ASEN 5050
SPACEFLIGHT DYNAMICS
Lambert's Problem

Objectives:

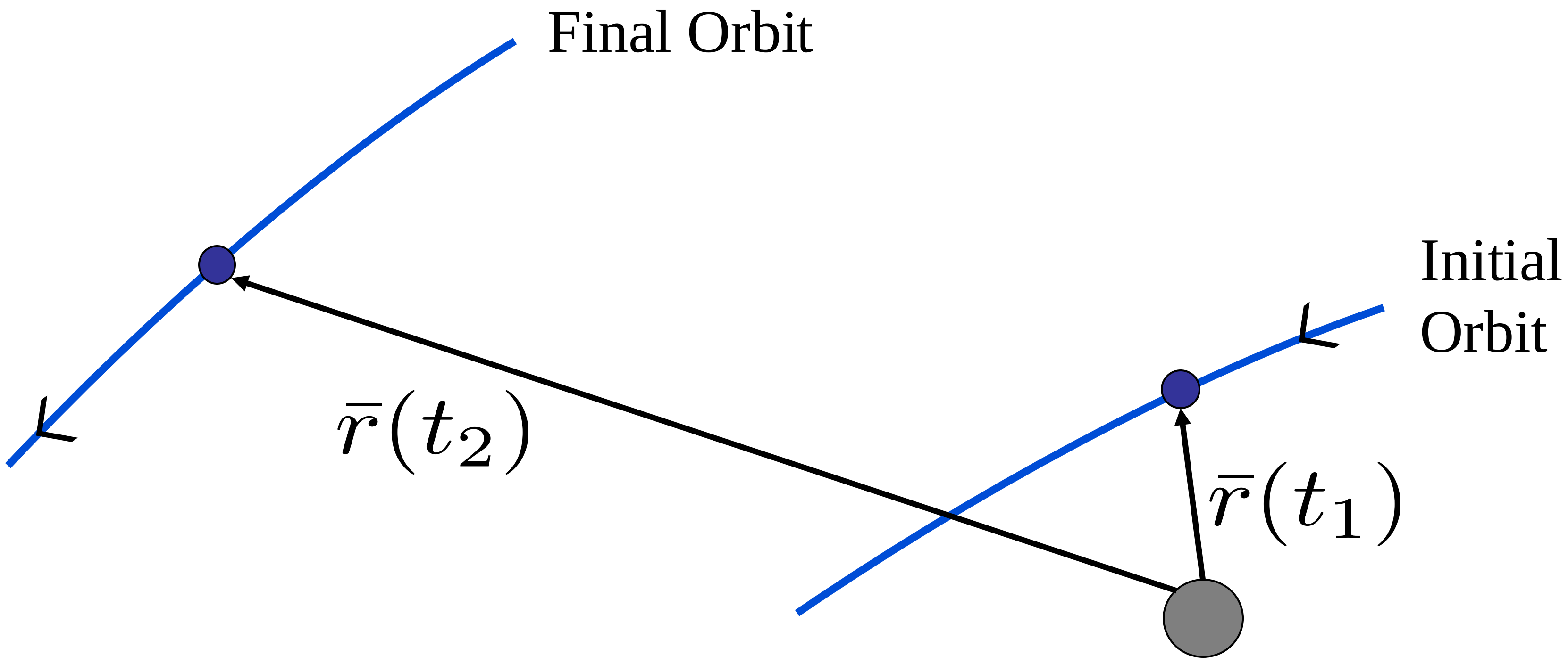
- Conceptual & geometric introduction to Lambert's problem
- Derive Lambert's equation for elliptical orbits
- Associate quadrant ambiguities with transfer solutions
- Present similar expressions for hyperbolic, parabolic arcs
- Present systematic procedure for solving Lambert's problem

Following derivation from Prussing & Conway, 2013, "Orbital Mechanics, 2nd edition"

Lambert's Problem

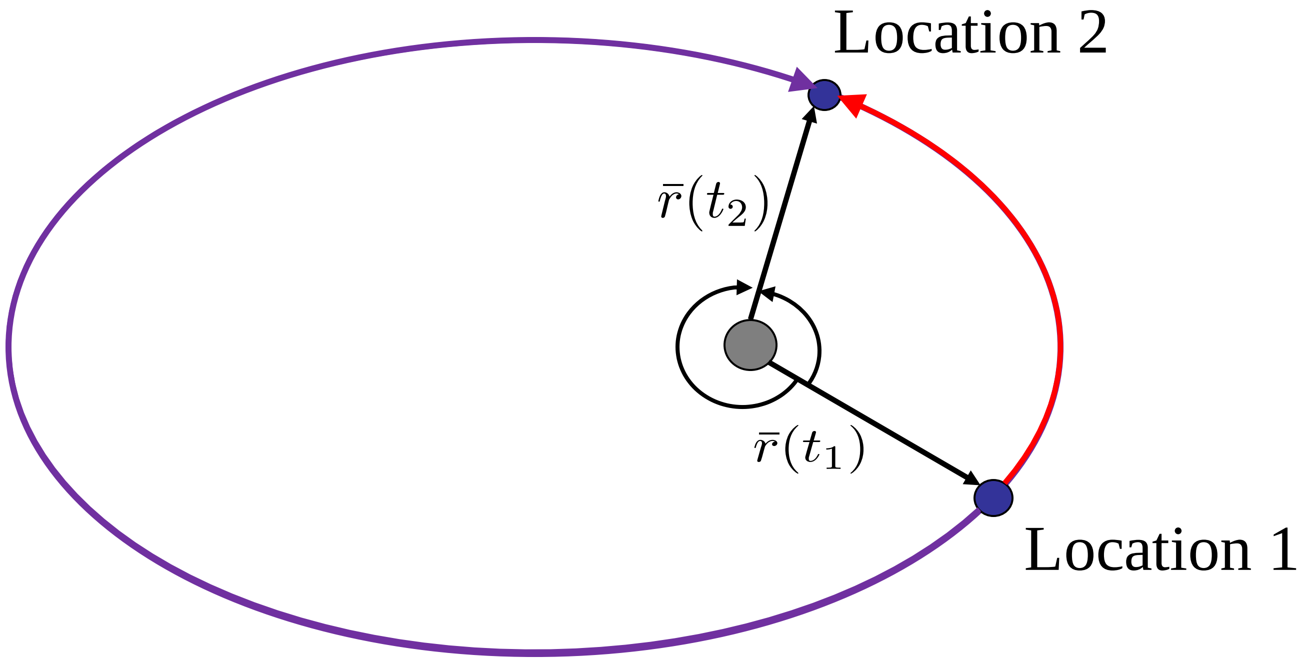
Given 2 position vectors, find connecting arc along a conic

→ Calculate orbital elements of that conic with a specific time of flight (TOF) and the two impulsive maneuvers required



Lambert's Problem

Transfer angle:



Lambert's Problem

There is also a variety of conics (via TOF) that may connect the two position vectors

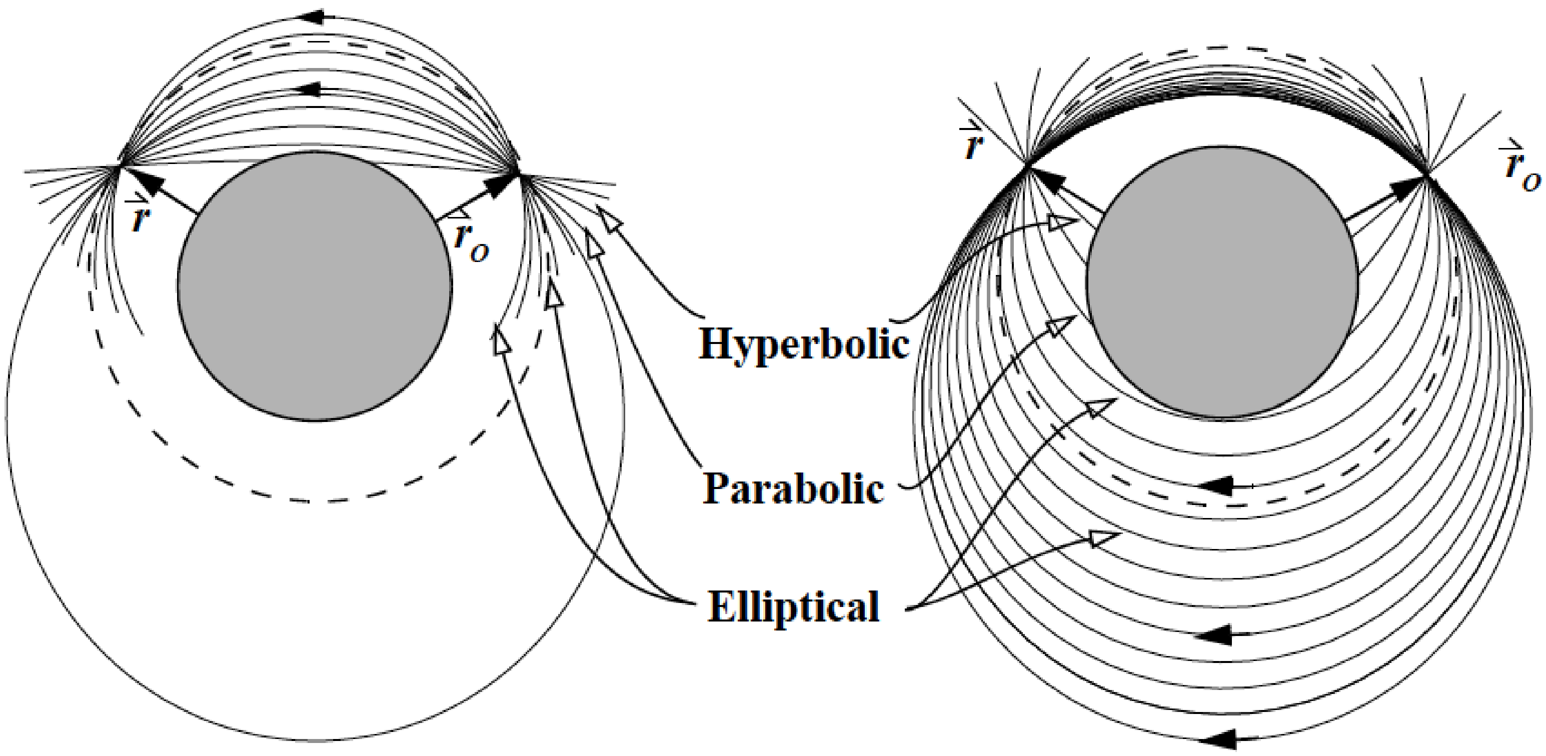
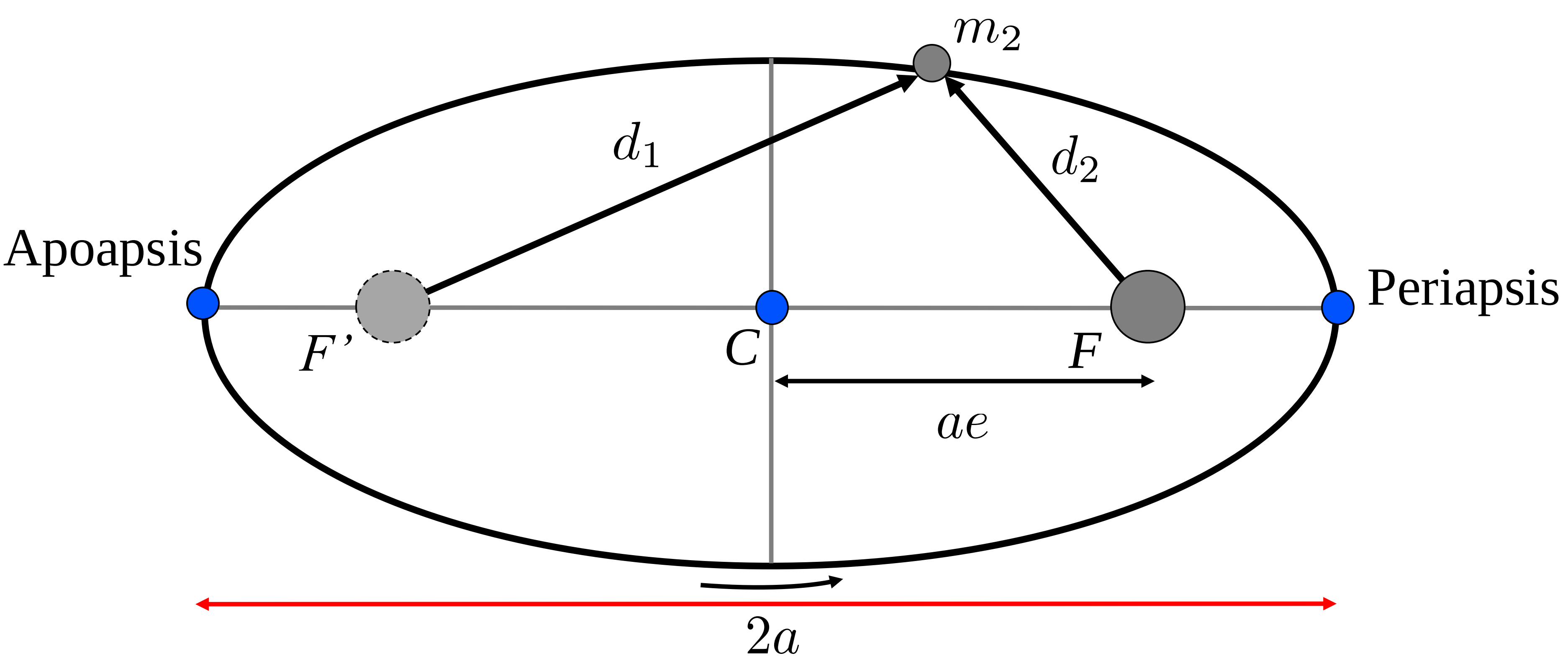


Image credit: Vallado 2013

Elliptical Orbit Geometry

Use geometry to construct a Lambert arc transfer.

