Optimal Control, Active Satellites and Space Situational Awareness

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Fundamental Principles

- To motivate an application of optimal control to SSA, a number of fundamental principles are assumed:

I. Space-based maneuvers are generally carried out in a fuel-optimal manner.

II. Given multiple correlation options, the one with the lowest propulsive cost is the most likely.

III. Ballistic arcs are more probable than thrusting arcs.

IV. For detection of a maneuver, it is not necessary to precisely reconstruct it.

- *The fundamental assumption is that satellite operators are "stingy" with their resources.*

Optimal Control and SSA

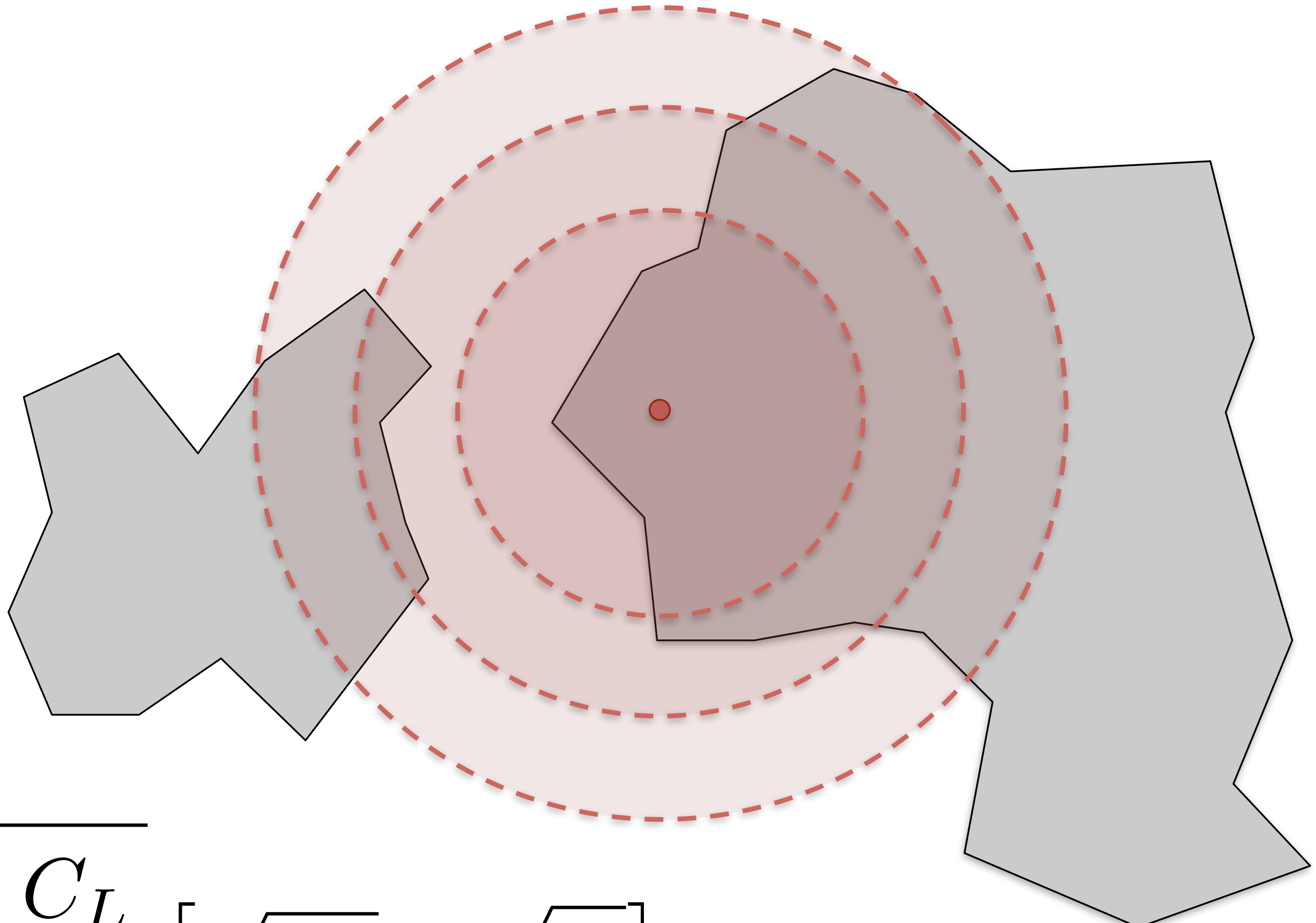
- With this backdrop, we can delve into specific questions that can be addressed using optimal control.
- Reachability:
  - What is the "range" of a satellite for a given amount of propellant?
- Correlation:
  - What is the likelihood that two objects on similar, but different, orbits are the same object?
- Maneuver Detection:
  - Has an object performed a maneuver since its last observation?
- Maneuver Reconstruction:
  - What core information can be determined for a thrusting satellite
- Propellant Usage:
  - How much propellant has an object expended?

Range and Reachability

M. Holzinger, D.J. Scheeres and R.S. Erwin. "On-Orbit Range Computation using Gauss' Variational Equations with J_2 Perturbations," *Journal of Guidance, Control and Dynamics 37(2)*: 2014.

Vehicle Range

- Mission planning / support
  - Range sets can reduce complicated differential problems to geometry
- Range is often ‘fuel-optimal’ and ‘fuel-constrained’
  - Natural independent variable changes from time to fuel
  - Breguet’s Aircraft Range Equation



$$R = \frac{2}{k}\sqrt{\frac{2}{\rho g S}\frac{C_L}{C_D^2}}\left[\sqrt{m_o} - \sqrt{m}\right]$$

- Goal:
  - Compute ΔV range sets for orbiting objects

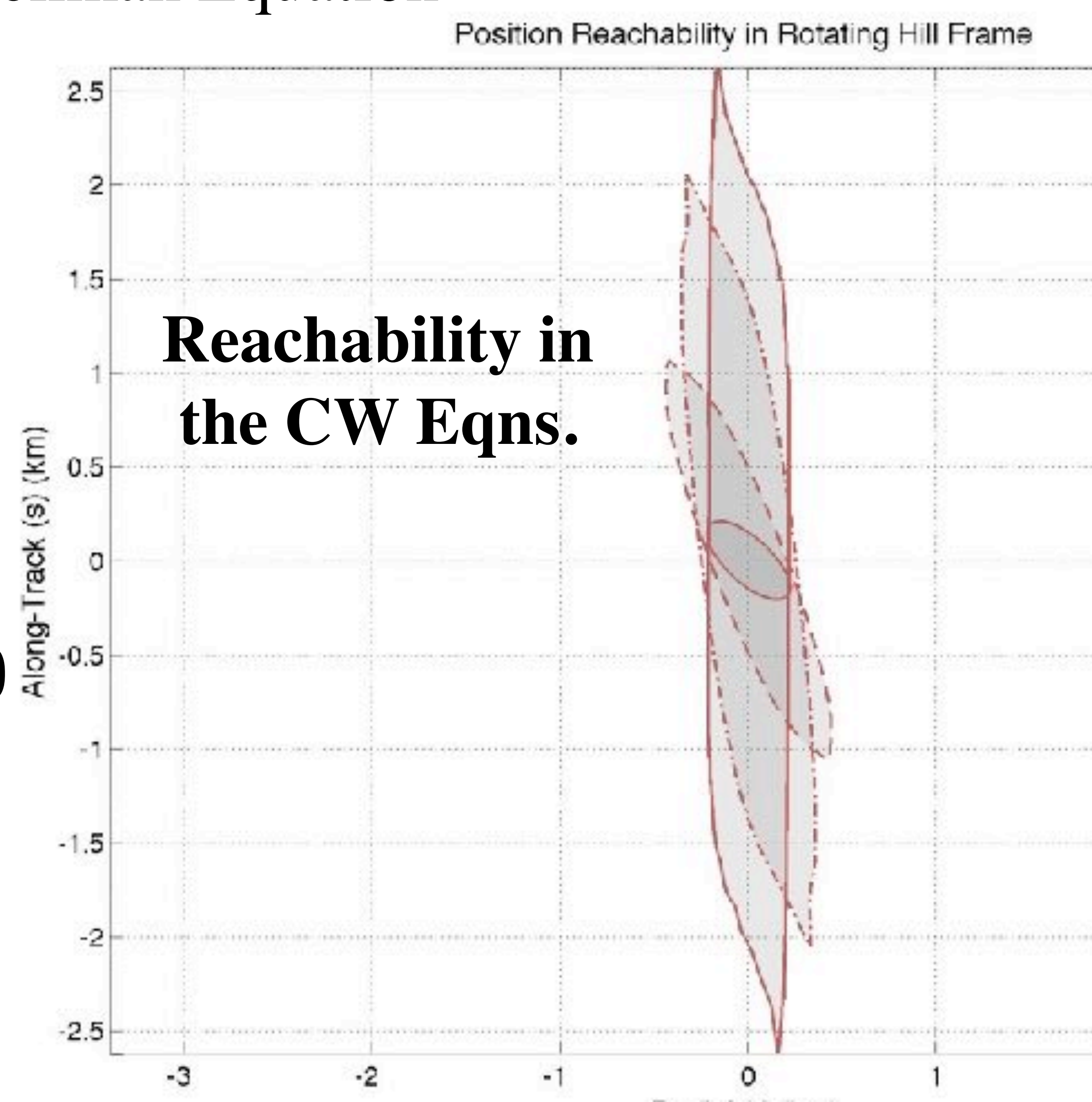
Classical Reachability

- The Reachable Set ≡ all positions/velocities that can be achieved with the given control over a ***set time period***
  - Defined by a Hamilton-Jacobi-Bellman Equation
  - Requires control limitations
  - A difficult computation that becomes harder as time grows

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + B \cdot \mathbf{u}$$

$$\frac{\partial \mathcal{V}}{\partial t} + \sup_{\mathbf{u} \in U} \left[\frac{\partial \mathcal{V}}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, t) + B \cdot \mathbf{u} \right] = 0$$

$$\mathcal{R}(t; \mathcal{V}(\mathbf{x}_0, t_0)) = \{\mathbf{x} | \mathcal{V}(\mathbf{x}, t) \leq 0\}$$



What is Orbit “Range”

- Classical Time-Reachability has limited applications
  - Makes the strong assumption that the object always thrusts
  - This is not fuel optimal, however, meaning that a satellite may “go farther” over a longer time by using a different sequence of thrusts
- How to make this analogous to the Breguet Range Equation: “How far” a satellite can travel?
  - Minimum ΔV transfers involve impulsive maneuvers
  - Fuel optimal transfers consist of thrust and coast arcs
  - Ideal transfers will:
    - Be measured in terms of ΔV
    - Measure range in orbit elements
    - Be time-free

