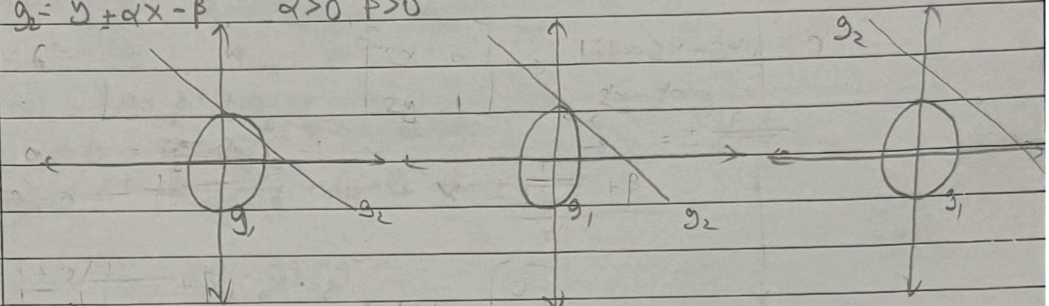
ASEN 6020 - HW 3
Spring 2025
Jash Bhalavat

Problem 1a

Problem 1 → $g_1 = x^2 + y^2 - 1$

a] $g_2 = y + \alpha x - \beta \qquad \alpha > 0 \;\; \beta > 0$



$\frac{\beta^2}{(1+\alpha^2)} < 1 \qquad\qquad \frac{\beta^2}{1+\alpha^2} = 1 \qquad\qquad \frac{\beta^2}{1+\alpha^2} > 1$

$\frac{\beta^2}{1+\alpha^2} < 1$ → In this case, g_2 intersects the circle (g_1) twice in which case the minimal distance between the 2 terminal manifolds is 0 at 2 points. This problem is not well posed

$\frac{\beta^2}{1+\alpha^2} = 1$ → In this case, g_2 intersects the circle (g_1) at one point. Here, the minimum distance between the 2 terminal conditions is 0. This problem does not need further solving

$\frac{\beta^2}{1+\alpha^2} > 1$ → This is a well-posed problem

$G = \begin{bmatrix} 2x & \alpha \\ 2y & 1 \end{bmatrix}$ → singular when $|G| = 0$

$|G| = 2x - 2\alpha y = 0 \rightarrow 2x = 2\alpha y \rightarrow x = \alpha y \rightarrow \frac{x}{y} = \alpha$

$g_1 = 0 = \alpha^2 y^2 + y^2 - 1 \rightarrow y^2(\alpha^2 + 1) = 1 \rightarrow \boxed{y_0 = \pm \frac{1}{\sqrt{1+\alpha^2}}}$

$g_1 = 0 = x_0^2 + \frac{1}{1+\alpha^2} - 1 \rightarrow \boxed{x_0 = \pm\sqrt{1 - \frac{1}{1+\alpha^2}}}$ ($\because \alpha, \beta > 0 \rightarrow x_0, y_0 \geq 0$) (1st quadrant)

$g_2 = 0 = y_1 + \alpha^2 y_1 - \beta \rightarrow \boxed{y_1 = \frac{\beta}{1+\alpha^2}}$

$g_2 = 0 = \frac{\beta}{1+\alpha^2} + \alpha x - \beta \rightarrow \boxed{x_1 = \frac{1}{\alpha}\left[\frac{-\beta}{1+\alpha^2} + \beta\right]}$

