ASEN 6020: Optimal Trajectories
Necessary Conditions for Optimal Control
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- State equations: $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t)$ where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{u} \in U \subset \mathbb{R}^m$ and $\boldsymbol{f} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^n$.

- Performance index:

$$J = K(\boldsymbol{x}_o, t_o, \boldsymbol{x}_f, t_f) + \int_{t_o}^{t_f} L(\boldsymbol{x}, \boldsymbol{u}, \tau) d\tau$$

  where K is a scalar function of the terminal states and times and L is a scalar function of the state, controls and time in the interval $[t_o, t_f]$.

- Terminal constraints: $\boldsymbol{g}(\boldsymbol{x}_o, t_o, \boldsymbol{x}_f, t_f) = \boldsymbol{0}$ where $\boldsymbol{g} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^l$, $l \leq 2n+2$.

- Hamiltonian:

$$H(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{u}, t) = L(\boldsymbol{x}, \boldsymbol{u}, t) + \boldsymbol{p} \cdot \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t)$$

  where H is a scalar function and $\boldsymbol{p} \in \mathbb{R}^n$ are the adjoints.

- Optimal control policy:

$$\left. \frac{\partial H}{\partial \boldsymbol{u}} \right|_{\boldsymbol{u}^*} = \boldsymbol{0} \text{ if } \boldsymbol{u} \text{ is in the interior of } U$$

  or

$$\boldsymbol{u}^*(\boldsymbol{x}, \boldsymbol{p}, t) = \arg \min_{\boldsymbol{u}} H(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{u}, t) \text{ if } \boldsymbol{u} \text{ is in the boundary or interior of } U$$

  Leading to the Hamiltonian:

$$H^*(\boldsymbol{x}, \boldsymbol{p}, t) = H(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{u}^*(\boldsymbol{x}, \boldsymbol{p}, t), t)$$

- Dynamics of an optimal control trajectory are defined by the differential equations:

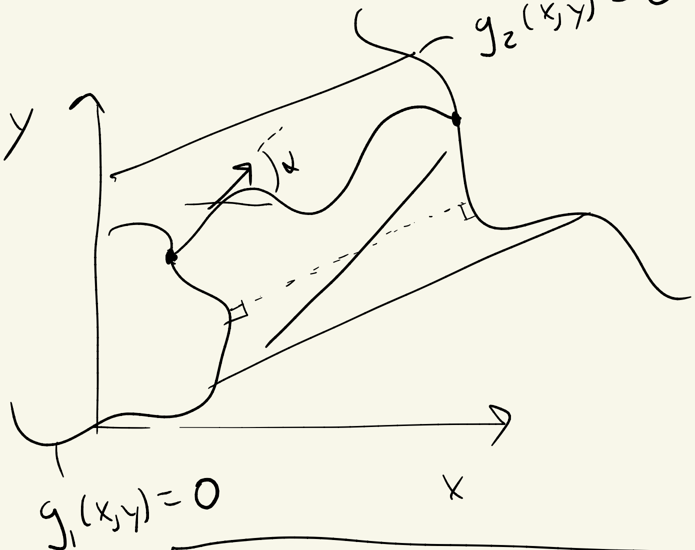
$$\begin{aligned} \dot{\boldsymbol{x}} &= \frac{\partial H^*}{\partial \boldsymbol{p}} \\ \dot{\boldsymbol{p}} &= -\frac{\partial H^*}{\partial \boldsymbol{x}} \end{aligned}$$

- Transversality Conditions:

$$\begin{aligned} \boldsymbol{p}_o &= -\frac{\partial K}{\partial \boldsymbol{x}_o} - \boldsymbol{\lambda} \cdot \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}_o} \\ H_o &= \frac{\partial K}{\partial t_o} + \boldsymbol{\lambda} \cdot \frac{\partial \boldsymbol{g}}{\partial t_o} \\ \boldsymbol{p}_f &= \frac{\partial K}{\partial \boldsymbol{x}_f} + \boldsymbol{\lambda} \cdot \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}_f} \\ H_f &= -\frac{\partial K}{\partial t_f} - \boldsymbol{\lambda} \cdot \frac{\partial \boldsymbol{g}}{\partial t_f} \end{aligned}$$

  where $\boldsymbol{\lambda} \in \mathbb{R}^l$ are the constant Lagrange multipliers associated with the constraints $\boldsymbol{g} = \boldsymbol{0}$.

Shortest line/distance between two manifolds.



Dynamics

$$\boxed{\frac{dx}{ds} = \cos\alpha \qquad \frac{dy}{ds} = \sin\alpha}$$

$dx = \cos\alpha \, ds \qquad dy = \sin\alpha \, ds$

$\boxed{\alpha \text{ is our control}}$

$\boxed{K = s_f - s_0 \Rightarrow \text{minimum path length}}$

$L = 0$

$$\boxed{H = \vec{p}\cdot\vec{F} = P_x \cos\alpha + P_y \sin\alpha}$$

Optimal Control condition --- $\frac{\partial H}{\partial \alpha} = 0 = -P_x \sin\alpha + P_y \cos\alpha$

$$\Rightarrow \boxed{\tan\alpha^* = P_y / P_x}$$

$$\boxed{\cos\alpha^* = \frac{\pm 1}{\sqrt{1+\tan^2\alpha^*}} \; ; \; \sin\alpha^* = \frac{\pm \tan\alpha^*}{\sqrt{1+\tan^2\alpha^*}}}$$

$$\boxed{H^*(x, y, P_x, P_y) = \pm\sqrt{P_x^2 + P_y^2}}$$; Time invariant $\Rightarrow$ $H^* =$ Constant.

$$\dot{x} = \frac{\partial H^*}{\partial P_x} = \frac{\pm P_x}{\sqrt{P_x^2 + P_y^2}} \qquad \dot{y} = \frac{\partial H^*}{\partial P_y} = \frac{\pm P_y}{\sqrt{P_x^2 + P_y^2}}$$

$$\dot{P}_x = -\frac{\partial H^*}{\partial x} = 0 \qquad \dot{P}_y = -\frac{\partial H^*}{\partial y} = 0 \Rightarrow$$ $\boxed{P_x \text{ \& } P_y \text{ are constants}}$

$$x = \frac{\pm P_x}{\sqrt{P_x^2 + P_y^2}}(s - s_0) + x_0 \quad ; \quad y = \frac{\pm P_y}{\sqrt{P_x^2 + P_y^2}}(s - s_0) + y_0$$

$$\frac{dy}{dx} = \frac{P_y}{P_x}$$ (= tan α^* ..... constant) $\Rightarrow$ $$\boxed{y = \left(\frac{P_y}{P_x}\right)(x - x_0) + y_0}$$

Trns. Cond.

$g_1(x_0, y_0) = 0 \qquad g_2(x_f, y_f) = 0$

$K = s_f - s_0$

$$P_{x_0} = -\frac{\partial K}{\partial x_0} - \lambda_1 \frac{\partial g_1}{\partial x_0} - \lambda_2 \frac{\partial g_2}{\partial x_0}$$

$$P_{x_0} = -\lambda_1 \left.\frac{\partial g_1}{\partial x_0}\right|_0 \; ; \; P_{x_f} = \lambda_2 \left.\frac{\partial g_2}{\partial x_f}\right|_f \; ; \; P_{y_0} = -\lambda_1 \frac{\partial g_1}{\partial y_0} \; ; \; P_{y_f} = \lambda_2 \frac{\partial g_2}{\partial y_f}$$

$$\boxed{\begin{aligned} H_0 &= \frac{\partial K}{\partial s_0} + \vec{\lambda} \cdot \frac{\partial \vec{g}}{\partial s_0} = -1 \\ H_f &= -\frac{\partial K}{\partial s_f} = -1 \end{aligned}}$$

$$H^* = \pm \sqrt{p_x^2 + p_z^2} = -1$$

$$P_{x_0} = \boxed{-\lambda_1 \frac{\partial g_1}{\partial x_0} = \lambda_2 \frac{\partial g_2}{\partial x_f}} = P_{x_f} \; ; \; P_{y_0} = \boxed{-\lambda_1 \frac{\partial g_1}{\partial y_0} = \lambda_2 \frac{\partial g_2}{\partial y_f}} = P_{y_f}$$