Problem Definition and Simplifying Assumptions

- Use Gauss' Variational Equations with classical orbit elements as constant of motion coordinates
  - Can afford to wait for Ω and ω to rotate under J_2
  - Can choose the true anomaly at which a maneuver occurs

$$\frac{da}{dt} = \frac{2a^{\frac{3}{2}} e \sin f}{\mu^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}} u_r + \frac{a^{\frac{3}{2}}(1+e\cos f)}{\mu^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}} u_\theta$$

$$\frac{de}{dt} = \frac{a^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}} \sin f}{\mu^{\frac{1}{2}}} u_r + \frac{a^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \left[\frac{e\cos^2 f + 2\cos f + e}{1+e\cos f}\right] u_\theta$$

$$\frac{di}{dt} = \frac{a^{\frac{1}{2}}}{\mu^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}} \left[\frac{(1-e^2)\cos(\omega+f)}{1+e\cos f}\right] u_h$$

- Accelerations are due only to control thrusting
- Thrust magnitude is unconstrained

Reachability: Orbit Element Space

- Restrict our notion of range to: $\text{æ} = (a, e, i)$
- Gauss Equations are linear in control:

$$\dot{\text{æ}} = \mathbf{F}(\text{æ}, t) \cdot \mathbf{u}$$

- Change independent parameter from time to DV:

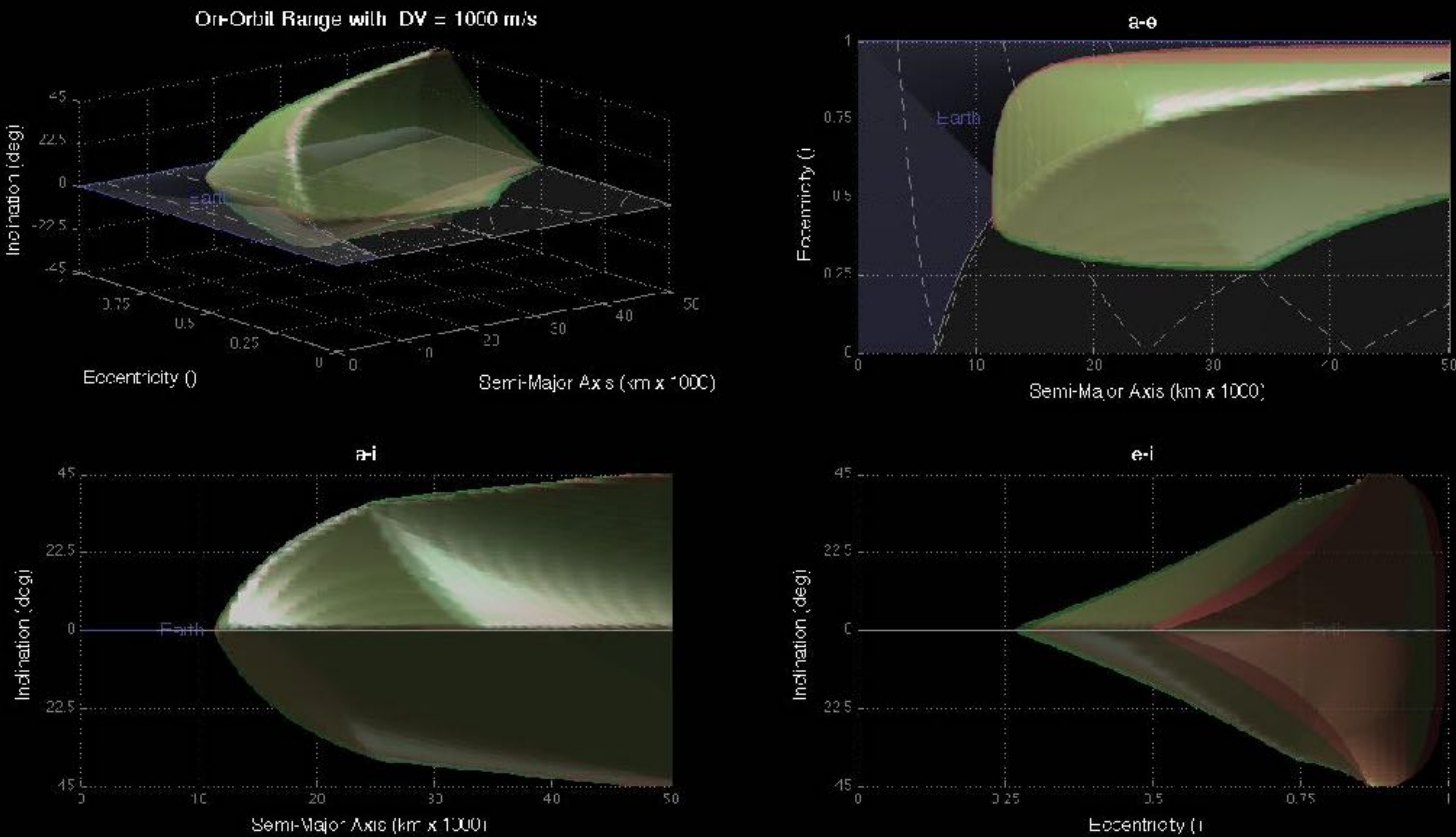
$$\frac{\partial \mathcal{V}}{\partial \Delta V} + \sup_{\mathbf{u} \in U} \left[\frac{\partial \mathcal{V}}{\partial \text{æ}} \mathbf{F}(\text{æ}, t) \cdot \frac{\mathbf{u}}{\Delta V} \right] = 0$$

Singularities of the resulting HJB Eqn. when $\Delta V = 0$ can be removed due to the linear control structure of the equations, can be solved using viscosity methods

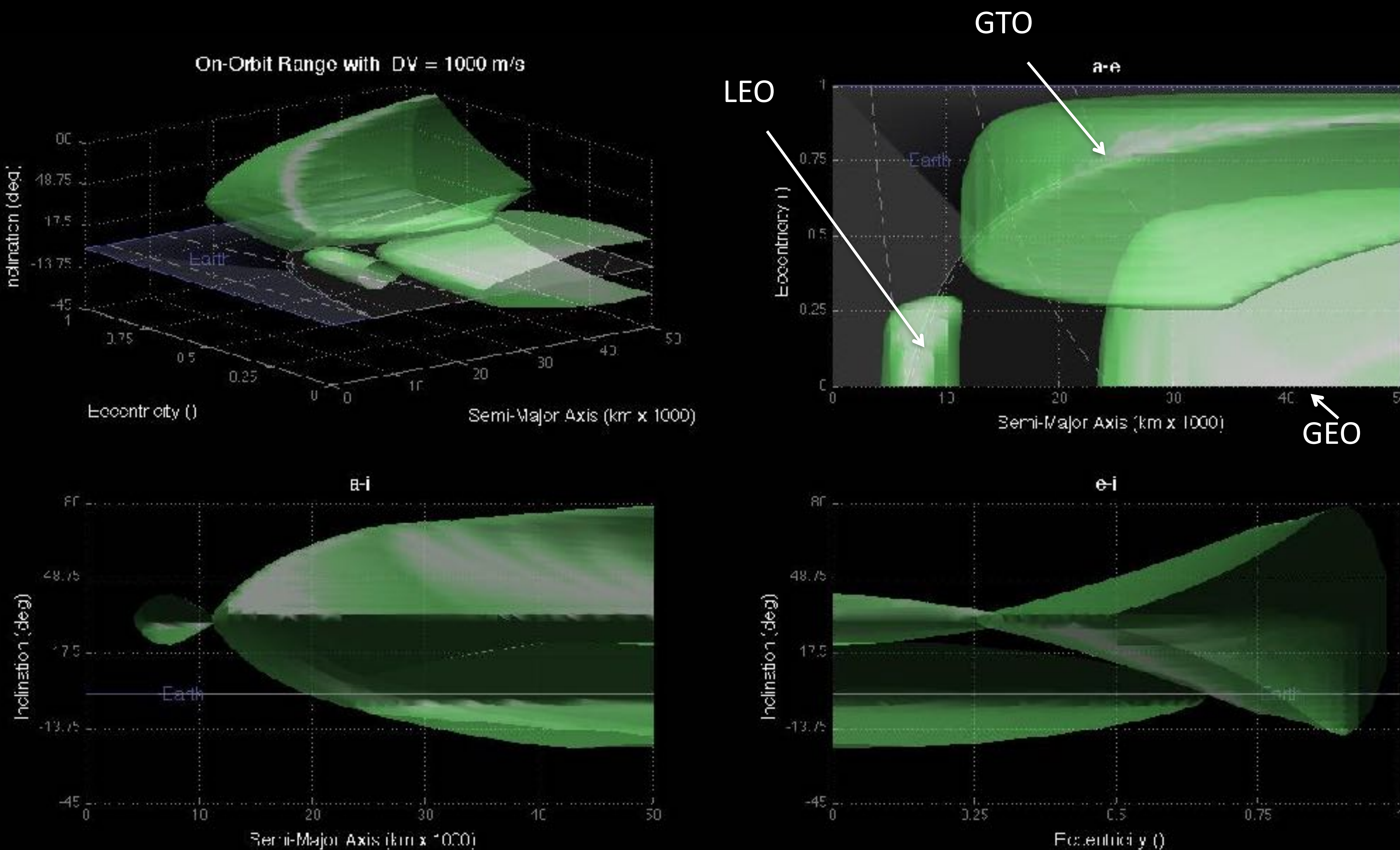
- Reachable set now defined as a function of DV, analogous to the Breguet Eqn.:

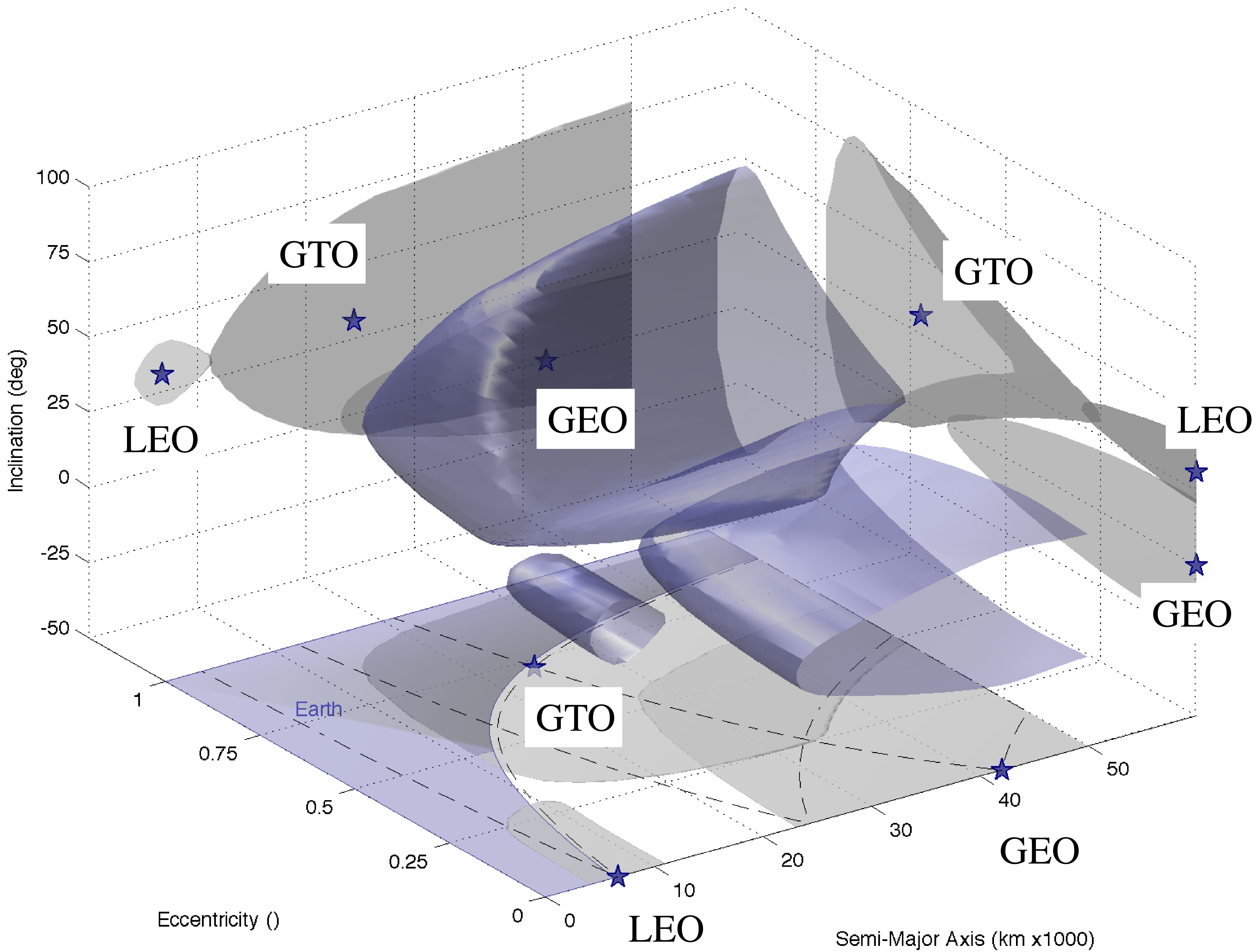
$$\mathcal{R}(\Delta V; \mathcal{V}(\text{æ}_0, \Delta V_0)) = \{\text{æ} | \mathcal{V}(\text{æ}, \Delta V) \leq 0\}$$

