ASEN 5044, Fall 2024
Statistical Estimation for Dynamical Systems

Lecture 32:
The Extended Kalman Filter

Prof. Nisar Ahmed (Nisar.Ahmed@Colorado.edu)

Tuesday 12/02/2024

Ann and H.J. Smead
Aerospace Engineering Sciences
UNIVERSITY OF COLORADO BOULDER

Announcements

- Midterm 2 grades to be posted tonight; HW 7+8 grades to post soon
- **Final project assignment tasks + Progress report 1 posted**
  - **Progress Report 1 due this Fri 12/06**
  - **Progress Report 2 out this Thurs 12/05, due next Fri 12/13**
  - **Final Project Report due in 2 weeks (Tues 12/17)**
- **Quiz 8 to be released Thurs 12/05, due Tues 12/10 (final quiz)**
- **Last lecture this Thurs 12/05 (no lectures or class recordings next week)**
- Course FCQ available online (link emailed to you all by CU): please be sure to fill out!

Overview

Last Time:

- DT Linearized Kalman filter (LKF)

Today:

- Quick overview of Final Project assignment
- DT Extended Kalman filter (EKF)
- Closer look at linearization and covariance approximations for LKF and EKF

for TMT (truth model tests):
given a "real" Q / $\tilde{W}$ (noise specs) for NL Dyn. process noise

in $\dot{x}_t = \underline{\underline{\mathcal{F}}}[x_t, u_t, \tilde{w}_t]$ $\quad \tilde{w} \sim$ AWGN w/ PSD $\tilde{W}_{NL}$

or

$x_{k+1} = f[x_k, u_k, \underline{\underline{w_k}}]$, $\underline{\underline{w_k}} \sim \underline{\underline{Q_{NL}}}$

= "ode45" of CT stoch. Dynamics

(RK integration)

input

$x_{k+1} = \text{ode45}(@(x,t), @\underline{\underline{\mathcal{F}}}[x_{t=k}, u_{t=k}, \underline{\underline{w_{t=k}}}])$

Last Time: The Linearized KF Algorithm

- So now we can estimate the total state as follows:

Computed offline — deterministic

random correction computed online

$$\hat{x}^+_{k+1} \approx x^*_{k+1} + \hat{\delta x}^+_{k+1}$$

where $x^*_{k+1} = x^*(t = t_{k+1})$ and $\hat{\delta x}_{k+1}$ is estimated using LTV KF for δx_{k+1} and δy_{k+1}:

Time update/prediction step for time k+1:

$$\hat{\delta x}^-_{k+1} = \tilde{F}_k \hat{\delta x}^+_k + \tilde{G}_k \delta u_k$$

$$P^-_{k+1} = \tilde{F}_k P^+_k \tilde{F}^T_k + \tilde{\Omega}_k Q_k \tilde{\Omega}^T_k$$

$$\delta u_{k+1} = u_{k+1} - u^*_{k+1}$$

Measurement update/correction step for time k+1:

$$\hat{\delta x}^+_{k+1} = \hat{\delta x}^-_{k+1} + K_{k+1}(\underline{\delta y_{k+1}} - \underbrace{\tilde{H}_{k+1}\hat{\delta x}^-_{k+1}}_{\hat{\delta y}_{k+1}})$$

$$P^+_{k+1} = (I - K_{k+1}\tilde{H}_{k+1})P^-_{k+1}$$

$$K_{k+1} = P^-_{k+1}\tilde{H}^T_{k+1}[\tilde{H}_{k+1}P^-_{k+1}\tilde{H}^T_{k+1} + R_{k+1}]^{-1}$$

$$\delta y_{k+1} = y_{k+1} - y^*_{k+1} = y_{k+1} - h(x^*_{k+1})$$

Actual received sensor measurement at time k+1

Computed nominal sensor measurement at time k+1

where $\tilde{F}_k, \tilde{G}_k, \tilde{\Omega}_k, \tilde{H}_k$ eval'd along (x^*, u^*) nom. sol'n at each time step k

Pros/Cons of the Linearized KF

- Pros:
  - Easy to program and numerically fast [can compute all required Jacobians offline]
  - Good for predictable systems with small/low process noise inputs

- Cons:
  - Will break if actual true system x(t) trajectory deviates too far from nominal x*(t)

(i.e. if $\delta x(t)$ and $\delta u(t)$ get too big $\Rightarrow \hat{\delta x}(t)$ will have large errors $\rightarrow$ possibly unrecoverable!!)

(DT Jacobians will be wrong
$\rightarrow$ P^+ & P^- covars will be wrong!)

[Filter Divergence]

- Alternative: what if we kept estimating total state (not just perturbation) using most recent online total state estimate as prior for linearization (instead of fixed nominal trajectory)?

The Extended Kalman Filter (EKF) Algorithm

- Step 1: Initialization: start with some initial estimate of total state and covariance

$$\hat{x}^{+}(0),\ \hat{P}^{+}(0)$$

- Step 2: set k=0

- Step 3: **Time update/prediction step for time k+1:**

ode45/R-K on NL ODE w/ $\hat{x}_k^+$ as IC [from $t=t_k \to t=t_{k+1}$] assuming $w_k=0$

$$\hat{x}_{k+1}^{-} = f[\hat{x}_k^{+}, u_k, w_k = 0]$$

(deterministic nonlinear DT dyn. fxn eval. on $\hat{x}_k^{+}$)

$$P_{k+1}^{-} = \tilde{F}_k P_k^{+} \tilde{F}_k^{T} + \tilde{\Omega}_k Q_k \tilde{\Omega}_k^{T},$$

(approx. predicted covar. via dyn. linearization about $\hat{x}_k^{+}$)

i.e. $\hat{x}_{k+1}^{-} \approx \hat{x}_k^{+} + \Delta T \cdot \mathcal{F}[\hat{x}_k^{+}, u_k]$

only for covar. approx.!

where $\tilde{F}_k|_{\hat{x}_k^{+}, u_k, w_k=0} \approx I + \Delta T \cdot \tilde{A}|_{(\hat{x}_k^{+}, u(t_k), w(t_k)=0)},$

$$\tilde{\Omega}_k| \approx \Delta T \cdot \Gamma(t)|_{(t=t_k)},$$

evaluate Jacobians about most recent "best" total state estimate!

The Extended Kalman Filter (EKF)

- Step 4: **Measurement update/correction step for time k+1:**

Compute:

$$\hat{y}^-_{k+1} = h[\hat{x}^-_{k+1}, v_{k+1} = 0] \quad \text{(deterministic nonlinear fxn evaluation)}$$

$$\tilde{H}_{k+1} = \frac{\partial h}{\partial x}\Big|_{\hat{x}^-_{k+1}} \quad \text{(meas. fxn Jacobian at predicted state)}$$

again: linearize about most recent "best" total state estimate!

$$\tilde{e}_{y_{k+1}} = y_{k+1} - \hat{y}^-_{k+1} \quad \text{(nonlinear meas. innovation: actual data minus predicted)}$$

$$\tilde{K}_{k+1} = P^-_{k+1}\tilde{H}^T_{k+1}[\tilde{H}_{k+1}P^-_{k+1}\tilde{H}^T_{k+1} + R_{k+1}]^{-1} \quad \text{(approx. KF gain from meas. linearization)}$$

$$\Rightarrow \quad \hat{x}^+_{k+1} = \hat{x}^-_{k+1} + \tilde{K}_{k+1}\tilde{e}_{y_{k+1}} \quad \text{(updated total state estimate)}$$

$$P^+_{k+1} = (I - \tilde{K}_{k+1}\tilde{H}_{k+1})P^-_{k+1} \quad \text{(approx. updated covar. via linearization)}$$

- Step 5: recursion: go back to step 3 and repeat cycle for next time step…

What do we mean by "nom[k]" for linearization?

- Remember, nom[k] means two different things depending on whether you are doing linearized KF or the EKF:

$$\tilde{F}_k \Big|_{nom[k]} = \begin{cases} \left[\dfrac{\partial f}{\partial x} \Big|_{(x_k^*,\, u_k^*,\, w_k = 0)} \right] & \text{for linearized KF} \\ \text{where } x_k^* = x^*(t = t_k) \;\; \& \;\; x^*(t) = x_{nom}(t) \\ \text{a priori/offline computed sol'n to } \dot{x} = f[x(t), u(t), w(t) = 0] \\ \left[\dfrac{\partial f}{\partial x} \Big|_{(\hat{x}_k^+,\, u(k),\, w(k) = 0)} \right] & \text{Extended KF} \\ \rightarrow x_{nom}[k] = \hat{x}_k^+ = \text{current best (total) state estimate} \\ \rightarrow \text{computed online} \end{cases}$$

The "1st Order" EKF Algorithm: Important Features

Useful to remember some key ideas for the EKF:

- Finding approx. Gaussian joint pdf for state and measurements from "best available guess" of total nonlinear system behavior/state at each time k
- Only use best available estimate of state at any point in time to compute required Jacobian matrices and nonlinear function evaluations at that time

  → do not need to know nominal trajectory for linearization in advance!!!

  (EKF figures it out online/"on the fly" using it's own estimates)

- We only need 1st order Taylor series/linearization of dynamics and measurements to get predicted covariance P^-_{k+1} , updated covariance P^+_{k+1}, and EKF gain $\tilde{K}_{k+1}$

  ➔ all of these <u>matrix quantities are obtained via Jacobians</u>

  (similar to vanilla KF, except now matrices are time-varying and depend on $\hat{x}^+_k$!)

- **<u>DO NOT</u> use linearization/Jacobians to get predicted state $\hat{x}^-_{k+1}$ or measurement $\hat{y}_{k+1}$**

  ➔ <u>predicted vectors come directly from integrating/evaluating nonlinear CT fxns!</u>

Origin of the Linearized Covariance Approximations

- Recall from Lec 31 that both linearized KF and EKF use similar expressions for covariance (main difference is in how Jacobians computed for each)
- e.g. for the prediction step at time k+1,

$$P_{k+1}^{-} \approx \tilde{F}_k P_k^{+} \tilde{F}_k^T + \tilde{\Omega}_k Q_k \tilde{\Omega}_k^T,$$

- But how are such covariance approximations mathematically justified?
- Consider for the EKF: idea: linearize $f(\dots)$ about current best state est. $\hat{x}_k^+$

$\rightarrow \hat{x}_{k+1}^- = E[\underline{f}(x(k), u(k), w(k)) \mid y_{1:k}]$

$\rightarrow$ Expand via Taylor series about $x(k) = \hat{x}_k^+$ & $w_k = 0$ & same given u_k:

$$\hat{x}_{k+1}^- = E\Big[f(\hat{x}^+(k), u(k), 0) + \underbrace{\frac{\partial f}{\partial x}\Big|_{(\hat{x}_k, u_k, 0)}}_{\tilde{F}_k|_{nom[k]}} \cdot (x_k - \hat{x}_k^+) + \underbrace{\frac{\partial f}{\partial w}\Big|_{nom[k]}}_{\tilde{\Omega}_k|_{nom[k]}} w_k + \underbrace{\frac{\partial f}{\partial u}\Big|_{nom[k]}}_{\tilde{G}_k|_{nom[k]}} (u_k - u_{nom}[k]) + \text{HOTS} \dots \Big| y_{1:k}\Big]$$

Origin of the Linearized Covariance Approximations

- So if if we neglect HOTs & hope that linearization is valid near likely values of $x(k)$ (i.e. w.r.t. $p(x_{k+1}|y_{1:k})$) we will get that $E[\cdots|y_{1:k}]$ works out to:

$$\hat{x}^-_{k+1} \approx \underline{\underline{f(\hat{x}^+_k, u_k, w_k=0)}} + \tilde{F}\big|_{k,\text{nom}[k]} \cdot \cancel{E[x_k - \hat{x}^+_k | y_{1:k}]}^{\,0} \text{ b/c } \hat{x}^+_k \triangleq E[x_k|y_{1:k}]$$

$$+ \underbrace{\cancel{\tilde{G}_{k|\text{nom}[k]} \cdot \delta u_k}}_{\text{assume no ctrls change}} + \underbrace{\cancel{\tilde{\Omega}_{k|\text{nom}[k]} \cdot E[w_k]}}_{\text{if AWGN for } w_k} = 0$$

