(1)

ASEN 5010
Spring 2025
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$\boxed{\text{HW \#4}}$

Problem 1 / 4.11 → a) $I_a = 2I_t$, $I_1 = I_a = 2I_t$, $I_2 = I_3 = I_t$

$$H^2 = H_1^2 + H_2^2 + H_3^2 = I_1^2\omega_1^2 + I_2^2\omega_2 + I_3^2\omega_3^2 = 4I_t^2\omega_1^2 + I_t^2\omega_2^2 + I_t^2(\omega_3^2)$$

$$= I_t^2(4\omega_1^2 + \omega_2^2 + \omega_3^2) \quad \text{— (1)}$$

$$T = \tfrac12 I_1\omega_1^2 + \tfrac12 I_2\omega_2^2 + \tfrac12 I_3\omega_3^2 = I_t\omega_1^2 + \tfrac12 I_t\omega_2^2 + \tfrac12 I_t\omega_3^2$$

$$= \frac{I_t}{2}(2\omega_1^2 + \omega_2^2 + \omega_3^2) \quad \text{— (2)}$$

Eqn → (1) → $H^2 = I_t^2(4\omega_1^2 + \omega_2^2 + \omega_3^2)$, Let $\boxed{\omega_2^2 + \omega_3^2 = z}$

$$\frac{H^2}{I_t^2} = 4\omega_1^2 + z \rightarrow \boxed{\omega_1^2 = \frac14\left(\frac{H^2}{I_t^2} - z\right)} \quad \text{— (3)}$$

Eqn → (2) → $T = \frac{I_t}{2}\left(\frac12\left(\frac{H^2}{I_t^2} - z\right) + z\right) = \frac14\frac{H^2}{I_t} + \frac{zI_t}{4}$ — (4)

Angle between $\vec{H}, \vec{\omega} \rightarrow \theta = \cos^{-1}\left(\frac{\vec{H}\cdot\vec{\omega}}{|\vec{H}||\vec{\omega}|}\right)$

$$\vec{H}\cdot\vec{\omega} = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = 2T$$

From eqn. 4 → $2T = \frac{H^2}{2I_t} + \frac{zI_t}{2} = \vec{H}\cdot\vec{\omega}$

$$|\vec{H}| = \sqrt{H^2} = |H| = H$$

$$|\vec{\omega}| = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \sqrt{\omega_1^2 + z} = \sqrt{\frac14\left(\frac{H^2}{I_t^2} - z\right) + z} = \sqrt{\frac{H^2}{4I_t^2} + \frac{3z}{4}} = \sqrt{\frac14\left(\frac{H^2 + 3zI_t^2}{I_t^2}\right)}$$

$$= \frac12\sqrt{\frac{H^2 + 3zI_t^2}{I_t^2}}$$

$$\cos\theta = \frac{\frac{H^2}{2I_t} + \frac{zI_t}{2}}{H\cdot\frac12\sqrt{\frac{H^2 + 3zI_t^2}{I_t^2}}} = \frac{\frac{H}{I_t} + \frac{zI_t}{H}}{\sqrt{\frac{H^2 + 3zI_t^2}{I_t^2}}} \quad \text{— (5)}$$

find extremum of $\cos\theta$ by taking $\frac{\partial\cos\theta}{\partial z} = 0$

$$\frac{\partial\cos\theta}{\partial z} = \frac{\sqrt{\frac{H^2 + 3zI_t^2}{I_t^2}}\left(\frac{I_t}{H}\right) - \left(\frac{H}{I_t} + \frac{zI_t}{H}\right)\frac12\left(\frac{H^2 + 3zI_t^2}{I_t^2}\right)^{-1/2}\cdot 3}{\frac{H^2 + 3zI_t^2}{I_t^2}}$$