Example 1: GTO w/ 1000m/s ΔV

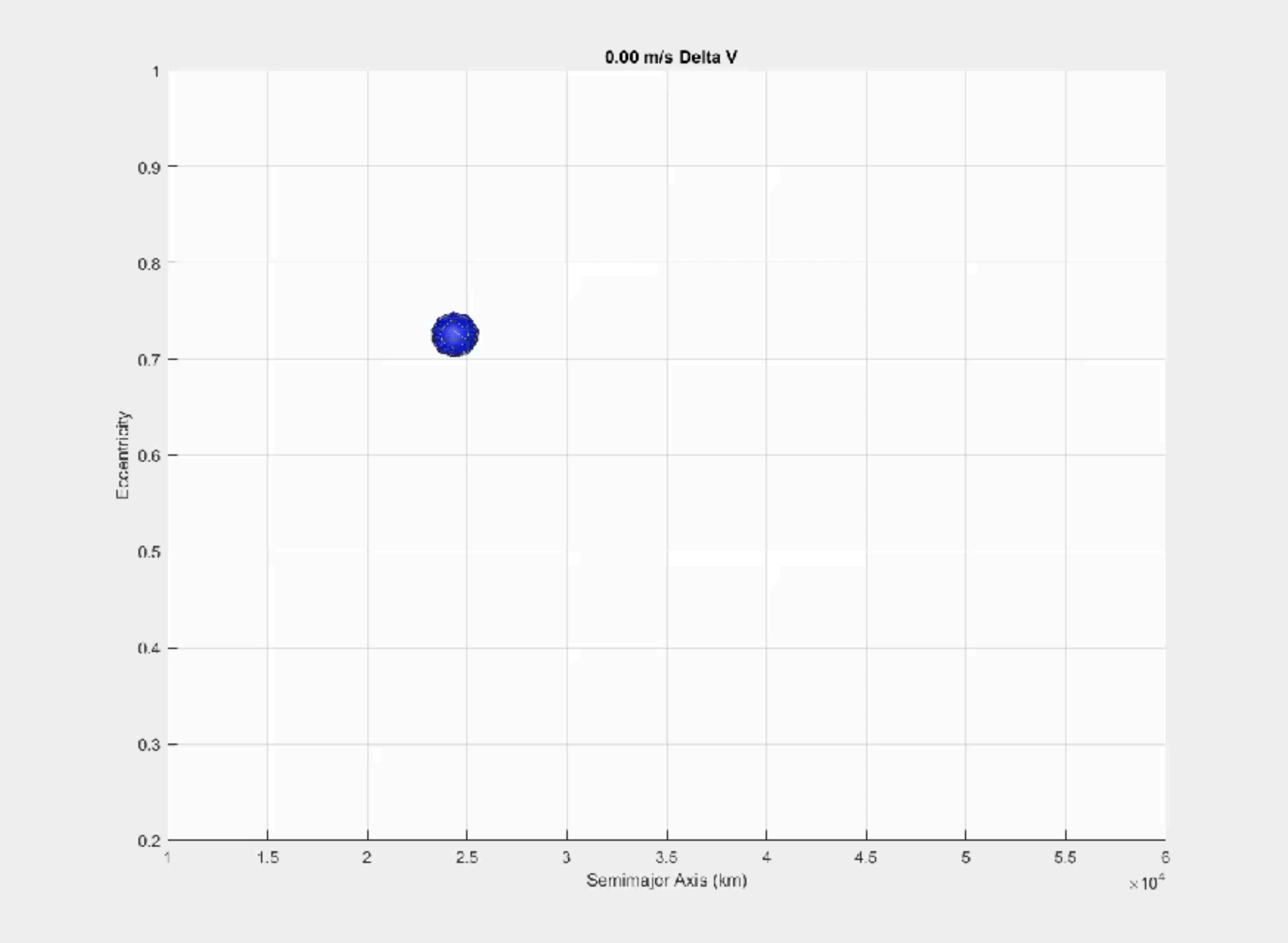


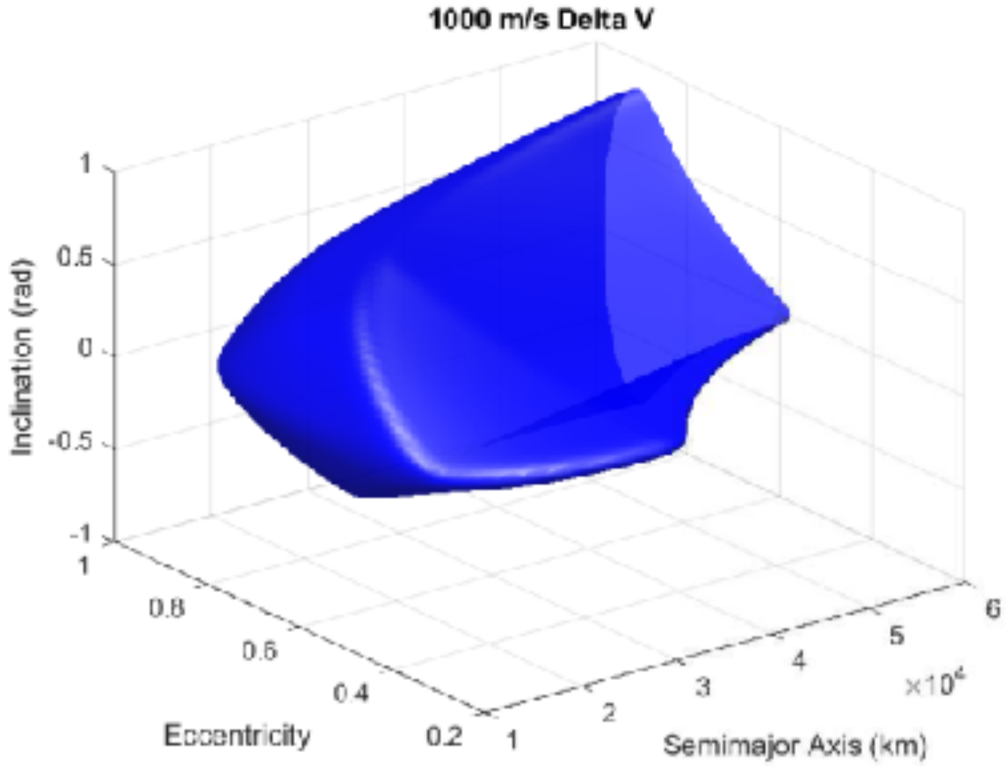
Example 2: Multiple Start Regions w/ 1000m/s ΔV

