ASEN 6060
ADVANCED ASTRODYNAMICS
Formulating the CR3BP

Objectives:

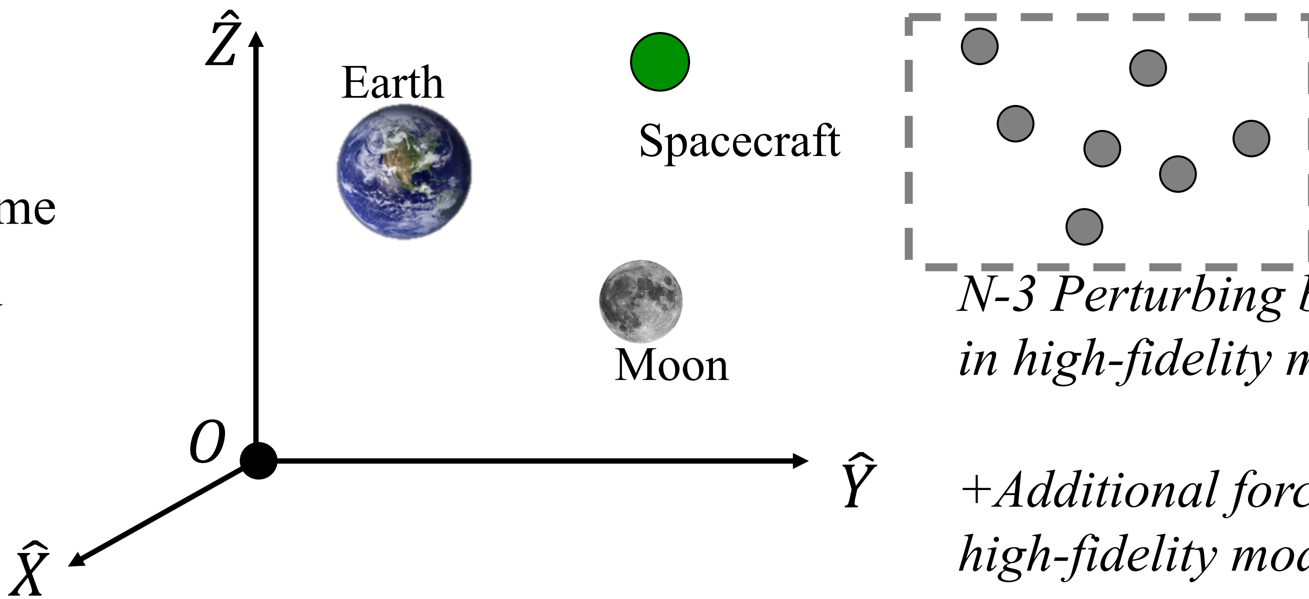
- Define fundamental assumptions and problem configuration for the CR3BP
- Define normalization scheme and incorporate assumptions
- Define rotating frame
- Derive the equations of motion for CR3BP
- Define the pseudo-potential function

Multi-Body Systems

Consider the following configuration of bodies, where each body can follow any general path