Numerator = 0 for $\frac{\partial\cos\theta}{\partial z} = 0 \rightarrow \sqrt{\frac{H^2 + 3zI_t^2}{I_t^2}}\left(\frac{I_t}{H}\right) = \frac{\left(\frac{H}{I_t} + \frac{zI_t}{H}\right)\cdot\frac32}{\sqrt{\frac{H^2 + 3zI_t^2}{I_t^2}}}$

$$\frac{H^2 + 3zI_t^2}{\cancel{I_t^2}}\cdot\frac{\cancel{I_t}}{\cancel{H}} = \frac32\left(\frac{H^2 + zI_t^2}{\cancel{HI_t}}\right) \rightarrow 2H^2 + 6zI_t^2 = 3H^2 + 3zI_t^2$$

$3zI_t^2 = H^2 \rightarrow \boxed{z = \left(H^2/3I_t^2\right)}$ — Plug into (5)

$$\cos\theta = \frac{\frac{H}{I_t} + \frac{H^2}{3I_t^2}\cdot\frac{\cancel{I_t}}{\cancel{H}}}{\sqrt{\frac{H^2 + H^2}{I_t^2}}} = \frac{\frac{H}{I_t}\left(1 + \frac13\right)}{\sqrt{\frac{2H^2}{I_t^2}}} = \frac{\frac{H}{I_t}\left(\frac43\right)}{\frac{|H|}{|I_t|}\sqrt2},$$

$H = \sqrt{H_1^2 + H_2^2 + H_3^2} > 0 \quad \because H_i^2 = I_i^2\omega_i^2 > 0$

$I_t > 0$ — Physical property

$$= \frac{\cancel{\left(\frac{H}{I_t}\right)}\frac43}{\cancel{\left(\frac{H}{I_t}\right)}\sqrt2} = \frac{4}{3\sqrt2} \rightarrow \theta = \cos^{-1}\left(\frac{4}{3\sqrt2}\right) = 0.339837 \text{ rad} = \boxed{19.4712^\circ}$$

$\therefore$ The largest angle between $\vec{H}$, $\vec{\omega}$ is $\boxed{19.4712^{\circ}}$

b) At largest angle $\rightarrow$ $Z = H^2/3I_t^2$

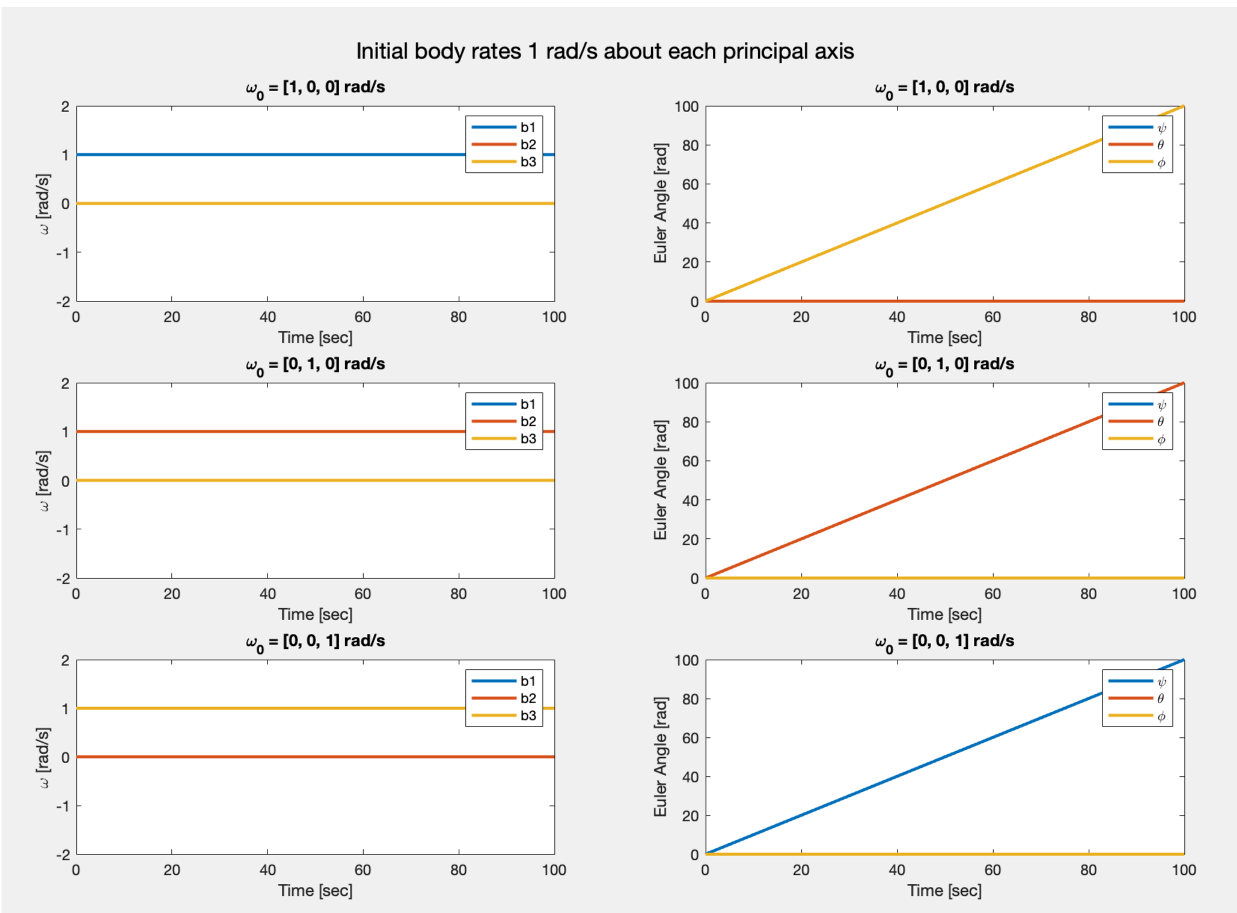
From eqn (4)

