ASEN 5044
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AQ2 → Show → $|e^{At}| = e^{tr(A)t}$

Assume A is diagonalizable. V is matrix where columns of V are eigenvectors of A → $V^{-1}AV = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ where λ_n is eigenvalue of A

$tr(A)t = tr\left(P\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}P^{-1}\right)t = (\lambda_1 + \dots + \dots \lambda_n)t$

$e^{tr(A)t} = e^{\lambda_1 t + \lambda_n t}$

$e^{At} = P\begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix}P^{-1} \rightarrow det(e^{At}) = \cancel{det(P)} \cdot \cancel{det(P^{-1})} \cdot [e^{\lambda_1 t + \lambda_n t}]$

$\therefore e^{tr(A)t} = |e^{At}|$

Even if A is not diagonalizable, the jordan form of A is

$\hat{A} = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_n \end{bmatrix}$ and $e^{\hat{A}t} = \begin{bmatrix} e^{\lambda_1 t} & x & 0 \\ & \ddots & x \\ 0 & & e^{\lambda_n t} \end{bmatrix}$

$\therefore e^{At} = Qe^{\hat{A}t}Q^{-1} \rightarrow |e^{At}| = \cancel{|Q|} \cdot |e^{\hat{A}t}| \cancel{|Q^{-1}|}$ $\left(\because |Q^{-1}| = \frac{1}{|Q|}\right)$

$|e^{At}| = |e^{\hat{A}t}| = e^{(\lambda_1 t + \lambda_2 t + \dots \lambda_n t)}$

$tr(A) = \sum_{i=1}^{n} \lambda_n \rightarrow tr(A)t = \lambda_1 t + \lambda_2 t + \dots \lambda_n t$

$e^{tr(A)t} = e^{(\lambda_1 t + \lambda_2 t + \dots \lambda_n t)}$

$\therefore \boxed{|e^{At}| = e^{tr(A)t}}$