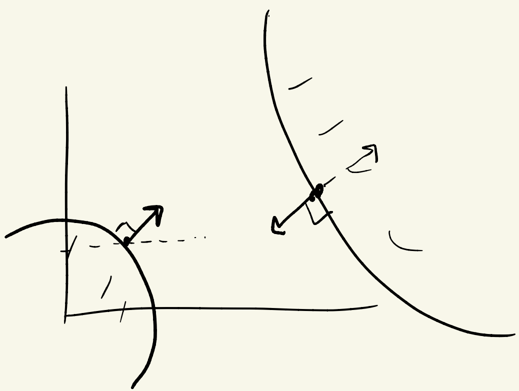
$$G \left\{ \begin{bmatrix} g_{1x_0} & g_{2x_f} \\ g_{1y_0} & g_{2y_f} \end{bmatrix} \right. \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

Either $\lambda_1 = \lambda_2 = 0$ ....

$$|G| = 0 \Rightarrow \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \in \mathrm{Null}(G)$$

Note $\frac{P_{y_0}}{P_{x_0}} = \frac{g_{1y_0}}{g_{1x_0}} = \frac{P_{yF}}{P_{xF}} = \frac{g_{2yF}}{g_{2xF}}$ ≡ slopes of the normal to the manifolds

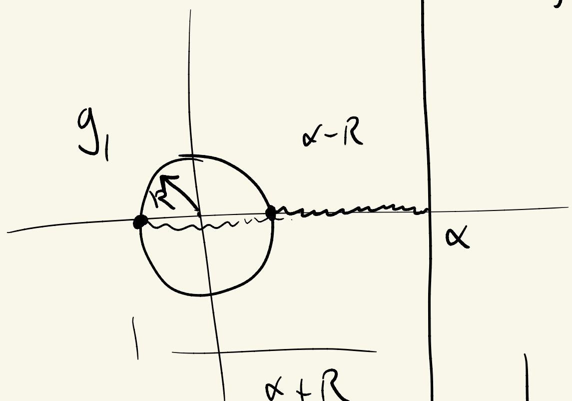
Must find the point on the manifolds where the normals are parallel, and point to each other too.



the extremal line will then connect these points.

$\frac{\partial g_1}{\partial (x,y)} \parallel \frac{\partial g_2}{\partial (x,y)} \rightarrow$ G is singular

Special Case



$$g_1 = \frac{1}{2}\left[x^2 + y^2 - R^2\right]$$

$$g_2 = (x - \alpha)$$

$$G = \begin{bmatrix} x_1 & 1_2 \\ y_1 & 0 \end{bmatrix}$$

$$|G| = 0 \Rightarrow |G| = -y = 0 \quad y = 0$$

$$x^2 = R^2 \Rightarrow \boxed{x = \pm R}$$

$$\begin{bmatrix} \pm R & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \boxed{\pm R\lambda_1 + \lambda_2 = 0}$$

$$\boxed{\lambda_2 = \mp R\lambda_1}$$

$$\frac{p_{y_f}}{p_{x_f}} = \frac{0}{1} = 0 \quad ; \quad H = -\sqrt{p_x^2 + p_y^2} = -|p_x| = -1$$

$$\boxed{p_x = \pm 1}$$

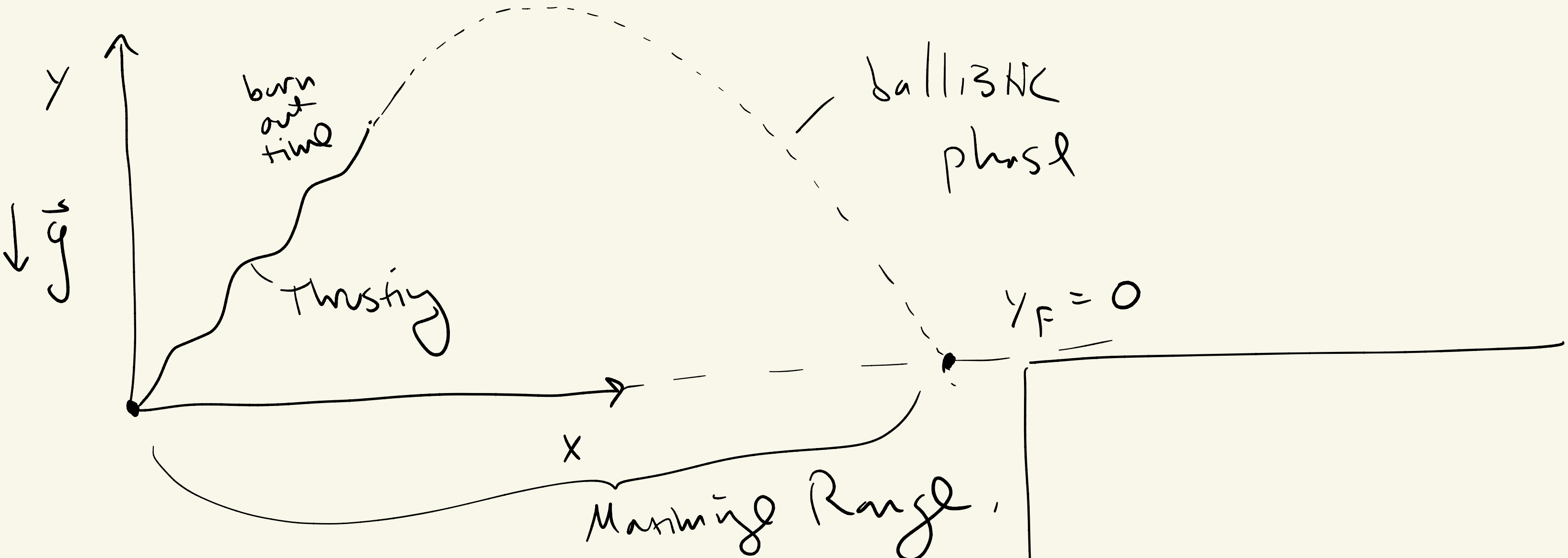
$$p_{x_0} = \pm 1 = \mp \lambda_1 R \Rightarrow \boxed{\lambda_1 = \frac{-1}{R}}$$

$$\lambda_2 = \pm 1 \qquad \boxed{R\lambda_1 + \lambda_2 = 0}$$

Min. is a straight line connecting $x = \alpha$ and $x = \pm R$

Cost function $\boxed{K = \alpha \mp R}$

Example: <u>Maximum Range for a Thrusting Rocket.</u>



Coords x, y
velocities u, v
Controls Thrust ≡ preprogrammed
Direction of the thrust.

Simple EOM

$$\boxed{\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= f(t)\cos\theta \\ \dot{v} &= F(t)\sin\theta - g \end{aligned}}$$

$$\left.\begin{aligned} m\dot{u} &= F(t)\cos\theta \\ m\dot{v} &= F(t)\sin\theta - mg \end{aligned}\right\} \quad \begin{aligned} \dot{u} &= \frac{F}{m}\cos\theta \\ \dot{v} &= \frac{F}{m}\sin\theta - g \end{aligned}$$

$$\frac{dm}{dt} = -\frac{F(t)}{c} \Rightarrow \frac{F(t)}{m} = \boxed{-\frac{c}{m}\frac{dm}{dt} = f(t)}$$

Initial Conditions: $x_0 = y_0 = u_0 = v_0 = t_0 = 0$

Boundary Conditions: $t_1 = t_{B/O}$, $\boxed{x_1, y_1, u_1, v_1 \text{ all Free}}$

when $\underline{t > t_{B/O}}$ problem is integrable

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= 0 \\ \dot{v} &= -g \end{aligned}$$

Let $\tau = t - t_{B/O}$ $\Rightarrow$

$$\boxed{\begin{aligned} x(\tau) &= x_1 + u_1\tau \\ y(\tau) &= y_1 + v_1\tau - \tfrac{1}{2}g\tau^2 \\ u(\tau) &= u_1 \\ v(\tau) &= v_1 - g\tau \end{aligned}}$$

