Question 1

Tuesday, March 1, 2022 13:34

DEFINE THE GATEAUX DERIVATIVE OF THE FUNCTION $f(x)$ AS:

$$df_+(\bar{x},\bar{u}) = \lim_{\lambda \to 0^+} \frac{1}{\lambda}\left[f(\bar{x}+\lambda\bar{u}) - f(\bar{x}) \right]$$

FOR $\bar{x}, \bar{u} \in \mathbb{R}^n$. USING THE NECESSARY CONDITION FOR A MINIMIZER $\bar{x}^*$ OF A NON-SMOOTH FUNCTION:

$$df_+(\bar{x}^*,\bar{u}) \geq 0 \quad \text{FOR ALL } \bar{u}$$

SHOW THAT THE FUNCTION

$$f(x) = \sqrt{\bar{x}\cdot\bar{x}}$$

IS MINIMIZED AT $\bar{x} = \bar{0}$. FIRST START WITH THE CASE OF $n=1$ AND THEN GENERALIZE TO $n>1$.

$\boxed{n=1}$

$\bar{x}^* = 0$ (scalar) u

$$f(0) = \sqrt{0\cdot 0} = 0$$

$$df_+(0,u) = \lim_{\lambda\to 0^+} \frac{1}{\lambda}\left[f(0+\lambda u) - 0\right]$$

$$= \lim_{\lambda\to 0^+} \frac{1}{\lambda} f(\lambda u) = \lim_{\lambda\to 0^+}\frac{1}{\lambda}\sqrt{(\lambda u)(\lambda u)}$$

$$= \lim_{\lambda\to 0^+} \frac{\sqrt{u^2 \overset{>0}{\lambda^2}}}{\lambda} = \lim_{\lambda\to 0^+} \frac{|u|\cancel{\lambda}}{\cancel{\lambda}} = |u|$$

$u^2 \geq 0 \to \sqrt{u^2} \geq 0 \Rightarrow u \geq 0$

$$\boxed{df_+(0,u) = |u| \geq 0 \quad \text{for all } u}$$

$\Rightarrow x^* = 0$ is a minimizer for the scalar case $(n=1)$

$\boxed{n>1}$

$$\bar{x}^* = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}$$

$$f(\bar{x}^*) = \sqrt{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}} = \sqrt{0}$$

$$df_+(\bar{x}^*,\bar{u}) = \lim_{\lambda\to 0^+}\frac{1}{\lambda}\left[f(\bar{x}^*+\lambda\bar{u}) - \overset{0}{\cancel{f(\bar{x}^*)}}\right]$$

$$= \lim_{\lambda\to 0^+}\frac{1}{\lambda}\left[f(\lambda\bar{u})\right]$$

$$= \lim_{\lambda\to 0^+}\frac{1}{\lambda}\left(\sqrt{\lambda\bar{u}\cdot\lambda\bar{u}}\right) \qquad \bar{u}\cdot\bar{u} = |\bar{u}|^2$$

$$= \lim_{\lambda\to 0^+}\frac{|\lambda\bar{u}|}{\lambda} = \lim_{\lambda\to 0^+}\frac{\cancel{\lambda}|\bar{u}|}{\cancel{\lambda}}$$

$$\boxed{df_+(\bar{x}^*,\bar{u}) = |\bar{u}| \geq 0}$$

MAGNITUDE ALWAYS ≥ 0

$\bar{x}^* = \bar{0}_{n\times 1}$ IS A MINIMIZER FOR ALL $n \geq 1$

Question 2

Tuesday, March 1, 2022 15:05

DEFINE A QUADRATIC FUNCTION AS:

$$f(x) = \sum_{i,j=1}^{n} q_{ij} x_i x_j = x^T Q x$$

WHERE $x \in \mathbb{R}^n$ AND IS A COLUMN VECTOR WITH ELEMENTS x_i AND $Q \in \mathbb{R}^{n \times n}$ AND IS A MATRIX WITH ELEMENTS q_{ij} AND THE $(\cdot)^T$ DENOTES A TRANSPOSE. SHOW

(a) $f(x) = \frac{1}{2} x^T (Q + Q^T) x = x^T Q x$

$$= \left[\frac{1}{2} x^T Q + \frac{1}{2} x^T Q^T\right] x = \underbrace{\frac{1}{2} \overset{1\times n \;\; n\times n \;\; n\times 1}{x^T Q x}}_{1\times 1} + \underbrace{\frac{1}{2} \overset{1\times n \;\; n\times n \;\; n\times 1}{x^T Q^T x}}_{1\times 1}$$

scalars $\hookrightarrow a = a^T$

$$= \frac{1}{2} x^T Q x + \frac{1}{2}(x^T Q^T x)^T$$

PROPERTY

$(\bar{a}^T B \bar{c})^T = (\bar{c}^T B^T \bar{a})$

if $\bar{a} = \bar{c}$

$\Rightarrow (\bar{a}^T B \bar{a})^T = \bar{a}^T B^T \bar{a}$

$$= \frac{1}{2} x^T Q x + \frac{1}{2} x^T \underbrace{(Q^T)^T}_{=Q} x$$

$$= \frac{1}{2} x^T Q x + \frac{1}{2} x^T Q x$$

$$\Rightarrow \boxed{f(x) = \frac{1}{2} x^T (Q + Q^T) x = x^T Q x}$$ ✓

(b) $\frac{\partial f}{\partial x} = (Q + Q^T) x$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[x^T Q x\right] = \frac{\partial}{\partial x}\left[\frac{1}{2} x^T (Q + Q^T) x\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{2} x^T Q x + \frac{1}{2} x^T Q^T x\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{2} x^T Q x\right) + \frac{\partial}{\partial x}\left(\frac{1}{2} x^T Q^T x\right)$$

PROPERTIES OF TRANSPOSE:

$$\frac{d}{dx}(\bar{y}^T B \bar{y}) = 2B\bar{y}$$

$$= \frac{1}{2}(2Qx) + \frac{1}{2}(2Q^T x)$$

$$= Qx + Q^T x$$

$$\boxed{\frac{\partial f}{\partial x} = (Q + Q^T) x}$$ ✓

Question 3

Tuesday, March 1, 2022 15:50

DEFINE AN OBJECTIVE FUNCTION TO BE:

$$f(x,y) = -\sin(x)\cos(y)$$

USING THE FIRST AND SECOND ORDER NECESSARY CONDITIONS FIND CANDIDATES FOR THE MINIMIZERS OF f. ARE ALL THE CANDIDATES LOCAL MINIMIZERS? WHAT ARE THE GLOBAL MINIMIZERS? HOW SHOULD THE DOMAIN OF (x,y) BE RESTRICTED FOR THERE TO BE A UNIQUE GLOBAL MINIMIZER?

