Rigid Body Kinematics

ASEN 5010

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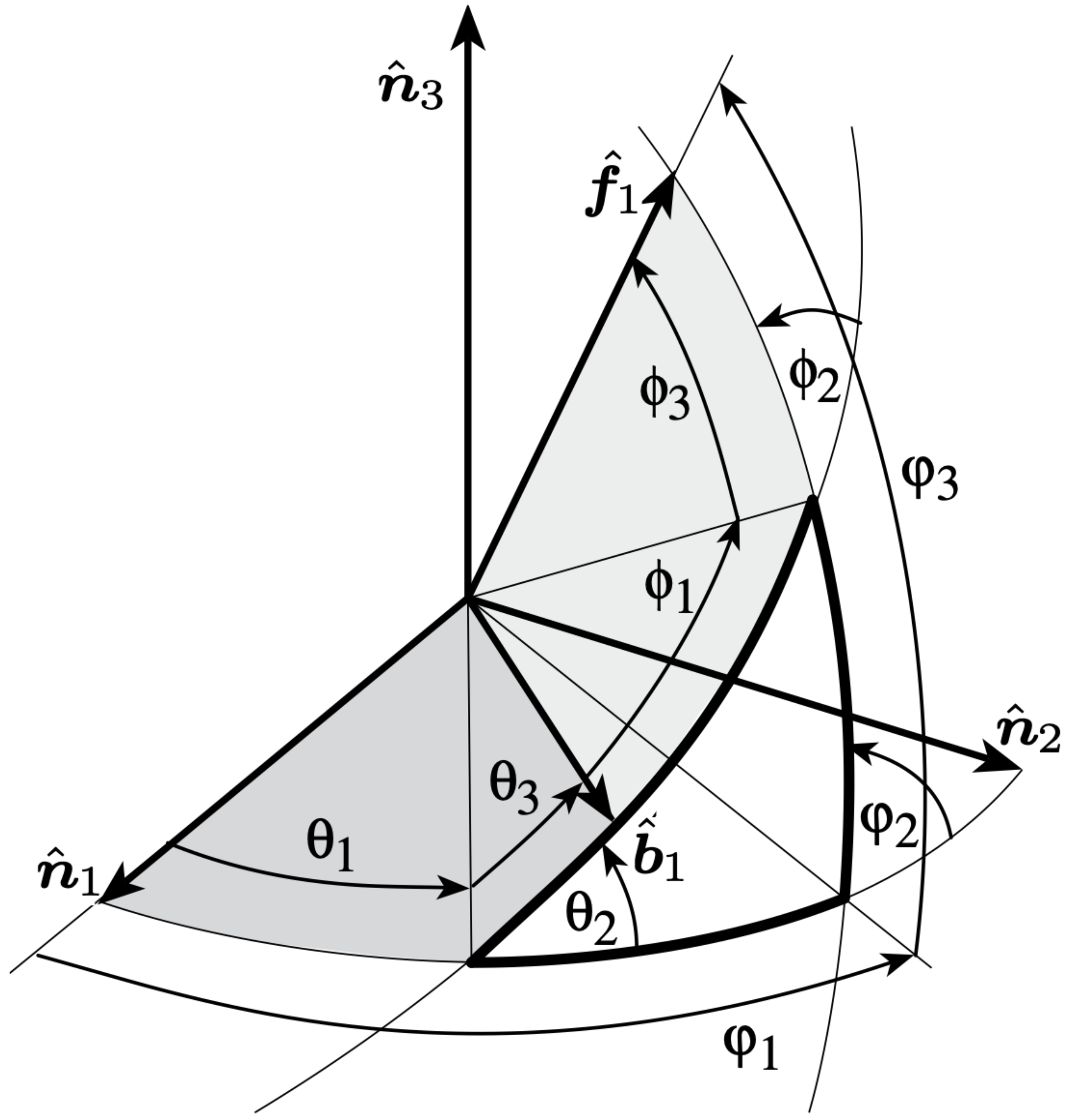
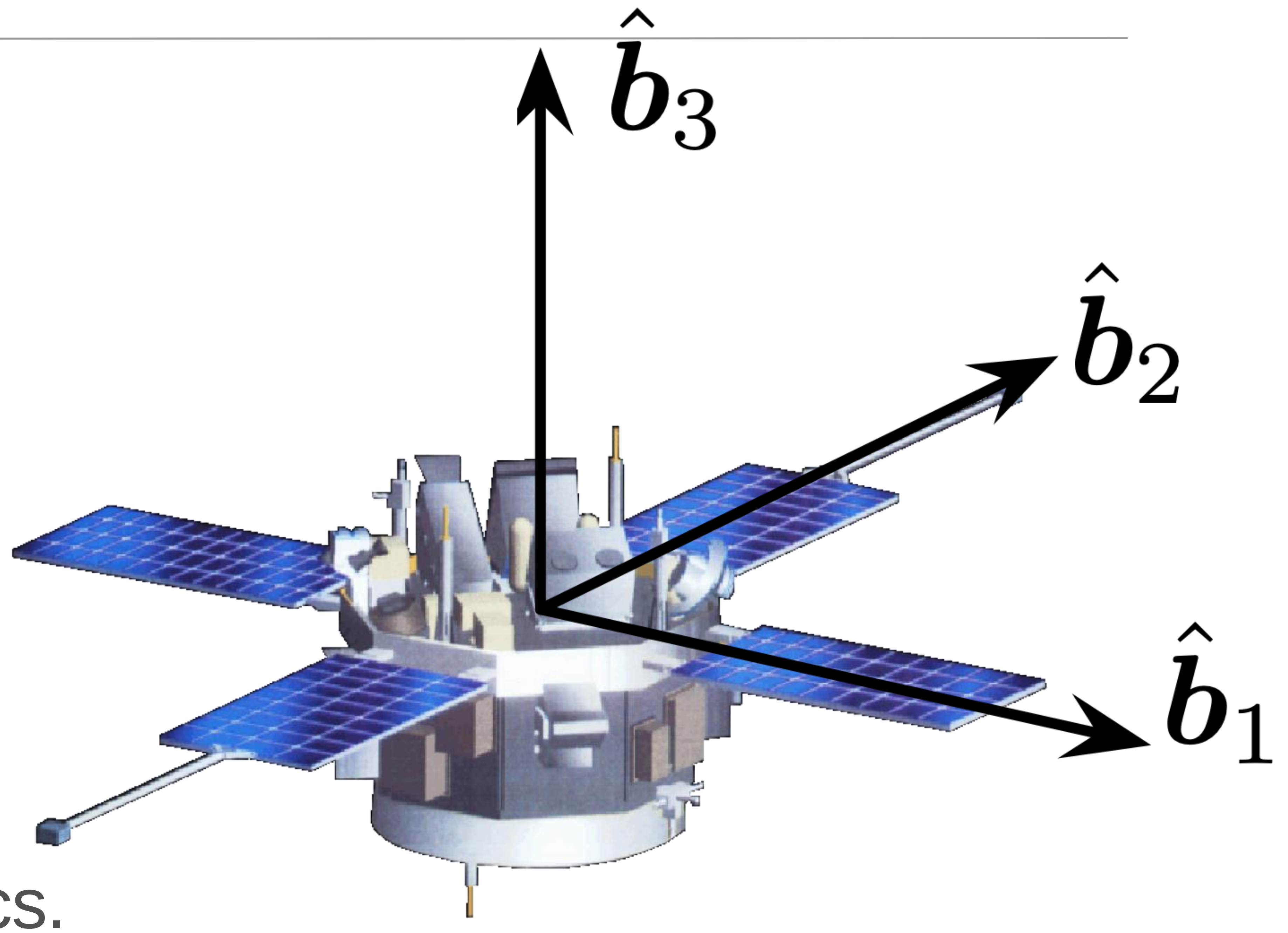
Outline

- Direction Cosine Matrix
- Euler Angle Sets
- Principal Rotation Parameters
- Euler Parameters (Quaternions)
- Classical Rodrigues Parameters
- Modified Rodrigues Parameters

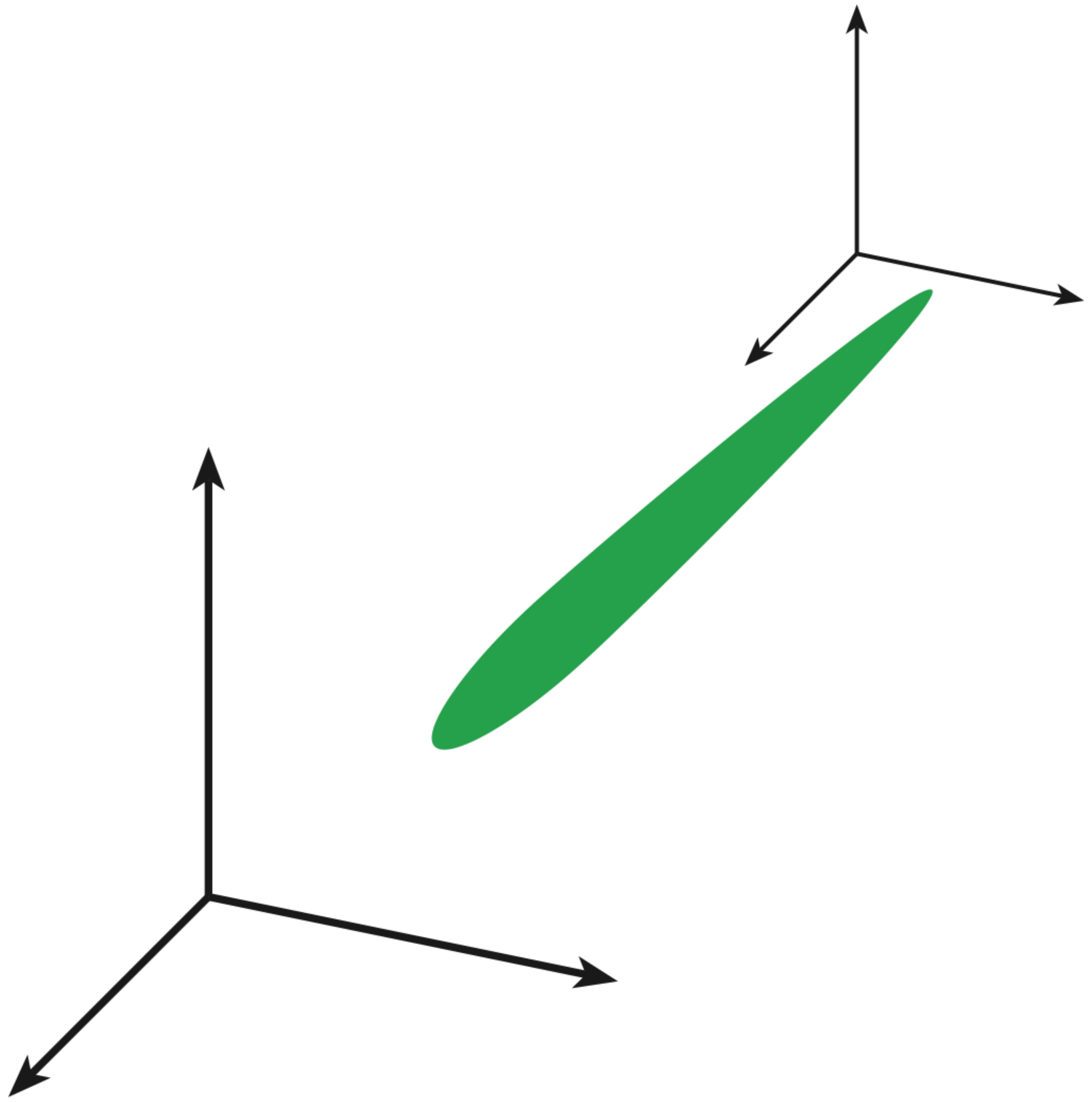


Introduction

- Attitude coordinates are set of coordinates that describe of both a rigid body or a reference frame
- An infinite number of coordinate choices exists, same as with position coordinates
- A good choice in attitude coordinates can greatly simplify the mathematics of the problem solving process
- A bad choice in attitude coordinates can introduce singularities in the attitude description, as well as highly nonlinear mathematics.



Relation to Position Coordinates



Translational errors can grow infinitely large!

Attitude errors can to grow to 180˚!

4 “Truths” about Attitude Coordinates

- A minimum of **three coordinates** is required to describe the relative angular displacement between two reference frames.
- Any minimal set of three coordinates will contain at least one geometrical orientation where the coordinates are **singular**, namely at least two coordinates are **undefined** or not unique.
- At or near such a geometric singularity, the corresponding **kinematic differential equations are also singular**.
- The geometric singularities and associated numerical difficulties can be avoided altogether through regularization. **Redundant sets of four or more coordinates exist that are universally valid**.

Direction Cosine Matrix

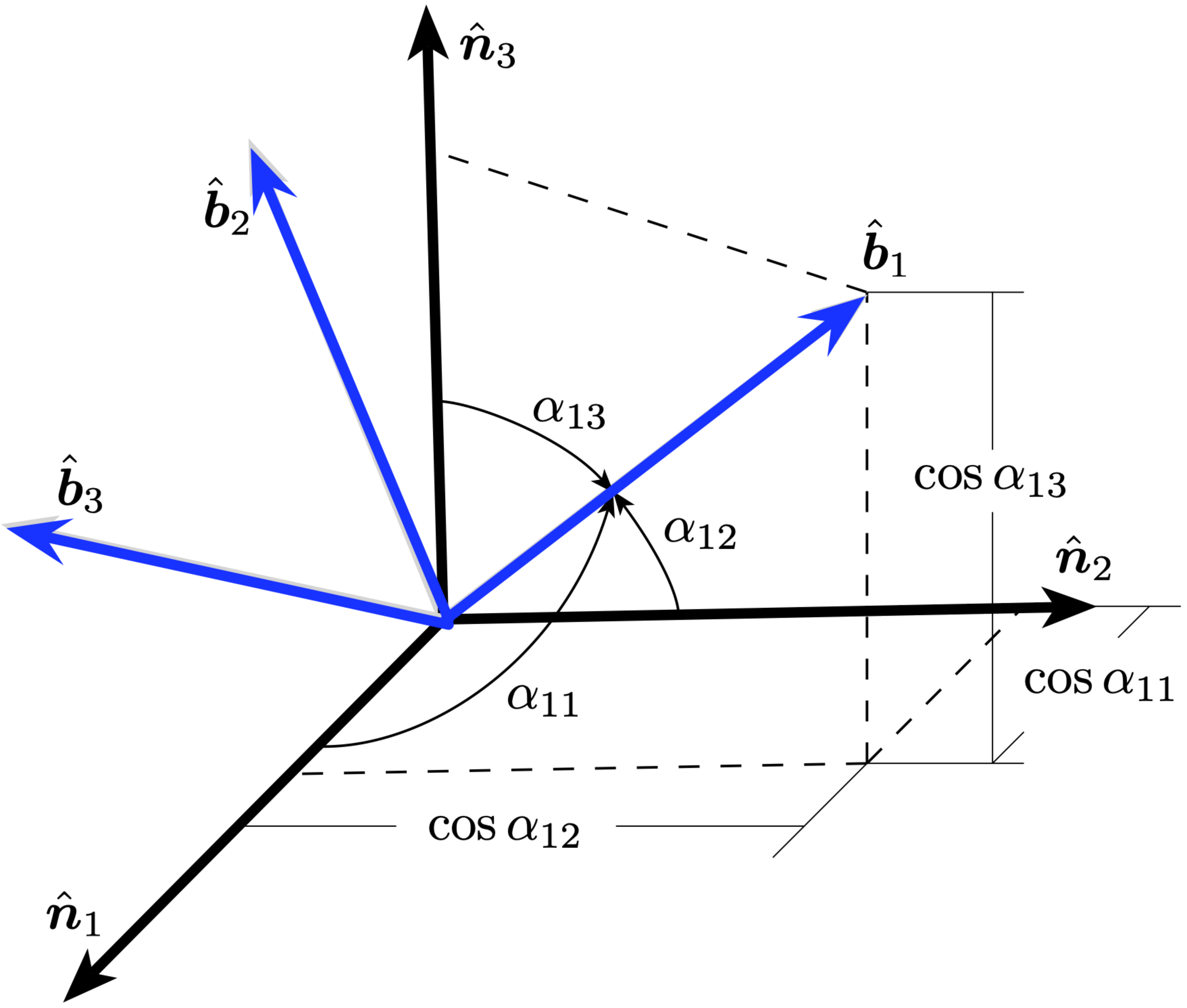
The mother of all attitude parameterizations...

Coordinate Frames

- A vectrix is a matrix of vectors.

$$\{\hat{\boldsymbol{n}}\} \equiv \begin{bmatrix} \hat{\boldsymbol{n}}_1 \\ \hat{\boldsymbol{n}}_2 \\ \hat{\boldsymbol{n}}_3 \end{bmatrix}$$

$$\{\hat{\boldsymbol{b}}\} \equiv \begin{bmatrix} \hat{\boldsymbol{b}}_1 \\ \hat{\boldsymbol{b}}_2 \\ \hat{\boldsymbol{b}}_3 \end{bmatrix}$$



Coordinate Frames

Frame base vectors are related through:

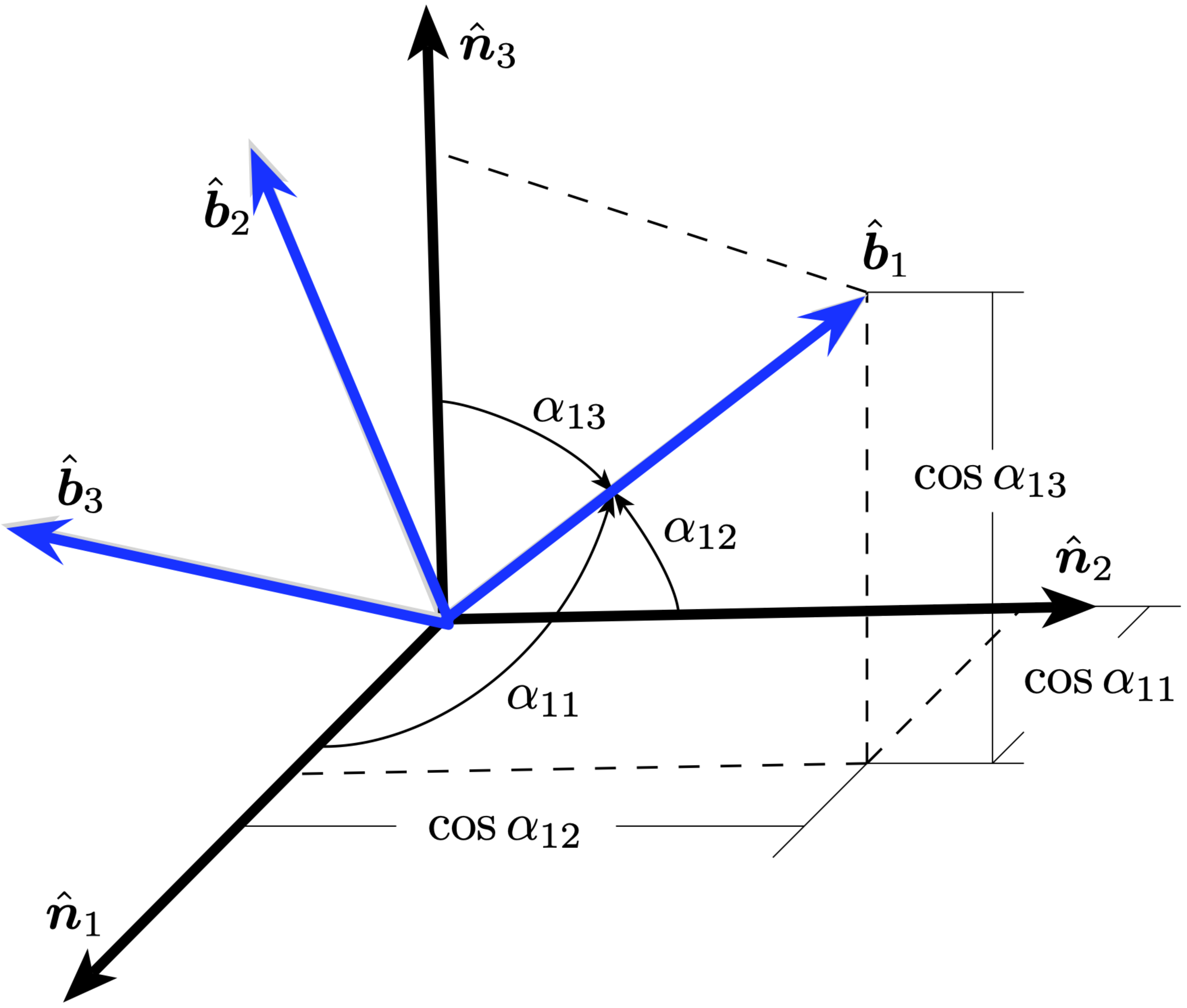
$$\hat{\boldsymbol{b}}_1 = \cos\alpha_{11}\hat{\boldsymbol{n}}_1 + \cos\alpha_{12}\hat{\boldsymbol{n}}_2 + \cos\alpha_{13}\hat{\boldsymbol{n}}_3$$
$$\hat{\boldsymbol{b}}_2 = \cos\alpha_{21}\hat{\boldsymbol{n}}_1 + \cos\alpha_{22}\hat{\boldsymbol{n}}_2 + \cos\alpha_{23}\hat{\boldsymbol{n}}_3$$
$$\hat{\boldsymbol{b}}_3 = \cos\alpha_{31}\hat{\boldsymbol{n}}_1 + \cos\alpha_{32}\hat{\boldsymbol{n}}_2 + \cos\alpha_{33}\hat{\boldsymbol{n}}_3$$

$$\{\hat{\boldsymbol{b}}\} = \begin{bmatrix} \cos\alpha_{11} & \cos\alpha_{12} & \cos\alpha_{13} \\ \cos\alpha_{21} & \cos\alpha_{22} & \cos\alpha_{23} \\ \cos\alpha_{31} & \cos\alpha_{32} & \cos\alpha_{33} \end{bmatrix} \{\hat{\boldsymbol{n}}\} = [C]\{\hat{\boldsymbol{n}}\}$$

Note that: $C_{ij} = \cos(\angle\, \hat{\boldsymbol{b}}_i, \hat{\boldsymbol{n}}_j) = \hat{\boldsymbol{b}}_i \cdot \hat{\boldsymbol{n}}_j$

Analogously, we can find:

$$\{\hat{\boldsymbol{n}}\} = \begin{bmatrix} \cos\alpha_{11} & \cos\alpha_{21} & \cos\alpha_{31} \\ \cos\alpha_{12} & \cos\alpha_{22} & \cos\alpha_{32} \\ \cos\alpha_{13} & \cos\alpha_{23} & \cos\alpha_{33} \end{bmatrix} \{\hat{\boldsymbol{b}}\} = [C]^T\{\hat{\boldsymbol{b}}\}$$



Matrix Inverse

Combining these two results, we find

$$\{\hat{\boldsymbol{b}}\} = [C][C]^T\{\hat{\boldsymbol{b}}\} \quad \longrightarrow \quad [C][C]^T = [I_{3\times 3}]$$

$$\{\hat{\boldsymbol{n}}\} = [C]^T[C]\{\hat{\boldsymbol{n}}\} \quad \longrightarrow \quad [C]^T[C] = [I_{3\times 3}]$$

Therefore, the inverse of a direction cosine matrix is simply the transpose operation.

$$\boxed{[C]^{-1} = [C]^T}$$

Coordinate Frame Transformation

- Let a vector have its components taken in the body frame *B* or the inertial frame *N*:

$$\boldsymbol{v} = v_{b_1}\hat{\boldsymbol{b}}_1 + v_{b_2}\hat{\boldsymbol{b}}_2 + v_{b_3}\hat{\boldsymbol{b}}_3 = \{v_b\}^T\{\hat{\boldsymbol{b}}\}$$

- we can now rearrange the vector expression as

$$\boldsymbol{v} = v_{n_1}\hat{\boldsymbol{n}}_1 + v_{n_2}\hat{\boldsymbol{n}}_2 + v_{n_3}\hat{\boldsymbol{n}}_3 = \{v_n\}^T\{\hat{\boldsymbol{n}}\}$$

- Equating components, we find that the two vector component sets must be related through

$$\boldsymbol{v} = \{v_n\}^T\{\hat{\boldsymbol{n}}\} = \{v_n\}^T[C]^T\{\hat{\boldsymbol{b}}\} = \{v_b\}^T\{\hat{\boldsymbol{b}}\}$$

- From here on, we will make use of the short-hand notation:

$$\boldsymbol{v}_b = [C]\boldsymbol{v}_n \qquad \boldsymbol{v}_n = [C]^T\boldsymbol{v}_b$$

$${}^{\mathcal{B}}\boldsymbol{v} \equiv \boldsymbol{v}_b \qquad {}^{\mathcal{N}}\boldsymbol{v} \equiv \boldsymbol{v}_n$$

DCM Determinant

- Let's find the determinant of the [*C*] by first evaluating

$$\det\left(CC^T\right) = \det\left([I_{3\times3}]\right) = 1$$

- Since [*C*] is a square matrix, we find that

$$\det(C)\det(C^T) = 1$$

- Because det([*C*]) is the same as det([*C*]^T), this is further reduced to

$$(\det(C))^2 = 1 \iff \det(C) = \pm 1$$

- Note that this is true for any orthogonal matrix.
- For a proper rotation matrix with right-handed coordinate system, then det(*C*) = +1.

