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Optimal Four-Impulse Fixed-Time Rendezvous in the Vicinity of a Circular Orbit

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Minimum-fuel, multiple-impulse orbital rendezvous is investigated for the case in which the transfer time is specified (time-fixed case). A method for obtaining optimal solutions which is applicable to rendezvous or orbit transfer between elliptical orbits of low eccentricity is presented. In this method, optimal solutions are constructed by satisfying the necessary conditions for the primer vector. It is assumed that the terminal orbits lie close enough to an intermediate circular reference orbit that the linearized equations of motion can be used to describe the transfer. The linear boundary-value problem for the rendezvous is then solved analytically. As an application of the method, optimal four-impulse, fixed-time rendezvous transfers between coplanar circular orbits are obtained for a range of transfer times. These linearized solutions offer physical insight into the problem and provide approximate initial conditions that can be used in an iterative numerical solution to the nonlinear optimal multiple-impulse problem.

Nomenclature†

$(\dot{\ })$ = first derivative of ( ) with respect to time. Time derivatives of vectors are taken with respect to an inertial reference frame.
$(\ )'$ = differentiation with respect to dimensionless time τ
G = gravity gradient matrix
$\mathbf{h}_j$ = vector defined by Eq. (16)
H = (4×4) matrix having columns $\mathbf{h}_j$
I = identity matrix
k = auxiliary variable defined by Eq. (26)
m = auxiliary variable defined by Eq. (25)
$\mathbf{p}$ = primer vector
q_j = auxiliary variable defined by Eq. (27)
$\mathbf{r}$ = position vector
δR = nondimensional difference between final and initial circular orbit radii
t = time
t_p = phasing time, used in definition of τ
$\mathbf{u}_j$ = thrust unit vector of jth impulse
$\mathbf{v}$ = velocity vector ($\mathbf{v} = \dot{\mathbf{r}}$)
$\mathbf{\Delta V}_j$ = vector velocity change due to jth thrust impulse
$\mathbf{\Delta V}$ = vector defined by Eq. (11)
$\mathbf{w}_j$ = vector defined by Eq. (9)
W = (4×4) matrix having columns $\mathbf{w}_j$
$\mathbf{x}$ = state vector
α_j = dimensionless time interval defined by Eq. (18)

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† A boldface lower case letter denotes a column vector. The lightface letter denotes its scalar magnitude.

δ = first variation of variable which it precedes
Δ = a change in variable in precedes
θ = central angle
λ = radial component of primer vector
μ = circumferential component of primer vector
ν = out-of-plane component of primer vector
τ = dimensionless time, defined prior to Eq. (2)
Φ_{ji} = state transition matrix between times t_i and t_j
ω = mean motion of reference orbit

Subscripts

0 = initial value
F = final value
1,2,3... = numerical subscript denotes the number of the thrust impulse (first, second, third...)
H = half-transfer time [Eq. (13)]
j = the time of jth impulse
r = radial component
θ = circumferential component

I. Introduction

ONE aspect of the general problem of minimum-fuel trajectories in an inverse square gravitational field assumes a variable thrust rocket having constant exhaust velocity and unbounded thrust magnitude. Because of the extensive theoretical work done by Lawden,[1-3] this problem has been called Lawden's Problem. This investigation of minimum-fuel, multiple-impulse, fixed-time rendezvous is a special case of Lawden's Problem.

In a paper by Edelbaum,[4] the known solutions to Lawden's Problem are cited and discussed. Many important results have been obtained for optimal multiple-impulse transfer. Examples include the work of Lawden,[2] Breakwell,[5] the extensions of the Hohmann transfer made by Shternfeld,[6] Edelbaum,[7] and Hoelker and Silber,[8] optimal transfer between coaxial ellipses by Marchal,[9] optimal transfer in the vicinity of a circular orbit by Edelbaum,[10] and optimal transfer between hyperbolas by Gobetz[11] and by Marchal.[12] These and other solutions have been obtained for the optimal time-open transfer, in which the transfer time is unspecified.

Fewer results, in comparison, have been obtained for the time-fixed case. The publications in this area are relatively recent and include theoretical studies by Lawden,[3] Lion and Handelsman,[13] solutions for long transfer times by Marec.[14]

The objective of this investigation is to describe a method for obtaining fixed-time, minimum-fuel, multiple-impulse transfers between close orbits, and to apply the method to the specific case of coplanar circle-to-circle four impulse rendezvous. The circle-to-circle coplanar rendezvous is of interest since it is the simplest case that demonstrates the existence of time-fixed optimal transfers requiring more than two impulses. The linearized results obtained offer some physical insight into the optimal fixed-time rendezvous and provide approximate initial conditions for a numerical solution to the nonlinear problem.

II. Necessary Conditions for an Optimal Impulsive Transfer

The necessary conditions for an optimal impulsive orbit transfer are expressed conveniently in terms of the primer vector,[3] the vector of adjoint variables associated with the vehicle velocity vector. From the definition of the adjoint equations and the fact that the gravity gradient matrix is symmetric, one can show that the primer vector $\mathbf{p}$ satisfies the same differential equation as the first-order variation in the position vector $\delta\mathbf{r}$,[13,15] namely,

$$\ddot{\mathbf{p}} = G(\mathbf{r})\mathbf{p} \tag{1}$$

where $G(\mathbf{r})$ is the gravity gradient matrix evaluated along a reference solution to the nonlinear orbit equations.