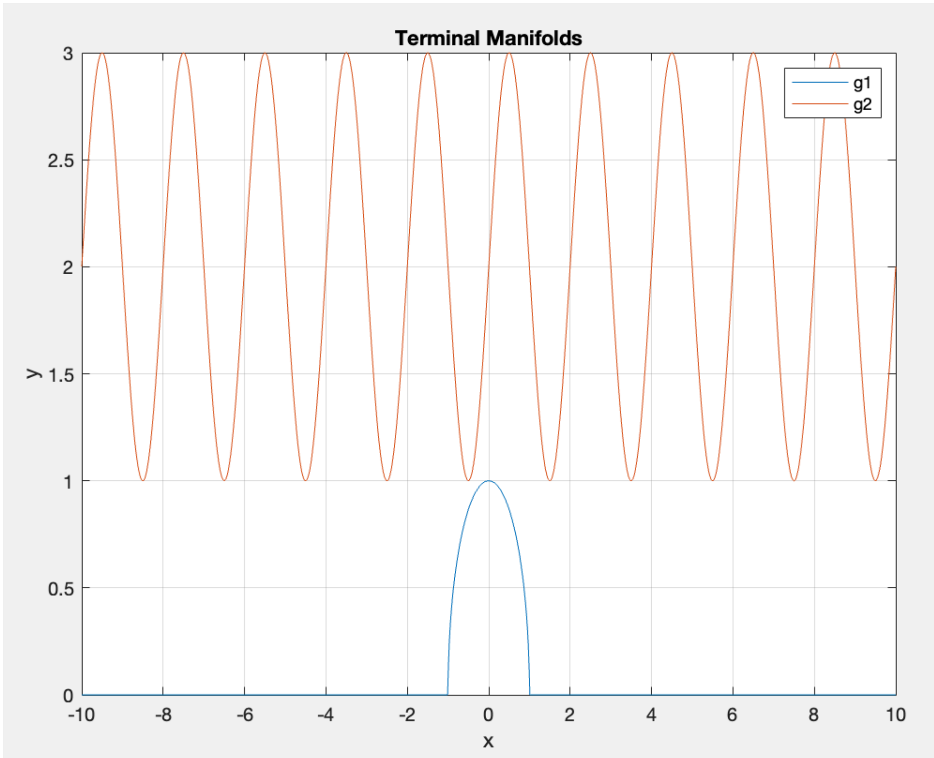
The minimal distance in this case is:

$$K = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

$$K = \sqrt{\left(\frac{1}{\alpha}\left(\frac{-\beta}{1+\alpha^2} + \beta\right) \mp \sqrt{1 - \frac{1}{1+\alpha^2}}\right)^2 + \left(\frac{\beta}{1+\alpha^2} \mp \frac{1}{\sqrt{1+\alpha^2}}\right)^2}$$

Problem 1b

This is what g1 and g2 look like:



(Note: g1 is a circle, but bottom half not shown because clearly the closest points from g2 will be on the top half)

ASEN 6020
Spring 2025
Jash Bhalavat

$\boxed{\text{HW \#3}}$

Problem 1 → b) $g_1 = x_1^2 + y_1^2 - 1 = 0$, $g_2 = y_2 - (2 + \sin(\pi x_2)) = 0$

$$G = \begin{bmatrix} 2x_1 & -\pi\cos(\pi x_2) \\ 2y_1 & 1 \end{bmatrix} \rightarrow \text{Singular} \rightarrow |G| = 0$$