Adding DCM's

- Assume three coordinate frames given: $\mathcal{N} : \{\hat{\boldsymbol{n}}\}$ $\mathcal{B} : \{\hat{\boldsymbol{b}}\}$ $\mathcal{R} : \{\hat{\boldsymbol{r}}\}$
- Let *N* and *B* frame orientation be related through $\{\hat{\boldsymbol{b}}\} = [C]\{\hat{\boldsymbol{n}}\}$
- Let *R* and *B* frame orientation be related through $\{\hat{\boldsymbol{r}}\} = [C']\{\hat{\boldsymbol{b}}\}$
- Then the *R* and *N* frame orientation are directly related through $\{\hat{\boldsymbol{r}}\} = [C'][C]\{\hat{\boldsymbol{n}}\} = [C'']\{\hat{\boldsymbol{n}}\}$
- Let us introduce the two-letter DCM notation [*NB*] as mapping from *B* to *N* frame, then the DCM addition is

$$\boxed{[RN] = [RB][BN]}$$

Kinematic Differential Equation

- What does this mean??
  - kinematic ➠ position description

what is $\boxed{[\dot{C}] = \frac{\mathrm{d}}{\mathrm{d}t}[C]}$

  - differential equation ➠ time rate equation
- How does the [*C*] direction cosine matrix evolve over time. The rotation rate of a rigid body is expressed through the body angular velocity vector:

$$\boldsymbol{\omega} = \omega_1 \hat{\boldsymbol{b}}_1 + \omega_2 \hat{\boldsymbol{b}}_2 + \omega_3 \hat{\boldsymbol{b}}_3$$

- This vector determines how a body will rotate, and thus also how the DCM describing the orientation will evolve.

Kinematic Differential Equation

- Let's study how the body frame vectors will evolve over time as seen by the inertial frame. To do so, we differentiate the vectrix of body frame orientation vectors.

$$\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{b}}_i = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{b}}_i + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \hat{\boldsymbol{b}}_i$$

- Let us introduce the matrix cross-product operator: $[\tilde{\boldsymbol{x}}] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$ where $\boldsymbol{x} \times \boldsymbol{y} \equiv [\tilde{\boldsymbol{x}}]\boldsymbol{y}$ and $[\tilde{\boldsymbol{x}}]^T = -[\tilde{\boldsymbol{x}}]$

- The body frame vectrix differential equation is then simply

$$\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\{\hat{\boldsymbol{b}}\} = -[\tilde{\boldsymbol{\omega}}]\{\hat{\boldsymbol{b}}\}$$

Kinematic Differential Equation

- Next take the inertial derivative of $\{\hat{\boldsymbol{b}}\} = [C]\{\hat{\boldsymbol{n}}\}$

$$\frac{{}^N\mathrm{d}}{\mathrm{d}t}\{\hat{\boldsymbol{b}}\} = \frac{{}^N\mathrm{d}}{\mathrm{d}t}\left([C]\{\hat{\boldsymbol{n}}\}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left([C]\right)\{\hat{\boldsymbol{n}}\} + [C]\frac{{}^N\mathrm{d}}{\mathrm{d}t}\left(\{\hat{\boldsymbol{n}}\}\right) = [\dot{C}]\{\hat{\boldsymbol{n}}\}$$

$$\frac{{}^N\mathrm{d}}{\mathrm{d}t}\{\hat{\boldsymbol{b}}\} = -[\tilde{\boldsymbol{\omega}}]\{\hat{\boldsymbol{b}}\} = -[\tilde{\boldsymbol{\omega}}][C]\{\hat{\boldsymbol{n}}\} = [\dot{C}]\{\hat{\boldsymbol{n}}\}$$

- This leads to $\left([\dot{C}] + [\tilde{\boldsymbol{\omega}}][C]\right)\{\hat{\boldsymbol{n}}\} = 0$
- Since this must be true for any *N* frame orientation, we find

$$\boxed{[\dot{C}] = -[\tilde{\boldsymbol{\omega}}][C]}$$

Kinematic Differential Equation

- An interesting fact is that this matrix differential equation holds for *any NxN orthogonal matrix*!

$$\frac{\mathrm{d}}{\mathrm{d}t}\left([C][C]^T\right) = [\dot{C}][C]^T + [C][\dot{C}]^T = 0$$

using the differential equation $[\dot{C}] = -[\tilde{\boldsymbol{\omega}}][C]$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left([C][C]^T\right) = -[\tilde{\boldsymbol{\omega}}][C][C]^T - [C][C]^T[\tilde{\boldsymbol{\omega}}]^T$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left([C][C]^T\right) = -[\tilde{\boldsymbol{\omega}}] + [\tilde{\boldsymbol{\omega}}] = 0$$

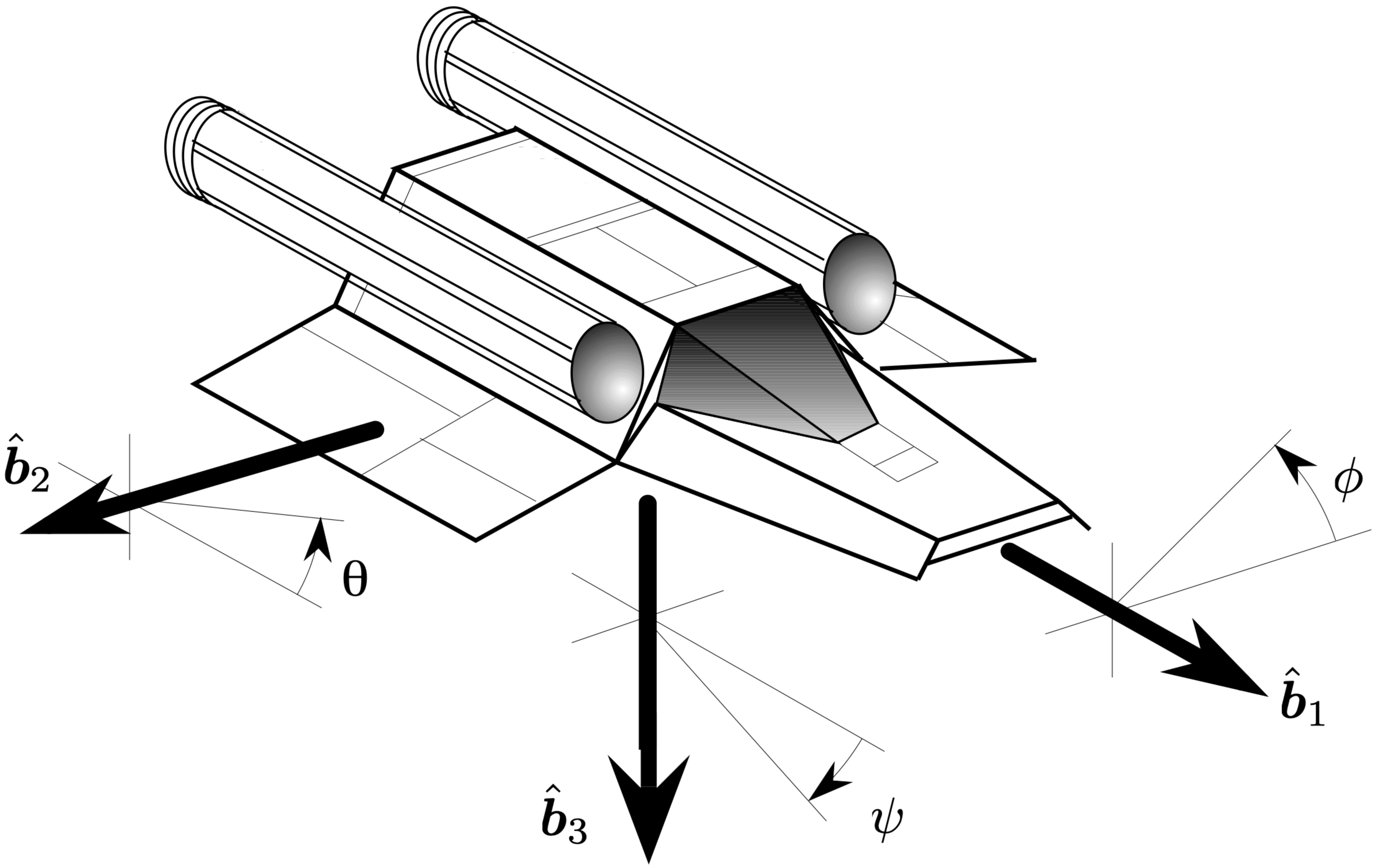
Euler Angles

The 101 of attitude coordinates...

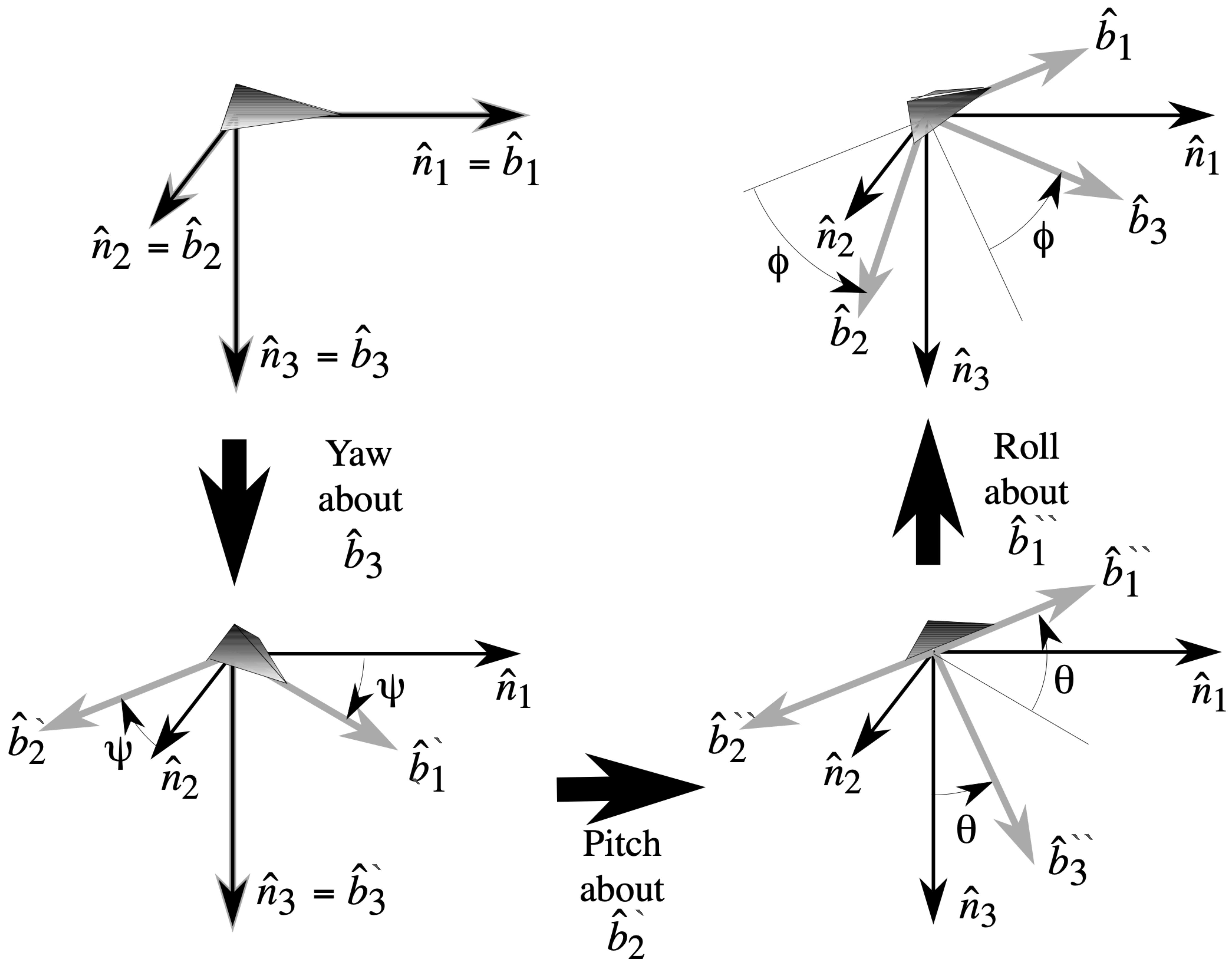
Description

- Most common set of attitude coordinates
- Describe the orientation between two frames using three *sequential* rotations
- Note that the order of rotation is important
- (i-j-k) Euler angles means we rotate first about the i^{th} axis, then about the j^{th} axis, and lastly about the k^{th} axis
- (3-2-1) Euler angles are the typical aircraft and spacecraft attitude angles
- Simple to visualize for small rotations

