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Global Solution for the Optimal Feedback Control of the Underactuated Heisenberg System

Chandeok Park, Daniel J. Scheeres, Vincent Guibout, and Anthony Bloch

***Abstract*—We present a global solution for an optimal *feedback* controller of the underactuated Heisenberg system or nonholonomic integrator. Employing a recently developed technique based on generating functions appearing in the Hamilton-Jacobi theory, we circumvent a singularity caused by underactuation to develop a nonlinear optimal feedback control in an implicitly analytical form. The systematic procedure to deal with underactuation indicates that generating functions should be effective tools for solving general underactuated optimal control problems.**

***Index Terms*—Generating function, Hamilton–Jacobi equation, Heisenberg system, nonholonomic integrator, optimal feedback control (OFC), underactuated system.**

I. Introduction

We study an optimal feedback control (OFC) problem for an underactuated Heisenberg system, using generating functions appearing in the Hamilton-Jacobi theory [1], [2]. Related work on the use of canonical transformations and their associated generating functions for solving optimal control problems may be found in Fraeijs de Veubeke [3], Powers and Tapley [4], Hagedorn [5], etc. However, these papers rely on using special canonical transformations only valid for their specific dynamical systems of interest, and thus their techniques are not directly applicable to general optimal control problems. We view a general two point boundary value problem (TPBVP) for a Hamiltonian system as the problem of finding a suitable canonical transformation [6], [7], and derive a generic procedure for solving underactuated optimal control problems.

By definition, a system is underactuated if the number of controls is less than the number of degrees of freedom. Such systems, which may be still shown to be controllable, can be frequently found in robotics [8], space flight orientation [9], aircraft maneuvers [10], etc. Among these, the Heisenberg system or nonholonomic integrator can be considered as a prototypical controllable kinematic system [11], [12], and thus has been located at the intersection of many different analysis: control theory, sub-Riemannian geometry, Hamilton-Jacobi theory, etc [13]–[16]; Recently Beals *et al.* studied the subelliptic geometry of Heisenberg groups in detail, and derived an analytical trajectory for the associated Hamiltonian [17]. See also references therein.

This work is an initial foray into research on underactuated optimal control systems using the generating function technique outlined in recent publications [18], [19]. We first incorporate the Heisenberg system into a typical optimal control formulation (Section II), and introduce a generic procedure for solving fully-actuated OFC problems with generating functions (Section III). Then we describe how this procedure should be adapted to deal with singularity originating from underactuation (Section IV). These processes demonstrate some advantages over the classical methods based on linearization or dynamic programming. Finally we end our discussion with concluding remarks (Section V).

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II. Problem Statement and Formulation

We study minimization of a quadratic performance index

$$J = \frac{1}{2}x^T(t_f)Q_f x(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\left[u_1^2(t) + u_2^2(t)\right]dt \qquad (1)$$

subject to the underactuated Heisenberg system

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}u_1(t)\\ u_2(t)\\ x_2(t)u_1(t) - x_1(t)u_2(t)\end{bmatrix}. \qquad (2)$$

Here $x(t) = [x_1(t)x_2(t)x_3(t)]^T : \mathbf{R} \rightarrow \mathbf{R}^3$ represents the state, $u(t) = [u_1(t)u_2(t)]^T : \mathbf{R} \rightarrow \mathbf{R}^2$ the control, $Q_f \in \mathbf{R}^{3\times 3} \geq 0$ the weight function for terminal cost function, and $t \in \mathbf{R}$ the general time index. This system is underactuated, yet can be easily shown to be controllable from a simple check of the controllability rank condition [11]; the 2 components of the control excite the 3 components of the state through the interconnection of its own dynamics.

The importance of this system is that its generalization

$$\begin{aligned}\dot{x} &= u, \quad x \in \mathbf{R}^n\\ \dot{Y} &= xu^T - ux^T, \quad u \in \mathbf{R}^n\end{aligned} \qquad (3)$$

represents a canonical system for a class of controllable systems of the form [11]

$$\dot{X} = B(X)u, \quad u \in \mathbf{R}^n,\ X \in \mathbf{R}^{n(n+1)/2}. \qquad (4)$$

There exist coordinates (x, u) in a neighborhood of a given point in x-domain such that a system of the form (4) can be converted into the canonical form (3) [12].

