ASEN 5010
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$\boxed{HW1}$

1. Given: ${}^B\vec{\omega}_{B/N}$, ${}^R\vec{\omega}_{R/N}$, $\delta\vec{\omega} = \vec{\omega}_{B/N} - \vec{\omega}_{R/N}$, To find: ${}^B\delta\vec{\omega}$

a) $${}^B\delta\dot{\vec{\omega}} = \frac{{}^B d}{dt}\delta\vec{\omega} = \frac{{}^B d}{dt}(\vec{\omega}_{B/N} - \vec{\omega}_{R/N}) = \frac{d}{dt}{}^B\vec{\omega}_{B/N} - \frac{d}{dt}{}^R\vec{\omega}_{R/N} = {}^B\dot{\vec{\omega}}_{B/N} - {}^R\dot{\vec{\omega}}_{R/N}$$

To get ${}^B\delta\vec{\omega}$ (which is in B frame) $\rightarrow$ $\boxed{{}^B\delta\dot{\vec{\omega}} = {}^B\vec{\omega}_{B/N} - [BR]\,{}^R\vec{\omega}_{R/N}}$

b) $\delta\vec{\omega} = {}^B\vec{\omega}_{B/N} - [BR]\,{}^R\vec{\omega}_{R/N}$

(a) — $$\delta\dot{\vec{\omega}} = \frac{{}^B d}{dt}{}^B\vec{\omega}_{B/N} - \frac{{}^B d}{dt}\left([BR]\,{}^R\vec{\omega}_{R/N}\right) = {}^B\dot{\vec{\omega}}_{B/N} - [BR]\frac{{}^B d}{dt}{}^R\vec{\omega}_{R/N} + [\tilde{\omega}_{B/R}][BR]\,{}^R\vec{\omega}_{R/N}$$

(b) — $$\frac{{}^B d}{dt}{}^R\vec{\omega}_{R/N} = {}^R\dot{\vec{\omega}}_{R/N} + \vec{\omega}_{R/N} \times \vec{\omega}_{R/N}$$ $(\because [\dot{BR}] = -[\tilde{\omega}_{B/R}][BR])$

Plug (b) in (a)

$$\delta\dot{\vec{\omega}} = {}^B\dot{\vec{\omega}}_{B/N} - [BR]\left({}^R\dot{\vec{\omega}}_{R/N} + \vec{\omega}_{R/N} \times \vec{\omega}_{R/N}\right) + [\tilde{\omega}_{B/R}][BR]\,{}^R\vec{\omega}_{R/N}$$

$$= {}^B\dot{\vec{\omega}}_{B/N} - [BR]\,{}^R\dot{\vec{\omega}}_{R/N} - \cancel{\vec{\omega}_{R/N} \times [BR]\,{}^R\vec{\omega}_{R/N} + [\tilde{\omega}_{B/R}][BR]\,{}^R\vec{\omega}_{R/N}}$$

$$\boxed{{}^B\delta\dot{\vec{\omega}} = {}^B\dot{\vec{\omega}}_{B/N} - [BR]\,{}^R\dot{\vec{\omega}}_{R/N}}$$

Now, inertial derivative of $\delta\omega \rightarrow \frac{{}^N d}{dt}\delta\vec{\omega}$, use transport theorem

$$\frac{{}^N d}{dt}\delta\vec{\omega} = \frac{{}^B d}{dt}\delta\vec{\omega} + \vec{\omega}_{B/N} \times \delta\vec{\omega} = {}^B\delta\dot{\vec{\omega}} + \vec{\omega}_{B/N} \times \delta\vec{\omega}$$

$$\boxed{\frac{{}^N d}{dt}\delta\vec{\omega} = {}^B\dot{\vec{\omega}}_{B/N} - [BR]\,{}^R\dot{\vec{\omega}}_{R/N} + \vec{\omega}_{B/N} \times \delta\vec{\omega}}$$