$$T = \frac{1}{4}\frac{H^2}{I_t} + \frac{Z I_t}{4} = \frac{1}{4}\frac{H^2}{I_t} + \frac{H^2}{3\cancel{I_t^2}} \cdot \frac{\cancel{I_t}}{4} = \frac{1}{4}\frac{H^2}{I_t} + \frac{H^2}{12 I_t} = \frac{3H^2 + H^2}{12 I_t} = \boxed{\frac{H^2}{3I_t} = T}$$

$$\boxed{T_{crit} = \frac{H^2}{3I_t}}$$ @ largest angle between $\vec{\omega}$, $\vec{H}$

Problem 2

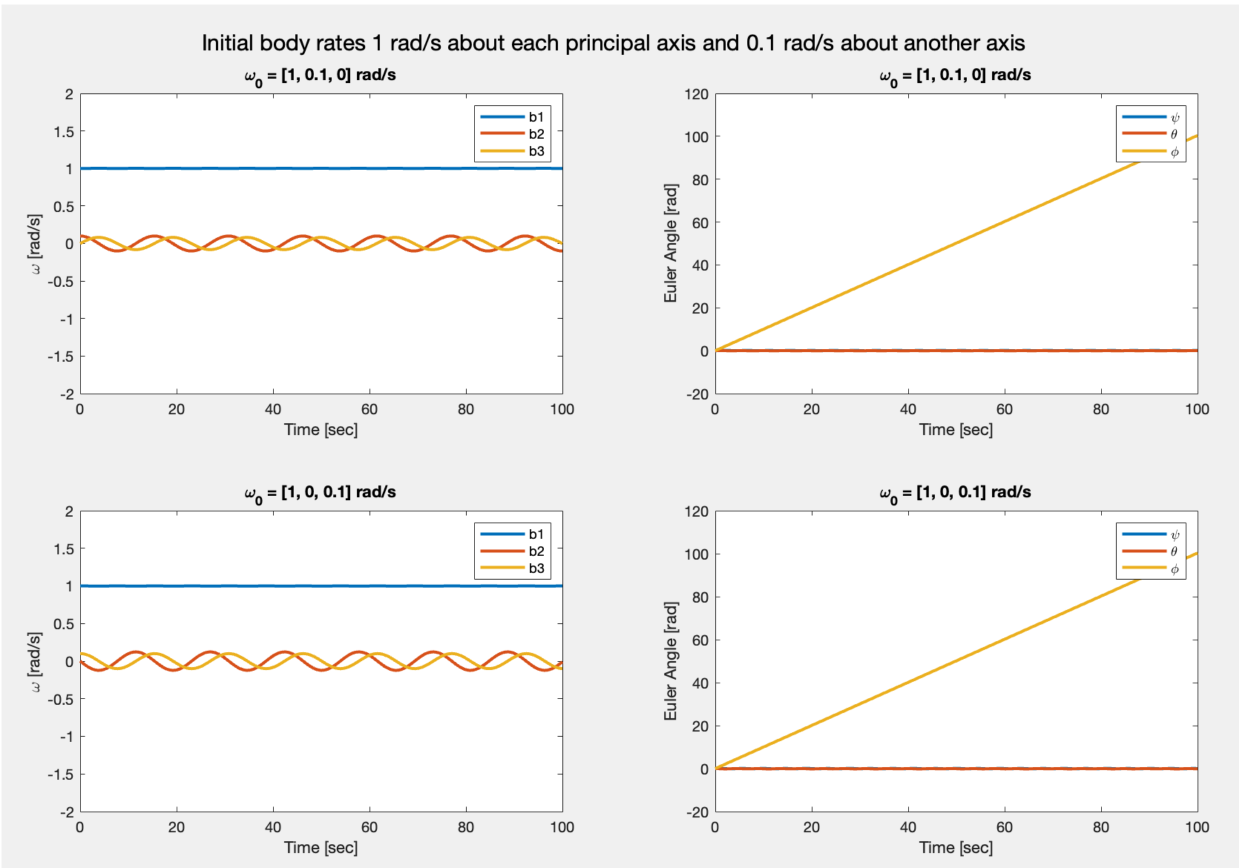
When the body rotates about its axis at 1 rad/sec, the rigid body's angular velocity and orientation are as follows:



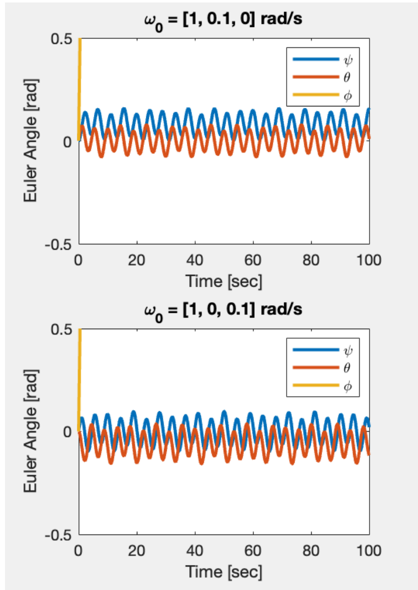
The orientation of the rigid body is represented by a 3-2-1 euler angle set (Ψ-Θ-ϕ). As can be seen, the pure spin rotation remains constant about each body axis for the whole time. For example, when the body initially rotates purely about the b1 axis, it only rotates about the b1 axis for the whole time. That's the reason why only one angle (ϕ) constantly increases. That's why this simulation is stable.

Similar observations can be made for the b2 and b3 pure spins. For b2, only one angle (Θ) constantly increases and for b3, only one angle (Ψ) constantly increases. Since, the rotation is purely about that axis and both the other angles stay constant, these are both stable rotations.