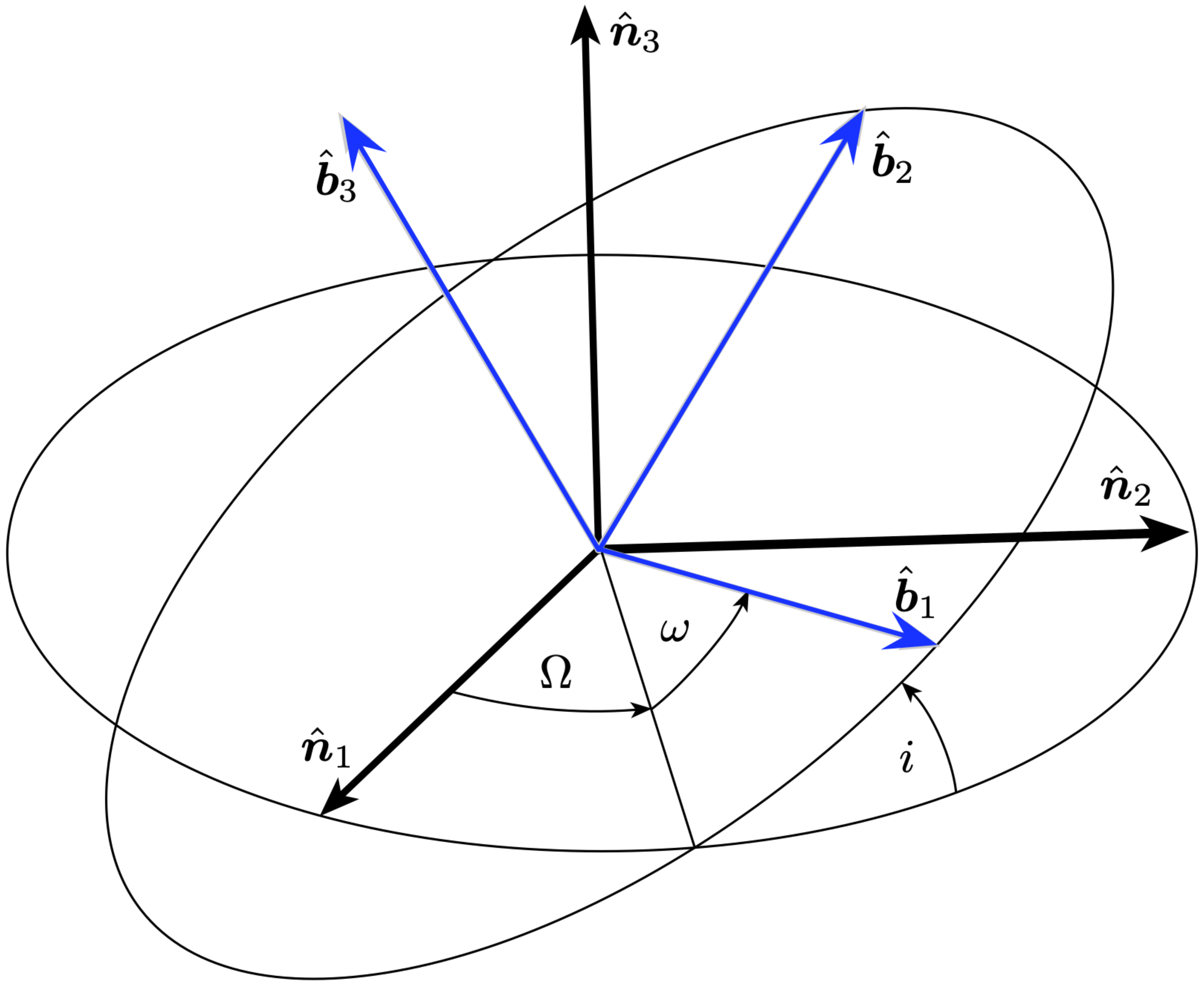
Aircraft/Spacecraft Orientation Angles



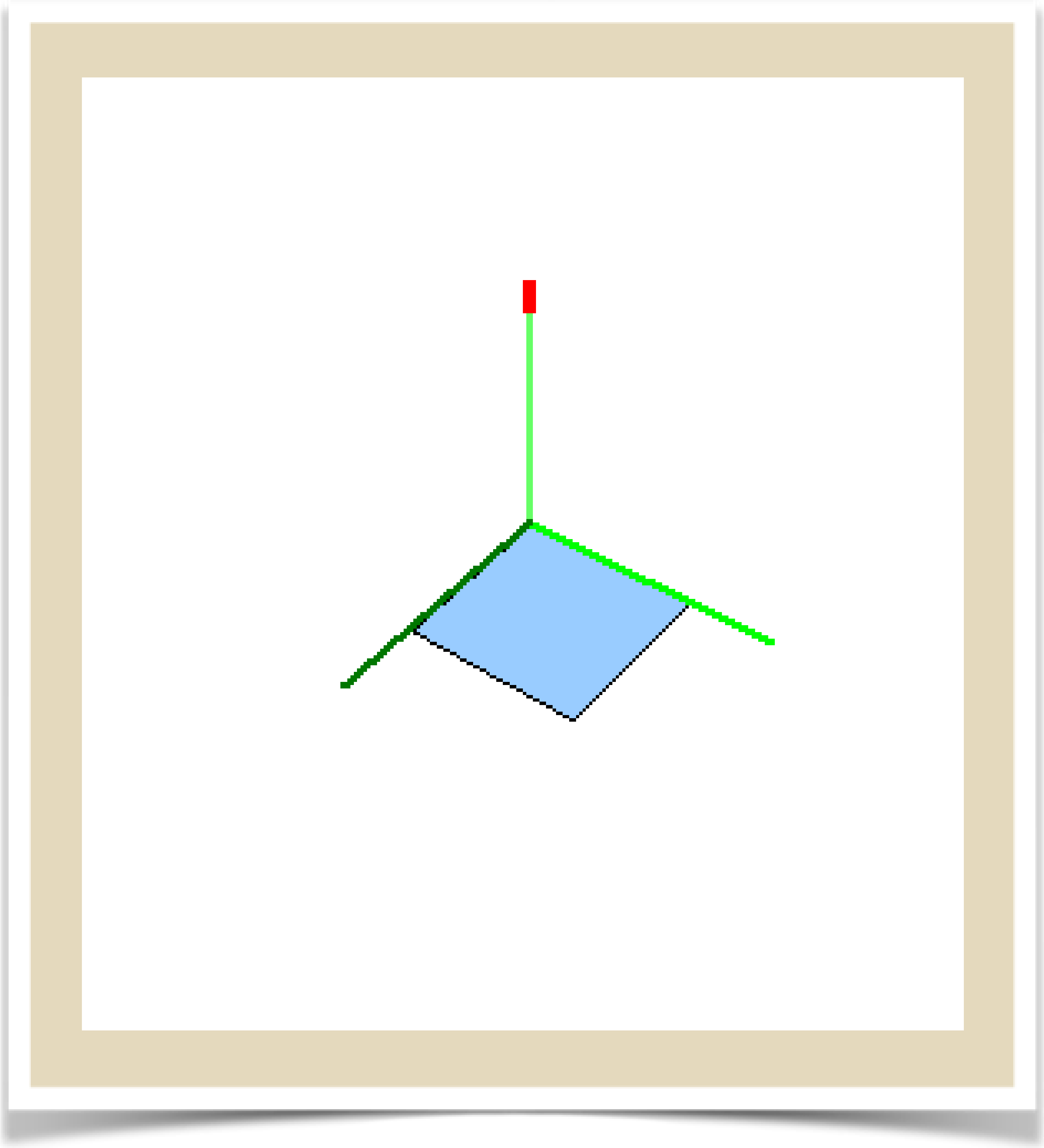
(3-2-1) Euler Angle Illustration



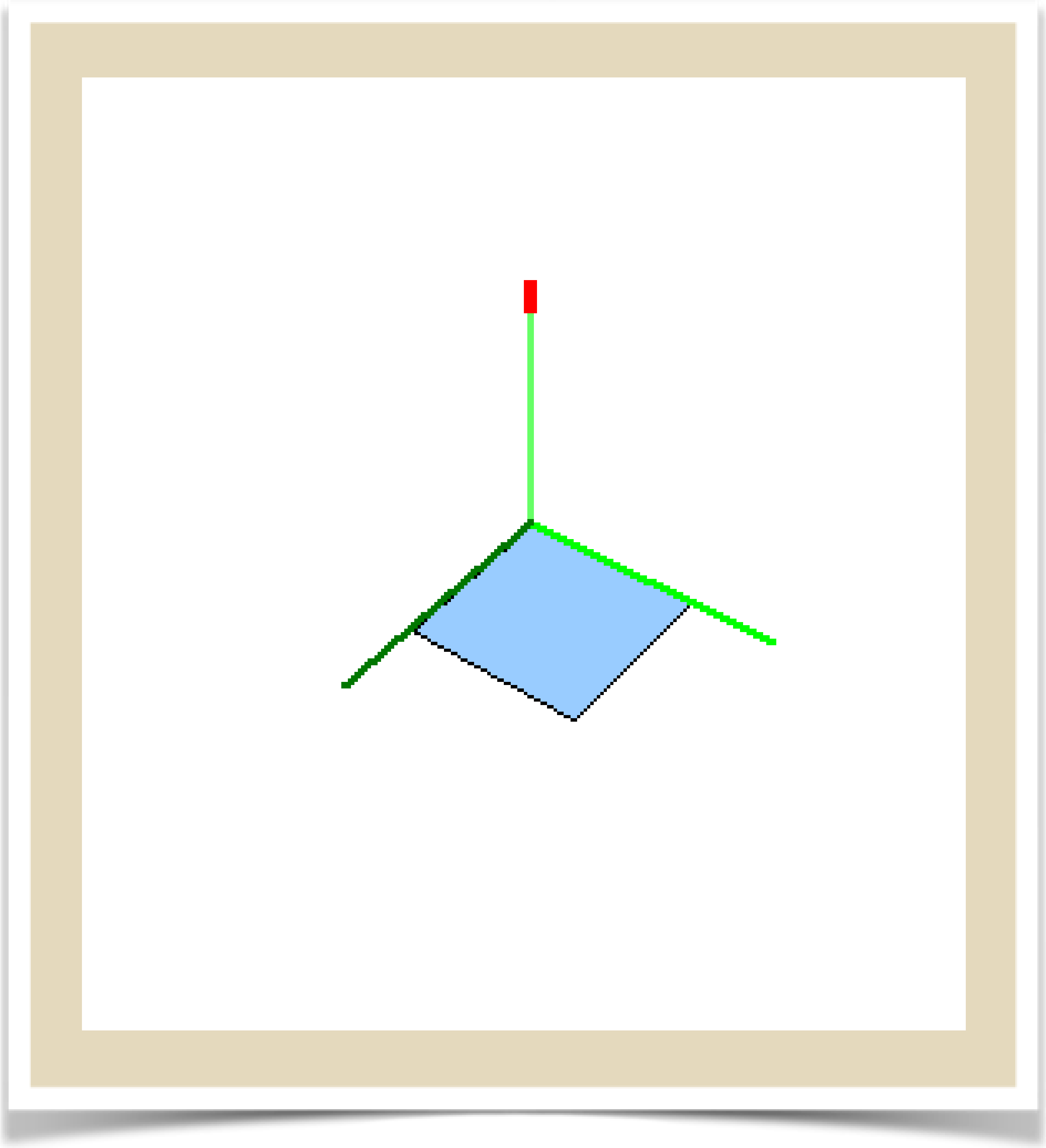
(3-1-3) Euler Angles



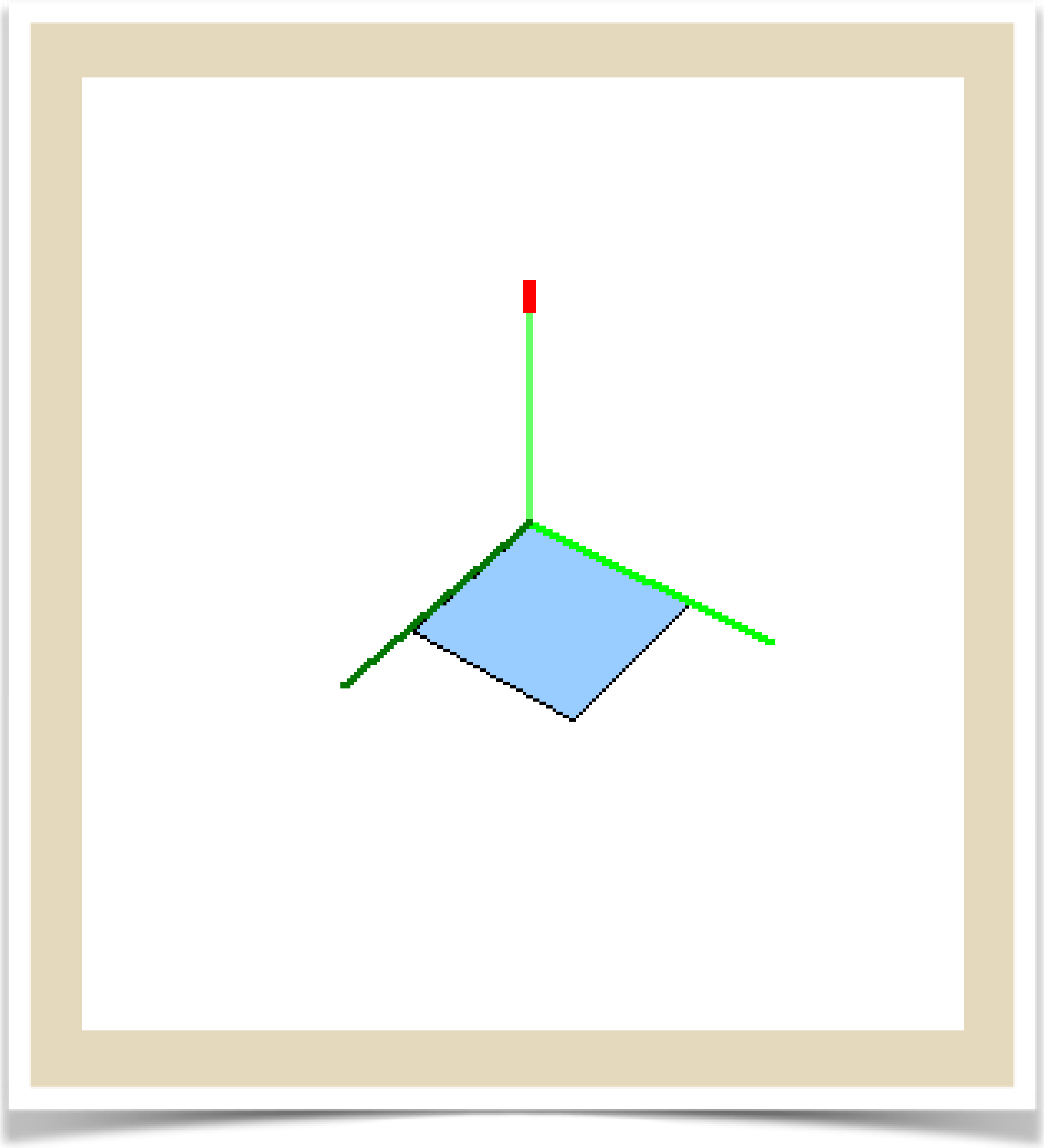
Commonly used to describe the orbit frame orientation relative to the inertial Frame *N*.



(3-2-1) Euler Angles
(60,50,70) Degrees



(3-1-3) Euler Angles
(60,50,70) Degrees



(3-2-1) Euler Angles
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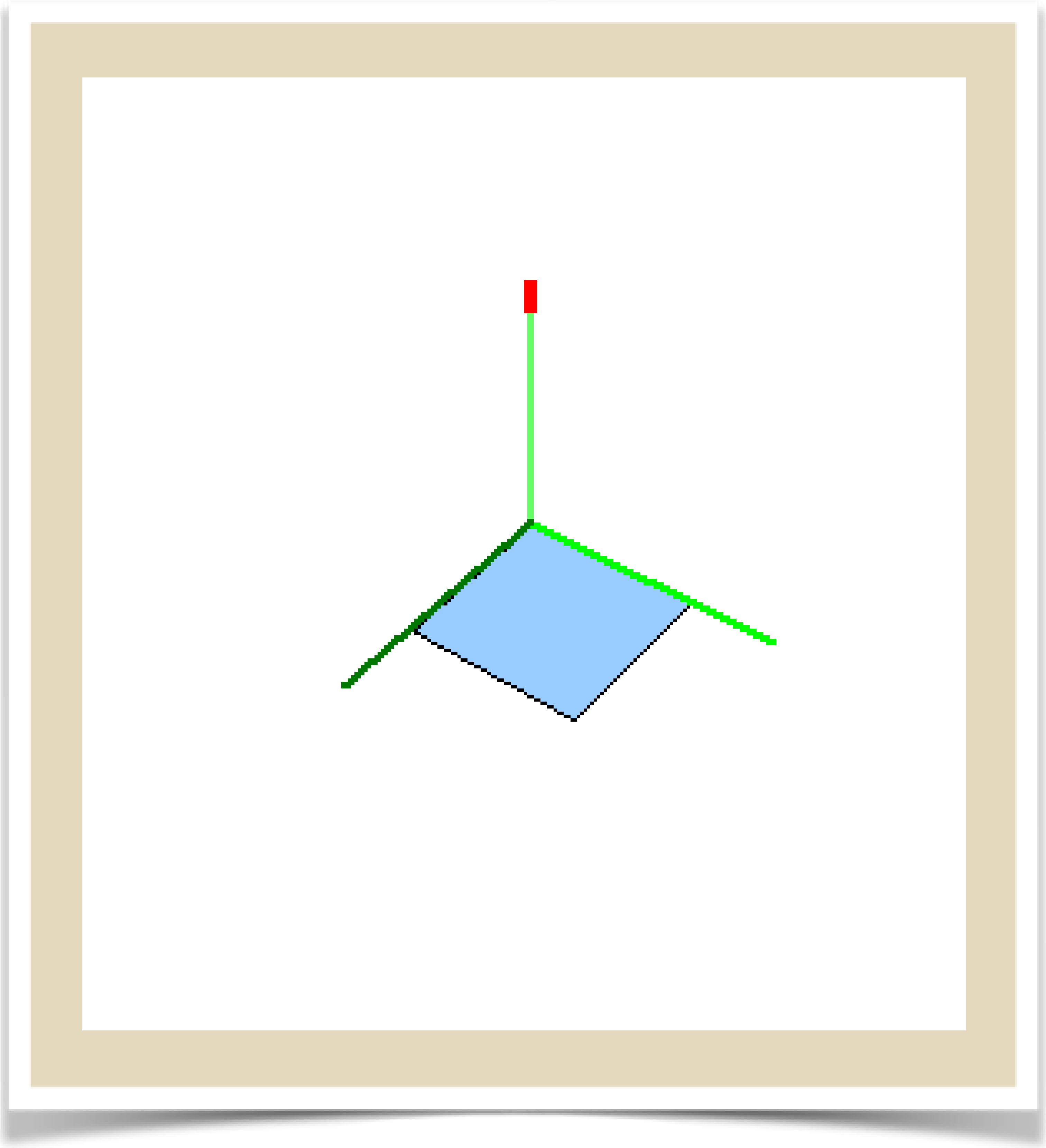
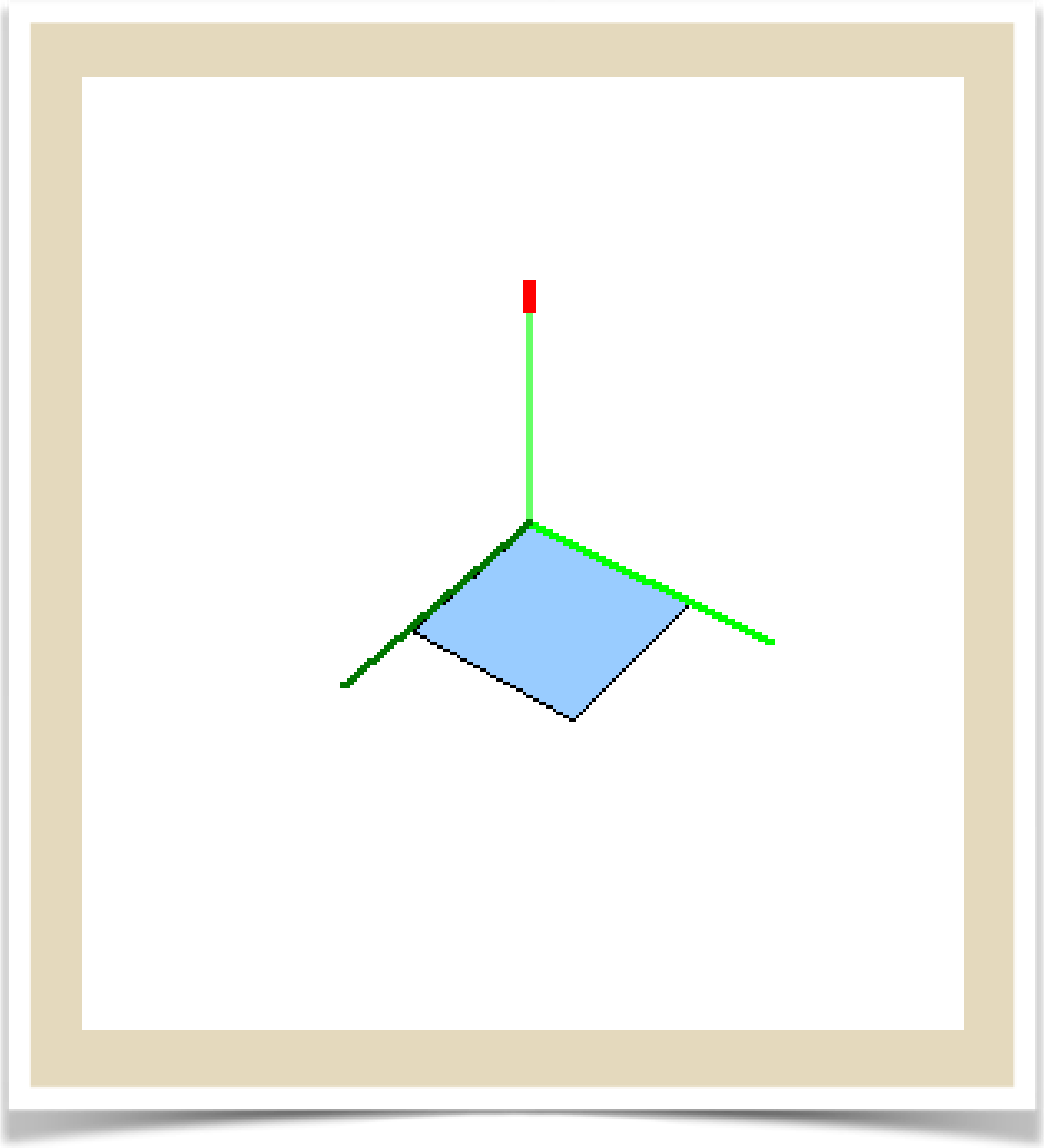
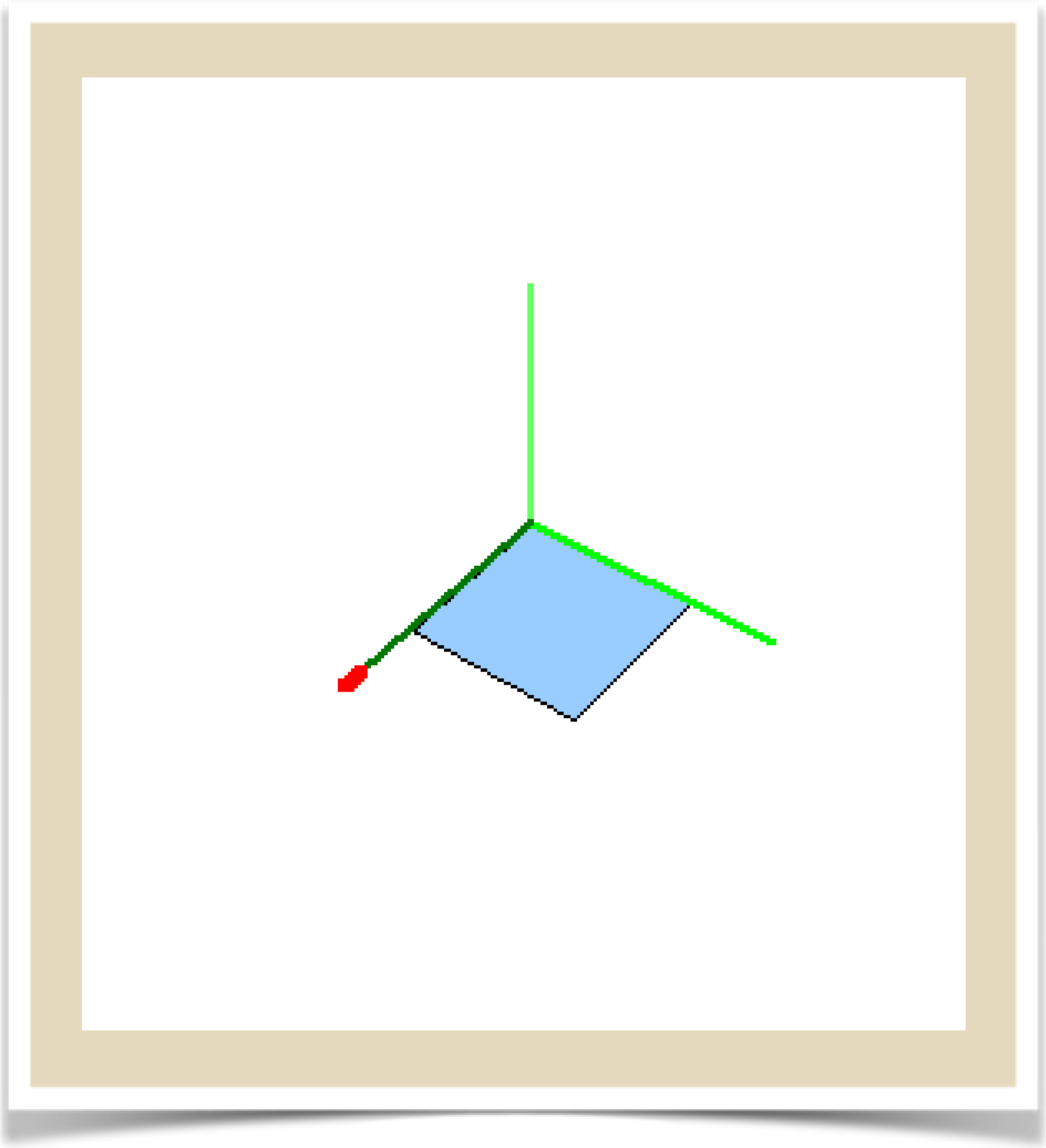


(3-1-3) Euler Angles
(75.6,77.3,-51.7) Degrees



(3-2-1) Euler Angles
(60,50,70) Degrees



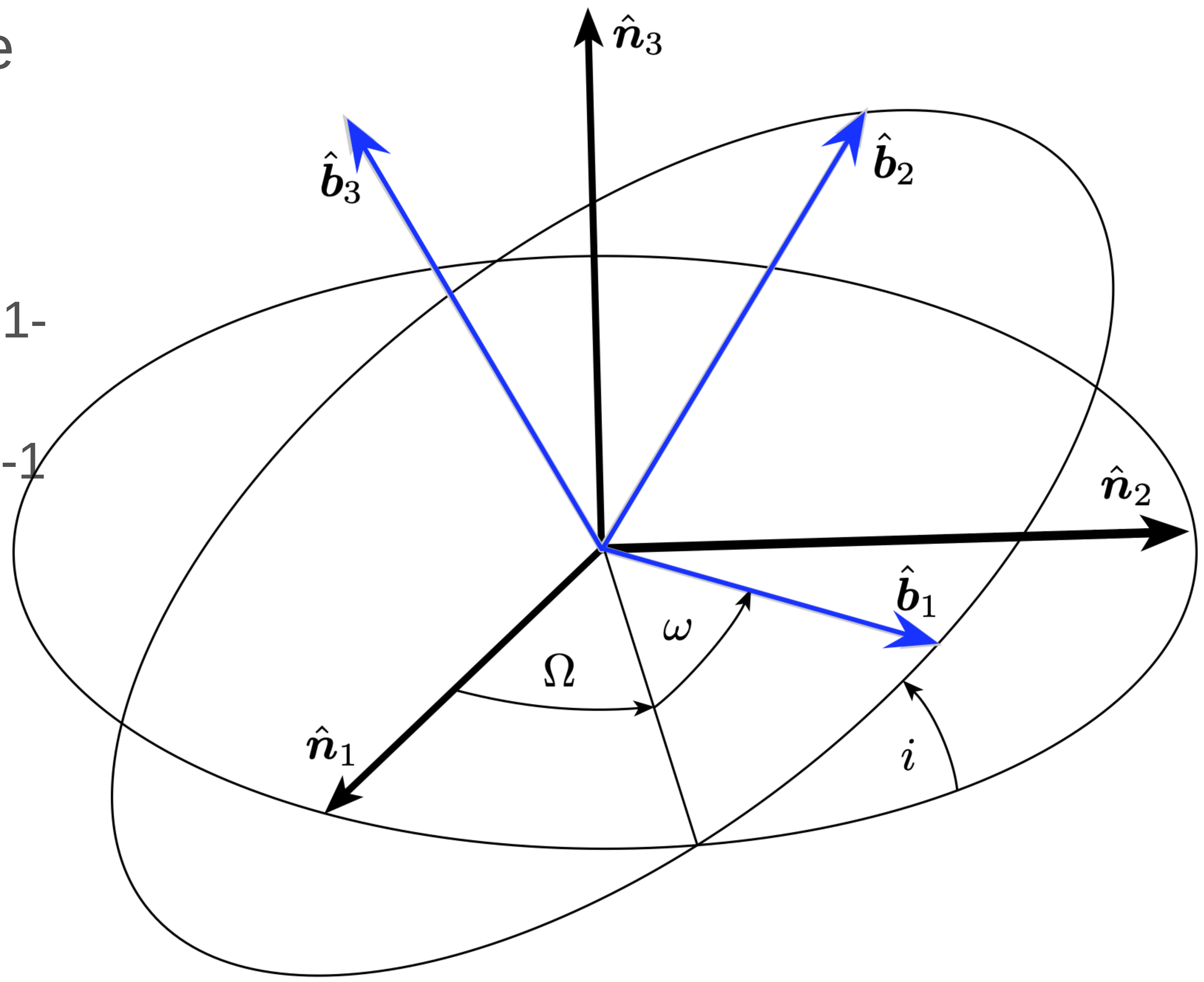
(1-3-2) Euler Angles
(37.2,-3.7,71.2) Degrees

Types of Euler Angles

- There are two types of Euler angles
  - Symmetric Set: Here the first and last rotation axis number is repeated. For example: 3-1-3 set used in astrodynamics to describe the orbit plane
  - Asymmetric Set: Here no axis rotation number is repeated. For example, the 3-2-1 (yaw-pitch-roll) angles used to describe many vehicles.
- Each type of Euler angles will have common mathematical properties and singularities.

Singularities

- Each set of Euler angles has a geometric singularity where two angles are not uniquely defined.
- It is always the second angle which causes trouble
  - Symmetric Set: 2nd angle is 0 or 180 degrees. For example, the 3-1-3 orbit angles with zero inclination.
  - Asymmetric Set: 2nd angle is +/- 90 degrees. For example, the 3-2-1 angle of an aircraft with 90 degrees pitch.



