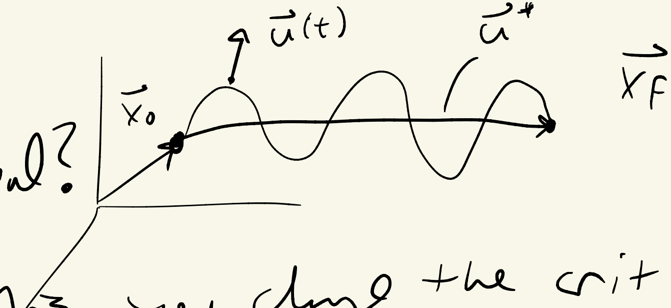
Functional Optimization



What makes a trajectory optimal?

- The cost function ⇒ change this you change the criterion.
- <u>Natural Trajectories</u> ⇒ Trajectories w/o control.

These are optimal in a deeper sense.

Define a hypothetical system comprised of n degrees of freedom q_i, $i=1,2,\dots,n$ along w/ their time derivatives, $\dot{q}_i = \frac{dq_i}{dt}$.

A natural dynamical system will evolve such that the "Action" is minimized.

The Action is the time integral of a functional

$L(\vec{q}, \dot{\vec{q}}, t) \equiv$ Kinetic Energy − Potential Energy

$$A = \int_{t_0}^{t_1} L(\vec{q}(\tau), \dot{\vec{q}}(\tau), \tau)\, d\tau \;;$$

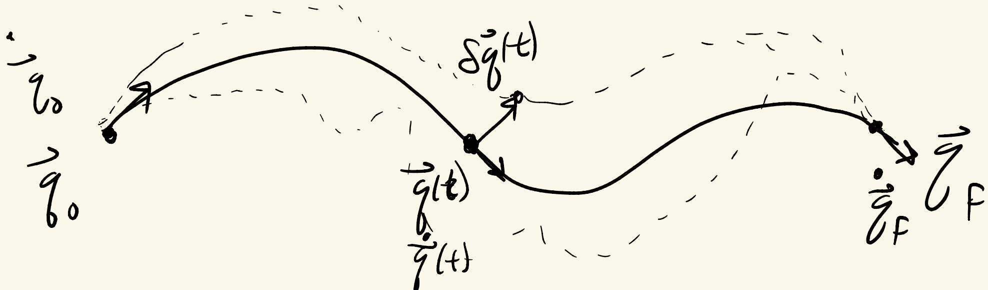
The natural trajectory minimizes A subject to terminal constraints

$$\boxed{\begin{array}{ll} \vec{q}(t_0) = \vec{q}_0 \;; & \vec{q}(t_F) = \vec{q}_F \\ \dot{\vec{q}}(t_0) = \dot{\vec{q}}_0 \;; & \dot{\vec{q}}(t_F) = \dot{\vec{q}}_F \end{array}}$$

What path $\vec{q}(t), \dot{\vec{q}}(t)$ minimizes A

Global Question → as usual its easier if we make it a Local Question.

Assume a solution & check in its neighborhood.



Consider a test solution.

To test it we need to consider local variations in this path.

Consider families of arbitrary variations from $\vec q(t)$, $\delta \vec q(t)$, that are smooth ($\delta \dot{\vec q}(t)$ exists) and which vanish at the end points.

$$\left\{ \delta \vec q(t_0) = \delta \vec q(t_f) = 0 \right\}$$

The deviations are "contemporaneous", meaning they are associated w/ $\vec q(t)$ at a specific time, $\delta \vec q(t)$.

$$J = \int_{t_0}^{t_f} L(\vec{q}(\tau), \dot{\vec{q}}(\tau), \tau)\, d\tau$$

$\vec{q}_0, \vec{q}_f, \dot{\vec{q}}_0, \dot{\vec{q}}_f$

$$J + \Delta J = \int_{t_0}^{t_f} L(\vec{q} + \delta\vec{q}, \dot{\vec{q}} + \delta\dot{\vec{q}}, t)\, d\tau$$

; $\delta\vec{q}$'s are contemporaneous

$$\delta\dot{\vec{q}} = \frac{d}{dt}\delta\vec{q}$$

If $\vec{q}(t)$ is the extremal, then

$$\Delta J > 0 \quad \text{for } |\delta\vec{q}| \neq 0.$$

In terms of variations

$$J + \Delta J = J + \delta J + \delta^2 J + \cdots O(\delta q^n) \Rightarrow \begin{matrix} \delta J = 0 \\ \delta^2 J > 0 \end{matrix}$$

From basic parametric optimization ideas......

$$\delta J = \delta \int_{t_0}^{t_1} L(\vec{q}, \dot{\vec{q}}, \tau)\, d\tau = \int_{t_0}^{t_1} \delta L(\vec{q}, \dot{\vec{q}}, \tau)\, d\tau$$

$$= \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial \vec{q}} \cdot \delta \vec{q} + \frac{\partial L}{\partial \dot{\vec{q}}} \cdot \delta \dot{\vec{q}} \right] d\tau \; ; \qquad \delta \dot{\vec{q}} = \delta \frac{d\vec{q}}{dt} = \frac{d}{dt}(\delta \vec{q})$$

Can't choose $\delta \vec{q}$ + $\delta \dot{\vec{q}}$ independently.

