ASEN 5044, Fall 2024
Statistical Estimation for Dynamical Systems

Lecture 27:
KF Gain and Steady State Behavior;
KF Consistency Evaluation

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Announcements

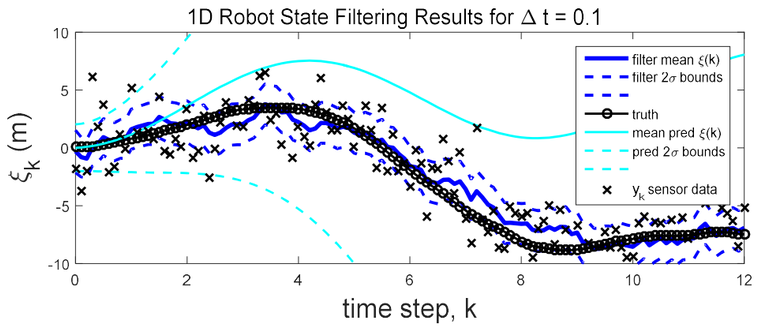
- **HW 7 due today**
- **Midterm 2: posted today, due Nov 14 at 11:59 pm on Gradescope (1 week)**
- **Final project partner sign up sheet on "Assignments" tab in Canvas**
  - Google docs sheet (**Due: Mon 11/18**) – please read + follow all instructions!!
  - Folks with a partner: enter group names (Groups = max of 3 students) (2 students groups ok too)
  - Folks without partner: start a new group or email each other to find a match
  - **Preview system descriptions posted on Canvas (each group must pick one)**
  - You may use Piazza to find partners (but restrict messaging to final project only during Midterm 2!)
  - **You must stay in your group for final assignment!!! (2 interim progress reports + group gets same grade)**
- **Final special topic lecture to be posted tomorrow (Bayesian derivation of KF)**

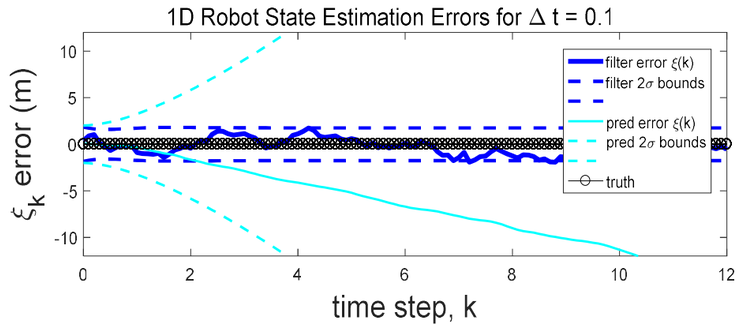
Last Time…

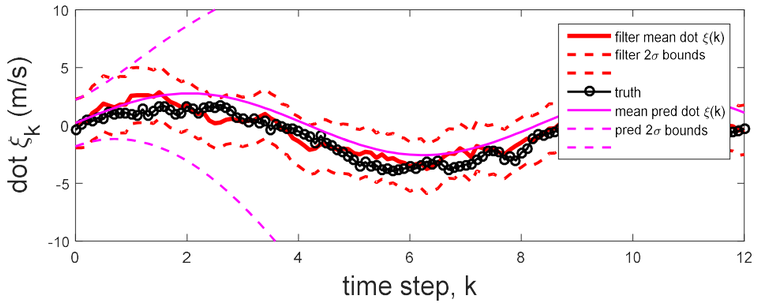
- **The Kalman Filter (KF)**: dynamic predictor-corrector state estimator

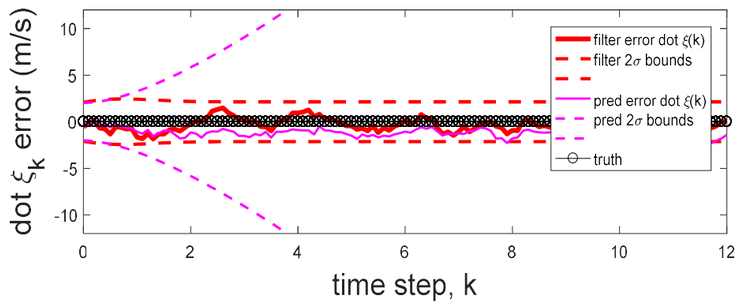
  ("dynamic RLLS": combine state prediction with RLLS updates at each time k – i.e. handle dynamics + process noise w_k + noisy measurements y_k)