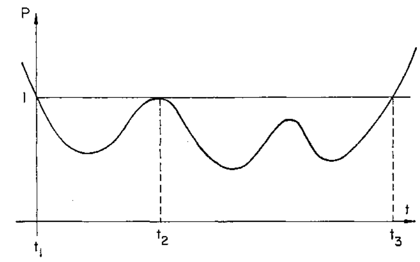

Fig. 1 Optimal three-impulse primer magnitude time history.

In terms of the primer vector the necessary conditions for an optimal impulsive transfer, first obtained by Lawden,[3] are: 1) the primer vector and its first time derivative must be continuous everywhere; 2) the thrust impulses must be applied in the direction of the primer vector at the times for which the magnitude p of the primer vector is unity; 3) the magnitude of the primer must not exceed unity during the transfer ($p \leq 1$); 4) as a consequence of these conditions, at impulse times which are not the initial or final time, $\dot{p} = 0$. A typical time history of a primer-vector magnitude for an optimal three-impulse transfer is shown in Fig. 1.

III. Application to Rendezvous between Neighboring Orbits

If one considers an orbital interception or rendezvous between orbits that lie close together, the vehicle's motion can be described by the equations of motion linearized about an intermediate reference orbit. In this case, the gravity gradient matrix (and therefore the primer vector) is evaluated along the reference orbit. To first order, the primer vector is then independent of the perturbed trajectory described by the linearized equations of motion. Therefore, the primer-vector equation and the linearized equation of motion can be solved separately, in contrast to a problem with a nonlinear state equation, in which the adjoint vector depends on the state.

Because of this separability, a solution of the primer-vector equation (1) which satisfies the necessary conditions for an optimum is independent of the boundary conditions for a specific interception or rendezvous. Once the optimal times of application and the directions of the thrust impulses are determined from the primer solution (necessary condition #2), the only additional information necessary to describe the transfer completely is the magnitude of each thrust impulse. These magnitudes depend on the specific boundary conditions of a transfer and are obtained by solving a linear boundary-value problem.

For a rendezvous between elliptical orbits of low eccentricity, a convenient reference trajectory is a circular orbit lying between the terminal orbits. For linear theory to be valid, both the inclination between the terminal orbits and the radial separation of the orbits relative to the semimajor axis of either orbit must be small. A circular reference is convenient because the linearized equations of motion and the primer-vector equations are analytically simple. However, closed-form analytical solutions for the linearized motion equations for elliptical reference orbits also are available[16] and could be useful in other applications.

IV. Construction of Primer Vector Solutions

Since the primer vector satisfies the same differential equation as the first-order variation in position [Eq. (1)], the primer vector along a circular orbit is described by the familiar circular-orbit variational equations (see Appendix). Let λ, μ, and ν denote the radial, circumferential, and out-

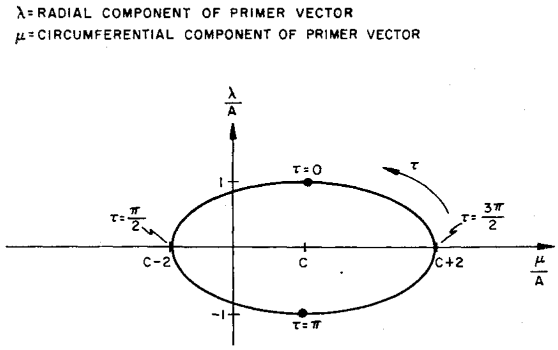

Fig. 2 Primer locus diagram for $B = 0$.

of-plane primer-vector components, respectively. Define a dimensionless time τ equal to $\omega(t - t_p)$ where ω is the mean motion of the circular reference orbit and t_p is an arbitrary constant. The primer vector along a circular orbit is then described by the equations

$$\lambda'' = 3\lambda + 2\mu' \tag{2}$$

$$\mu'' = -2\lambda' \tag{3}$$

$$\nu'' = -\nu \tag{4}$$

where a prime denotes differentiation with respect to τ.

The out-of-plane component ν is uncoupled from the in-plane components and satisfies a linear oscillator equation. The solution for the in-plane components can be expressed in the form

$$\lambda = A(\cos\tau + 2B) \tag{5}$$

$$\mu = A(-2\sin\tau - 3B\tau + C) \tag{6}$$

where A,B,C are arbitrary constants. The fourth arbitary constant t_p is included in the definition of τ.

Equations (5) and (6) form a parametric description of a locus in the $\lambda - \mu$ plane. For $B = 0$, the locus is a 1×2 ellipse shown in Fig. 2. For $B \neq 0$ the locus is the cycloid-like curve shown in Fig. 3. The arbitrary constants in Eqs. (5) and (6) characterize the locus as follows: A determines the scale of the plot, B characterizes the shape of the locus, and C positions the locus along the μ axis. Since the transfer is assumed to begin at $t = 0$, t_p corresponds to the initial value of $\tau(\tau_0 = \omega t_p)$.

For a planar problem, solutions to the primer-vector equation which satisfy the necessary conditions for an optimal transfer can be constructed geometrically using the primer locus diagram. For a specified fixed nondimensional transfer time $\tau_F - \tau_0$ the necessary conditions (Sec. II) require that the locus be smooth, that the locus lie on or inside a unit circle centered at the origin of the $\lambda - \mu$ plane ($p \leq 1$ for all $\tau \epsilon [\tau_0, \tau_F]$), and that the locus be tangent to the unit circle for impulses not at the terminal times ($p' = 0$). The optimum times to apply the impulses are those times for which

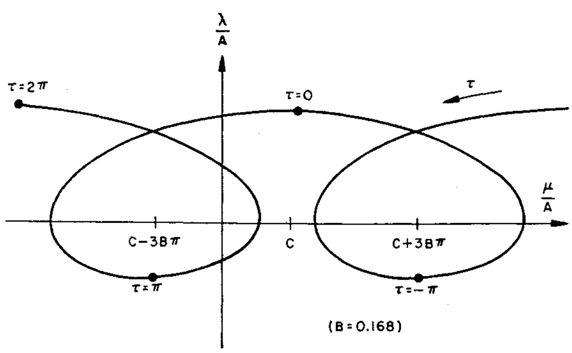

Fig. 3 Primer locus diagram for $B > 0$.