Single-Axis DCM

- The rotation matrix $[M_i]$ for a single axis rotation about the i^{th} body axis is given by

$$[M_1(\theta)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

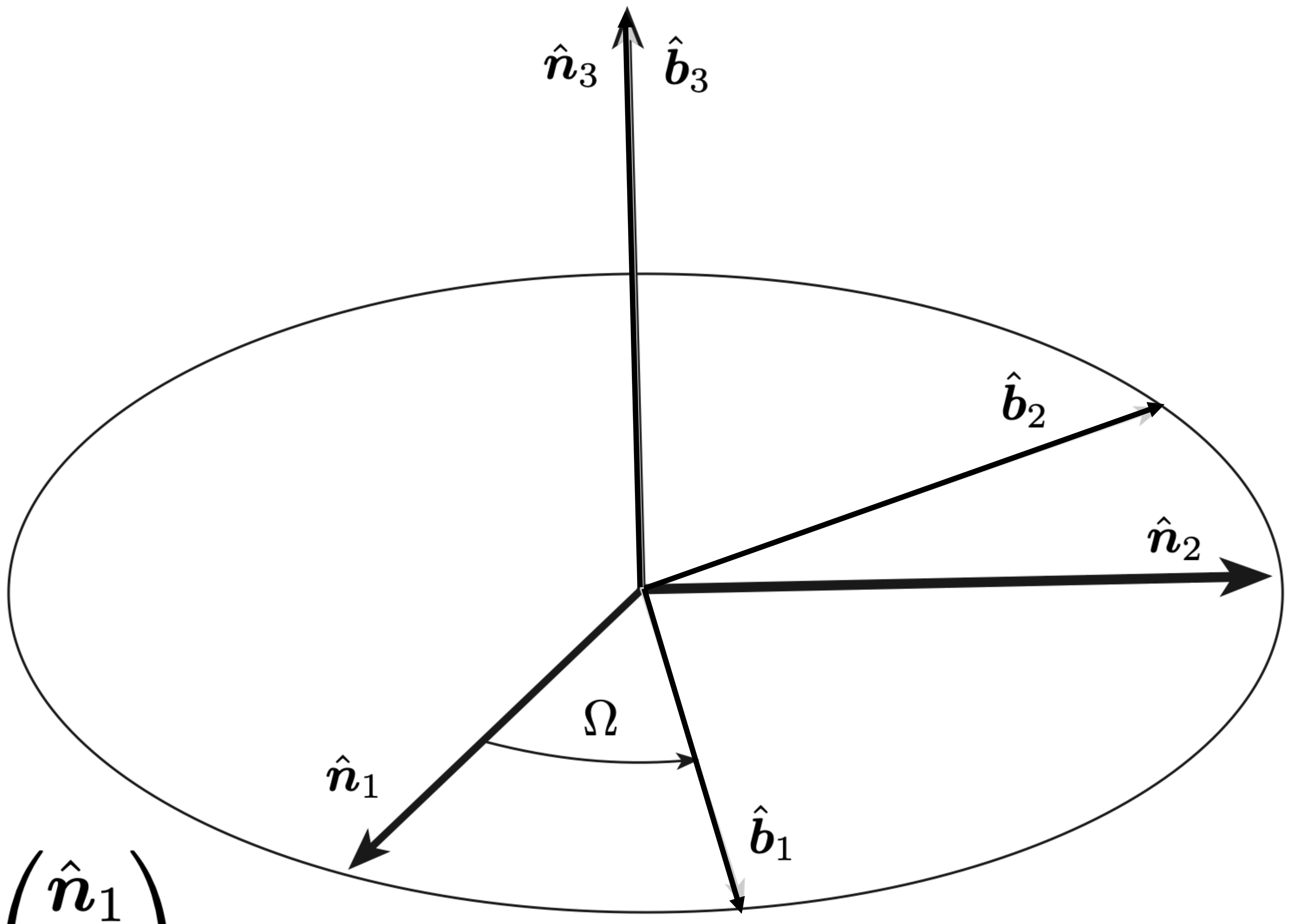
$$[M_2(\theta)] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$[M_3(\theta)] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

- Consider the 3-axis rotation using Ω
- The *B* and *N* frame axis are related through

$$\hat{\boldsymbol{b}}_1 = \cos\Omega \hat{\boldsymbol{n}}_1 + \sin\Omega \hat{\boldsymbol{n}}_2$$
$$\hat{\boldsymbol{b}}_2 = -\sin\Omega \hat{\boldsymbol{n}}_1 + \cos\Omega \hat{\boldsymbol{n}}_2$$
$$\hat{\boldsymbol{b}}_3 = \hat{\boldsymbol{n}}_3$$



This allows us to write

$$\begin{pmatrix} \hat{\boldsymbol{b}}_1 \\ \hat{\boldsymbol{b}}_2 \\ \hat{\boldsymbol{b}}_3 \end{pmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \hat{\boldsymbol{n}}_1 \\ \hat{\boldsymbol{n}}_2 \\ \hat{\boldsymbol{n}}_3 \end{pmatrix}$$

Mapping Euler Angles to Rotation Matrix

- Let the (α, β, γ) Euler angle sequence be $(\theta_1, \theta_2, \theta_3)$ To obtain the final rotation matrix [*BN*] which maps inertial frame vector components to body frame vector components, we make use of the composite rotation matrix property [*RN*]=[*RB*][*BN*].

$$[C(\theta_1, \theta_2, \theta_3)] = [M_\gamma(\theta_3)][M_\beta(\theta_2)][M_\alpha(\theta_1)]$$

- Carrying out this matrix algebra, we can find formulas which will map any Euler angle set to the corresponding rotation matrix.

3-2-1 Euler Angles

- Given the yaw, pitch and roll angles, we can compute the DCM using the three elemental rotation matrices:

$$[BN] = [M_1(\theta_3)][M_2(\theta_2)][M_3(\theta_1)] = [M_1(\phi)][M_2(\theta)][M_3(\psi)]$$

$$[BN] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3-2-1 Euler Angles

- Forward mapping is given by:

$$[BN] = \begin{bmatrix} c\theta_2 c\theta_1 & c\theta_2 s\theta_1 & -s\theta_2 \\ s\theta_3 s\theta_2 c\theta_1 - c\theta_3 s\theta_1 & s\theta_3 s\theta_2 s\theta_1 + c\theta_3 c\theta_1 & s\theta_3 c\theta_2 \\ c\theta_3 s\theta_2 c\theta_1 + s\theta_3 s\theta_1 & c\theta_3 s\theta_2 s\theta_1 - s\theta_3 c\theta_1 & c\theta_3 c\theta_2 \end{bmatrix}$$

- Inverse mapping back to Euler angles is found by examining the matrix element entries.

$$\psi = \theta_1 = \tan^{-1}\left(\frac{C_{12}}{C_{11}}\right)$$

$$\theta = \theta_2 = -\sin^{-1}(C_{13})$$

$$\phi = \theta_3 = \tan^{-1}\left(\frac{C_{23}}{C_{33}}\right)$$

Note that the quadrants must be checked with the inverse tangent function!

3-1-3 Euler Angles

- Forward mapping is given by:
- Inverse mapping back to Euler angles is found by examining the matrix element entries.

$$[BN] = \begin{bmatrix} c\theta_3 c\theta_1 - s\theta_3 c\theta_2 s\theta_1 & c\theta_3 s\theta_1 + s\theta_3 c\theta_2 c\theta_1 & s\theta_3 s\theta_2 \\ -s\theta_3 c\theta_1 - c\theta_3 c\theta_2 s\theta_1 & -s\theta_3 s\theta_1 + c\theta_3 c\theta_2 c\theta_1 & c\theta_3 s\theta_2 \\ s\theta_2 s\theta_1 & -s\theta_2 c\theta_1 & c\theta_2 \end{bmatrix}$$

$$\Omega = \theta_1 = \tan^{-1}\left(\frac{C_{31}}{-C_{32}}\right)$$

$$i = \theta_2 = \cos^{-1}\left(C_{33}\right)$$

$$\omega = \theta_3 = \tan^{-1}\left(\frac{C_{13}}{C_{23}}\right)$$

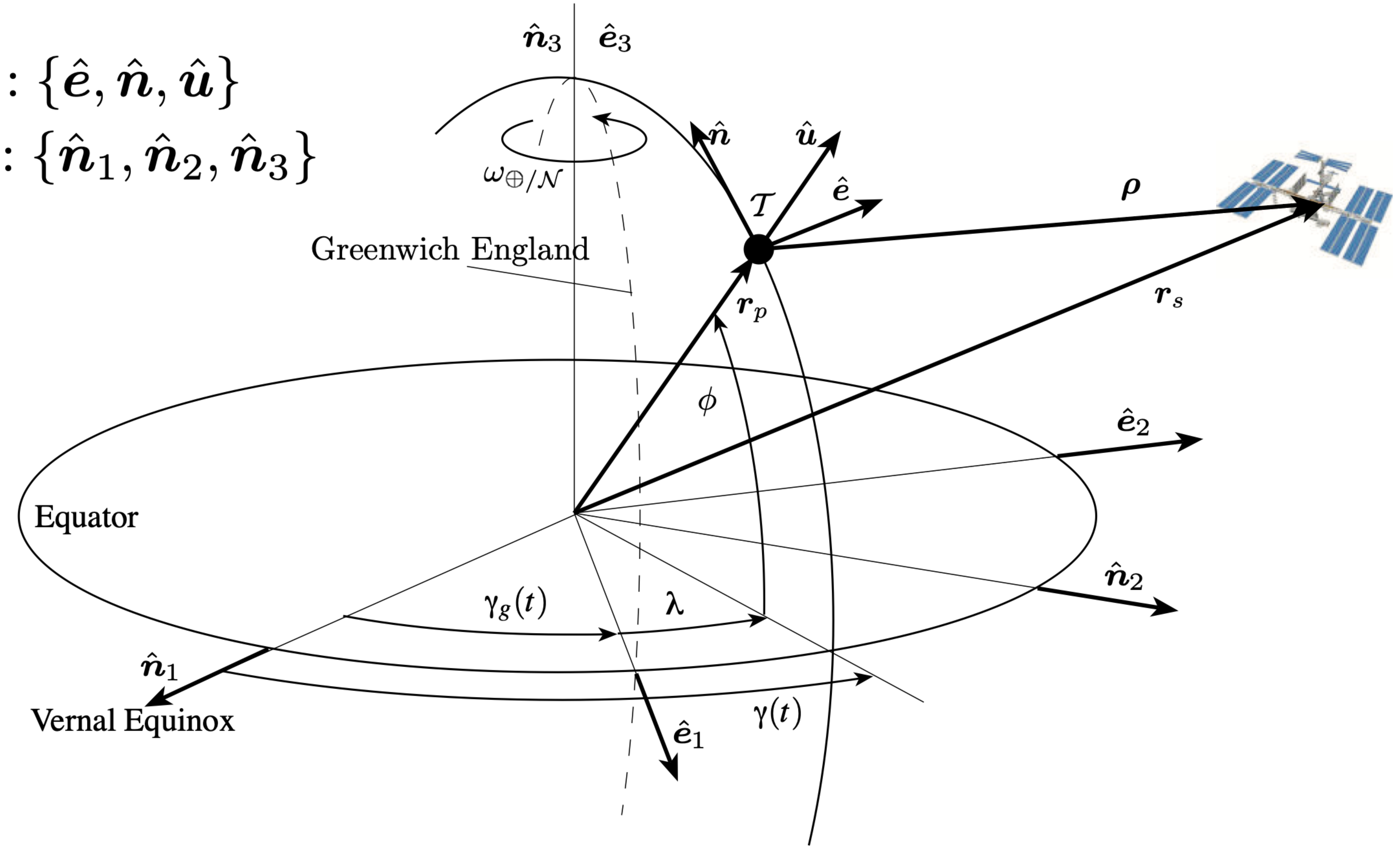
Note that the quadrants must be checked with the inverse tangent function!

Example

- Consider the astrodynamics problem, where the topographic frame (surface frame) *T* is defined as shown in the figure below.

$$\mathcal{T} : \{\hat{\boldsymbol{e}}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{u}}\}$$

$$\mathcal{N} : \{\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}$$



- Here the rotation matrix [*TN*] was given as

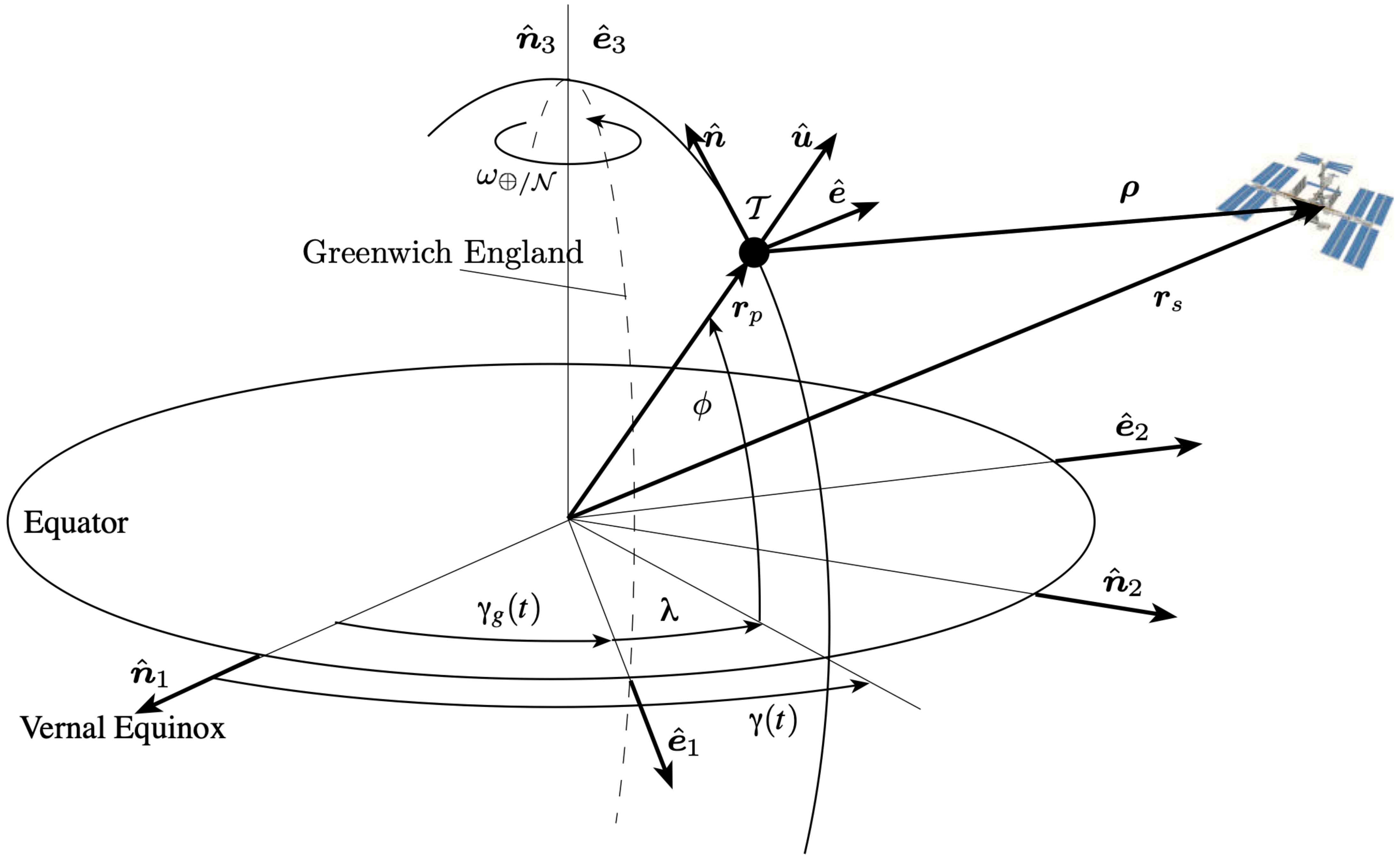
$$[TN] = \begin{bmatrix} -\sin\gamma(t) & \cos\gamma(t) & 0 \\ -\cos\gamma(t)\sin\phi & -\sin\gamma(t)\sin\phi & \cos\phi \\ \cos\gamma(t)\cos\phi & \sin\gamma(t)\cos\phi & \sin\phi \end{bmatrix}$$

- Let's derive this rotation matrix expression. To go from the *N* frame to the *T* frame, the first rotation is a 3-axis rotation by the angle Ω.

$$[M_3(\gamma(t))] = \begin{bmatrix} \cos\gamma(t) & \sin\gamma(t) & 0 \\ -\sin\gamma(t) & \cos\gamma(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- The next rotation is about the 2-axis with the angle Φ.

$$[M_2(-\phi)] = \begin{bmatrix} \cos(-\phi) & 0 & -\sin(-\phi) \\ 0 & 1 & 0 \\ \sin(-\phi) & 0 & \cos(-\phi) \end{bmatrix}$$

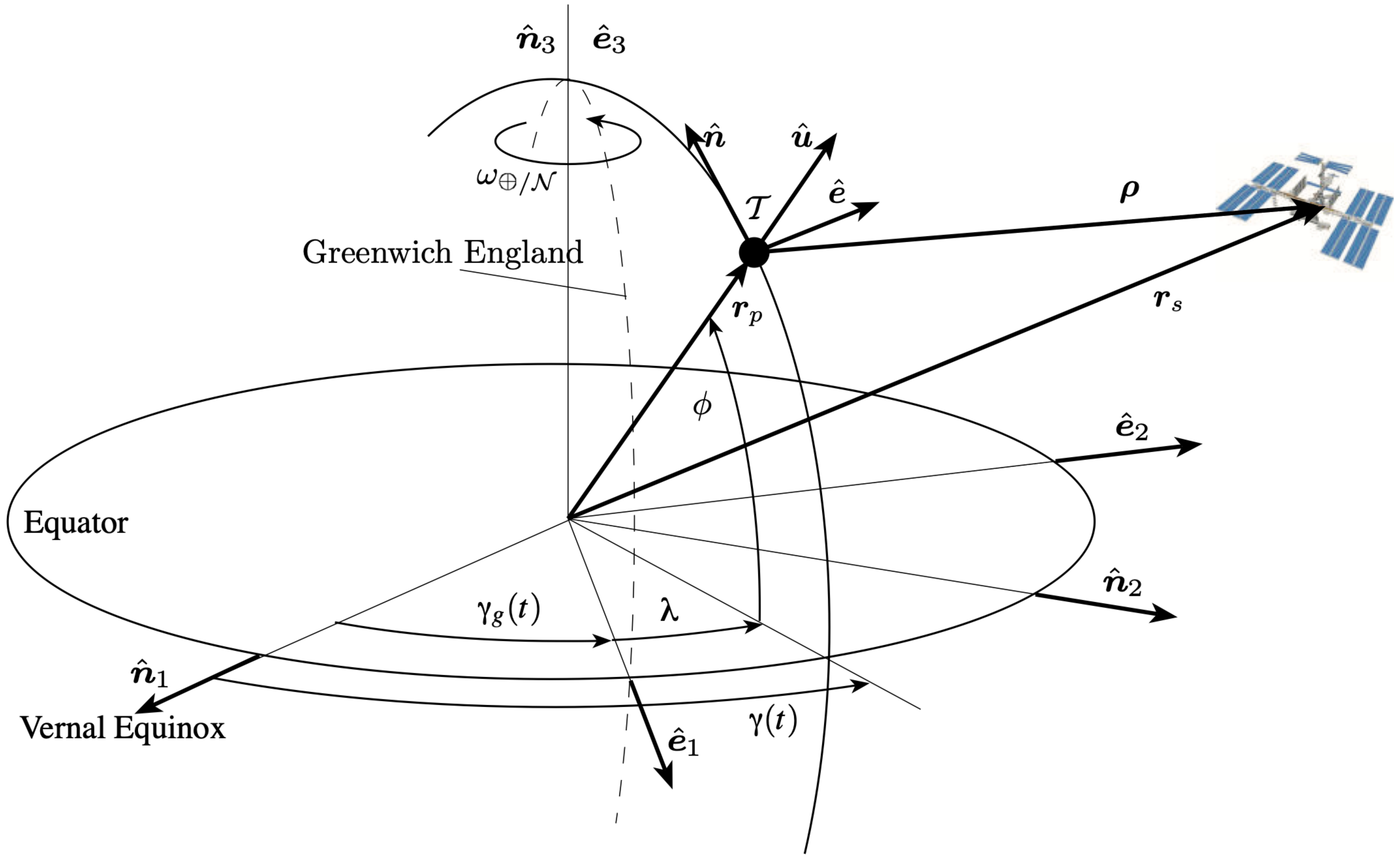


- However, we are not yet done. We still need to align the 1,2 and 3 axis of our current frame to that of the *T* frame. First we correct the 1-axis by doing a 90 degree rotation about our current 3-axis

- Next, we fix both the 2 and 3 axis orientation by doing 90 degree rotation about the current 1-axis.

$$[M_3(90^o)] = \begin{bmatrix} \cos 90^o & \sin 90^o & 0 \\ -\sin 90^o & \cos 90^o & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[M_1(90^o)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 90^o & \sin 90^o \\ 0 & -\sin 90^o & \cos 90^o \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$



- Finally, we add up all these rotation matrices to find the desired [*TN*] direction cosine matrix:

$$[TN] = [M_1(90^o)][M_3(90^o)][M_2(-\phi)][M_3(\gamma(t))]$$

$$[TN] = \begin{bmatrix} -\sin\gamma(t) & \cos\gamma(t) & 0 \\ -\cos\gamma(t)\sin\phi & -\sin\gamma(t)\sin\phi & \cos\phi \\ \cos\gamma(t)\cos\phi & \sin\gamma(t)\cos\phi & \sin\phi \end{bmatrix}$$

Rotation Addition

- Assume we have a yaw-pitch-roll rotation defined from the inertial frame *N* to the reference frame *R* through

$$\boldsymbol{\theta}_{RN} = \{\psi_{RN}, \theta_{RN}, \phi_{RN}\}$$

- Assume we also know the yaw-pitch-roll rotation defined from the reference frame *R* to the body frame *B* through

$$\boldsymbol{\theta}_{BR} = \{\psi_{BR}, \theta_{BR}, \phi_{BR}\}$$

- The question is, what are the yaw-pitch-roll angles that will take us directly from the inertial frame *N* to the body frame *B*.

$$\boldsymbol{\theta}_{BN} = \{\psi_{BN}, \theta_{BN}, \phi_{BN}\}$$

- Note that

$$\boldsymbol{\theta}_{BN} \neq \boldsymbol{\theta}_{BR} + \boldsymbol{\theta}_{RN}$$

Rotation Addition

- To add two Euler angle rotations, we go back to the rotation matrix addition property. First, we find:

$$\boldsymbol{\theta}_{BR} \quad \Rightarrow \quad [BR(\boldsymbol{\theta}_{BR})] \qquad \boldsymbol{\theta}_{RN} \quad \Rightarrow \quad [RN(\boldsymbol{\theta}_{RN})]$$

- Then, we compute [*BN*] using:
- Last, we find the desired 3-2-1 Euler angles using the inverse mapping:

$$[BN(\boldsymbol{\theta}_{BN})] = [BR(\boldsymbol{\theta}_{BR})][RN(\boldsymbol{\theta}_{RN})]$$

$$[BN(\boldsymbol{\theta}_{BN})] \quad \Rightarrow \quad \boldsymbol{\theta}_{BN} = \{\psi_{BN}, \theta_{BN}, \phi_{BN}\}$$

Rotation Subtraction

- Similarly, assume that we are given: $\boldsymbol{\theta}_{BN} = \{\psi_{BN}, \theta_{BN}, \phi_{BN}\}$

- In this case we would like to find the attitude tracking error of body *B* relative to the reference orientation *R*.

$$\boldsymbol{\theta}_{RN} = \{\psi_{RN}, \theta_{RN}, \phi_{RN}\}$$

$$\boldsymbol{\theta}_{BN} \quad \Rightarrow \quad [BN(\boldsymbol{\theta}_{BN})]$$

$$\boldsymbol{\theta}_{RN} \quad \Rightarrow \quad [RN(\boldsymbol{\theta}_{RN})]$$

$$[BR(\boldsymbol{\theta}_{BR})] = [BN(\boldsymbol{\theta}_{BN})][RN(\boldsymbol{\theta}_{RN})]^T$$

$$[BR(\boldsymbol{\theta}_{BR})] \quad \Rightarrow \quad \boldsymbol{\theta}_{BR} = \{\psi_{BR}, \theta_{BR}, \phi_{BR}\}$$

Example 3.2

- Let the orientation of two spacecraft *B* and *F* relative to an inertial frame *N* be given through the (3-2-1) Euler angles:
- The orientation matrices of these Euler angles are found using Eq. (3.20):

$$\boldsymbol{\theta}_{\mathcal{B}} = (30^o, -45^o, 60^o)^T \qquad \boldsymbol{\theta}_{\mathcal{F}} = (10^o, 25^o, -15^o)^T$$

$$[BN] = \begin{bmatrix} 0.612372 & 0.353553 & 0.707107 \\ -0.78033 & 0.126826 & 0.612372 \\ 0.126826 & -0.926777 & 0.353553 \end{bmatrix}$$

$$[FN] = \begin{bmatrix} 0.892539 & 0.157379 & -0.422618 \\ -0.275451 & 0.932257 & -0.234570 \\ 0.357073 & 0.325773 & 0.875426 \end{bmatrix}$$