Given the above problem statement, our goal is to find the optimal feedback control (OFC) law for an arbitrary initial point $(x, t) \in \mathbf{R}^3 \times \mathbf{R}$, which is compatible with the system dynamics and the terminal constraints if they exist. We consider two extreme cases of terminal boundary conditions separately for convenience of analysis:

- **Hard Constraint Problem (HCP)**: Terminal boundary condition is completely specified to a fixed point in $\mathbf{R}^3$, i.e., $x(t_f) = x_f$. It can be stated that $Q_f = 0$ without loss of generality.
- **Soft Constraint Problem (SCP)**: Terminal boundary condition is completely unspecified, and is indirectly determined by minimizing the terminal cost function $(1/2)x^T(t_f)Q_f x(t_f)$.

As a first step, we recast our OFC problem within the framework of Hamiltonian system theory. Define the pre-Hamiltonian $\bar{H}$ as

$$\bar{H}(x,\lambda,u,t)=\frac{1}{2}\left(u_1^2+u_2^2\right)\cdots+\lambda_1 u_1+\lambda_2 u_2+\lambda_3(x_2 u_1-x_1 u_2) \tag{5}$$

where $\lambda = [\lambda_1\lambda_2\lambda_3]$ represents the costate adjoint to the system dynamics. Referring to the 1st order necessary conditions for optimality [20], [21], we obtain the costate equations and the optimality conditions:

$$\begin{bmatrix}\dot{\lambda}_1\\ \dot{\lambda}_2\\ \dot{\lambda}_3\end{bmatrix}=\begin{bmatrix}\lambda_3 u_2\\ -\lambda_3 u_1\\ 0\end{bmatrix} \tag{6}$$

$$\begin{bmatrix}u_1\\ u_2\end{bmatrix}=\begin{bmatrix}-\lambda_1-\lambda_3 x_2\\ -\lambda_2+\lambda_3 x_1\end{bmatrix}. \tag{7}$$

Then substituting (7) into (2),(5), and (6) leads to a standard Hamiltonian system for states and costates:

$$H(x,\lambda) = -\frac{1}{2}\left[(x_1\lambda_3-\lambda_2)^2+(x_2\lambda_3+\lambda_1)^2\right] \tag{8}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-\lambda_1-\lambda_3 x_2\\ -\lambda_2+\lambda_3 x_1\\ -\lambda_1 x_2-\lambda_3 x_2^2+\lambda_2 x_1-\lambda_3 x_1^2\end{bmatrix} \tag{9}$$

$$\begin{bmatrix}\dot{\lambda}_1\\ \dot{\lambda}_2\\ \dot{\lambda}_3\end{bmatrix}=\begin{bmatrix}-\lambda_2\lambda_3+\lambda_3^2 x_1\\ \lambda_1\lambda_3+\lambda_3^2 x_2\\ 0\end{bmatrix}. \tag{10}$$

For the HCP, the initial and terminal states are explicitly given, whereas the initial and terminal costates are to be determined to satisfy the optimality conditions. For the SCP only the initial states are given, whereas all the other quantities are not known. In this case, however, the well-known transversality condition provides additional boundary conditions [20]:

$$\lambda(t_f) = Q_f x(t_f). \tag{11}$$

Hence in both cases, our OFC problem is reduced to a two point boundary value problem (TPBVP) for a Hamiltonian system.

III. Optimal Feedback Control Law Developed From Generating Functions

There exist diverse numerical techniques for solving the above TPBVP. However, they usually yield the open loop solution for a specific boundary condition, which does not fit into our goal of obtaining the OFC law on a given domain of interest.

