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ASEN 5044
Fall 2024

$\boxed{\text{HW 2}}$

Problem 1 → $\ddot{r} = r\dot{\theta}^2 - \frac{k}{r^2} + u_1(t)$, $\ddot{\theta} = -\frac{2\dot{\theta}\dot{r}}{r} + \frac{1}{r}u_2(t)$, $x = [r, \dot{r}, \theta, \dot{\theta}]^T$

a] $$\dot{x} = \begin{bmatrix} \dot{r} \\ \ddot{r} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 x_4^2 - k/x_1^2 + u_1(t) \\ x_4 \\ -\frac{2x_4x_2}{x_1} + \frac{1}{x_1}u_2(t) \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = F(x,u)$$ where $x = [r, \dot{r}, \theta, \dot{\theta}]^T = [x_1, x_2, x_3, x_4]^T$

b] $r(t) = r_0$, $\dot{r}(0) = 0$, $\theta(t) = \omega_0 t + \text{constant}$, $\dot{\theta}(t) = \omega_0$, $u_1(t) = u_2(t) = 0$, $\omega_0 = \sqrt{\frac{k}{r^3}}$

$x_{nom} = [r_0, 0, \omega_0 t + \text{constant}, \omega_0]^T$, $u_{nom} = [0, 0]^T$

$$\left[\frac{\partial F}{\partial x}\right]_{x_{nom}, u_{nom}} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} & \frac{\partial F_1}{\partial x_4} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} & \frac{\partial F_2}{\partial x_4} \\ \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_2} & \frac{\partial F_3}{\partial x_3} & \frac{\partial F_3}{\partial x_4} \\ \frac{\partial F_4}{\partial x_1} & \frac{\partial F_4}{\partial x_2} & \frac{\partial F_4}{\partial x_4} & \frac{\partial F_4}{\partial x_4} \end{bmatrix}_{x_{nom}, u_{nom}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ x_4^2 + \frac{2k}{x_1^3} & 0 & 0 & 2x_1x_4 \\ 0 & 0 & 0 & 1 \\ +\frac{2x_4x_2}{x_1^2} - \frac{u_2(t)}{x_1^2} & -\frac{2x_4}{x_1} & 0 & -\frac{2x_2}{x_1} \end{bmatrix}_{x_{nom}, u_{nom}}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \omega_0^2 + \frac{2k}{r_0^3} & 0 & 0 & 2r_0\omega_0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2\omega_0}{r_0} & 0 & 0 \end{bmatrix} = \bar{A}\big|_{nom} \text{ s.t. } \dot{\bar{x}} = \bar{A}\big|_{nom}\bar{x} + \bar{B}\big|_{nom}\bar{u}$$

$$\left[\frac{\partial F}{\partial u}\right]_{x_{nom}, u_{nom}} = \begin{bmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} \\ \frac{\partial F_3}{\partial u_1} & \frac{\partial F_3}{\partial u_2} \\ \frac{\partial F_4}{\partial u_1} & \frac{\partial F_4}{\partial u_2} \end{bmatrix}_{x_{nom}, u_{nom}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{x_1} \end{bmatrix}_{x_{nom}, u_{nom}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{r_0} \end{bmatrix} = \bar{B}\big|_{nom} \text{ s.t. } \dot{\bar{x}} = \bar{A}\big|_{nom}\bar{x} + \bar{B}\big|_{nom}\bar{u}$$

$y(t) = [r(t), \theta(t)]^T$
$= [x_1, x_3]^T = h(x,u)$

$$\left[\frac{\partial h}{\partial x}\right]_{nom} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \frac{\partial h_1}{\partial x_3} & \frac{\partial h_1}{\partial x_4} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \frac{\partial h_2}{\partial x_3} & \frac{\partial h_2}{\partial x_4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \bar{C}\big|_{nom}$$

$$\left[\frac{\partial h}{\partial u}\right]_{nom} = \begin{bmatrix} \frac{\partial h_1}{\partial u_1} & \frac{\partial h_1}{\partial u_2} \\ \frac{\partial h_2}{\partial u_1} & \frac{\partial h_2}{\partial u_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \bar{D}\big|_{nom}$$

s.t. $\bar{y} = \bar{C}\big|_{nom}\bar{x} + \bar{D}\big|_{nom}\bar{u}$

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(code attached at the end)

$\boxed{\text{HW 2}}$

Problem 1] c] $\hat{A} = \begin{bmatrix} \bar{A} & \bar{B} \\ 0 & 0 \end{bmatrix} \rightarrow e^{\hat{A}t} = \begin{bmatrix} F & G \\ 0 & I \end{bmatrix}$, doing this via matlab

$F = \begin{bmatrix} 1.002 & 9.9998 & 0 & 772.5749 \\ 0 & 0.9999 & 0 & 154.5133 \\ 0 & 0 & 1 & 0.9991 \\ 0 & 0 & 0 & 0.9997 \end{bmatrix}, \quad G = \begin{bmatrix} 49.9994 & 0.3856 \\ 9.9998 & 0.1157 \\ -0.0001 & 0.0075 \\ 0 & 0.0015 \end{bmatrix}$

$H = \bar{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad M = \bar{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

d] $\because$ F is calculated using $\bar{A}, \bar{B}$ which are perturbations of the system, the columns of F can be used to interpret the perturbation dynamics. 1st col. indicates that perturbation in distance (δr) depends only on itself. 2nd col. denotes that perturbation change in distance ($\delta\dot{r}$) depends on δr and $\delta\dot{r}$. 3rd col. denotes that perturbation θ depends only on $\delta\theta$. 4th col. denotes that perturbation in $\dot{\theta}$ is coupled with all perturbation states.