- The rotation matrix relating the *B* and *F* frames is found to be

$$[BF] = [BN][FN]^T = \begin{bmatrix} 0.303372 & -0.0049418 & 0.952859 \\ -0.935315 & 0.1895340 & 0.298769 \\ -0.182075 & -0.9818620 & 0.052877 \end{bmatrix}$$

- Using the transformations in Eq. (3.34), the Euler angles are computed using

$$\psi = \tan^{-1}\left(\frac{-0.0049418}{0.303372}\right) = -0.933242 \text{ deg}$$
$$\theta = -\sin^{-1}(0.952859) = -72.3373 \text{ deg}$$
$$\phi = \tan^{-1}\left(\frac{0.298769}{0.052877}\right) = 79.9636 \text{ deg}$$

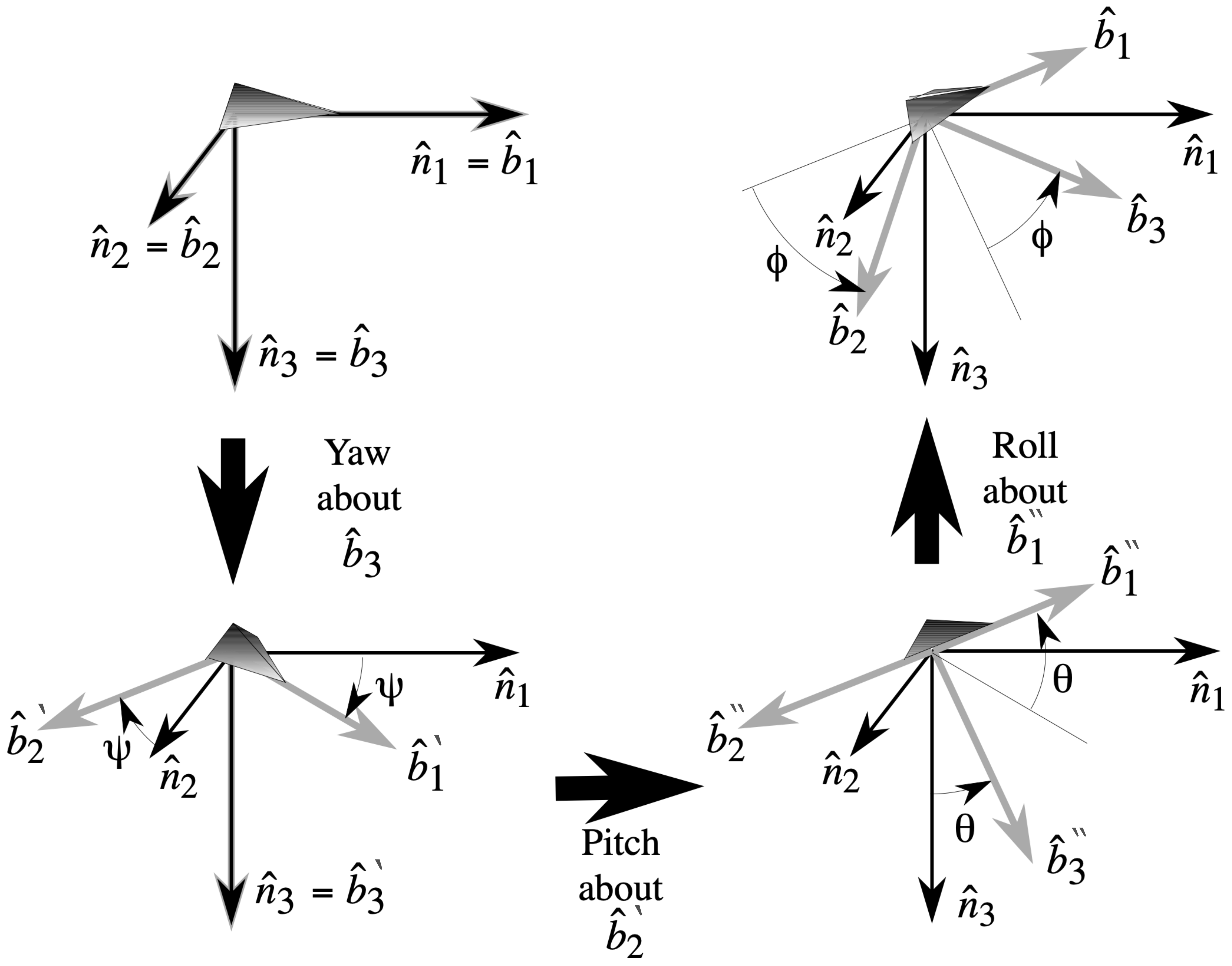
(3-2-1) Kinematic Differential Equation

- We would like to find the differential equations of the Euler angles (i.e. yaw, pitch and roll angles).

$$\dot{\psi}(t) \qquad \dot{\theta}(t) \qquad \dot{\phi}(t)$$

- The angular rotation rate is not measured as yaw, pitch and roll rates, but rather through the body angular velocity vector
- We need to find out how these Euler angle rates and the body angular velocity components are related.

$$\boldsymbol{\omega} = \omega_1 \hat{\boldsymbol{b}}_1 + \omega_2 \hat{\boldsymbol{b}}_2 + \omega_3 \hat{\boldsymbol{b}}_3$$



Using the above figure, it is evident that $\boldsymbol{\omega} = \dot{\psi}\hat{\boldsymbol{n}}_3 + \dot{\theta}\hat{\boldsymbol{b}}_2' + \dot{\phi}\hat{\boldsymbol{b}}_1$

Recall that angular velocity vectors are truly vectors and can be simply added up.

- Next, we need to express the $\hat{\boldsymbol{b}}_2'$ in terms of $\{\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$ vectors:

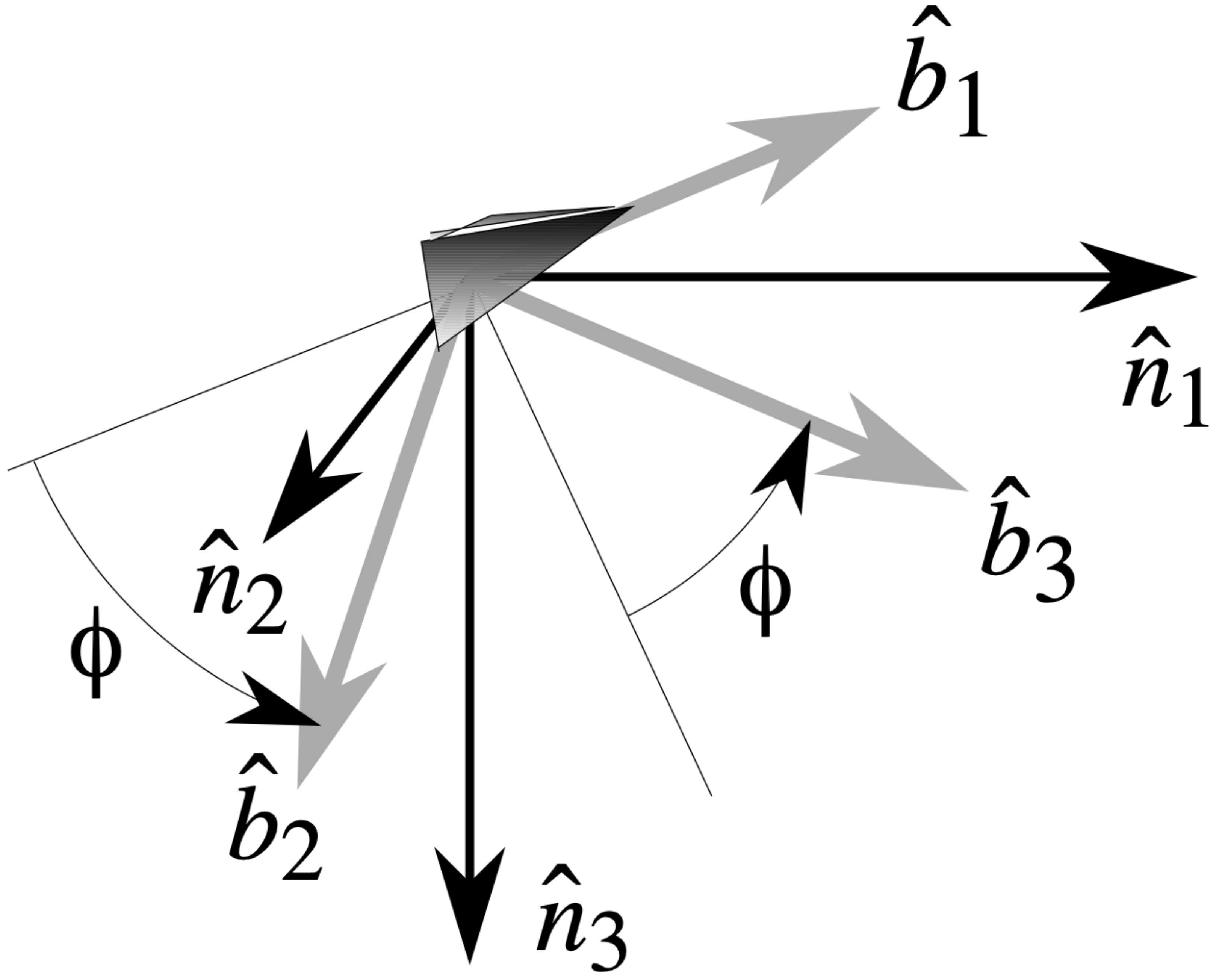
$$\hat{\boldsymbol{b}}_2' = \cos\phi\hat{\boldsymbol{b}}_2 - \sin\phi\hat{\boldsymbol{b}}_3$$

- To write the $\hat{\boldsymbol{n}}_3$ in terms of $\{\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$ vector, we use the mapping between the (3-2-1) Euler angles and [*BN*]:

$$\hat{\boldsymbol{n}}_3 = -\sin\theta\hat{\boldsymbol{b}}_1 + \sin\phi\cos\theta\hat{\boldsymbol{b}}_2 + \cos\phi\cos\theta\hat{\boldsymbol{b}}_3$$

- The last step is to equate the vector components by setting

$$\boldsymbol{\omega} = \omega_1\hat{\boldsymbol{b}}_1 + \omega_2\hat{\boldsymbol{b}}_2 + \omega_3\hat{\boldsymbol{b}}_3 = \dot{\psi}\hat{\boldsymbol{n}}_3 + \dot{\theta}\hat{\boldsymbol{b}}_2' + \dot{\phi}\hat{\boldsymbol{b}}_1$$



- Finally, we can relate the Euler angle rates and the body angular velocity vector components through:

- The inverse relationship (the kinematic differential equation of the (3-2-1) Euler angles) is found to be

$$
{}^{\mathcal{B}}\boldsymbol{\omega} = {}^{\mathcal{B}}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{bmatrix} -\sin\theta & 0 & 1 \\ \sin\phi\cos\theta & \cos\phi & 0 \\ \cos\phi\cos\theta & -\sin\phi & 0 \end{bmatrix} \begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix}
$$

$$
\begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{\cos\theta} \begin{bmatrix} 0 & \sin\phi & \cos\phi \\ 0 & \cos\phi\cos\theta & -\sin\phi\cos\theta \\ \cos\theta & \sin\phi\sin\theta & \cos\phi\sin\theta \end{bmatrix} {}^{\mathcal{B}}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}
$$

$$
= [B(\psi, \theta, \phi)]\, {}^{\mathcal{B}}\boldsymbol{\omega}
$$

(3-1-3) Kinematic Differential Eqn

- Similarly, the body angular velocity vector is written in terms of the (3-1-3) Euler angles as
- with the inverse transformation (the kinematic differential equation of the Euler angles) being

$$
{}^{\mathcal{B}}\boldsymbol{\omega} = \begin{bmatrix} \sin\theta_3 \sin\theta_2 & \cos\theta_3 & 0 \\ \cos\theta_3 \sin\theta_2 & -\sin\theta_3 & 0 \\ \cos\theta_2 & 0 & 1 \end{bmatrix} \begin{pmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \end{pmatrix}
$$

$$
\begin{aligned}
\begin{pmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \end{pmatrix} &= \frac{1}{\sin\theta_2} \begin{bmatrix} \sin\theta_3 & \cos\theta_3 & 0 \\ \cos\theta_3 \sin\theta_2 & -\sin\theta_3 \sin\theta_2 & 0 \\ -\sin\theta_3 \cos\theta_2 & -\cos\theta_3 \cos\theta_2 & \sin\theta_2 \end{bmatrix} {}^{\mathcal{B}}\boldsymbol{\omega} \\
&= [B(\boldsymbol{\theta})]^{\mathcal{B}}\boldsymbol{\omega}
\end{aligned}
$$

Comments

- Note that it is always the second Euler angle which causes the kinematic differential equations to become singular.
- As with the Euler angle geometric singularities, we find that for
  - Asymmetric Euler angles: differential equations are singular at $\theta_2 = \pm 90^{\mathrm{o}}$
  - Symmetric Euler angles: differential equations are singular at $\theta_2 = 0^{\mathrm{o}} \text{ or } 180^{\mathrm{o}}$
- With Euler angles, one is never more than a 90 degree removed from a singularity. This makes these attitude coordinates less attractive for large reorientations.

Principal Rotation Vector

The building block of many advanced attitude coordinates...

Theorem 3.1 (Euler's Principal Rotation): A rigid body or coordinate reference frame can be brought from an arbitrary initial orientation to an arbitrary final orientation by a single rigid rotation through a principal angle Φ about the principal axis $\hat{e}$; the principal axis is a judicious axis fixed in both the initial and final orientation.

That's great!! But, what does this mean?

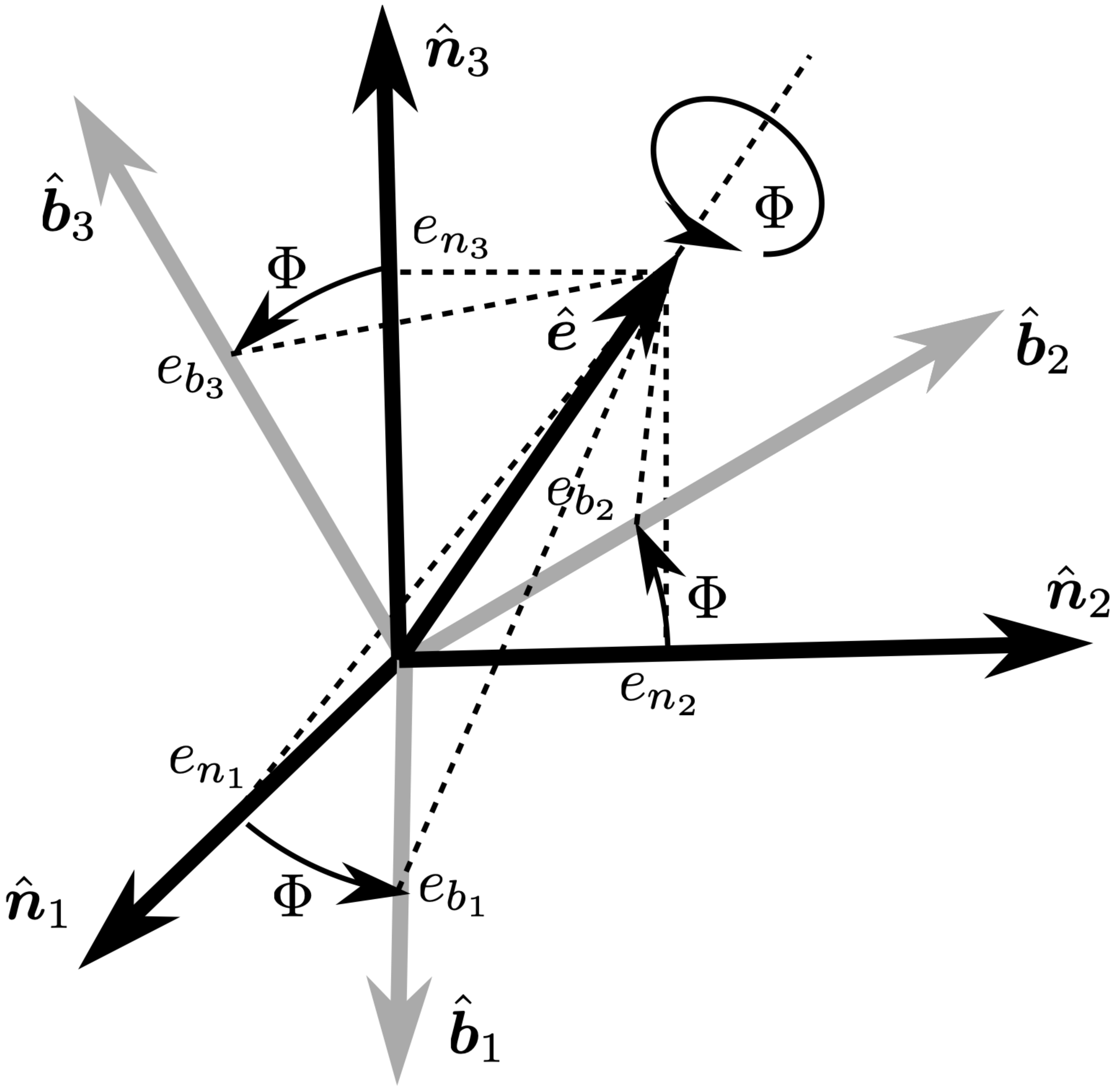
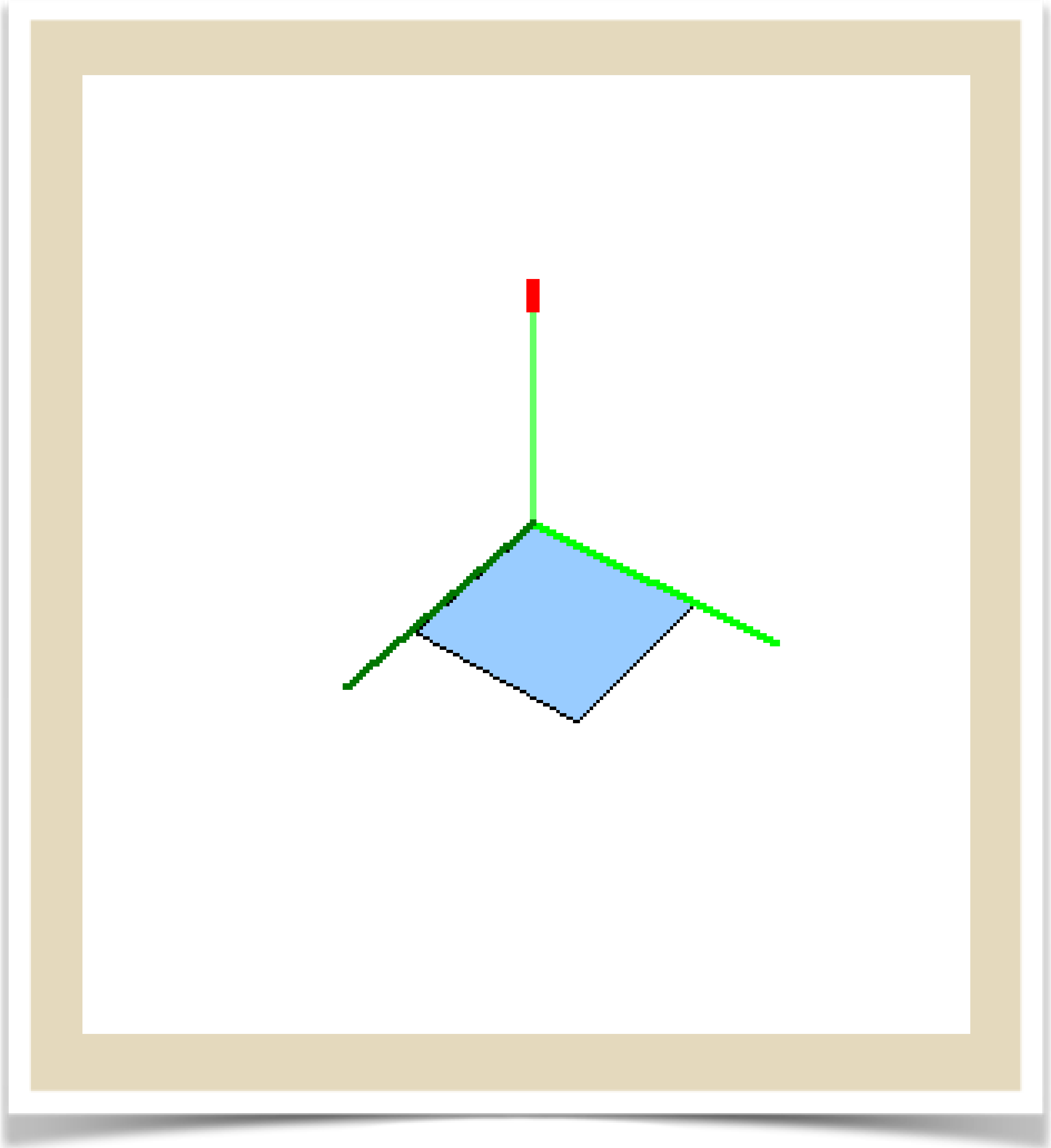
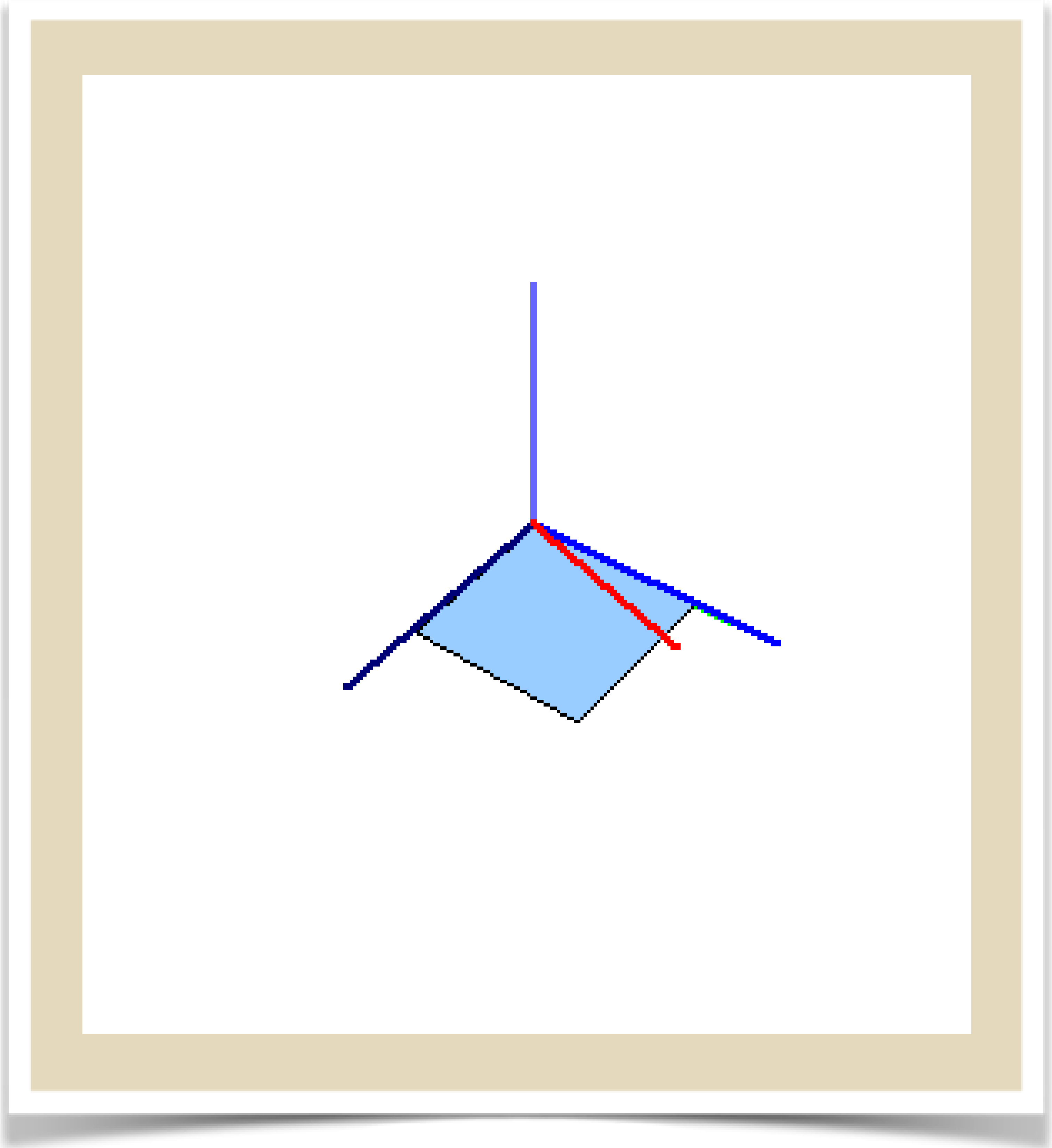