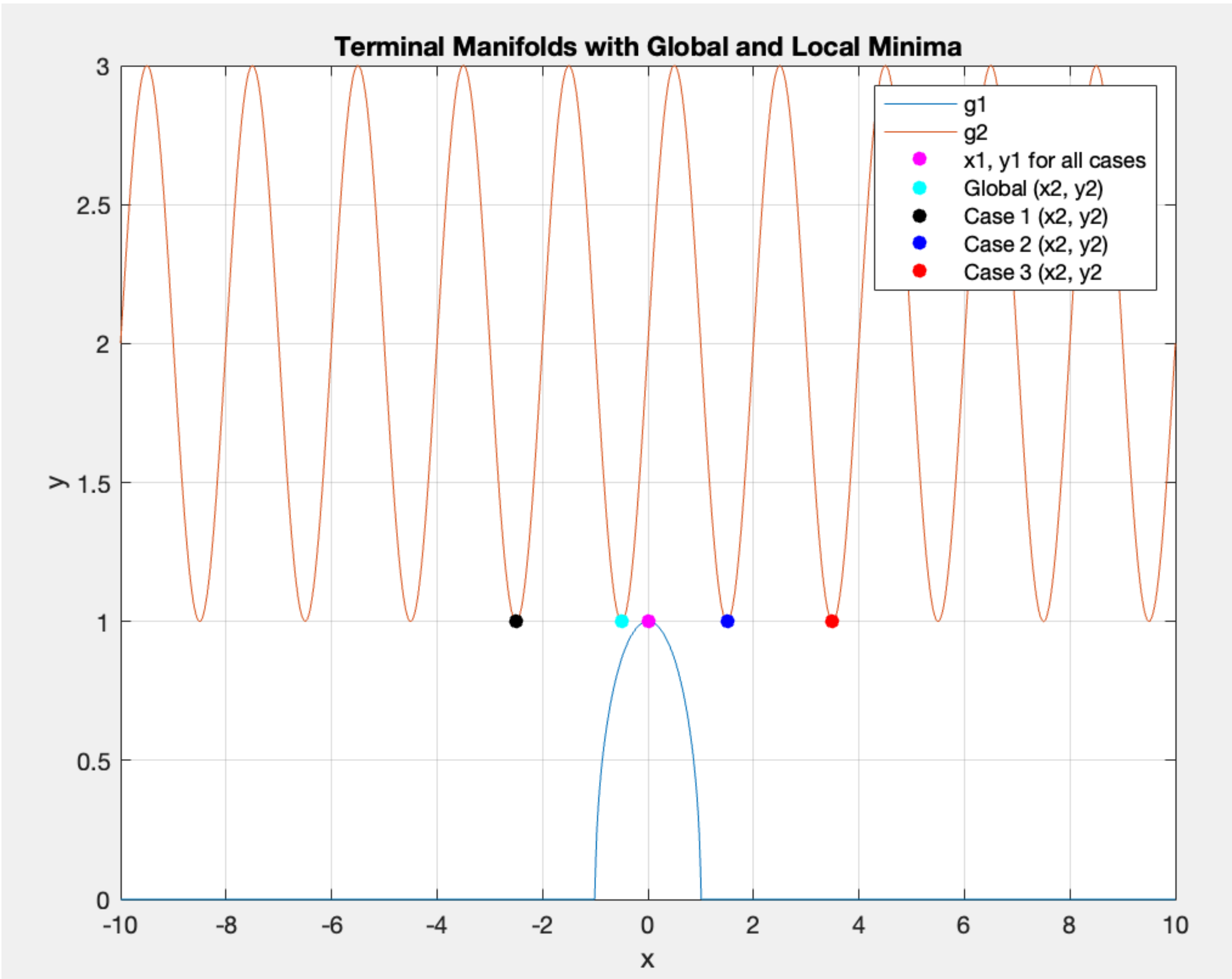
$$|G| = 2x_1 + 2y_1\pi\cos(\pi x_2) = 0$$

Using Matlab's fsolve() function to solve this for zero using the following initial conditions (where the initial conditions have the following form - [x1_0, y1_0, x2_0]:

1. Global - [0, 1, -0.5]

2. Local - [0, 1, 0]
3. Local - [0, 1, -1]
4. Local - [1, 0, 1.5]

All of these are found by visually inspecting the above graph. Plugging into Matlab, the following global and local minimal points are found



	x1	y1	x2	y2	Distance
Global	0	1	-0.5	1	0.5
Case 1	0	1	2.5	1	2.5
Case 2	0	1	1.5	1	1.5
Case 3	0	1	3.5	1	3.5

As expected the global distance is the lowest among all the other local distances.

Problem 2

Problem 2 → $\dot{x} = u$, $\dot{y} = v$, $\dot{u} = f(t)\cos\theta$, $\dot{v} = f(t)\sin\theta - g$

Performance index: $J = K + L$ ($L \equiv 0$, $K \equiv x_f$) → $J = x_f$

Hamiltonian: $H = \cancel{L}^{0} + p \cdot f = p_x u + p_y v + p_u f(t)\cos\theta + p_v[f(t)\sin\theta - g]$

$$f(t) = \begin{cases} >0 & 0 \le t \le t_{B/O} \\ 0 & t > t_{B/O} \end{cases}$$

Initial conditions: $x_0, y_0, u_0, v_0, t_0 = 0$

Optimal control policy → $\frac{\partial H}{\partial \theta} = 0 = -p_u f(t)\sin\theta + p_v f(t)\cos\theta$

$p_u \cancel{f(t)} \sin\theta = p_v \cancel{f(t)} \cos\theta$ (assume $f(t) \neq 0$)

$\boxed{\tan\theta^* = \frac{p_v}{p_u}}$ $\quad \tan^2\theta^* = \frac{p_v^2}{p_u^2}$

$H^* = H|_{\tan\theta^* = p_v/p_u} \to H^* = p_x u + p_y v + p_u f(t)\cos\theta^* + p_v f(t)\sin\theta^* - p_v g$