FIRST ORDER $\frac{\partial f}{\partial x} = \bar{0}$ SECOND ORDER $\frac{\partial^2 f}{\partial \bar{x}^2} > 0$

1ST ORDER

$$\bar{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \frac{\partial f}{\partial \bar{x}} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} -\cos(x)\cos(y) & \sin(x)\sin(y) \end{bmatrix}$$

Candidates

$$\boxed{\begin{array}{ll} x = \pi k + \frac{\pi}{2},\ y = \pi \ell & k \,\&\, \ell \text{ integers} \quad (1) \\ x = \pi k,\ y = \pi\ell + \frac{\pi}{2} & (2) \end{array}}$$

2ND ORDER $\frac{\partial^2 f}{\partial \bar{x}^2} > 0$

$$\frac{\partial^2 f}{\partial \bar{x}^2} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial^2 y} \end{bmatrix} = \begin{bmatrix} \sin x \cos y & \cos x \sin y \\ \cos x \sin y & \sin x \cos y \end{bmatrix}$$

TEST SOLUTION

(1) $(x,y) = (\pi k + \frac{\pi}{2}, \pi \ell)$ $\quad \frac{\partial^2 f}{\partial \bar{x}^2} = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$

NEED $\sin x$ & $\cos y$ EITHER BOTH $+1$ OR BOTH -1 TO MAKE $\frac{\partial^2 f}{\partial \bar{x}^2}$ POSITIVE DEFINITE

$+1$: $x = \frac{\pi}{2}, \frac{5\pi}{2} \rightarrow 2\pi k + \frac{\pi}{2}$; $y = 2\pi\ell$

-1 $x = \frac{3\pi}{2}, \frac{7\pi}{2} \rightarrow 2\pi k + \frac{3\pi}{2}$; $y = 2\pi \ell + \pi$

$$\boxed{\text{MINIMIZERS: } (x,y) = \begin{cases} (2\pi k + \frac{\pi}{2}, 2\pi \ell) \\ (2\pi k + \frac{3\pi}{2}, 2\pi\ell + \pi) \end{cases} \quad k, \ell : \text{integers}}$$

TEST SOLUTION

(2) $(x,y) = (\pi k, \pi \ell + \frac{\pi}{2})$ $\quad \frac{\partial^2 f}{\partial \bar{x}^2} = \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix}$

REGARDLESS OF CHOICES OF k & ℓ $\frac{\partial^2 f}{\partial \bar{x}^2}$ CANNOT BE MADE POSITIVE DEFINITE OR POSITIVE SEMI-DEFINITE

eigenvalues of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \lambda = -1, 1$

" " $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \rightarrow \lambda = -1, 1$

$\Rightarrow$ TEST SOLUTION IS NOT VALID

ARE ALL CANDIDATES LOCAL MINIMIZERS?

$\rightarrow$ No, all candidates are not local minimizers (2) gives saddle points and only a subset of (1) gives minimizers, others are maximizers.

WHAT ARE THE GLOBAL MINIMIZERS?

$\rightarrow$ There are no global minimizers unless the domain is restricted such that only one local minimizer (of the unrestricted domain) appears

OR $\rightarrow$ All local minimizers are global minimizers, although they are not unique.

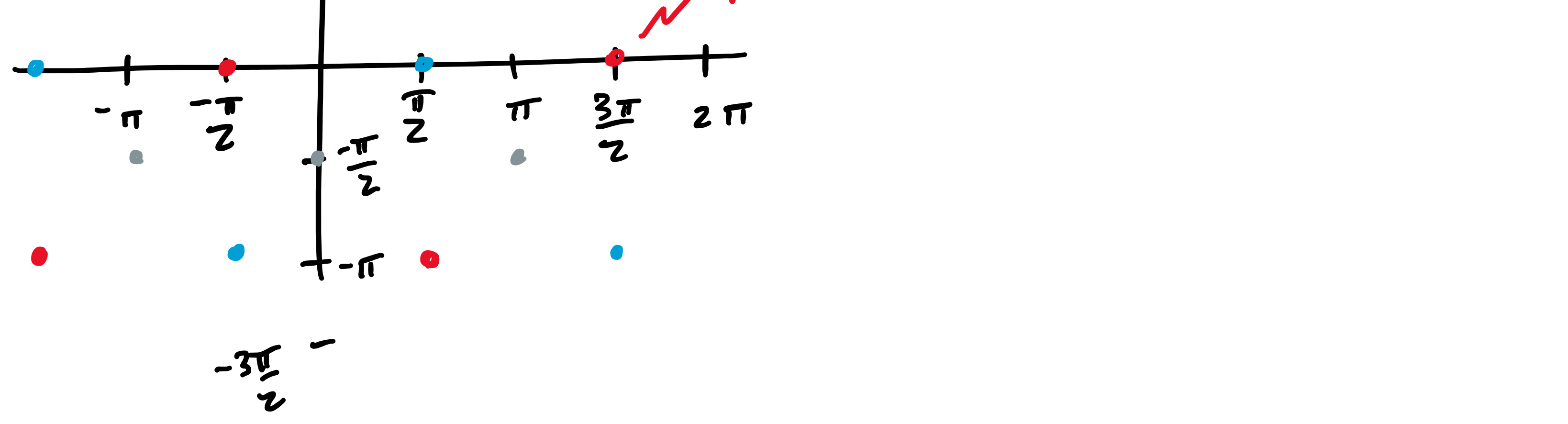
HOW SHOULD THE DOMAIN BE RESTRICTED FOR THERE TO BE A UNIQUE GLOBAL MINIMIZER?