  - Algorithm
  - Example + basic intuition behind behavior vs <u>assumed</u> proc/meas noise



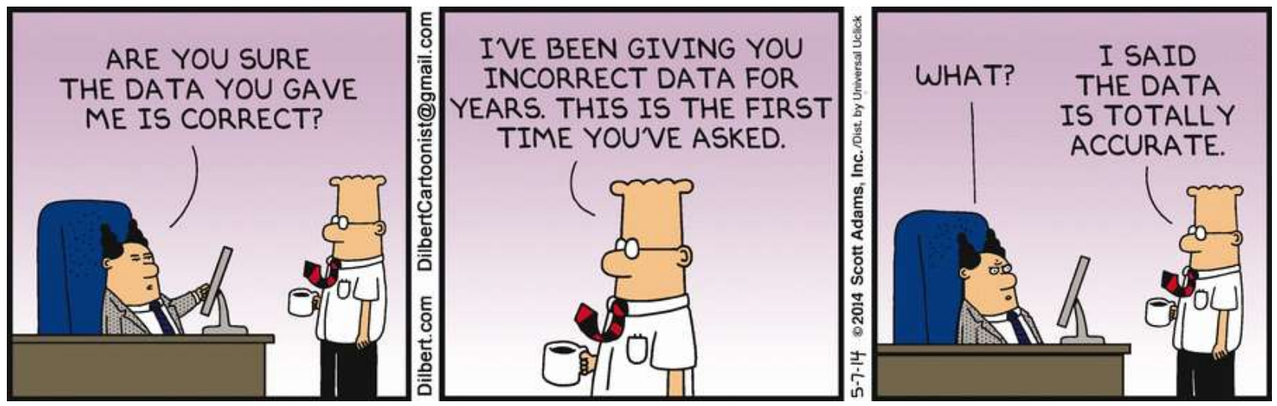






Today…

- Useful property of KFs: **Kalman gain behavior** + **steady state behavior**
  - Process noise vs. measurement noise tradeoffs (balancing model trust vs. sensor trust)
  - Riccati equation, steady state error covariance, and steady state Kalman gain
- **How to tell if your (linear) KF is <u>actually</u> working correctly???**
  - **Want to avoid GIGO systems (Garbage Input, Garbage Output)**



- **KF dynamic consistency analysis and "Truth Model Testing" (TMT)**
- **Chi-square tests (NEES/NIS)** – check if KF's state errors/measurement residuals make sense for given system + measurement + noise models
  - **Do actual state errors/meas. residuals agree with KF's estimated error covariances?**
  - Formal statistical tests to examine this question

Basic Properties of the Kalman Gain K_{k+1}

- Dimension: n x p (generally non-square matrix)
- Kalman gain K_{k+1} represents mapping of new information from space of p-dimensional sensor information (i.e. p x 1 "innovation vector") to the n x 1 state vector x_{k+1}
- "Magnitude" / "strength": generally time-varying: depends on $P^-_{k+1},\ Q,\ R,\ H_{k+1}$

actual physical meas. ↑ predicted meas. ↑

$(y_{k+1} - \hat{y}^-_{k+1})$ where $\hat{y}^-_{k+1} = H_{k+1}\hat{x}^-_{k+1}$

$$\text{Consider:}\quad \hat{x}^+_{k+1} = \hat{x}^-_{k+1} + K_{k+1}(y_{k+1} - H_{k+1}\hat{x}^-_{k+1})$$

$$K_{k+1} = P^-_{k+1}H^T_{k+1}[H_{k+1}P^-_{k+1}H^T_{k+1} + R]^{-1}$$

Recall: Scalar analysis of RLS gain!

