q}} = -[\dot{M}(\boldsymbol{q}, \dot{\boldsymbol{q}})]\dot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T[M_{\boldsymbol{q}}(\boldsymbol{q})]\dot{\boldsymbol{q}} + \boldsymbol{Q}$$

If we use the kinetic energy T as the Lyapunov function of this system, then we find:

$$V(\dot{\boldsymbol{q}}) = T = \frac{1}{2}\dot{\boldsymbol{q}}^T[M(\boldsymbol{q})]\dot{\boldsymbol{q}}$$

Using the work/energy relationship, we can write

$$\dot{V} = \sum_{i=1}^{N} \dot{q}_i Q_i$$

Setting the control equal to $Q_i = -K_i \dot{q}_i$

yields $$\dot{V} = \sum_{i=1}^{N} -K_i \dot{q}_i^2 < 0$$

Next, what if the control authority magnitude is limited?

1st approach:

We could reduce our feedback gains K_i such that

$$|Q_i| = |K_i \dot{q}_i| \leq Q_{i_{\max}}$$

is true for all state errors $\dot{q}_i$ considered.

$\Downarrow$

reduced control performance!

2nd approach:

We would like to be able to handle control saturation without sacrificing performance (as much) throughout the maneuver!

Let us treat the saturated control problem as an optimization problem. For stability, we require the Lyapunov rate to be negative semi-definite:

$$\dot{V} \leq 0$$

Thus, we define a cost function which is equal to this Lyapunov rate, and aim to minimize it!

$$\boxed{J = \dot{V} = \sum_{i=1}^{n} \dot{q}_i Q_i}$$

We define a ***Lyapunov optimal control law*** to be one that minimizes the Lyapunov rate function.

Given a limited amount of control authority, the Lyapunov optimal rate control is simply

$$Q_i = -Q_{i_{\max}} \operatorname{sgn}(\dot{q}_i)$$

which yields

$$J = \dot{V} = \sum_{i=1}^{N} -Q_{i_{\max}} \dot{q}_i \operatorname{sgn}(\dot{q}_i)$$

Note that this direct implementation will have chatter issues around the target state values.

The following control is only Lyapunov optimal during saturation periods, but avoids the zero crossing chatter.

$$Q_i = \begin{cases} -K_i \dot{q}_i & \text{for } |K_i \dot{q}_i| \leq Q_{i_{\max}} \\ -Q_{i_{\max}} \operatorname{sgn}(\dot{q}_i) & \text{for } |K_i \dot{q}_i| > Q_{i_{\max}} \end{cases}$$

Note that individual control components can be saturated, while others are not!

Saturated Attitude Control

- Next we study attitude control laws when the external control torque is saturated in one or more of its components.

Case 1: Attitude Tracking Problem

Un-saturated control found previously:

$$\boldsymbol{u}_{\text{us}} = -K\boldsymbol{\sigma} - [P]\delta\boldsymbol{\omega} + [I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\boldsymbol{\omega}}]\boldsymbol{\omega}_r\right) + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} - \boldsymbol{L}$$

Corresponding Lyapunov rate:

$$\dot{V} = \delta\boldsymbol{\omega}^T\left(-[\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} + \boldsymbol{u} - [I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\boldsymbol{\omega}}]\boldsymbol{\omega}_r\right) + K\boldsymbol{\sigma} + \boldsymbol{L}\right)$$

Lyapunov optimal saturated control strategy:

$$u_i = \begin{cases} u_{\text{us}_i} & \text{for } |u_{\text{us}_i}| \le u_{\text{max}_i} \\ u_{\text{max}_i} \cdot \text{sgn}(u_{\text{us}_i}) & \text{for } |u_{\text{us}_i}| > u_{\text{max}_i} \end{cases}$$

Conservative stability boundary (sufficient condition):

$$\left|\left([I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\boldsymbol{\omega}}]\boldsymbol{\omega}_r\right) + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} - K\boldsymbol{\sigma} - \boldsymbol{L}\right)_i\right| \le u_{\text{max}_i}$$

If this is violated, we don't necessarily have instability!