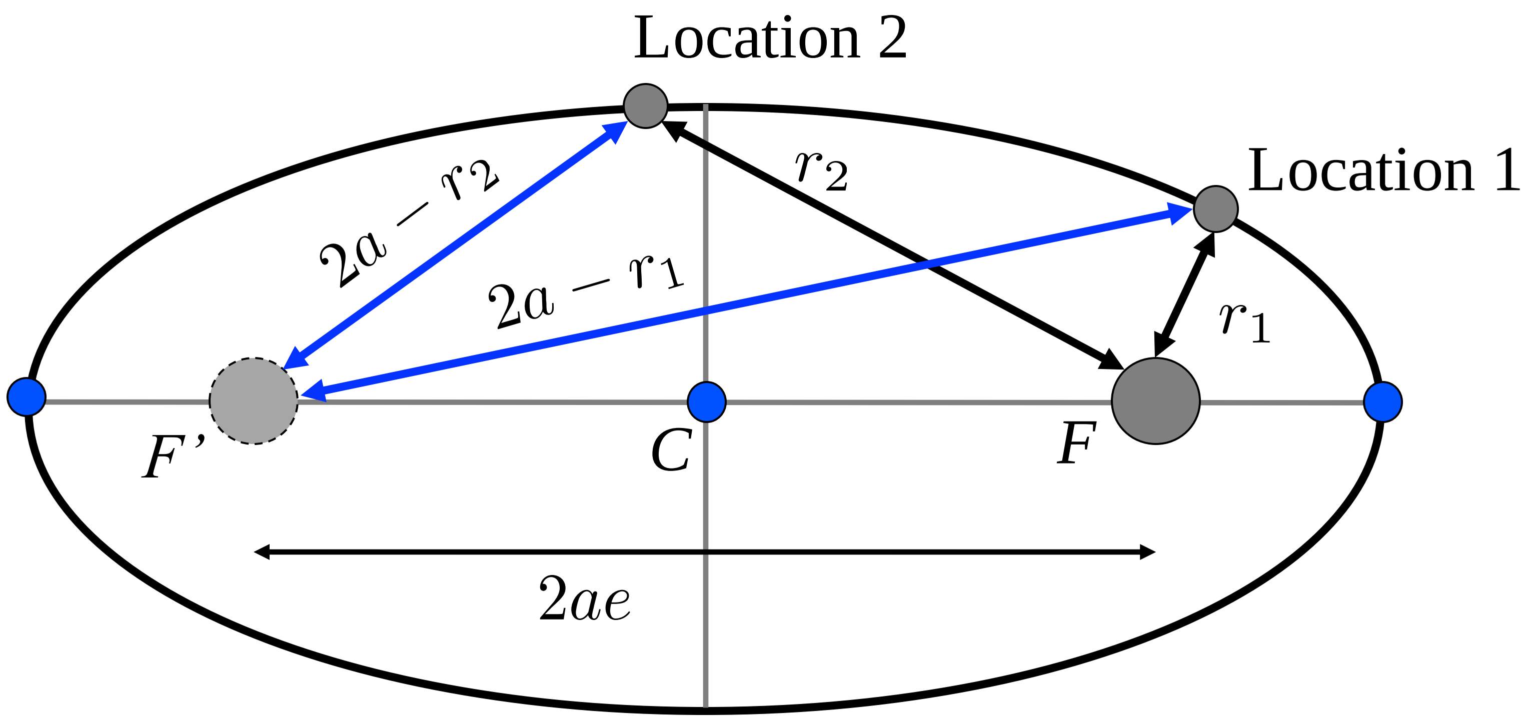
Recall the following relation:



Locating the Vacant Focus

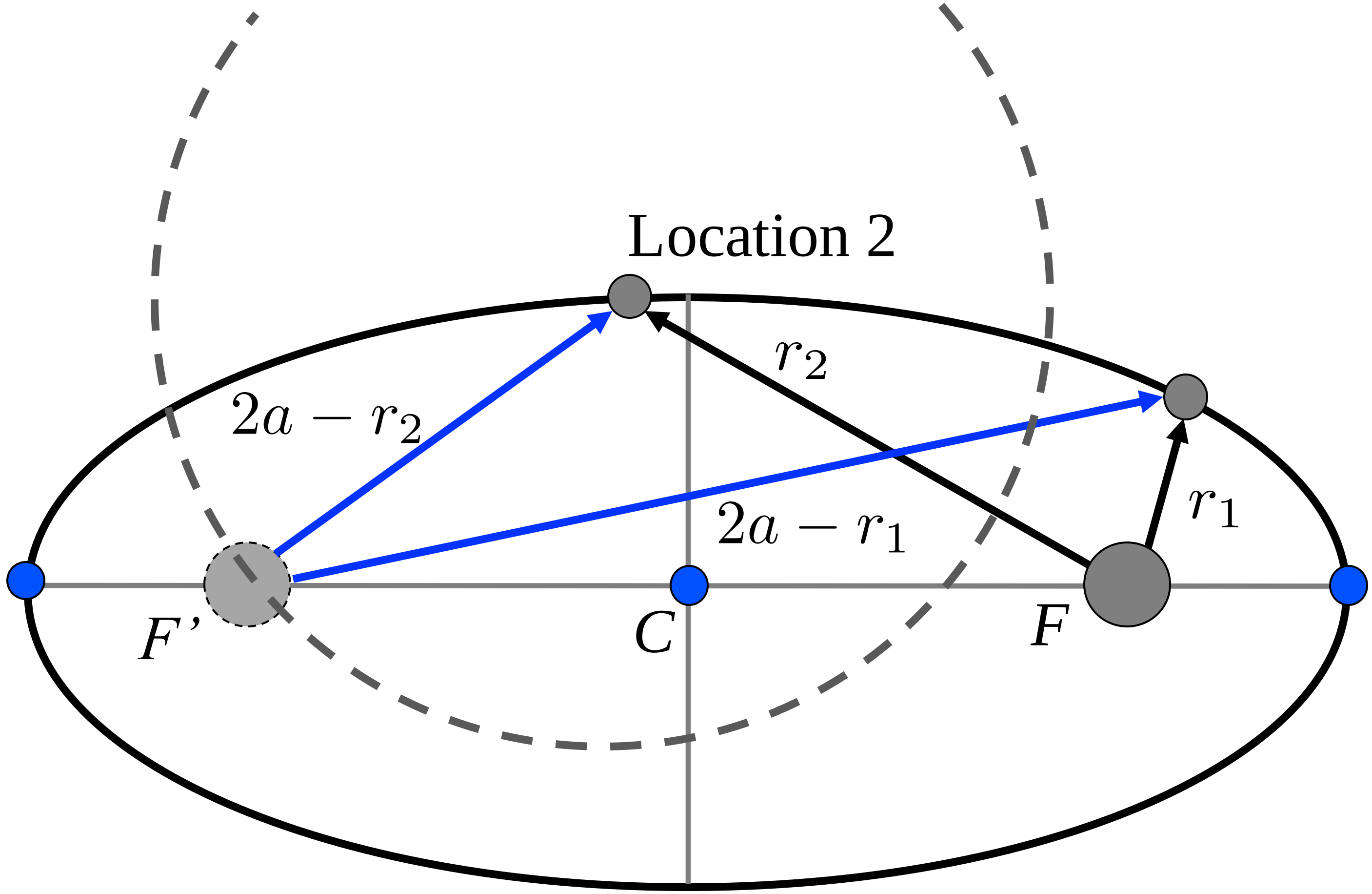
To identify the transfer conic, need to locate the vacant focus