$\rightarrow$ As previously stated, there will be a unique global minimizer if the domain is restricted to contain only one local minimizer. There are infinitely many ways to restrict the domain to make this occur.

One possibility:

$$x \in (\pi, 2\pi)$$
$$y \in (\tfrac{\pi}{2}, \tfrac{3\pi}{2})$$

$(x,y) = (\frac{3\pi}{2}, \pi)$ would become a global minimizer with $f(\frac{3\pi}{2}, \pi) = -1$ the minimum value



Question 4

Tuesday, March 1, 2022 16:53

CONSIDER THE GENERAL QUADRATIC FUNCTION

$$f(\bar{x}) = a + \bar{b}^T\bar{x} + \bar{x}^T Q \bar{x}$$

WHERE $\bar{x} \in \mathbb{R}^n$, $a \in \mathbb{R}$, $b \in \mathbb{R}^n$ and $Q \in \mathbb{R}^{n\times n}$ and $Q^T = Q$. GIVE THE NECESSARY CONDITIONS FOR OPTIMALITY (i.e. MINIMA) IN $\mathbb{R}^n$ TO EXIST. GIVE A SUFFICIENT CONDITION FOR OPTIMALITY TO EXIST. GIVE AN EXAMPLE WHERE $Q \geq 0$ (i.e. Q IS SEMI-DEFINITE) AND A unique MINIMUM DOES NOT EXIST.

NECESSARY CONDITIONS FOR $\bar{x}^*$ TO BE A MINIMIZER

$$\left.\frac{\partial f}{\partial \bar{x}}\right|_{\bar{x}^*} = 0$$

$$\frac{\partial f}{\partial \bar{x}} = \frac{\partial}{\partial x}\left[a + \bar{b}^T\bar{x} + \bar{x}^T Q \bar{x}\right]$$

$$= \bar{b} + 2Q\bar{x} = 0$$

$$2Q\bar{x} = -\bar{b}$$

$$Q\bar{x} = -\tfrac{1}{2}\bar{b}$$

$$\boxed{\bar{x} = -\tfrac{1}{2}Q^{-1}\bar{b}}$$

SUFFICIENCY CONDITION

$$\frac{\partial^2 f}{\partial x^2} > 0 \qquad \frac{\partial^2 f}{\partial x^2} = 2Q > 0$$

$$\boxed{Q \text{ MUST BE POSITIVE DEFINITE}}$$

OK TO SAY SEMI-DEFINITE, JUST NOT SUFFICIENT FOR UNIQUE SOLUTION

↳ eigenvalues > 0

of all $z \in \mathbb{R}^n$ $\quad \bar{z}^T Q \bar{z} > 0$

If $Q = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 0 \end{bmatrix}$, $a = 0$, $\bar{b} = \bar{0}$, then $f(\bar{x}) = x_1^2$. The minimum will be $x_1 = 0$, but $\bar{x}_{2:n}$ are all free $\bar{x}_{2:n} \in \mathbb{R}$, so there are infinite minima at $\bar{x}^* = \left\{ \begin{matrix} x_1 = 0 \\ \vdots \\ x_n \in \mathbb{R} \end{matrix} \right\}$

Question 5

Wednesday, March 2, 2022 16:47

FIND THE MINIMUM DISTANCE BETWEEN THE PLANE P AND THE ORIGIN OF A COORDINATE SYSTEM. THE PLANE P IS DEFINED BY THE EQUATION

$$\bar{b}\cdot\bar{x} + a = 0$$

WHERE $\bar{x} \in \mathbb{R}^3$, $\bar{b} \in \mathbb{R}^3$ IS NORMAL TO THE PLANE AND $a \in \mathbb{R}$. FOR THE COST FUNCTION USE BOTH $J = \frac{1}{2}\bar{x}\cdot\bar{x}$ AND $J = \sqrt{\bar{x}\cdot\bar{x}}$ AND SHOW THE MINIMUM POINT IS THE SAME. COMMENT ON ANY DIFFERENCES IN THE SOLUTION PROCEDURE.

$\boxed{J = \frac{1}{2}\bar{x}\cdot\bar{x}}$

$$L(\bar{x},\lambda) = \underbrace{J(\bar{x})}_{\frac{1}{2}\bar{x}\cdot\bar{x}} + \lambda \underbrace{g(x)}_{\bar{b}\cdot\bar{x}+a}$$

$$L = \frac{1}{2}\bar{x}\cdot\bar{x} + \lambda(\bar{b}\cdot\bar{x} + a)$$

$$\frac{\partial L}{\partial x} = \frac{1}{2}(\bar{x}+\bar{x}) + \lambda\bar{b} = 0$$

$$= \bar{x} + \lambda\bar{b} = 0$$

$$\boxed{\bar{x} = -\lambda\bar{b}}$$

$$g(\bar{x}) = 0 \quad\Rightarrow\quad \bar{b}\cdot(-\lambda\bar{b}) + a = 0$$

$$-\lambda\underbrace{(\bar{b}\cdot\bar{b})}_{|b|^2} + a = 0$$

$$\lambda = \frac{a}{|b|^2}$$

$$\Rightarrow \boxed{\bar{x} = -\frac{a\bar{b}}{|b|^2}}$$

$\boxed{J = \sqrt{\bar{x}\cdot\bar{x}}}$

$$L(\bar{x},\lambda) = \sqrt{\bar{x}\cdot\bar{x}} + \lambda(\bar{b}\cdot\bar{x} + a)$$