Maximum Range: $K = x_f$ cost func

Terminal Manifold: $y_f = 0$; $\boxed{u_f, v_f, t_f \equiv \text{free}}$

Pre-Hamiltonian: $H = p_x u + p_y v + p_u F \cdot \cos\theta + p_v (F \sin\theta - g)$

$$F(t) = \begin{cases} > 0 & 0 \le t \le t_{B/O} \\ 0 & t > t_{B/O} \end{cases}$$

Control: $\theta^* = \underset{\theta}{\arg\max}\, H$; $H^* = \underset{\theta}{\max}\, H$

Can compute $\frac{\partial H}{\partial \theta} = \boxed{0 = -p_u F \sin\theta + p_v F \cos\theta}$ $F \neq 0$

$$\boxed{\tan\theta^* = \frac{p_v}{p_u}}$$

$$H^* = H\Big|_{\tan\theta^* = \frac{P_v}{P_u}}$$

$$\cos^2\theta = \frac{1}{1+t^2\theta} = \frac{P_u^2}{P_u^2+P_v^2}$$

$$\sin^2\theta = \frac{P_v^2}{P_u^2+P_v^2}$$

$$H^* = P_x u + P_y v + F\underbrace{\left[\frac{P_u^2}{\sqrt{P_u^2+P_v^2}} + \frac{P_v^2}{\sqrt{P_u^2+P_v^2}}\right]}_{\sqrt{P_u^2+P_v^2}} - P_v g$$

$$\boxed{H^* = P_x u + P_y v + F(t)\sqrt{P_u^2+P_v^2} - P_v g}$$

State $\dot{x} = \frac{\partial H^*}{\partial P_x} = u$; $\dot{y} = \frac{\partial H^*}{\partial P_y} = v$; $\dot{u} = \frac{\partial H^*}{\partial P_u} = \frac{F P_u}{\sqrt{P_u^2+P_v^2}}$

$$\dot{v} = \frac{\partial H^*}{\partial P_v} = \frac{F P_v}{\sqrt{P_u^2+P_v^2}} - g$$

Adjoint EOM:

$$\dot{P}_x = -\frac{\partial H^*}{\partial x} = 0 \qquad \dot{P}_y = -\frac{\partial H^*}{\partial y} = 0$$

$$\boxed{P_x, P_y \text{ are constant.}}$$

$$\dot{P}_u = -\frac{\partial H^*}{\partial u} = -\underline{P_x} \qquad \dot{P}_v = -\frac{\partial H^*}{\partial v} = -\underline{P_y}$$

$$\Rightarrow \boxed{P_u = P_{u_0} - P_{x_0} \cdot t \qquad P_v = P_{v_0} - P_{y_0} \cdot t}$$

$$\Rightarrow \boxed{\tan\theta^* = \frac{P_{v_0} - P_{y_0} \cdot t}{P_{u_0} - P_{x_0} \cdot t}}$$

- <u>Bi-Linear Control Law For thrust</u>
- Can show that this is monotonic
- <u>Was initially Classified</u>

Transversality Conditions

$$K = x_f \qquad \vec{g}_0 = \begin{bmatrix} x_0 \\ y_0 \\ u_0 \\ v_0 \\ \vdots \\ t_0 \end{bmatrix} \qquad \vec{g}_f = [y_f]$$

$$\vec{p}_0 = -\frac{\partial K}{\partial \vec{x}_0} - \vec{\lambda}_0 \cdot \frac{\partial \vec{g}_0}{\partial \vec{x}_0} = -\vec{\lambda}_0 \cdot I = -\vec{\lambda}_0 \Rightarrow$$

$\vec{p}_0$ are arbitrary but satisfy conditions

$H_0 = \lambda_{t_0} \Rightarrow$ unconstrained.

$$p_{x_f} = 1 = p_{x_0} \,; \quad p_{y_f} = \lambda_{y_f} = \lambda \,; \quad \boxed{p_{u_f} = p_{v_f} = 0}$$

$$\boxed{H_f = 0}$$

$$\boxed{\tan\theta^* = \frac{p_{v_0} - p_{y_0} t}{p_{u_0} - t}}$$

Choose $\boxed{p_{u_0}, p_{y_0}, p_{v_0}}$ + integrate ....

success occurs when

$$\boxed{\begin{aligned} p_{u_f} &= 0 \\ p_{v_f} &= 0 \\ H_f &= 0 \end{aligned}}$$

$\boxed{\text{3 unknowns} \quad \text{3 conditions}}$

"Simplification" ... can relate the state at time t_f back to burnout time $t_{B/O}$.

Terminal Conditions: $\quad x_f = x_1 + u_1(t_f - t_{1/0})$

$$\Rightarrow y_f = 0 = y_1 + v_1(t_f - t_{1/0}) - \frac{1}{2}g(t_f - t_{1/0})^2$$

Solve for

$$(t_f - t_{1/0}) = \frac{v_1}{g} + \frac{1}{g}\sqrt{v_1^2 + 2gy_1}$$

$$\boxed{x_f = x_1 + \frac{u_1}{g}\left[v_1 + \sqrt{v_1^2 + 2gy_1}\right]}$$

Terminal Constraints at $t_{1/0} = t_1$

$$\boxed{K = x_1 + \frac{u_1}{g}\left[v_1 + \sqrt{v_1^2 + 2gy_1}\right]}$$

$$\boxed{g_{t_{1/0}} \equiv 0}$$

$$P_{x_1} = 1 \qquad P_{u_1} = \frac{1}{g}\left[v_1 + \sqrt{v_1^2 + 2g y_1}\right]$$

$$P_{y_1} = \frac{u_1}{\sqrt{v_1^2 + 2g y_1}} \qquad P_{v_1} = \frac{u_1}{g}\left[1 + \frac{v_1}{\sqrt{v_1^2 + 2g y_1}}\right]$$

$$P_u = -P_x t + P_{u_0} \qquad @\ t_{B/O} \qquad P_u = P_{u_1}$$

$$P_v = -P_y t + P_{v_0} \qquad P_v = P_{v_1}$$

$$\frac{1}{g}\left[v_1 + \sqrt{v_1^2 + 2g y_1}\right] = -t_{B/O} + P_{u_0}$$

$$\frac{u_1}{g}\left[1 + \frac{v_1}{\sqrt{v_1^2 + 2g y_1}}\right] = \frac{-u_1}{\sqrt{v_1^2 + 2g y_1}}\, t_{B/O} + P_{v_0}$$

$$P_{u_0} = t_{B/0} + \frac{1}{g}\left[v_1 + \sqrt{v_1^2 + 2gy_1} \right]$$

$$P_{v_0} = \frac{u_1}{\sqrt{v_1^2 + 2gy_1}}\, t_{B/0} + \frac{u_1}{g}\left[1 + \frac{v_1}{\sqrt{v_1^2 + 2gy_1}} \right]$$

$$\tan\theta^* = \frac{-\dfrac{u_1}{\sqrt{v_1^2 + 2gy_1}}\, t + \dfrac{u_1}{\sqrt{v_1^2 + 2gy_1}}\, t_{B/0} + \dfrac{u_1}{g}\left[1 + \dfrac{v_1}{\sqrt{v_1^2 + 2gy_1}} \right]}{-t + t_{B/0} + \dfrac{1}{g}\left[v_1 + \sqrt{v_1^2 + 2gy_1} \right]}$$

Cristo!

$$\tan\theta^* = \frac{u_1}{\sqrt{v_1^2 + 2gy_1}} \left[\frac{-t + t_{B/0} + \frac{1}{g}\left[v_1 + \sqrt{v_1^2 + 2gy_1} \right]}{- - - - - - -} \right] = \frac{u_1}{\sqrt{v_1^2 + 2gy_1}}$$

How to solve?

Choose $\theta \equiv$ constant, integrate till $t_{B/O}$

Compare $\boxed{\tan\theta - \frac{u_1}{\sqrt{v_1^2+2gy_1}}}$ & adjust θ till this equals zero.