Rewrite as

$$\delta J = \int_{t_0}^{t_f} \left[\frac{\partial L}{\partial \vec{q}} \cdot \delta \vec{q} + \underbrace{\frac{\partial L}{\partial \dot{\vec{q}}} \cdot \frac{d}{d\tau}(\delta \vec{q})}_{\frac{\partial L}{\partial \dot{\vec{q}}} \cdot d(\delta \vec{q})} \right] d\tau$$

Recall Integration by parts.

$$\int u\, dv = uv - \int v\, du$$

$$u = \frac{\partial L}{\partial \dot{\vec{q}}} \qquad dv = d(\delta \vec{q})$$

$$du = \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\vec{q}}}\right) d\tau \qquad v = \delta \vec{q}$$

$$\delta J = \int_{t_0}^{t_f} \left[\frac{\partial L}{\partial \vec{q}} \cdot \delta \vec{q} \right] d\tau + \left(\frac{\partial L}{\partial \dot{\vec{q}}} \right) \cdot \delta \vec{q} \Big|_{t_0}^{t_f} - \int_{t_0}^{t_f} \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\vec{q}}} \right) \cdot \delta \vec{q} \, d\tau$$

$$= \left(\frac{\partial L}{\partial \dot{\vec{q}}} \right) \cdot \delta \vec{q} \Big|_{t_f} - \left(\frac{\partial L}{\partial \dot{\vec{q}}} \right) \cdot \delta \vec{q} \Big|_{t_0} + \int_{t_0}^{t_f} \left[\underline{\frac{\partial L}{\partial \vec{q}} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\vec{q}}} \right)} \right] \cdot \delta \vec{q} \, d\tau$$

Note $\delta \vec{q}_0 = \delta \vec{q}_f = 0$ by definition

$$\frac{\partial L}{\partial \vec{q}} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\vec{q}}} \right) = 0 \implies \text{Lagrange's Eqns of Motion}$$

$$\boxed{\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\vec{q}}} \right) = \frac{\partial L}{\partial \vec{q}}}$$

For Mechanical Systems $L = \underset{K.E.}{T} - \underset{P.E.}{U}$

$$A = \int_{t_0}^{t_1} (T - U)\, dt \qquad \vec{r} \in \mathbb{R}^3, \ \dot{\vec{r}} \text{ defined}$$

$$\left.\begin{array}{l} T = \frac{1}{2} m \dot{\vec{r}} \cdot \dot{\vec{r}} \\ U(\vec{r}) = -\frac{GM}{|\vec{r}|} \end{array}\right\} \Rightarrow \boxed{L = \frac{1}{2} m \dot{\vec{r}} \cdot \dot{\vec{r}} + \frac{GM}{|\vec{r}|}}$$

$$\frac{d}{dt}\left(m\dot{\vec{r}}\right) = \frac{\partial}{\partial \vec{r}} L(\vec{r}, \dot{\vec{r}}) = -\frac{GM}{|\vec{r}|^3} \vec{r}$$

$$m\ddot{\vec{r}} = \frac{-GM}{|\vec{r}|^3} \vec{r}$$

Hamilton's Equation

Given the Lagrangian $L(\vec{q}, \dot{\vec{q}}, t)$, Action is minimized

if the path satisfies $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{q}}}\right) - \frac{\partial L}{\partial \vec{q}} = 0$ at any point

Now consider a restatement of the problem (Legendre Transformation)

Define $\vec{p} = \frac{\partial L}{\partial \dot{\vec{q}}}$; assume $\left| \frac{\partial^2 L}{\partial \dot{\vec{q}}^2} \right| \neq 0 \Rightarrow$

$$\boxed{\dot{\vec{q}}(\vec{q}, \vec{p}, t)}$$

Define the functional

Hamiltonian

$$\boxed{H(\vec{q}, \vec{p}, t) = \vec{p} \cdot \dot{\vec{q}}(\vec{q}, \vec{p}, t) - L(\vec{q}, \dot{\vec{q}}(\vec{q}, \vec{p}, t), t)}$$