Problem 2 → $p = p_0 + \dot{p}_0 t + \ddot{p}_0 \frac{1}{2} t^2$, $x = [p, \dot{p}, \ddot{p}]^T$

a) $\dot{p} = \dot{p}_0 + \ddot{p}_0 t$, $\boxed{\ddot{p} = \ddot{p}_0}$, $\dddot{p} = 0 \longrightarrow \dot{p} = \dot{p}_0 + \ddot{p} t \rightarrow \boxed{\dot{p}_0 = (\dot{p} - \ddot{p} t)}$

$p = p - \cancel{\ddot{p} t} + \cancel{\ddot{p} t}$, $\ddot{p} = \ddot{p}_0$

$$\dot{x} = \begin{bmatrix} \dot{p} \\ \ddot{p} \\ \dddot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \\ \ddot{p} \end{bmatrix} = Ax$$

b) $$e^{At} = \sum_{j=0}^{\infty} \frac{(At)^j}{j!} = I + At + (At)^2/2! + \dots$$

$$(At)^2 = (At)(At) = \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(At)^3 = (At)^2 (At) = \begin{bmatrix} 0 & 0 & t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$(At)^n$ where $n \geq 3$ = zeros($n \times n$)

$$\therefore e^{At} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & t^2/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} = e^{At}}$$

$$e^{At} = Q e^{\hat{A}t} Q^{-1} \rightarrow \lambda(A) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} \Rightarrow 0 = -\lambda \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = -\lambda^3 = 0$$

$\boxed{\lambda = 0}$

$\therefore \hat{A} = A$ where $\hat{A}$ is the jordan form of A

$$\therefore e^{\hat{A}t} = e^{At} = \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore e^{At} = Q e^{\hat{A}t} Q^{-1} = \boxed{\boxed{\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}} = e^{At}}$$

c) $$\underset{(t=0)}{e^{A0}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{I = e^{A0}}$$

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HW2

Problem 3 → $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $s(t) = \begin{bmatrix} e^{t} & 0 \\ 0 & 2e^{-t} \end{bmatrix}$

$$AS(t) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} e^{t} & 0 \\ 0 & 2e^{-t} \end{bmatrix} = \begin{bmatrix} e^{t} & 0 \\ 0 & -2e^{-t} \end{bmatrix} = \boxed{\dot{s}(t) = AS(t)}$$

$$e^{At} = I + (At) + \frac{(At)^2}{2!} + \dots$$

$$(At)^2 = (At)(At) = \begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix}\begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix} = \begin{bmatrix} t^2 & 0 \\ 0 & t^2 \end{bmatrix}$$

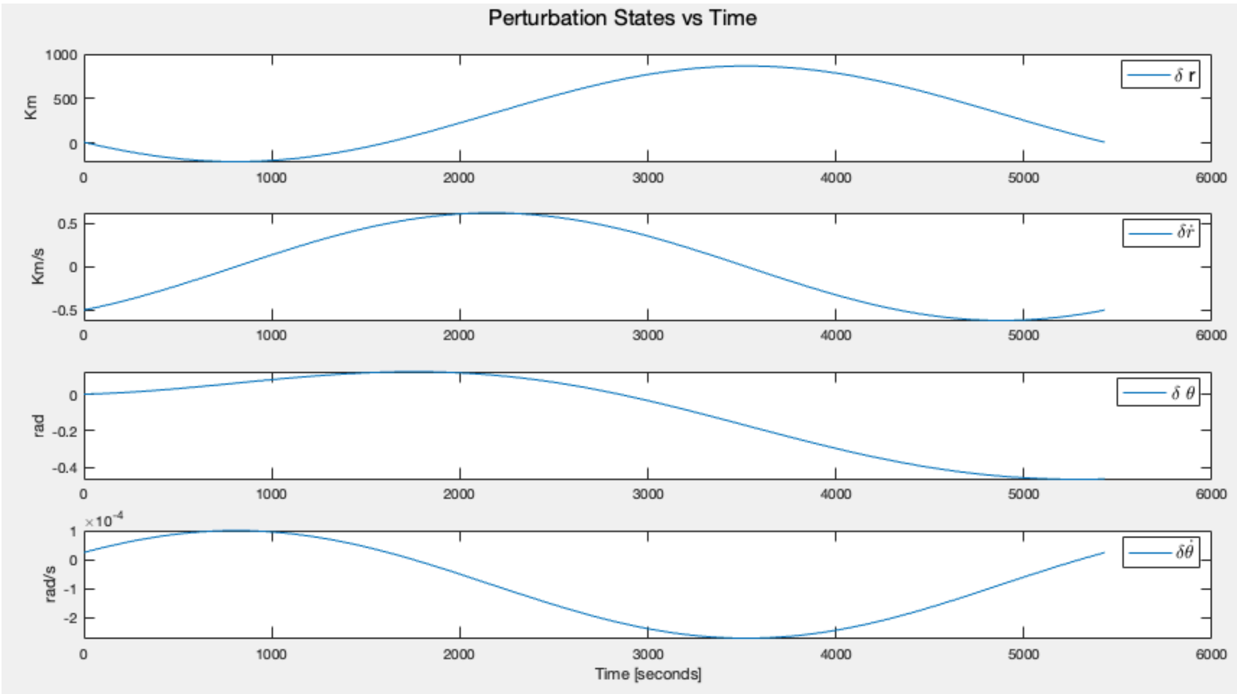
$$(At)^3 = (At)^2(At) = \begin{bmatrix} t^2 & 0 \\ 0 & t^2 \end{bmatrix}\begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix} = \begin{bmatrix} t^3 & 0 \\ 0 & -t^3 \end{bmatrix}$$