Instead, we view the Hamiltonian phase flow (8)–(10) as a transformation between moving terminal coordinates $(x(t),\lambda(t))$ and *fixed* initial coordinates $(x_0(t),\lambda_0(t)) \equiv$ constant. Then, the latter set of coordinates becomes a Hamiltonian system with its Hamiltonian trivially defined as $H_0(x_0(t),\lambda_0(t),t) \equiv 0$. As the transformation between these two coordinate systems is canonical by definition, there exist generating functions that can have one of the four classical forms [6], [18]:[1]

$$F_1(x,x_0,t,t_0),\quad F_2(x,\lambda_0,t,t_0)$$
$$F_3(\lambda,x_0,t,t_0),\quad F_4(\lambda,\lambda_0,t,t_0).$$

These generating functions are functions of initial and terminal coordinates of equal quantities and provide the relations between the initial and terminal states and costates. For example, the following relations hold for the F_1 and F_2, which we extensively use in the next section:

$$\lambda = \frac{\partial F_1(x,x_0,t,t_0)}{\partial x} \tag{12}$$

$$\lambda_0 = -\frac{\partial F_1(x,x_0,t,t_0)}{\partial x_0} \tag{13}$$

$$0 = H(x,\lambda,t)+\frac{\partial F_1(x,x_0,t,t_0)}{\partial t} \tag{14}$$

$$\lambda = \frac{\partial F_2(x,\lambda_0,t,t_0)}{\partial x} \tag{15}$$

$$x_0 = \frac{\partial F_2(x,\lambda_0,t,t_0)}{\partial \lambda_0} \tag{16}$$

$$0 = H(x,\lambda,t)+\frac{\partial F_2(x,\lambda_0,t,t_0)}{\partial t}. \tag{17}$$

The partial differential equations (PDEs) found by substituting for λ in (14) and (17) are referred to as the Hamilton-Jacobi equation (HJE). A crucial property is that these generating functions are mutually linked with each other by the Legendre transformation, which assumes the following form for the F_1 and F_2:

$$F_2(x,\lambda_0,t,t_0) = F_1(x,x_0,t,t_0)+\lambda_0^T x_0. \tag{18}$$

This relation holds as long as the implicit function theorem is satisfied for the associate relations (12), (13) and (15), (16). Among all kinds of generating functions, F_1 is a special quantity, as it provides the OFC law as well as the sufficient condition for optimality by the following theorem.

Theorem 3.1 (Sufficient Condition for Optimality): Let x_f be the *fixed* terminal state at t_f and x be the *moving* initial state at t. Then for both the HCP and SCP, the function

$$V(x,t) = -F_1(x_f,x,t_f,t)+\phi(x_f,t_f)$$

satisfies the Hamilton–Jacobi–Bellman equation (HJBE) and its associated boundary condition

$$\frac{\partial V}{\partial t}(x,t)+\min_u \bar{H}\left(x,\frac{\partial V}{\partial x},u,t\right)=0$$
$$V(x=x_f,t=t_f)=\phi(x_f,t_f).$$

Thus, it is the optimal cost function. Furthermore, at *any initial* time $t \le t_f$ the optimal feedback control can be expressed as

$$u(t) = arg\min_{\bar{u}} \bar{H}\left(x,\frac{\partial V(x,t)}{\partial x},\bar{u},t\right).$$

For the SCP where x_f and t_f are *implicit*, they are *explicitly* determined by the relations (11)–(13).

Proof: Refer to Park and Scheeres [18], [22].

It is important to note that the above theorem is not just a restatement of the sufficient condition from dynamic programming. In contrast to the dynamic programming problem solely relying on the HJBE to determine the optimal cost function, we can solve the HJE for any kind of generating function, and convert it into the F_1 *algebraically* by the Legendre transformation. This distinctive characteristics is taken advantage of to solve our OFC problem for the Heisenberg system.

