ASEN 5044, Fall 2024

Statistical Estimation for Dynamical Systems

Lecture 02:
Rapid Linear Algebra Refresher

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Thursday 08/29/2024

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Today

- Start reviewing important mathematical tools and concepts
- Quick refresher of linear algebra
  - highlights of n-dimensional matrix-vector concepts

START READING: Chapters 1.1 and 1.2 in Simon book;

Quiz 1 to be posted on Canvas tomorrow, due Tues

(out Fri 8/30/24 at 9 am, due Tues 9/03/24 at 10 am)

Vectors and vector operations in n-dimensions

- **Vectors:** *represented as* ordered list of elements (with respect to some basis set)

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \qquad v^T = \begin{bmatrix} v_1, & v_2, & \cdots & v_n \end{bmatrix},$$

v_j: j^{th} scalar element

$v_j \in \mathbb{R}$, $v \in \mathbb{R}^n \to [\mathbb{R}^n]$

$(\mathbb{R}^n \leftrightarrow \mathbb{R}^{n\times 1})$

- **Inner (dot) product:** is scalar $\alpha = a^T b$ for vectors $a, b \in \mathbb{R}^n$

$$\text{notation: } < a, b >= \alpha = \textstyle\sum_{j=1}^{n} a_j b_j = a_1 b_1 + \cdots + a_j b_j$$

- **Outer product:** way of describing alignment of elements of vectors:

if $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^m$ ($m \neq n$ possibly),

then the outer product of b and c is defined as $A = bc^T = \begin{bmatrix} b_1c_1 & b_1c_2 & \cdots & b_1c_m \\ b_2c_1 & b_2c_2 & \cdots & b_2c_m \\ \vdots & \vdots & \vdots & \vdots \\ b_nc_1 & b_nc_2 & \cdots & b_nc_m \end{bmatrix}$

Matrices

- Matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}] \in \mathbb{R}^{m \times n}$$

If $A = BC$, then $B \in \mathbb{R}^{m \times p}$ and $C \in \mathbb{R}^{p \times n}$, where $p \geq 1$

(i.e. inner dimensions of B and C **MUST** match! And outer dims must make sense to get A)

Also: $A^T = (BC)^T = C^T B^T$

(recall: $BC \neq CB$ in general, i.e. non-commutative)

- Trace: sum of diagonal entries for <u>square</u> <u>n x n</u> matrix A:

$$\text{tr}(A) = \sum_{i=1}^{n} a_{ii} \longrightarrow \text{scalar!}$$

Note: if $A \in \mathbb{R}^{n \times p}$ and $B \in \mathbb{R}^{p \times n}$, then $\text{tr}(AB) = \text{tr}(BA)$

Note: in general: $\text{tr}(ABC) = \text{tr}(CAB) = \text{tr}(BCA)$ for compatible A, B, C matrices

- Symmetric matrix: if A is $n \times n$, then A is symmetric if $A = A^T$

Linear dependence/independence, rank

- Set of vectors $\{v_1, v_2, \ldots, v_n\}$ is said to be **linearly dependent** if

$$\exists \text{ scalars } \alpha_j \neq 0, \; j = 1, \ldots, n, \text{ s.t. } v_i = \sum_{j \neq i} \alpha_j v_j \text{ for at least } 1 \; i = 1, \ldots, n$$

(i.e. at least one vector in the set equals a non-trivial **linear combination** of other vectors in the set)

- Vectors $\{v_1, v_2, \ldots, v_n\}$ are **linearly independent (LI)** if they are not linearly dependent

- Square matrix A is rank = n (full rank) if all its column vecs v_i (or row vecs r_i) are LI:

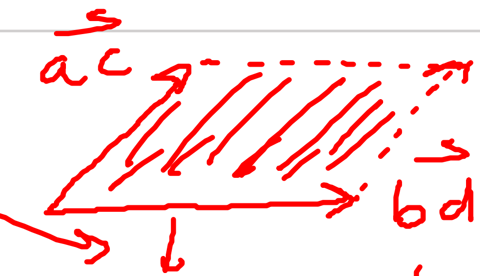
$$A = [v_1, v_2, \cdots, v_n] = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \qquad v_i \in \mathbb{R}^{n \times 1} \qquad r_i \in \mathbb{R}^{1 \times n}$$

for non-square $A \in \mathbb{R}^{m \times n}$, $\text{rank}(A) \leq \min(m, n)$

(i.e. square A is just a stacked set of nx1 column vectors or 1xn row vectors; if rank(A)=n, then vectors LI)