E.g., assume a specified semi-major axis, *a*



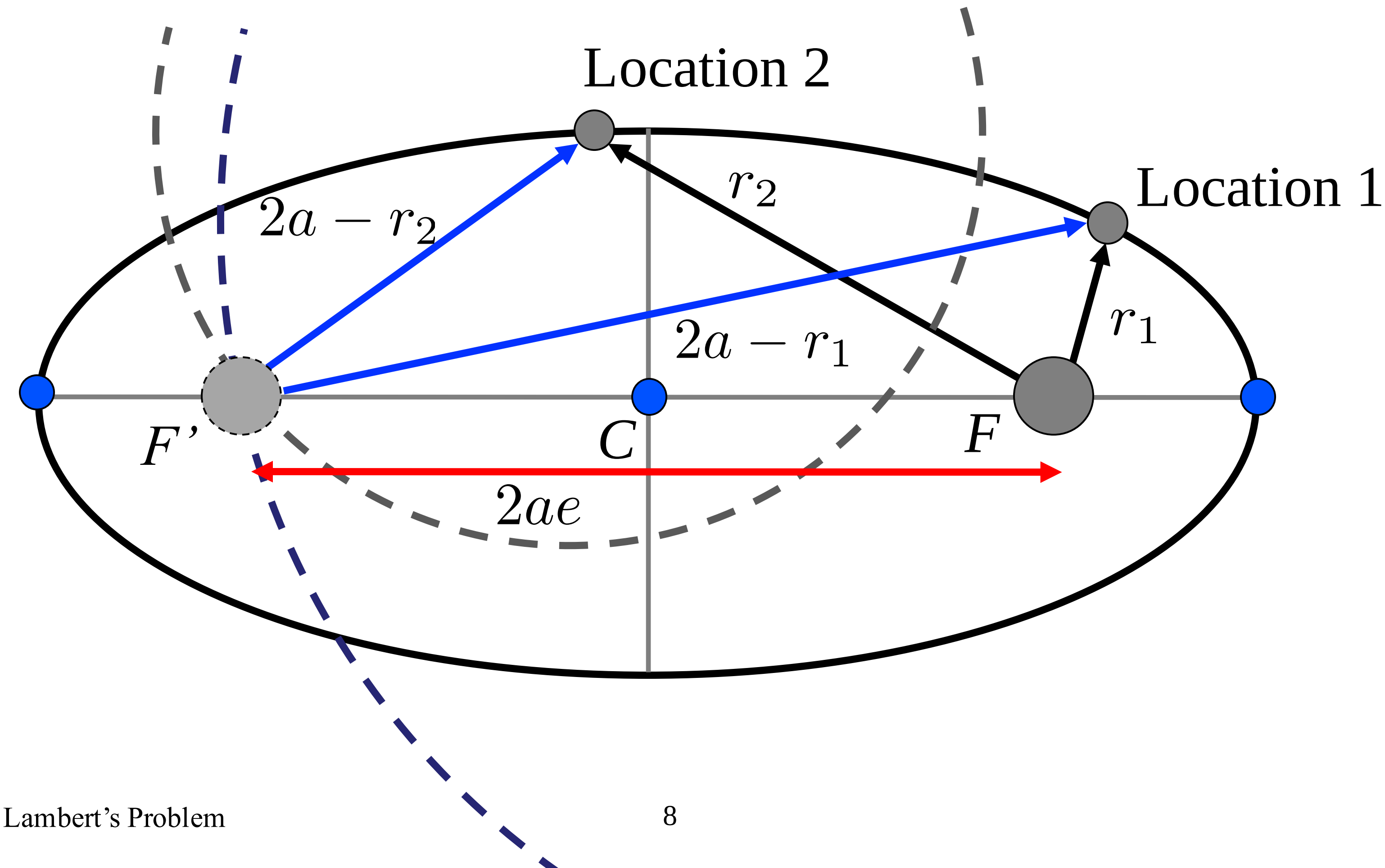
Locating the Vacant Focus

To identify the transfer conic, need to locate the vacant focus



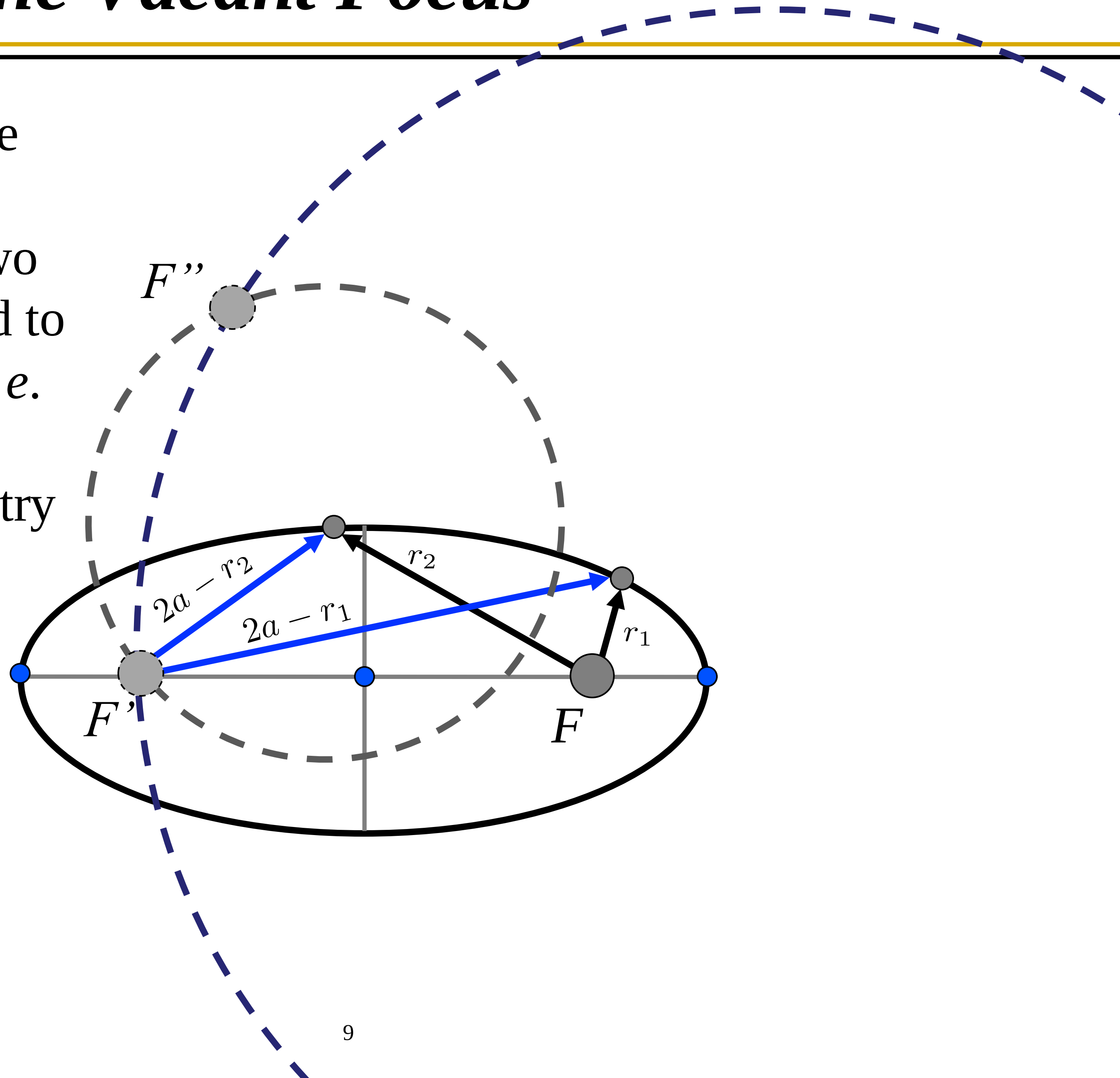
Locating the Vacant Focus

For a given transfer orbit semi-major axis, locate the vacant focus at the intersection of the two circles around each location *i* with radius $2a\text{-}r_i$



Locating the Vacant Focus

For a given *a*, there are two possible vacant foci. The two choices correspond to different values of *e*.
→ Also influences transfer arc geometry



Locating the Vacant Focus

Locus of possible vacant foci lies on one half of a hyperbola

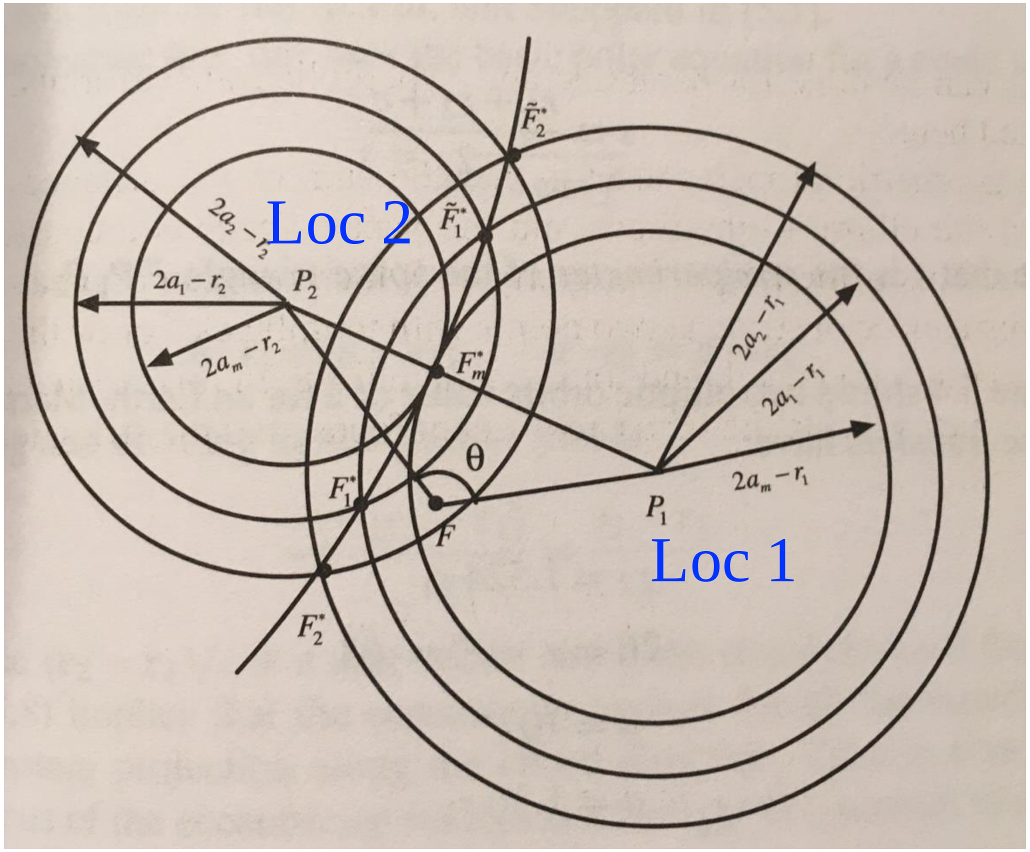
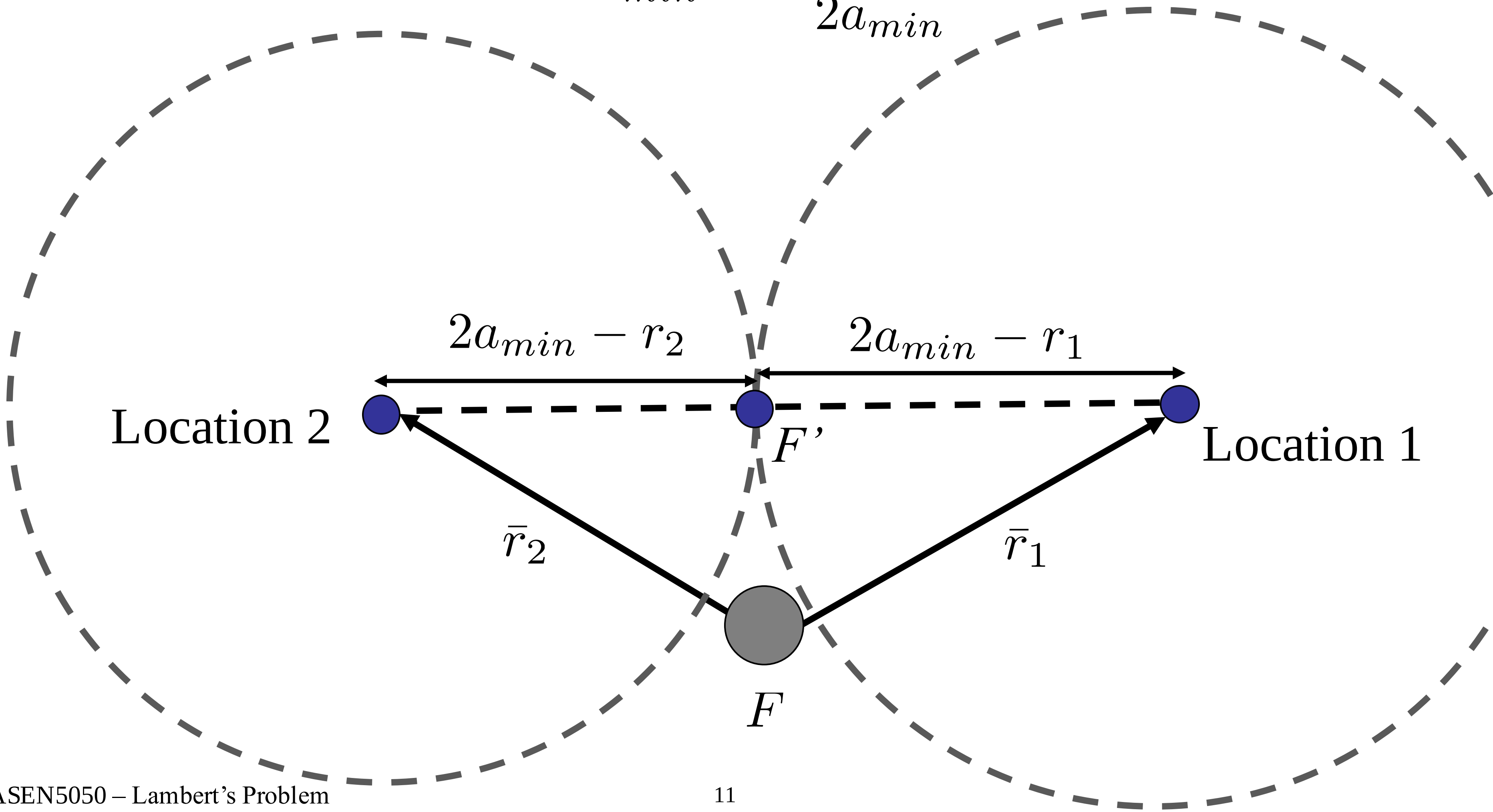