IV. Optimal Feedback Control Law for the Heisenberg System

Given Theorem 3.1 in the previous section, we need to develop the F_1 generating function by solving the HJE (14), which becomes a nonlinear partial differential equation (PDE) for the associated Hamiltonian (8). In general, it is very difficult to solve this PDE, as it is difficult to obtain at least one initial/boundary condition due to an inherent singularity [18], [22]. However, as we have emphasized earlier, we can

[1]The generating functions, which appear in the Hamilton-Jacobi theory, are functions providing relationship for a canonical transformation. We refer to Goldstein [1] and Greenwood [2] for comprehensive exposition of this rich and mature theory in analytical mechanics.

solve for any other kind of generating function which may possess a well-defined initial/boundary condition, and then recover the F_1 generating function by the Legendre transformation (18). For example, the functional form

$$F_2(x_f, \lambda_0, t_f = t_0, t_0) = x_f^T \lambda_0 \tag{19}$$

provides a well-defined initial condition for the F_2 generating function by the relation (15), (16). Using this initial condition, we have established a compact global solution for the F_2 by solving the associated HJE (17):

Proposition 4.1 (F_2 in Global Form): For the Hamiltonian (8), the F_2 generating function of the following form:

$$\begin{aligned} F_2(x_f, \lambda_0, t_f, t_0) =& x_{1f}\lambda_{10} + x_{2f}\lambda_{20} + x_{3f}\lambda_{30} \cdots \\ &+ \frac{1}{2}\left[(x_{1f}\lambda_{30} - \lambda_{20})^2 + (x_{2f}\lambda_{30} + \lambda_{10})^2\right] \\ &\cdots \frac{\tan\left[(t_f - t_0)\lambda_{30}\right]}{\lambda_{30}} \end{aligned} \tag{20}$$

satisfies the HJE (17) globally and its initial condition $F_2(x_f, \lambda_0, t_f = t_0, t_0) = x_f^T \lambda_0$.

Proof: It is trivial to show that the above F_2 satisfies the initial condition. In order to show that it satisfies the HJE (17), we first observe

$$\frac{\partial F_2}{\partial t_f} = \frac{1}{2}\left[(x_{1f}\lambda_{30} - \lambda_{20})^2 + (x_{2f}\lambda_{30} + \lambda_{10})^2\right] \times \sec^2\left[(t_f - t_0)\lambda_{30}\right]. \tag{21}$$

Then considering the F_2-associated relation (15), we can obtain

$$\lambda_{1f} = \lambda_{10} + (x_{1f}\lambda_{30} - \lambda_{20})\tan\left[(t_f - t_0)\lambda_{30}\right] \tag{22}$$

$$\lambda_{2f} = \lambda_{20} + (x_{2f}\lambda_{30} + \lambda_{10})\tan\left[(t_f - t_0)\lambda_{30}\right] \tag{23}$$

$$\lambda_{3f} = \lambda_{30}. \tag{24}$$

Introducing the above relations into the Hamiltonian (8) yields

$$\begin{aligned} &H\left(x_f, \lambda_f(\lambda_0, x_f), t_f\right) \\ &= -\frac{1}{2}\left[x_{1f}\lambda_{30} - \lambda_{20} - (x_{2f}\lambda_{30} + \lambda_{10})\tan\left[(t_f - t_0)\lambda_{30}\right]\right]^2 \\ &\quad - \frac{1}{2}\left[x_{2f}\lambda_{30} + \lambda_{10} + (x_{1f}\lambda_{30} - \lambda_{20})\tan\left[(t_f - t_0)\lambda_{30}\right]\right]^2 \end{aligned}$$

which can be reduced to

$$H(x_f, \lambda_0, t_f) = -\frac{1}{2}\left[(x_{1f}\lambda_{30} - \lambda_{20})^2 + (x_{2f}\lambda_{30} + \lambda_{10})^2\right] \times \sec^2\left[(t_f - t_0)\lambda_{30}\right]. \tag{25}$$

Finally introduction of (21) and (25) into the HJE (17) yields an identity, which completes the proof. $(\mathbf{Q.E.D})$

Applying our multi-variable polynomial series expansion technique introduced in Guibout and Scheeres [6], [7] to solve the HJE (17) for F_2,[2] we were able to solve for the general series form of the F_2 generating function, and to subsequently note its equivalence to the tangent function in the last term of (20).