$\hat{X}\hat{Y}\hat{Z}$: Inertial frame

O, Inertially fixed basepoint



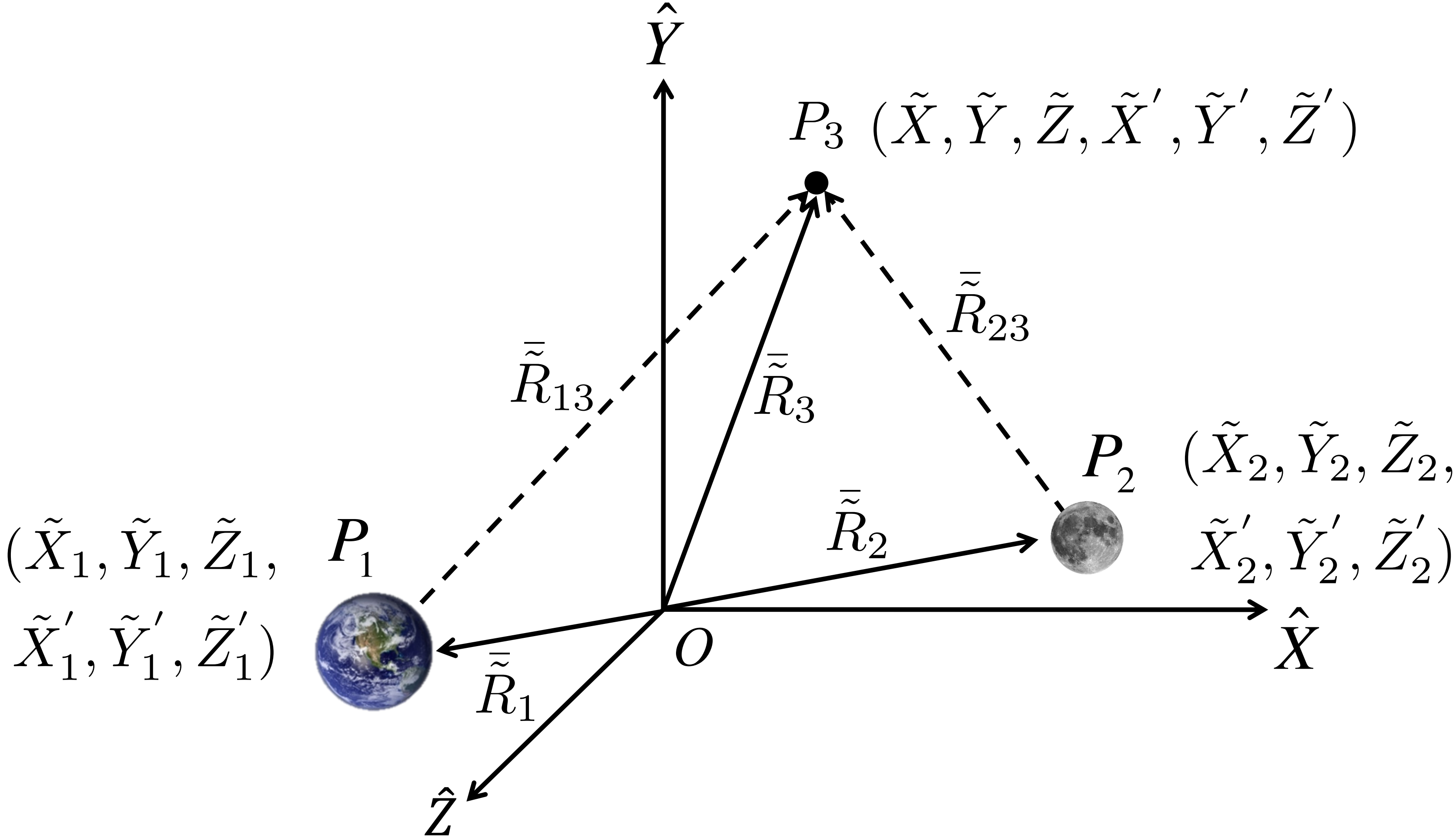
N-3 Perturbing bodies in high-fidelity model

+Additional forces in high-fidelity model

Interested in motion of spacecraft within Earth-Moon system

System Configuration

Reduce the model complexity by assuming only three bodies, modeled as point masses with constant mass, in the system:



Deriving EOMs for Three-Body Problem

Derive equations of motion using the potential function for Body 3, per unit mass $\tilde{M}_3$, due to the gravity of two point masses:

Where the tilde denotes dimensional quantities and

$$\bar{\tilde{R}}_3'' = \tilde{X}''\hat{X} + \tilde{Y}''\hat{Y} + \tilde{Z}''\hat{Z}$$

$$\tilde{R}_{13} = \sqrt{\left(\tilde{X} - \tilde{X}_1\right)^2 + \left(\tilde{Y} - \tilde{Y}_1\right)^2 + \left(\tilde{Z} - \tilde{Z}_1\right)^2}$$

$$\tilde{R}_{23} = \sqrt{\left(\tilde{X} - \tilde{X}_2\right)^2 + \left(\tilde{Y} - \tilde{Y}_2\right)^2 + \left(\tilde{Z} - \tilde{Z}_2\right)^2}$$

To derive the equations of motion (EOMs), note the force per unit mass (i.e., acceleration) acting on Body 3 is written as:

Deriving EOMs for Three-Body Problem

Taking these derivatives:

$$\frac{\partial \tilde{U}_3}{\partial \tilde{X}} = \frac{\partial}{\partial \tilde{X}} \left(\frac{\tilde{G}\tilde{M}_1}{\tilde{R}_{13}} + \frac{\tilde{G}\tilde{M}_2}{\tilde{R}_{23}} \right)$$