Image credit: Prussing & Conway, 2013

Minimum Energy Transfer

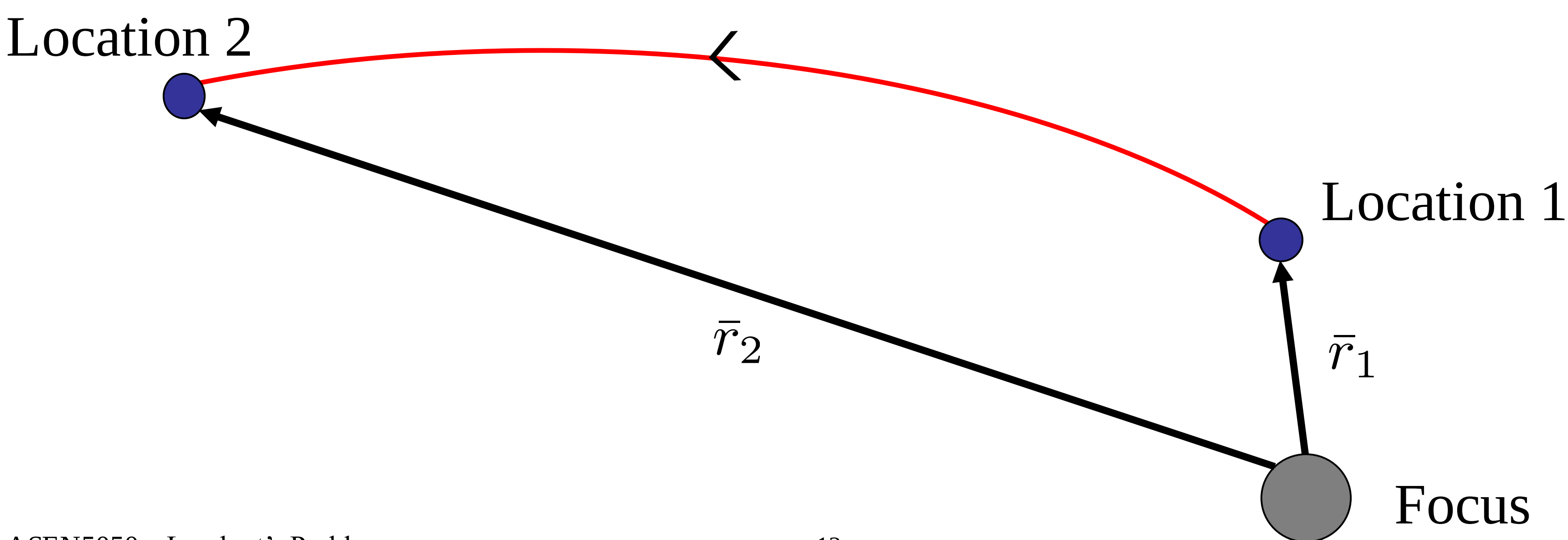
The minimum value of *a* corresponds to the transfer with the minimum specific energy $\mathcal{E}_{min} = -\dfrac{\mu}{2a_{min}}$



Defining the Space Triangle

Define a chord length, *c*, calculated using cosine law:

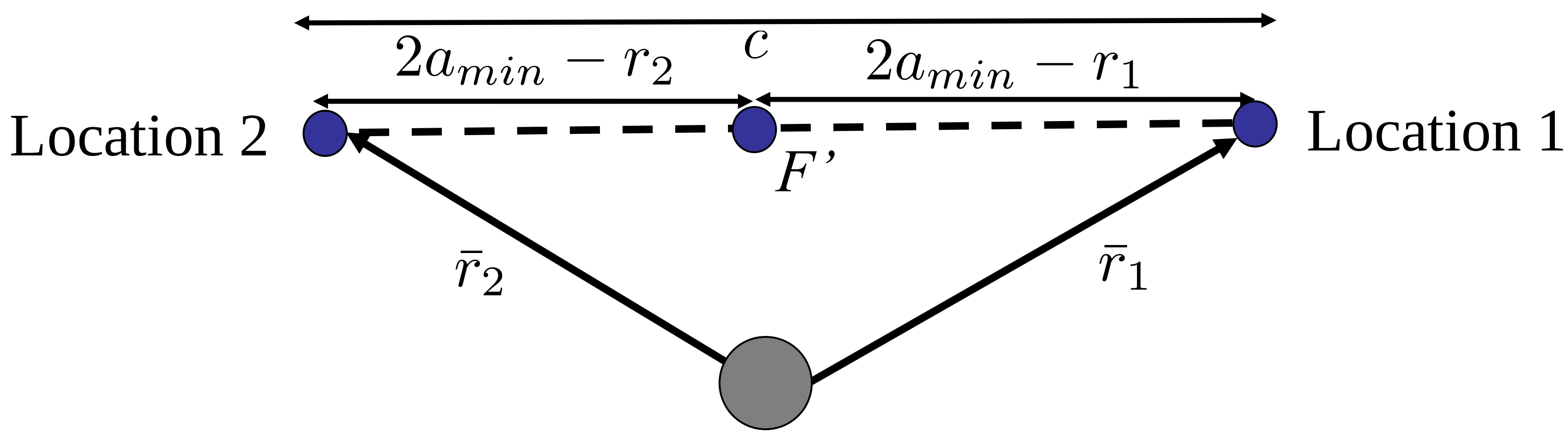
Then, a semi-perimeter, *s*, half the sum of lengths of all 3 sides of the triangle:



Defining the Space Triangle

For the minimum energy ellipse with a_{min}, write in terms of s

$$c = (2a_{min} - r_2) + (2a_{min} - r_1) = 4a_{min} - r_1 - r_2$$



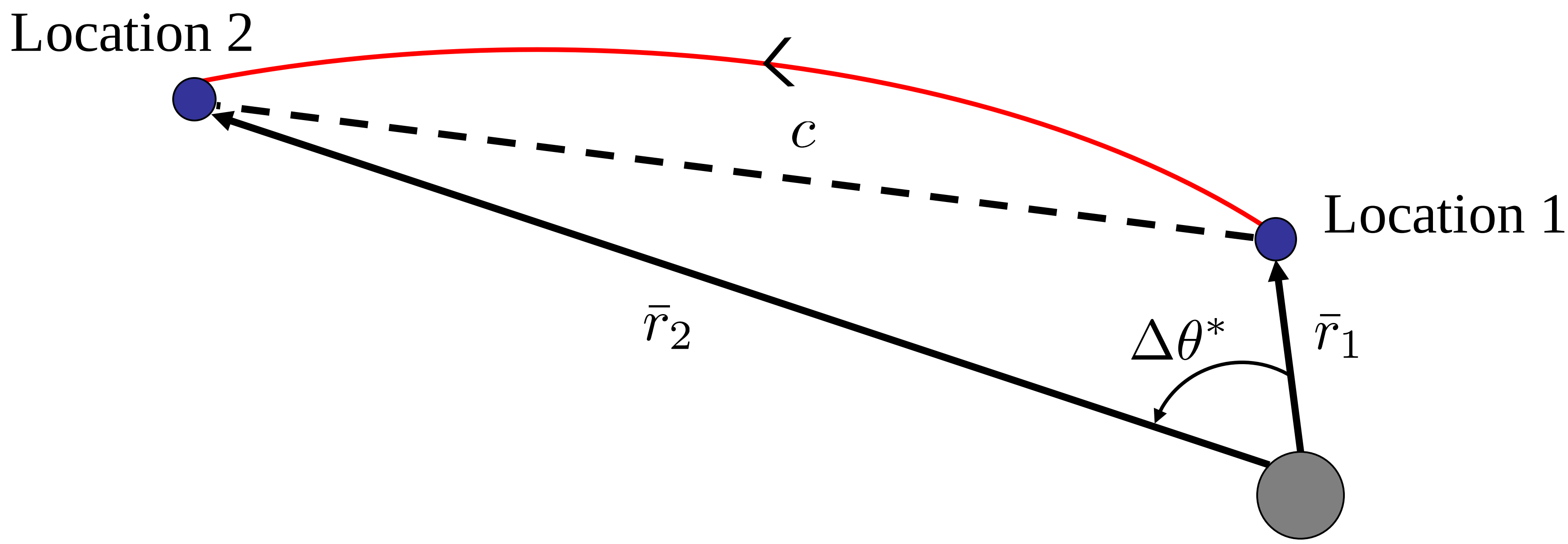
$$cos(\Delta\theta^*) = \frac{\bar{r}_2 \cdot \bar{r}_1}{|\bar{r}_2||\bar{r}_1|}$$

Design choice:

$\Delta\theta^* < 180^\circ$

$\Delta\theta^* > 180^\circ$

Lambert's Equation



The time of flight from location 1 to location 2 is calculated using:

- Semi-major axis, *a*, of the transfer ellipse
- The chord length, *c*
- The sum $r_1 + r_2$

Deriving Lambert's Equation

Recall from Kepler's equation:

$$TOF = t_2 - t_1 = \frac{1}{n}\left[(E_2 - e\sin(E_2)) - (E_1 - e\sin(E_1))\right]$$

However, need *a* and *e* to evaluate this. To turn this into a useable expression, define two mathematical quantities:

Such that:

Typo note: in the next few slides, $E_M = E_m$

Deriving Lambert's Equation

Recall the following relation: $r = a(1 - e\cos(E))$

Then: $r_1 + r_2 = a\left[2 - e\left(\cos(E_1) + \cos(E_2)\right)\right]$

And in terms of E_p and E_m:

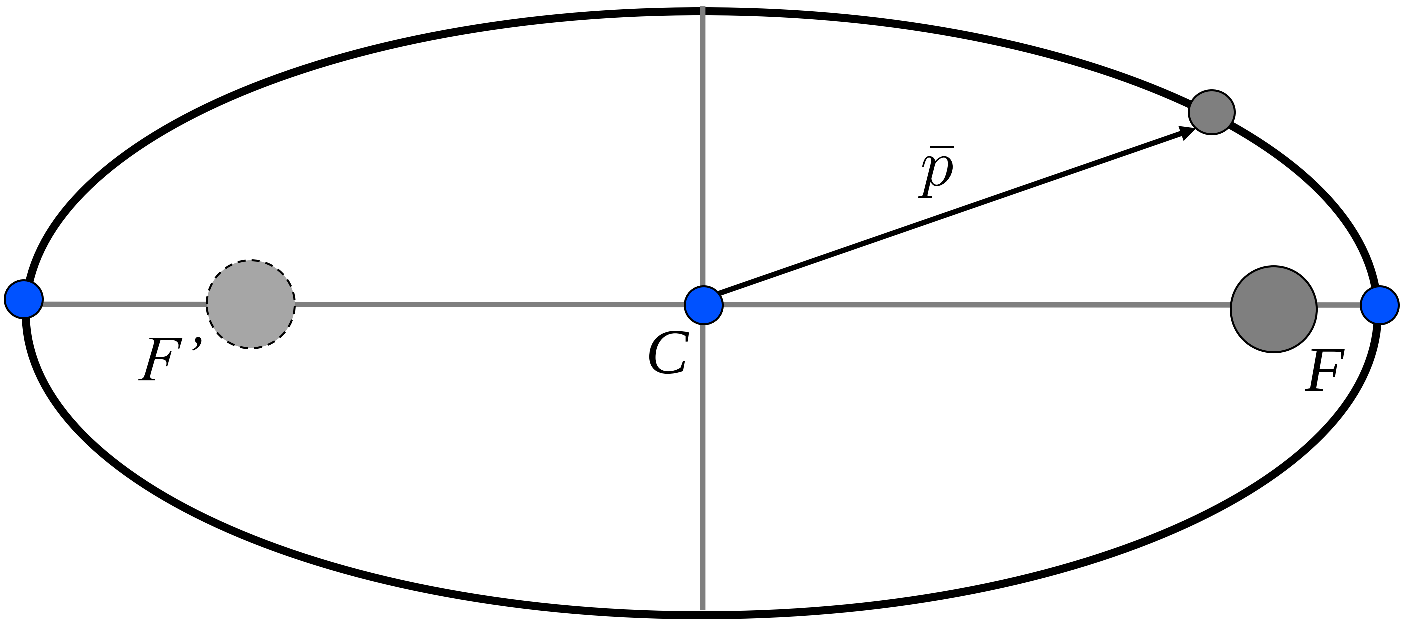
$$r_1 + r_2 = a\left[2 - e\left(\cos(E_p - E_m) + \cos(E_p + E_m)\right)\right]$$

