ASEN 6060
ADVANCED ASTRODYNAMICS
State Transition Matrix

Objectives:

- Define the State Transition Matrix (STM) in the CR3BP
- Present mathematical properties of the STM

Studying the Neighborhood of a Trajectory

- Given a reference trajectory $\bar{x}_R(t)$, often interested in properties of nearby trajectories $\bar{x}(t)$
- One approach to studying the solutions in the neighborhood of this reference path leverages first-order variational equations

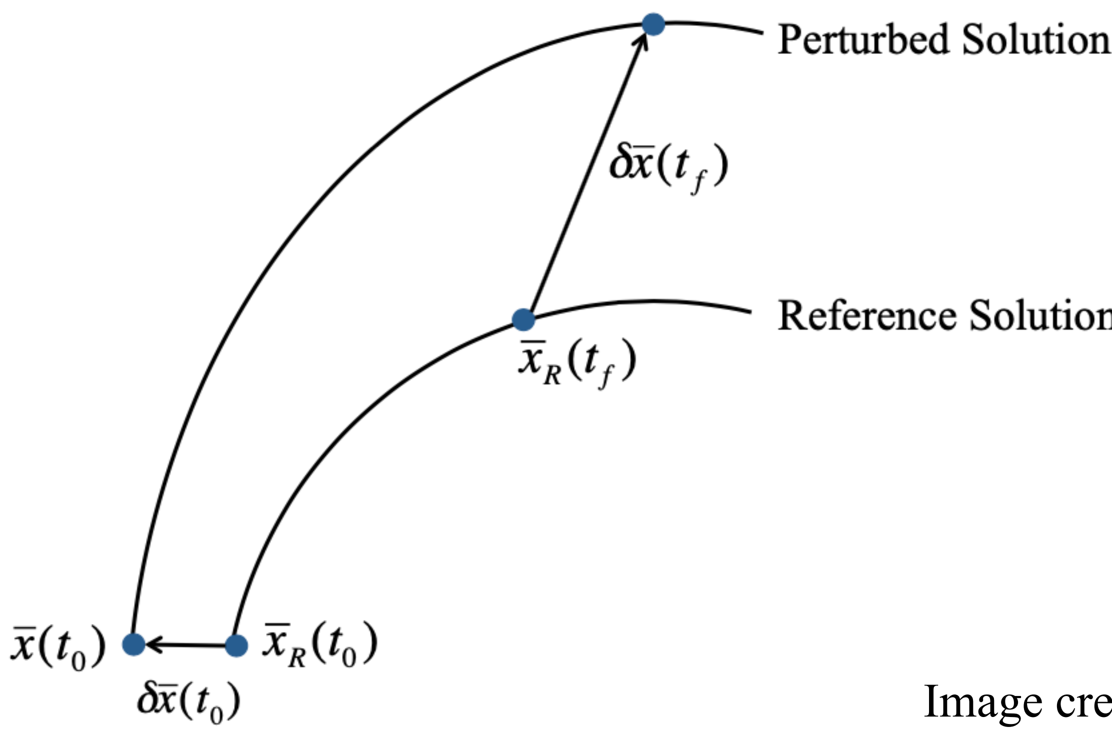


Image credit: Bosanac 2016

Applying Taylor Series Expansion

Introducing a variation $\delta\bar{x}(t_0)$ in the initial state at t_0 and propagating to a time t in the nonlinear system

$$\dot{\bar{x}} = \bar{f}(\bar{x})$$

produces a perturbed trajectory:

where

Applying Taylor Series Expansion

Simplifying (by noting that the reference trajectory is a solution to the nonlinear system) yields the linear vector variational equation:

Commonly denote the coefficient using the matrix ***A(t)*** to produce

Solutions to this linear, time-varying system take the form

State Transition Matrix

The STM is governed by a first-order matrix differential equation, derived by substituting this solution into the variational equations

$$\delta\dot{\bar{x}}(t) = \mathbf{A}(t)\delta\bar{x}(t) \qquad \delta\bar{x}(t) = \mathbf{\Phi}(t, t_0)\delta\bar{x}(t_0)$$