$\cos^2\theta = \frac{1}{1+\tan^2\theta} = \frac{1}{1+p_v^2/p_u^2} = \frac{p_u^2}{p_u^2+p_v^2}$

$\cos^2\theta + \sin^2\theta = 1 \to \sin^2\theta = 1 - \frac{p_u^2}{p_u^2+p_v^2} = \frac{p_v^2}{p_u^2+p_v^2}$

$H^* = p_x u + p_y v + f(t)\left[\frac{p_u^2}{\sqrt{p_u^2+p_v^2}} + \frac{p_v^2}{\sqrt{p_u^2+p_v^2}}\right] - p_v g = p_x u + p_y v + f(t)\sqrt{p_u^2+p_v^2} - p_v g$

State → $\dot{x} = \frac{\partial H^*}{\partial p_x} = u$, $\dot{y} = \frac{\partial H^*}{\partial p_y} = v$, $\dot{u} = \frac{\partial H^*}{\partial p_u} = \frac{f(t) p_u}{\sqrt{p_u^2+p_v^2}}$, $\dot{v} = \frac{\partial H^*}{\partial p_v} = \frac{f(t) p_v}{\sqrt{p_u^2+p_v^2}} - g$

Adjoint EOM → $\dot{p}_x = -\frac{\partial H^*}{\partial x} = 0$, $\dot{p}_y = -\frac{\partial H^*}{\partial y} = 0$, $\dot{p}_u = -\frac{\partial H^*}{\partial u} = -p_x$

$\dot{p}_v = -\frac{\partial H^*}{\partial v} = -p_y$ — p_x, p_y are constant

$p_u = \int -p_x\,dt = -p_{x_0}t + p_{u_0}$ → $\boxed{\tan\theta^* = \frac{p_{v_0} - p_{y_0}t}{p_{u_0} - p_{x_0}t}} = f(t)$

$p_v = \int -p_y\,dt = -p_{y_0}t + p_{v_0}$

Taking the derivative → $f'(t) = \frac{df(t)}{dt}$ $\frac{(p_{u_0} - p_{x_0}t)(-p_{y_0}) - (p_{v_0} - p_{y_0}t)(-p_{x_0})}{(p_{u_0} - p_{x_0}t)^2}$

$f'(t) = \frac{-p_{u_0}p_{y_0} + \cancel{p_{x_0}p_{y_0}t} + p_{v_0}p_{x_0} - \cancel{p_{x_0}p_{y_0}t}}{(p_{u_0} - p_{x_0}t)^2} = \boxed{\frac{p_{v_0}p_{x_0} - p_{u_0}p_{y_0}}{(p_{u_0} - p_{x_0}t)^2}}$

There is a discontinuity when $p_{u_0} = p_{x_0}t$.

But, otherwise $f'(t)$ is always +ve/-ve depending on $p_{v_0}, p_{x_0}, p_{u_0}, p_{y_0}$

∵ the numerator $(p_{u_0} - p_{x_0}t)^2 \ge 0$. ∴ $f'(t)$ is always either +ve/-ve (or undefined, when there is a discontinuity). ∴ $f(t)$ is a monotonic function.

Problem 3a

Problem 3 → $\dot{\vec{x}} = A\vec{x} + B\vec{u}$, $J = \frac{1}{2}\vec{x}_f^T S \vec{x}_f + \int_0^{t_f} \frac{1}{2}[\vec{x}^T Q \vec{x} + \vec{u}^T R \vec{u}] d\tau = K + \int_0^{t_f} L\, d\tau$

a) t_f constrained $\vec{x}_0, t_f$ constrained

$L = \frac{1}{2}[\vec{x}^T Q \vec{x} + \vec{u}^T R \vec{u}]$, $\vec{p} = \vec{\lambda}$, $f(\vec{x}) = \dot{\vec{x}}$

$$\boxed{H = L + \vec{p} \cdot f = \frac{1}{2}[\vec{x}^T Q \vec{x} + \vec{u}^T R \vec{u}] + \lambda^T [A\vec{x} + B\vec{u}]}$$

$$\frac{\partial H}{\partial u} = 0 = R\vec{u} + B^T \lambda \rightarrow R\vec{u} = -B^T \lambda \rightarrow \boxed{\vec{u} = -R^{-1} B^T \lambda}$$