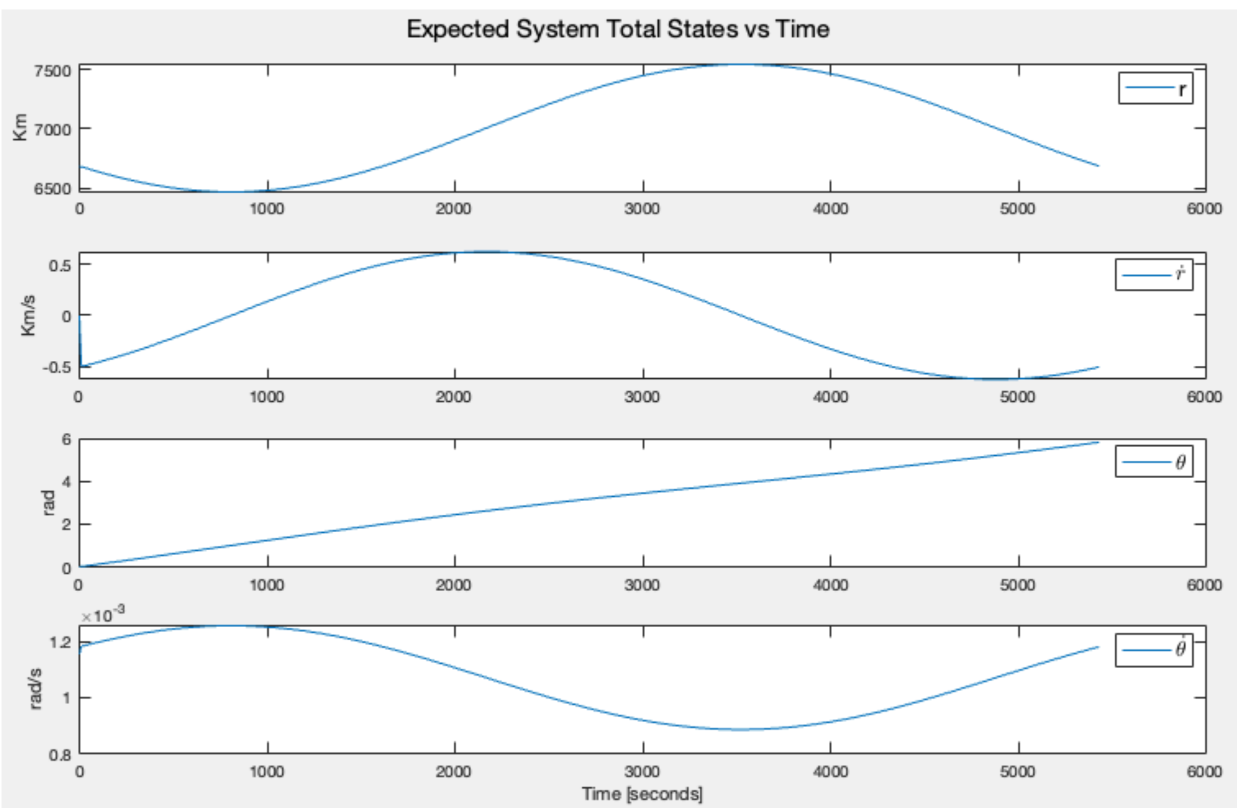
$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} t^2 & 0 \\ 0 & t^2 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} t^3 & 0 \\ 0 & -t^3 \end{bmatrix} + \dots$$

$$= \begin{bmatrix} 1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\dots & 0 \\ 0 & 1-t+\frac{t^2}{2!}-\frac{t^3}{3!}-\dots \end{bmatrix} = \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix} = \Phi(t)$$