Determinants of Square Matrices

- 2x2 case: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $det(A) = |A| = ad - bc$



det(A) = volume of parallelopiped (in n-dims)

$$\rightarrow A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ if } |A| \neq 0$$

- General case: define the **cofactor:** $c_{ij} = (-1)^{i+j}|M_{ij}|$ (determinant of minor)

  where the **minor** is: $M_{ij} = A$ with row i and column j removed

→ so $det(A) = |A| = \sum_{i=1}^{n} a_{ij}c_{ij}, \ \forall j = 1, ..., n$ (cofactor expansion)

3x3 example:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 5 & 6 \end{bmatrix}$$

expand along 1st column:

$$|A| = 0 \cdot det(\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}) - (1) \cdot det(\begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}) + (0) \cdot det(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix})$$

$$= 4$$

Singular/Non-Singular Square Matrices

- A is **singular** if $|A| = 0$ (some rows/cols are LD)
- A is **non-singular** if $|A| \neq 0$ (i.e. if all rows/cols of A are linearly indep)
- Also, if $|A| = 0$, then $\exists x \neq 0$ such that $Ax = 0$

  But if $|A| \neq 0$, then $Ax = 0$ if and only if $x = 0$ (only trivial solution)

Vocabulary:

- **Singular** = **non-invertible** = **rank deficient** (i.e. rank(A) < n)
- **Non-singular** = **invertible** = **full-rank**

→ If A is non-singular, then $|A| \neq 0$ and $\exists\ A^{-1} \in \mathbb{R}^{n \times n}$ s.t. $AA^{-1} = A^{-1}A = I$

where the inverse of A is $A^{-1} = \frac{adj(A)}{det(A)} = \frac{C_A^T}{\underbrace{det(A)}_{|A|}}$ (C_A^T is transpose of matrix of cofactors)

adjugate

Solutions to "Nice" Linear Systems of Equations

- If $A \in \mathbb{R}^{n \times n}$ and $|A| \neq 0$, and $b \in \mathbb{R}^n$, then we can solve

$$Ax = b \text{ for } x \in \mathbb{R}^n$$

$$\rightarrow \boxed{x = A^{-1}b}$$

Recall: this tells us that x is the unique solution in R^n, because:

- A represents a "1 to 1" and "onto" linear transformation from R^n to R^n

>>**Range space** of A is R^n (if and only if = iff $|A| \neq 0$)

$$\text{Range}(A) = \{y \in \mathbb{R}^n |\ \exists x \in \mathbb{R}^n \ s.t. \ Ax = y\}$$

>>**Null space** of A is trivial (i.e. Null(A) only contains x=0) (iff: $|A| \neq 0$)

$$\text{Null}(A) = \{x \in \mathbb{R}^n |\ Ax = 0\}$$

Solutions to "Not Nice" Linear Systems of Eqs.?

- Consider an **"overdetermined"** system of eqs. (i.e. more rows than cols):

$$y = Mx, \qquad \text{where } y \in \mathbb{R}^m,\ x \in \mathbb{R}^n,\ M \in \mathbb{R}^{m\times n},\ m > n$$

$\rightarrow$ if $\text{rank}(M) = n$ (full col rank), then easy to show that the (square) **Gram matrix** G

$$G = M^T M \in \mathbb{R}^{n\times n} \text{ also has } \text{rank}(G) = n$$

$\rightarrow G$ is invertible (i.e. $|G| \neq 0$) $\rightarrow G^{-1}$ exists $\rightarrow G^{-1} = (M^T M)^{-1}$

$\rightarrow$ how to solve for x in original sys of eqs? First multiply $y = Mx$ by M^T on both sides:

$$M^T y = M^T M x \ \rightarrow M^T y = Gx \quad (\text{since } G = M^T M)$$

Now if $\text{rank}(M) = n$, then $G^{-1} = (M^T M)^{-1}$

So multiply by G^{-1} on both sides: $G^{-1} M^T y = G^{-1} G x \ \rightarrow G^{-1} M^T y = x$

therefore: $\boxed{x = G^{-1} M^T y = (M^T M)^{-1} M^T y,}$ $\boxed{\text{where } (M^T M)^{-1} M^T = M_L^+ \text{ is left pseudo-inverse}}$