Deriving EOMs for Three-Body Problem

Using Newton's 2nd law, EOMs for the three-body problem:

$$\tilde{X}_3'' =$$

$$\tilde{Y}_3'' =$$

$$\tilde{Z}_3'' =$$

Simplifying Assumptions

To reduce the complexity of the dynamical model, we will use all the following assumptions:

Nondimensionalization

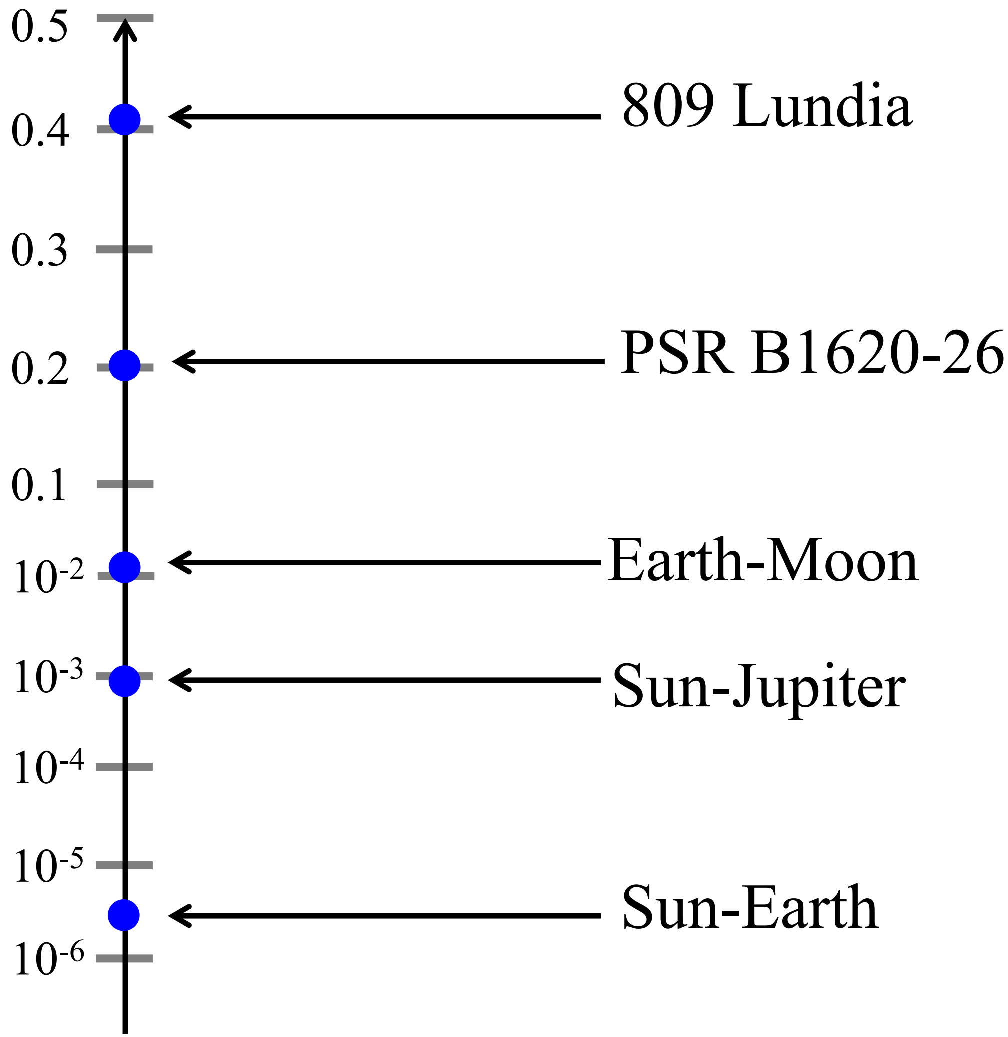
In practice, time, mass and length quantities can have very different orders of magnitude. Also useful to compare quantities with similar ratios of the mass of the primaries

Introduce characteristics quantities m^*, l^*, t^* and nondimensional quantities have no tilde.