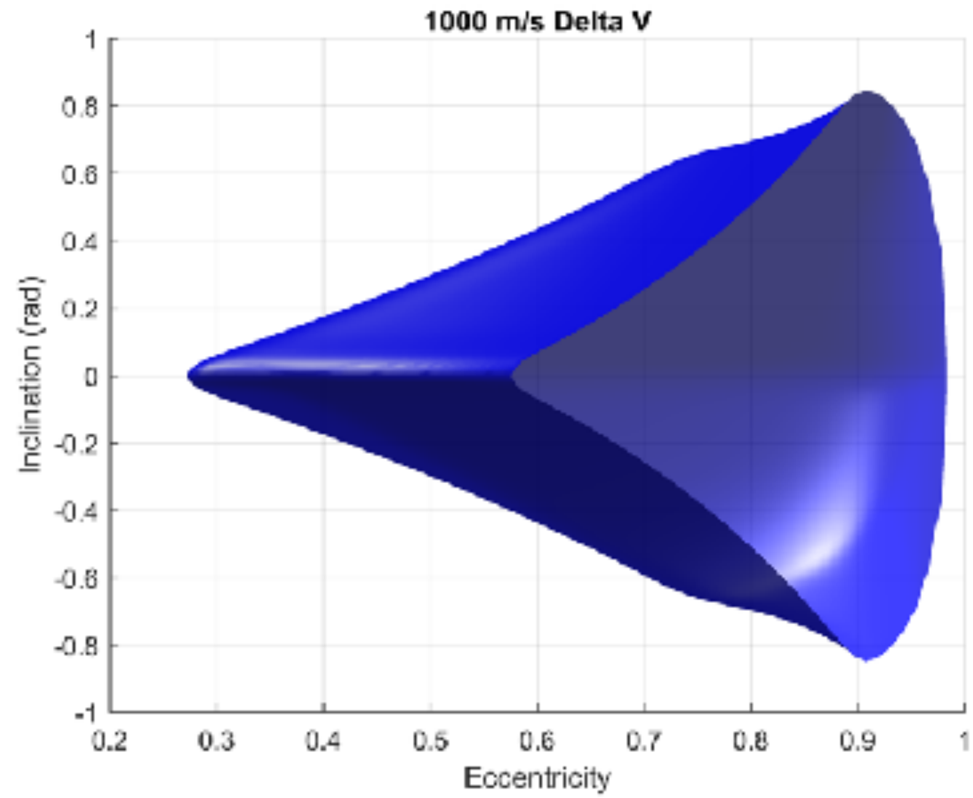


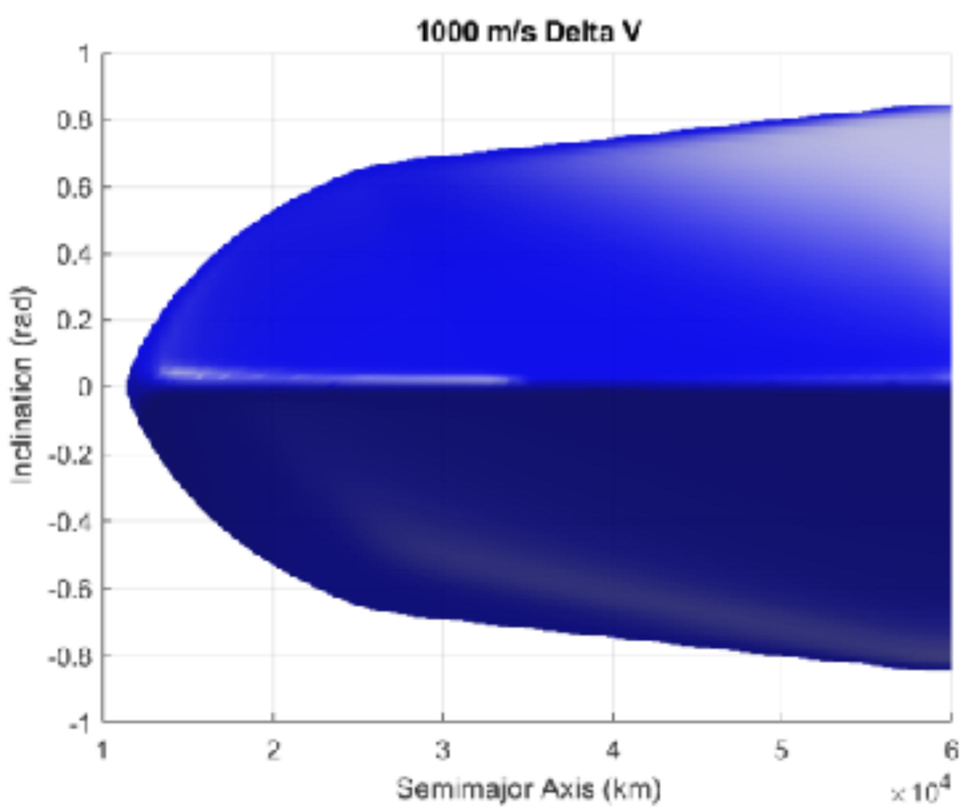


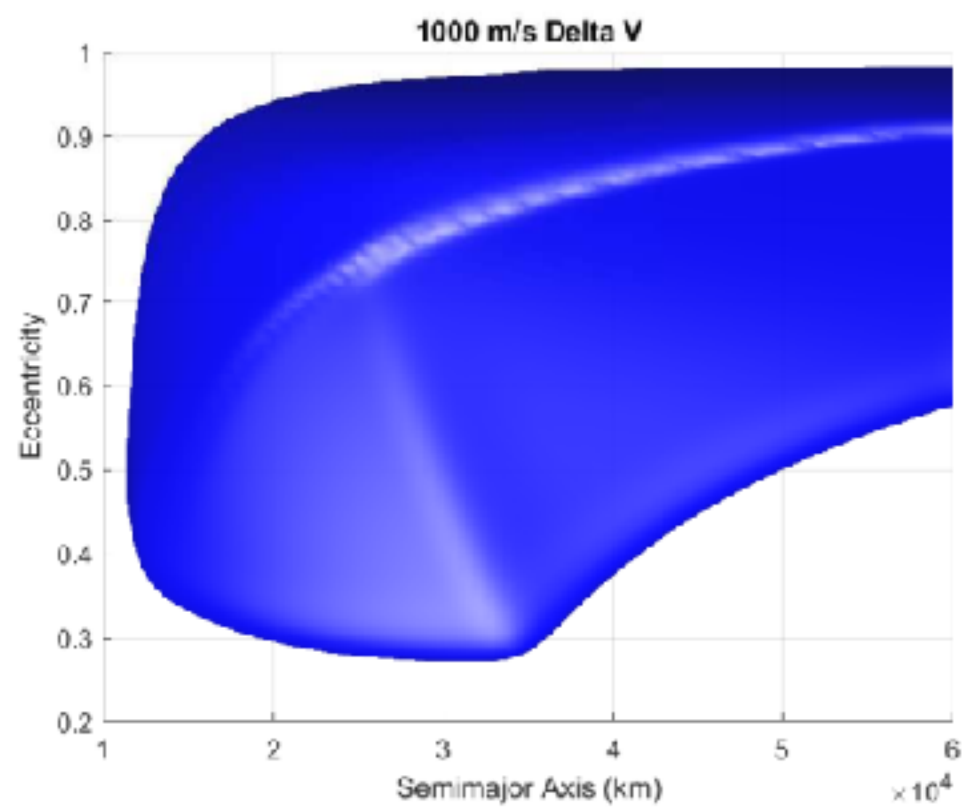
GTO
GTO
GTO
GEO
GEO
GEO
LEO
LEO
LEO
Earth
Inclination (deg)
100
75
50
25
0
-25
-50
Eccentricity ()
1
0.75
0.5
0.25
0
Semi-Major Axis (km x1000)
0
10
20
30
40
50


0.00 m/s Delta V
Eccentricity
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
Semimajor Axis (km)
$\times 10^4$

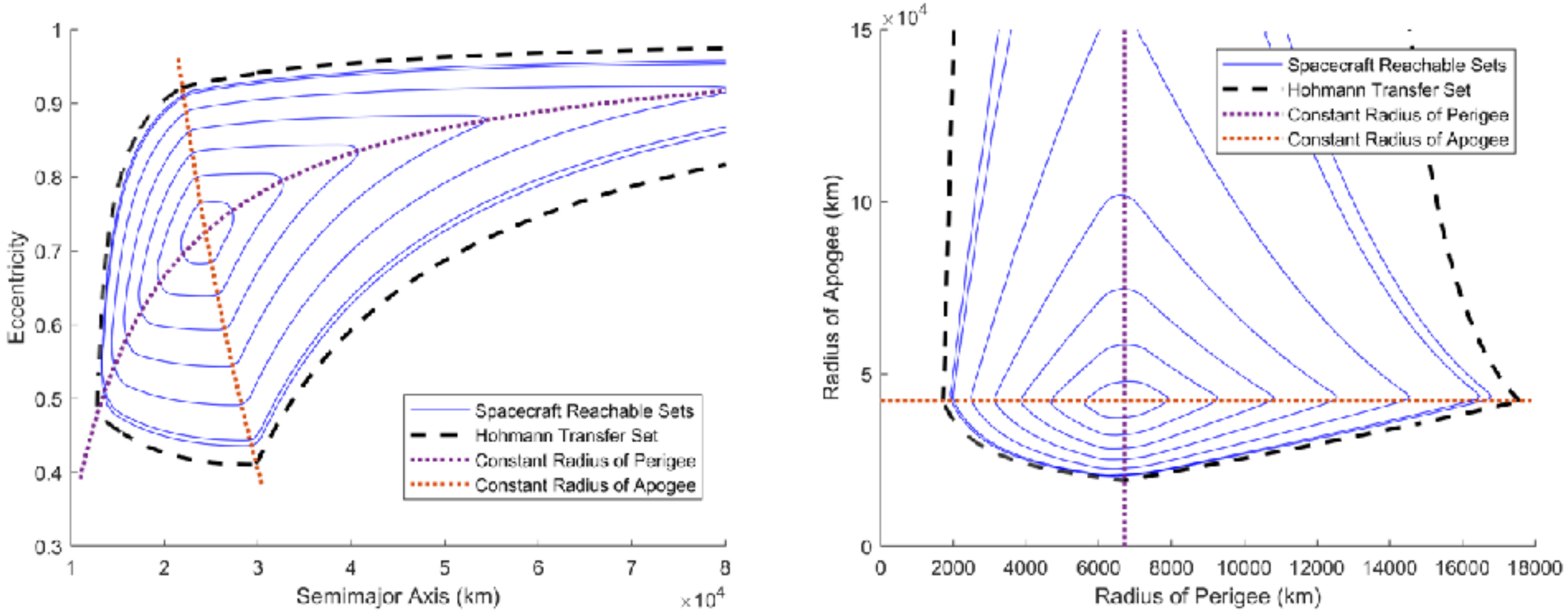

1000 m/s Delta V
Inclination (rad)
Eccentricity
Semimajor Axis (km)
$\times 10^4$