Specific Example: Let $\boxed{F = \text{constant}}$

$$\ddot{x} = F\cos\theta^*$$
$$\ddot{y} = F\sin\theta^* - g$$

$F\cos\theta^* + F\sin\theta^* \equiv$ constant

$$\boxed{\begin{aligned} \dot{x} &= u = f\cos\theta^* t \\ \dot{y} &= v = (f\sin\theta^* - g)t \\ x &= \frac{1}{2} f\cos\theta^* t^2 \\ y &= \frac{1}{2}(f\sin\theta^* - g)t^2 \end{aligned}}$$

At $t = t_{B/O} = T$ we get

$$\boxed{\begin{aligned} u_1 &= f\cos\theta^* T \\ v_1 &= (f\sin\theta^* - g)T \\ x_1 &= \frac{1}{2} f\cos\theta^* T^2 \\ y_1 &= \frac{1}{2}(f\sin\theta^* - g)T^2 \end{aligned}}$$

$$\tan\theta^* = \frac{u_1}{\sqrt{v_1^2 + 2g y_1}}$$

$$= \frac{f\cos\theta^* T}{\sqrt{\underbrace{(f\sin\theta^* - g)^2 T^2 + g(f\sin\theta^* - g)T^2}_{f^2 s^2\theta^* - 2f s\theta^* g + \cancel{g^2} + g f s\theta^* - \cancel{g^2}}}}$$

$\Rightarrow$ Cancel T; $\boxed{n = \frac{f}{g} = \frac{\text{thrust}}{\text{weight}}}$

$$\Rightarrow \boxed{\tan\theta^* = \frac{\cos\theta^*}{\sqrt{\sin^2\theta^* - \frac{1}{n}\sin\theta^*}}}$$

$$\boxed{\sin\theta^* \sqrt{\sin^2\theta^* - \frac{1}{n}\sin\theta^*} = \cos^2\theta^*}$$

$$\sin^2\theta^*\left[\sin^2\theta^* - \frac{1}{n}\sin\theta^*\right] = \cos^4\theta^*$$

$$\sin^4\theta^* - \cos^4\theta^* = (\sin^2\theta^* - \cos^2\theta^*)\underbrace{(\sin^2\theta^* + \cos^2\theta^*)}_{1} = \boxed{-\cos 2\theta^* = \frac{1}{n}\sin^3\theta^*}$$

$$\boxed{\sin^3\theta^* + n\cos 2\theta^* = 0} \Rightarrow \text{solve to get } \theta^*(n)$$

$$\boxed{\sin^3\theta^* - 2n\sin^2\theta^* + n = 0} \qquad n \geq 1$$

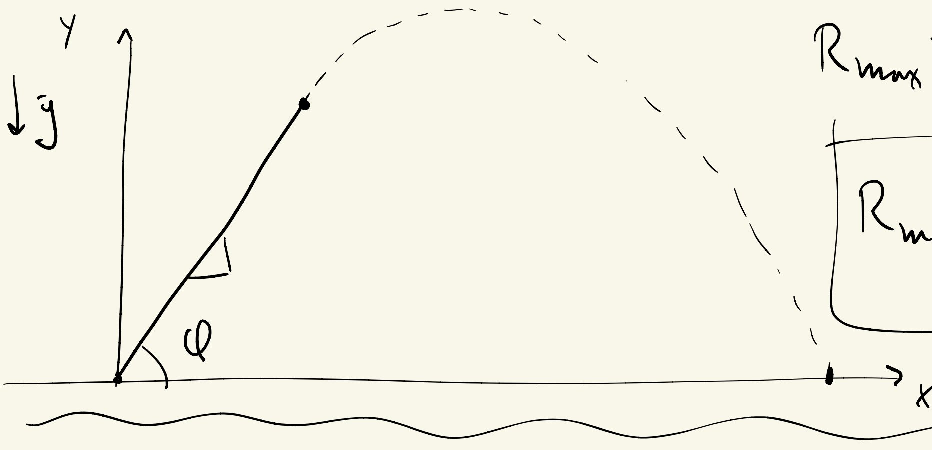
n	θ^*	
1	90°	⇒ hovering
1.1	68.60°	
1.6	55.08°	
4.0	47.94°	
⋮		
∞	45°	⇒ Impulsive Limit

⇒ Trajectory

$$x = f\cos\theta^* \,\frac{t^2}{2}$$

$$y = (f\sin\theta^* - g)\,\frac{t^2}{2}$$

$$\Rightarrow \quad \frac{y}{x} = \boxed{\frac{\sin\theta^* - 1/n}{\cos\theta^*} = \tan\varphi} = \underline{\text{constant}}$$



$$R_{max} = x_1 + \frac{u_1}{g}\left[v_1 + \sqrt{v_1^2 + 2gy_1}\right]$$

$$\boxed{R_{max} = f\,T^2\cos\theta^*\left[\frac{n}{\sin\theta^*} - \frac{1}{2}\right]}$$

Linear Quadratic Problem (solution)

$$\dot{x} = A(t)\, x + Bu \quad ; \quad x \in \mathbb{R}^n \; ; \; A \in \mathbb{R}^{n\times n} \; ; \; u \in \mathbb{R}^m , \; B \in \mathbb{R}^{n\times m}$$

Dynamics

Cost funcion

$$J = \frac{1}{2} x_f^T S x_f + \int_{t_0}^{t_f} \frac{1}{2}\left[x^T Q x + u^T R u \right] d\tau$$

$S, Q \in \mathbb{R}^{n\times n}$; $R \in \mathbb{R}^{m\times m}$ | $Q > 0$; $R > 0$; $S \geq 0$

Terminal conditions, $x(t_0) = x_0$; $t_0 = 0$; t_f specified

x_f free

Hamiltonian

$\lambda \in \mathbb{R}^n$, "adjoints"

$$\boxed{H = \frac{1}{2}\left[x^T Q x + u^T R u\right] + \lambda^T \left[Ax + Bu\right]}$$

$$\frac{\partial H}{\partial u} = Ru + B^T \lambda = 0 \Rightarrow \boxed{u = -R^{-1} B^T \lambda(t)}$$

$$\underset{m\times 1}{u} \quad \underset{m\times m}{R^{-1}} \quad \underset{m\times n}{B^T} \quad \underset{n\times 1}{\lambda}$$

Optimal Control Dynamics

$$\boxed{\begin{aligned} \dot{x} &= Ax - BR^{-1}B^T\lambda \\ \dot{\lambda} &= -Qx - A^T\lambda \end{aligned}} \Rightarrow \begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \left[\begin{array}{c|c} A & -BR^{-1}B^T \\ \hline -Q & -A^T \end{array}\right] \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

Holds for $A(t)$ or A T.I.

Solution exists using notation

$$\boxed{\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \Phi(t, t_0) \begin{bmatrix} x_0 \\ \lambda_0 \end{bmatrix}}$$

$$\Phi(t,t_0) = \left[\begin{array}{c|c} \varphi_{xx}(t,t_0) & \varphi_{x\lambda}(t,t_0) \\ \hline \varphi_{\lambda x}(t,t_0) & \varphi_{\lambda\lambda}(t,t_0) \end{array}\right];$$

Transversality

x_0 + t_0 given $\Rightarrow$ λ_0 arbitrary

H_0 "

$$\lambda_f = \frac{\partial K}{\partial x_f} = \boxed{S x_f = \lambda_f}$$

$$\begin{bmatrix} x_f \\ \lambda_f \end{bmatrix} = \Phi(t_f, t_0) \begin{bmatrix} x_0 \\ \lambda_0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_0 \\ \lambda_0 \end{bmatrix} = \Phi(t_0, t_f) \begin{bmatrix} x_f \\ \lambda_f \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \Phi(t, t_f) \begin{bmatrix} x_f \\ \lambda_f \end{bmatrix}$$

$$\Phi(t_f, t_0)^{-1} = \Phi(t_0, t_f)$$

$$x(t) = \phi_{xx}(t, t_f)\, x_f + \phi_{x\lambda}(t, t_f)\, \lambda_f$$

$$\lambda(t) = \phi_{\lambda x}(t, t_f)\, x_f + \phi_{\lambda\lambda}(t, t_f)\, \lambda_f$$