Case 2: Attitude Regulator Problem

In this case the unsaturated control torque on the previous slide simplifies to:

$$\boldsymbol{u}_{\text{us}} = -K\boldsymbol{\sigma} - [P]\boldsymbol{\omega}$$

while the Lyapunov rate expression reduces to:

$$\dot{V} = \boldsymbol{\omega}^T \left(\boldsymbol{u} + K\boldsymbol{\sigma}\right)$$

A conservative stability boundary which guarantees that $\dot{V} \leq 0$ is

$$K|\sigma_i| \leq u_{\max_i}$$

However, note that the MRP attitude error are typically bounded by switching between the original and shadow sets!

$$|\boldsymbol{\sigma}| \leq 1$$

$$\Downarrow$$

$$K \leq u_{\max_i}$$

Case 3: Rate Regulator Problem

A common situation just requires the current spacecraft spin to be arrested. Essentially, the final attitude is irrelevant and we set $K = 0$.

The Lyapunov optimal saturated control strategy in this case reduces to:

$$u_i = \begin{cases} -P_{ii}\omega_i & \text{for } |P_{ii}\omega_i| \leq u_{\max_i} \\ -u_{\max_i} \cdot \operatorname{sgn}(\omega_i) & \text{for } |P_{ii}\omega_i| > u_{\max_i} \end{cases}$$

The corresponding Lyapunov rate function is

$$\dot{V}(\boldsymbol{\omega}) = -\sum_{i=1}^{M} P_{ii}\omega_i^2 - \sum_{i=M+1}^{N} \omega_i u_{\text{maxi}} \cdot \operatorname{sgn}(\omega_i) < 0$$

Note, this function is ***negative definite***, and thus globally asymptotically stabilizing!

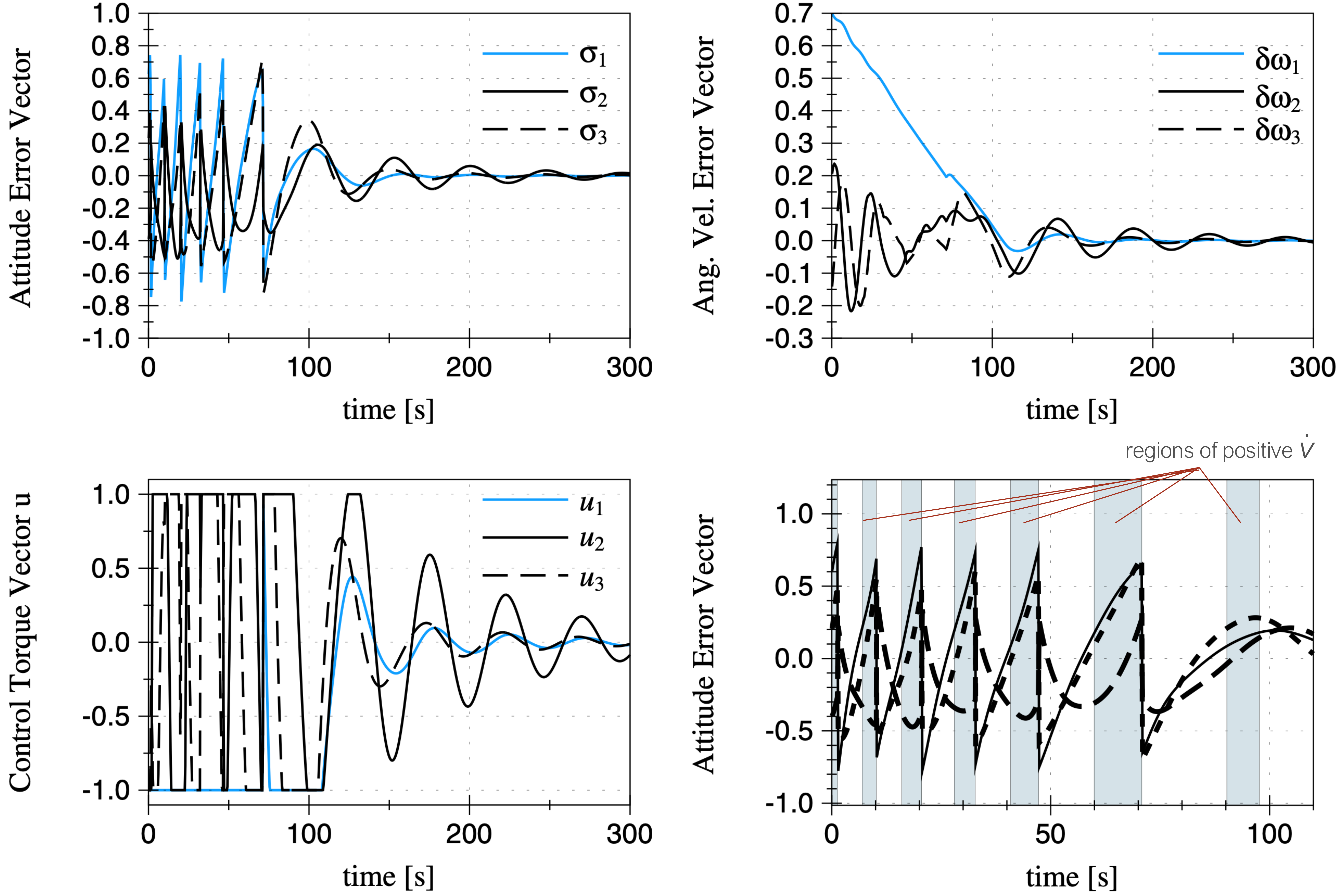
Further, because no inertia terms are used in this saturated control, it is very robust to modeling errors.