1000 m/s Delta V
Inclination (rad)
Eccentricity


1000 m/s Delta V
Inclination (rad)
Semimajor Axis (km)
$\times 10^4$


1000 m/s Delta V
Eccentricity
Semimajor Axis (km)
$\times 10^4$

Set Reparameterization

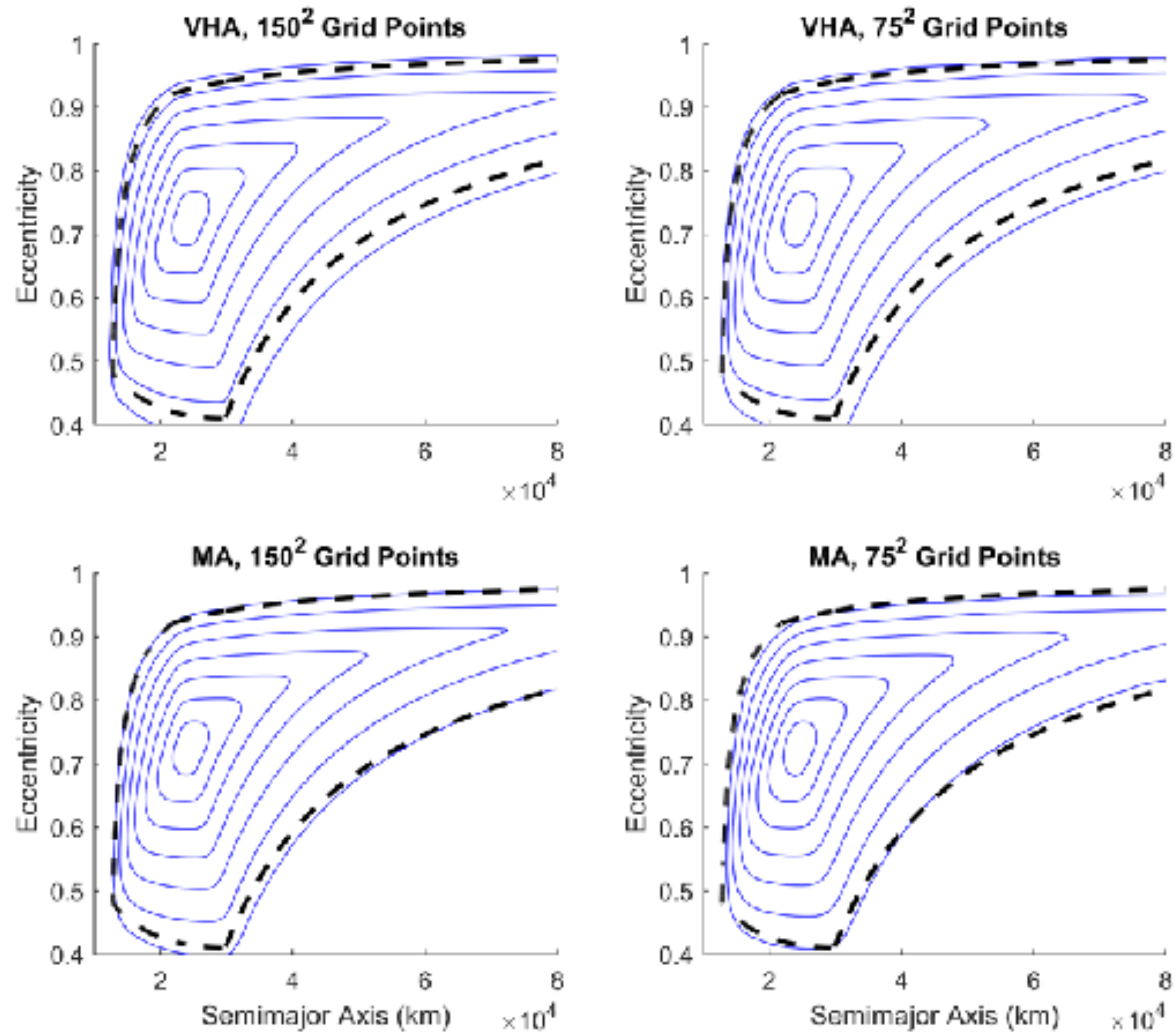


Computational Accuracy

150^2 Grid Points | 75^2 Grid Points

“Very High” Accuracy

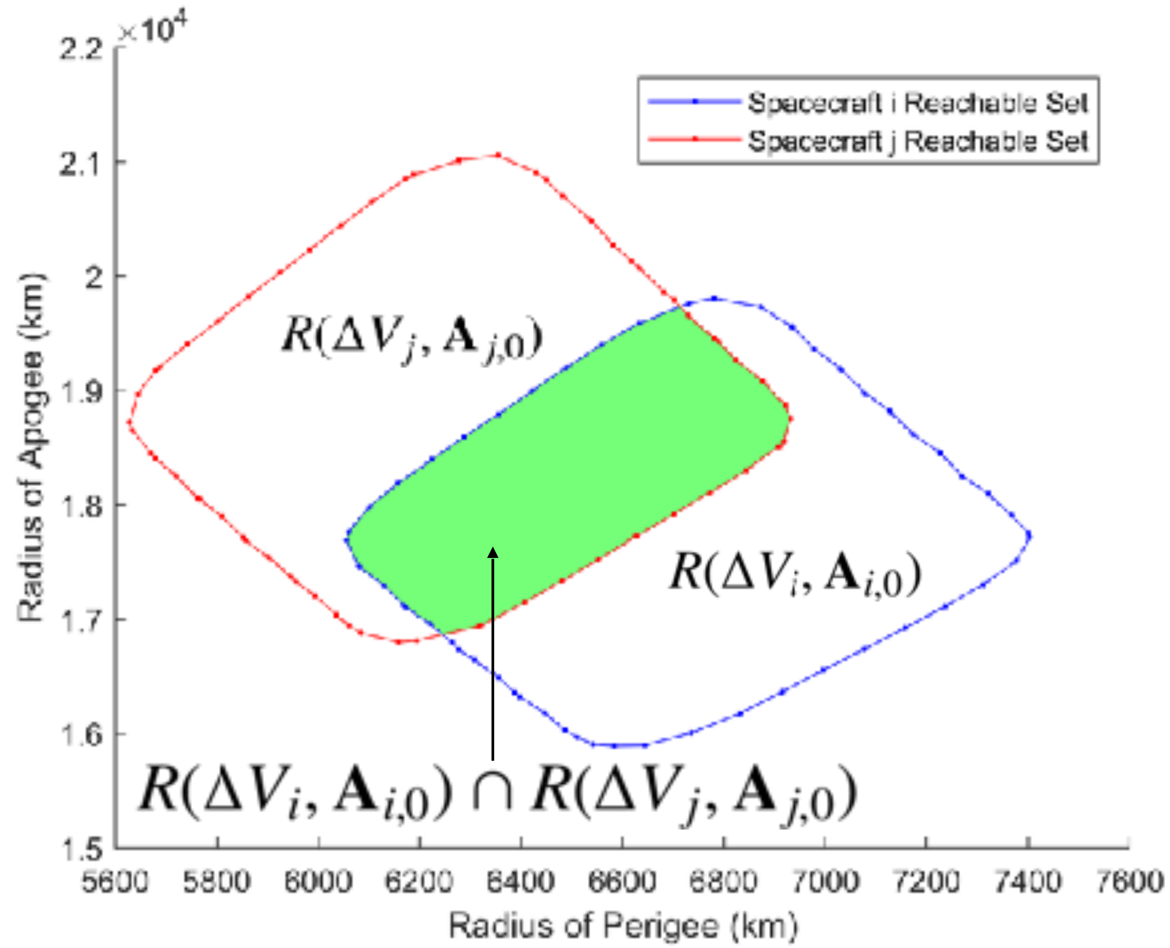
“Medium” Accuracy



Reachable Set Applications to Multi-Spacecraft Scenarios

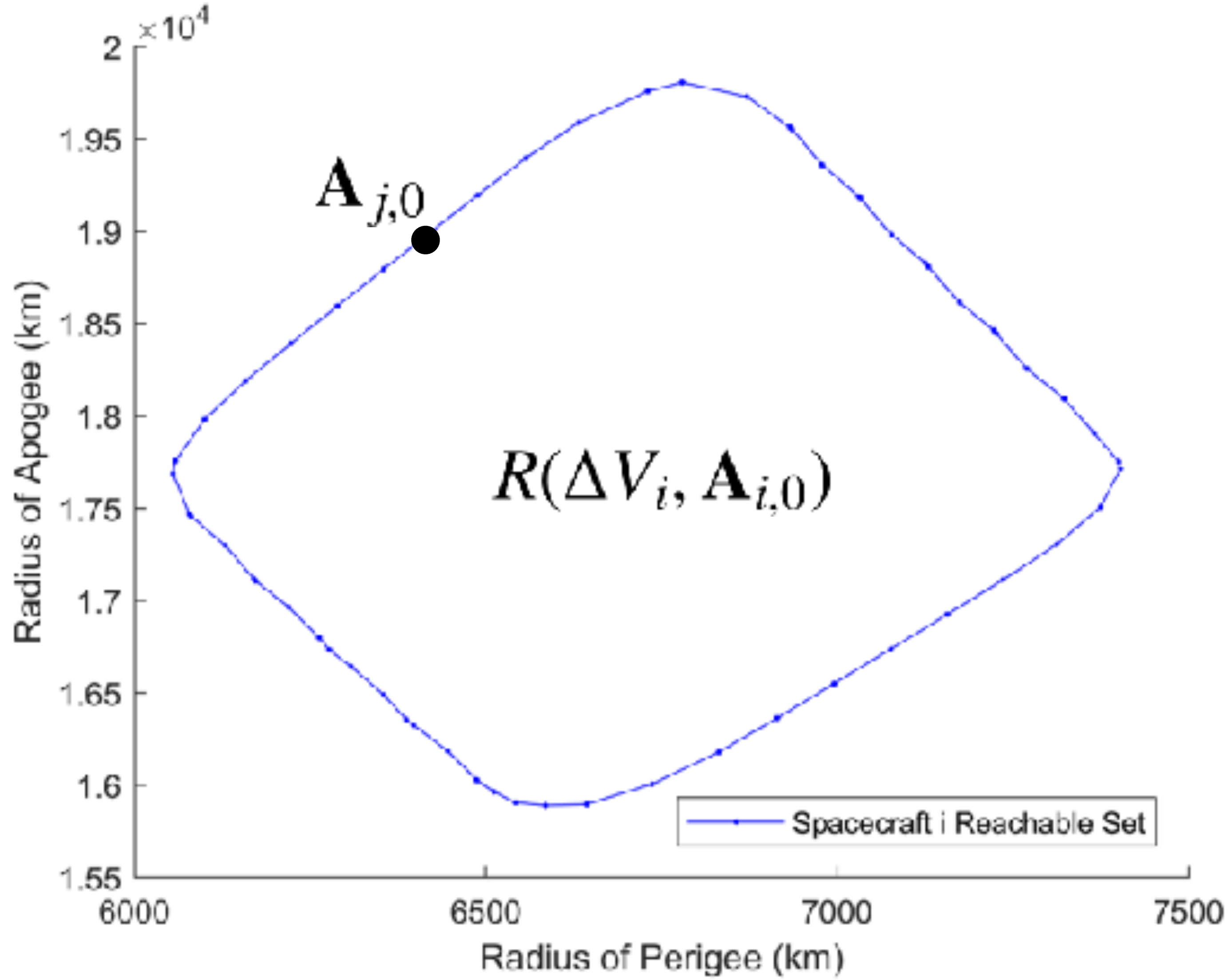
Rendezvous

- Defined here as matching orbit element sets for two separate spacecraft
- Define the reachable set as $R(\Delta V_i, \mathbf{A}_{i,0})$, a function of available ΔV_i and initial orbit $\mathbf{A}_{i,0}$