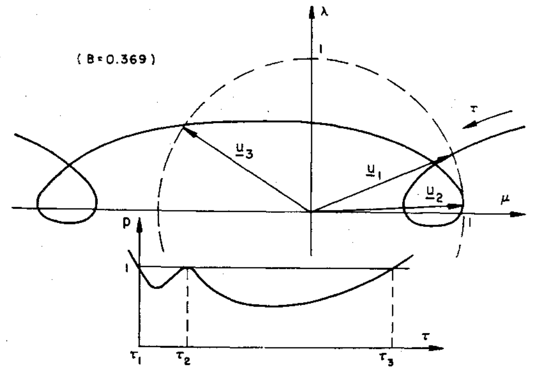

Fig. 4 Primer locus construction for three-impulse optimal transfer.

the locus intersects the unit circle ($p = 1$); the thrust directions are given by the λ and μ components at these times.

Example constructions of three- and four-impulse optimal primer vector solutions are shown in Figs. 4 and 5, respectively. The $\mathbf{u}_i$ are the unit vectors in the direction of the thrust. The subscript i refers to the number of the impulse (first, second, etc.).

V. Rendezvous Boundary-Value Problem

In terms of the linearized model for the motion of the vehicle, the effect of several thrust impulses during a transfer is obtained by superposition. Define a six-component state variation $\delta\mathbf{x}$, composed of the three components of the vehicle's position variation $\delta\mathbf{r}$, and the three components of its velocity variation $\delta\mathbf{v}$

$$\delta\mathbf{x} \triangleq \begin{bmatrix} \delta\mathbf{r} \\ \delta\mathbf{v} \end{bmatrix} \tag{7}$$

Let $\delta\mathbf{x}_0$ denote the state variation of the vehicle at the initial time, $t = 0$; let $\delta\mathbf{x}_F$ denote the final state variation,

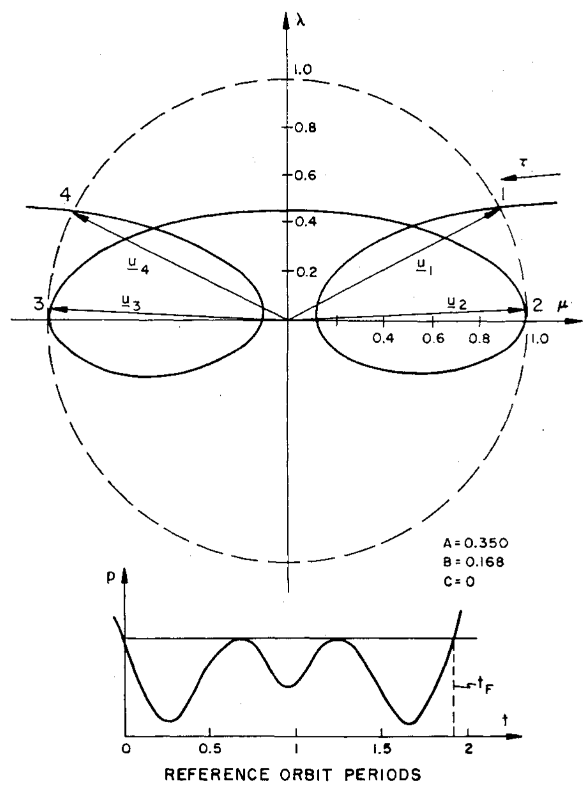

Fig. 5 "Two-loop" four-impulse primer solution.

which for rendezvous is the state variation of the target at the final time. In terms of the n velocity changes, $\mathbf{\Delta V}_j$, caused by the thrust impulses at times t_j ($j = 1,2,\ldots,n$) the boundary-value equation is

$$\delta \mathbf{x}_F = \Phi_{F0}\delta \mathbf{x}_0 + W(t_j, \mathbf{u}_j)\mathbf{\Delta V} \tag{8}$$

where Φ_{F0} is the state transition matrix of the system, W is a matrix, the jth column of which is

$$\mathbf{w}_j = \Phi_{Fj}\begin{bmatrix} 0 \\ I \end{bmatrix}\mathbf{u}_j \tag{9}$$

I is a 3×3 identity matrix and the $\mathbf{u}_j$ are thrust unit vectors

$$\mathbf{u}_j = \mathbf{\Delta V}_j / \Delta V_j \tag{10}$$

$\mathbf{\Delta V}$ is a vector having as components the magnitudes of the n velocity changes;

$$\mathbf{\Delta V} = \begin{bmatrix} \Delta V_1 \\ \Delta V_2 \\ \cdot \\ \cdot \\ \cdot \\ \Delta V_n \\ \cdot \end{bmatrix} \tag{11}$$

The matrix W is a function of the times of the impulses and their directions. These t_j and $\mathbf{u}_j$ are obtained directly from the primer-locus construction. The vector $\mathbf{w}_j$ has the physical interpretation of being the change in the final state variation vector due to a unit-magnitude velocity change at the optimal time t_j in the optimal direction $\mathbf{u}_j$ (the direction of the primer vector).

For a specific transfer time the initial state variation and the desired final state variation are known. The W matrix is constructed from the optimization. The solution to the boundary-value problem is then the $\mathbf{\Delta V}$, which will accomplish the optimal rendezvous.