$$\frac{\partial L}{\partial x} = \frac{1}{2}\left[\bar{x}\cdot\bar{x}\right]^{-1/2}\left[2\bar{x}\right] + \lambda\bar{b} = 0$$

$$= \frac{\bar{x}}{\sqrt{\bar{x}\cdot\bar{x}}} + \lambda\bar{b} = 0 \quad\Rightarrow\quad \boxed{\bar{x} = -\lambda\bar{b}\sqrt{\bar{x}\cdot\bar{x}}}$$

$$-\lambda\bar{b}\sqrt{\bar{x}\cdot\bar{x}} = \bar{x}$$ SQUARE BOTH SIDES

$$\lambda^2(\bar{b}\cdot\bar{b})(\cancel{\bar{x}\cdot\bar{x}}) = \cancel{\bar{x}\cdot\bar{x}}$$

$$\lambda^2(\bar{b}\cdot\bar{b}) = 1 \quad\Rightarrow\quad \boxed{\lambda = \frac{1}{\sqrt{\bar{b}\cdot\bar{b}}} = \frac{1}{|b|}}$$

$$\bar{x} = -\frac{1}{\sqrt{\bar{b}\cdot\bar{b}}}\,\bar{b}\sqrt{\bar{x}\cdot\bar{x}}$$

$$g(\bar{x}) = 0 \quad\Rightarrow\quad \bar{b}\cdot\bar{x} + a = 0$$

$$\bar{b}\cdot\left[-\frac{1}{\sqrt{\bar{b}\cdot\bar{b}}}\,\bar{b}\sqrt{\bar{x}\cdot\bar{x}}\right] + a = 0$$

$$-\frac{\bar{b}\cdot\bar{b}}{\sqrt{\bar{b}\cdot\bar{b}}}\sqrt{\bar{x}\cdot\bar{x}} + a = 0$$

$$a = \sqrt{\bar{b}\cdot\bar{b}}\sqrt{\bar{x}\cdot\bar{x}}$$

$$\sqrt{\bar{x}\cdot\bar{x}} = \frac{a}{\sqrt{\bar{b}\cdot\bar{b}}}$$

$$\bar{x} = -\frac{1}{\sqrt{\bar{b}\cdot\bar{b}}}\,\bar{b}\left(\frac{a}{\sqrt{\bar{b}\cdot\bar{b}}}\right)$$

$$\bar{x} = \frac{-a\bar{b}}{\bar{b}\cdot\bar{b}} = \frac{-a\bar{b}}{|b|^2}$$

$$\boxed{\bar{x} = \frac{-a\bar{b}}{|b|^2}}$$

BOTH METHODS YIELD THE SAME RESULT, HOWEVER USING THE COST FUNCTION $J = \sqrt{\bar{x}\cdot\bar{x}}$ PROVED MORE COMPLICATED TO SOLVE. THIS SHOWS HOW A CAREFUL CHOICE OF THE COST FUNCTION, J, CAN SIMPLIFY THE SOLUTION PROCEDURE.

Question 6

Wednesday, March 2, 2022 16:52

REPEAT THE PROCEDURE FROM QUESTION 5, NOW FINDING THE MINIMUM DISTANCE TO THE ORIGIN UNDER THE INEQUALITY CONSTRAINT

$$\bar{b}\cdot\bar{x} + a \geq 0$$

CONSIDER ALL POSSIBLE SOLUTIONS AS A FUNCTION OF a AND $\bar{b}$.

$\boxed{J = \tfrac{1}{2}\bar{x}\cdot\bar{x}}$ $\quad h(\bar{x}) \geq 0$

$\underbrace{}_{f(\bar{x})}$ $\quad \bar{b}\cdot\bar{x} + a \geq 0$

no equality constraints exist — inequality constraint

$$L = \lambda_0 f(\bar{x}) + \cancel{\lambda\cdot g(\bar{x})} + \sigma\cdot h(\bar{x})$$

$$L = \lambda_0\left[\tfrac{1}{2}\bar{x}\cdot\bar{x}\right] + \sigma\cdot(\bar{b}\cdot\bar{x} + a)$$

$\lambda_0 = 1 \qquad L = \tfrac{1}{2}\bar{x}\cdot\bar{x} + \sigma\cdot(\bar{b}\cdot\bar{x} + a)$

NECESSARY CONDITIONS

$(\lambda_0, \bar{\lambda}, \bar{\sigma}) \neq \bar{0}, \quad \lambda_0 \geq 0, \quad \sigma_j \leq 0 \rightarrow$ since $\sigma \in \mathbb{R}^{1\times 1} \rightarrow \sigma \neq 0$ $\boxed{\sigma < 0}$

$$\left.\frac{\partial L}{\partial x}\right|_* = \bar{0} \qquad g(\bar{x}^*) = \bar{0} \qquad h(\bar{x}^*) \geq 0$$

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x}\left[\tfrac{1}{2}\bar{x}\cdot\bar{x} + \sigma\cdot(\bar{b}\cdot\bar{x} + a)\right] = 0$$

$$= \bar{x} + \sigma\bar{b} = 0$$

$$\bar{x} = -\sigma\bar{b}$$

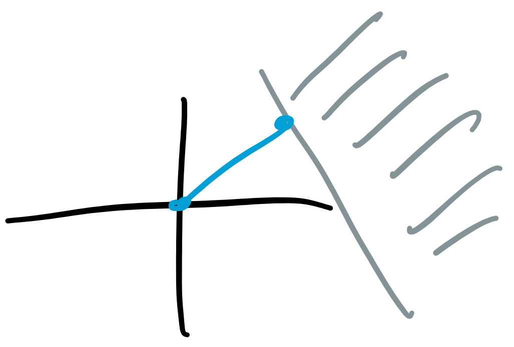
$h(\bar{x}^*) \geq 0 \qquad \bar{b}\cdot\bar{x} + a \geq 0$

$$\bar{b}\cdot(-\sigma\bar{b}) + a \geq 0$$

consider $h=0$ case $(\sigma < 0)$

$$-\sigma\bar{b}\cdot\bar{b} = -a$$

$$\sigma\underbrace{\bar{b}\cdot\bar{b}}_{>0} = a$$



2D REPRESENTATION OF $a<0$

$\sigma = \dfrac{a}{\bar{b}\cdot\bar{b}}$ but $\boxed{a<0}$ to make $\sigma < 0$

$\frac{\partial L}{\partial x} = 0$

$\bar{x} + \dfrac{a}{\bar{b}\cdot\bar{b}}\bar{b} = 0 \Rightarrow \boxed{\bar{x}^* = -\dfrac{a\bar{b}}{\bar{b}\cdot\bar{b}}}$ $\Rightarrow$ same result as Q5, but with added condition that $a<0$

consider $h>0$ $(\sigma = 0)$ — INEQUALITY CONSTRAINT INACTIVE → FEASIBLE SET CONTAINS THE ORIGIN