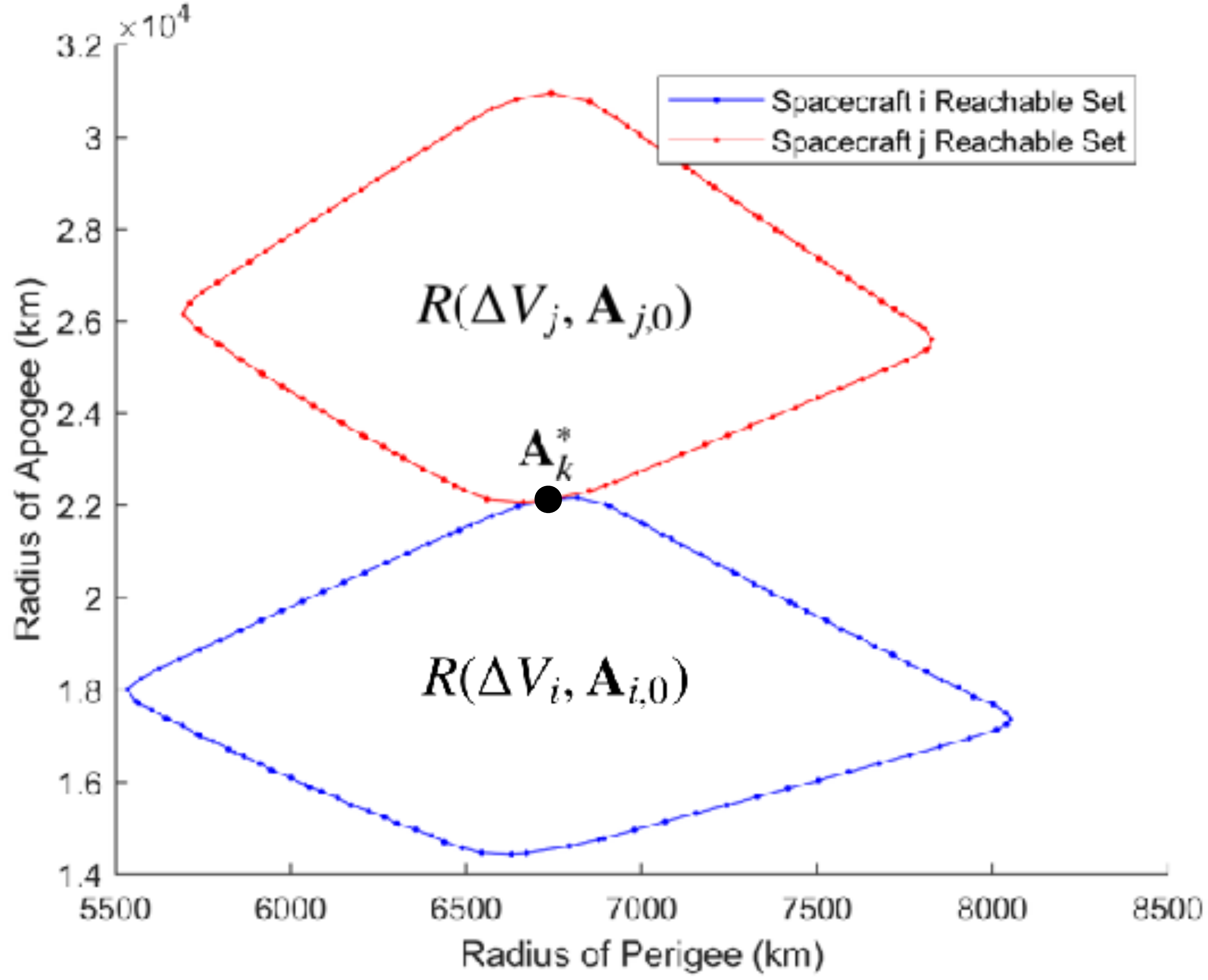
Minimum ΔV Rendezvous

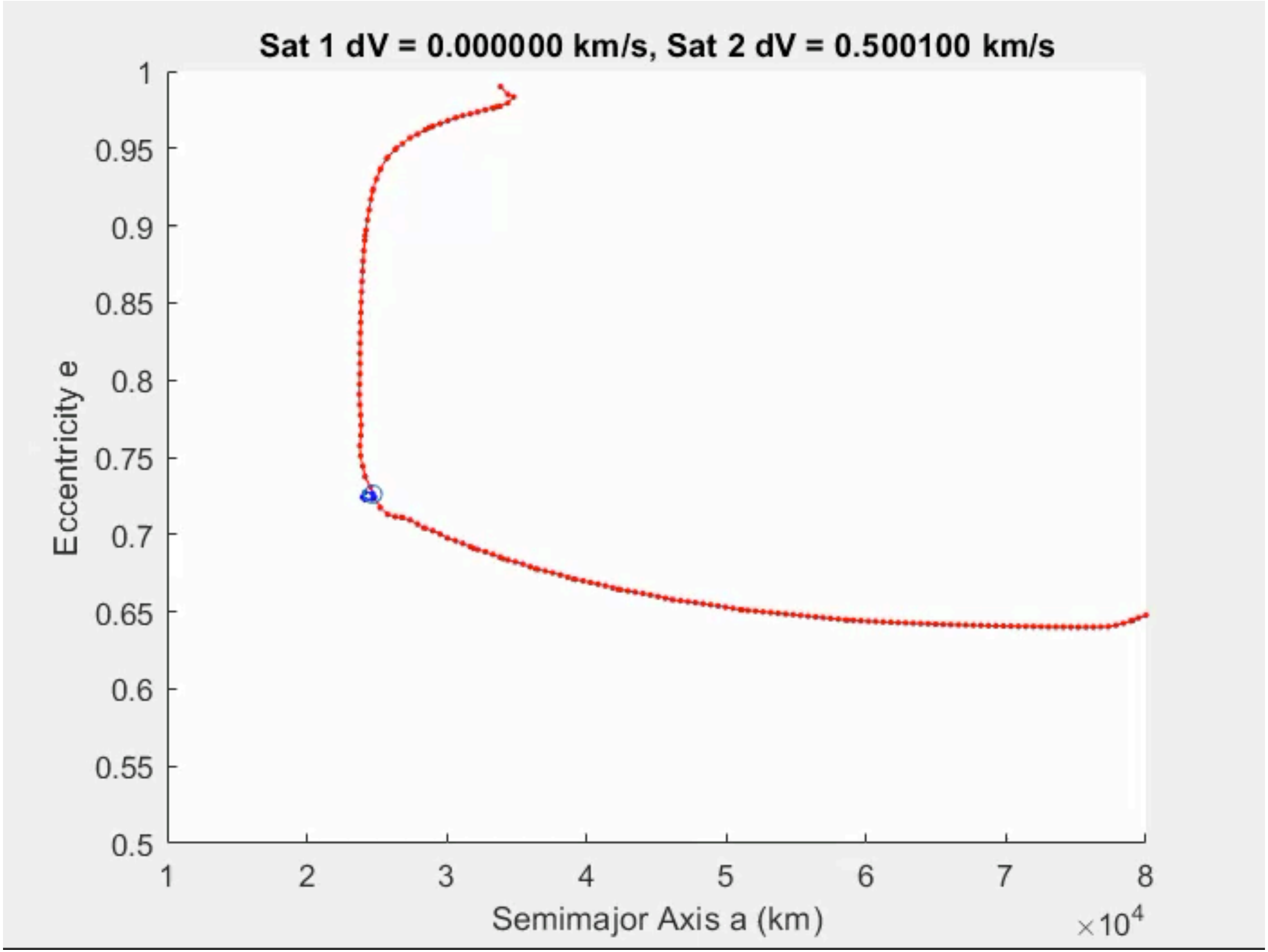
$$\Delta V^*_{i,j} = \underset{\Delta V}{\text{argmin}} \left\{ \Delta V | \mathbf{A}_{j,0} \in R(\Delta V, \mathbf{A}_{i,0}) \right\}$$



Minimum ΔV Rendezvous

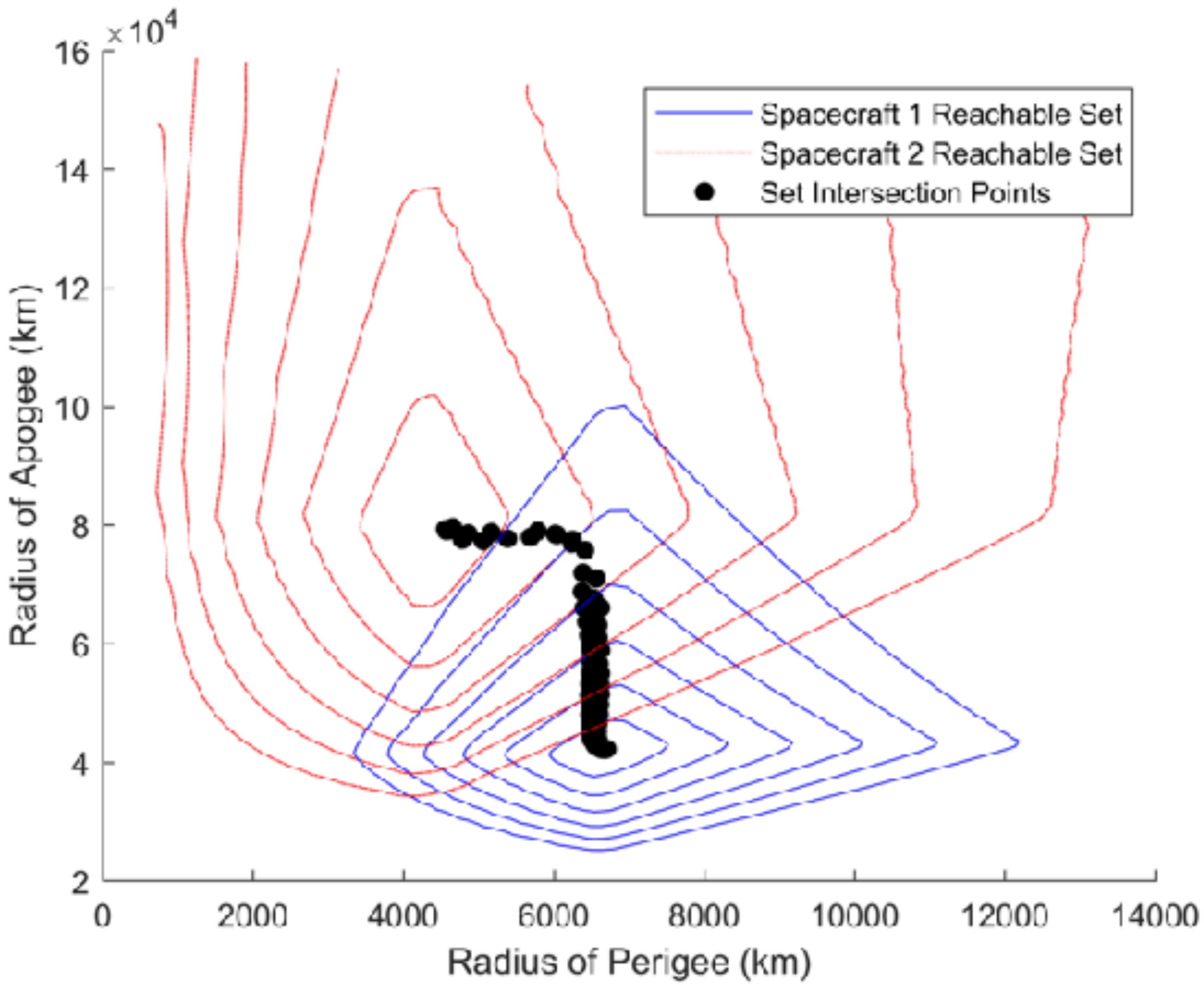
$$\exists! \mathbf{A}_k^* \in R(\Delta V_i, \mathbf{A}_{i,0}) \cap R(\Delta V_j, \mathbf{A}_{j,0})$$