Saturated Attitude Control Example:

Parameter	Value	Units
I_1	140.0	kg-m^2
I_2	100.0	kg-m^2
I_3	80.0	kg-m^2
$\boldsymbol{\sigma}(t_0)$	$[0.60\ \ -0.40\ 0.20]$	
$\boldsymbol{\omega}(t_0)$	$[0.70\ 0.20\ \ -0.15]$	rad/sec
$[P]$	$[18.67\ 2.67\ 10.67]$	kg-m^2/sec
K	7.11	kg-m^2/sec^2

Torque saturation level: $$u_{\max_i} = 1 \text{ N m}$$

Unsaturated control law: $$\boldsymbol{u} = -K\boldsymbol{\sigma} - [P]\boldsymbol{\omega}$$



Attitude Error Vector
time [s]
σ_1
σ_2
σ_3
Ang. Vel. Error Vector
time [s]
$\delta\omega_1$
$\delta\omega_2$
$\delta\omega_3$
Control Torque Vector u
time [s]
u_1
u_2
u_3
regions of positive $\dot{V}$
Attitude Error Vector
time [s]

Linear Closed-Loop Dynamics

Surprisingly elegant inverse-kinematic solutions...

The idea...

- An extensive body of literature exists on the behavior of ***linear*** closed-loop dynamics (CLD) of the form:

$$\ddot{\boldsymbol{\epsilon}} + P\dot{\boldsymbol{\epsilon}} + K\boldsymbol{\epsilon} = 0$$

PD-feedback control

$$\ddot{\boldsymbol{\epsilon}} + P\dot{\boldsymbol{\epsilon}} + K\boldsymbol{\epsilon} + K_i \int_0^t \boldsymbol{\epsilon} \mathrm{d}t = 0$$

PID-feedback control

Note: the proposed CLD doesn't require any knowledge of the system mass or inertia properties compared to

$$[I]\delta\dot{\boldsymbol{\omega}} + [P]\delta\boldsymbol{\omega} + K\boldsymbol{\sigma} = 0$$

$$[\dot{B}]^{-1}\dot{\boldsymbol{\sigma}} + [B]^{-1}\ddot{\boldsymbol{\sigma}} \qquad [B]^{-1}\dot{\boldsymbol{\sigma}}$$

With the complicated differential kinematic equations, what a mess this will be!

...or will it?

Setup

- Let us solve for the linear CLD using Euler parameters (β_1,β_2,β_3).*

Desired CLD:

$$\boxed{\ddot{\boldsymbol{\epsilon}} + P\dot{\boldsymbol{\epsilon}} + K\boldsymbol{\epsilon} = 0}$$

Find $\boldsymbol{u}$ to achieve CLD

EOM:

$$[I]\dot{\boldsymbol{\omega}} + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} = \boxed{\boldsymbol{u}}$$

Attitude State Vector:

$$\boldsymbol{\epsilon} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \sin\left(\frac{\Phi}{2}\right)\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

Differential Kinematic Equations:

$$\dot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\boldsymbol{\omega}$$

$$[T(\beta_0, \boldsymbol{\epsilon})] = \begin{bmatrix} \beta_0 & -\beta_3 & \beta_2 \\ \beta_3 & \beta_0 & -\beta_1 \\ -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} = \beta_0[I_{3\times 3}] + [\tilde{\boldsymbol{\epsilon}}]$$