Now that we have obtained the F_2 generating function, it remains to recover the F_1 by the Legendre transformation (18). First rearranging for F_1, we have

$$F_1(x_f, x_0, t_f, t_0) = F_2(x_f, \lambda_0, t_f, t_0) - \lambda_0^T x_0.$$

[2]Guibout has developed a Mathematica code to perform these computational processes involving multi-variable series expansion, extraction of the recursive ODEs, and their numerical solution. Once the Hamiltonian is given as a polynomial series, all these symbolic/numeric manipulations are performed automatically.

Here observe that the arguments of the F_1 and the F_2 are different from each other; we need to find a relation $\lambda_0 = \lambda_0(x_f, x_0, t_f, t_0)$ to be replaced on the right hand side, which is accomplished by rearranging the relation (16) for λ_0 using a series inversion technique. To be more specific, evaluating (16) for (20), we obtain

$$x_{10} = x_{1f} + (x_{2f}\lambda_{30} + \lambda_{10})\frac{\tan\left[(t_f - t_0)\lambda_{30}\right]}{\lambda_{30}} \tag{26}$$

$$x_{20} = x_{2f} - (x_{1f}\lambda_{30} - \lambda_{20})\frac{\tan\left[(t_f - t_0)\lambda_{30}\right]}{\lambda_{30}} \tag{27}$$

$$\begin{aligned} x_{30} =& x_{3f} + \left[\left(x_{1f}^2 + x_{2f}^2\right)\lambda_{30} - (x_{1f}\lambda_{20} - x_{2f}\lambda_{10})\right] \\ &\times \frac{\tan\left[(t_f - t_0)\lambda_{30}\right]}{\lambda_{30}} \\ &+ \frac{1}{2}\left[(x_{1f}\lambda_{30} - \lambda_{20})^2 + (x_{2f}\lambda_{30} + \lambda_{10})^2\right] \cdots \\ &\times \frac{\left[(t_f - t_0)\lambda_{30}\right]\sec^2\left[(t_f - t_0)\lambda_{30}\right] - \tan\left[(t_f - t_0)\lambda_{30}\right]}{\lambda_{30}^2}. \end{aligned} \tag{28}$$

In general for problems subject to full actuation, the algebraic inversion of the related equations expressed as a power series is justified under the mild assumptions of the implicit function theorem. Thus it can be performed systematically by symbolic manipulations. Unfortunately, this general situation does not apply to our specific problem due to the underactuation. Observe that, unlike the first two equations where there exist linear λ_{10} and λ_{20} terms respectively, the third equation does not possess a linear λ_{30} term. This is caused by the lack of control component in the x_3-direction, and violates the assumptions of the implicit function theorem, as the Jacobian matrix of $x_0 = \partial F_2/\partial\lambda_0$ could be singular, i.e., $|\partial^2 F_2/\partial\lambda_0^2|$ could vanish.

In an attempt to resolve this peculiar impediments caused by underactuation, we now deal with the HCP and the SCP separately for convenience of analysis.

A. Hard Constraint Problem

The unfavorable singularity related to underactuation suggests that the determination of λ_{30} should be pivotal for solving this particular OFC problem. Thus, we attempt to find a relation $\lambda_{30} = \lambda_{30}(x_f, x_0, t_f, t_0)$. First rearranging (26) and (28) for λ_{10} and λ_{20}, respectively, leads to

$$\lambda_{10} = \frac{(x_{10} - x_{1f})\lambda_{30}}{\tan\left[(t_f - t_0)\lambda_{30}\right]} - x_2\lambda_{30} \tag{29}$$

$$\lambda_{20} = \frac{(x_{20} - x_{2f})\lambda_{30}}{\tan\left[(t_f - t_0)\lambda_{30}\right]} + x_1\lambda_{30}. \tag{30}$$