At time t_0, the initial condition for the STM is $\mathbf{\Phi}(t_0, t_0) = \mathbf{I}_{6\times 6}$, the 6x6 identity matrix because

$$\delta\bar{x}(t_0) = \mathbf{\Phi}(t_0, t_0)\delta\bar{x}(t_0) = \mathbf{I}_{\mathbf{6}\times\mathbf{6}}\delta\bar{x}(t_0)$$

Integrating the STM

Append STM to state vector and simultaneously integrate with the state vector by appending matrix differential equations to EOMs, using initial state vector (specified) and initial STM (identity matrix)

$$\begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \\ \Phi_{11} \\ \Phi_{12} \\ \Phi_{13} \\ \vdots \\ \Phi_{66} \end{bmatrix}$$

Unpack $\boldsymbol{\Phi}$ (from $\Phi_{11}, \ldots, \Phi_{66}$)

$$\begin{aligned} \dot{x} &= \ldots \\ \dot{y} &= \ldots \\ \dot{z} &= \ldots \\ \ddot{x} &= \ldots \\ \ddot{y} &= \ldots \\ \ddot{z} &= \ldots \\ \dot{\Phi}_{11} &= \ldots \\ \dot{\Phi}_{12} &= \ldots \\ \dot{\Phi}_{13} &= \ldots \\ &\vdots \\ \dot{\Phi}_{66} &= \ldots \end{aligned}$$

Unpack $\dot{\boldsymbol{\Phi}} = \mathbf{A}\boldsymbol{\Phi}$

State Transition Matrix

State Transition Matrix (STM):

$$\delta\bar{x}(t) = \mathbf{\Phi}(t, t_0)\delta\bar{x}(t_0)$$

- Governed by the first-order matrix differential equation

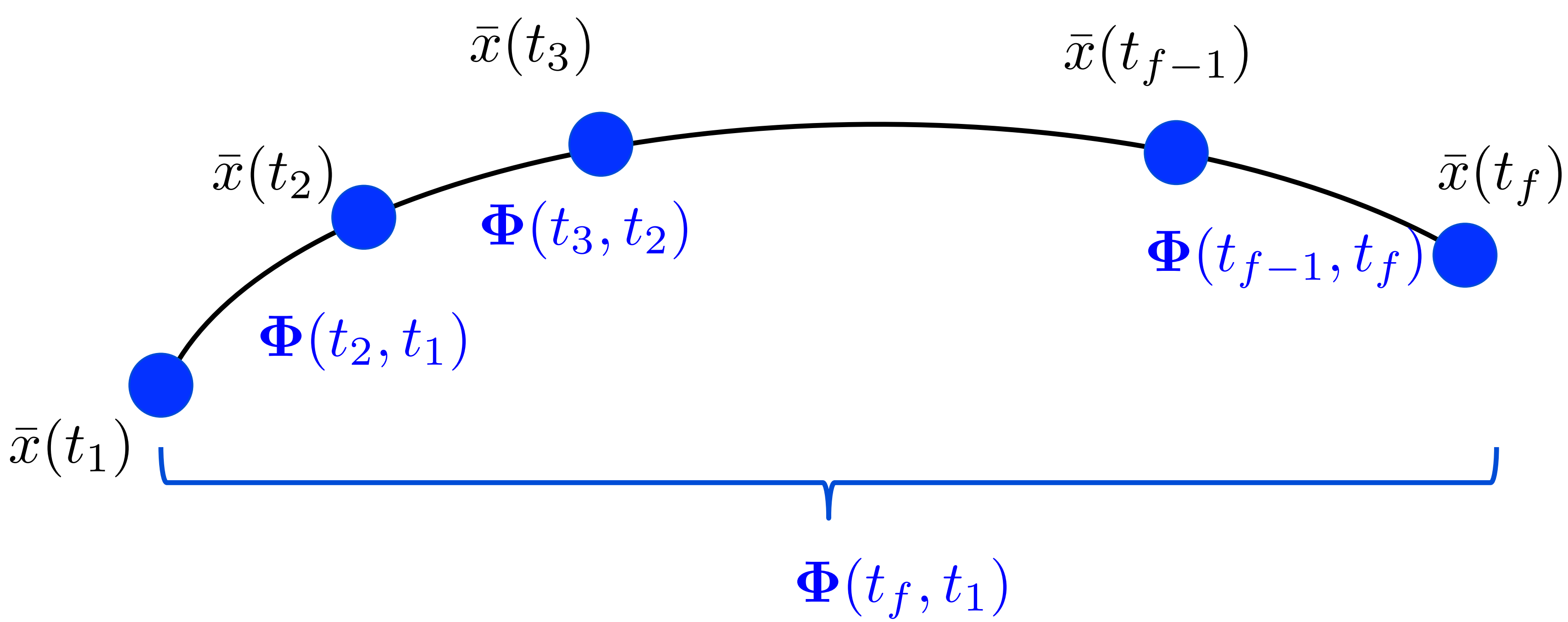
$$\dot{\mathbf{\Phi}}(t, t_0) = \mathbf{A}(t)\mathbf{\Phi}(t, t_0)$$