But

$$\lambda_f = S\, x_f$$

$$x(t) = [\phi_{xx} + \phi_{x\lambda} S]\, x_f$$

$$\lambda(t) = [\phi_{\lambda x} + \phi_{\lambda\lambda} S]\, x_f$$

$$x_f = [\phi_{xx} + \phi_{x\lambda} S]^{-1}\, x$$

$$\lambda(t) = [\phi_{\lambda x} + \phi_{\lambda\lambda} S] \cdot [\phi_{xx} + \phi_{x\lambda} S]^{-1} \cdot x(t)$$

$$u(t) = -R^{-1} B^T \underbrace{[\phi_{\lambda x} + \phi_{\lambda\lambda} S][\phi_{xx} + \phi_{x\lambda} S]^{-1}}_{K(t, t_f)} x(t)$$

$$\dot{x} = Ax + Bu$$

$$K(t, t_f) \Rightarrow \lambda(t) = K(t, t_f)\, x(t)$$

$$K(t_f, t_f) = [0 + S] \cdot [I + 0]^{-1} = S$$

whats the differential Equation related $K(t, t_f)$

$$\lambda = K(t, t_f)\, x \quad ; \quad K(t_f, t_f) = S$$

$$\dot{\lambda} = \dot{K} x + K \dot{x} = -Qx - A^T \lambda$$

$$\dot{x} = Ax - BR^{-1}B^T \lambda$$

$$\dot{K} x = -KAx + KBR^{-1}B^T\lambda - Qx - A^T\lambda \quad ; \text{ But } \lambda = Kx$$

$$\dot{K} x = [-KA + KBR^{-1}B^T K - Q - A^T K]\, x \quad \forall x$$

Ricatti Differential Equation

$$\boxed{\dot{K} = -KA - A^T K + KBR^{-1}B^T K - Q}$$

$K(t_f, t_f) = S = S^T$

$K(t, t_f) = K^T$

solving $K(t, t_f) \Rightarrow$ solution to the problem

$$\boxed{u(t) = -R^{-1}B^TK(t,t_f)\,x(t)}$$ Optimal Feedback Control Law .....

$$\boxed{\dot{x} = \left[A - BR^{-1}B^TK(t,t_f)\right]x}$$

Solving form x_0

Time Varying Linear Dynamical System.

Must solve for $\underbrace{K(t,t_f)}_{\Phi(t,t_f)}$ before we generate the optimal solution

Primer Vector Theory:

Dynamics:

$$\dot{\vec{r}} = \vec{V}$$

$$\dot{\vec{V}} = \vec{g}(\vec{r}, t) + \frac{\vec{T}(t)}{m}$$

$$\dot{m} = -b(m, \vec{r}, t)$$ mass flow rate function $= b$

$\vec{r}$ position
$\vec{V}$ velocity
m mass
$\vec{T}$ thrust

Chemical Thrust:

$\vec{T}(t) = \dot{m}\vec{V}_e$; $\dot{m} < 0$ $\quad \vec{V}_e \equiv$ exhaust velocity.

$\dot{m} = -b(m, \vec{r}, t)$; $\Rightarrow m$ = can be a function of tank pressure, represented by m.

Usually....

- $b(m, t) \sim b(t)$ $\quad t \equiv$ control the throttle.
- Assume $b(t) \leq b_0$, max flow rate.

$c = |\vec{V}_e|$ = characteristic speed, or specific Impulse

$\hat{u} = \dfrac{-\vec{V}_e}{|\vec{V}_e|}$ = controlled thrust direction

So total thrust accel. ...

$$\vec{u} = \frac{\vec{T}}{m} = \frac{\dot{m}\vec{V}_e}{m} = \frac{-b(t)\,c(-\hat{u})}{m} = \boxed{\frac{c\,b(t)}{m}\hat{u} = \vec{u}(t)}$$

Controls are $b(t)$ & $\hat{u}$

Primer Vector Theory

"Low Thrust" Propulsion System

$$\vec{T}(t) = k_1(t)\cdot k_2(t,\vec{r})\cdot T_0\ \hat{u}$$

$k_1(t) \equiv$ Throttle ; $0 \le k_1 \le 1$

$k_2 \equiv$ Power scaling ... $\begin{cases} \text{nuclear} \sim \text{const.} \\ \text{SEP} \sim \frac{1}{r^2} \end{cases}$

$$T_0 = \frac{2P_{jet}}{u_e} = \frac{2\eta P_{elec}}{u_e\left(1+\frac{2k_i\xi}{u_e^2}\right)}$$

P_{jet} = "delivered power"

P_{elec} = "generated power"

η = efficiency, u_e = exhaust speed, $k_i\xi$ ionization energies

---

Mass flow rate: $\dot{m} = -\frac{k_1k_2T_0}{u_e} = \frac{-2k_1k_2P_{jet}}{u_e^2} \equiv -b(t)$

$$\vec{u} = \frac{k_1k_2 2P_{jet}}{m\,u_e}\hat{u} = \frac{b(t)u_e}{m}\hat{u} \qquad u_e \equiv c$$

$$\boxed{\vec{u} = \frac{b(t)c}{m}\hat{u}} \Rightarrow \text{generic form as chemical}$$

General Formulation

Usual goal is to maximize the final mass, to make that a min. problem we take

$$K = -m_f \qquad (L \equiv 0)$$

State: $\vec{X} = \begin{bmatrix} \vec{r} \\ \vec{V} \\ m \end{bmatrix}$; $\dot{\vec{X}} = \begin{bmatrix} \vec{V} \\ \vec{g}(\vec{r},t) + \frac{bc}{m}\hat{u} \\ -b \end{bmatrix} = \vec{F}(\vec{X},t)$

Adjoints: $\vec{P} = \begin{bmatrix} \vec{P}_r \\ \vec{P}_V \\ P_m \end{bmatrix}$

$$\boxed{H = \vec{P} \cdot \vec{F}(\vec{X},t)}$$

$$\boxed{H = \vec{P}_r \cdot \vec{V} + \vec{P}_v \cdot \left[\vec{g}(\vec{r},t) + \frac{bc}{m}\hat{u}\right] - P_m b}$$

Note, if $\vec{g}(\vec{r},t) = \vec{g}(\vec{r}) \Rightarrow$ then $H \equiv$ constant over the optimal trajectory, even though $b(t)$, & $\hat{u}(t)$

Find $\underset{}{\text{argmin}}\ H(\vec{x}, \vec{p}, b, \hat{u}, t)$

$$H = [\vec{p}_r \cdot \vec{v} + \vec{p}_v \cdot \vec{g}] + \underbrace{b\left[\frac{c}{m} \vec{p}_v \cdot \hat{u} - p_m\right]}$$

① Select $\hat{u}$, since $c, m \,\&\, b > 0$, H will be minimized when $\hat{u}^* = -\frac{\vec{p}_v}{|\vec{p}_v|} = -\hat{p}_v$; $\vec{p}_v \cdot \hat{u}^* = |\vec{p}_v|$

$\boxed{\vec{p}_v \equiv \text{primer vector.}}$

$$\hat{u}^* = -\hat{p}_v$$

$$H\big|_{\hat{u}^*} = [\vec{p}_r \cdot \vec{v} + \vec{p}_v \cdot \vec{g}] - b \underbrace{\left[\frac{c}{m} p_v + p_m\right]}$$

$\boxed{K(p_m, p_v, c, m) \equiv \text{Switching Function}}$

② Select $b(t)$, mass flow rate.

$$\boxed{H = [\vec{p}_r \cdot \vec{v} + \vec{p}_v \cdot \vec{g}] - b K(p_m, p_v, c, m)}$$

$$\text{If} \begin{cases} K > 0 & b = b_{max} & \text{Max Thrust} \\ K < 0 & b = 0 & \text{Null Thrust} \\ K = 0 & 0 \le b \le b_{max} & \text{Intermediate or Singular Arc.} \end{cases}$$

Next Consider the adjoint Dynamics

$$\dot{\vec{p}}_r = \frac{-\partial H^*}{\partial \vec{r}} = -\vec{p}_v \cdot \frac{\partial \vec{g}}{\partial \vec{r}}$$

$$\dot{\vec{p}}_v = \frac{-\partial H^*}{\partial \vec{v}} = -\vec{p}_r$$

$$\ddot{\vec{p}}_v = -\dot{\vec{p}}_r = \vec{p}_v \cdot \left(\frac{\partial \vec{g}}{\partial \vec{r}}\right)$$

$$\boxed{\ddot{\vec{p}}_v = \left(\frac{\partial \vec{g}}{\partial \vec{r}}\right)^T \cdot \vec{p}_v}$$

$$\dot{p}_m = \frac{-\partial H^*}{\partial m} = \frac{-bc}{m^2} p_v$$

Dynamics Become:

$$\dot{\vec{r}} = \frac{\partial H^*}{\partial \vec{p}_r} = \vec{v}$$

$$\dot{\vec{v}} = \frac{\partial H^*}{\partial \vec{p}_v} = \vec{g}(\vec{r}, t) - \frac{bc}{m} \hat{p}_v$$

$$\boxed{\ddot{\vec{r}} = \vec{F}(\vec{r}, t, \vec{p}_v)}$$

$$\dot{m} = \frac{\partial H^*}{\partial p_m} = -b$$

b is chosen according to

$$\boxed{K(p_m, p_v, c, m)}$$

Properties of the solutions:

$\vec{r}(t)$ is continuous function.