Plug $\vec{u}$ back into $\dot{\vec{x}} = A\vec{x} + B\vec{u}$

$$\dot{\vec{x}} = A\vec{x} - BR^{-1}B^T \lambda$$
$$\dot{\lambda} = -\frac{\partial H}{\partial \vec{x}} = -Q\vec{x} - A^T \lambda \rightarrow \boxed{\begin{bmatrix} \dot{\vec{x}} \\ \dot{\lambda} \end{bmatrix} = \left[\begin{array}{c|c} A & -BR^{-1}B^T \\ -Q & -A^T \end{array}\right] \begin{bmatrix} \vec{x} \\ \lambda \end{bmatrix}}$$

$$\Phi(t, t_0) = \left[\begin{array}{c|c} A & -BR^{-1}B^T \\ -Q & -A^T \end{array}\right] := \begin{bmatrix} \Phi_{xx}(t,t_0) & \Phi_{x\lambda}(t,t_0) \\ \Phi_{\lambda x}(t,t_0) & \Phi_{\lambda\lambda}(t,t_0) \end{bmatrix}$$

Transversality conditions — x_0, t_0 are given, λ_0, H_0 arbitrary

$\because$ final state is free → $\boxed{\lambda_f = \frac{\partial K}{\partial x_f} = S\vec{x}_f}$

$$\begin{bmatrix} x_f \\ \lambda_f \end{bmatrix} = \Phi(t_f, t_0) \begin{bmatrix} \vec{x}_0 \\ \lambda_0 \end{bmatrix}$$
$$\begin{bmatrix} x_0 \\ \lambda_0 \end{bmatrix} = \Phi(t_0, t_f) \begin{bmatrix} \vec{x}_f \\ \lambda_f \end{bmatrix} \rightarrow \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \Phi(t, t_f) \begin{bmatrix} \vec{x}_f \\ \lambda_f \end{bmatrix}$$

$$\vec{x}(t) = \Phi_{xx}(t,t_f)\vec{x}_f + \Phi_{x\lambda}(t,t_f)\lambda_f = [\Phi_{xx}(t,t_f) + \Phi_{x\lambda}(t,t_f)]\vec{x}_f$$
$$\lambda(t) = \Phi_{\lambda x}(t,t_f)\vec{x}_f + \Phi_{\lambda\lambda}(t,t_f)\lambda_F = [\Phi_{\lambda x}(t,t_f) + \Phi_{\lambda\lambda}(t,t_f)S]\vec{x}_f$$

$$\vec{x}_f = [\Phi_{xx} + \Phi_{x\lambda}S]^{-1} x$$
$$\boxed{\lambda(t) = [\Phi_{\lambda x} + \Phi_{\lambda\lambda}S] \cdot [\Phi_{xx} + \Phi_{x\lambda}S]^{-1} \cdot \vec{x}(t)}$$

$$\boxed{u(t) = -R^{-1}B^T[\Phi_{\lambda x} + \Phi_{\lambda\lambda}S][\Phi_{xx} + \Phi_{x\lambda}S]^{-1} x(t)}$$

$$\boxed{\lambda_0 = -\frac{\partial K}{\partial x_0} = 0} \quad \boxed{H_0 = \frac{\partial K}{\partial t_0} = 0} \quad \boxed{H_f = -\frac{\partial K}{\partial t_f} = 0}$$

Using Riccati Diff Eq → $\dot{K} = -KA - A^T K + KBR^{-1}B^T K - Q$

$$\boxed{\begin{aligned} u(t) &= -R^{-1}B^T K(t, t_f) x(t) \\ \dot{x} &= [A - BR^{-1}B^T K(t, t_f)]\vec{x} \end{aligned}}$$

Problem 3b

b) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $R = 1$, $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$H = \cancel{\frac{1}{2} \vec{x}^T \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{x}}^{\,0} + u^2 + \vec{\lambda}^T \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \right)$$

$$u^* = -\frac{1}{1} \begin{bmatrix} 0 & 1 \end{bmatrix} \vec{\lambda} = \boxed{\begin{bmatrix} 0 & -1 \end{bmatrix} \vec{\lambda}(t) = u(t)^*}$$