$$K_{k+1} = \frac{P^-_{k+1}h_{k+1}}{h^2_{k+1}P^-_{k+1} + r} \quad \left(P^-_{k+1} = \sigma^2_{k+1} = \text{prior variance of scalar } x \text{ state}\right)$$

- $\lim_{P^-_{k+1}\to 0}$: K_{k+1} small ($\to 0$) : very certain of prior state info → don't adjust much due to $y_{k+1} - \hat{y}_{k+1}$

- $\lim_{P^-_{k+1}\to \infty}$: K_{k+1} approaches upper limit of $\frac{1}{h}$: no useful a priori state info [meas. y_{k+1} has all info]

More generally: Non scalar vector-matrix case

- If P^-_{k+1} "very small" (ie $HP^-_{k+1}H^T << R$ in PosDef sense) then K_{k+1} also "small", ie $\hat{x}^+_{k+1} \approx \hat{x}^-_{k+1}$ {KF trusts the dynamics prediction more than y data}

- IF P^-_{k+1} "very large" (ie $HP^-_{k+1}H^T >> R$ in pos def sense) then K_{k+1} approaches "upper limit" (KF trusts y_{k+1} data more than the pure model pred. $\hat{x}^-_{k+1}$)

But also recall! $P^-_{k+1} = F\underline{P^+_k}F^T + Q_{(KF)}$

Steady-state Properties of the KF

- KF gives state estimate along with update of estimation error covariance (P_k^+)
- What is the "smallest/best possible" covariance? (ie what is smallest possible cost $J_k = tr(P_k^+)$? "How much juice can be squeezed from $y_1 \to y_k$?")

Recall: KF pred.: $P_{k+1}^- = F P_k^+ F^T + Q$

KF meas. update: $P_{k+1}^+ = P_{k+1}^- - P_{k+1}^- H_{k+1}^T \underbrace{\left[H_{k+1} P_{k+1}^- H_{k+1}^T + R \right]^{-1}}_{K_{k+1}\ (\text{KF gain})} H_{k+1} P_{k+1}^-$

but: @ time k: $P_k^+ = P_k^- - P_k^- H_k^T [\cdots] H_k P_k^-$

↳ So plug into P_{k+1}^- expression:

$$P_{k+1}^- = F \left[P_k^- - P_k^- H_k^T \left[H_k P_k^- H_k^T + R \right]^{-1} H_k P_k^- \right] F^T + Q$$

(*) → Non-linear DT "one-step" update to get P_{k+1}^- from P_k^- (* assuming meas. update occurs @ time k)

known as **DT Matrix Riccati Equn. (DT MRE)** (*)

The Algebraic Riccati Equation (ARE)

Jacopo Riccati (1676 - 1754)

- Special case for LTI DT systems:
  - if process noise w(k) hits every state
  - AND if (F,H) is observable

then DT MRE implies convergence to a steady state a priori $P_\infty^- > 0 \ (\text{posdef})$

AND obtain the **DT Algebraic Riccati Equation (ARE)**, which is easier to solve (though still non-linear in P_∞^-):

$$P_\infty^- = F(P_\infty^- - P_\infty^- H^T [H P_\infty^- H^T + R]^{-1} H P_\infty^-) F^T + Q$$

Algebraic Riccati Equation (ARE)

Steady-State KF Gain

- Suppose the conditions for the ARE hold, so that

$$P_\infty^- = F(P_\infty^- - P_\infty^- H^T[HP_\infty^- H^T + R]^{-1} HP_\infty^-)F^T + Q$$

- Since the Kalman gain is

$$K_{k+1} = P_{k+1}^- H^T[HP_{k+1}^- H^T + R]^{-1}$$

→ it follows that there must also exist a steady state Kalman gain K_∞

$$K_\infty = P_\infty^- H^T[HP_\infty^- H^T + R]^{-1}$$

→ K_∞ often used in practice to save computation at each time step

(since $K_{k+1} \to K_\infty$ quickly anyway, there is generally little performance loss)

*In Matlab: can use the "**`dlqe.m`**" (discrete linear quadratic estimator) command to find steady state KF gain, along with steady state a priori and a posteriori covariances

$$[K_\infty, P_\infty^-, P_\infty^+] = \text{dlqe}[F, \text{eye}(n), H, Q, R]$$

$tr(P_\infty^+)$ = smallest possible cost $J(u)$!