- Initial condition: $\mathbf{\Phi}(t_0, t_0) = \mathbf{I}_{6\times 6}$
- Approximates sensitivity of solution at a time *t* to perturbations in its initial state
  – Eigenvalues of STM reveal information about nearby trajectories
- Useful mathematical properties

STM Property #1

The STM from t_0 to t can be decomposed into the product of multiple STMs, generated over successive subsets of time interval of interest

$$t_1 < t_2 < \ldots < t_{f-1} < t_f$$



STM Property #1

The STM from t_0 to t can be decomposed into the product of multiple STMs, generated over successive subsets of time interval of interest

Consider the STM from time t_0 to t_f: $\delta\bar{x}(t_f) = \mathbf{\Phi}(t_f, t_0)\delta\bar{x}(t_0)$

Applying the chain rule and substituting the variations at intermediate times where $t_0 < t_1 < \ldots < t_{f-1} < t_f$

$$\delta\bar{x}(t_f) = \mathbf{\Phi}(t_f, t_{f-1})\delta\bar{x}(t_{f-1}) = \mathbf{\Phi}(t_f, t_{f-1})\mathbf{\Phi}(t_{f-1}, t_{f-2})\delta\bar{x}(t_{f-2}) = \ldots$$

$$\delta\bar{x}(t_f) = \mathbf{\Phi}(t_f, t_{f-1})\mathbf{\Phi}(t_{f-1}, t_{f-2})\ldots\mathbf{\Phi}(t_2, t_1)\mathbf{\Phi}(t_1, t_0)\delta\bar{x}(t_0)$$

Thus:

STM Property #2

The STM from t_0 to t is related to the STM from t to t as

One example:

STM Property #3

The STM is a symplectic matrix

A *2n* x *2n* matrix, **S**, is symplectic if it satisfies the condition

$$\mathbf{S}^T\mathbf{J}\mathbf{S} = \mathbf{J}$$

where $$\mathbf{J} = \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} \\ -\mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix}$$

A symplectic transformation matrix is useful for understanding properties of the solution space that are preserved through the transformation, e.g., the volume of the phase space

STM Property #3

To prove that the STM is symplectic at any time *t*, note:

1) The STM is symplectic at time t_0

2) The time derivative of the symplectic condition is 0 for all time

Because the initial STM is symplectic, it is symplectic for all time

→This is useful information because it tells us that the determinant of the STM is always equal to unity

(See proof on pg 106 of Koon, Lo, Marsden & Ross, "Dynamical Systems, the Three-Body Problem and Space Mission Design" that initially uses Hamiltonian formulation → then, use Legendre transform to prove STM symplectic in Lagrangian formulation)

STM Property #3

Write the CR3BP in Hamiltonian form with $\bar{z} = \begin{bmatrix} \bar{q} \\ \bar{p} \end{bmatrix}$

Then, $\dot{\bar{z}} = J \nabla H(\bar{z})$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad\qquad p_i = \frac{\partial L}{\partial \dot{q}_i}$$