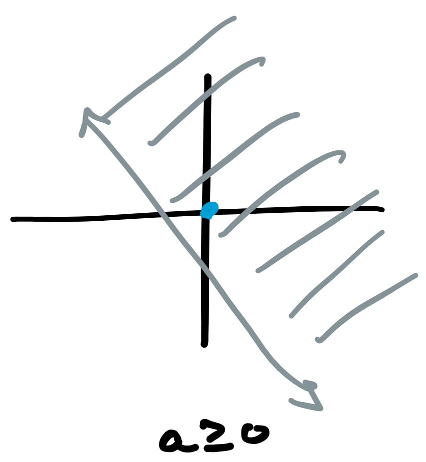
$$\frac{\partial L}{\partial x} = \bar{x} + \overset{0}{\cancel{\sigma}}\bar{b} = 0$$

$$\boxed{\bar{x}^* = \bar{0}}$$

check $h(\bar{x}^*) \geq 0$

$$\bar{b}\cdot\bar{0} + a \geq 0$$

$$\boxed{a \geq 0}$$



$a \geq 0$

Question 7 (2025)

Monday, March 10, 2025 14:08

MINIMIZE $f(\vec r) = \frac{1}{2}\vec r \cdot Q \cdot \vec r$ SUBJECT TO THE TWO CONSTRAINTS

$$g_1(\vec r) = \hat a_1 \cdot \vec r + b_1 = 0$$
$$g_2(\vec r) = \hat a_2 \cdot \vec r + b_2 = 0$$

IN THE ABOVE, $\vec r, \hat a_1, \hat a_2 \in \mathbb{R}^3$, $|\hat a_1| = |\hat a_2| = 1$, $b_1, b_2 \in \mathbb{R}$ AND

$$Q = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

(a) UNDER WHAT CONSTRAINTS ON $\hat a_1, \hat a_2, b_1$ & b_2 WILL A SOLUTION NOT EXIST ?

A solution will not exist if $\hat a_1 \parallel \hat a_2$, but $b_1 \neq b_2$, similarly if $\hat a_1 = -\hat a_2$, and $b_1 \neq -b_2$, then a solution will not exist. Essentially, if the planes do not intersect, then both constraints cannot be satisfied simultaneously.

(b) ASSUMING THAT A SOLUTION EXISTS, GIVE A FORM OF THE OPTIMAL SOLUTION FOR $\vec r$ AS A FUNCTION OF $\hat a_i$ & b_i

DEFINE LAGRANGIAN

$$L = \frac{1}{2}\vec r^T Q \vec r + \underbrace{\begin{bmatrix}\lambda_1 \\ \lambda_2\end{bmatrix}^T}_{1\times 2} \overbrace{\underbrace{\begin{bmatrix}\hat a_1^T \vec r + b_1 \\ \hat a_2^T \vec r + b_2\end{bmatrix}}_{2\times 1}}^{\vec g(\vec r)\ 3\times 1} \qquad \vec b = \underbrace{\begin{bmatrix} b_1 \\ b_2\end{bmatrix}}_{2\times 1}$$

DEFINE $\Lambda = \underbrace{\begin{bmatrix}\lambda_1 \\ \lambda_2\end{bmatrix}}_{2\times 1}$ $\qquad A = \underbrace{\begin{bmatrix}\hat a_1^T \\ \hat a_2^T\end{bmatrix}}_{2\times 3}$

$$L = \frac{1}{2}\vec r^T Q \vec r + \Lambda^T (A\vec r + \vec b)$$

NECESSARY CONDITION (1^{ST} ORDER) $\rightarrow \frac{\partial L}{\partial \vec r} = 0$

$$\frac{\partial L}{\partial \vec r} = \frac{1}{2}\underbrace{(Q + Q^T)}_{=2\mathbb{I}}\vec r + \underbrace{\Lambda^T}_{1\times 2}\underbrace{A}_{2\times 3}$$

$$0 = \vec r + \Lambda^T A \Rightarrow \boxed{\vec r = -\Lambda^T A}\ (\star) \Rightarrow -\lambda_1 \hat a_1 - \lambda_2 \hat a_2$$

$\hookrightarrow = -A^T \Lambda$

USE CONSTRAINTS TO SOLVE FOR Λ

$$\vec g(\vec r) = A\vec r + \vec b = 0$$
$$A(-A^T\Lambda) + \vec b = 0$$
$$\boxed{\Lambda = (AA^T)^{-1}\vec b}\ (+)$$

SUBSTITUTE $(+)$ INTO $(\star)$

$$\boxed{\vec r = -\underbrace{A^T}_{3\times 2}\underbrace{(AA^T)^{-1}}_{2\times 2}\underbrace{\vec b}_{2\times 1}}$$

$$\vec r = -\underbrace{[\hat a_1\ \ \hat a_2]}_{3\times 2}\underbrace{\begin{bmatrix} \hat a_1^T \hat a_1 & \hat a_1^T \hat a_2 \\ \hat a_2^T \hat a_1 & \hat a_2^T \hat a_2\end{bmatrix}^{-1}}_{2\times 2}\vec b$$