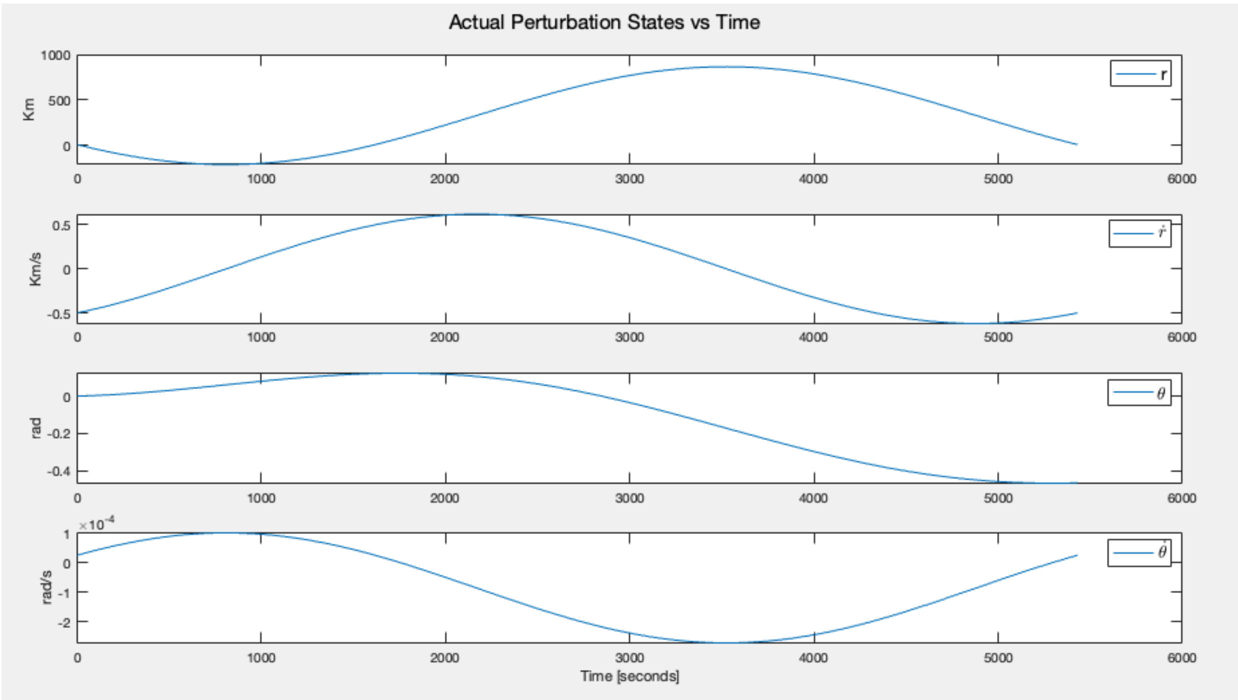
$\therefore$ s(t) is not the state transition matrix of the system.