In terms of the n components of $\mathbf{\Delta V}$ and a q-component state vector, Eq. (8) represents q equations in the n unknowns.

However, not all $\mathbf{\Delta V}$ solutions are admissible, since some components of $\mathbf{\Delta V}$ may be negative. Since the line in space along which the thrust is to be directed is constrained by the primer vector direction, a negative component of $\mathbf{\Delta V}$ requires that the thrust be directed in a sense opposite to the primer vector. This solution will satisfy the boundary conditions, but violate a necessary condition for an optimal transfer and is therefore inadmissible. The condition for an admissible solution is then

$$\Delta V_j \geq 0, \quad j = 1,2,\ldots,n \tag{12}$$

VI. Optimal Four-Impulse Coplanar Rendezvous

As an application of the outlined method of constructing optimal transfers, consider a four-impulse transfer between close coplanar orbits. Optimal four-impulse solutions are of interest since four impulses is the maximum number necessary to realize a minimum-fuel solution to the linearized, coplanar, time-fixed rendezvous problem. This stems from a result obtained by Neustadt[17] and by Potter,[18] that for a linear system the maximum number of impulses necessary to realize an optimal transfer is the number of constraints on the state variables at the specified final time. For a two-dimensional rendezvous transfer, all four vehicle state variables (two components of position, two of velocity) are constrained to be those of the target at the final time. Therefore, four impulses is the maximum number necessary to perform a minimum-fuel coplanar rendezvous. For a three-dimensional rendezvous as many as six impulses may be required.

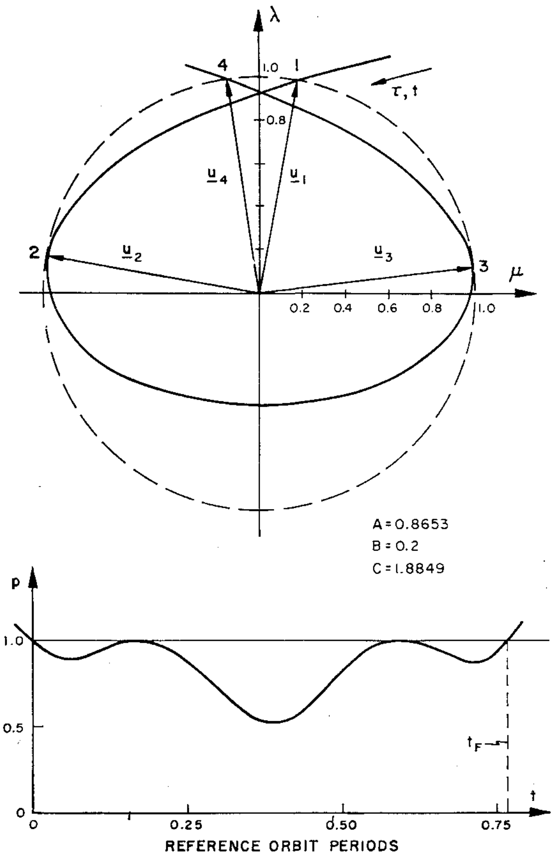

Fig. 6 "One-loop" four-impulse primer solution.

Figures 5 and 6 demonstrate two types of primer-locus constructions that satisfy the necessary conditions. To obtain the necessary tangencies of the locus with the unit circle, the center of the circle must lie at a point on the μ axis, which is either between two loops of the locus (Fig. 5) or in the center of a loop (Fig. 6). This geometrical consideration results in the following symmetry properties for optimal four-impulse solutions: 1) the impulse times are symmetric about the time halfway through the transfer; 2) the first and fourth impulse directions have equal radial (λ) components and equal but opposite circumferential (μ) components. The second and third impulse directions also satisfy this property.

The four-impulse primer solutions obtained exist in families characterized by the number of loops of the locus contained within the unit circle in the $\lambda - \mu$ plane. A computer program was written to construct primer solutions for two of these families, the "one-loop" family (Fig. 6) and the "two-loop" family (Fig. 5). A given family of solutions was generated by choosing the appropriate value of the constant C in Eqs. (5) and (6), and incrementing the shape constant B to cover all cases of interest; in each construction the value of the scaling constant A is chosen to make the tangency circle have unit magnitude. For a given family of solutions, the value of the constant B corresponds to the value of the fixed transfer time. The value of the transfer time is then an output of the construction process.

Each family of primer-vector solutions exists for a range of transfer times t_F. (The final time t_F is the transfer time since the initial time is chosen to be $t = 0$.) One-loop solutions exist for $0.46 \leq t_F \leq 1.5$ reference-orbit periods. Two-loop solutions exist for $1 \leq t_F \leq 2.5$ reference-orbit periods.

For these solutions Fig. 7 displays the times of the impulses for an optimal fixed-time transfer. Figures 8 and 9 show the elevation angles of the optimal thrust directions. The elevation angle is measured with respect to the positive μ

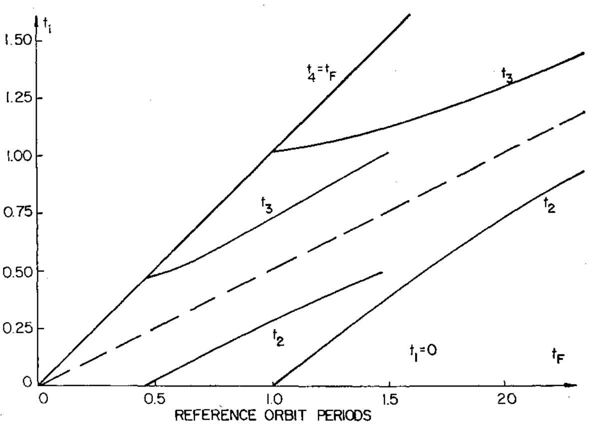

Fig. 7 Impulse times for two families of four-impulse solutions.

axis (local horizontal) and has a positive sense in the first quadrant of the $\lambda - \mu$ plane.