$\hookrightarrow \hat a_1^T \hat a_2 = \hat a_2^T \hat a_1$

$$\vec r = -[\hat a_1\ \ \hat a_2]\begin{bmatrix} 1 & \hat a_1^T \hat a_2 \\ \hat a_1^T \hat a_2 & 1\end{bmatrix}^{-1}\vec b \qquad c = \hat a_1^T \hat a_2$$

$$\vec r = -[\hat a_1\ \ \hat a_2]\frac{1}{1-c^2}\underbrace{\begin{bmatrix} 1 & -c \\ -c & 1\end{bmatrix}}_{2\times 2}\underbrace{\vec b}_{2\times 1}$$

$$= -\frac{1}{1-c^2}[\hat a_1\ \ \hat a_2]\begin{bmatrix} b_1 - c b_2 \\ b_2 - c b_1\end{bmatrix}$$

$$\boxed{\vec r^* = -\left(\frac{b_1 - c b_2}{1-c^2}\right)\hat a_1 - \left(\frac{b_2 - c b_1}{1-c^2}\right)\hat a_2}$$

$$\frac{\partial^2 L}{\partial \vec r^2} = \mathbb{I} > 0 \Rightarrow \vec r^* \text{ IS A MINIMIZER}$$

Question 8 (2025)

Thursday, March 3, 2022 14:16

FROM LAWDEN pg. 22, PROBLEM 3: GIVEN A SET OF n NUMBERS x_i, $i = 1, 2, \dots, n$, THE GEOMETRIC AND ARITHMETIC MEANS ARE COMPUTED AS

$$GM(x) = (x_1 x_2 \cdots x_n)^{1/n}$$
$$AM(x) = \frac{1}{n}[x_1 + x_2 + \dots x_n]$$

RESPECTIVELY. LET $J = AM(x)$ BE THE COST FUNCTION WITH THE CONSTRAINT $GM(x) = g$, AN ARBITRARY POSITIVE CONSTANT VALUE. SHOW THAT THE OPTIMAL SOLUTION FOR MINIMIZING J SUBJECT TO THE CONSTRAINT OCCURS WHEN THE ELEMENTS x_i ARE ALL EQUAL.

$$J = AM(\bar{x}) = \frac{1}{n}[x_1 + x_2 + \dots x_n]$$
$$g(\bar{x}) = g - GM(\bar{x}) = 0 = g - (x_1 x_2 \cdots x_n)^{1/n}$$

$$L(\bar{x}, \lambda) = J(\bar{x}) + \lambda g(\bar{x})$$
$$= \frac{1}{n}[x_1 + x_2 + \dots x_n] + \lambda[g - (x_1 x_2 \cdots x_n)^{1/n}]$$

$$\frac{\partial L}{\partial x_i} = \frac{1}{n} + \lambda\left[\frac{1}{n}(x_1 x_2 \cdots x_n)^{1/n - 1}\left(\frac{1}{x_i}(x_1 x_2 \cdots x_n)\right)\right] = 0$$

$$= \frac{1}{n} + \lambda\left[\frac{1}{n}\left[\frac{(x_1 x_2 \cdots x_n)^{1/n}}{(x_1 \cancel{x_2} \cdots x_n)}\right]\right]\left[\frac{1}{x_i}(\cancel{x_1 + x_2} \cdots x_n)\right] = 0$$

$$= \frac{1}{n} + \lambda\left[\frac{1}{n}\underbrace{(x_1 x_2 \cdots x_n)^{1/n}}_{GM(\bar{x})}\right]\left(\frac{1}{x_i}\right) = 0$$

$$= \frac{1}{n} + \lambda \frac{1}{n} \frac{1}{x_i} GM(\bar{x}) = 0$$

$$\cancel{\frac{1}{n}} = -\lambda \cancel{\frac{1}{n}} \frac{1}{x_i} GM(\bar{x})$$

$$1 = -\lambda \frac{1}{x_i} GM(\bar{x})$$

$$x_i = \underline{-\lambda GM(\bar{x})} \Rightarrow$$ all values x_i $i = 1, \dots, n$ must be the same

$$GM(\bar{x}) = \frac{x_i}{-\lambda}$$ $-\lambda g \to$ a constant

SUB INTO CONSTRAINT EQUATION

$$0 = g - GM(\bar{x}) = g + \frac{x_i}{\lambda}$$

$$g = -\frac{x_i}{\lambda} \Rightarrow x_i = -\lambda g$$

$$GM(\bar{x}) = [(-\lambda g)^n]^{1/n} = [(-\lambda)^n g^n]^{1/n} = g$$

GM must be a positive number

$$\Rightarrow \boxed{\lambda = -1 \quad x_i = g}$$

Question 9 (2025)

Monday, March 7, 2022 10:33

CONSIDER THE OBJECTIVE FUNCTION

$$f(\bar{x}) = (x_1-2)^2 + x_2^2 + x_3^2$$

SUBJECT TO THE TWO CONSTRAINT EQUATIONS

$$g_1(\bar{x}) = x_1^2 + x_2^2 + x_3^2 - 2 = 0$$
$$g_2(\bar{x}) = x_1^2 + x_2^2 - 1 = 0$$

- APPLY THE NECESSARY CONDITIONS TO THIS PROBLEM AND FIND ALL POSSIBLE SOLUTIONS. CHARACTERIZE THE SOLUTIONS AS MAXIMA, MINIMA, OR SADDLE POINTS

$$L(\bar{x},\bar{\lambda}) = \left[(x_1-2)^2 + x_2^2 + x_3^2\right] + \lambda_1\left[x_1^2 + x_2^2 + x_3^2 - 2\right] + \lambda_2\left[x_1^2 + x_2^2 - 1\right]$$