*Paielli, R. A. and Bach, R. E., "Attitude Control with Realization of Linear Error Dynamics," Journal of Guidance, Control and Dynamics, Vol. 16, No. 1, Jan.–Feb. 1993, pp. 182–189.

$$\boxed{\dot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\boldsymbol{\omega}}$$

$$\Big\Downarrow \frac{\mathrm{d}}{\mathrm{d}t}$$

$$\ddot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\dot{\boldsymbol{\omega}} + \frac{1}{2}\boxed{[\dot{T}]\boldsymbol{\omega}}$$

$$[T]\boldsymbol{\omega} = \beta_0\boldsymbol{\omega} + [\tilde{\boldsymbol{\epsilon}}]\boldsymbol{\omega}$$

$$\Big\Downarrow \frac{\mathrm{d}}{\mathrm{d}t}$$

$$\boxed{[\dot{T}]\boldsymbol{\omega}} = \boxed{\dot{\beta}_0}\boldsymbol{\omega} - [\tilde{\boldsymbol{\omega}}]\boxed{\dot{\boldsymbol{\epsilon}}}$$

$$\boxed{\dot{\beta}_0 = -\frac{1}{2}\boldsymbol{\epsilon}^T\boldsymbol{\omega}}$$

Note:

$$[\tilde{\boldsymbol{\omega}}][T]\boldsymbol{\omega} = [\tilde{\boldsymbol{\omega}}]\left(\beta_0[I_{3\times3}] + [\tilde{\boldsymbol{\epsilon}}]\right)\boldsymbol{\omega}$$

$$= \cancel{[\tilde{\boldsymbol{\omega}}]\boldsymbol{\omega}}\beta_0 + [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\epsilon}}]\boldsymbol{\omega}$$

$$= -[\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}]\boldsymbol{\epsilon}$$

$$[\tilde{\boldsymbol{a}}][\tilde{\boldsymbol{a}}] = \boldsymbol{a}\boldsymbol{a}^T - \boldsymbol{a}^T\boldsymbol{a}I_{3\times3}$$

$$[\tilde{\boldsymbol{\omega}}][T]\boldsymbol{\omega} = -\left(\boldsymbol{\omega}\boldsymbol{\omega}^T - \omega^2[I_{3\times3}]\right)\boldsymbol{\epsilon}$$

$$= \boxed{-\boldsymbol{\epsilon}^T\boldsymbol{\omega}\boldsymbol{\omega} + \omega^2\boldsymbol{\epsilon}}$$

$$\ddot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\dot{\boldsymbol{\omega}} - \frac{1}{4}\left(\boldsymbol{\epsilon}^T\boldsymbol{\omega}\boldsymbol{\omega} + \boxed{[\tilde{\boldsymbol{\omega}}][T]\boldsymbol{\omega}}\right) \Rightarrow \ddot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\dot{\boldsymbol{\omega}} - \frac{1}{4}\omega^2\boldsymbol{\epsilon}$$

$$\ddot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\dot{\boldsymbol{\omega}} - \frac{1}{4}\omega^2\boldsymbol{\epsilon} \qquad \dot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\boldsymbol{\omega}$$

$$\ddot{\boldsymbol{\epsilon}} + P\dot{\boldsymbol{\epsilon}} + K\boldsymbol{\epsilon} = 0$$

$$[T]\left(\underbrace{\dot{\boldsymbol{\omega}} + P\boldsymbol{\omega} + [T]^{-1}\left(-\frac{1}{2}\omega^2\boldsymbol{\epsilon} + 2K\boldsymbol{\epsilon}\right)}_{\text{must be zero}}\right) = 0$$

Can [*T*] be inverted?

$$[T]^{-1} = [T]^T + \frac{1}{\beta_0}\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T$$

Always possible except 180º case.

$$[T] = \beta_0 [I_{3\times 3}] + [\tilde{\boldsymbol{\epsilon}}]$$

$$[T]^{-1} = [T]^T + \frac{1}{\beta_0}\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T$$

$$\dot{\boldsymbol{\omega}} + P\boldsymbol{\omega} + [T]^{-1}\left(-\tfrac{1}{2}\omega^2\boldsymbol{\epsilon} + 2K\boldsymbol{\epsilon}\right) = 0$$