Due to the symmetry properties of the four-impulse primer solutions, it is convenient to measure time with respect to the half-transfer time t_H

where

$$t_H \triangleq t_F/2 \tag{13}$$

Then the boundary-value equation (8) appears as

$$\delta \mathbf{x}_H - \Phi_{H0}\delta\mathbf{x}_0 = H\Delta\mathbf{V} \tag{14}$$

where $\delta\mathbf{x}_H$ is the state variation of the target at time t_H

$$\delta\mathbf{x}_H = \Phi_{HF}\delta\mathbf{x}_F \tag{15}$$

and the matrix H is formed by columns

$$\mathbf{h}_j = \begin{bmatrix} h_{1j} \\ h_{2j} \\ h_{3j} \\ h_{4j} \end{bmatrix}, \qquad \Phi_{Hj} = \begin{bmatrix} 0 \\ I \end{bmatrix} \mathbf{u}_j \tag{16}$$

For a circular reference orbit, substitution of the appropriate partitions of the state transition matrix (see Appendix) yields

$$\mathbf{h}_j = \begin{bmatrix} \sin\alpha_j & 2(1-\cos\alpha_j) \\ -2(1-\cos\alpha_j) & 4\sin\alpha_j - 3\alpha_j \\ \cos\alpha_j & 2\sin\alpha_j \\ -2\sin\alpha_j & 4\cos\alpha_j - 3 \end{bmatrix} \begin{bmatrix} u_{jr} \\ u_{j\theta} \end{bmatrix} \tag{17}$$

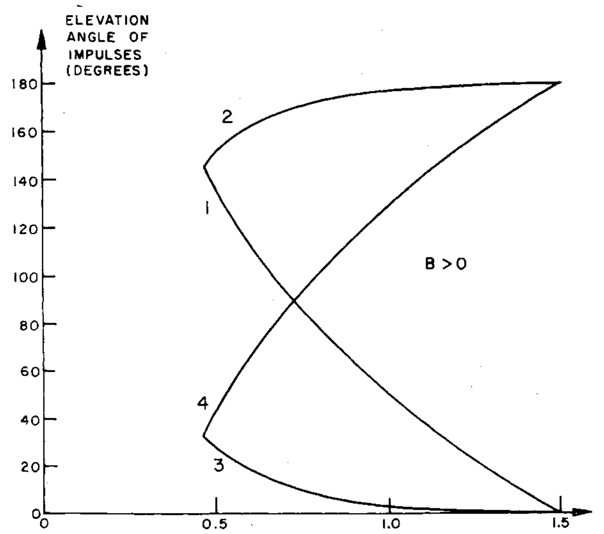

Fig. 8 Impulse elevation angle for one-loop solutions.

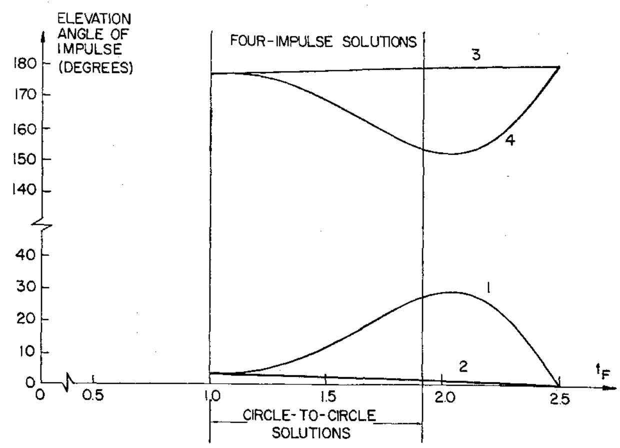

Fig. 9 Impulse elevation angle for two-loop solutions.

where the subscripts r and θ refer to the thrust-unit vector components in the radial and circumferential directions, respectively, and

$$\alpha_j = \tau_H - \tau_j = \omega(t_H - t_j) \tag{18}$$

In these variables the symmetry properties of the four-impulse primer solutions can be expressed as

$$\alpha_4 = -\alpha_1, \qquad \alpha_3 = -\alpha_2 \tag{19}$$

$$\mathbf{u}_4 = \begin{bmatrix} u_{4r} \\ u_{4\theta} \end{bmatrix} = \begin{bmatrix} u_{1r} \\ -u_{1\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{u}_1 \tag{20}$$

$$\mathbf{u}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{u}_2 \tag{21}$$

These properties result in the simplified form for H

$$H = \begin{bmatrix} h_{11} & h_{12} & -h_{12} & -h_{11} \\ h_{21} & h_{22} & h_{22} & h_{21} \\ h_{31} & h_{32} & h_{32} & h_{31} \\ h_{41} & h_{42} & -h_{42} & -h_{41} \end{bmatrix} \tag{22}$$

VII. Circle-to-Circle Coplanar Four-Impulse Solutions

As shown in the Appendix, for a circular reference orbit (of unit radius and unit mean motion) lying halfway between the terminal circular orbits, the final and initial nondimensional state variations are given by

$$\delta\mathbf{x}_F = \begin{bmatrix} \delta R/2 \\ \delta\theta_F \\ 0 \\ -\frac{3}{4}\delta R \end{bmatrix}, \quad \delta\mathbf{x}_0 = \begin{bmatrix} -\delta R/2 \\ 0 \\ 0 \\ \frac{3}{4}\delta R \end{bmatrix} \tag{23}$$

Where δR is the nondimensional algebraic difference between the final and initial terminal orbit radii, and $\delta\theta_F$ is the final variation in central angle (see Fig. 10).