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \vec{\lambda}$$

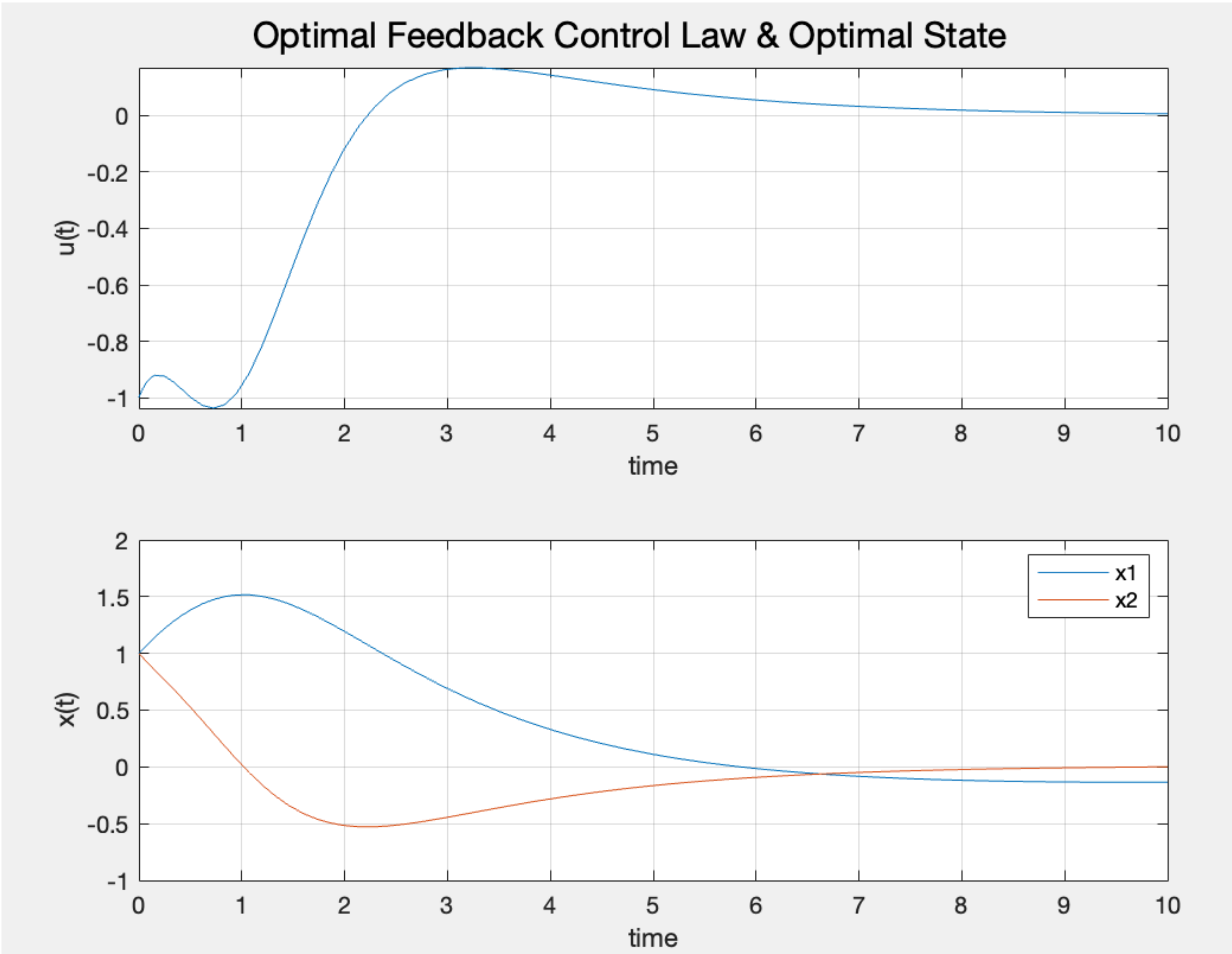
$$\dot{\vec{\lambda}} = \cancel{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{x}}^{\,0} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \vec{\lambda} \rightarrow \Phi(t, t_0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\dot{K} = -KA - A^T K + K B R^{-1} B^T K - \cancel{Q}^{\,0}$$

$$\dot{K} = -K \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} K + K \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} K$$

$K_0 = K(t_f, t_f) = S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, Using matlab ODE45() to solve this:

After getting K, x*(t) and u*(t) can also be found. x*(t) is computed by using ODE45() as well. The plot below shows the optimal feedback control law and the optimal state starting from x0 = [1, 1]^T.