Illustration of Euler's Principal Rotation Theorem



(3-2-1) Euler Angles
(60,50,70) Degrees



Principal Rotation Vector

$$\Phi = 80.3385^o$$

$$\hat{\boldsymbol{e}} = (0.429577, 0.867729, 0.250019)^T$$

- Let's study the last statement of this theorem first: "the principal axis is a judicious axis fixed in both the initial and final orientation"

$$\hat{\boldsymbol{e}} = e_{b_1}\hat{\boldsymbol{b}}_1 + e_{b_2}\hat{\boldsymbol{b}}_2 + e_{b_3}\hat{\boldsymbol{b}}_3$$

$$\hat{\boldsymbol{e}} = e_{n_1}\hat{\boldsymbol{n}}_1 + e_{n_2}\hat{\boldsymbol{n}}_2 + e_{n_3}\hat{\boldsymbol{n}}_3$$

$$\Rightarrow \quad e_{b_i} = e_{n_i} = e_i$$

- This means that the principal axis unit vector will have the same vector components in the initial (i.e. inertial) and the final frame (i.e. body frame)

- Using the rotation matrix [*C*], the $\hat{\boldsymbol{e}}$ frame vector components in *B* and *N* frame can be related through

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = [C] \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

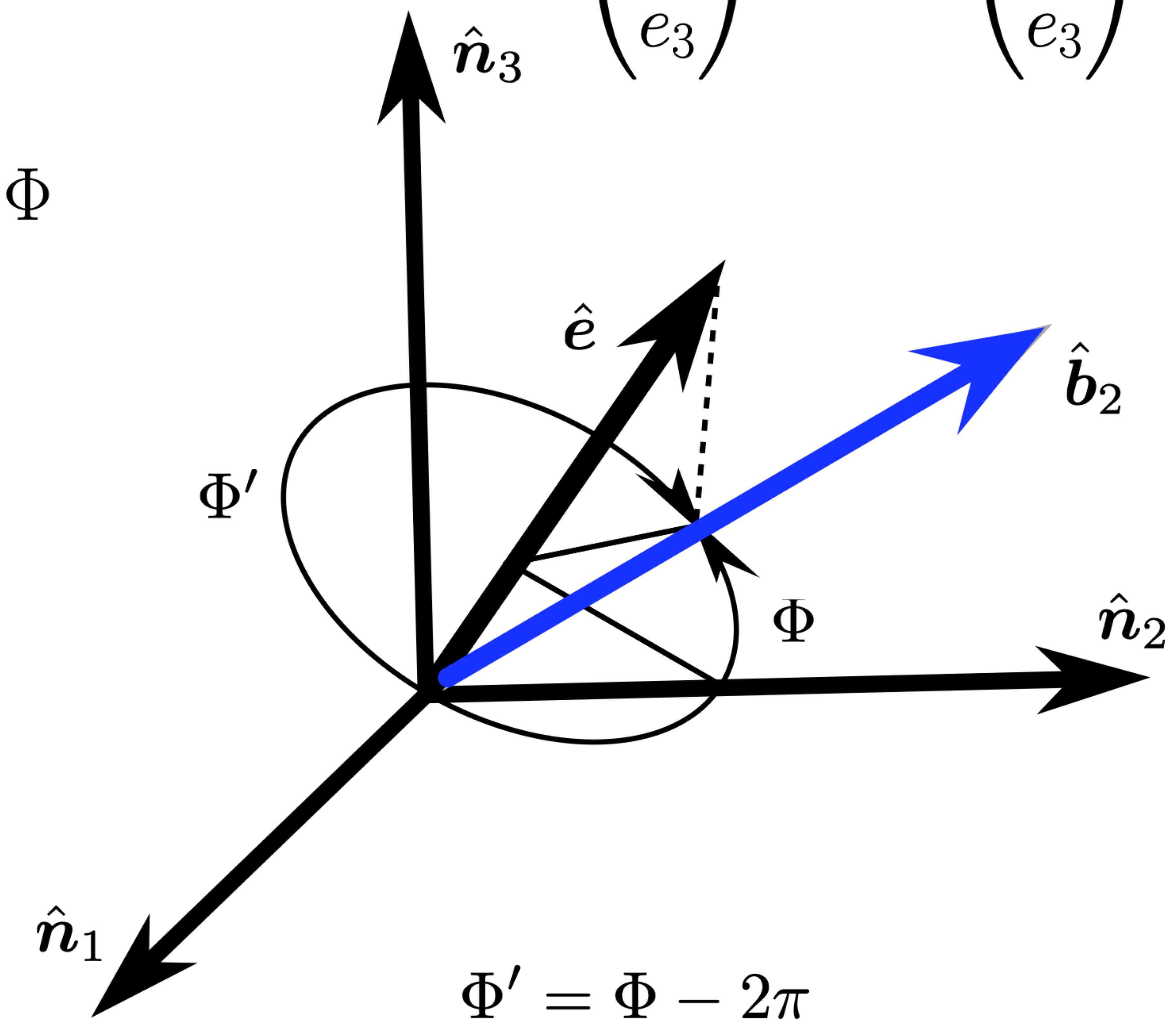
- From this last equation, it is evident that $\hat{\boldsymbol{e}}$ must be an eigenvector of [C] with an eigenvalue of +1.
- This eigenvector is unique to within a sign of or .
- The vector is not defined for a zero rotation!
- There are four possible principal rotations:

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = [C] \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

$\hat{\boldsymbol{e}}$ Φ

$\hat{\boldsymbol{e}}$

$$\begin{gathered}(\hat{\boldsymbol{e}}, \Phi) \\ (-\hat{\boldsymbol{e}}, -\Phi) \\ (\hat{\boldsymbol{e}}, \Phi') \\ (-\hat{\boldsymbol{e}}, -\Phi')\end{gathered}$$



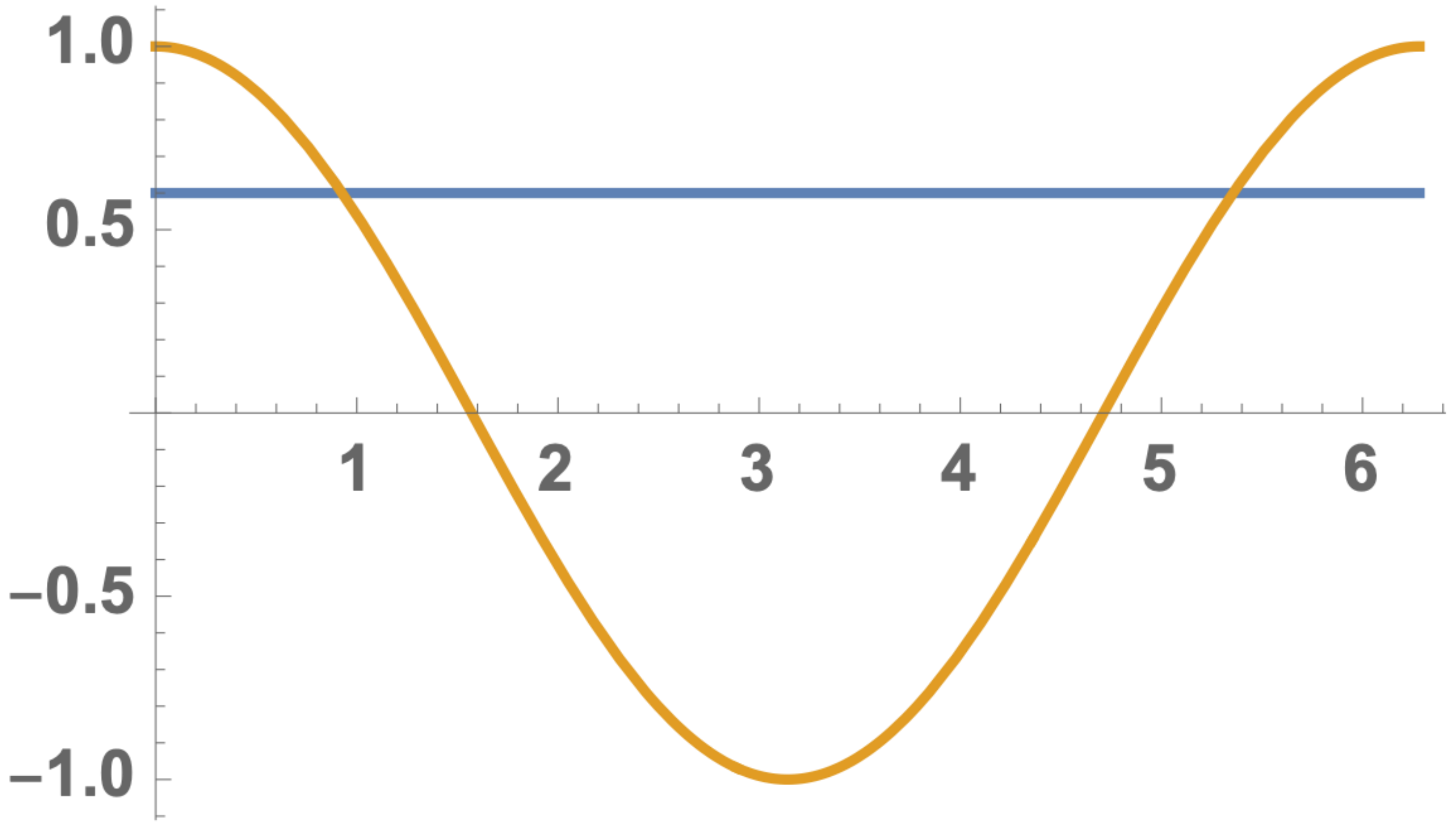
$$\Phi' = \Phi - 2\pi$$

Relationship to DCM

- We can express the $[C]$ matrix in terms of PRV components as
- The inverse transformation from $[C]$ to PRV is found by inspecting the matrix structure:

$$[C] = \begin{bmatrix} e_1^2\Sigma + c\Phi & e_1e_2\Sigma + e_3s\Phi & e_1e_3\Sigma - e_2s\Phi \\ e_2e_1\Sigma - e_3s\Phi & e_2^2\Sigma + c\Phi & e_2e_3\Sigma + e_1s\Phi \\ e_3e_1\Sigma + e_2s\Phi & e_3e_2\Sigma - e_1s\Phi & e_3^2\Sigma + c\Phi \end{bmatrix}$$

$$\Sigma = 1 - \mathrm{c}\Phi$$



$$\cos\Phi = \frac{1}{2}(C_{11} + C_{22} + C_{33} - 1) \qquad \Phi' = \Phi - 2\pi$$

$$\hat{\boldsymbol{e}} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \frac{1}{2\sin\Phi}\begin{pmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{pmatrix}$$

PRV Addition

- DCM method:

$$[FN(\Phi, \hat{\boldsymbol{e}})] = [FB(\Phi_2, \hat{\boldsymbol{e}}_2)][BN(\Phi_1, \hat{\boldsymbol{e}}_1)]$$

- Direct method:

$$\Phi = 2\cos^{-1}\left(\cos\frac{\Phi_1}{2}\cos\frac{\Phi_2}{2} - \sin\frac{\Phi_1}{2}\sin\frac{\Phi_2}{2}\hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_2\right)$$

$$\hat{\boldsymbol{e}} = \frac{\cos\frac{\Phi_2}{2}\sin\frac{\Phi_1}{2}\hat{\boldsymbol{e}}_1 + \cos\frac{\Phi_1}{2}\sin\frac{\Phi_2}{2}\hat{\boldsymbol{e}}_2 + \sin\frac{\Phi_1}{2}\sin\frac{\Phi_2}{2}\hat{\boldsymbol{e}}_1 \times \hat{\boldsymbol{e}}_2}{\sin\frac{\Phi}{2}}$$

PRV Subtraction

- DCM method:

$$[FB(\Phi_2, \hat{\boldsymbol{e}}_2)] = [FN(\Phi, \hat{\boldsymbol{e}})][BN(\Phi_1, \hat{\boldsymbol{e}}_1)]^T$$

- Direct method:

$$\Phi_2 = 2\cos^{-1}\left(\cos\frac{\Phi}{2}\cos\frac{\Phi_1}{2} + \sin\frac{\Phi}{2}\sin\frac{\Phi_1}{2}\hat{\boldsymbol{e}}\cdot\hat{\boldsymbol{e}}_1\right)$$

$$\hat{\boldsymbol{e}}_2 = \frac{\cos\frac{\Phi_1}{2}\sin\frac{\Phi}{2}\hat{\boldsymbol{e}} - \cos\frac{\Phi}{2}\sin\frac{\Phi_1}{2}\hat{\boldsymbol{e}}_1 + \sin\frac{\Phi}{2}\sin\frac{\Phi_1}{2}\hat{\boldsymbol{e}}\times\hat{\boldsymbol{e}}_1}{\sin\frac{\Phi_2}{2}}$$

PRV Differential Kinematic Equation

- Mapping from body angular velocity vector to PRV rates:

- Mapping from PRV rates to body angular velocity vector:

$$\dot{\gamma} = \left[[I_{3\times 3}] + \frac{1}{2}[\tilde{\gamma}] + \frac{1}{\Phi^2}\left(1 - \frac{\Phi}{2}\cot\left(\frac{\Phi}{2}\right)\right)[\tilde{\gamma}]^2 \right] {}^{\mathcal{B}}\boldsymbol{\omega}$$

$${}^{\mathcal{B}}\boldsymbol{\omega} = \left[[I_{3\times 3}] - \left(\frac{1-\cos\Phi}{\Phi^2}\right)[\tilde{\gamma}] + \left(\frac{\Phi - \sin\Phi}{\Phi^3}\right)[\tilde{\gamma}]^2 \right] \dot{\gamma}$$

Conclusion

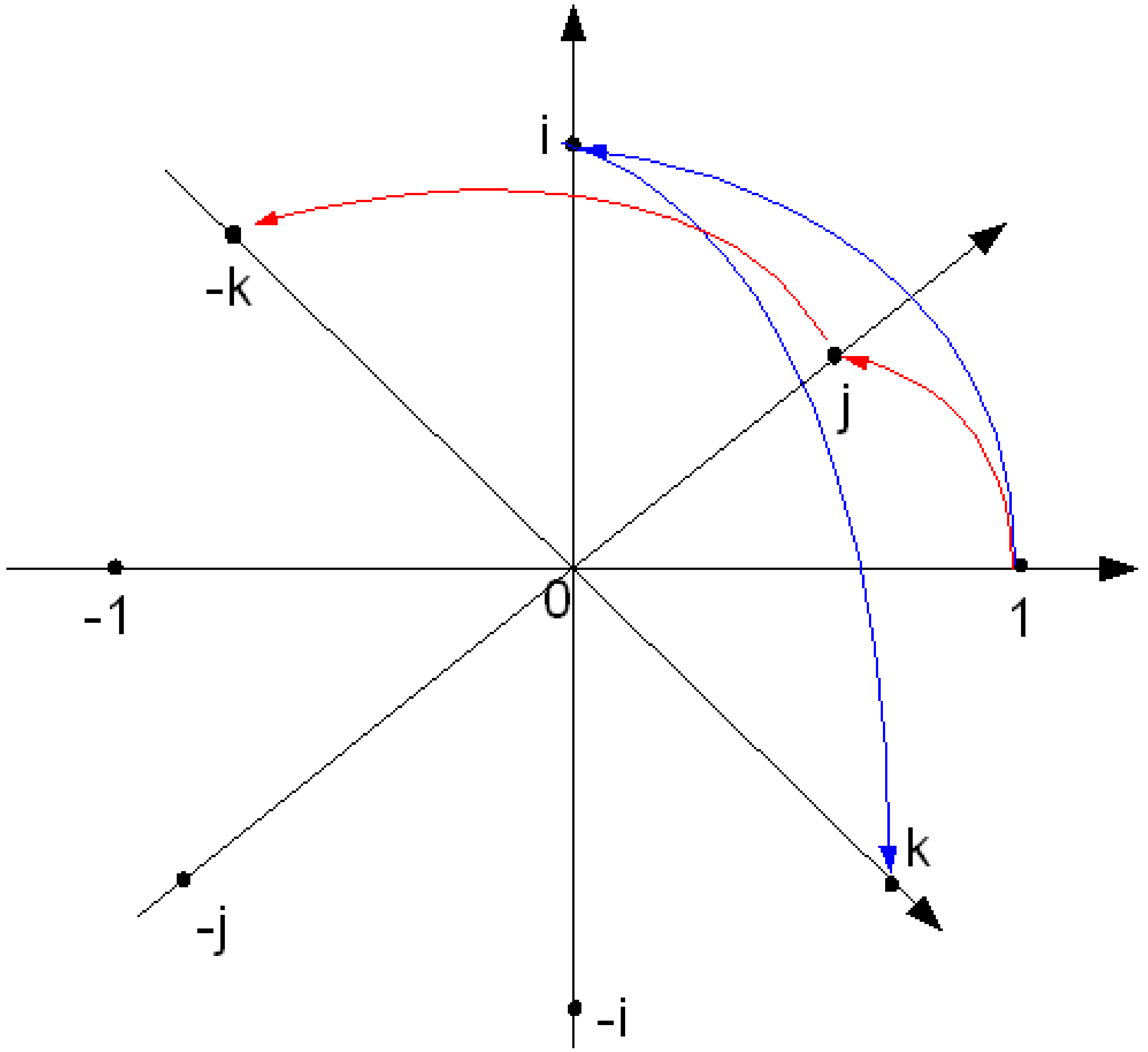
- PRV is based on a very fundamental rotation/orientation property called Euler's principal rotation theorem
- Singular for zero-rotation
- PRVs form the basis for many other attitude coordinates which are very useful for large angle rotations

Euler Parameters (Quaternions)

Voted most popular attitude coordinates in the non-singular category...

Introduction

- Very popular redundant set of attitude coordinates
- Are called either Euler Parameters (EPs) or quaternions
- Major benefits:
  - Non-singular attitude description
  - Linear differential kinematic equation
  - Works well for small and large rotations
- Drawbacks:
  - Constraint equation must be identified as all times
  - Not as simple to visualize



https://en.wikipedia.org/wiki/Quaternion

Definition of EP

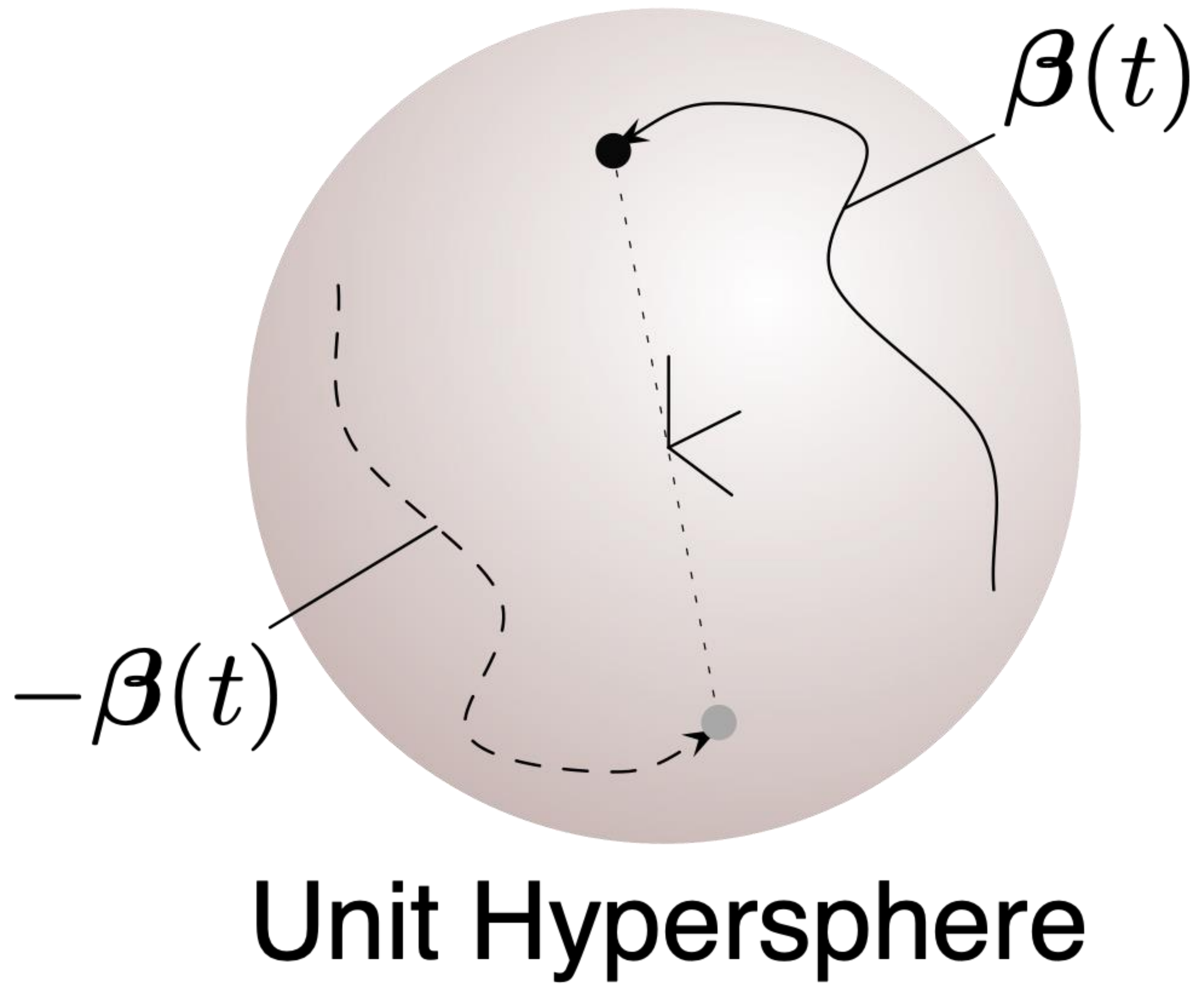
- The redundant Euler Parameters are defined using the principal rotation components as

$$\begin{aligned}\beta_0 &= \cos(\Phi/2)\\ \beta_1 &= e_1 \sin(\Phi/2)\\ \beta_2 &= e_2 \sin(\Phi/2)\\ \beta_3 &= e_3 \sin(\Phi/2)\end{aligned}$$

Constraints:

$$e_1^2 + e_2^2 + e_3^2 = 1$$

$$\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = 1$$



- Note that the 4-coordinate set has a single constraint equation! All EPs must lie on the three-dimensional surface of a 4-dimensional hypersphere.