Using a trigonometric identity: $\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$

$$r_1 + r_2 = 2a[1 - e\cos(E_p)\cos(E_M)]$$

Deriving Lambert's Equation

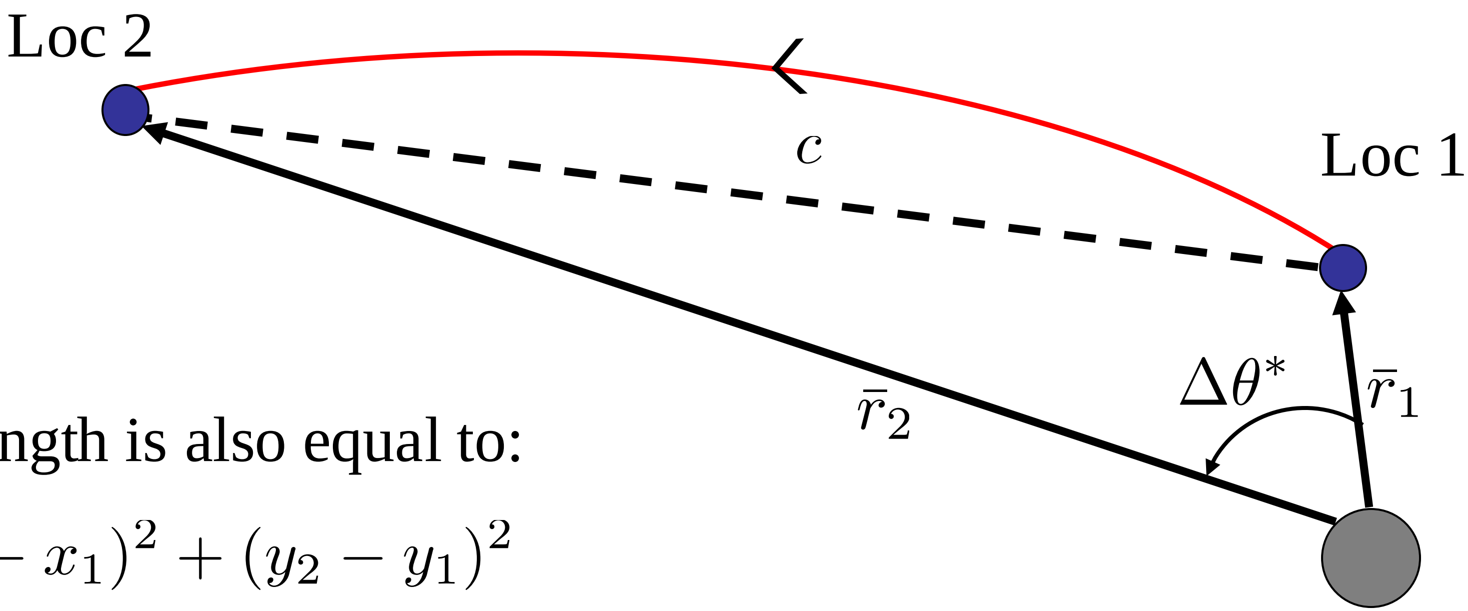
And defining Cartesian coordinates locating each point on an ellipse with respect to the center, C:



Recall that:

Where $b = a\sqrt{1 - e^2}$

Deriving Lambert's Equation



The chord length is also equal to:

$$c^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$c^2 = (a\cos(E_2) - a\cos(E_1))^2 + (b\sin(E_2) - b\sin(E_1))^2$$

$$c^2 = a^2\left[(\cos(E_p + E_m) - \cos(E_p - E_m))^2 + (1 - e^2)(\sin(E_p + E_m) - \sin(E_p - E_m))^2\right]$$

Employing trigonometric identities:

$$c^2 = a^2\left[(2\sin(E_p)\sin(E_m)^2 + (1 - e^2)(2\cos(E_p)\sin(E_m))^2\right]$$

Deriving Lambert's Equation

Rearranging:

$$c^2 = a^2 \left[(2\sin(E_p)\sin(E_m))^2 + (1-e^2)(2\cos(E_p)\sin(E_m))^2 \right]$$

$$c^2 = 4a^2 \sin^2(E_m) \left[(\sin(E_p))^2 + (1-e^2)(\cos(E_p))^2 \right]$$

$$c^2 = 4a^2 \sin^2(E_m) \left[1 - e^2 \cos^2(E_p) \right]$$

Then, because $e < 1$ for an ellipse, define the following variable:

$$\cos\xi = e\cos E_p$$

Substitute into above expression:

$$c = 2a\sin(E_M)\sin(\xi)$$

Deriving Lambert's Equation

Then, recall previous expression:

$$r_1 + r_2 = 2a\left[1 - e\cos(E_p)\cos(E_m)\right] = 2a\left[1 - \cos(\xi)\cos(E_m)\right]$$

And define two additional variables:

Such that:

Then:

$$r_1 + r_2 + c = 2a\left[1 - \cos(\xi)\cos(E_m) + \sin(E_m)\sin(\xi)\right]$$

$$2s = 2a[1 - \cos(\xi + E_m)] = 2a[1 - \cos\alpha]$$

$$2s = 4a\sin^2\left(\frac{\alpha}{2}\right) \longrightarrow \sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{s}{2a}}$$

Deriving Lambert's Equation

And:

$$r_1 + r_2 - c = 2a\left[1 - \cos(\xi)\cos(E_m) - \sin(E_m)\sin(\xi)\right]$$

$$2s - 2c = 2a[1 - \cos(\xi - E_m)] = 2a[1 - \cos(\beta)] = 4a\sin^2\left(\frac{\beta}{2}\right)$$

$$\sin\left(\frac{\beta}{2}\right) = \sqrt{\frac{s-c}{2a}}$$

Back to Kepler's equation:

$$TOF = t_2 - t_1 = \frac{1}{n}[(E_2 - e\sin(E_2)) - (E_1 - e\sin(E_1))]$$

$$TOF = \frac{1}{n}(2E_m - e\sin(E_p + E_m) + e\sin(E_p - E_m))$$

Substituting trigonometric identities:

$$\mathrm{TOF} = \frac{2}{n}[E_M - e\cos E_p \sin E_M] = \frac{2}{n}[E_M - \cos(\xi)\sin E_M]$$

Lambert's Equation

Rewrite in terms of α and β via a trigonometric identity, to recover **Lambert's equation**:

where:

Note: this expression holds only when using less than one revolution along an **elliptical orbit**

But there is an ambiguity in the quadrants of the angles α and β!

Multiple Revolution Arcs along Ellipses

Can perform *N* additional revolutions along an ellipse.

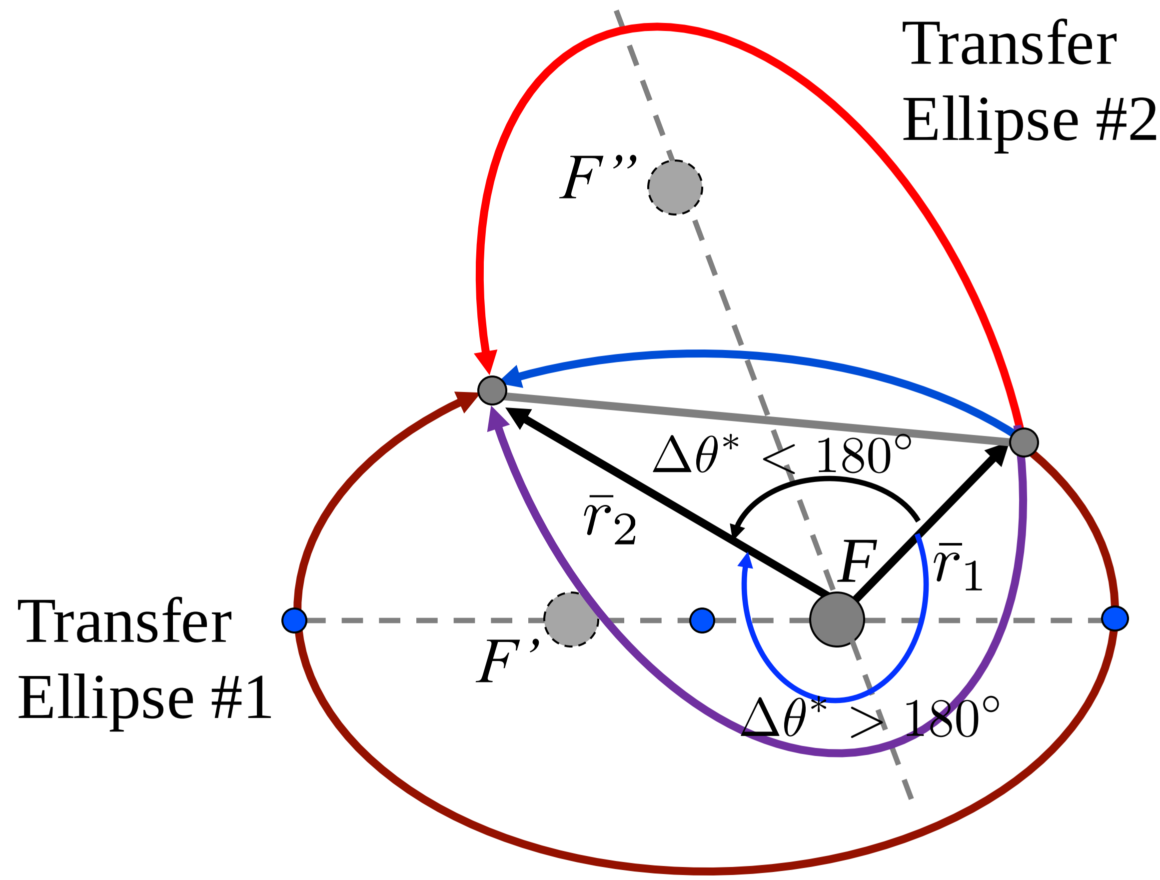
Then, TOF expression becomes:

$$TOF = \frac{1}{n}\left(2N\pi + \alpha - \beta - (\sin\alpha - \sin\beta)\right)$$

Corresponding to an additional *N* orbital periods, increasing TOF

Transfer Arcs for Given a Along Ellipses

For a given set of position vectors and semi-major axis, $a > a_{min}$, there are four possible transfer arcs. (not drawn to scale!)



Quadrant Ambiguities - Ellipses

For a given value of *a*, these four arcs are distinguished by different combinations of the angles α and β

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{s}{2a}} \qquad\qquad \sin\left(\frac{\beta}{2}\right) = \sqrt{\frac{s-c}{2a}}$$

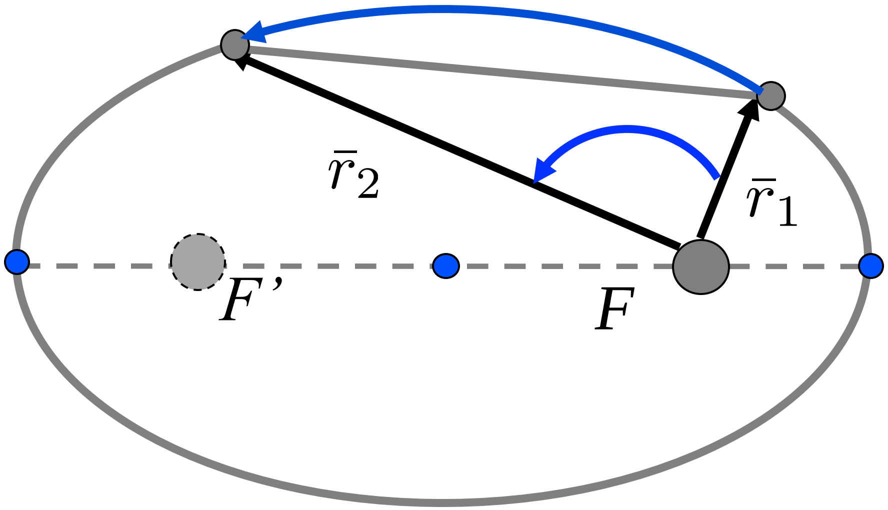
Because

Label α_0, β_0 the principal values, defined for $[0,\pi]$ rad

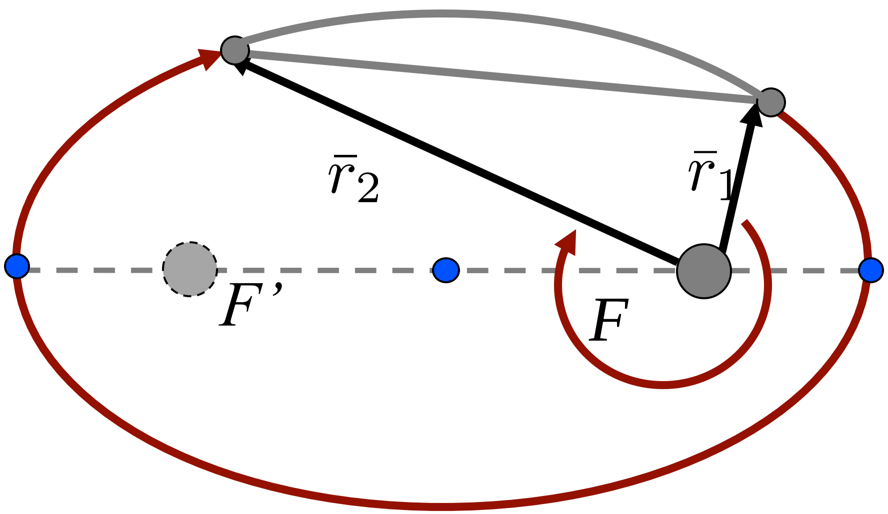
Must choose correct combination to ensure that $\alpha - \beta = E_2 - E_1$ and the correct transfer arc is calculated