$g_2:\ x_1^2 + x_2^2 = 1$

$g_1:\ \underbrace{(x_1^2+x_2^2)}_{=1} + x_3^2 - 2 = 0$

$x_3^2 = 1 \Rightarrow \boxed{x_3 = \pm 1}$

$$\frac{\partial L}{\partial x_1} = 0 = 2(x_1-2) + 2\lambda_1 x_1 + 2\lambda_2 x_1 = 0$$
$$2x_1 - 4 + 2\lambda_1 x_1 + 2\lambda_2 x_1 = 0$$
$$2x_1(1+\lambda_1+\lambda_2) = 4$$
$$x_1 = 2/(1+\lambda_1+\lambda_2)$$

$$\frac{\partial L}{\partial x_2} = 2x_2 + 2\lambda_1 x_2 + 2\lambda_2 x_2 = 0$$
$$2x_2(1+\lambda_1+\lambda_2) = 0$$

$$\frac{\partial L}{\partial x_3} = 2x_3 + 2\lambda_1 x_3 = 0$$
$$\lambda_1 = -\frac{2x_3}{2x_3} = -1 \qquad \boxed{\lambda_1 = -1}$$

$$2x_2(1-1+\lambda_2) = 0$$
$$2x_2\lambda_2 = 0 \quad x_2 = 0 \text{ or } \lambda_2 = 0$$

$$2x_1(1-1+\lambda_2) = 4$$
$$x_1\lambda_2 = 2 \Rightarrow x_1 = \frac{2}{\lambda_2},\ \lambda_2 = \frac{2}{x_1}$$

↳ shows that $\lambda_2 \neq 0 \Rightarrow \boxed{x_2 = 0}$

$g_1:\ x_1^2 + x_2^2 = 1$

$$\left(\frac{2}{\lambda_2}\right)^2 = 1 \Rightarrow 4 = \lambda^2 \Rightarrow \boxed{\lambda_2 = \pm 2}$$

$$x_1 = \frac{2}{\lambda_2} \Rightarrow \boxed{x_1 = \pm 1}$$

CANDIDATE OPTIMAL POINTS

a) (1, 0, 1)
b) (−1, 0, 1)
c) (1, 0, −1)
d) (−1, 0, −1)

$\delta^2 L = \frac{\partial^2 L}{\partial \bar{x}^2}\Big|_{x^*}$

SECOND VARIATION OF $L \Rightarrow$ min, max, saddle

pos def ⇒ global min
neg def ⇒ global max
indefinite ⇒ saddle

pos semi-def ⇒ local min
neg semi-def ⇒ local max

$$\frac{\partial^2 L}{\partial x_1^2} = 2 + 2\lambda_1 + 2\lambda_2$$
$$\cancel{2+2}(-1)+2(\pm 2) = \pm 4$$

$$\frac{\partial^2 L}{\partial x_2^2} = 2 + 2\lambda_1 + 2\lambda_2 = \pm 4$$

$$\frac{\partial^2 L}{\partial x_3^2} = 2 + 2\lambda_1 = 2 - 2 = 0$$

$$\frac{\partial^2 L}{\partial x_1 \partial x_2} = \frac{\partial^2 L}{\partial x_1 \partial x_3} = \frac{\partial^2 L}{\partial x_2 \partial x_3} = 0$$

$$\frac{\partial^2 L}{\partial \bar{x}^2} = \begin{bmatrix} \pm 4 & 0 & 0 \\ 0 & \pm 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) $\frac{\partial^2 L}{\partial x}\Big|_{x^*} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \geq 0$

(b) $\begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leq 0$

(c) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \geq 0$

(d) $\begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leq 0$

$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ eig(·) = $\begin{matrix} 0 \\ 4 \\ 4 \end{matrix}$ → pos-semi-def

$\begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ eig(·) = $\begin{matrix} 0 \\ -4 \\ -4 \end{matrix}$ → neg-semi-def

(a) & (c) LOCAL MINIMA
(b) & (d) LOCAL MAXIMA

Question 10 (2025)

Monday, March 7, 2022 11:19

USING THE FRITZ-JOHN CONDITIONS, SOLVE THE FOLLOWING OPTIMIZATION PROBLEM:

MAXIMIZE: $f(x,y,z) = y^2 + z^2$

SUBJECT TO:

$x=R, y=0, z=0$

$x^2+y^2+z^2=R^2$

Equality Constraint: $g(x,y,z) = x^2+y^2+z^2-R^2=0$

Inequality Constraint: $h(x,y,z) = x-R \geq 0$ $x \geq R$

ARE THE CONSTRAINTS LINEARLY INDEPENDENT AT THE SOLUTIONS TO THE NECESSARY CONDITIONS? WHAT SHOULD THE VALUE OF λ_0 BE?

Problem asks for maximization – repose problem as minimization in order to employ Fritz-John conditions.

★ ASSUMING $R>0$ ★

Maximization of $f(x,y,z) = y^2+z^2$ is equivalent to the minimization of $-f(x)$, so our problem becomes min of $f'(x,y,z) = -y^2-z^2$

However, by inspection one can see that in order to satisfy both constraints simultaneously

$x \geq R$ and $x^2+y^2+z^2=R^2$

the only solution max/min is $(R,0,0)$

We will proceed as a minimization problem...

NECESSARY CONDITIONS: $\left.\frac{\partial L}{\partial \bar{x}}\right|_* = 0$, $g(\bar{x}^*)=0$, $h(\bar{x}^*) \geq 0$, $\lambda_0 \geq 0$, $\sigma_j \leq 0$

$L(\bar{x},\bar{\lambda},\bar{\sigma}) = \lambda_0(y^2+z^2) + \lambda_1(x^2+y^2+z^2-R^2) + \sigma(x-R)$

$\frac{\partial L}{\partial x} = 2\lambda_1 x + \sigma = 0$ $\quad x = -\frac{\sigma}{2\lambda_1}$ $\quad \sigma \leq 0 \rightarrow$ makes this term >0 as long as $\lambda_1 > 0$

$\frac{\partial L}{\partial y} = 2\lambda_0 y + 2\lambda_1 y = 0$ $\quad \lambda_0 y = -\lambda_1 y \rightarrow \lambda_0 = -\lambda_1$