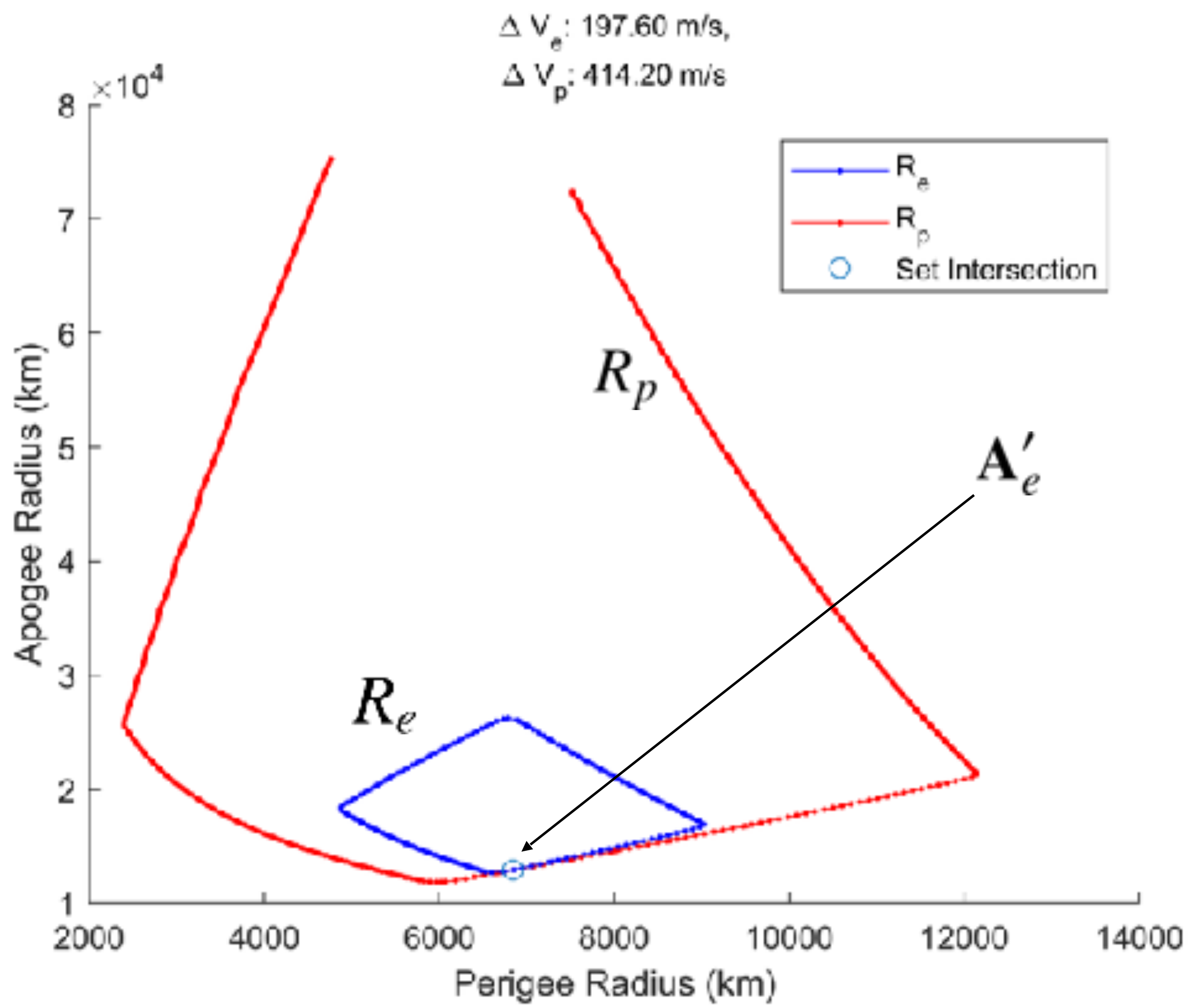
[Need to reparametrize this]

Minimum ΔV Rendezvous



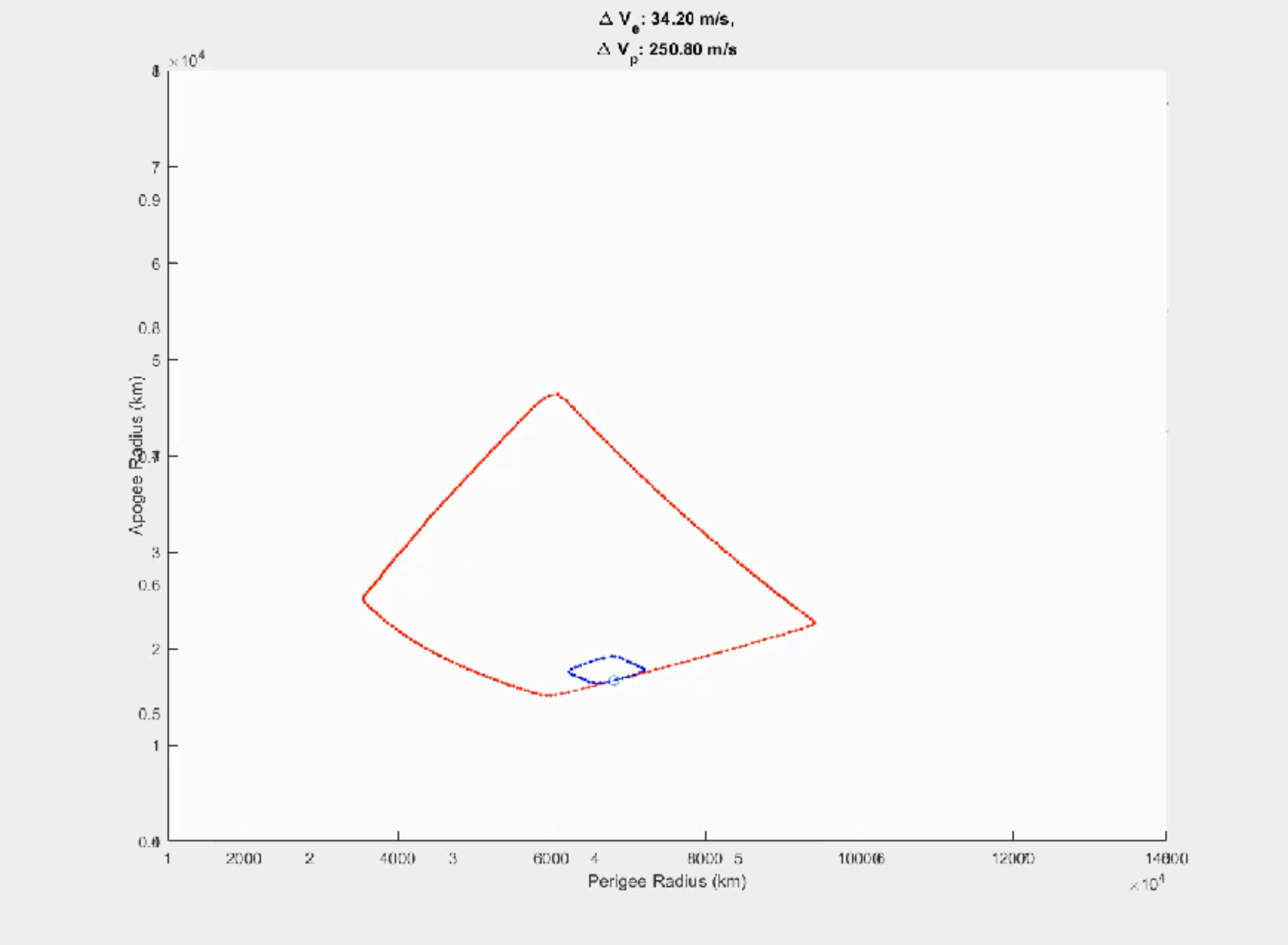
Pursuit/Evasion

- Pursuing spacecraft attempts to rendezvous, evading spacecraft attempts to avoid rendezvous/capture



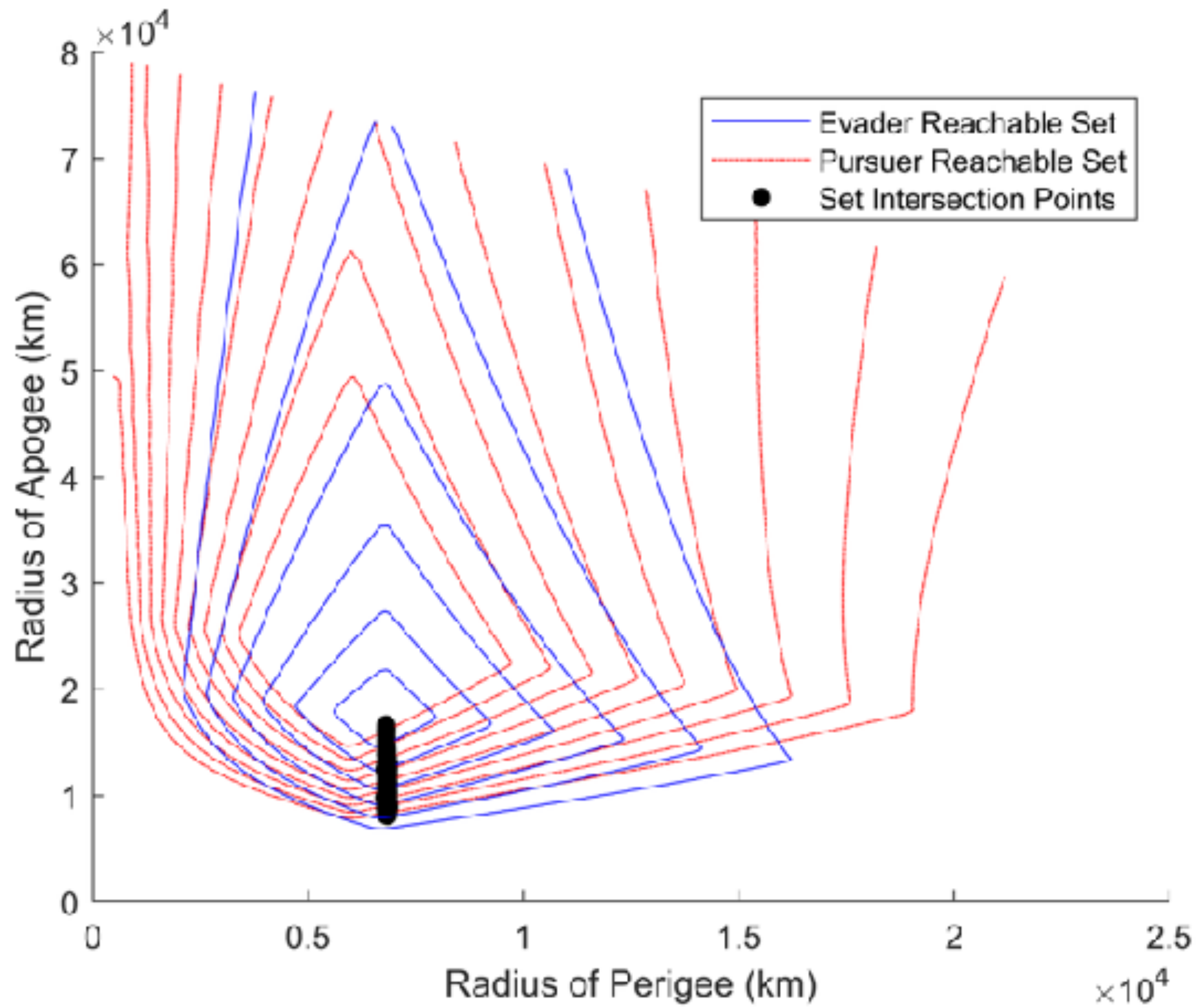
$$\exists! \mathbf{A}'_e \quad \text{where}$$

$$\mathbf{A}'_e \in R_e(\Delta V_{e,max}, \mathbf{A}_{e,0})$$
$$\text{and}$$
$$\mathbf{A}'_e \notin R_p(\Delta V'_p, \mathbf{A}_{p,0})$$



Δ V_e: 34.20 m/s,
Δ V_p: 250.80 m/s
×10^4
Apogee Radius (km)
Perigee Radius (km)
×10^4

Evasion Strategy



Control Distance and Correlation

M.J. Holzinger, D.J. Scheeres and K.T. Alfriend. 2012. “Object Correlation, Maneuver Detection, and Maneuver Characterization Using Control Distance Metrics,” *Journal of Guidance, Control and Dynamics* 35(4): 1312-1325.