Then substituting these two equations into (28) and performing some algebra, we obtain an algebraic transcendental equation for λ_{30}:

$$\begin{aligned} &2\left[1 - \cos\left[2(t_f - t_0)\lambda_{30}\right]\right] - \\ &\quad r(x_f, x_0)\left[2(t_f - t_0)\lambda_{30} - \sin\left[2(t_f - t_0)\lambda_{30}\right]\right] = 0 \end{aligned} \tag{31}$$

where

$$r(x_f, x_0) = \frac{(x_{10} - x_{1f})^2 + (x_{20} - x_{2f})^2}{x_{30} - x_{3f} + x_{1f}x_{20} - x_{2f}x_{10}}. \tag{32}$$

Solving this equation for λ_{30} leads to the relation $\lambda_{30} = \lambda_{30}(x_f, x_0, t_f, t_0)$. If this is introduced back into (29), (30), we can explicitly determine the preferred relation $\lambda_0 = \lambda_0(x_f, x_0, t_f, t_0)$.

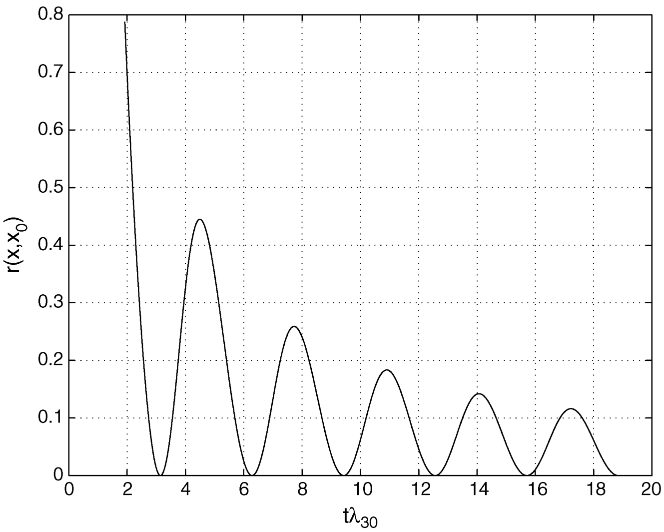

Fig. 1. $r(x_f, x_0) = f[(t_f - t_0)\lambda_{30}]$.

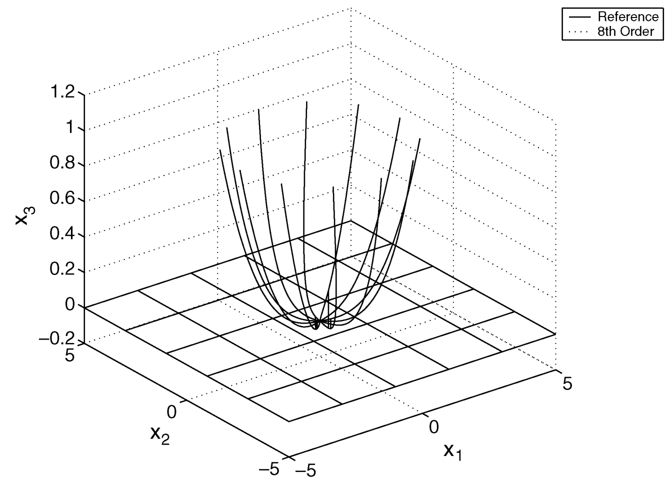

Fig. 2. Example 1: optimal trajectories for $x_0 = [3\cos\theta 3\sin\theta 1]$, $0 \leq \theta \leq 2\pi$.

Then we execute the Legendre transformation (18) to obtain the F_1 generating function:

$$\begin{aligned} F_1 &(x_f, x_0, \lambda_{30}(x_f, x_0), t_f, t_0) \\ &= -\frac{1}{2}\left[(x_{10} - x_{1f})^2 + (x_{20} - x_{2f})^2\right]\frac{\lambda_{30}}{\tan\left[(t_f - t_0)\lambda_{30}\right]} \\ &\quad - (x_{30} - x_{3f} + x_{1f}x_{20} - x_{2f}x_{10})\lambda_{30}. \end{aligned} \tag{33}$$

Finally Theorem 3.1 provides the optimal cost function as $V(x,t) = -F_1(x_f, x, t_f, t)$, and the associated OFC law (7) as

$$\begin{bmatrix} u_1(x_f, x) \\ u_2(x_f, x) \end{bmatrix} = \begin{bmatrix} -(x_1 - x_{1f})\frac{\lambda_3}{\tan\left[(t_f - t)\lambda_3\right]} \\ -(x_2 - x_{2f})\frac{\lambda_3}{\tan\left[(t_f - t)\lambda_3\right]} \end{bmatrix}. \tag{34}$$

Note that (x_0, t_0) have been replaced by (x, t) to emphasize that the initial condition is arbitrary.