Optimal Feedback Control Law & Optimal State
u(t)
0
-0.2
-0.4
-0.6
-0.8
-1
0 1 2 3 4 5 6 7 8 9 10
time
x(t)
2
1.5
1
0.5
0
-0.5
-1
x1
x2
0 1 2 3 4 5 6 7 8 9 10
time

Problem 4

problem 4 → $J = \int_0^T |u|\, d\tau$, $\dot{x} = -x + u$, $|u| \le 1$, $x_0 = 1$, $x(T) = 0$, $t_0 = 0$, $t_f = T$

a) given J, $K = 0$, $L = |u|$, $n = 1$, $m = 1$

$H = L + p \cdot f = \boxed{|u| + p(-x+u) = H(x,p,u)}$

b) Pontryagin Principle → choose $\boxed{u^* = -p}$

c) $\boxed{\begin{array}{l} H = |p| + p(-x-p) \\ \text{If } p < 1 \to |u| = 0 \\ \text{If } p > 1 \to u = \pm 1 \end{array}}$ $\boxed{H^* = |p| - p^2 - px}$

$\boxed{\text{If } p < 1 \to \text{singular arc}}$

d) $\boxed{\dot{x} = \frac{\partial H^*}{\partial p} = 1 - 2p - x}$

$\boxed{\dot{p} = -\frac{\partial H^*}{\partial x} = p}$

d) $p(t) = \int_0^T \dot{p}\, dt = \boxed{p(T) - p_0} = p(t)$

$x(t) = \int_0^T \dot{x}\, dt = t - 2pt - x(t)\Big|_0^T = T - 2p(T) - \cancel{x(T)}^{0} + x(0) = \boxed{T - 2p(T) + 1 = x(t)}$

e) $\boxed{u^* = -p(t)} = -p(T) + p_0$

$\boxed{J^* = \int_0^T |u^*|\, d\tau} = \boxed{\int_0^T |-p(t)|\, d\tau}$

f) $\boxed{\begin{array}{l} u^* = -p(\ln(2)) = -p(T) + p_0 \\ J^* = \int_0^{\ln 2} |-p(T) + p_0|\, d\tau \end{array}}$

g) $\boxed{\begin{array}{l} u^* = -p(T) + p_0 \\ J^* = \int_0^T |-p(T) + p_0|\, d\tau \end{array}}$

Problem 5

$\boxed{\text{HW \#3}}$

Problem 5 → $t=0$, $\ddot{x}=u$, $|u|\le 1$, $x_f=0$, $\dot{x}_f=0$, $L=0$, $K=t_f$

$$\vec{x}=\begin{bmatrix} x \\ \dot{x} \end{bmatrix} \rightarrow \dot{\vec{x}}=\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}=\begin{bmatrix} \dot{x} \\ u \end{bmatrix}=f(\vec{x},u)$$

$$\dot{\vec{x}}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix}+\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} u \end{bmatrix}=A\vec{x}+Bu$$

$$J=K+\cancel{\int_0^T L\,dt}^{\,0}=t_f=K$$

$$H=\cancel{L}^{\,0}+p\cdot f=p_{\dot{x}}\,\dot{x}+p_u\cdot u$$

Optimal control → Using Pontryagin principle → choose $\boxed{u^*=-p_u}$

$$\boxed{\begin{array}{l} \text{If } p_u>0 \rightarrow u^*=-1 \\ \text{If } p_u<0 \rightarrow u^*=1 \\ H^*=p_{\dot{x}}\dot{x}+p_u\cdot u^* \end{array}}$$

If $p_u=0$ – $u^*=0$ – singular arc

$$\boxed{\begin{array}{l} \dot{\vec{x}}=\dfrac{\partial H^*}{\partial p_{\dot{x}}}=\dot{x} \\ \dot{\ }=\dfrac{\partial H^*}{\partial p_u}=u^* \end{array}}
\qquad
\boxed{\begin{array}{l} \dot{p}=\dfrac{-\partial H^*}{\partial x}=0 \\ \quad -\dfrac{\partial H^*}{\partial \dot{x}}=-p_x \end{array}}$$

$$\boxed{\begin{array}{ll} p_0=-\dfrac{\partial K}{\partial x_0}=0 & p_f=\dfrac{\partial K}{\partial x_f}=0 \\ H_0=\dfrac{\partial K}{\partial t_0}=0 & H_f=\dfrac{\partial K}{\partial t_f}=1 \end{array}}$$