$\vec{v}(t)$ is continuous as ~~de~~ long as $b_{max} < \infty$

DiffEq for $\vec{p}_v(t)$ is not a linear ODE! when thrusting occurs

It is during a Null Thrust Arc.

$\vec{P}_v$ is smooth, because discontinuities only arise

in $\dddot{\vec{P}}_v$ if Impulsive maneuvers occur,

or $\ddddot{\vec{P}}_v$ if $b_{max} < \infty$

"$\vec{P}_v$" is well behaved.

Transversality conditions on $P_m \Rightarrow \quad K = -m_f$

$P_{m_f} = \frac{\partial K}{\partial m_f} = -1$, dynamics of P_m?

$\dot{P}_m = \frac{-bc}{m} P_v$; $c, m, P_v > 0$
$b \geq 0$

$\dot{P}_m \leq 0$

Switching function $\quad > 0$

$K = P_m + \frac{c P_v}{m} \Rightarrow$ If $P_m > 0 \Rightarrow b = b_{max}$

$\dot{P}_m < 0$

K can't become ≤ 0 until $P_m < 0$.

Tells us that $P_{m_0} \geq -1$, often $P_{m_0} < 0$ unless we have an initial thrust.

What are the control implications ....

1. $\hat{u} = \dfrac{-\vec{p}_v}{|\vec{p}_v|}$, Since $\vec{p}_v$ is smooth, $\hat{u}$ is smooth as well $\Rightarrow$ do not expect discontinuities in thrust direction $\Rightarrow$ unless $p_v \to 0$.

2. $$b = \begin{cases} b_{max} & K > 0 \\ 0 & K < 0 \\ [0, b_{max}] & K = 0 \end{cases} \qquad \boxed{K = p_m + \frac{c}{m} p_v}$$

$$\left(\dot{p}_v = \frac{\vec{p}_v \cdot \dot{\vec{p}}_v}{|\vec{p}_v|} \right)$$

Study the dynamics of K,

$$\dot{K} = \underbrace{\dot{p}_m - \frac{c}{m^2} p_v \dot{m}} + \frac{c}{m} \dot{p}_v \qquad \dot{\vec{p}}_v = -\vec{p}_r$$

$$\dot{p}_m = -\frac{bc}{m^2} p_v \qquad \boxed{-\frac{bc}{m^2} p_v + \frac{c}{m^2} p_v b \equiv 0}$$

$$\dot{m} = -b$$

$$\boxed{\dot{K} = \frac{c}{m} \dot{p}_v} = -\frac{c}{m} \frac{\vec{p}_v \cdot \vec{p}_r}{p_v}$$

During thrusting have $m(t) \Rightarrow$ not trivial variation in $\dot{K}$.

During NT arcs, $m = \text{const}$.

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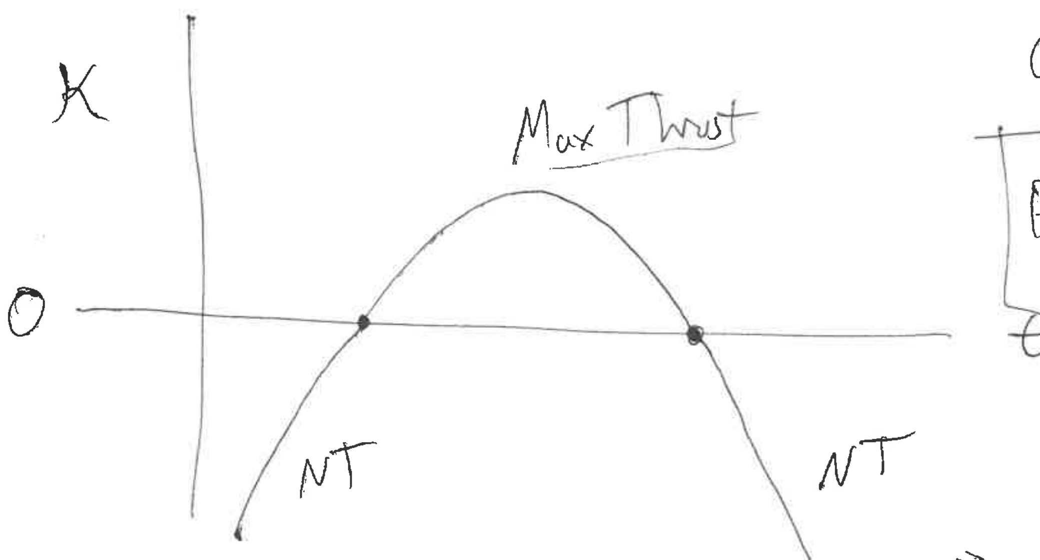
$$\frac{dK}{dt} = \frac{c}{m}\frac{dP_v}{dt} \Rightarrow$$ Let $K<0, b=0,$ $m = \text{const.}$

$$\int dK = \frac{c}{m}\int dP_v$$

$$\boxed{K = \frac{c}{m}P_v + \text{Const.}}$$; P_v still changes, so K is not constant....

Switching Function Trajectory



Occurs when

$$\boxed{P_v > -\frac{m}{c}P_m}$$

Once thrusting starts $\dot{\vec{P}}_v = \vec{P}_v \cdot \frac{\partial \vec{g}}{\partial \vec{r}}$ are not much changed.

But $\dot{P}_m = \frac{-b_{max}c}{m^2}P_v$, rapidly changes.

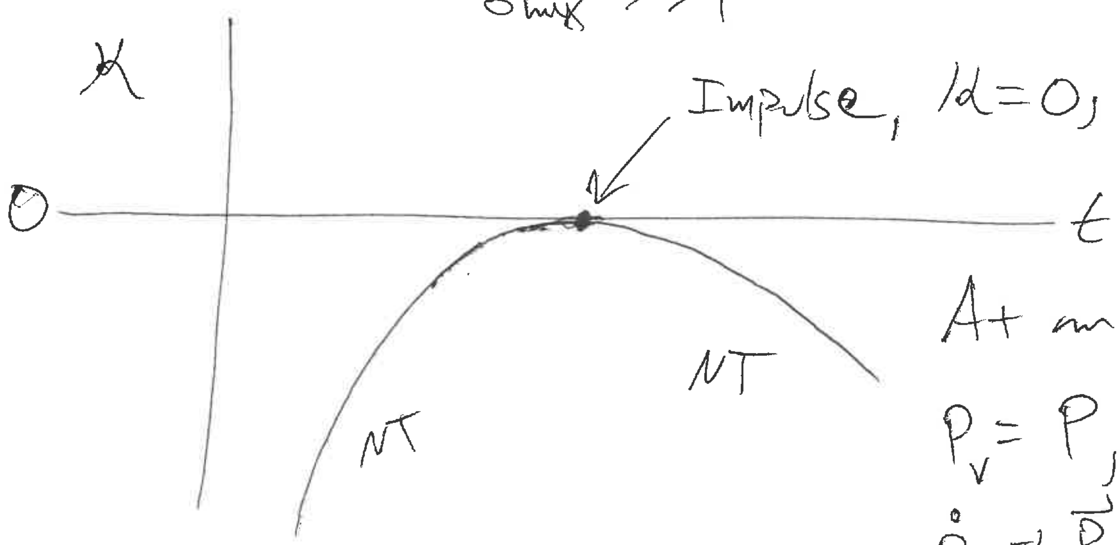
$$P_m \sim P_{m_0} - \frac{b_{max}c}{m^2}P_v\Delta t \Rightarrow$$ Leads to short thrust periods.