Bunch of algebra...

$$\boxed{\dot{\boldsymbol{\omega}} = -P\boldsymbol{\omega} - 2\left(K - \frac{\omega^2}{4}\right)\frac{\boldsymbol{\epsilon}}{\beta_0}}$$

Remarkably simply required angular acceleration expression!

$$[I]\dot{\boldsymbol{\omega}} + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} = \boldsymbol{u}$$

$$\boxed{\boldsymbol{u} = [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} + [I]\left(-P\boldsymbol{\omega} - 2\left(K - \frac{\omega^2}{4}\right)\frac{\boldsymbol{\epsilon}}{\beta_0}\right)}$$

Not that even though we started out with the non-singular Euler parameters, the attitude feedback terms is the Gibbs vector which is singular for 180º rotations!

Comparison to Gibbs Feedback

Paielli/Back feedback control law:

$$\boldsymbol{u} = [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} + [I]\left(-P\boldsymbol{\omega} - 2\left(K - \frac{\omega^2}{4}\right)\frac{\boldsymbol{\epsilon}}{\beta_0}\right)$$

Gibbs-vector Feedback

$$\boldsymbol{u} = [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} - P\boldsymbol{\omega} - K\boldsymbol{q}$$

Integral Feedback

- By starting out with a different desired linear CLD, we can also include an integral feedback term:

$$\ddot{\boldsymbol{\epsilon}} + P\dot{\boldsymbol{\epsilon}} + K\boldsymbol{\epsilon} + K_i \int_0^t \boldsymbol{\epsilon}\mathrm{d}t = 0$$

$$\Big\Downarrow \ \text{Bunch of more algebra following earlier steps...}$$

$$\dot{\boldsymbol{\omega}} = -P\boldsymbol{\omega} - 2\left(K - \frac{\omega^2}{4}\right)\frac{\boldsymbol{\epsilon}}{\beta_0} - 2K_i\left([T]^T + \frac{1}{\beta_0}\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T\right)\int_0^t \boldsymbol{\epsilon}\mathrm{d}t$$

$$\Big\downarrow$$

$$[I]\dot{\boldsymbol{\omega}} + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} = \boldsymbol{u}$$

Tracking Problem

- This linear CLD behavior can also be achieved for an attitude tracking problem with a time-varying reference attitude *R*.

Kinematic expressions:

$$\dot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\delta\boldsymbol{\omega}$$

$$\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_r$$

Even more algebra of the same sort...

$$\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_r - P\delta\boldsymbol{\omega} - 2\left(K - \frac{\delta\omega^2}{4}\right)\frac{\boldsymbol{\epsilon}}{\beta_0}$$

$$[I]\dot{\boldsymbol{\omega}} + [\tilde{\omega}][I]\boldsymbol{\omega} = \boldsymbol{u}$$

Linear MRP CLD

- The previous linear CLD development only yielded elegant results because of some special properties of Euler parameter kinematic equations. However, the resulting feedback was singular at 180º.

$$[T]^{-1} = [T]^T + \frac{1}{\beta_0}\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T$$

Important property for simple linear EP CLD.

MRP Kinematic Equations:

$$\dot{\boldsymbol{\sigma}} = \frac{1}{4}[B(\boldsymbol{\sigma})]\boldsymbol{\omega}$$

$$[B]^{-1} = \frac{1}{(1+\sigma^2)^2}[B]^T$$

elegant near-orthogonal inverse property.

Desired Linear MRP CLD:

$$\ddot{\boldsymbol{\sigma}} + P\dot{\boldsymbol{\sigma}} + K\boldsymbol{\sigma} = 0$$

