ASEN 6060
ADVANCED ASTRODYNAMICS
Alternative Derivations

Objectives:

- Brief overview of alternative approaches to deriving the EOMs for the CR3BP

Lagrangian Approach

Consider generalized coordinates $\bar{q}$ and the associated velocities $\dot{\bar{q}}$
Describe an autonomous dynamical system via Lagrangian function:

$$L(\bar{q}, \dot{\bar{q}}) = T - V$$

T = Kinetic energy, V = Potential energy

Integrating the continuous Lagrangian along a path from t_0=0 to t yields an action integral:

$$A = \int_{t_0}^{t} L(\bar{q}, \dot{\bar{q}}) dt$$

Lagrangian Approach

By Hamilton's principle, a path in a holonomic system corresponds to a stationary action integral with respect to path variations with fixed endpoints.

In an autonomous system, this is written as:

$$\delta A = \delta \int_{t_0}^{t} L(\bar{q}, \dot{\bar{q}}) dt = \int_{t_0}^{t} \sum_{i=1}^{m} \left[\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right) \delta q_i \right] dt = 0$$

The solution for a holonomic continuous-time system is the standard form of Euler-Lagrange equations:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0$$

Lagrangian Approach

In the rotating frame, the Lagrangian for the CR3BP is formulated by writing *T, V* in terms of nondimensional rotating frame coordinates to produce:

$$L = \frac{1}{2}\left((\dot{x} - y)^2 + (\dot{y} + x)^2 + \dot{z}^2\right) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

Evaluate the Euler-Lagrange equations in rotating frame, eg for *x*:

$$\frac{\partial L}{\partial x} = \dot{y} + x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3}$$

$$\frac{\partial L}{\partial \dot{x}} = \dot{x} - y \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \ddot{x} - \dot{y}$$

$$\ddot{x} = 2\dot{y} + x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3}$$

Hamiltonian Approach

Can formulate canonical equations by transforming the Lagrangian form into a Hamiltonian form via Legendre's transformation.

First, consider *m* associated conjugate momenta p_i defined as:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

And define a scalar Hamiltonian function as:

$$H = \sum_{i=1}^{m} p_i \dot{q}_i - L$$

Express velocities as a function of configuration space variables and conjugate momenta to rewrite $H(\bar{q}, \bar{p}, t)$

Hamiltonian Approach

The canonical equations for the Hamiltonian system are derived via:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Applying this approach to the CR3BP, the conjugate momenta are:

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} - y \qquad p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y} + x \qquad p_z = \frac{\partial L}{\partial \dot{z}} = \dot{z}$$

Then, the Hamiltonian is written as:

$$H(x, y, z, p_x, p_y, p_z) = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + yp_x - xp_y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}$$

Hamiltonian Approach

Then derive canonical equations as:

$$\dot{x} = \frac{\partial H}{\partial p_x} = p_x + y \qquad \dot{y} = \frac{\partial H}{\partial p_y} = p_y - x \qquad \dot{z} = \frac{\partial H}{\partial p_z} = p_z$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = p_y - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = -p_x - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3}$$