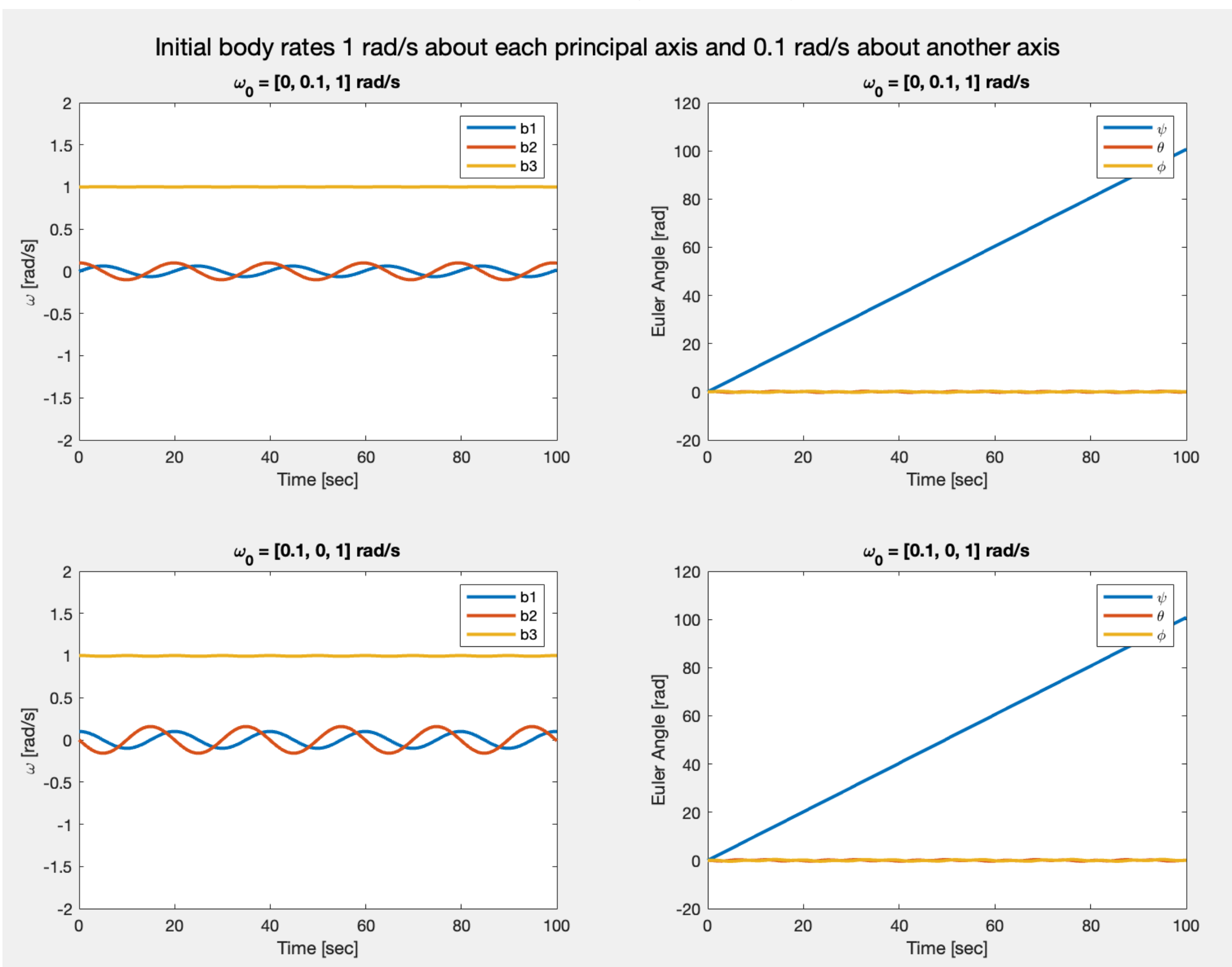
Then, let's deviate from the pure spin by adding a small rotation about other principal axes. The plots below show the angular velocities and orientation of the rigid body (represented by 3-2-1 euler angles [Ψ-Θ-ϕ]) when b1 has 1 rad/s rotation but b2 and b3 are also perturbed with 0.1 rad/s initial rotation making this NOT a pure spin:



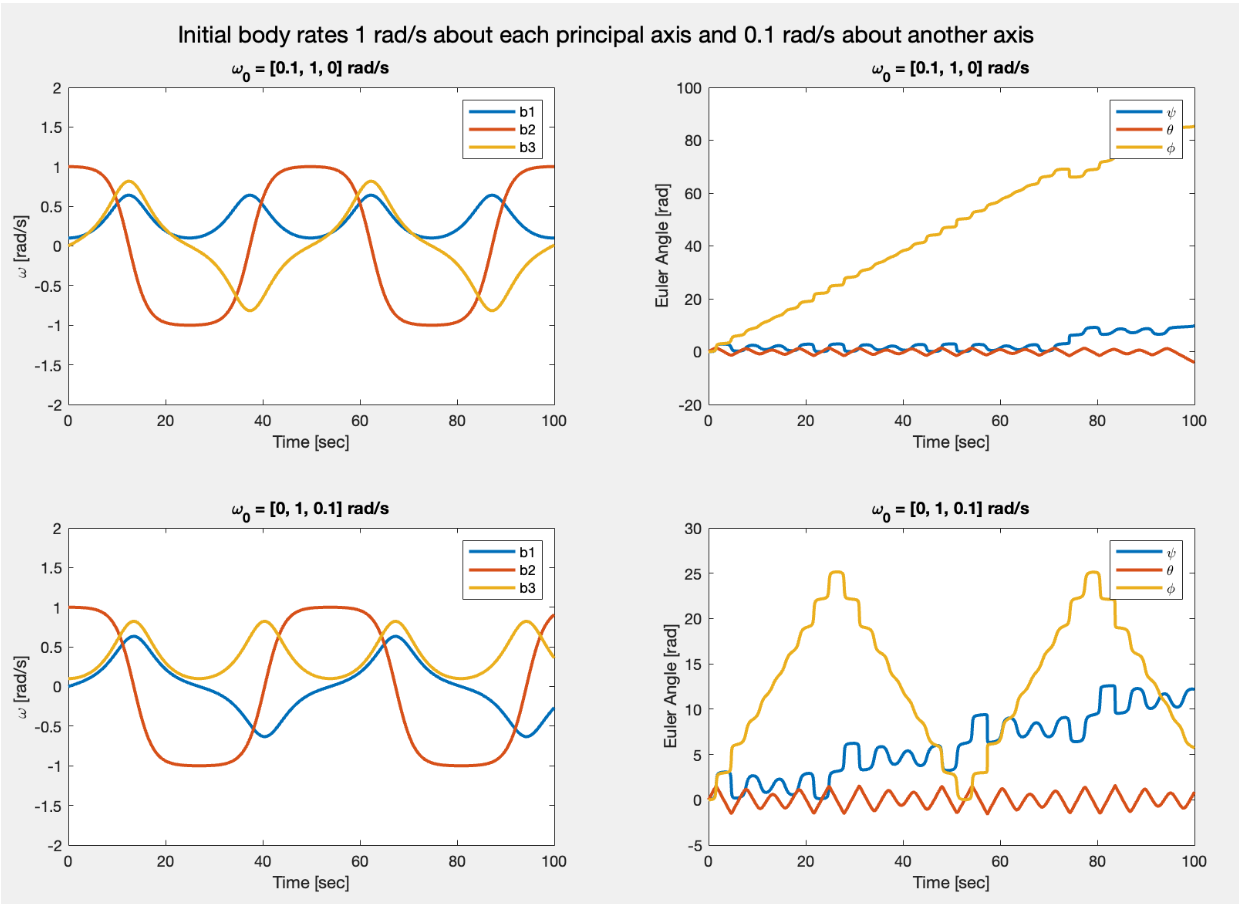
The angular velocities are similar but the key difference is that the spin about the non-principal axis is non-zero and oscillatory. That makes sense because initial spin isn't purely about the principal axis. Additionally, since the spin is about the major moment of inertia, the rigid body maintains the initial spin and does not spin chaotically. Zooming into the plots on the right, it can be seen that the rigid body orientation in the b2 and b3 axis is oscillatory and relatively minor. Hence, this is stable.



Similar results are seen when the rotation is about the b3 axis (minor moment of inertia). Here, b1 and b2 are perturbed with a 0.1 rad/s initial angular velocity. This is also stable.



However, when the initial rotation is about the b2 axis (1 rad/s) and perturbations about the b1 and b3 axes (0.1 rad/s), the rigid body experiences a very chaotic motion as seen below:



This can be attributed to the fact that b2 is the intermediate moment of inertia axis. That's why the orientation of the rigid body is chaotic. This is unstable.