If $T \sim \vec{g}(\vec{r},t) \Rightarrow$ thrust durations can become long .... due to larger change in the trajectory.

What happens if $b_{max} >> 1$

Can have impulsive change in P_m, during which P_v = constant.

$b_{max} >> 1$



At an impulse juncture

$P_v = P$,

$\dot{P}_v \propto \vec{P}_v \cdot \vec{P}_r = 0$

$\vec{P}_r \perp \vec{P}_v$ at an impulse.

We can find that

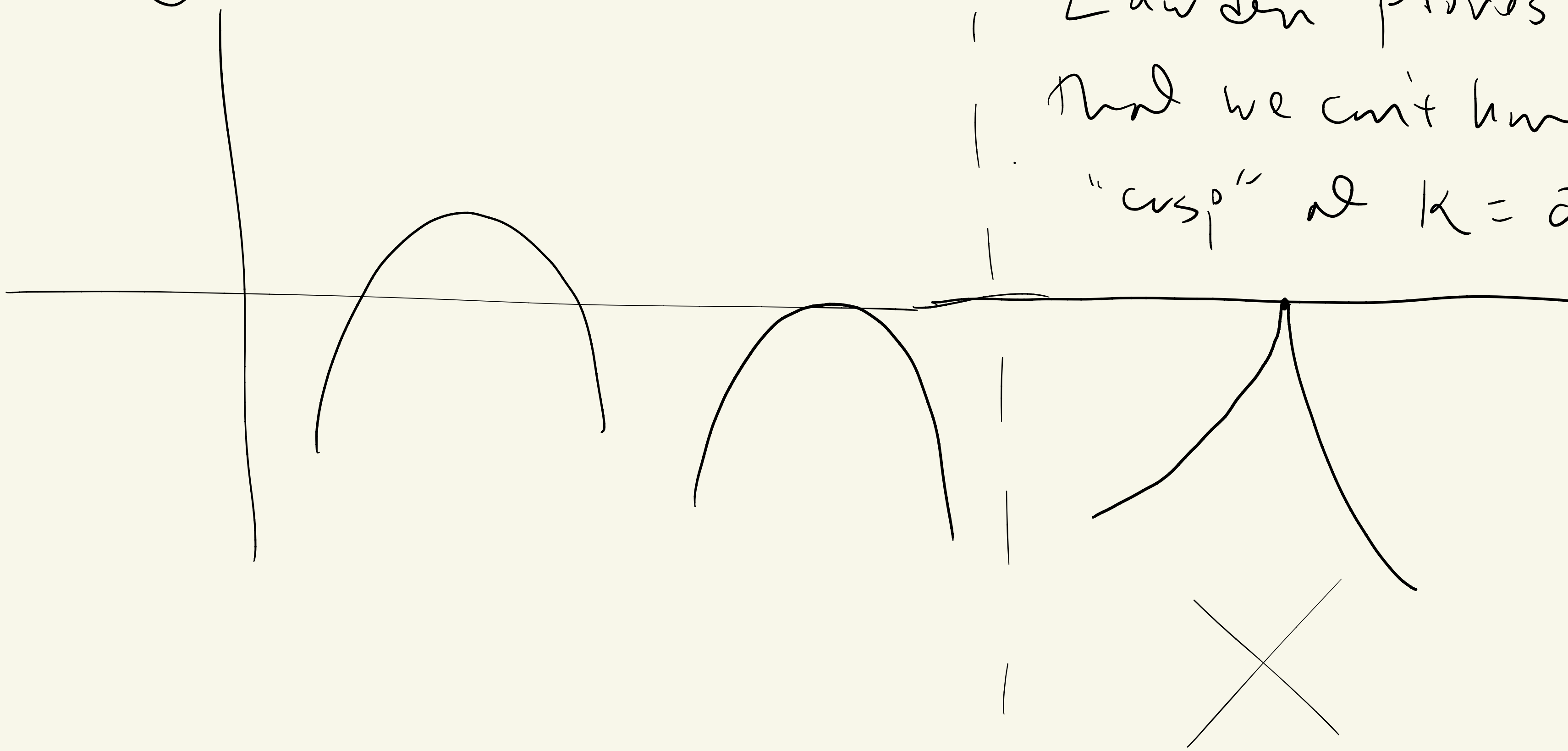
$$\kappa = \frac{c}{m}(P_v - P) \; ; \quad P = \text{constant value}$$

$P_v \le P$ for impulsive maneuvers.

$\Rightarrow$ Singular Arcs

Switching function dynamics...

Lawden "proves" that we can't have a "cusp" at $k = 0$



Singular Arcs: IF $K \equiv 0$ over an interval of time, then we have an indeterminant or singular arc.

bK, b is seemingly arbitrary.

Some immediate implications:

$$K = 0 \Rightarrow \mathring{K} = 0 \; ; \; \mathring{K} = \frac{c}{m} \mathring{p}_v \Rightarrow \mathring{p}_v = 0$$

Thus p_v has a constant magnitude, however p_m + m may still vary. What is the optimal control?

Is it arbitrary? $\Rightarrow$ No

2 perspectives: Out of all the arbitrary controls, there may be only 1, or a subset, that satisfy the trans. conditions.

(2) The singular arc can be enforced by proper choice of control.

$$\dot{p}_V = -\frac{\vec{p}_V \cdot \vec{p}_r}{p_V} = 0 \; ; \quad \ddot{p}_V = \dddot{p}_V = \ddddot{p}_V = \cdots = 0$$

Eventually we find a constraint on the control to enforce this condition (assume $p_V \neq 0$)

$$\dot{p}_V \propto -\vec{p}_V \cdot \vec{p}_r \; ; \quad \ddot{p}_V \propto -\dot{\vec{p}}_V \cdot \vec{p}_r - \vec{p}_V \cdot \dot{\vec{p}}_r = +\left(\vec{p}_r \cdot \vec{p}_r + \vec{p}_V \cdot \frac{d\vec{g}}{d\vec{r}} \cdot \vec{p}_V\right)$$

$$\dddot{p}_V = -2\vec{p}_V \cdot \vec{g}_{\vec{r}} \cdot \vec{p}_r + \vec{p}_r \cdot \vec{g}_{\vec{r}} \cdot \vec{p}_r + \vec{p}_V \cdot \underbrace{\left(\frac{d^2\vec{g}}{d\vec{r}^2} \cdot \dot{\vec{r}}\right)}\cdot \vec{p}_V = 0$$

$$\ddddot{p}_V = \text{- - - - - - - - - - - -} \; \vec{p}_V \cdot \left(\frac{d^2\vec{g}}{d\vec{r}^2} \cdot \underbrace{(\vec{g} + \vec{u})}_{\ddot{\vec{r}}}\right) \cdot \vec{p}_V \cdots = 0$$

Alternate Formulations (Simplified)

$$\dot{\vec{r}} = \vec{V}$$

$$\dot{\vec{V}} = \vec{g} + \vec{u} \quad ; \quad |\vec{u}| \le u_{max}$$

Minimum $|\Delta V|$ problem.

$$J = \int_{t_0}^{t_f} |\vec{u}|\, d\tau \quad \Rightarrow \text{ neglect mass rate of change.}$$

$$H = \vec{P}_r \cdot \vec{V} + \vec{P}_V \cdot \vec{g} + \underbrace{\vec{P}_V \cdot \vec{u} + |\vec{u}|}_{|\vec{u}|(1 + \vec{P}_V \cdot \hat{u})}$$

$\hat{u}$ direction should be $\boxed{\hat{u} = -\hat{P}_V}$ $\Rightarrow |\vec{u}|(\underbrace{1 - P_V}_{\text{switching function}})$