Transfer Arcs and Quadrant Ambiguities

$$\alpha - \beta = E_2 - E_1$$



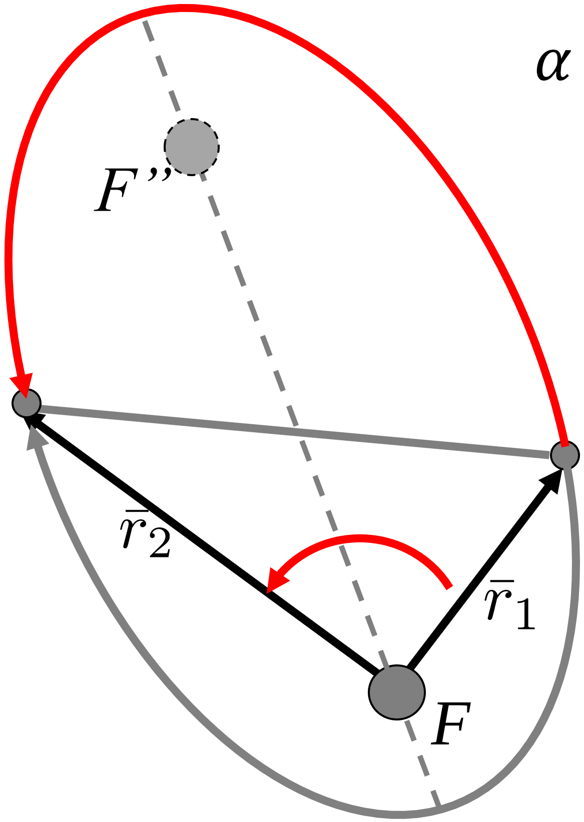
$$\Delta\theta^* < 180^\circ$$
$$\alpha = \alpha_0$$
$$\beta = \beta_0$$



$$\Delta\theta^* > 180^\circ$$
$$\alpha = 2\pi - \alpha_0$$
$$\beta = -\beta_0$$

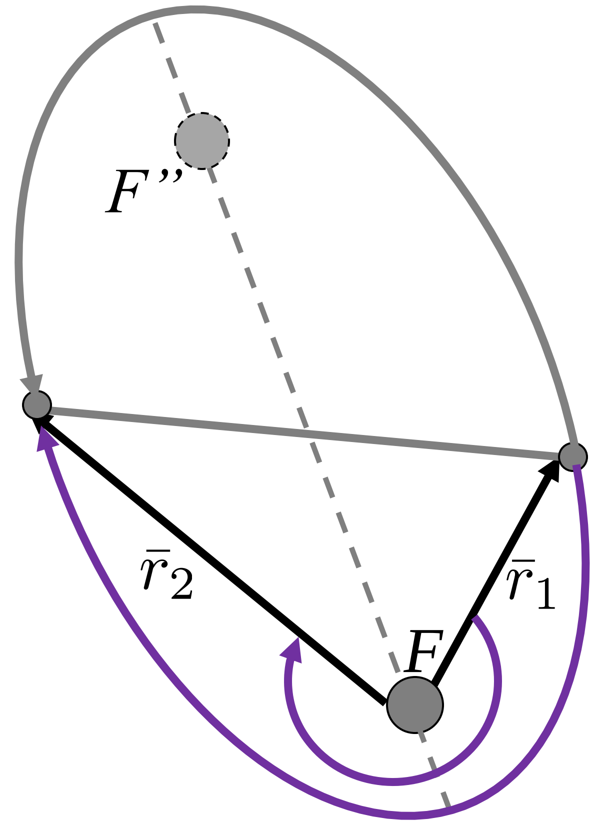
Transfer Arcs and Quadrant Ambiguities

$$\alpha - \beta = E_2 - E_1$$



$$\Delta\theta^* < 180^\circ$$

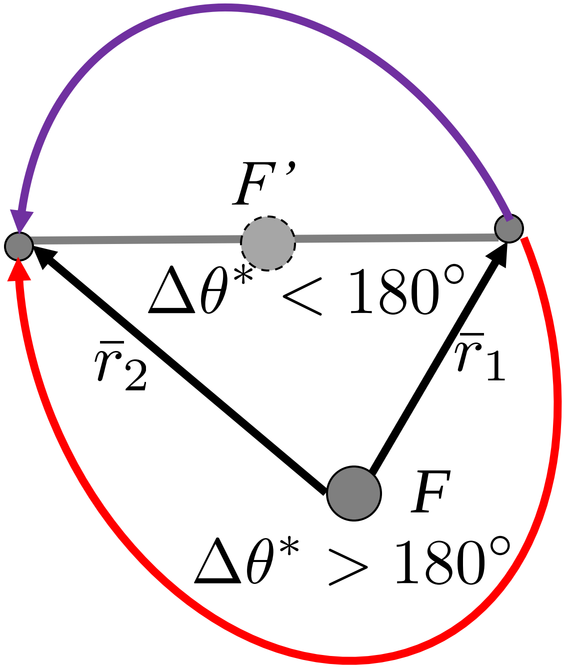
$$\alpha = 2\pi - \alpha_0 \qquad \beta = \beta_0$$



$$\Delta\theta^* > 180^\circ$$

$$\alpha = \alpha_0 \qquad \beta = -\beta_0$$

Minimum Energy Transfer Along Ellipse



At the minimum value of semi-major axis, a_{min}:

$$a_{min} = \frac{s}{2}$$

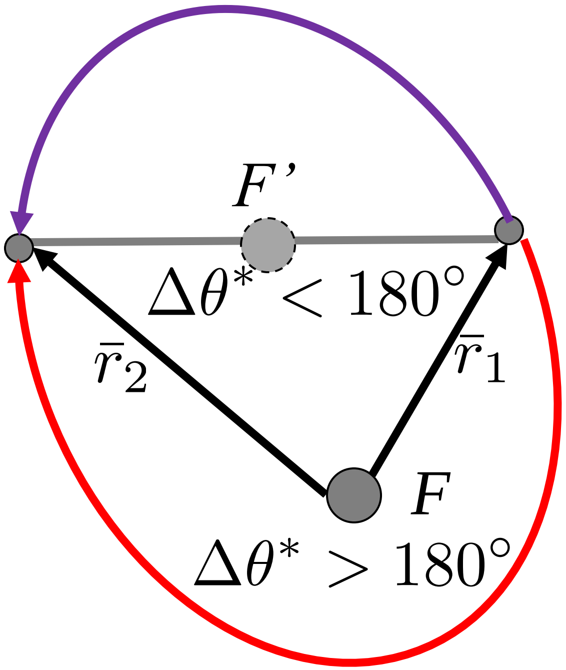
$$\alpha - \beta = E_2 - E_1$$

$$\sin\left(\frac{\alpha_m}{2}\right) = \sqrt{\frac{s}{2a_{min}}}$$

$$\sin\left(\frac{\beta_m}{2}\right) = \sqrt{\frac{s-c}{2a_{min}}} = \sqrt{\frac{s-c}{s}}$$

If: $\Delta\theta^* < 180°$ $\Delta\theta^* > 180°$

Minimum Energy Transfer Along Ellipse



The TOF along each of these two arcs is useful:

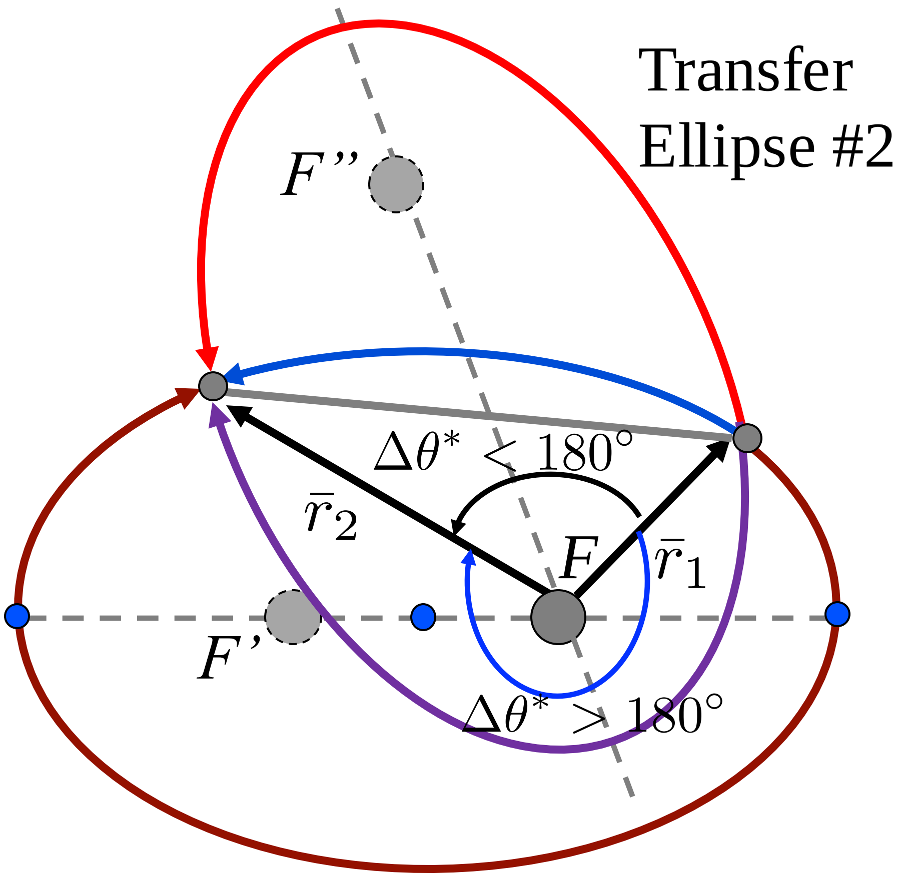
$$TOF_{min} = \frac{1}{n}((\alpha_m - \beta_m) - (\sin(\alpha_m) - \sin(\beta_m)))$$

Evaluate TOF_{min} expression using α_m, β_m that correspond to selected transfer angle

For each transfer angle on a general transfer that uses an arc along an ellipse:

Transfer Arcs for Given a Along Ellipses

For a given set of position vectors and semi-major axis, $a > a_{min}$, there are four possible transfer arcs.



Note: Conceptual diagrams not to scale!