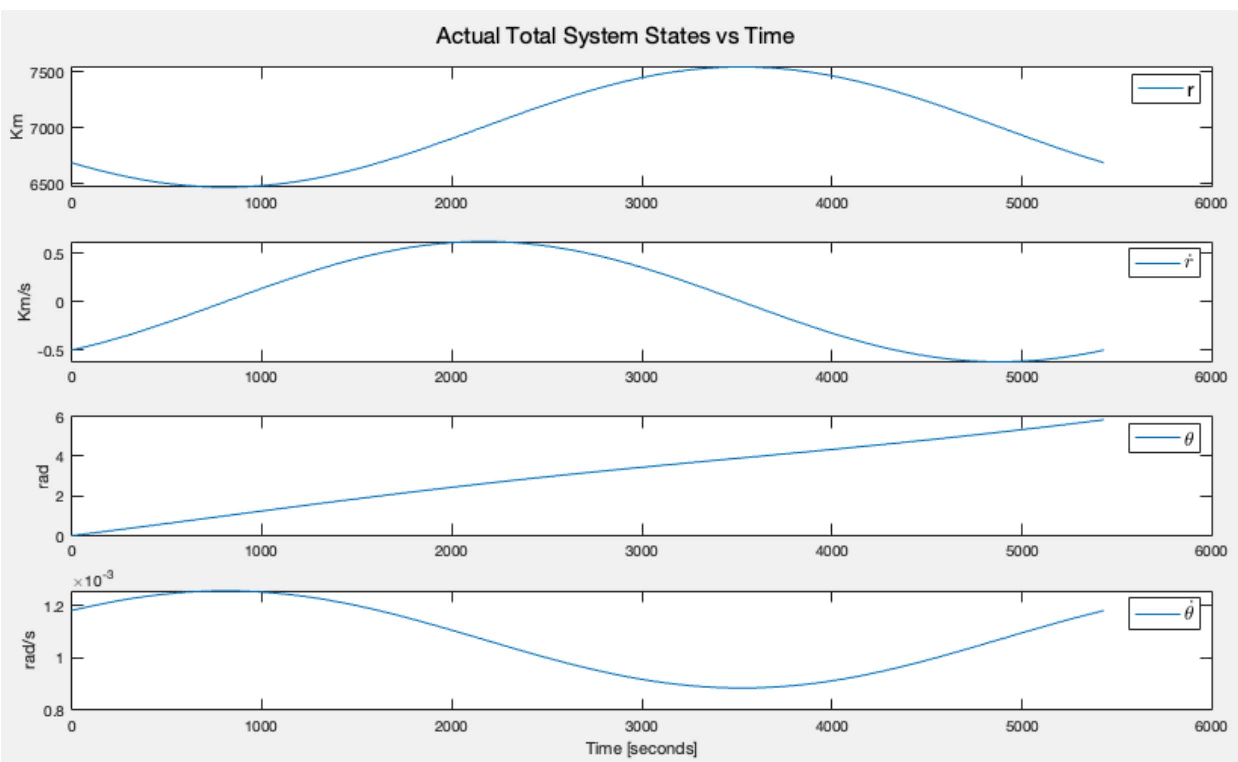
- Problem 4, part a





- Problem 4, part b

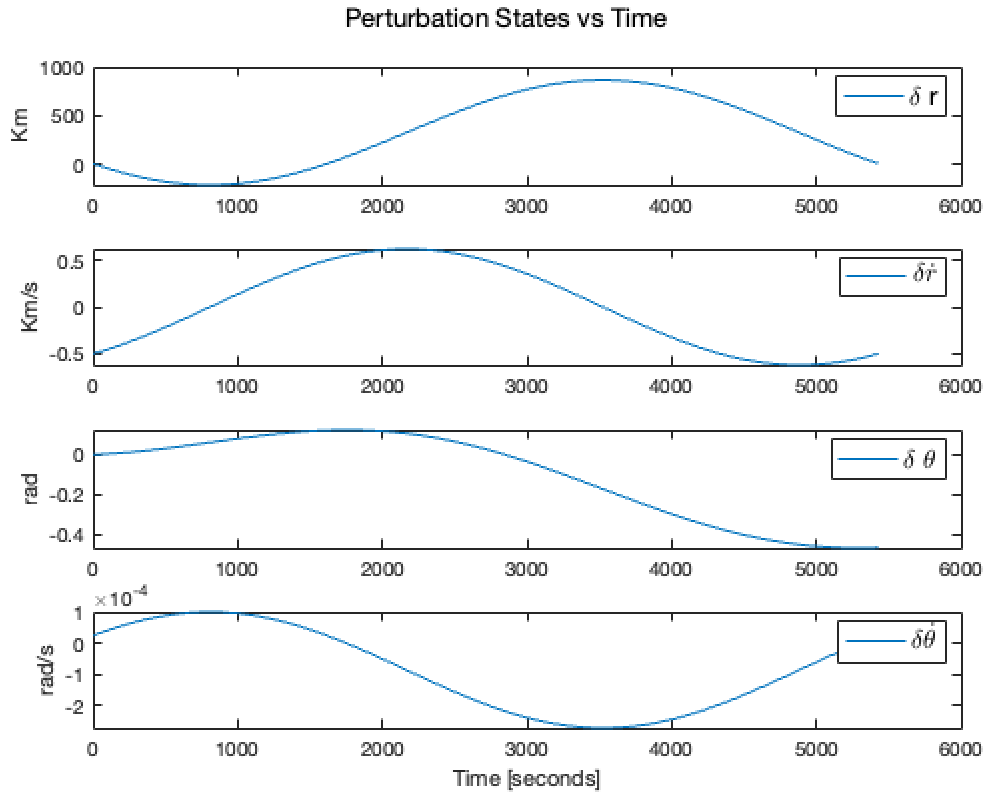


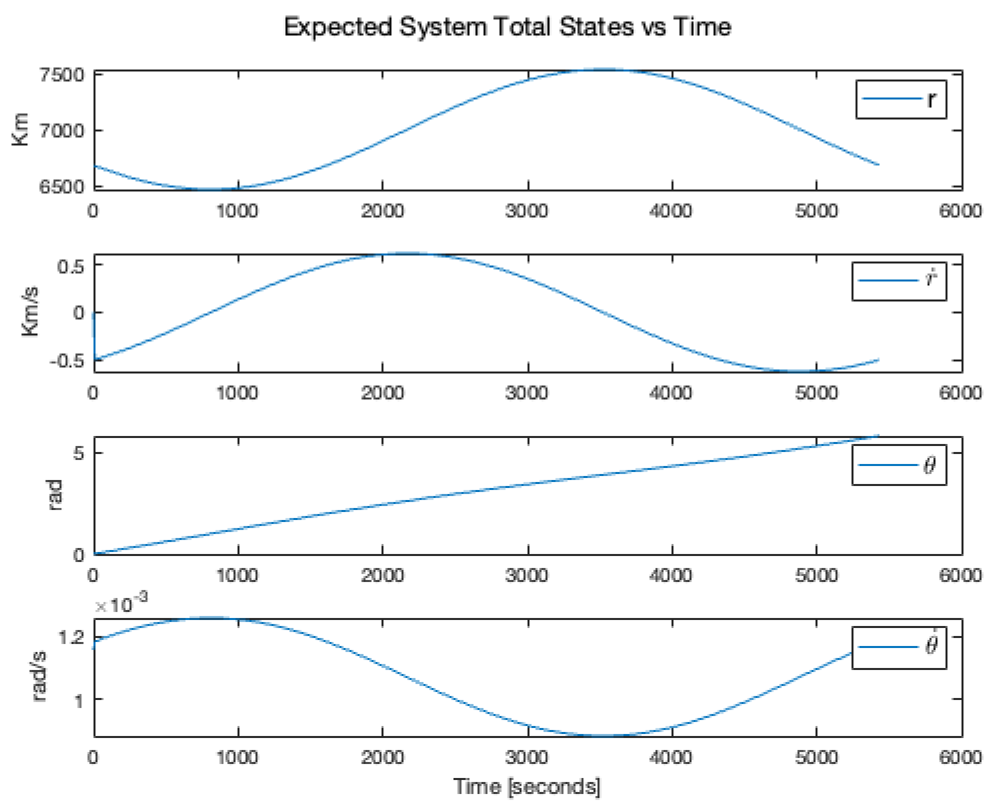


The linearized DT model provides a very good approximation of the nonlinear system's behavior. That may be because the nonlinear system is modeled around a nominal trajectory. Additionally, the time step is small compared to the period of an orbit. That's why the linearized DT model provides a good approximation.

```
clear; clc; close all;

delta_t = 10;
k = 398600;
r0 = 6678;
omega_0 = sqrt(k/r0^3);
orbital_period = 2*pi * sqrt(r0^3/k);

A_bar = [0 1 0 0; omega_0^2+2*k/r0^3 0 0 2*r0*omega_0; 0 0 0 1; 0
-2*omega_0/r0 0 0];
B_bar = [0 0; 1 0; 0 0; 0 1/r0];

% A is always square matrix
n = length(A_bar);
m = size(B_bar,2);

A_bar_hat = [A_bar B_bar; zeros([n - m, n+m])];

syms t

stm = expm(A_bar_hat*t);
stmf(t) = stm;
f(t) = stm(1:4, 1:4);
g(t) = stm(1:4, 5:6);

% Problem 1, part c
out = double(stmf(10));

% Problem 4, part a
r_dot_0 = 0;
theta_0 = omega_0 * t;
theta_dot_0 = omega_0;
x_nom(t) = [r0, r_dot_0, theta_0, theta_dot_0]';
x_nom_0 = double(x_nom(0));

delta_r0 = 10;
delta_r_dot = -0.5;
delta_theta = 0;
delta_theta_dot = 2.5e-5;
delta_x_0 = [delta_r0, delta_r_dot, delta_theta, delta_theta_dot]';

u1 = 0;
u2 = 0;