1. Mass:

Common Mass Ratios

$$\mu = \frac{\tilde{M}_2}{\tilde{M}_1 + \tilde{M}_2}$$



Nondimensionalization

2. Length

3. Time

Recall from 2BP:

Nondimensional gravitational constant: $G = \frac{\tilde{G} m^* (t^*)^2}{(l^*)^3} = 1$

Writing Nondimensional EOMs

Relating dimensional and nondimensional components

$$X = \frac{\tilde{X}}{l*} \qquad Y = \frac{\tilde{Y}}{l*} \qquad Z = \frac{\tilde{Z}}{l*}$$

$$R_{13} = \frac{\tilde{R}_{13}}{l*} \qquad R_{23} = \frac{\tilde{R}_{23}}{l*}$$

$$\tilde{X}'' = \frac{d}{d\tau}\left(\frac{d\tilde{X}}{d\tau}\right) = \frac{d}{d(tt^*)}\left(\frac{d(Xl^*)}{d(tt^*)}\right) = \frac{l^*}{(t^*)^2}\frac{d^2X}{dt^2}$$

$$\tilde{Y}'' = \frac{l^*}{(t^*)^2}\frac{d^2Y}{dt^2} \qquad \tilde{Z}'' = \frac{l^*}{(t^*)^2}\frac{d^2Z}{dt^2}$$

Writing Nondimensional EOMs

$$\tilde{X}'' = -\frac{\tilde{G}\tilde{M}_1(\tilde{X}-\tilde{X}_1)}{\tilde{R}_{13}^3} - \frac{\tilde{G}\tilde{M}_2(\tilde{X}-\tilde{X}_2)}{\tilde{R}_{23}^3}$$

$$\frac{l*}{(t*)^2}X'' = -\frac{\tilde{G}(1-\mu)m^*l^*(X-X_1)}{(l^*)^3R_{13}^3} - \frac{\tilde{G}(\mu)m^*l^*(X-X_2)}{(l^*)^3R_{23}^3}$$

$$X'' = \frac{\tilde{G}(t^*)^2m^*}{(l^*)^3}\left[-\frac{(1-\mu)(X-X_1)}{R_{13}^3} - \frac{\mu(X-X_2)}{R_{23}^3}\right]$$

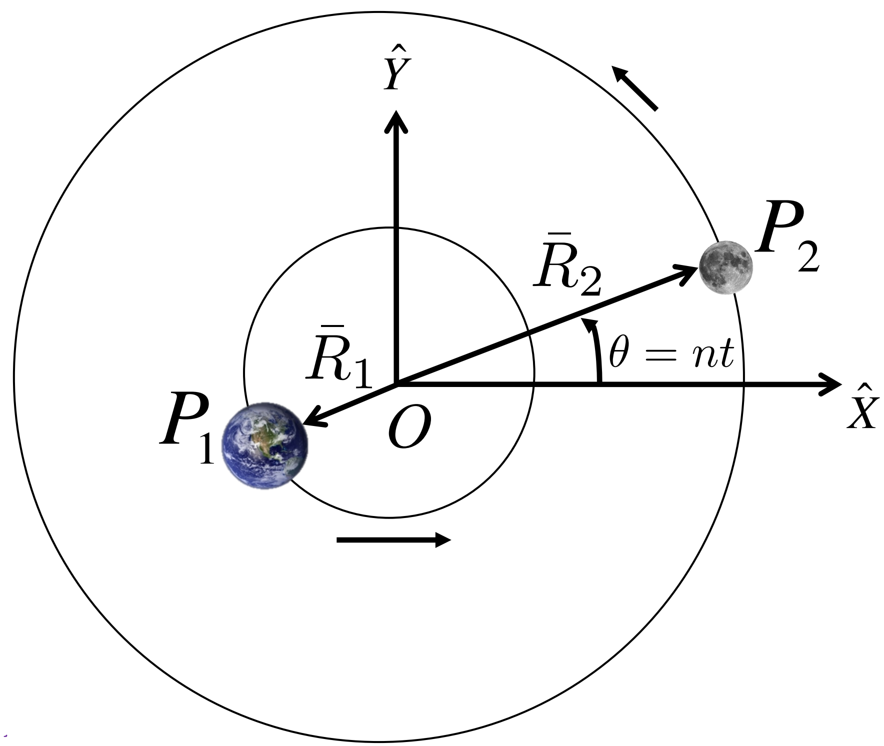
Writing Nondimensional EOMs

$$X^{''} = -\frac{(1-\mu)(X - X_1)}{R_{13}^3} - \frac{\mu(X - X_2)}{R_{23}^3}$$

$$\Big\downarrow \quad \text{Repeat for } Y, Z \text{ equations}$$

$$\bar{R}^{''} = -\frac{(1-\mu)}{R_{13}^3}(\bar{R} - \bar{R}_1) - \frac{\mu}{R_{23}^3}(\bar{R} - \bar{R}_2)$$