For these initial and final state variations, Eq. (14) can be solved for the required velocity changes to yield

$$\Delta\mathbf{V} = \begin{bmatrix} \Delta V_1 \\ \Delta V_2 \\ \Delta V_3 \\ \Delta V_4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} q_2/m & -2(h_{32}/k) \\ -q_1/m & 2(h_{31}/k) \\ q_1/m & 2(h_{31}/k) \\ -q_2/m & -2(h_{32}/k) \end{bmatrix} \begin{bmatrix} \delta R \\ \delta\theta_F \end{bmatrix} \tag{24}$$

$$m = h_{11}h_{42} - h_{12}h_{41} \tag{25}$$

$$k = h_{22}h_{31} - h_{21}h_{32} \tag{26}$$

$$q_j = 2h_{4j} + 3h_{1j}, \qquad (j = 1,2) \tag{27}$$

The solution of Eq. (24) exists and is unique if and only if the matrix H is nonsingular. The determinant of H, equal to $4mk$, is found to vanish only at the maximum and minimum

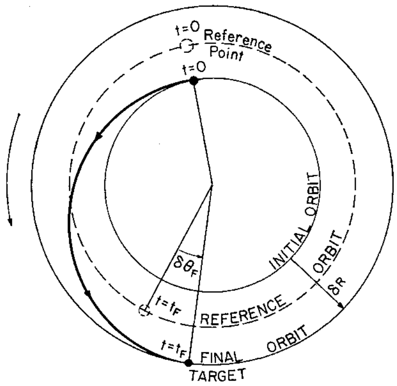

Fig. 10 Definition of state variables.

transfer times for each of the two families of primer solutions investigated.

Of the solutions given by Eq. (24) the one-loop primer solutions violate the admissibility condition of Eq. (12). However, two-loop solutions for $1 \leq t_F \leq 1.92$ reference-orbit periods yield admissible solutions, as shown in Fig. 9.

A simple test for admissibility can be made without first solving the boundary-value equation for $\mathbf{\Delta V}$. For the H matrix of Eq. (22) and the terminal state variations of Eq. (23), the third row of matrix Eq. (14) becomes

$$0 = h_{31}\Delta V_1 + h_{32}\Delta V_2 + h_{32}\Delta V_3 + h_{31}\Delta V_4 \qquad (28)$$

For an admissible solution it is necessary that

$$\operatorname{sgn}(h_{31}) = -\operatorname{sgn}(h_{32}) \qquad (29)$$

The admissibility condition also restricts the range of values of δR and $\delta\theta_F$ for which admissible circle-to-circle four-impulse rendezvous are possible. Figure 11 displays those final states, in terms of δR and $\delta\theta_F$, which are attainable by optimal four-impulse rendezvous transfers. Due to the symmetry introduced by choosing the reference orbit to lie half-way between the terminal orbits $|\delta\theta_F/\delta R|$ is plotted as a function of the fixed transfer time. The dashed vertical lines represent values of the transfer time which are boundaries for these four-impulse transfer rendezvous. These boundary transfer times permit optimal transfer only if $\delta R = 0$.

In Fig. 11 lines of constant $\Delta V/\delta R$ are shown. ΔV is the total fuel cost of the rendezvous (nondimensionalized by being expressed in fractions of reference orbital velocity).

Figure 12 shows a comparison between an optimal four-impulse rendezvous transfer and the two-impulse transfer corresponding to the same transfer time ($t_F = 1.65$ reference-orbit periods). The trajectories are shown in a rotating frame in which the target is fixed at the origin.

The four-impulse trajectory in this figure is characterized by a primer vector of the form of Fig. 5, with the modification that the thrust unit vectors are the negatives of those shown in Fig. 5. Thus, the first two impulses are primarily retro-impulses, directed contrary to the direction of motion (negative μ-direction). Similarly, the third and fourth thrust impulses are predominantly in the direction of motion.

The physical effect of these impulses is shown in Fig. 12. The first impulse slows the vehicle down in its orbital motion, causing it to fall inside its initial orbit. The second impulse slows it down further causing it to remain inside rather than be thrown out to a larger radius by the increased centrifugal acceleration. In this particular example, the second impulse has very nearly circularized the trajectory inside the initial orbit. The third impulse, occurring after a coasting period that decreases the phase angle between the vehicle and target, speeds up the vehicle causing it to proceed to a larger radius and intercept the target. The fourth impulse speeds up the vehicle further, matching the vehicle orbital velocity with that of the target, to effect rendezvous.

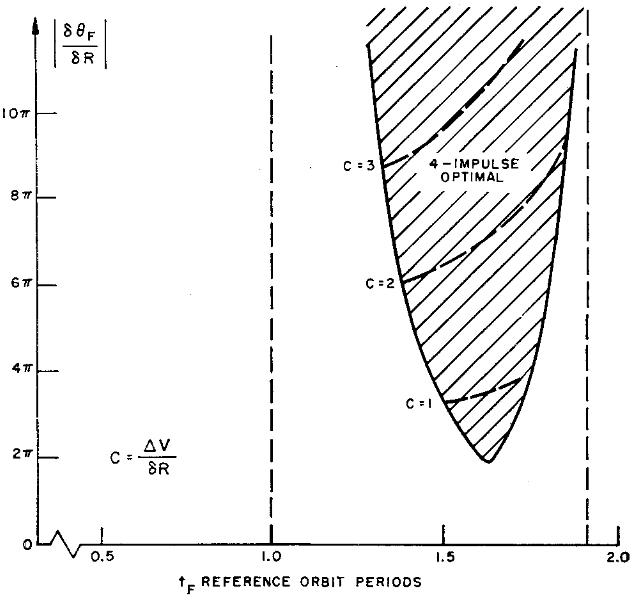

Fig. 11 Reachable final state variations for optimal four-impulse rendezvous.