- Since the PRV components are not unique, we find that the EP also isn't unique:
- Note that the alternate EP set corresponds to performing the larger principal rotation angle (i.e., rotating the long way round)

$$(-\hat{\boldsymbol{e}}, -\Phi) \qquad \beta_0' = \cos\left(-\frac{\Phi}{2}\right) = \cos\left(\frac{\Phi}{2}\right) = \beta_0$$

$$\beta_i' = -e_i \sin\left(-\frac{\Phi}{2}\right) = e_i \sin\left(\frac{\Phi}{2}\right) = \beta_i$$

$$(\hat{\boldsymbol{e}}, \Phi') \qquad \beta_0' = \cos\left(\frac{\Phi'}{2}\right) = \cos\left(\frac{\Phi}{2} - \pi\right) = -\cos\left(\frac{\Phi}{2}\right) = -\beta_0$$

$$\beta_i' = e_i \sin\left(\frac{\Phi'}{2}\right) = e_i \sin\left(\frac{\Phi'}{2} - \pi\right) = -e_i \sin\left(\frac{\Phi}{2}\right) = -\beta_i$$

Euler Parameter to DCM Relationship

- The rotation matrix can be expressed in terms or EPs as:
- The inverse relationship is found by inspection to be

$$[C] = \begin{bmatrix} \beta_0^2+\beta_1^2-\beta_2^2-\beta_3^2 & 2\left(\beta_1\beta_2+\beta_0\beta_3\right) & 2\left(\beta_1\beta_3-\beta_0\beta_2\right) \\ 2\left(\beta_1\beta_2-\beta_0\beta_3\right) & \beta_0^2-\beta_1^2+\beta_2^2-\beta_3^2 & 2\left(\beta_2\beta_3+\beta_0\beta_1\right) \\ 2\left(\beta_1\beta_3+\beta_0\beta_2\right) & 2\left(\beta_2\beta_3-\beta_0\beta_1\right) & \beta_0^2-\beta_1^2-\beta_2^2+\beta_3^2 \end{bmatrix}$$

$$\beta_0 = \pm\frac{1}{2}\sqrt{C_{11}+C_{22}+C_{33}+1}$$

$$\beta_1 = \frac{C_{23}-C_{32}}{4\beta_0}$$

$$\beta_2 = \frac{C_{31}-C_{13}}{4\beta_0}$$

$$\beta_3 = \frac{C_{12}-C_{21}}{4\beta_0}$$

Singular if: $\beta_0 \rightarrow 0$

- Sheppard's method is a robust method to compute the EP from a rotation matrix:

1st step: Find largest value of

$$\beta_0^2 = \frac{1}{4}\left(1 + \mathrm{trace}\left([C]\right)\right) \qquad \beta_2^2 = \frac{1}{4}\left(1 + 2C_{22} - \mathrm{trace}\left([C]\right)\right)$$

$$\beta_1^2 = \frac{1}{4}\left(1 + 2C_{11} - \mathrm{trace}\left([C]\right)\right) \qquad \beta_3^2 = \frac{1}{4}\left(1 + 2C_{33} - \mathrm{trace}\left([C]\right)\right)$$

2nd step: Compute the remaining EPs using

$$\beta_0\beta_1 = (C_{23} - C_{32})/4 \qquad \beta_1\beta_2 = (C_{12} + C_{21})/4$$

$$\beta_0\beta_2 = (C_{31} - C_{13})/4 \qquad \beta_3\beta_1 = (C_{31} + C_{13})/4$$

$$\beta_0\beta_3 = (C_{12} - C_{21})/4 \qquad \beta_2\beta_3 = (C_{23} + C_{32})/4$$

Adding Euler Parameters

- A very useful advantage of EPs is how you can add or subtract two orientations using them. Using DCMs, we can add two rotations using:

$$[FN(\boldsymbol{\beta})] = [FB(\boldsymbol{\beta}'')][BN(\boldsymbol{\beta}')]$$

$$\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{bmatrix} \beta_0'' & -\beta_1'' & -\beta_2'' & -\beta_3'' \\ \beta_1'' & \beta_0'' & \beta_3'' & -\beta_2'' \\ \beta_2'' & -\beta_3'' & \beta_0'' & \beta_1'' \\ \beta_3'' & \beta_2'' & -\beta_1'' & \beta_0'' \end{bmatrix} \begin{pmatrix} \beta_0' \\ \beta_1' \\ \beta_2' \\ \beta_3' \end{pmatrix}$$

- However, using EPs directly, we find the elegant result:
- Note that this matrix is orthogonal!

- By reshuffling the terms (i.e. permutation) in the last EP addition equation, we can also write this as

$$\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{bmatrix} \beta_0' & -\beta_1' & -\beta_2' & -\beta_3' \\ \beta_1' & \beta_0' & -\beta_3' & \beta_2' \\ \beta_2' & \beta_3' & \beta_0' & -\beta_1' \\ \beta_3' & -\beta_2' & \beta_1' & \beta_0' \end{bmatrix} \begin{pmatrix} \beta_0'' \\ \beta_1'' \\ \beta_2'' \\ \beta_3'' \end{pmatrix}$$

- To subtract two orientations described through EPs, we can use the last two equations and exploit the orthogonality property of the 4x4 matrix to invert it and solve for either $\boldsymbol{\beta}'$ or $\boldsymbol{\beta}''$.

Euler Parameter Differential Equation

- Using the differential equation of DCMs, and the relationship between EPs and DCMs, we can derive the differential kinematic equations of Euler parameters.
- However, this is a rather lengthy and algebraically complex task. The end result is the amazingly simple bi-linear result:

$$\begin{pmatrix} \dot\beta_0 \\ \dot\beta_1 \\ \dot\beta_2 \\ \dot\beta_3 \end{pmatrix} = \frac{1}{2} \begin{bmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} \begin{pmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

- Rearranging the terms on the right hand side of this differential equation, we can also write this as

$$\begin{pmatrix} \dot{\beta}_0 \\ \dot{\beta}_1 \\ \dot{\beta}_2 \\ \dot{\beta}_3 \end{pmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

- Having the differential equation depend linearly on the EPs is important in estimation theory. This makes the EPs ideal coordinate candidates to be used in a Kalman filter (Spacecraft orientation estimator).

2^{nd} Euler Parameter Differential Kinematic Eqs.

- The EP differential equations can also be written in the following convenient form for numerical integration:

$$\dot{\boldsymbol{\beta}} = \frac{1}{2}[B(\boldsymbol{\beta})]\boldsymbol{\omega} \qquad [B(\boldsymbol{\beta})] = \begin{bmatrix} -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_0 & -\beta_3 & \beta_2 \\ \beta_3 & \beta_0 & -\beta_1 \\ -\beta_2 & \beta_1 & \beta_0 \end{bmatrix}$$

- The [*B*] matrix satisfies the following useful identities:

$$[B(\boldsymbol{\beta})]^T \boldsymbol{\beta} = \mathbf{0}$$
$$[B(\boldsymbol{\beta})]^T \boldsymbol{\beta}' = -[B(\boldsymbol{\beta}')]^T \boldsymbol{\beta}$$

3rd Euler Parameter Differential Kinematic Eqs.

- In control applications, the scalar and vector components of the Euler parameters are sometimes treated separately.

Define: $$\boldsymbol{\epsilon} \equiv (\beta_1, \beta_2, \beta_3)^T$$

Define: $$[T(\beta_0, \boldsymbol{\epsilon})] = \beta_0 [I_{3\times 3}] + [\tilde{\boldsymbol{\epsilon}}]$$

Differential Equation:
$$\dot{\beta}_0 = -\frac{1}{2}\boldsymbol{\epsilon}^T \boldsymbol{\omega} = -\frac{1}{2}\boldsymbol{\omega}^T \boldsymbol{\epsilon}$$
$$\dot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\boldsymbol{\omega}$$

Conclusion

- Non-singular, redundant set of attitude coordinates
- Euler parameter vector must abide by the unit length constraint
- There are two sets of EPs that describe a particular orientation (short and long way round)
- Convenient method to add two EP vectors
- Linear differential kinematic equations

Classical Rodrigues Parameters (Gibbs Vector or CRPs)

Popular coordinates for large rotations and robotics....

CRP Definitions

Euler parameter relationship:

$$q_i = \frac{\beta_i}{\beta_0} \qquad i = 1, 2, 3$$

Singular if 0
(±180° case)

$$\beta_0 = \frac{1}{\sqrt{1 + \boldsymbol{q}^T \boldsymbol{q}}}$$

$$\beta_i = \frac{q_i}{\sqrt{1 + \boldsymbol{q}^T \boldsymbol{q}}} \qquad i = 1, 2, 3$$

Singular if ∞
(±180° case)

Principal rotation parameter relationship:

$$\boldsymbol{q} = \tan \frac{\Phi}{2} \hat{\boldsymbol{e}}$$

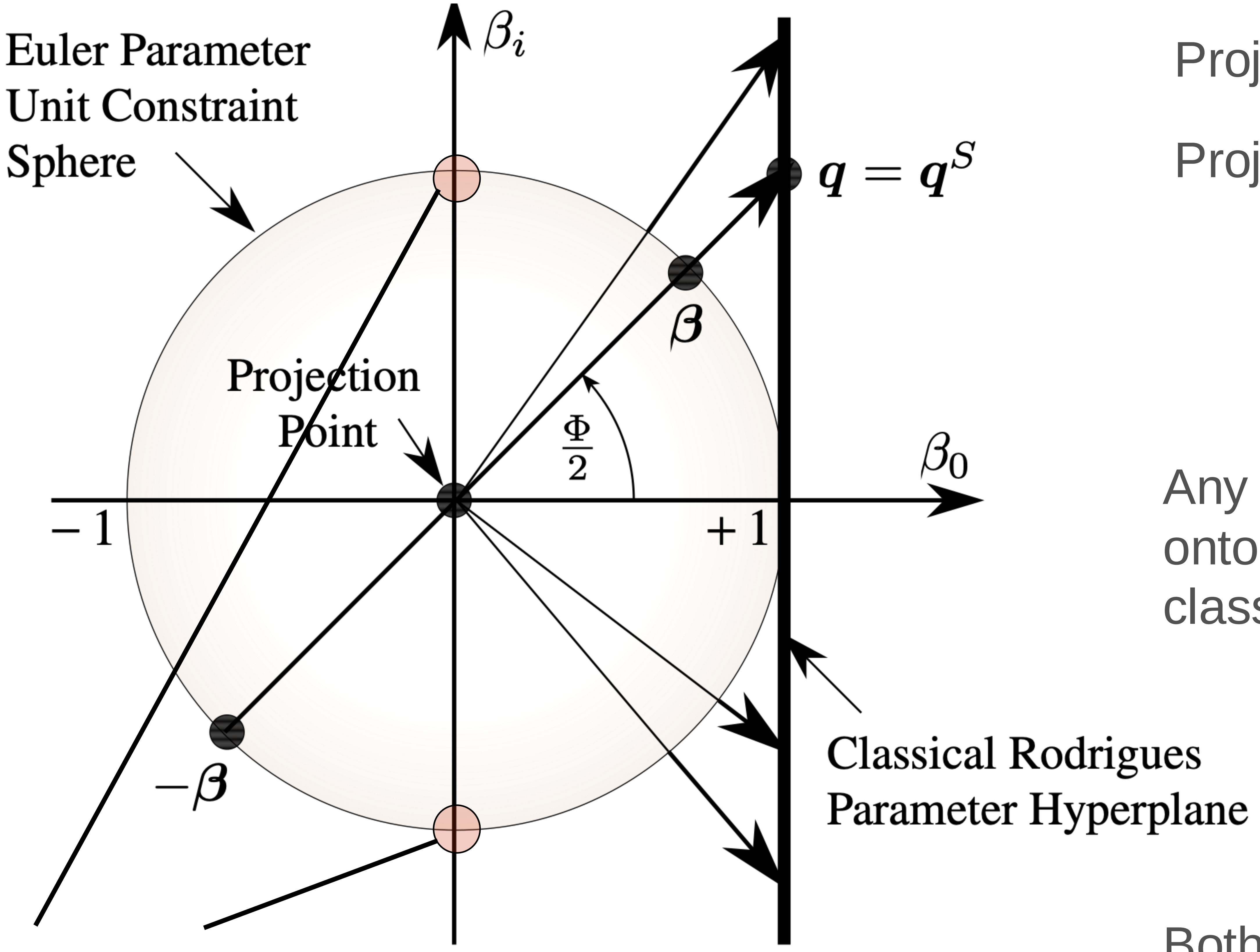
Singular for ±180°

$$\boldsymbol{q} \approx \frac{\Phi}{2} \hat{\boldsymbol{e}}$$

Linearizes to
angles over 2.

These parameters are much better suited for large spacecraft rotations than Euler angles, while remaining a minimal coordinate set.
Only the upside down description is singular.

Stereographic Projection



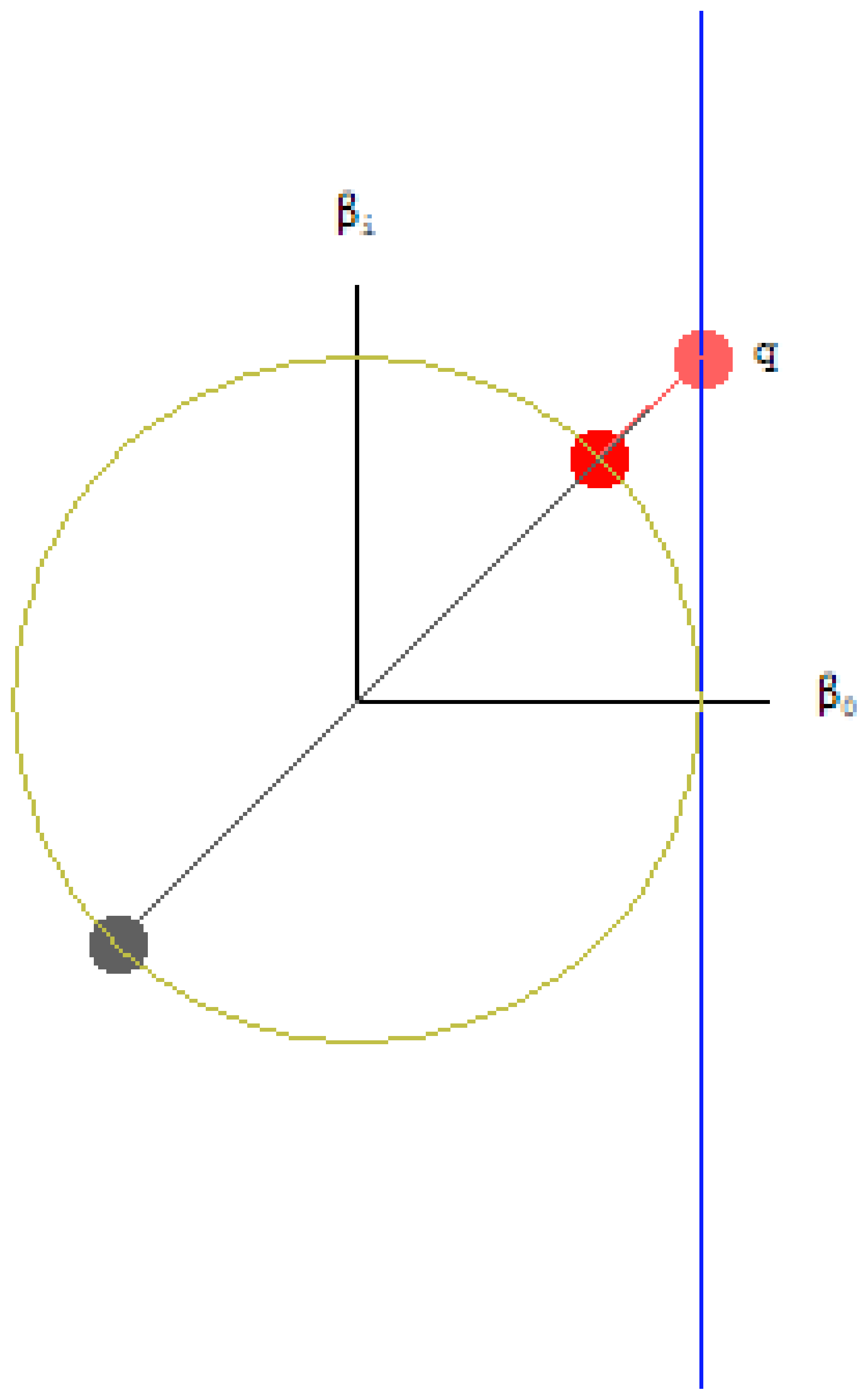
Singular Projection at 180º rotation!

Projection Point: $(0, 0, 0, 0)$

Projection Plane: $\beta_0 = +1$

Any attitude (surface point) is projected onto the hyper-plane to form the classical Rodrigues parameters.

Both EP sets yield the identical CRP coordinates.



See Mathematica Demo

Shadow CRP Set

- Using the alternate set of Euler parameters, we can find the "shadow" set of CRP parameters:

$$q_i^S = \frac{-\beta_i}{-\beta_0} = q_i$$

- For the case of CRPs, the shadow set and the original set of attitude parameters are identical. Thus, the shadow set cannot be used to avoid the 180º singularity.

Direction Cosine Matrix

Matrix components:

$$[C] = \frac{1}{1+\boldsymbol{q}^T\boldsymbol{q}}\begin{bmatrix} 1+q_1^2-q_2^2-q_3^2 & 2\left(q_1q_2+q_3\right) & 2\left(q_1q_3-q_2\right) \\ 2\left(q_2q_1-q_3\right) & 1-q_1^2+q_2^2-q_3^2 & 2\left(q_2q_3+q_1\right) \\ 2\left(q_3q_1+q_2\right) & 2\left(q_3q_2-q_1\right) & 1-q_1^2-q_2^2+q_3^2 \end{bmatrix}$$

$$[C] = \frac{1}{1+\boldsymbol{q}^T\boldsymbol{q}}\left(\left(1-\boldsymbol{q}^T\boldsymbol{q}\right)[I_{3\times 3}] + 2\boldsymbol{q}\boldsymbol{q}^T - 2[\tilde{\boldsymbol{q}}]\right)$$

$$[C(\boldsymbol{q})]^{-1} = [C(\boldsymbol{q})]^T = [C(-\boldsymbol{q})]$$

Attitude Addition/Subtraction

- DCM method:

$$[FN(\boldsymbol{q})] = [FB(\boldsymbol{q}'')][BN(\boldsymbol{q}')]$$

- Direct method:

$$\boldsymbol{q} = \frac{\boldsymbol{q}'' + \boldsymbol{q}' - \boldsymbol{q}'' \times \boldsymbol{q}'}{1 - \boldsymbol{q}'' \cdot \boldsymbol{q}'}$$

Attitude Addition

$$\boldsymbol{q}'' = \frac{\boldsymbol{q} - \boldsymbol{q}' + \boldsymbol{q} \times \boldsymbol{q}'}{1 + \boldsymbol{q} \cdot \boldsymbol{q}'}$$

Relative Attitude (Subtraction)

Note: Using $\delta\boldsymbol{q} = \boldsymbol{q} - \boldsymbol{q}'$ to compute the relative attitude, or attitude error, still yields a result that is a proper CRP attitude measure. However, also note that the approximation $\delta\boldsymbol{q} \approx \boldsymbol{q}''$ only holds for small attitude differences.

Differential Kinematic Equations

Matrix components:

$$\dot{\boldsymbol{q}} = \frac{1}{2} \begin{bmatrix} 1+q_1^2 & q_1 q_2 - q_3 & q_1 q_3 + q_2 \\ q_2 q_1 + q_3 & 1+q_2^2 & q_2 q_3 - q_1 \\ q_3 q_1 - q_2 & q_3 q_2 + q_1 & 1+q_3^2 \end{bmatrix} {}^{\mathcal{B}}\!\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

Vector computation:

$$\dot{\boldsymbol{q}} = \frac{1}{2} \left[[I_{3\times3}] + [\tilde{\boldsymbol{q}}] + \boldsymbol{q}\boldsymbol{q}^T \right] {}^{\mathcal{B}}\boldsymbol{\omega}$$

$${}^{\mathcal{B}}\boldsymbol{\omega} = \frac{2}{1+\boldsymbol{q}^T\boldsymbol{q}} \left([I_{3\times3}] - [\tilde{\boldsymbol{q}}] \right) \dot{\boldsymbol{q}}$$

Note: Only contains quadratic nonlinearities, but is singular for Φ =180º.

Cayley Transform

- Amazingly elegant matrix transformation, that allows us to use attitude parameters in higher dimensional spaces.

☺

- Let [*Q*] be a skew-symmetric matrix, [*C*] be a proper orthogonal matrix, and [*I*] be a identity matrix. These matrices can be of any dimension *N*. The Cayley Transform is then defined as:

$$\boxed{[C] = ([I]-[Q])\,([I]+[Q])^{-1} = ([I]+[Q])^{-1}\,([I]-[Q])}$$

$$[Q] = ([I]-[C])\,([I]+[C])^{-1} = ([I]+[C])^{-1}\,([I]-[C])$$

Note: Both the forward and backwards mapping between [*Q*] and [*C*] has the same algebraic form!

Example:

- For 3D space, the proper orthogonal [*C*] matrix is also a rotation or direction cosine matrix. In this case we find that

$$[Q] = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}$$

where the unique matrix elements are the CRP!

$$[C] = \begin{bmatrix} 0.813797 & 0.296198 & -0.5 \\ 0.235888 & 0.617945 & 0.75 \\ 0.531121 & -0.728292 & 0.433012 \end{bmatrix}$$

$$\Downarrow$$

$$[Q] = \begin{bmatrix} 0 & -0.021052 & 0.359933 \\ 0.021052 & 0 & -0.516027 \\ -0.359933 & 0.516027 & 0 \end{bmatrix} \Rightarrow \boldsymbol{q} = \begin{pmatrix} 0.516027 \\ 0.359933 \\ 0.021052 \end{pmatrix}$$

CRP vector

- Higher Dimensional Example:

$$[C] = \begin{bmatrix} 0.505111 & -0.503201 & -0.215658 & 0.667191 \\ 0.563106 & -0.034033 & -0.538395 & -0.626006 \\ 0.560111 & 0.748062 & 0.272979 & 0.228387 \\ -0.337714 & 0.431315 & -0.767532 & 0.332884 \end{bmatrix}$$

$$[Q] = \begin{bmatrix} 0 & 0.5 & 0.2 & -0.3 \\ -0.5 & 0 & 0.7 & -0.6 \\ -0.2 & -0.7 & 0 & -0.4 \\ 0.3 & -0.6 & 0.4 & 0 \end{bmatrix}$$

4D space CRP

Note: The *N*-dimensional proper orthogonal matrices can be parameterized with higher dimensional attitude coordinates.

That's nice, but is there also a higher dimensional equivalent to the differential kinematic equations to solve [*Q(t)*]?

- Recall that regardless of the dimensionality of the orthogonal matrix [*C*(*t*)], it must evolve according to:

$$[\dot{C}] = -[\tilde{\boldsymbol{\omega}}][C]$$

- These higher-dimensional "body angular velocities" can be related to the higher dimensional CRPs using:

$$[\dot{Q}] = \frac{1}{2}\left([I] + [Q]\right)[\tilde{\boldsymbol{\omega}}]\left([I] - [Q]\right)$$
$$[\tilde{\boldsymbol{\omega}}] = 2\left([I] + [Q]\right)^{-1}[\dot{Q}]\left([I] - [Q]\right)^{-1}$$

- Thus, can solve for the [*C*(*t*)] using a reduced coordinate set.