Consider a variation of H wrt $\vec{p}$, $\delta\vec{p}$, $(\delta\vec{q} = \vec{0})$

$$\delta_p H = \frac{\partial H}{\partial \vec{p}} \cdot \delta\vec{p} = \boxed{\dot{\vec{q}} \cdot \delta\vec{p}} + \underbrace{\vec{p} \cdot \delta_p \dot{\vec{q}} - \frac{\partial L}{\partial \dot{\vec{q}}} \cdot \delta_p \dot{\vec{q}}}_{\to\, 0}$$

$$\delta_p \dot{\vec{q}} = \frac{\partial \dot{\vec{q}}}{\partial \vec{p}} \cdot \delta\vec{p} \quad \ldots \text{ but } \ldots \quad \vec{p} = \frac{\partial L}{\partial \dot{\vec{q}}}$$

So for arbitrary $\delta\vec{p}$

$$\left(\frac{\partial H}{\partial \vec{p}} - \dot{\vec{q}}\right) \cdot \delta\vec{p} = 0 \Rightarrow \boxed{\dot{\vec{q}} = \frac{\partial H}{\partial \vec{p}}},$$

Legendre Transform, if repeated gives us back the Lagrange dyn. eqns.

$$\boxed{L = \vec{p} \cdot \dot{\vec{q}} - H(\vec{q}, \vec{p}, t)}$$

$$\boxed{J = A = \int_{t_0}^{t_f} \left[\vec{p}\cdot\dot{\vec{q}} - H(\vec{q},\vec{p},\tau) \right] d\tau}$$

Consider the stationary condition again, $\boxed{\delta J = 0}$

$$\delta J = \int_{t_0}^{t_f} \left[\delta\vec{p}\cdot\dot{\vec{q}} + \vec{p}\cdot\delta\dot{\vec{q}} - \frac{\partial H}{\partial \vec{q}}\cdot\delta\vec{q} - \frac{\partial H}{\partial \vec{p}}\cdot\delta\vec{p} \right] d\tau = 0$$

$$\int_{t_0}^{t_f} \vec{p}\cdot\delta\dot{\vec{q}}\, d\tau = \int_{t_0}^{t_f} \vec{p}\, \frac{d}{d\tau}(\delta\vec{q})\, d\tau = \overset{0}{\left. \vec{p}\cdot\delta\vec{q} \right|_{t_0}^{t_f}} - \int \delta\vec{q}\cdot\dot{\vec{p}}\, d\tau$$

$\delta q_0 = \delta q_f = 0$

$$\delta J = \int_{t_0}^{t_f} \left[\left(\dot{\vec{q}} - \frac{\partial H}{\partial \vec{p}} \right)\cdot\delta\vec{p} - \left(\dot{\vec{p}} + \frac{\partial H}{\partial \vec{q}} \right)\cdot\delta\vec{q} \right] d\tau = 0$$

Note, $\{p\}$'s are not nec. independent, but by def.

Hamilton's Eqns of Motion

$$\boxed{\dot{\vec{q}} = \frac{\partial H}{\partial \vec{p}}} \qquad \boxed{\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{q}}} \; ; \quad H = \dot{\vec{q}} \cdot \vec{p} - L(\vec{q}, \dot{\vec{q}}, t)$$

$$\dot{\vec{q}}(\vec{q}, \vec{p})$$

For a Newtonian Dyns Problem

$$L = T - U \qquad (\vec{r} = \vec{q})$$

$$L = \frac{1}{2} m \dot{\vec{r}} \cdot \dot{\vec{r}} - U(\vec{r}) \; ; \quad \vec{p} = \frac{\partial L}{\partial \dot{\vec{r}}} = m \dot{\vec{r}} \Rightarrow \boxed{\dot{\vec{r}} = \frac{1}{m} \vec{p}}$$

$$H = \frac{\vec{p} \cdot \vec{p}}{m} - \frac{1}{2} \frac{1}{m} \vec{p} \cdot \vec{p} + U(\vec{r}) = \boxed{\frac{1}{2} \frac{\vec{p} \cdot \vec{p}}{m} + U(\vec{q}) = H}$$

$$\dot{\vec{q}} = \frac{\partial H}{\partial \vec{p}} = \frac{1}{m} \vec{p}$$

$$\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{q}} = -\frac{\partial U}{\partial \vec{q}}$$

The Hamiltonian is more meaningful than the Lagrangian...

$$\frac{d}{dt}H(\vec{q},\vec{p},t) = \frac{\partial H}{\partial \vec{q}}\cdot\dot{\vec{q}} + \frac{\partial H}{\partial \vec{p}}\cdot\dot{\vec{p}} + \frac{\partial H}{\partial t} = \cancel{\frac{\partial H}{\partial \vec{q}}\cdot\frac{\partial H}{\partial \vec{p}} - \frac{\partial H}{\partial \vec{p}}\cdot\frac{\partial H}{\partial \vec{q}}} + \frac{\partial H}{\partial t}$$