2. Define tensor T in A, B frames as: where $B: \{\hat{b}_1, \hat{b}_2, \hat{b}_3\}$ $A: \{\hat{a}_1, \hat{a}_2, \hat{a}_3\}$

$${}^A T = \begin{bmatrix} T_{11}\hat{a}_1\hat{a}_1 & T_{12}\hat{a}_1\hat{a}_2 & T_{13}\hat{a}_1\hat{a}_3 \\ T_{21}\hat{a}_2\hat{a}_1 & T_{22}\hat{a}_2\hat{a}_2 & T_{23}\hat{a}_2\hat{a}_3 \\ T_{31}\hat{a}_3\hat{a}_1 & T_{32}\hat{a}_3\hat{a}_2 & T_{33}\hat{a}_3\hat{a}_3 \end{bmatrix} \quad \& \quad {}^B T = \begin{bmatrix} t_{11}\hat{b}_1\hat{b}_1 & t_{12}\hat{b}_1\hat{b}_2 & t_{13}\hat{b}_1\hat{b}_3 \\ t_{21}\hat{b}_2\hat{b}_1 & t_{22}\hat{b}_2\hat{b}_2 & t_{23}\hat{b}_2\hat{b}_3 \\ t_{31}\hat{b}_3\hat{b}_1 & t_{32}\hat{b}_3\hat{b}_2 & t_{33}\hat{b}_3\hat{b}_3 \end{bmatrix}$$

where T_{xx} & t_{xx} are arbitrary values.

Let $\begin{bmatrix}\hat{b}_1\\ \hat{b}_2\\ \hat{b}_3\end{bmatrix} = \{\hat{b}\}$ & $\begin{bmatrix}\hat{a}_1\\ \hat{a}_2\\ \hat{a}_3\end{bmatrix} = \{\hat{a}\}$, $\{\hat{b}\} = [C]\{\hat{a}\} = [BA]\{\hat{a}\}$

$\{\hat{b}\}\cdot\{\hat{b}\}^T$ = coordinate set for tensors (same goes for $\{\hat{a}\}\cdot\{\hat{a}\}^T$)

$$\{\hat{b}\}\cdot\{\hat{b}\}^T = [C]\{a\}\cdot([C]\{a\})^T = [C]\{a\}\{a\}^T[C]^T$$

$$\boxed{\therefore {}^B[T] = [C]\,{}^A[T]\,[C]^T}$$

3] As seen in 2], tensors' coordinates are matrix multiplication of two vectores.
such as $\rightarrow$ $\{\hat{b}\} \cdot \{\hat{b}\}^T$,

An element of a tensor has 3 components $\rightarrow$ $T_{ij} \; \hat{b}_i \hat{b}_j^T$

so, a derivative of a tensor in a different frame would use the following transport theorem $\rightarrow$ ${}^B[T] = [C]\,{}^A[T][C]^T$

$$\frac{d}{dt}\left([C]\,{}^A[T][C]^T\right) = \frac{d}{dt}{}^B[T] + \vec{\omega}_{B/A} \times {}^B[T]$$

For a tensor $\rightarrow$ $\vec{\omega}_{B/A} \times T_{ij} \, \hat{b}_i \hat{b}_j^T = \cancel{\vec{\omega}_{B/A} \times T_{ij}}^{\,0} + \vec{\omega}_{B/A} \times \hat{b}_i \, T_{ij} \, \hat{b}_j^T + T_{ij} \, \hat{b}_i \, \hat{b}_j^T \times \vec{\omega}_{B/A}^T$

$$= [\tilde{\omega}_{B/A}]\,{}^B[T] + {}^B[T][\omega_{B/A}]^T$$

$$[C]\frac{d}{dt}{}^A[T][C]^T = \frac{d}{dt}{}^B[T] + [\tilde{\omega}_{B/A}]\,{}^B[T] + {}^B[T][\tilde{\omega}_{B/A}]^T$$

(Skew-symmetric matrix $\rightarrow$ $\tilde{\omega}^T = -\tilde{\omega}$

$$\therefore \boxed{[C]\frac{d}{dt}{}^A[T][C]^T = \frac{d}{dt}{}^B[T] + [\tilde{\omega}_{B/A}]\,{}^B[T] - {}^B[T][\tilde{\omega}_{B/A}]}$$