Example: 1D Robot: Part Trois: Régime Permanent

- Same DT model as before:

$$x(k) = [\xi(k), \dot{\xi}(k)]^T$$
$$u(k) = 2\cos(0.75 t_k) \text{ (ZOH)}$$
$$w(k) \sim \mathcal{N}(0, Q) \text{ (AWGN)}$$
$$x(0) \sim \mathcal{N}(\mu_0, P_0), \text{ where } \mu_0 = [0,0]^T,\ P_0 = I_{2\times 2}$$

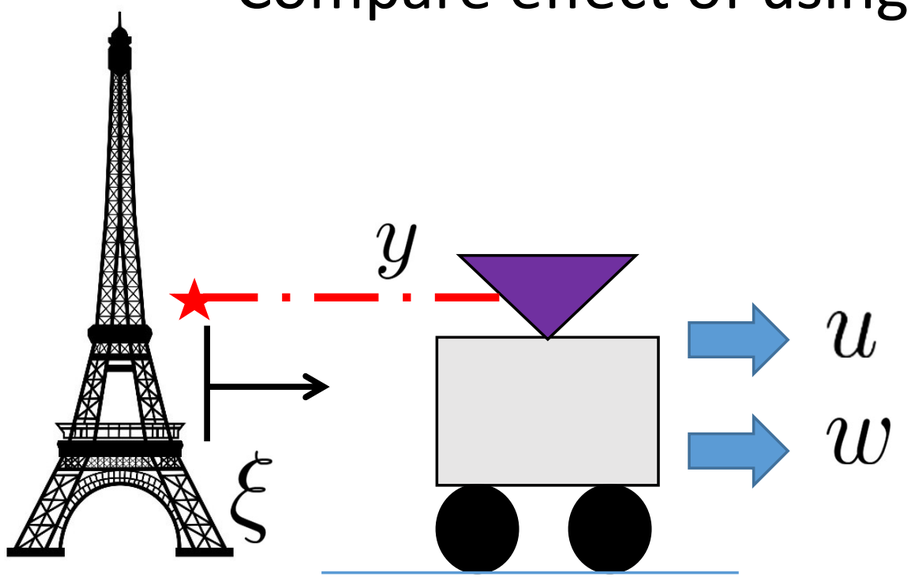
$$x(k+1) = Fx(k) + Gu(k) + w(k)$$

$$F = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \quad G = \begin{bmatrix} 0.5\Delta t^2 \\ \Delta t \end{bmatrix} \quad \Delta t = 0.1 \text{ sec}$$

$$Q = 1 \text{ (m/s)}^2,\ R = 0.5 \text{ m}^2$$

use Van Loan's to get Q

- Compare effect of using dynamic KF gain to steady-state KF gain (using dlqe.m in Matlab)