With all these procedures justified by Theorem 3.1, it remains to actually solve the governing equation (31) for $\lambda_{30} = \lambda_{30}(x_f, x_0, t_f, t_0)$. We first rearrange for $r(x_f, x_0)$

$$r(x_f, x_0) = \frac{2\left[1 - \cos\left[2(t_f - t_0)\lambda_{30}\right]\right]}{\left[2(t_f - t_0)\lambda_{30} - \sin\left[2(t_f - t_0)\lambda_{30}\right]\right]}$$

which is an odd function and is plotted for $(t_f - t_0)\lambda_{30} \geq 0$ in Fig. 1. If the magnitude of r is sufficiently small/large, the solution to our transcendental equation (31) becomes multiple/unique respectively. It turns out that λ_{30} obtained from the interval $|(t_f - t_0)\lambda_{30}| \leq \pi$ provides the true optimal solution. Solutions obtained from $|(t_f - t_0)\lambda_{30}| > \pi$ describe the non-optimal transfer, simply in the sense that they do not minimize the performance criterion (1).

We now test the validity of our solution by a specific numerical example. Demonstrating the feedback nature, we first choose a set of initial conditions parameterized by $x_0 = [3\cos\theta, 3\sin\theta, 1]$, $0 \leq \theta \leq 2\pi$, and solve the governing equation (31) by a least squares method implemented in Matlab. Fig. 2 shows the actual trajectories for some selected initial conditions on the circle, which almost coincide with reference solutions from all phase angles. The reference solutions have been developed for comparison by solving the relevant TPBVP numerically using the forward shooting method.

B. Soft Constraint Problem

Unfortunately for the SCP, we cannot use the governing transcendental equation (31) to determine λ_{30}, as $r(x_f, x_0)$ is not determined a priori due to the unknown terminal states. Instead we must consider simultaneously 9 equations (26)–(28), (22)–(24), and (11)) for 9 unknowns $(x_f, \lambda_f, \lambda_0)$.

Assuming that $Q_f = diag(q_1, q_2, q_3)$ is a diagonal matrix, we can easily reduce the above 9 equations to 6 equations for (x_f, λ_0)

$$\begin{aligned} x_{10} &= x_{1f} + (x_{2f}\lambda_{30} + \lambda_{10})\frac{\tan\left[(t_f - t_0)\lambda_{30}\right]}{\lambda_{30}} \\ x_{20} &= x_{2f} - (x_{1f}\lambda_{30} - \lambda_{20})\frac{\tan\left[(t_f - t_0)\lambda_{30}\right]}{\lambda_{30}} \\ x_{30} &= x_{3f} + \left[\left(x_{1f}^2 + x_{2f}^2\right)\lambda_{30} - (x_{1f}\lambda_{20} - x_{2f}\lambda_{10})\right] \\ &\quad \times \frac{\tan\left[(t_f - t_0)\lambda_{30}\right]}{\lambda_{30}} \\ &\quad + \frac{1}{2}\left[(x_{1f}\lambda_{30} - \lambda_{20})^2 + (x_{2f}\lambda_{30} + \lambda_{10})^2\right]\cdots \\ &\quad \frac{\left[(t_f - t_0)\lambda_{30}\right]\sec^2\left[(t_f - t_0)\lambda_{30}\right] - \tan\left[(t_f - t_0)\lambda_{30}\right]}{\lambda_{30}^2} \\ x_{1f} &= \frac{\lambda_{10} - \lambda_{20}\tan\left[(t_f - t_0)\lambda_{30}\right]}{q_1 + \lambda_{30}\tan\left[(t_f - t_0)\lambda_{30}\right]} \\ x_{2f} &= \frac{\lambda_{20} + \lambda_{10}\tan\left[(t_f - t_0)\lambda_{30}\right]}{q_2 + \lambda_{30}\tan\left[(t_f - t_0)\lambda_{30}\right]} \\ x_{3f} &= \frac{\lambda_{30}}{q_3}. \end{aligned}$$

Solving these algebraic equations simultaneously corresponds to determining the initial costates λ_0 as well as the OFC law.

As a numerical verification, we choose the terminal performance weight function to be

$$Q_f = \mathrm{diag}(10, 10, 100)$$

and solve the above 6 equations for (x, λ_0). A set of initial conditions are parameterized by $x_0 = [\cos\theta \sin\theta 3]$, $0 \leq \theta \leq 2\pi$. Fig. 3 shows the actual trajectories for some selected initial conditions on the circle.

V. Conclusion

We have adapted a previously developed generating function technique to derive a global solution for the optimal feedback control of the underactuated Heisenberg system in the context of Hamilton–Jacobi theory. Solving the associated Hamilton–Jacobi equation for a generating function, we have developed a unique methodology in terms of algebraic equations related with the underactuated variable. Their solution has led to the optimal cost function and the associated feedback

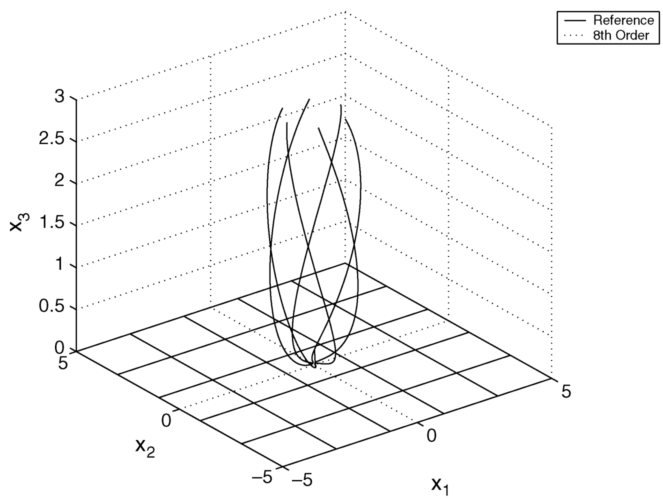

Fig. 3. Example 2: optimal trajectories for $x_0 = [\cos\theta \sin\theta 3]$, $0 \leq \theta \leq 2\pi$.

control law by a series of algebraic processes. While we carried out the above procedure for the Heisenberg system, we expect this approach will be generalizable to other underactuated controllable systems and we hope to pursue this as future research. Also with the above unique technical features, this approach can be favorably compared with the dynamic programming method or linear quadratic formulation.

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On the Nonexistence of Quadratic Lyapunov Functions for Consensus Algorithms

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***Abstract*—We provide an example proving that there exists no quadratic Lyapunov function for a certain class of linear agreement/consensus algorithms, a fact that had been numerically verified in [6]. We also briefly discuss sufficient conditions for the existence of such a Lyapunov function.**

***Index Terms*—Consensus algorithms, Lyapunov theory, multi-agent systems.**

I. INTRODUCTION

We examine a class of algorithms that can be used by a group of agents (e.g., UAVs, nodes of a communication network, etc.) in order to reach consensus on a common opinion (represented by a scalar or vector), starting from different initial opinions, and possibly in the presence of severe restrictions on inter-agent communications.

We focus on a particular algorithm, whereby, at each time step, every agent averages its own opinion with received messages containing the current opinions of some other agents. While this algorithm is known to converge under mild conditions, convergence proofs usually rely on the "span norm" of the vector of opinions. In this note, we address the question of whether convergence can also be established using a quadratic Lyapunov function. Among other reasons, this question is of interest because of its potential implications on convergence time analysis. A negative answer to this question was provided in [6], where the nonexistence of a quadratic Lyapunov function was verified numerically. In this paper, we provide an explicit example and proof of this fact.

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