delta_u = [u1, u2]';

delta_x = [delta_x_0];
x_nominal = [x_nom_0];

time = 0:10:orbital_period;

for i = 2:length(time)
    delta_x(:,i) = double(f(time(i))) * delta_x_0 + double(g(time(i))) *
```

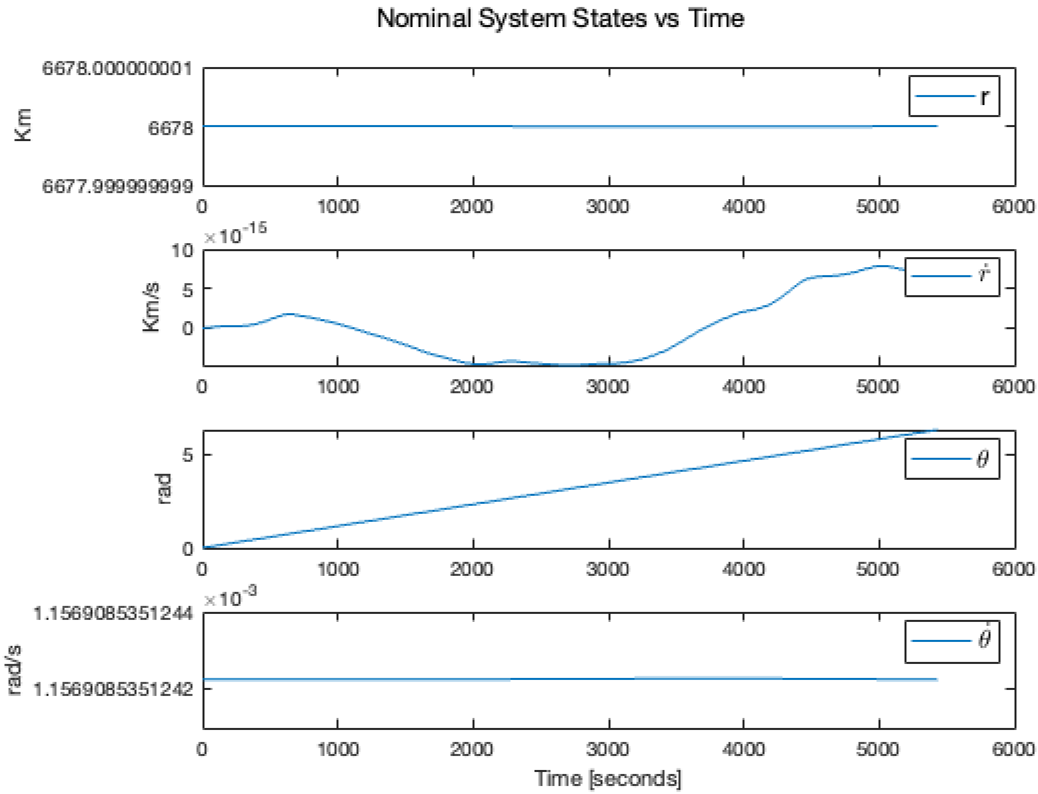
```
delta_u;
    x_nominal(:,i) = double(x_nom(time(i))) + delta_x(:,i);
end

figure()
subplot(4,1,1)
plot(time, delta_x(1,:))
legend("\delta r", 'FontSize', 13)
ylabel("Km")

subplot(4,1,2)
plot(time, delta_x(2,:))
legend("$\delta \dot{r}$", 'Interpreter', 'latex', 'FontSize', 13)
ylabel("Km/s")

subplot(4,1,3)
plot(time, delta_x(3,:))
legend("\delta \theta", 'FontSize', 13)
ylabel("rad")

subplot(4,1,4)
plot(time, delta_x(4,:))
legend("$\delta \dot{\theta}$", 'Interpreter', 'latex', 'FontSize', 13)
ylabel("rad/s")
xlabel("Time [seconds]")

sgtitle("Perturbation States vs Time")


figure()
subplot(4,1,1)
plot(time, x_nominal(1,:))
legend("r", 'FontSize', 13)
ylabel("Km")

subplot(4,1,2)
plot(time, x_nominal(2,:))
legend("$\dot{r}$", 'Interpreter', 'latex', 'FontSize', 13)
ylabel("Km/s")

subplot(4,1,3)
plot(time, x_nominal(3,:))
legend("\theta", 'FontSize', 13)
ylabel("rad")

subplot(4,1,4)
plot(time, x_nominal(4,:))
legend("$\dot{\theta}$", 'Interpreter', 'latex', 'FontSize', 13)
ylabel("rad/s")
xlabel("Time [seconds]")

sgtitle("Expected System Total States vs Time")
```