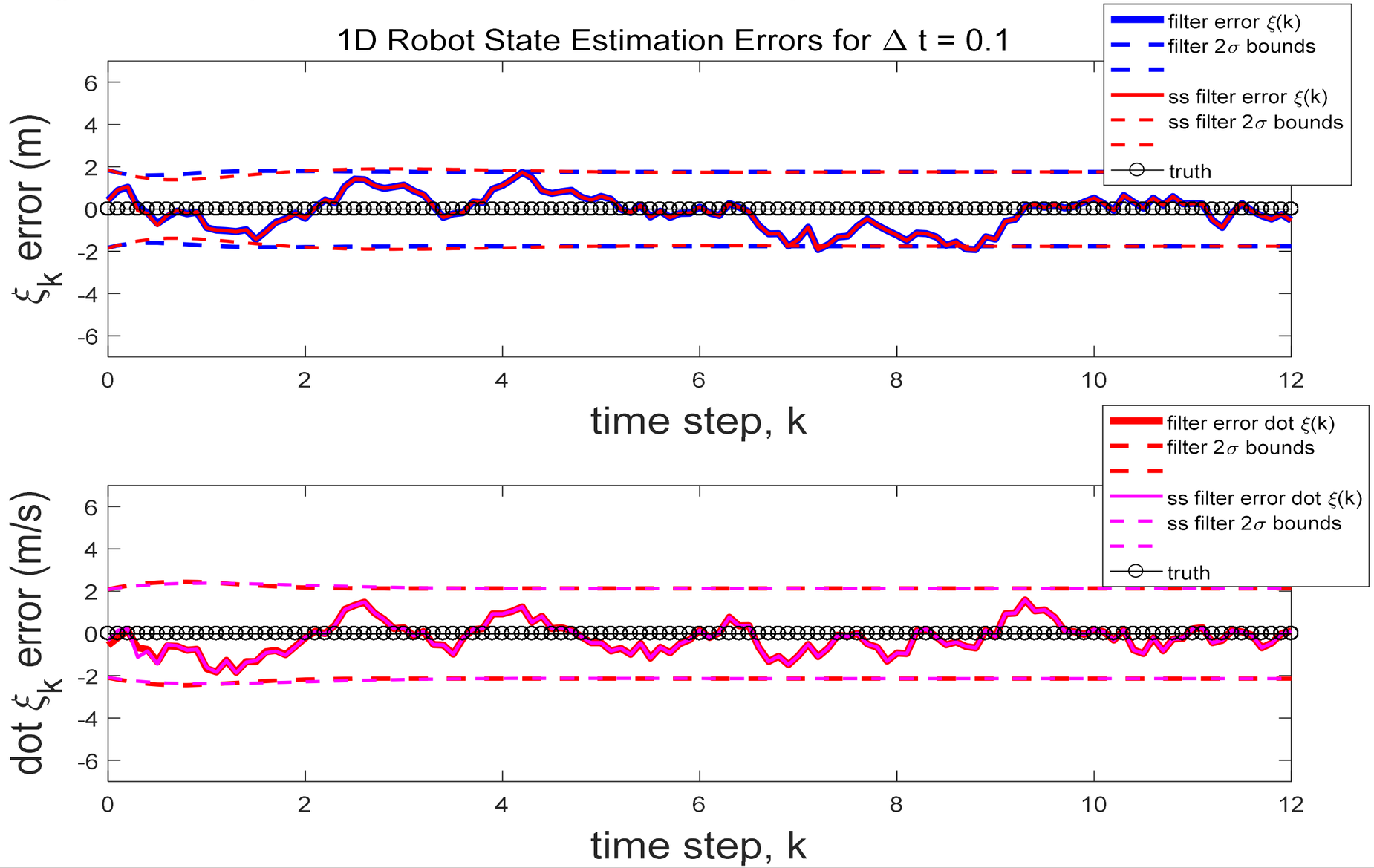
$$K_\infty = \begin{bmatrix} 0.1548 \\ 0.1300 \end{bmatrix}$$

$$P_\infty^- = \begin{bmatrix} 0.9157 & 0.7691 \\ 0.7691 & 1.2406 \end{bmatrix}$$

$$P_\infty^+ = \begin{bmatrix} 0.7740 & 0.6501 \\ 0.6501 & 1.1406 \end{bmatrix}$$

Results: Dynamic KF Gain vs. Steady-state KF Gain

- Not a significantly noticeable difference in performance



KF Consistency Analysis

- Like batch LLS and RLLS: estimates produced by the KF are random vectors
- For KF: this is due to uncertainties from process and measurement noise
- KF recursively assesses estimation uncertainty via error covariance matrix
- BUT: first need to set tuning parameters in Q_{KF} – generally not obvious!
- Other possible latent practical issues: approximate (F,G,H,M) model, unmodeled state dynamics, non-white noise, …
- So: how to know if KF estimates & covariances are correct for given system?

  i.e. do error statistics provided by KF reflect the actual error statistics?

Evaluating Errors in a KF State Estimator

- KF is itself a dynamical system (defined in software) that tries to track an actual physical system: