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AsEN 5044
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Fall 2024

HW1

AQ1 →

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -40/3 \\ 0 & 5 & 0 \end{bmatrix}, \text{ let } a = \frac{40}{3}, b = 5$$

$$\Phi(t, t_0) = e^{A \Delta t} = \sum_{i=0}^{\infty} \frac{A^i (\Delta t)^i}{i!}, A^0 = I, A^1 = A$$

$$A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & b & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & b & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -ab & 0 \\ 0 & 0 & -ab \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -ab & 0 \\ 0 & 0 & -ab \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & b & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & a^2 b \\ 0 & 0 & a^2 b \\ 0 & -ab^2 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a^2 b \\ 0 & -ab^2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & b & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a^3 b & 0 \\ 0 & 0 & a^3 b^2 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a^3 b & 0 \\ 0 & 0 & a^3 b^2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & b & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -a^4 b \\ 0 & -a^4 b^2 & 0 \\ 0 & a^5 b^3 & 0 \end{bmatrix}$$

$$\Phi(t, t_0) = I + A \Delta t + A^2 \frac{\Delta t^2}{2} + A^3 \frac{\Delta t^3}{6} + A^4 \frac{\Delta t^4}{24} + A^5 \frac{\Delta t^5}{120}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & b & 0 \end{bmatrix} \Delta t + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{ab}{2} & 0 \\ 0 & 0 & -\frac{ab}{2} \end{bmatrix} \Delta t^2 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{a^2 b}{6} \\ 0 & -\frac{ab^2}{6} & 0 \end{bmatrix} \Delta t^3 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{a^3 b}{24} & 0 \\ 0 & 0 & \frac{a^3 b^2}{24} \end{bmatrix} \Delta t^4 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{a^4 b}{120} & 0 \\ 0 & \frac{a^5 b^3}{120} & 0 \end{bmatrix} \Delta t^5$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \left[1 - \frac{ab}{2!} \Delta t^2 + \frac{(ab)^2}{4!} \Delta t^4 - \dots \right] & \left[a \Delta t + \frac{a^2 b}{3!} \Delta t^3 - \frac{a^3 b^2}{5!} \Delta t^5 + \dots \right] \\ 0 & \left[b \Delta t - \frac{ab^2}{3!} \Delta t^3 + \frac{a^2 b^3}{5!} \Delta t^5 - \dots \right] & \left[1 - \frac{ab}{2!} \Delta t^2 + \frac{(ab)^2}{4!} \Delta t^4 - \dots \right] \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\sqrt{ab} \Delta t) & -\frac{\sqrt{a}}{\sqrt{b}} \sin(\sqrt{ab} \Delta t) \\ 0 & \frac{\sqrt{b}}{\sqrt{a}} \sin(\sqrt{ab} \Delta t) & \cos(\sqrt{ab} \Delta t) \end{bmatrix}$$

$$\Phi(t, t_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\sqrt{\frac{200}{3}} \Delta t\right) & -\frac{\sqrt{40}}{\sqrt{5}} \sin\left(\sqrt{\frac{200}{3}} \Delta t\right) \\ 0 & \sqrt{\frac{15}{40}} \sin\left(\sqrt{\frac{200}{3}} \Delta t\right) & \cos\left(\sqrt{\frac{200}{3}} \Delta t\right) \end{bmatrix}$$