Perturbation States vs Time
Km
1000
500
0
δ r
Km/s
0.5
0
-0.5
δṙ
rad
0
-0.2
-0.4
δ θ
×10^-4
rad/s
1
0
-1
-2
δθ̇
0
1000
2000
3000
4000
5000
6000
Time [seconds]

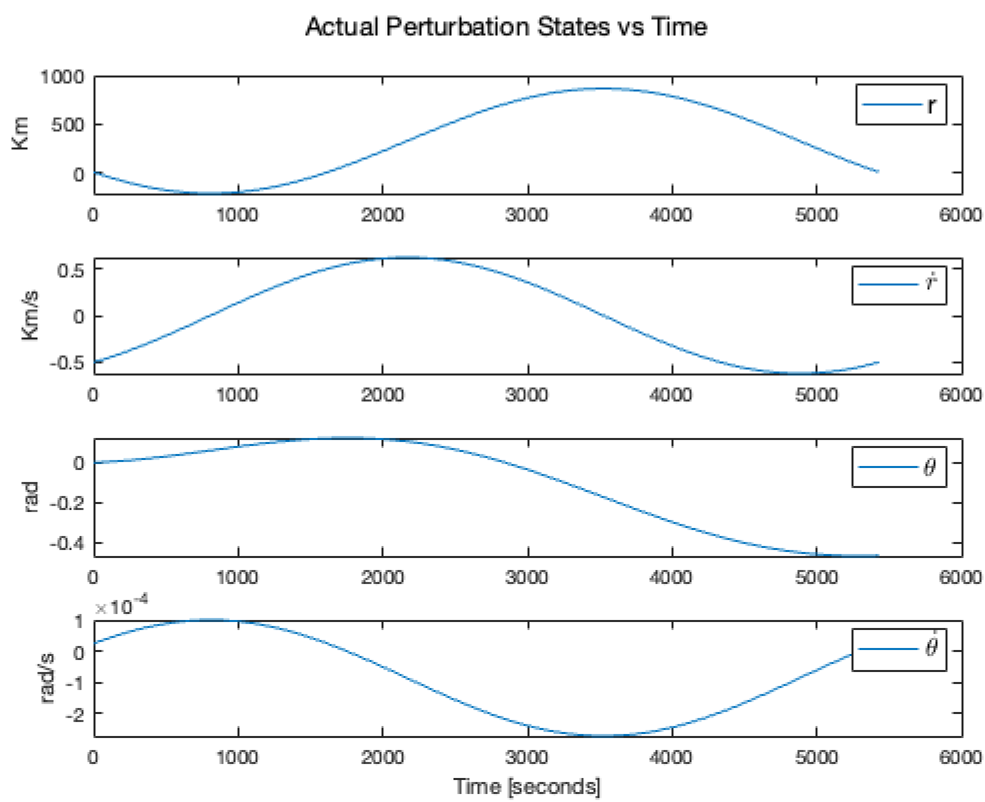

Expected System Total States vs Time
Km
7500
7000
6500
r
Km/s
0.5
0
-0.5
ṙ
rad
5
0
θ
×10^-3
rad/s
1.2
1
θ̇
0
1000
2000
3000
4000
5000
6000
Time [seconds]

```
clear; clc; close all;

k = 398600;
r0 = 6678;
omega_0 = sqrt(k/r0^3);
r_dot_0 = 0;
theta_0 = 0;
theta_dot_0 = omega_0;
u1 = 0;
u2 = 0;

x0 = [r0, r_dot_0, theta_0, theta_dot_0]';

orbital_period = 2*pi * sqrt(r0^3/k);
time = 0:10:orbital_period;

x_nom = @(t,x) [x(2); x(1)*x(4)^2 - k/x(1)^2 + u1; x(4); -2*x(4)*x(2)/x(1) +
1/x(1)*u2];

% Perturbations
A_bar = [0 1 0 0; omega_0^2+2*k/r0^3 0 0 2*r0*omega_0; 0 0 0 1; 0
-2*omega_0/r0 0 0];
B_bar = [0 0; 1 0; 0 0; 0 1/r0];

dx0 =  [10; -0.5; 0; 2.5e-5];
x_perturb = @(t, dx) [dx(2); (omega_0^2+2*k/r0^3)*dx(1) + 2*r0*omega_0*dx(4);
dx(4); -2*omega_0/r0*dx(2)];

[t_nom, x_nominal] = ode45(x_nom, time, x0);
[t_perturb, x_perturb] = ode45(x_perturb, time, dx0);

x = x_nominal + x_perturb;


figure()
subplot(4,1,1)
plot(time, x_nominal(:,1))
legend("r", 'FontSize', 13)
ylabel("Km")

subplot(4,1,2)
plot(time, x_nominal(:,2))
legend("$\dot{r}$", 'Interpreter', 'latex', 'FontSize', 13)
ylabel("Km/s")

subplot(4,1,3)
plot(time, x_nominal(:,3))
legend("\theta", 'FontSize', 13)
ylabel("rad")

subplot(4,1,4)
plot(time, x_nominal(:,4))
legend("$\dot{\theta}$", 'Interpreter', 'latex', 'FontSize', 13)
ylabel("rad/s")
```