Eigenvalues and Eigenvectors of Square Matrices

- **Given** $A \in \mathbb{R}^{n \times n}$, $\exists$ scalars λ_i (eigenvalues) (possibly complex numbers)

$$\text{such that } \exists \text{ associated eigenvectors } v_i \in \mathbb{R}^n \text{ (possibly complex, if } \lambda_i \text{ complex)}$$

$$\text{where } Av_i = \lambda_i v_i, \text{ where } v_i \neq 0 \text{ (by def.)}$$

$$\rightarrow \text{can solve for these via } (A - \lambda_i I)v_i = 0$$

$$\rightarrow \text{for non-trivial } v_i, \text{ want matrix } (A - \lambda_i I) = Q(\lambda_i) \text{ to be singular,}$$

$$\text{i.e. want } Q(\lambda_i)v_i = 0 \text{ for } v_i \neq 0$$

$$\rightarrow det(Q(\lambda_i)) = det(A - \lambda_i I) = 0$$

$\rightarrow$ gives the **characteristic polynomial** for A (polynomial in λ of order n)
$\rightarrow$ roots of the characteristic polynomial = eigenvalues of A
$\rightarrow$ n (complex conjugate) eigenvalues always exist

Handy Dandy Facts About E'vals/E'vecs

- FACT 1: For real-valued <u>symmetric</u> square matrices

  i.e. if $A = A^T$,

  then n eigenvalues are all real

  AND n eigenvectors exist which are all linearly independent and orthogonal

- FACT 2: For <u>any</u> square matrix A

$$tr(A) = \sum_{i=1}^{N} \lambda_i \text{ (trace is the sum of e'vals)}$$

$$det(A) = |A| = \prod_{i=1}^{N} \lambda_i \text{ (determinant is product of e'vals)}$$

so if $A \in \mathbb{R}^{n \times n}$ is singular, *at least* one e'val is 0

and corresponding e'vecs are basis for Null(A)!

Positive Definite (Symmetric) Matrices

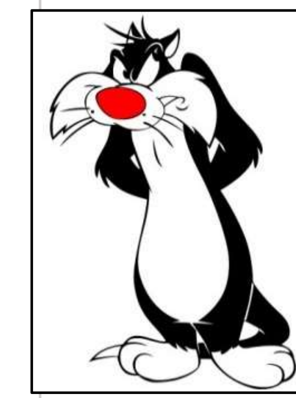
- **Definition:** Matrix $P \in \mathbb{R}^{n \times n}$ is positive definite (posdef) if:

$$x^T P x > 0 \quad \text{for all} \quad x \neq 0 \in \mathbb{R}^n$$

If P symmetric and posdef, then all e'vals of P positive : $\lambda_i(P) > 0, \; i = 1, \cdots, n$

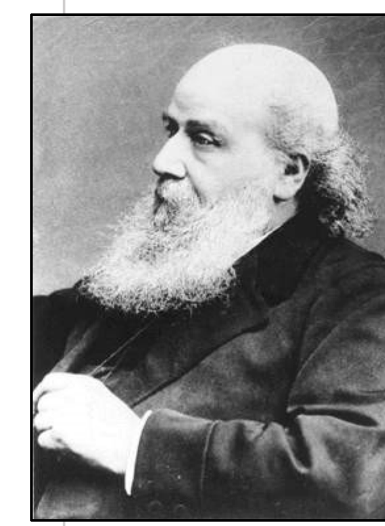
$\rightarrow$ We often need to verify that a matrix is posdef in computation,

but we don't necessarily want to compute all the e'vals of P to do so (expensive)!



Sylvester's method: check pos def'ness of *symmetric* P by examining if the

principal minors of P are all positive $\rightarrow$ if so, then P is posdef



i.e. let

$$P = P^T = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \rightarrow \text{to see if posdef, check:}$$

$p_{11} >_? 0,$ (1st principal minor)

$det(\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}) >_? 0,$ (2nd principal minor)

$\cdots$

$det(P) >_? 0,$ (nth principal minor)