Path of Primaries



Center of mass lies along line between primaries:

Write coordinates of P_1, P_2 in inertial frame, relative to barycenter

Update EOMs

Plug in known coordinates of primaries:

$$X'' = -\frac{(1-\mu)(X+\mu\cos(t))}{R_{13}^3} - \frac{\mu(X-(1-\mu)\cos(t))}{R_{23}^3}$$

$$Y'' = -\frac{(1-\mu)(Y+\mu\sin(t))}{R_{13}^3} - \frac{\mu(Y-(1-\mu)\sin(t))}{R_{23}^3}$$

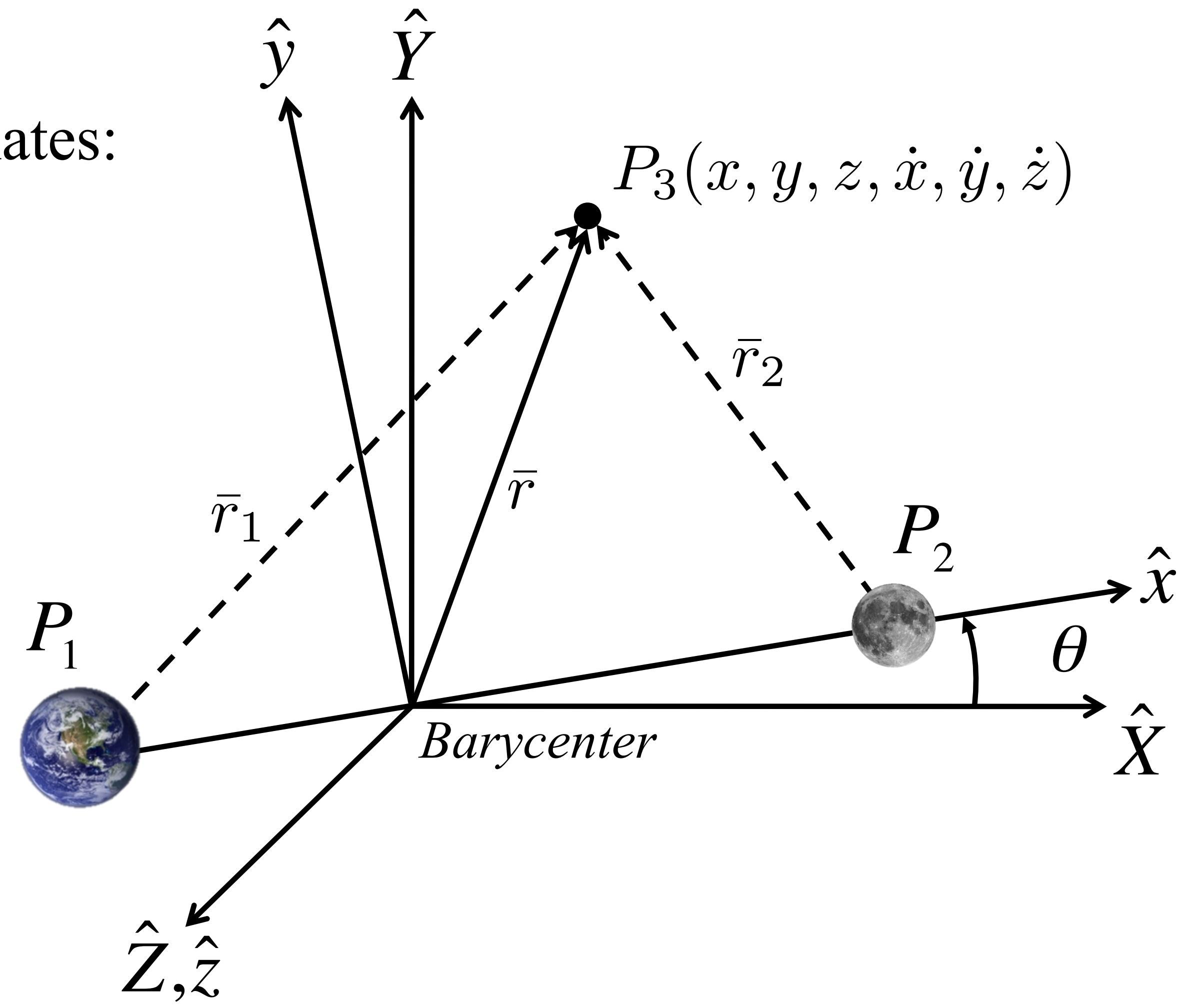
$$Z'' = -\frac{(1-\mu)Z}{R_{13}^3} - \frac{\mu Z}{R_{23}^3}$$