In this Hamiltonian system, denote STM as $\mathbf{\Psi}(t, t_0)$

Where $\quad \dot{\mathbf{\Psi}} = \mathbf{J} H_{\bar{z}\bar{z}} \mathbf{\Psi}$

And $\quad \mathbf{\Psi}(t_0, t_0) = \mathbf{I}_{2n \times 2n}$

To prove that the STM is symplectic at any time *t*, let's prove:

1) The STM is symplectic at time t_0

2) The time derivative of the symplectic condition is 0 for all time

STM Property #3

1) At time t_0: $\mathbf{S}^T\mathbf{J}\mathbf{S} = \mathbf{\Psi}(t_0,t_0)^T\mathbf{J}\mathbf{\Psi}(t_0,t_0) = \mathbf{I}_{6\times 6}^T\mathbf{J}\mathbf{I}_{6\times 6} = \mathbf{J}$

2) Time derivative of the LHS of the symplectic condition is:

$$\frac{d}{dt}\left(\mathbf{\Psi}(t,t_0)^T\mathbf{J}\mathbf{\Psi}(t,t_0)\right) = \dot{\mathbf{\Psi}}(t,t_0)^T\mathbf{J}\mathbf{\Psi}(t,t_0) + \mathbf{\Psi}(t,t_0)^T\mathbf{J}\dot{\mathbf{\Psi}}(t,t_0)$$

$$\frac{d}{dt}\left(\mathbf{\Psi}(t,t_0)^T\mathbf{J}\mathbf{\Psi}(t,t_0)\right) = \mathbf{\Psi}(t,t_0)^T H_{\bar{z}\bar{z}}^T\mathbf{J}^T\mathbf{J}\mathbf{\Psi}(t,t_0) + \mathbf{\Psi}(t,t_0)^T\mathbf{J}\mathbf{J}H_{\bar{z}\bar{z}}\mathbf{\Psi}(t,t_0)$$

$$\frac{d}{dt}\left(\mathbf{\Psi}(t,t_0)^T\mathbf{J}\mathbf{\Psi}(t,t_0)\right) = \mathbf{\Psi}(t,t_0)^T\left[H_{\bar{z}\bar{z}}^T - H_{\bar{z}\bar{z}}\right]\mathbf{\Psi}(t,t_0)$$

$$\frac{d}{dt}\left(\mathbf{\Psi}(t,t_0)^T\mathbf{J}\mathbf{\Psi}(t,t_0)\right) = \mathbf{0}$$

Since $H_{\bar{z}\bar{z}}$ symmetric in Hamiltonian system

STM Property #3

Relate the STM $\mathbf{\Psi}(t, t_0)$ in the Hamiltonian problem with $\bar{z} = \begin{bmatrix} \bar{q} \\ \bar{p} \end{bmatrix}$

to the STM $\mathbf{\Phi}(t, t_0)$ in the Lagrangian problem with $\bar{x} = \begin{bmatrix} \bar{q} \\ \dot{\bar{q}} \end{bmatrix}$

via Legendre transformation.

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} - y \qquad p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y} + x \qquad p_z = \frac{\partial L}{\partial \dot{z}} = \dot{z}$$

$$\mathbf{\Phi}(t, t_0) = \mathbf{S}(t)^{-1}\mathbf{\Psi}(t, t_0)\mathbf{S}(t_0)$$

If interested, see Tsuda and Scheeres, 2009, “Computation and Applications of an Orbital Dynamics Symplectic State Transition Matrix”, JGCD, Vol. 32, No. 4

Importance of STM Symplectic Property

- This symplectic property also supplies useful insight into the eigenvalues of a general STM
- If an STM satisfies the symplectic condition,

- Define an eigenvalue of an STM as λ_i
- Then, the eigenvalues of the inverse of the STM are $1/\lambda_i$
- However, similar matrices possess equivalent eigenvalues.

- Legendre transformation to Lagrangian formulation corresponds to a symplectic transformation. Thus, observations above hold for associated STM in Lagrangian formulation.