Assigning Angle Quadrants

For Lambert's equation, given a value of *a*, solve iteratively:

$$TOF = \frac{1}{n}\left(\alpha - \beta - (\sin(\alpha) - \sin(\beta))\right)$$

Plug in the correct combination of α and β, in terms of the principal values, that corresponds to the desired transfer type

	Shorter TOF **TOF < TOF_{min}**	**Longer TOF** **TOF > TOF_{min}**
$\Delta\theta^* < 180°$	Blue arc $\alpha = \alpha_0$ $\beta = \beta_0$	Red arc $\alpha = 2\pi - \alpha_0$ $\beta = \beta_0$
$\Delta\theta^* > 180°$	Purple arc $\alpha = \alpha_0$ $\beta = -\beta_0$	Brown arc $\alpha = 2\pi - \alpha_0$ $\beta = -\beta_0$

Solving Lambert's Equation

Solve Lambert's equation iteratively using a numerical method:

Recall:

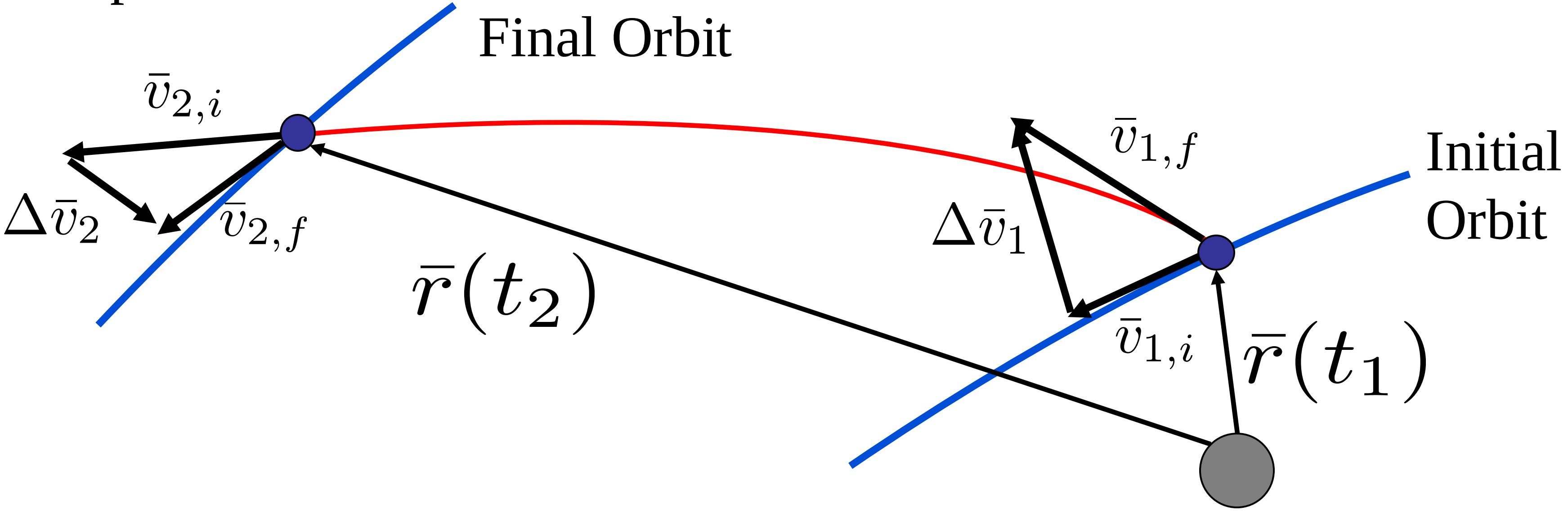
$$TOF = \frac{1}{n}\left((\alpha - \beta) - (\sin(\alpha) - \sin(\beta))\right)$$

where

$$n = \sqrt{\frac{\mu}{a^3}} \qquad \sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{s}{2a}} \qquad \sin\left(\frac{\beta}{2}\right) = \sqrt{\frac{s-c}{2a}}$$

Solving Lambert's Problem for Ellipses

If using a Lambert's arc for a transfer, need to calculate the orbital elements of the conic (*a*, *e*) connecting two position vectors and then the velocity at each of the two locations to compute the two maneuvers



Calculate *a* from Lambert's equation.

Calculate *e* from:

$$p = \frac{4a(s-r_1)(s-r_2)}{c^2}\sin^2\left(\frac{\alpha+\beta}{2}\right) = a(1-e^2)$$

Calculating Transfer Arc Along Hyperbola

For $\Delta\theta^* < 180^\circ$

$$TOF_h = \sqrt{\frac{a^3}{\mu}}\left(\sinh(\alpha_h) - \alpha_h - (\sinh(\beta_h) - \beta_h)\right)$$

For $\Delta\theta^* > 180^\circ$

$$TOF_h = \sqrt{\frac{a^3}{\mu}}\left(\sinh(\alpha_h) - \alpha_h + (\sinh(\beta_h) - \beta_h)\right)$$

Where:

$$\alpha_h = 2\sinh^{-1}\sqrt{\frac{s}{2|a|}} \qquad \beta_h = 2\sinh^{-1}\sqrt{\frac{s-c}{2|a|}}$$

Calculating Transfer Arc Along Parabola

For $\Delta\theta^* < 180^\circ$

$$TOF_p = \frac{1}{3}\sqrt{\frac{2}{\mu}}\left(s^{3/2} - (s-c)^{3/2}\right)$$

For $\Delta\theta^* > 180^\circ$

$$TOF_p = \frac{1}{3}\sqrt{\frac{2}{\mu}}\left(s^{3/2} + (s-c)^{3/2}\right)$$

Example

Let's design a transfer with a TOF of 5 hrs to connect the following two states in Earth-centered inertial coordinate system:

$$\bar{R}_1 = -654\hat{X} + 13605\hat{Y} + 1997\hat{Z} \quad \text{km}$$
$$\bar{V}_1 = -5.53\hat{X} + 0.849\hat{Y} + 0.6830\hat{Z} \quad \text{km/s}$$

$$\bar{R}_2 = 7284\hat{X} - 19341\hat{Y} - 3264\hat{Z} \quad \text{km}$$
$$\bar{V}_2 = 3.07\hat{X} + 2.63\hat{Y} + 0.444\hat{Z} \quad \text{km/s}$$

For a transfer angle greater than 180 degrees (design choice)

Example

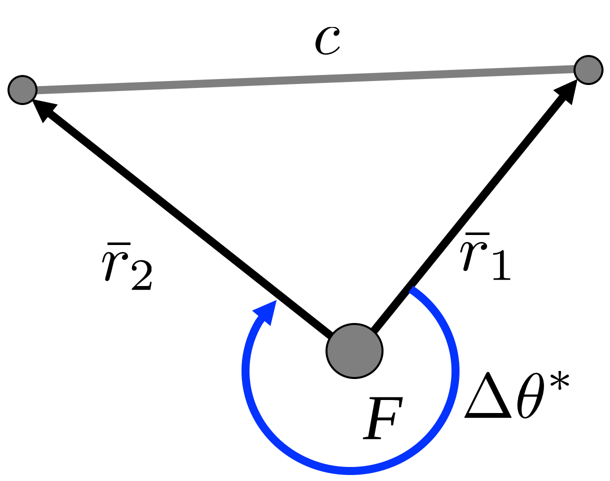
Step 1: Calculate the transfer angle

$$\Delta\theta^* = \pm\cos^{-1}\left(\frac{\bar{r}_1 \cdot \bar{r}_2}{|\bar{r}_1||\bar{r}_2|}\right) = 197.7^\circ = 360^\circ - 162.31^\circ$$

Step 2: Calculate geometric quantities

$$c = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\Delta\theta^*)} = 34{,}295 \text{ km}$$

$$s = 0.5(r_1 + r_2 + c) = 34{,}492 \text{ km}$$



Example

Step 3: Determine if transfer uses elliptical or hyperbolic orbit

Compare to parabolic transfer TOF for $\Delta\theta^* > 180^\circ$

$$TOF_p = \frac{1}{3}\sqrt{\frac{2}{\mu}}\left(s^{3/2} + (s-c)^{3/2}\right) = 1.329 \text{ hrs}$$

↳ Since desired TOF > TOF_p, a 5hr transfer between the 2 position vectors leverages a segment along an elliptical orbit

Example

Step 4: Determine if transfer requires shorter or longer TOF via comparison to TOF along the minimum energy transfer

Semi-major axis for min. energy transfer:

$$a_m = \frac{s}{2} = 17{,}246 \text{ km} \qquad n_m = \sqrt{\frac{\mu}{a_m^3}}$$

$$\alpha_m = \pi \qquad \beta_{m,0} = 2\sin^{-1}\sqrt{\frac{s-c}{s}}$$

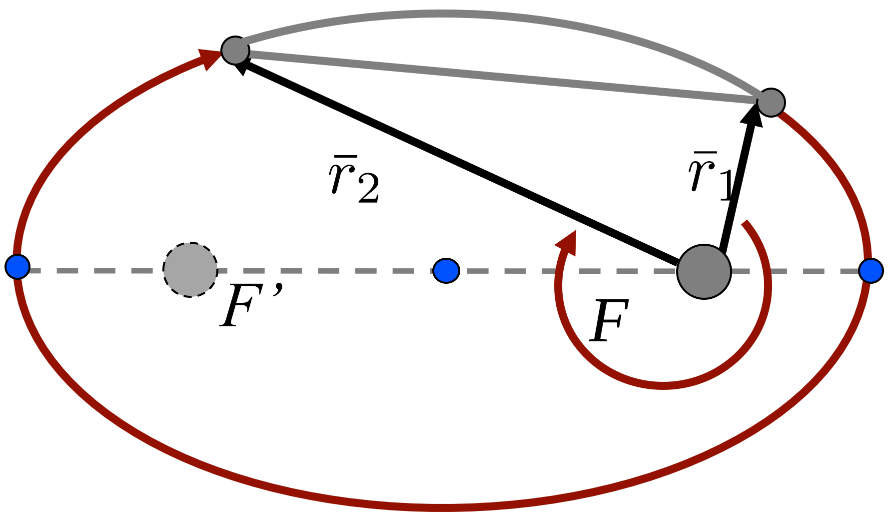
Since $\Delta\theta^* > 180°$ $\qquad \beta_m = -\beta_{m,0} = -0.1515$

$$TOF_{min} = \frac{1}{n_m}((\alpha_m - \beta_m) - (\sin(\alpha_m) - \sin(\beta_m))) = 3.13 \text{ hrs}$$

Since TOF > TOF_{min}, we can set correct quadrant of α

Example

	Shorter TOF TOF < TOF_{min}	Longer TOF TOF > TOF_{min}
$\Delta\theta^* < 180°$	$\alpha = \alpha_0 \quad \beta = \beta_0$	$\alpha = 2\pi - \alpha_0 \quad \beta = \beta_0$
$\Delta\theta^* > 180°$	$\alpha = \alpha_0 \quad \beta = -\beta_0$	$\boxed{\alpha = 2\pi - \alpha_0 \quad \beta = -\beta_0}$



$\Delta\theta^* > 180°$

$\alpha = 2\pi - \alpha_0$

$\beta = -\beta_0$

Long TOF

Example

Step 5: Iteratively solve the TOF equation to calculate *a*

- Iterate until $|TOF-TOF_i| <$ tol
- Initial guess a_m $+ \Delta a$ → what is a good value of Δa?
- Use correct α and β for this transfer type in Lambert's equation

$$\alpha_0 = 2\sin^{-1}\left(\sqrt{\frac{s}{2a}}\right) \qquad \beta_0 = 2\sin^{-1}\left(\sqrt{\frac{s-c}{2a}}\right)$$

- For this transfer example, $\alpha = 2\pi - \alpha_0$ $\quad \beta = -\beta_0$

$$TOF = \frac{1}{n}\left(\alpha - \beta - (\sin(\alpha) - \sin(\beta))\right)$$

Example

Step 6: After iterating,

$a =$ 19,001 km

Double check that this value is correct, by plugging into TOF eq:

$TOF =$ 4.9998 hrs

↳ ~10^{-4} hrs different from desired TOF – is that sufficiently accurate?

Example

Step 7: Find eccentricity via *p*

evaluate using $a = 19{,}001$ km

$$p = \frac{4a(s-r_1)(s-r_2)}{c^2}\sin^2\left(\frac{\alpha+\beta}{2}\right) = 17{,}175 \text{ km}$$

$$e = \sqrt{1-\frac{p}{a}} = 0.31$$

($e \geq 0$ so no sign check needed)

Example

Step 8: Find true anomaly at each location along transfer orbit

$$\theta_1^* = \pm\cos^{-1}\left(\frac{1}{e}\left(\frac{p}{r_1} - 1\right)\right) = \pm 37^\circ$$

$$\theta_2^* = \pm\cos^{-1}\left(\frac{1}{e}\left(\frac{p}{r_2} - 1\right)\right) = \pm 125.3^\circ$$

Check all combinations of $\theta_2^* - \theta_1^*$ to identify combination that equals transfer angle

$$\theta_2^* - \theta_1^* = \Delta\theta^* = 197.7^\circ$$

True only for: $\theta_1^* = 37^\circ$, $\theta_2^* = -125.3^\circ, 234.7^\circ$

Example

Step 9: Calculate the velocity vectors at each of t_1, t_2 before and after the maneuvers in a common frame

$$\vec{r}_2 = f\vec{r}_1 + g\vec{v}_1$$

$$\vec{v}_2 = \dot{f}\vec{r}_1 + \dot{g}\vec{v}_1$$

$$\vec{v}_{1,f} = -6.0331\,\hat{x} + 0.5490\,\hat{y} + 0.4824\,\hat{z} \text{ km/s}$$

$$\vec{v}_{2,i} = 3.2735\,\hat{x} + 2.5273\,\hat{y} + 0.1439\,\hat{z} \text{ km/s}$$