$\boxed{\dot{H} = \frac{\partial H}{\partial t}}$ $\Rightarrow$ If the system is time invariant, $\boxed{H(\vec{q},\vec{p}) \equiv \text{Constant}}$

Newtonian, $\boxed{H = \text{Total Energy} = T + U}$

Consider the Action evaluated along the dyn. equs.

$$J^* = J\Big|_{\left[\substack{q_0,\, q_F \\ \dot{\vec{p}} = -\frac{\partial H}{\partial \vec{q}} \\ \dot{\vec{q}} = \frac{\partial H}{\partial \vec{p}}}\right.} = \underline{W(\vec{q}_0, \vec{q}_F, t_0, t_F)}$$

Hamilton's Principal Function.

$$\boxed{W(\vec{q}_0, \vec{q}_f, t_0, t_f) = \int_{t_0}^{t_f} \left[\dot{\vec{q}} \cdot \vec{p} - H(\vec{q}, \vec{p}, \tau) \right] d\tau}$$

Hamilton's Principal Function

How does W vary with $\vec{q}_0$ & $\vec{q}_f$?

q_0 $\qquad$ q_F

Recall $\delta q_0 = 0$

$\delta q_0 \neq 0 \Rightarrow \delta J \neq 0 \Rightarrow \delta W \neq 0.$

Vary δq_F

$$\delta_{q_F} W = \frac{\partial W}{\partial \vec{q}_F} \cdot \delta q_F = \vec{p}_F \cdot \delta \vec{q}_F + \delta\left(\cancel{\quad}\right)^{\equiv 0} \Rightarrow \boxed{\vec{p}_F = \frac{\partial W}{\partial \vec{q}_F}}$$

$$\text{Vary } \delta q_0 \Rightarrow \delta_{q_0} W = \frac{\partial W}{\partial \vec{q}_0} \cdot \delta \vec{q}_0 = -\vec{p}_0 \cdot \delta \vec{q}_0 \Rightarrow \boxed{\vec{p}_0 = -\frac{\partial W}{\partial \vec{q}_0}}$$

How does $W(\vec{q}_0, \vec{q}_f, t_0, t_f)$ evolve forwards + backwards in time?

$$t_f \to t_f + \cancel{\delta t_f} = t_f + dt_f$$

$$\vec{p}_f(t_f + dt_f) = \vec{p}_f(t_f) + \frac{d}{dt}\vec{p}_f \cdot dt_f + \dots$$

$$\vec{q}_f(t_f + dt_f) = \vec{q}_f + \frac{d}{dt}\vec{q}_f \cdot dt_f + \dots$$

$$d_{t_f} W = \frac{\partial W}{\partial \vec{q}_0} \cdot \frac{d\vec{q}_0}{dt_f} dt_f + \frac{\partial W}{\partial \vec{q}_f} \cdot \frac{d\vec{q}_f}{dt_f} dt_f + \frac{\partial W}{\partial t_0} \cdot \frac{dt_0}{dt_f} dt_f + \frac{\partial W}{\partial t_f} \cdot \frac{dt_f}{dt_f} dt_f \quad ; \quad \frac{d\vec{q}_0}{dt_f} = 0$$

$$\frac{dt_0}{dt_f} = 0$$

$$\frac{dt_f}{dt_f} = 1$$

$$d_{t_f} W = \frac{\partial W}{\partial \vec{q}_f} \cdot \frac{dq_f}{dt} \cdot dt_f + \frac{\partial W}{\partial t_f} dt_f = \int_{t_0}^{t_f + dt_f} [\quad] d\tau - \int_{t_0}^{t_f} [\quad] d\tau$$

$$= \left[\dot{q}_f \cdot p_f - H(q_f, p_f, t_f) \right] dt_f$$

$$\cancel{\frac{\partial W}{\partial \vec{q}_f}} \cdot \frac{d\vec{q}_f}{dt_f} dt_f + \frac{\partial W}{\partial t_f} dt_f = \cancel{\frac{\partial W}{\partial \vec{q}_f}} \cdot \dot{\vec{q}}_f \, dt_f - H(q_f, p_f, t_f) \, dt_f$$

$$\vec{p}_f = \frac{\partial W}{\partial \vec{q}_f}$$

$$\left[\frac{\partial W}{\partial t_f} + H\left(\vec{q}_f, \frac{\partial W}{\partial \vec{q}_f}, t_f\right)\right] dt_f = 0$$

$$\boxed{\frac{\partial W}{\partial t_f} + H\left(\vec{q}_f, \frac{\partial W}{\partial \vec{q}_f}, t_f\right) = 0}$$

Partial Diff. Equation

Hamilton - Jacobi Equation

For dt_0

$$\boxed{\frac{\partial W}{\partial t_0} - H\left(\vec{q}_0, -\frac{\partial W}{\partial \vec{q}_0}, t_0\right) = 0}$$