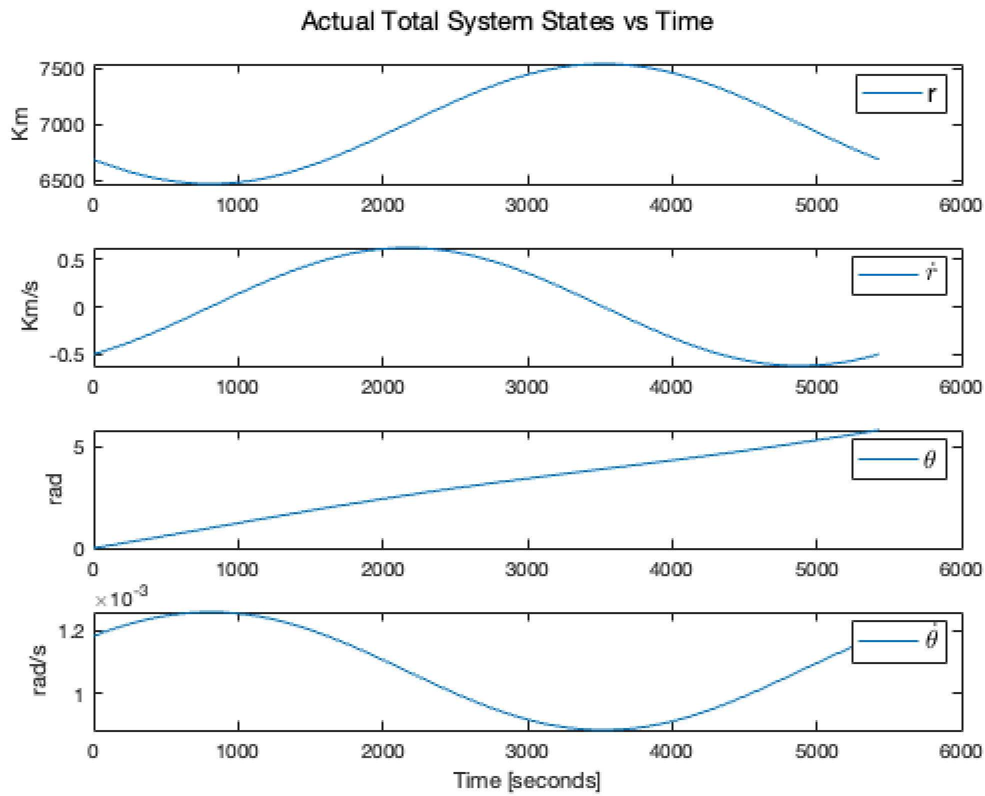
```
xlabel("Time [seconds]")

sgtitle("Nominal System States vs Time")


figure()
subplot(4,1,1)
plot(time, x_perturb(:,1))
legend("r", 'FontSize', 13)
ylabel("Km")

subplot(4,1,2)
plot(time, x_perturb(:,2))
legend("$\dot{r}$", 'Interpreter', 'latex', 'FontSize', 13)
ylabel("Km/s")

subplot(4,1,3)
plot(time, x_perturb(:,3))
legend("\theta", 'FontSize', 13)
ylabel("rad")

subplot(4,1,4)
plot(time, x_perturb(:,4))
legend("$\dot{\theta}$", 'Interpreter', 'latex', 'FontSize', 13)
ylabel("rad/s")
xlabel("Time [seconds]")

sgtitle("Actual Perturbation States vs Time")


figure()
subplot(4,1,1)
plot(time, x(:,1))
legend("r", 'FontSize', 13)
ylabel("Km")

subplot(4,1,2)
plot(time, x(:,2))
legend("$\dot{r}$", 'Interpreter', 'latex', 'FontSize', 13)
ylabel("Km/s")

subplot(4,1,3)
plot(time, x(:,3))
legend("\theta", 'FontSize', 13)
ylabel("rad")

subplot(4,1,4)
plot(time, x(:,4))
legend("$\dot{\theta}$", 'Interpreter', 'latex', 'FontSize', 13)
ylabel("rad/s")
xlabel("Time [seconds]")

sgtitle("Actual Total System States vs Time")
```



Nominal System States vs Time
Km
6678.000000001
6678
6677.99999999
r
Km/s
×10^-15
$\dot{r}$
rad
θ
rad/s
×10^-3
1.1569085351244
1.1569085351242
$\dot{\theta}$
Time [seconds]



Actual Perturbation States vs Time
Km
r
Km/s
$\dot{r}$
rad
θ
rad/s
×10^-4
$\dot{\theta}$
Time [seconds]



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