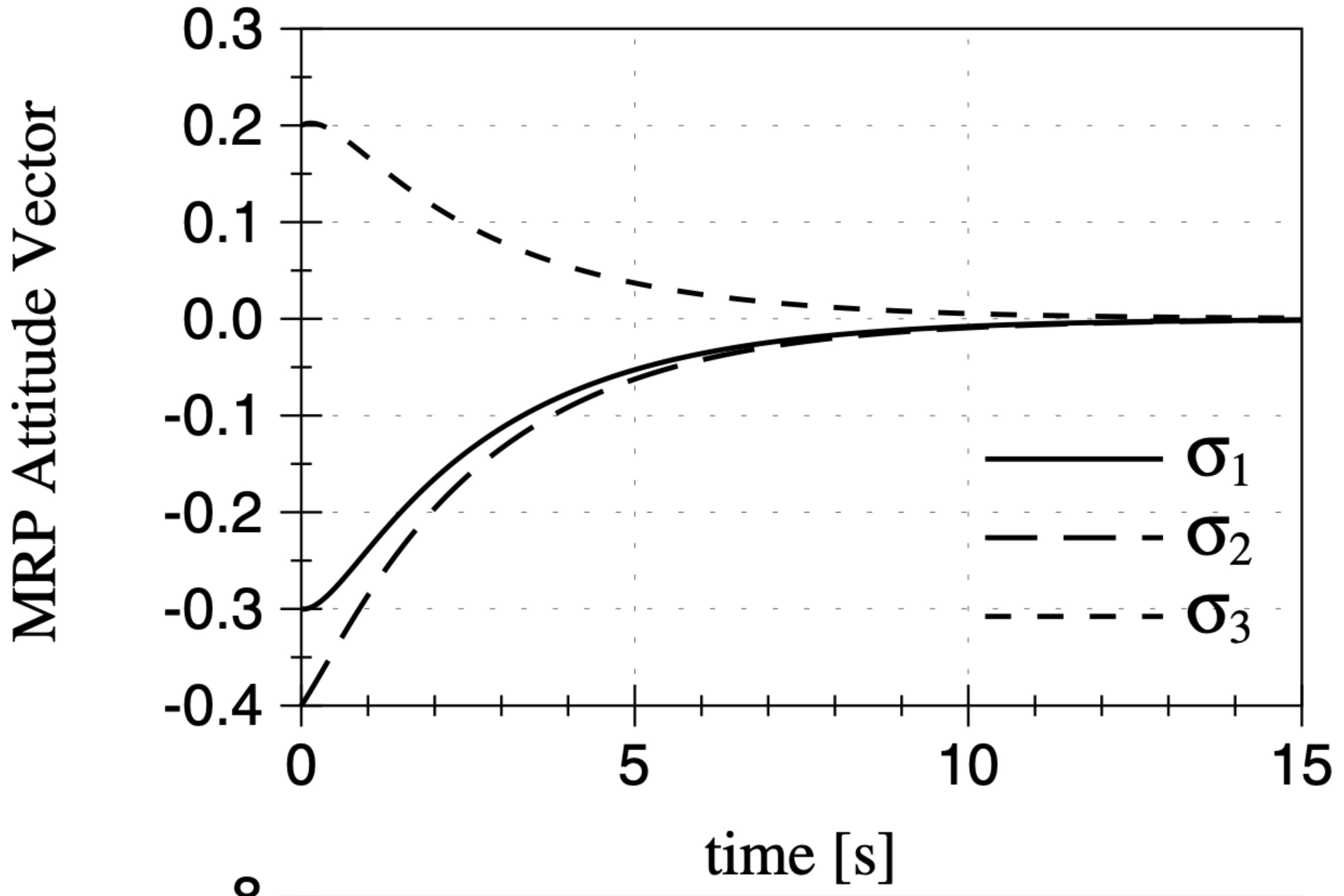
MRP Feedback Control

$$\dot{\boldsymbol{\omega}} = -P\boldsymbol{\omega} - \left(\boldsymbol{\omega}\boldsymbol{\omega}^T + \left(\frac{4K}{1+\sigma^2} - \frac{\omega^2}{2}\right)[I_{3\times3}]\right)\boldsymbol{\sigma}$$

This feedback is non-singular at 180º and globally valid by switching to the shadow set!

Linear MRP CLD Example:

Parameter	Value	Units
I_1	30.0	kg-m^2
I_2	20.0	kg-m^2
I_3	10.0	kg-m^2
$\boldsymbol{\sigma}(t_0)$	$[-0.30\ -0.40\ 0.20]$	
$\boldsymbol{\omega}(t_0)$	$[0.20\ 0.20\ 0.20]$	rad/sec
$[P]$	3.0	kg-m^2/sec
K	1.0	kg-m^2/sec^2



Goal: Regulator problem which arrests satellite motion at the zero attitude.