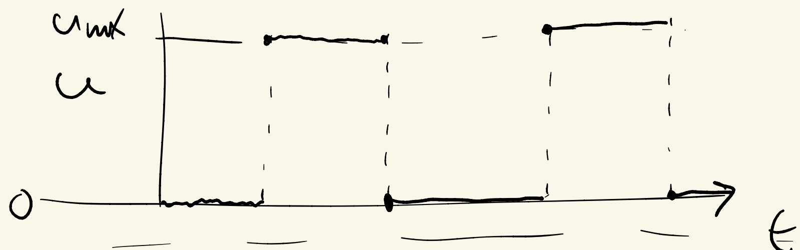
$$|\vec{u}|^* = \begin{cases} P_V < 1 & u = 0 \\ P_V > 1 & u = u_{max} \\ P_V = 1 & \text{singular control} \end{cases}$$

Nec. Conditions

$$\dot{\vec{p}}_v = -\vec{p}_r$$
$$\dot{\vec{p}}_r = -\vec{p}_v \cdot \frac{\partial \vec{g}}{\partial r}$$

$$\boxed{\ddot{\vec{p}}_v = \vec{p}_v \cdot \frac{\partial \vec{g}}{\partial r}}$$

$$\ddot{\vec{r}} = \vec{g} + \underbrace{\vec{u}^*}_{-u^* \hat{p}_v}$$



"Minimum Energy" Trajectory....

$$J = \frac{1}{2}\int_{t_0}^{t_f} \vec{u}\cdot\vec{u}\, d\tau \Rightarrow$$

$$H = \vec{p}_r \cdot \vec{v} + \vec{p}_v \cdot \vec{g} + \underbrace{\vec{p}_v \cdot \vec{u} + \frac{1}{2}\vec{u}\cdot\vec{u}}_{\text{Analytic} \Rightarrow \frac{\partial H}{\partial \vec{u}} = \vec{0}} \quad ; \ |\vec{u}| \le u_{max}$$

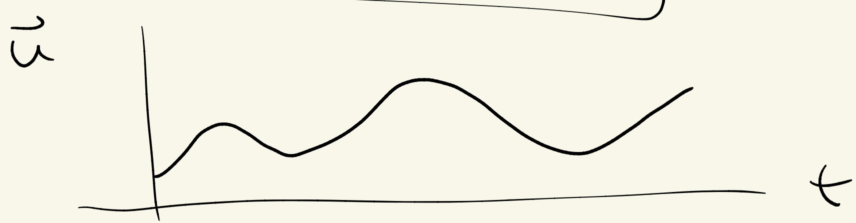
$$\frac{\partial H}{\partial \vec{u}} = \vec{p}_v + \vec{u} = 0 \Rightarrow \boxed{\vec{u}^* = -\vec{p}_v} \Rightarrow$$

$\vec{p}_v$ is smooth.....

$\vec{u}^*$ is smooth!

$$\boxed{H^* = \vec{p}_r \cdot \vec{v} + \vec{p}_v \cdot \vec{g} \underbrace{- \vec{p}_v \cdot \vec{p}_v + \frac{1}{2}\vec{p}_v \cdot \vec{p}_v}_{-\frac{1}{2}\vec{p}_v \cdot \vec{p}_v}}$$

$\vec{u}^*$ is smooth, $\neq \vec{0}$



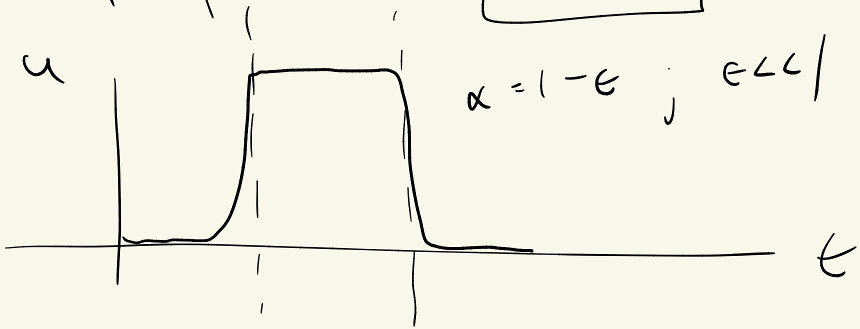
Development of "Homotopy" Methods ...

$$\boxed{L = \underbrace{\alpha|\vec{u}|}_{} + (1-\alpha)\frac{1}{2}\vec{u}\cdot\vec{u}}$$

Solve the O.C. problem for $\alpha = 0$

Increment $\alpha \neq 0$, $\alpha \ll 1$ $\Rightarrow$ resolve the problem.

Repeat process $\Rightarrow$ $\boxed{\alpha \rightarrow 1}$



$\alpha = 1-\epsilon$; $\epsilon \ll 1$

2-Body Problem Primer Vector Theory

$$\vec{g} = -\frac{\mu}{r^3}\vec{r}$$

$$\frac{d\vec{g}}{d\vec{r}} = -\frac{\mu}{r^3} I_{3\times3} + \frac{3\mu}{r^5}\vec{r}\vec{r}^T = -\frac{\mu}{r^3}\left[I_{3\times3} - 3\hat{r}\hat{r}^T \right]$$

$$H^* = \vec{P}_r \cdot \vec{V} + \vec{P}_V \cdot \left(-\frac{\mu}{r^3}\vec{r}\right) - b\left(P_m + \frac{c}{m}P_V\right)$$

$$\boxed{\ddot{\vec{P}}_V = -\dot{\vec{P}}_r = -\frac{\mu}{r^3}\left[I_{3\times3} - 3\hat{r}\hat{r}^T \right] \cdot \vec{P}_V}$$

$$\dot{P}_m = -\frac{bc}{m^2}P_V$$

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r} - \frac{bc}{m}\hat{P}_V$$

$$\dot{m} = -b$$

Transversality Conditions

Let $\vec{r}_0, \vec{v}_0, t_0 = 0, m_0$ be specified.

Then $\vec{P}_{r_0}, \vec{P}_{v_0}, H_0, P_{m_0}$ are arbitrary

Let $\vec{r}_f, \vec{v}_f, t_f$ be "free" $\equiv$ no constraints

$\Rightarrow \vec{P}_{r_f} = \vec{P}_{v_f} = \vec{0}$; $H_f = 0$; $P_{m_f} = -1$

For the 2BP H is Time Invariant, $H =$ constant, $\boxed{H = 0}$

$P_{m_f} = -1$, so $K \neq 0, < 0$.

If $\vec{P}_{r_f} = \dot{\vec{P}}_{v_f} = 0$ & $\vec{P}_{v_f} = 0$, and $\sigma_f = 0$ at the terminal time,

$\ddot{\vec{P}}_{v_f} = \vec{0}$, integrating backwards $\Rightarrow$ $\boxed{\vec{P}_v = \vec{P}_r \equiv \vec{0}}$

$\boxed{\text{Optimal to do Nothing}}$

Relaxing $L = |\vec{u}| \Rightarrow$ minimum ΔV

$L = \frac{1}{2}\vec{u}\cdot\vec{u} \Rightarrow$ " Energy

Cauchy-Schwarz Inequality

$$\left[\int_a^b f(t)\cdot g(t)\,dt\right] \leq \left[\int_a^b f^2(t)\,dt \cdot \int_a^b g^2(t)\,dt\right]^{1/2}$$

Let $f(t) = \sqrt{\vec{u}\cdot\vec{u}}$; $g(t) = 1$

$$\Delta V = \int_a^b |\vec{u}|\,dt \leq \left[\underbrace{\left(\int_a^b \vec{u}\cdot\vec{u}\,dt\right)}_{2J}\cdot(b-a)\right]^{1/2}$$

$$\boxed{J = \frac{1}{2}\int_a^b \vec{u}\cdot\vec{u}\,dt}$$

$$\boxed{b-a = T}$$

$$\boxed{\Delta V(T) \leq \sqrt{T \cdot 2\, J(T)}}$$ $\Rightarrow$ gives a convenient relationship between min ΔV + min Energy cost functions.

$\Downarrow$

$$\Delta V_b^*(T) \leq \sqrt{T \cdot 2 \cdot J^*(T)}$$

ΔV is computed assuming $|\vec{u}| \leq b_{max}$

From the principle of constraint, if I increase $b_{max} \uparrow$, my costs will go down, $\Delta V_b \downarrow$ $b \to \infty \Rightarrow$ Impulsive mnvrs.

$$\boxed{\Delta V_\infty^* \leq \Delta V_b^* \leq \sqrt{2T J^*}}$$ ☆