$\frac{\partial L}{\partial z} = 2\lambda_0 z + 2\lambda_1 z = 0$ $\quad \lambda_0 z = -\lambda_1 z$

gives no information about y^*, z^*

Check to see if constraints are linearly independent

$\frac{\partial g}{\partial \bar{x}} = [2x \quad 2y \quad 2z]$

$\frac{\partial h}{\partial \bar{x}} = [1 \quad 0 \quad 0]$

NOT linearly independent when $y^* = z^* = 0$

$$\boxed{\begin{array}{l}\Rightarrow \lambda_0 = 0, \text{ since } g(\bar{x}) \text{ and } h(\bar{x}) \text{ are not linearly} \\ \text{independent and therefore represent a situation} \\ \text{where the constraints alone control the candidate} \\ \text{points}\end{array}}$$

$\frac{\partial L}{\partial y} = 2\lambda_1 y = 0$

$\frac{\partial L}{\partial z} = 2\lambda_1 z = 0$

$\Rightarrow y = z = 0$

$g(\bar{x}^*) = 0 = x^2 - R^2$

$\Rightarrow x = \pm R$, but $h(\bar{x}^*) \geq 0$

$x - R \geq 0$

$x \geq R$

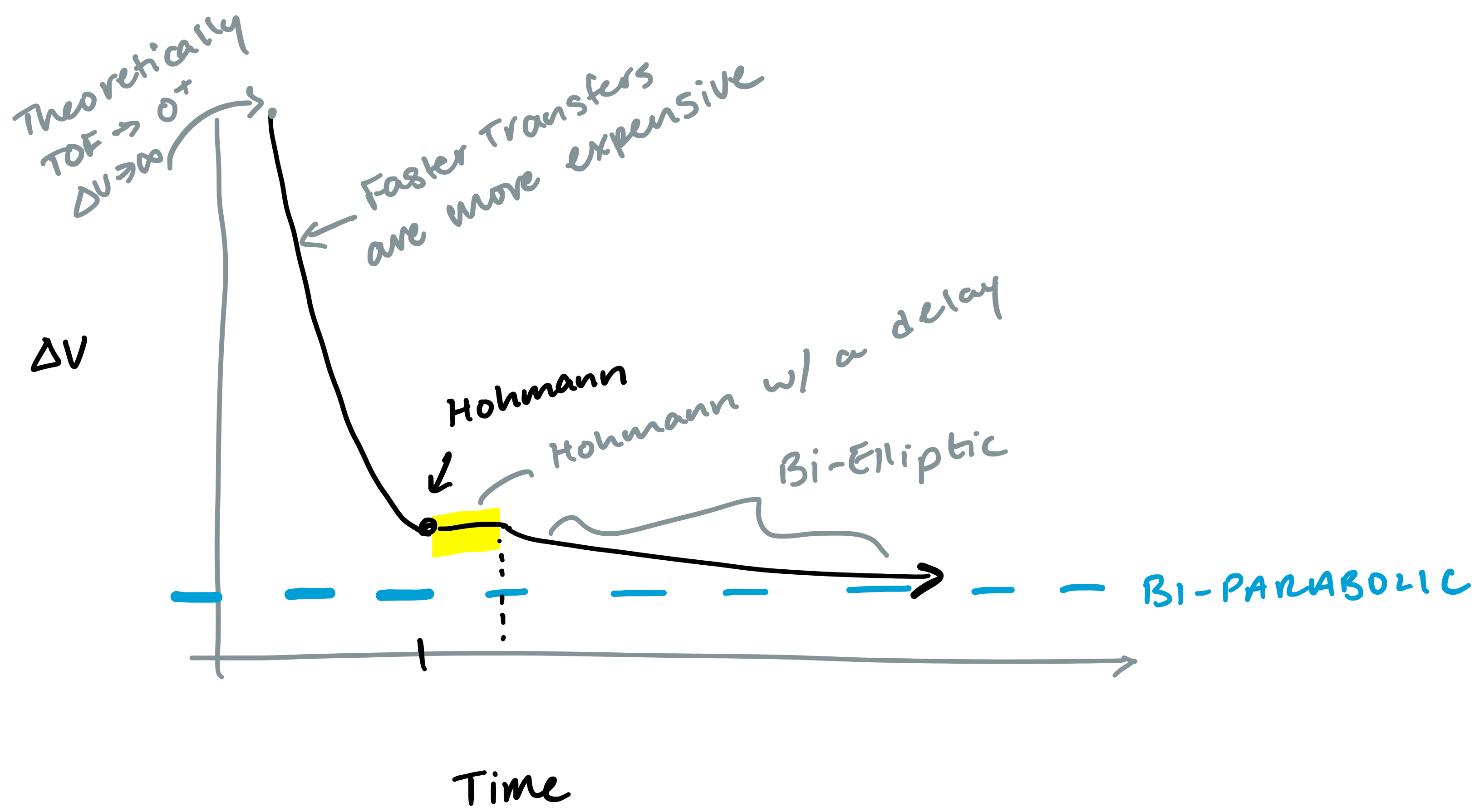
$\Rightarrow x = R$

$$\boxed{\bar{x}^* = \begin{bmatrix} x^* \\ y^* \\ z^* \end{bmatrix} = \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix}}$$

Question 11 (2025)

Monday, March 10, 2025 14:43

SKETCH OUT THE SOLUTION TO THE TIME - ΔV OPTIMAL TRANSFER PROBLEM BETWEEN TWO CIRCULAR ORBITS WITH A RADIUS RATIO ARBITRARILY > 11.94. SPECIFICALLY, FIND THE "PARETO FRONT" THAT DELINEATES THE OPTIMAL ΔV SOLUTION GIVEN A SPECIFIED TRANSFER TIME. CONSIDER TRANSFER TIMES FROM $T = 0^+ \to \infty$ — DON'T WORRY ABOUT THE SPEED OF LIGHT LIMIT!



Students should also provide some explanation / justification of why the Pareto front takes this shape.