As shown in Fig. 12, the two-impulse solution corresponding to the same transfer time requires thrust impulses that are directed predominantly along the local vertical direction. This results in an expensive transfer in terms of fuel; as shown for this specific transfer time, the optimal four-impulse solution uses 55% less ΔV than the two-impulse solution based on the same linear model.

VIII. Degenerate Primer-Vector Solutions

Another class of primer-vector solutions to be considered is the case for which the primer-vector magnitude is identically unity. This case most easily is described by writing the primer-vector solution [Eqs. (5) and (6)] in the alternative form:

$$\lambda = A^{\#} \cos\omega t + B^{\#} \sin\omega t + 2C^{\#} \qquad (30)$$

$$\mu = 2B^{\#}\cos\omega t - 2A^{\#}\sin\omega t - 3C^{\#}\omega t + D^{\#} \qquad (31)$$

where the arbitrary constants of Sec. IV are related to the

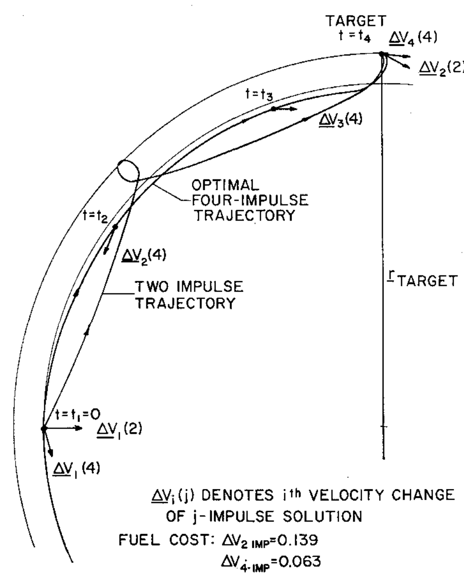

Fig. 12 Comparison between optimal four- and non-optimal two-impulse rendezvous.

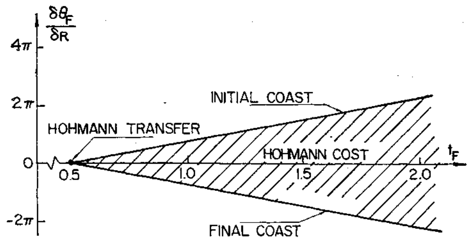

Fig. 13 Reachable final states for degenerate solutions.

foregoing constants by

$$A = (A^{\#2} + B^{\#2})^{1/2} \tag{32}$$

$$B = C^{\#}/A \tag{33}$$

$$t_p = (1/\omega)\tan^{-1}(B^{\#}/A^{\#}) \tag{34}$$

$$C = (D^{\#} - 3C^{\#}\omega t_p)/A \tag{35}$$

By choosing the arbitrary constants such that

$$A^{\#} = B^{\#} = C^{\#} = 0 \tag{36}$$

$$D^{\#} = 1 \tag{37}$$

one obtains a unit magnitude primer-vector solution, which is represented by a point in the $(\lambda - \mu)$ plane located at (0, 1).

This primer solution satisfies the necessary conditions of Sec. II, but the number of impulses and the optimal times of the impulses are not determined by the primer vector. The direction of the impulses, however, is always along the local horizontal.

$$\mathbf{u}_j = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{all } j \tag{38}$$

Solutions for which the unit vectors are the negative of those of Eq. (38) also satisfy the necessary conditions.

To facilitate obtaining four-impulse solutions corresponding to this type of primer-vector solution, the two-impulse solution is first examined. For the unit vectors of Eq. (38) and the boundary conditions of Eq. (23) the boundary-value equation (8) yields the admissible solution for the velocity change magnitudes,

$$\Delta V_1 = \Delta V_2 = \delta R/4 \tag{39}$$

for which

$$\delta\theta_F = 0 \tag{40}$$

$$t_F = (2l + 1)\,\pi, \qquad l = 0,1,2\ldots \tag{41}$$

For $l = 0$ this is the linearized form of the Hohmann transfer. Equation (40) results from the fact that the period of the Hohmann-transfer ellipse is the same as the reference-orbit period. $l = 1,2,3,\ldots$ corresponds to waiting to apply the second impulse $1,2,3,\ldots$ full reference-orbit periods after the time of the second impulse for a Hohmann transfer.

Figure 13 shows the final states reached by the linearized Hohmann transfer. Since the magnitude of the primer vector never exceeds unity for this type of transfer, unlimited coasting periods before the first impulse and after the last impulse are fuel optimal. By coasting in the initial circular orbit before performing a Hohmann transfer, any point on the line labelled "initial coast" can be reached. The slope of this line is $\frac{3}{4}$, the value of $\delta\dot{\theta}/\delta R$ for a body in the initial orbit. A similar line exists for a coast in the final orbit, as shown in the figure.

Any point in the wedge formed by the two coasting lines can be reached by a unique combination of initial coast, Hohmann transfer, final coast. All rendezvous lying in this wedge are achieved by the fuel cost of a Hohmann transfer. This wedge-shaped region is the same as that shown in Fig. 7 of Ref. 13.