$\longrightarrow$ EKF state prediction @ time k+1:

$$\boxed{\hat{x}^-_{k+1} = f(\hat{x}^+_k, u_k, w_k=0)}$$

Origin of the Linearized Covariance Approximations

- So therefore, $P^-_{k+1} = E[\{x_{k+1} - \hat{x}^-_{k+1}\}\{\dots\}^T \mid y_{1:k}]$

$\rightarrow$ But since $x(k+1) \approx \hat{x}^-_{k+1} + \tilde{F}_k|_{nom}[x_k - \hat{x}^+_k] + \tilde{\Omega}_k|_{nom} w_k + \tilde{G}_k|_{nom} \delta u_k$

(by Taylor Series)

$\rightarrow$ square up the x_{k+1} approximation after sub'ing into $E[\{\dots\}\{\dots\}^T \mid y_{1:k}]$ to get:

$$\boxed{P^-_{k+1} = \tilde{F}_k|_{nom} P^+_k \tilde{F}_k|^T_{nom} + \tilde{\Omega}_k|_{nom} Q_k \tilde{\Omega}^T_k|_{nom}}$$

EKF covar prediction ✓

(can apply the same logic to get matrix covar approx. for EKF measurements & linearized KF updates)

Initializing and Tuning the EKF (and Linearized KF)

- One major issues for the EKF (and Linearized KF):
  - how to pick initial guess for state estimate and error covariance?
  - how to tune DT process noise covariance parameters, Q_{EKF} (Q_{LKF})?

- Poor choices can lead to filter inconsistent behavior and/or divergence!

- For nonlinear systems, cannot guarantee convergence to steady state covariance using a linearized approximation
  - different from KF for truly LTI systems, which is more "forgiving" as long as we have right model

Initializing and Tuning the EKF (and Linearized KF)

- Initialization: no silver bullet, but some general ideas and considerations (incomplete/non-exhaustive list…):
  - "Inflated" diagonal initial covariance – tricky to use for certain types of problems (e.g. how big to set Euler angle errors? Quaternions?)
  - Batch data processing to warm start, e.g. "static" initialization with linearized least squares or non-linear least squares
  - Sometimes can use LKF to initialize a "well-known"/well-observed portion of trajectory, before switching to EKF for remainder
  - Control can help quite a lot! (e.g. aircraft/spacecraft system stabilization)

Initializing and Tuning the EKF (and Linearized KF)

- Process noise tuning and filter error compensation: no silver bullet, but some general ideas and considerations (incomplete/non-exhaustive list…):
  - Tuning Q_{EKF}/Q_{LKF} not too different than from linear KF, but can be non-intuitive
    - no obvious way to generalize Van Loan's method to convert CT → DT AWGN in nonlinear case
    - can still apply chi-square NEES/NIS tests to validate filter (even more important to apply NEES/NIS tests to EKF and Linearized KF – these should pass in order to validate that linearization is acceptable!)
  - Deal with presence of <u>biases</u> -- tuning Q_{EKF}/Q_{LKF} generally will <u>not be enough</u>!
    - Significant persistent biases can show up from neglected higher order terms/dynamics (linearization errors always lead to some bias…these may or may not be acceptably small…)
    - If biases are observable, can augment state vector to include and estimate/remove online
    - Depending on nature of non-linearities: can sometimes explicitly compute and remove biases (e.g. converting polar range + bearing observations to Cartesian x-y "pseudo-measurements")
  - "Desperate last resorts"/band-aids/ad-hockery to cope with covariance approximations and other stubborn covariance issues:
    - add artificial "pseudo-noise" to Q_{EKF} / Q_{LKF} to compensate for errors and tweak filter gains
    - add "fudge" terms to selectively "jack-up" predicted state covariance: $\breve{P}_{k+1}^{-} = \Upsilon P_{k+1}^{-} \Upsilon^T$