Rotating Frame

Introduce a frame that rotates with the two primaries

Lowercase coordinates: correspond to rotating frame

No tilde: nondimensional quantities



Rotating Frame

Transformation written as simple rotation:

In the rotating frame, the primaries are located at:

P₁: *P₂*:

Then: $\bar{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$$\bar{r}_1 = (x + \mu)\hat{x} + y\hat{y} + z\hat{z}$$

$$\bar{r}_2 = (x - 1 + \mu)\hat{x} + y\hat{y} + z\hat{z}$$

Rewriting EOMs in Rotating Frame

First step is to convert the acceleration to use rotating frame components and an observer in the rotating frame

$$\frac{{}^{I}d\bar{r}}{dt} = \frac{{}^{R}d\bar{r}}{dt} + {}^{I}\bar{\omega}^{R} \times \bar{r}$$

$$\frac{{}^{I}d\bar{r}}{dt} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z} + \hat{z} \times [x\hat{x} + y\hat{y} + z\hat{z}]$$

$$\frac{{}^{I}d\bar{r}}{dt} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z} + x\hat{y} - y\hat{x}$$

$$\frac{{}^{I}d\bar{r}}{dt} = (\dot{x} - y)\hat{x} + (\dot{y} + x)\hat{y} + \dot{z}\hat{z}$$

Rewriting EOMs in Rotating Frame

$$\frac{{}^I d\bar{v}}{dt} = \frac{{}^R d\bar{v}}{dt} + {}^I \bar{\omega}^R \times \bar{v}$$

$$\frac{{}^I d\bar{v}}{dt} = (\ddot{x} - \dot{y})\hat{x} + (\ddot{y} + \dot{x})\hat{y} + \ddot{z}\hat{z} + \hat{z} \times [(\dot{x} - y)\hat{x} + (\dot{y} + x)\hat{y} + \dot{z}\hat{z}]$$

$$\frac{{}^I d\bar{v}}{dt} = (\ddot{x} - 2\dot{y} - x)\hat{x} + (\ddot{y} + 2\dot{x} - y)\hat{y} + \ddot{z}\hat{z}$$

Recall: $\bar{R}'' = -\frac{(1-\mu)}{R_{13}^3}(\bar{R} - \bar{R}_1) - \frac{\mu}{R_{23}^3}(\bar{R} - \bar{R}_2)$

Circular Restricted Three-Body Problem

Nondimensional equations of motion for spacecraft in the rotating frame, using coordinates relative to the barycenter:

$$\ddot{x} = 2\dot{y} + x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3}$$

$$\ddot{y} = -2\dot{x} + y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}$$

$$\ddot{z} = -\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3}$$

Where:

$$r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}$$
$$r_2 = \sqrt{(x-1+\mu)^2 + y^2 + z^2}$$

Pseudo-Potential Function

Define a pseudo-potential function, which is similar to a potential function and depends only on position variables but incorporates an additional term due to the rotation of the rotating frame

$$\ddot{x} = 2\dot{y} + x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3}$$

$$\ddot{y} = -2\dot{x} + y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}$$

$$\ddot{z} = -\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3}$$

Definition of Pseudo-Potential Function

The pseudo-potential function which satisfies these constraints is:

Producing compact equations of motion:

$$\ddot{x} - 2\dot{y} = \frac{\partial U^*}{\partial x}$$

$$\ddot{y} + 2\dot{x} = \frac{\partial U^*}{\partial y}$$

$$\ddot{z} = \frac{\partial U^*}{\partial z}$$