By analogy with the two-impulse (Hohmann) transfer corresponding to an identically unity magnitude primer vector, one investigates four-impulse solutions for which the impulses are separated by π radians in nondimensional time, with thrust unit vectors given by Eq. (38). The nondimensional transfer time is then 3π or 1.5 reference-orbit periods. For this case the boundary-value equation (8) has the form

$$\begin{bmatrix} \delta R \\ \delta\tilde{\theta}_F \\ 0 \\ -\frac{3}{2}\delta R \end{bmatrix} = \begin{bmatrix} 4 & 0 & 4 & 0 \\ -9\pi & -6\pi & -3\pi & 0 \\ 0 & 0 & 0 & 0 \\ -7 & 1 & -7 & 1 \end{bmatrix} \begin{bmatrix} \Delta V_1 \\ \Delta V_2 \\ \Delta V_3 \\ \Delta V_4 \end{bmatrix} \tag{42}$$

where

$$\delta\tilde{\theta}_F \triangleq \delta\theta_F - \tfrac{3}{4}\delta R t_F$$

Since the foregoing coefficient matrix is singular, the solution for the velocity changes is nonunique. The solution can be expressed in terms of an arbitrary constant c as

$$\Delta V_1 = c \tag{43}$$

$$\Delta V_2 = \delta R/4 - c - \delta\theta_F/6\pi \tag{44}$$

$$\Delta V_3 = \delta R/4 - c \tag{45}$$

$$\Delta V_4 = c + \delta\theta_F/6\pi \tag{46}$$

The admissibility condition [Eq. (12)] requires that

$$0 \le c \le \delta R/4 \tag{47}$$

and, therefore,

$$-(3\pi/2)\delta R \le \delta\theta_F \le \tfrac{3}{2}\,\pi\delta R \tag{48}$$

Since that nondimensional transfer time for this maneuver is 3π, transfers that satisfy Eq. (48) lie within the Hohmann wedge of Fig. 13. In addition, the cost of an admissible four-impulse rendezvous is the cost of a Hohmann transfer, $\delta R/2$.

Physically the first and third impulses (applied a full reference orbit apart) together supply a full-strength Hohmann impulse of magnitude $\delta R/4$. The second and fourth impulses together also constitute a full-strength Hohmann impulse and provide the correct phasing, in the $\delta\theta_F$ term, to allow rendezvous.

Summary

A method for constructing linearized, fixed-time, minimum-fuel, multiple-impulse trajectories has been presented and applied to the case of circle-to-circle, coplanar, four-impulse rendezvous. A family of four-impulse solutions, representing the maximum number of impulses necessary to effect minimum-fuel rendezvous for the linearized coplanar problem, exists for transfer times between 1 and 1.92 reference orbit periods. The reachable final states for these rendezvous are shown in Fig. 11 and an example trajectory is displayed in Fig. 12. In addition, a class of degenerate optimal four-impulse solutions exists which can also be accomplished with the same fuel cost and transfer time by a two-impulse Hohmann transfer using coasting periods in the initial and final orbits.

Appendix

The circular-orbit variational equations are well documented (e.g., Ref. 20). The motion of a vehicle that experiences small deviations from a circular orbit of radius a and mean motion ω is described by the equations

$$\delta\ddot{r} = 3\omega^2\delta r + 2a\omega\delta\dot{\theta} \tag{A1}$$

$$a\delta\ddot{\theta} = -2\omega\delta\dot{r} \quad \text{(A2)}$$

$$\delta\ddot{z} = -\omega^2\delta z \quad \text{(A3)}$$

where δr, $a\delta\theta$, and δz are the first-order variations in the radial, circumferential, and out-of-plane components of position variation. In order for linear theory to be valid, the velocity variations, as well as the position components $\delta r/a$ and δz must be small. $\delta\theta$ is unrestricted since the gravitational attraction is independent of θ. Thus, for the linear approximations to be valid, the motion is confined to a torus about the reference orbit.

These equations are conveniently nondimensionalized by choosing units such that $a = \omega = 1$. Distance then is measured in fractions of reference orbit radius, and velocity is measured in fractions of reference orbital velocity. In terms of the foregoing variables, a state vector [Eq. (7)] can be defined as

$$\delta\mathbf{x} = \begin{bmatrix} \delta\mathbf{r} \\ \delta\mathbf{v} \end{bmatrix} = \begin{bmatrix} \delta r \\ \delta\theta \\ \delta z \\ \delta\dot{r} \\ \delta\dot{\theta} \\ \delta\dot{z} \end{bmatrix} \quad \text{(A4)}$$

The solution to Eqs. (A1–A3) can then be expressed in terms of the state transition matrix, $\Phi_{\tau\tau_0}$

$$\delta\mathbf{x}(\tau) = \Phi_{\tau\tau_0}\delta\mathbf{x}(\tau_0) \quad \text{(A5)}$$

The elements of this matrix are given in Ref. 20.

The state vector of a vehicle in a circular orbit of dimensionless radius $1 + \delta a$ coplanar with the reference orbit is obtained by finding a solution to Eqs. (A1) and (A2) for which δr and $\delta\dot{\theta}$ are constants. This results in

$$\delta r = \delta a, \qquad \delta\dot{\theta} = -\tfrac{3}{2}\delta a$$

The planar state variation of a body in a circular orbit of radius $1 + \delta a$ is then

$$\delta\mathbf{x}(\tau) = \begin{bmatrix} \delta a \\ \delta\theta(\tau) \\ 0 \\ -\frac{3}{2}\delta a \end{bmatrix}$$

This result is used in Sec. VII to describe the terminal states for a circle-to-circle transfer.

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