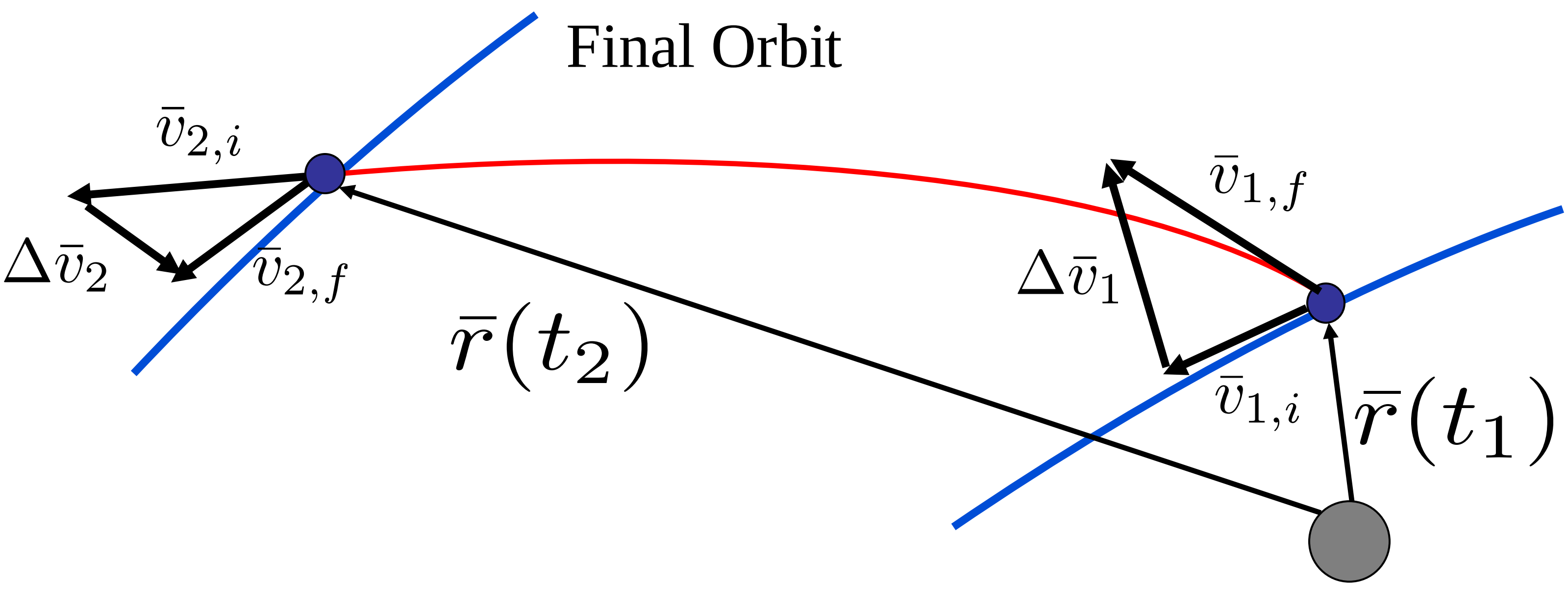
Optional: can also calculate other orbital elements when needed

Example

Step 10: Find the maneuvers required for the transfer

$\Delta\vec{v}_1 = \vec{v}_{1,f} - \vec{v}_{1,i} \rightarrow \Delta v_1 = 0.616$ km/s

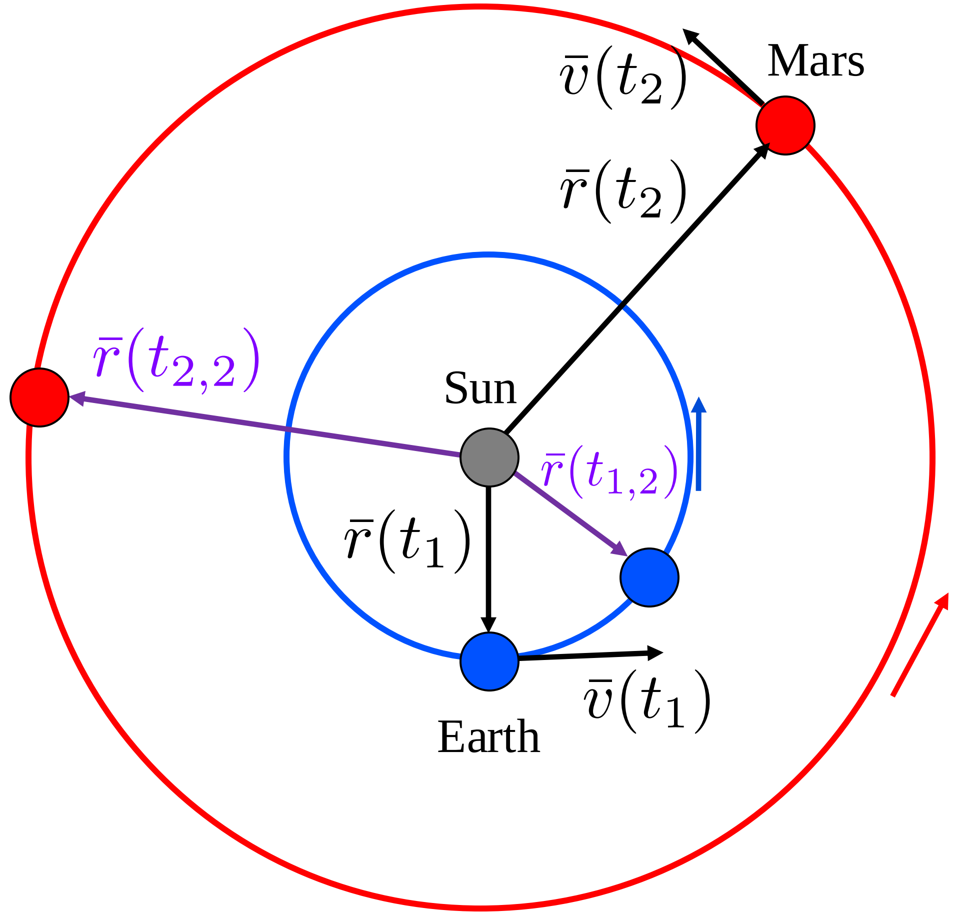
$\Delta\vec{v}_2 = \vec{v}_{2,f} - \vec{v}_{2,i} \rightarrow \Delta v_2 = 0.3742$ km/s



Interplanetary Transfers

Simplified analysis: match states of bodies in heliocentric orbits

Transfer design space between two orbits considers broader array of initial/final state vectors and, potentially, flight times



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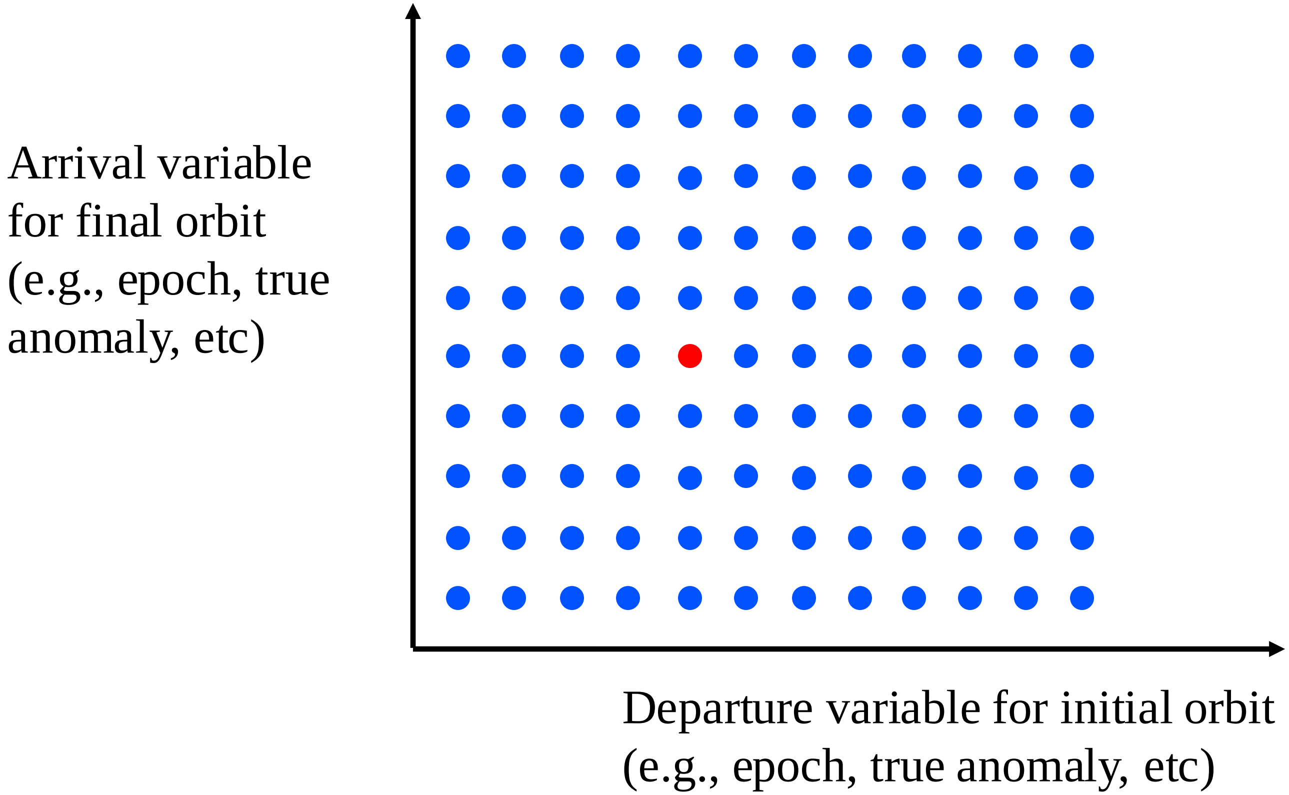
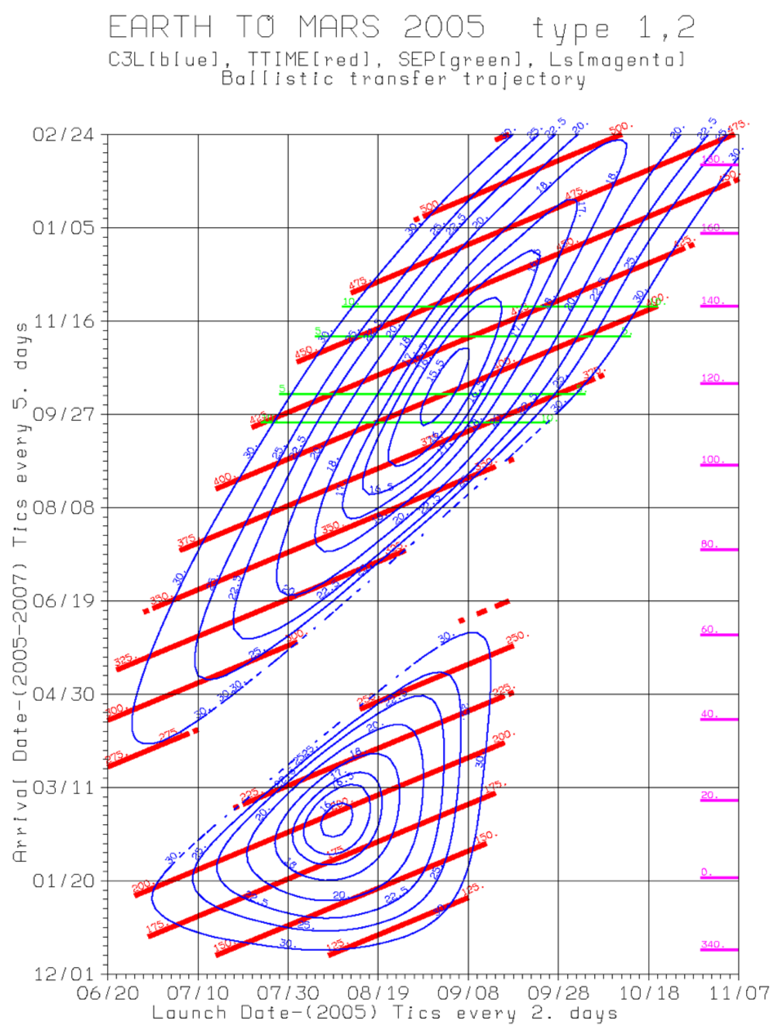


Characterizing Transfers

Properties of interest:

Porkchop Plot



Credit: NASA

Transfer Boundary Conditions