- This parameterization is singular whenever a principal rotation of 180º is performed.

- **Physical Example:**

Consider a typical mechanical system. The EOM can be written in the form

$$[M(\boldsymbol{x}, t)]\ddot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)$$

To solve this system for the state accelerations, the system positive definite system mass matrix must be inverted, a numerically expensive operations for large dimensions.

This inverse could be avoided by using the spectral decomposition:

$$[M] = [V][D][V]^T \qquad\qquad [M]^{-1} = [V]^T[D]^{-1}[V]$$

where [*V*] is a proper orthogonal eigenvector matrix and [*D*] is a diagonal eigenvalue matrix. To determine [*V*(*t*)] the Cayley transform could be used to track a reduced parameter set:

$$[Q] = ([I] - [V])([I] + [V])^{-1}$$

Modified Rodrigues Parameters (MRPs)

The “cool” new attitude coordinates...

MRP Definitions

Euler parameter relationship:

$$\sigma_i = \frac{\beta_i}{1+\beta_0} \qquad i = 1,2,3$$

Singular if -1
(±360° case)

$$\beta_0 = \frac{1-\sigma^2}{1+\sigma^2}$$

$$\beta_i = \frac{2\sigma_i}{1+\sigma^2} \qquad i = 1,2,3$$

Singular if ∞
(±360° case)

PRV relationship:

$$\boldsymbol{\sigma} = \tan\frac{\Phi}{4}\hat{\boldsymbol{e}}$$

Singular for ±360°

$$\boldsymbol{\sigma} \approx \frac{\Phi}{4}\hat{\boldsymbol{e}}$$

Linearizes to
angles over 4.

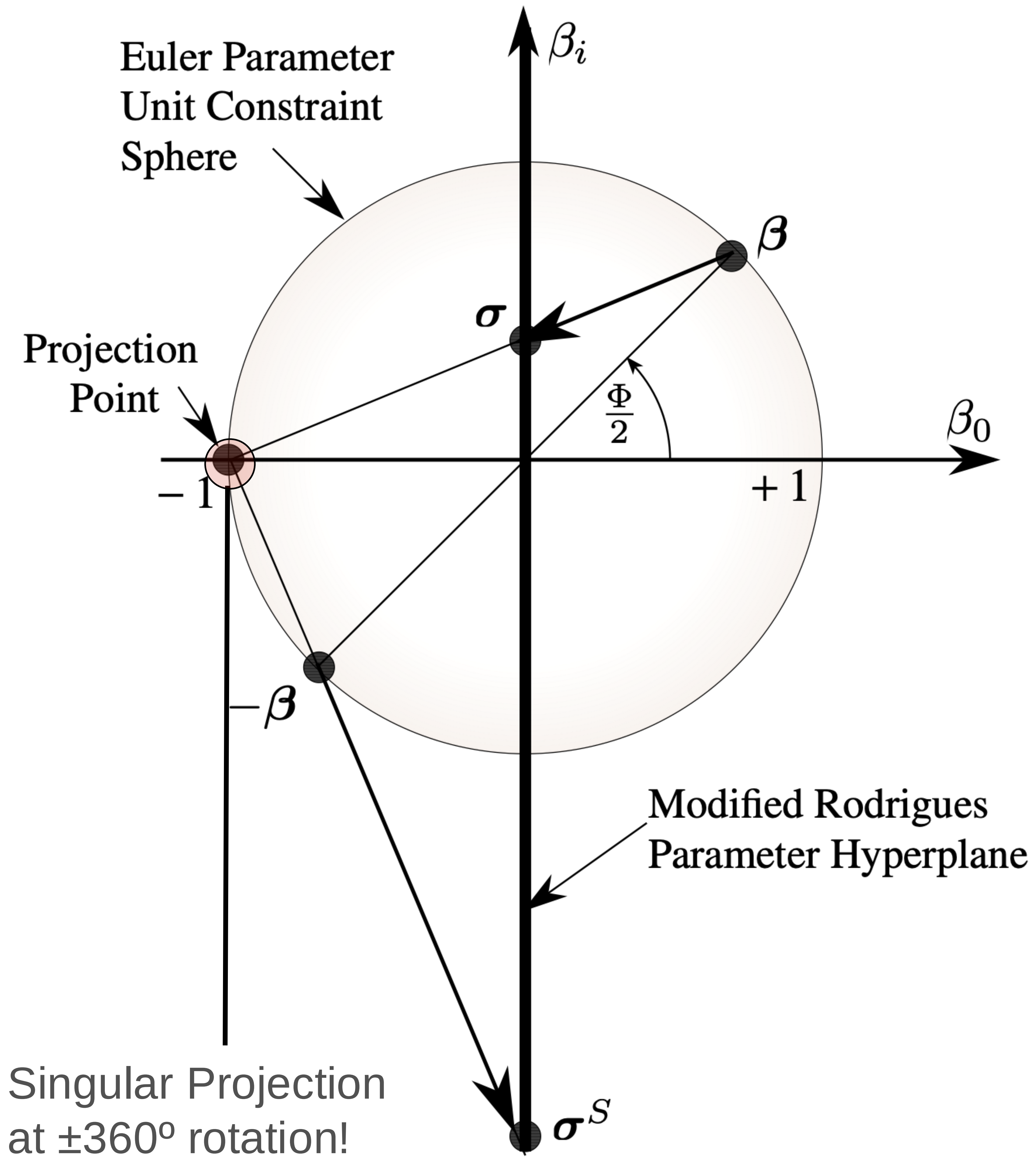
(Show Mathematica Example)

CRP relationship:

$$\boldsymbol{q} = \frac{2\boldsymbol{\sigma}}{1-\sigma^2}$$

$$\boldsymbol{\sigma} = \frac{\boldsymbol{q}}{1+\sqrt{1+\boldsymbol{q}^T\boldsymbol{q}}}$$

Stereographic Projection

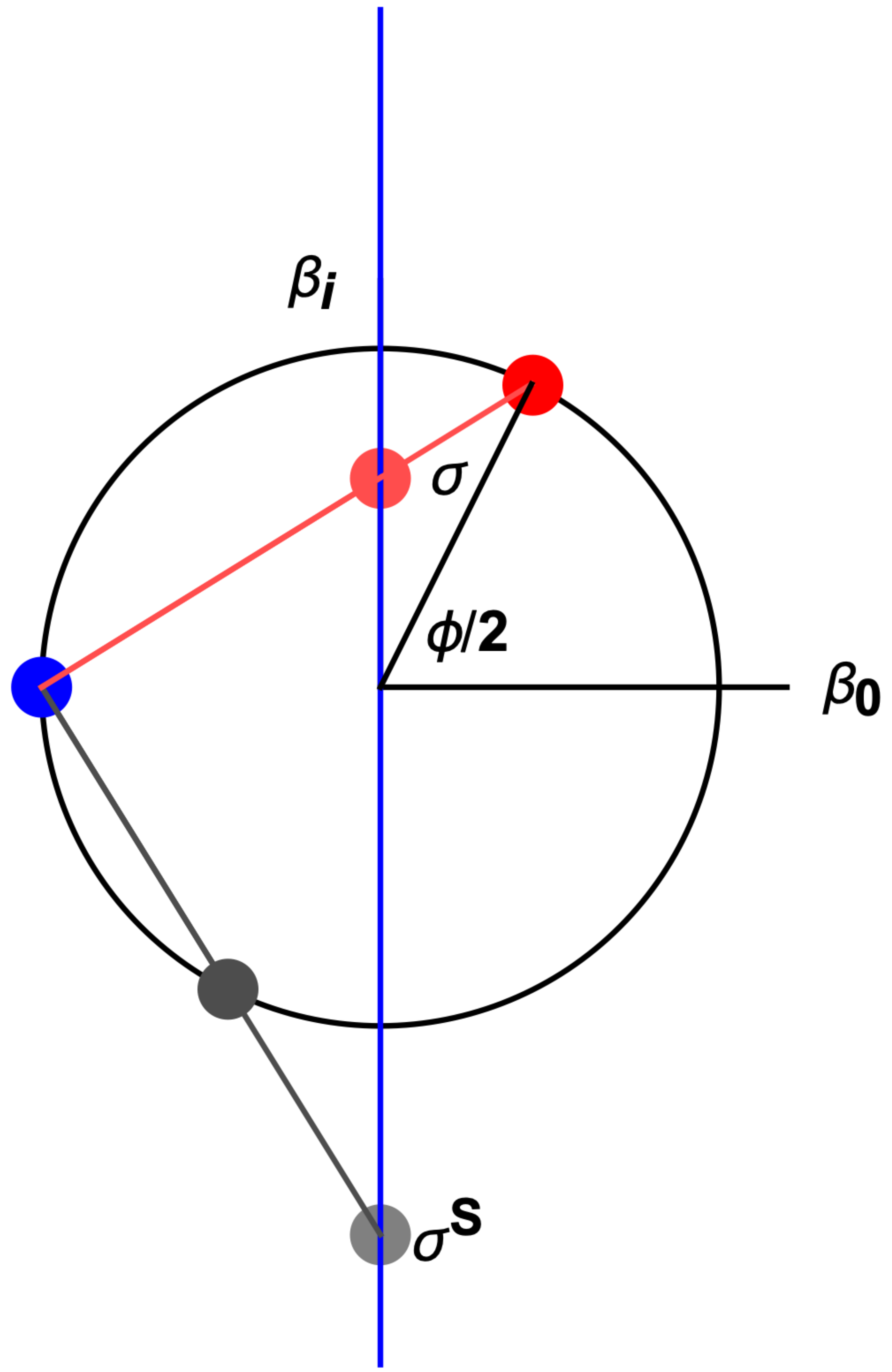


Projection Point: $(-1, 0, 0, 0)$

Projection Plane: $\beta_0 = 0$

Any attitude (surface point) is projected onto the hyper-plane to form the modified Rodrigues parameters.

The two EP sets yield *distinct* MRP coordinate values with different singular behaviors.



See Mathematica Demo

Shadow MRP Set

- Using the alternate set of Euler parameters, we can find the "shadow" set of MRP parameters:

$$\sigma_i^S = \frac{-\beta_i}{1-\beta_0} = \frac{-\sigma_i}{\sigma^2} \quad i = 1, 2, 3$$

Unique MRP Parameters

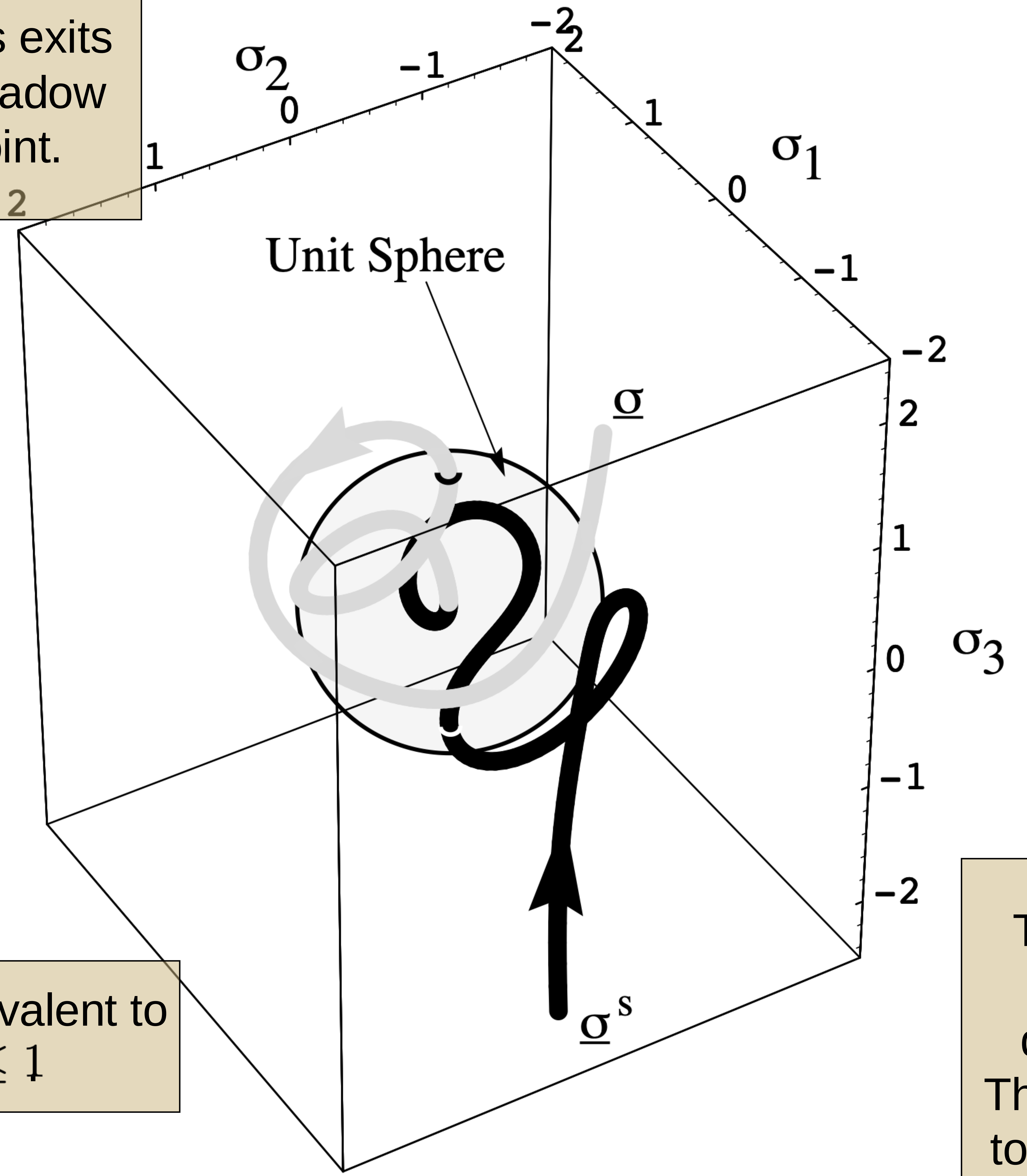
A common switching surface is $\sigma^2 = \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} = 1$. Note that

$$\begin{aligned} |\boldsymbol{\sigma}| &\leq 1 \quad \text{if} \quad \Phi \leq 180^{\mathrm{o}} \\ |\boldsymbol{\sigma}| &\geq 1 \quad \text{if} \quad \Phi \geq 180^{\mathrm{o}} \\ |\boldsymbol{\sigma}| &= 1 \quad \text{if} \quad \Phi = 180^{\mathrm{o}} \end{aligned}$$

$$\boldsymbol{\sigma}^S = \tan\left(\frac{\Phi - 2\pi}{4}\right)\hat{\boldsymbol{e}}$$

$$\boldsymbol{\sigma}^S = \tan\left(\frac{\Phi'}{4}\right)\hat{\boldsymbol{e}}$$

As one set of MRP coordinates exits the unit sphere surface, the shadow set enters at the opposite point.



Setting $\beta_0 \geq 0$ is equivalent to enforcing $|\boldsymbol{\sigma}| \leq 1$

The original shadow set of MRPs are convenient to describe tumbling bodies. The coordinates always point to the zero attitude along the shortest rotational path

Direction Cosine Matrix

Matrix components:

$$[C] = \frac{1}{(1+\sigma^2)^2} \begin{bmatrix} 4\left(\sigma_1^2 - \sigma_2^2 - \sigma_3^2\right) + (1-\sigma^2)^2 & 8\sigma_1\sigma_2 + 4\sigma_3(1-\sigma^2) & \cdots \\ 8\sigma_2\sigma_1 - 4\sigma_3(1-\sigma^2) & 4\left(-\sigma_1^2 + \sigma_2^2 - \sigma_3^2\right) + (1-\sigma^2)^2 & \cdots \\ 8\sigma_3\sigma_1 + 4\sigma_2(1-\sigma^2) & 8\sigma_3\sigma_2 - 4\sigma_1(1-\sigma^2) & \cdots \end{bmatrix}$$

$$\cdots \begin{matrix} 8\sigma_1\sigma_3 - 4\sigma_2(1-\sigma^2) \\ 8\sigma_2\sigma_3 + 4\sigma_1(1-\sigma^2) \\ 4\left(-\sigma_1^2 - \sigma_2^2 + \sigma_3^2\right) + (1-\sigma^2)^2 \end{matrix}$$

Vector computation:

$$[C] = [I_{3\times 3}] + \frac{8[\tilde{\boldsymbol{\sigma}}]^2 - 4\left(1-\sigma^2\right)[\tilde{\boldsymbol{\sigma}}]}{\left(1+\sigma^2\right)^2}$$

Interesting property:

$$[C(\boldsymbol{\sigma})]^{-1} = [C(\boldsymbol{\sigma})]^T = [C(-\boldsymbol{\sigma})]$$

Attitude Addition/Subtraction

- DCM method:

$$[FN(\boldsymbol{\sigma})] = [FB(\boldsymbol{\sigma}'')][BN(\boldsymbol{\sigma}')]$$

- Direct method:

$$\boldsymbol{\sigma} = \frac{(1 - |\boldsymbol{\sigma}'|^2)\boldsymbol{\sigma}'' + (1 - |\boldsymbol{\sigma}''|^2)\boldsymbol{\sigma}' - 2\boldsymbol{\sigma}'' \times \boldsymbol{\sigma}'}{1 + |\boldsymbol{\sigma}'|^2|\boldsymbol{\sigma}''|^2 - 2\boldsymbol{\sigma}' \cdot \boldsymbol{\sigma}''}$$

Attitude Addition

$$\boldsymbol{\sigma}'' = \frac{(1 - |\boldsymbol{\sigma}'|^2)\boldsymbol{\sigma} - (1 - |\boldsymbol{\sigma}|^2)\boldsymbol{\sigma}' + 2\boldsymbol{\sigma} \times \boldsymbol{\sigma}'}{1 + |\boldsymbol{\sigma}'|^2|\boldsymbol{\sigma}|^2 + 2\boldsymbol{\sigma}' \cdot \boldsymbol{\sigma}}$$

Relative Attitude (Subtraction)

Differential Kinematic Equations

Matrix components:

$$\dot{\boldsymbol{\sigma}} = \frac{1}{4}\begin{bmatrix} 1-\sigma^2+2\sigma_1^2 & 2\left(\sigma_1\sigma_2-\sigma_3\right) & 2\left(\sigma_1\sigma_3+\sigma_2\right) \\ 2\left(\sigma_2\sigma_1+\sigma_3\right) & 1-\sigma^2+2\sigma_2^2 & 2\left(\sigma_2\sigma_3-\sigma_1\right) \\ 2\left(\sigma_3\sigma_1-\sigma_2\right) & 2\left(\sigma_3\sigma_2+\sigma_1\right) & 1-\sigma^2+2\sigma_3^2 \end{bmatrix} {}^{\mathcal{B}}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

Vector computation:

$$\dot{\boldsymbol{\sigma}} = \frac{1}{4}\left[\left(1-\sigma^2\right)[I_{3\times 3}] + 2[\tilde{\boldsymbol{\sigma}}] + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right] {}^{\mathcal{B}}\boldsymbol{\omega} = \frac{1}{4}[B(\boldsymbol{\sigma})]\,{}^{\mathcal{B}}\boldsymbol{\omega}$$

Note: Only contains quadratic nonlinearities, but is singular for Φ =±360º.

- Now, let's invert the differential kinematic equation and find:

$$\boldsymbol{\omega} = 4[B]^{-1}\dot{\boldsymbol{\sigma}}$$

- Note the near-orthogonal property of the [*B*] matrix:

$$[B]^{-1} = \frac{1}{\left(1+\sigma^2\right)^2}[B]^T$$

You can proof this by investigating $[B][B]^T$.

- This leads to the elegant inverse transformation

$$\boldsymbol{\omega} = \frac{4}{\left(1+\sigma^2\right)^2}[B]^T\dot{\boldsymbol{\sigma}}$$

$$\boldsymbol{\omega} = \frac{4}{\left(1+\sigma^2\right)^2}\left[\left(1-\sigma^2\right)[I_{3\times3}] - 2[\tilde{\boldsymbol{\sigma}}] + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right]\dot{\boldsymbol{\sigma}}$$

Cayley Transform

- Let [*S*] be a skew-symmetric matrix, [*C*] be a proper orthogonal matrix, and [*I*] be a identity matrix. These matrices can be of any dimension *N*. The **extended Cayley Transform** is then defined as:

$$\boxed{[C] = ([I] - [S])^2([I] + [S])^{-2} = ([I] + [S])^{-2}([I] - [S])^2}$$

Unfortunately no equivalent inverse transformation exists. Instead, we define [*W*] to be the “square root” of [*C*]:

$$[C] = [W][W]$$

$$[C] = [V][D][V]^*$$

Adjoint Operator

- The “matrix square root” can then be defined as

$$[W] = [V] \begin{bmatrix} \ddots & & 0 \\ & \sqrt{[D]_{ii}} & \\ 0 & & \ddots \end{bmatrix} [V]^*$$

$$[W] = [V] \begin{bmatrix} e^{+i\frac{\theta_1}{2}} & 0 & \cdots & & & 0 \\ 0 & e^{-i\frac{\theta_1}{2}} & \cdots & & & 0 \\ \vdots & \vdots & \ddots & & & 0 \\ & & & e^{+i\frac{\theta_{N-1}}{2}} & 0 & 0 \\ & & & 0 & e^{-i\frac{\theta_{N-1}}{2}} & 0 \\ & & & 0 & 0 & +1 \end{bmatrix} [V]^*$$ Odd dimension

$$[W] = [V] \begin{bmatrix} e^{+i\frac{\theta_1}{2}} & 0 & \cdots & & 0 \\ 0 & e^{-i\frac{\theta_1}{2}} & \cdots & & 0 \\ \vdots & \vdots & \ddots & & 0 \\ & & & e^{+i\frac{\theta_{N-1}}{2}} & 0 \\ & & & 0 & e^{-i\frac{\theta_{N-1}}{2}} \end{bmatrix} [V]^*$$ Even dimension

- The standard Cayley transform can now be used to map between the skew-symmetric [S] matrix and the orthogonal [W] matrix:

$$[W] = ([I] - [S])([I] + [S])^{-1} \qquad = ([I] + [S])^{-1}([I] - [S])$$

$$[S] = ([I] - [W])([I] + [W])^{-1} \qquad = ([I] + [W])^{-1}([I] - [W])$$

- As with the CRP coordinates, for the 3D case the [S] matrix elements are MRP attitude coordinates. For higher dimensional cases, this allows us to parameterize N-dimensional proper orthogonal matrices using higher dimensional MRP coordinates.

- Recall that regardless of the dimensionality of the orthogonal matrix [$W(t)$], it must evolve according

$$[\dot{W}] = -[\tilde{\mathbf{\Omega}}][W]$$

These higher-dimensional "body angular velocities" can be related to the higher dimensional MRPs using:

$$[\tilde{\boldsymbol{\omega}}] = [\tilde{\mathbf{\Omega}}] + [W][\tilde{\mathbf{\Omega}}][W]^T$$

$$[\dot{S}] = \frac{1}{2}\left([I] + [S]\right)[\tilde{\mathbf{\Omega}}]\left([I] - [S]\right)$$

- This parameterization is singular whenever a principal rotation of 360º is performed.

- If these higher dimensional MRPs are singular for ±360º rotations, can this singularity be avoided by switching to “higher-dimensional shadow” set?

- This question was raised by some structures engineers trying to apply this extended Cayley transform to parameterize a proper orthogonal matrix in their problem.

- This is still an unsolved problem, is waiting to be investigated by some enterprising graduate student...