M.J. Holzinger and D.J. Scheeres. 2012. “Analytical Reachability Solutions for a Class of Nonlinear Systems with Ellipsoidal Initial Sets,” *IEEE Transactions on Aerospace and Electronic Systems* 48(2): 1583-1600.

Correlation of Maneuvering Objects

- We are interested in measuring the "distance" between two orbit states, quantified by control effort.
- Optimal Control theory plays a fundamental role – it can ideally identify the minimum level of control effort needed to connect two orbit states.
- The "Control Distance" provides this measure, with additional benefits:
  - It is a metric and thus can be rigorously used to measure distances between states in terms of control effort
  - It can be generalized to stochastic values
  - It can be defined for partial state measurements
  - It can be evaluated for a range of possible cost metrics

Control Distance

- The "control distance" is defined as the value function of the optimal control law connecting two states.

$$d_c(\mathbf{x}_0, t_0; \mathbf{x}_f, t_f) = \inf_{\mathbf{u}\in U} \left[\int_{t_0}^{t_f} L(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau) d\tau \right]$$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + B \cdot \mathbf{u}$$

Dynamics

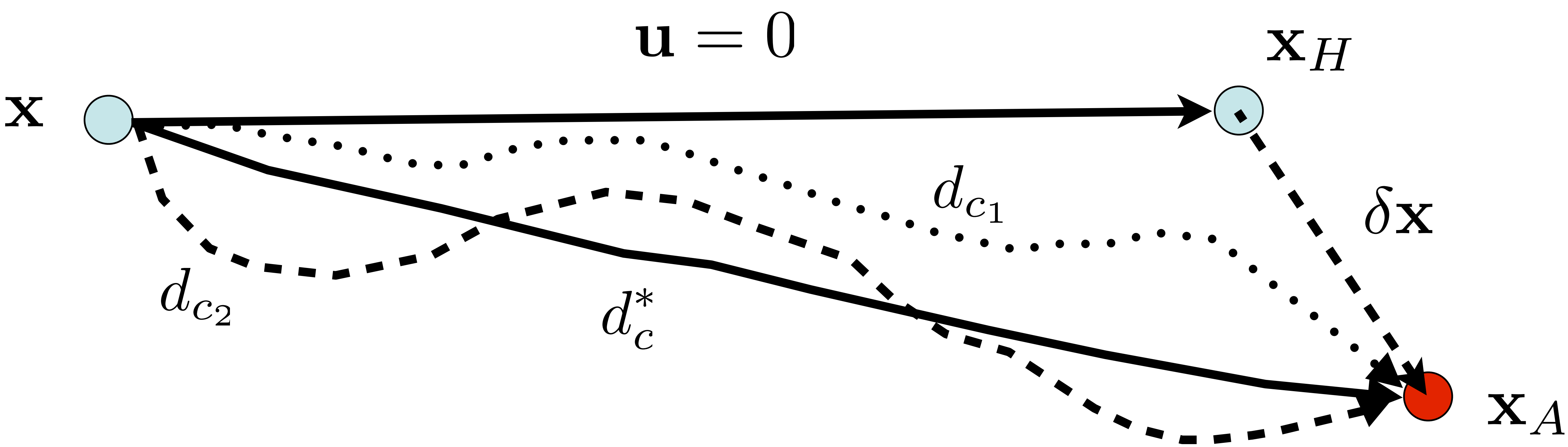
$$\mathbf{g}(\mathbf{x}_0, t_0; \mathbf{x}_f, t_f) = 0$$

Terminal Manifold Constraints

– Optimal Control Policy:

$$\mathbf{u}^* = \mathrm{argmin}_{\mathbf{u}\in U} \left[L(\mathbf{x}, \mathbf{u}, t) - \mathbf{p} \cdot (\mathbf{f}(\mathbf{x}, t) + B \cdot \mathbf{u}) \right]$$

Control Distance



The Control Distance provides a rigorous measure of the level of effort to connect trajectories.

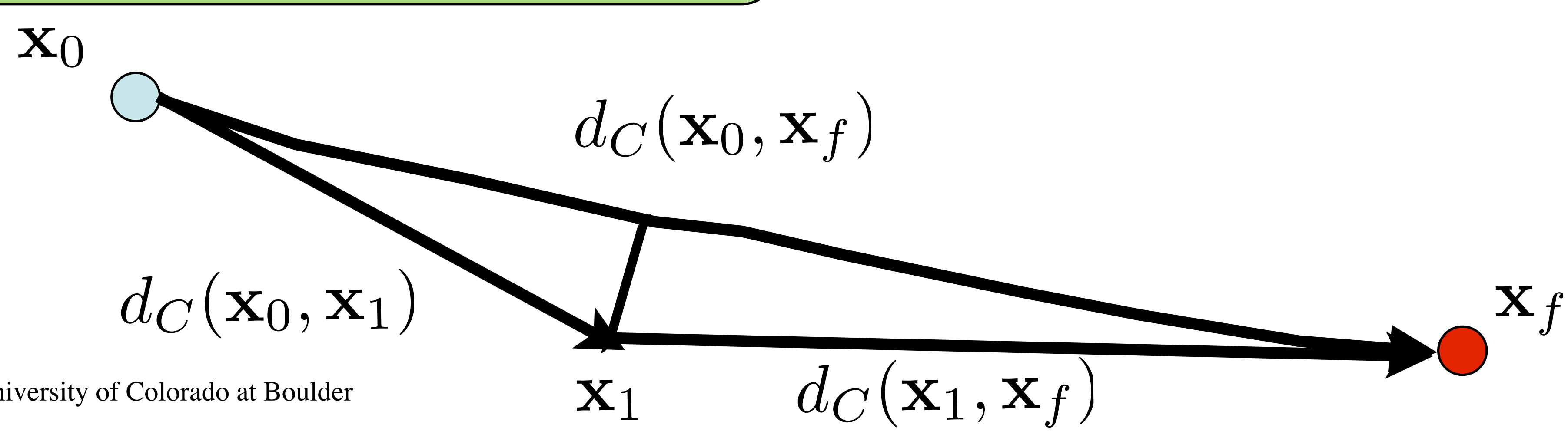
$$d_c^* \leq \{d_{c_1}, d_{c_2}, \ldots\}$$

Control Distance as a Metric

- Positivity: $d_c \geq 0$
  - Guaranteed by choice of $L(\cdot,\cdot,\cdot) \geq 0$
- Strictly Positive:
  - $d_c = 0 \Leftrightarrow (\mathbf{x}_0, t_0; \mathbf{x}_f, t_f)$ are connected by a ballistic trajectory
- Symmetry: $d_c(\mathbf{x}_0, t_0; \mathbf{x}_f, t_f) = d_c(\mathbf{x}_f, t_f; \mathbf{x}_0, t_0)$
  - Due to reversible nature of the dynamics
- Triangle Inequality:

$$d_c(\mathbf{x}_0, t_0; \mathbf{x}_f, t_f) \leq d_c(\mathbf{x}_0, t_0; \mathbf{x}_1, t_1) + d_c(\mathbf{x}_1, t_1; \mathbf{x}_f, t_f)$$

  - Satisfied by principle of optimality



Other Distance Metrics

- Euclidean Distance (d_E)
  - Standard distance metric
  - Does not account for uncertainty
- Mahalanobis Distance
  - Does the new observation match the expected distribution?

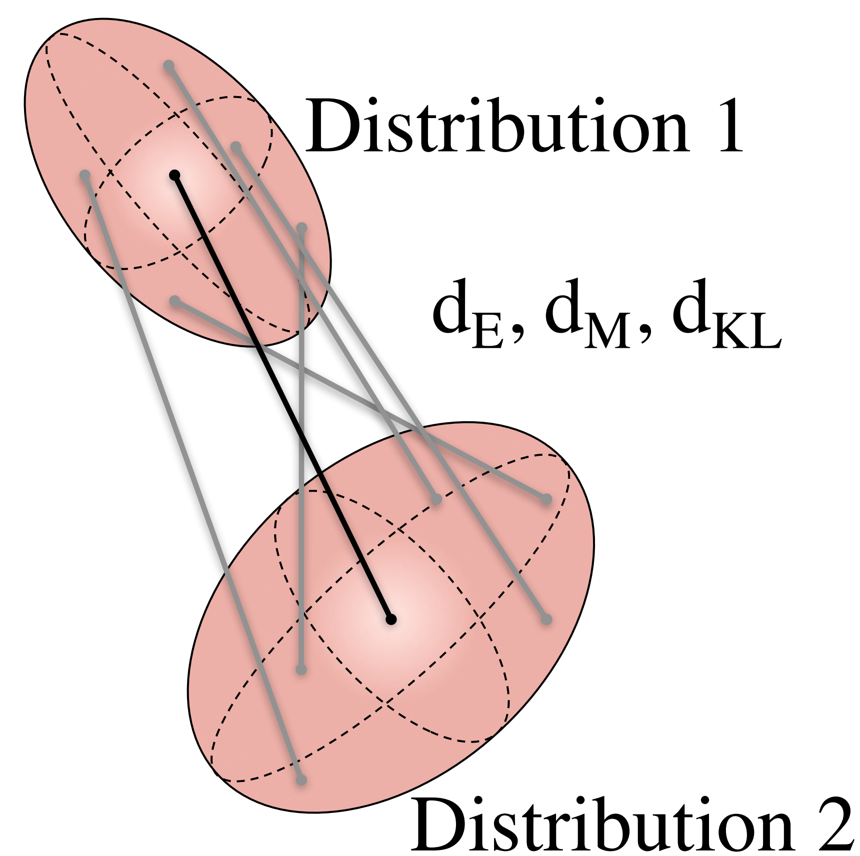
$$d_M(\mathbf{x}) = \sqrt{(\mathbf{x}-\mu_x)^T \mathbf{\Sigma}^{-1} (\mathbf{x}-\mu_x)}$$

- Kullback Leibler Distance (KLD) [Kullback 1959]
  - A measure of the 'distance' between members of populations f_1 and f_2 with the smallest directed divergence
  - Not quite a metric (violates symmetry)

$$d_{KL}(f_1(\mathbf{x}), f_2(\mathbf{x})) = \int f_2(\mathbf{x}) \log \frac{f_2(\mathbf{x})}{f_1(\mathbf{x})} d\mathbf{x}$$

$$N(\mu_i, \mathbf{\Sigma}_i),\ i = 1, 2$$

  - Can capture effects of process noise on distribution propagation



Control Cost Function

- What control cost to use?
  - Minimum Fuel: Directly relates to propellant usage

$$L_{u_M} = \|\mathbf{u}\| \leq u_M \qquad \mathbf{u}^* = \begin{cases} -u_M \hat{\mathbf{p}} & \|\mathbf{p}\| \geq 1 \\ 0 & \|\mathbf{p}\| < 1 \end{cases} \qquad \boxed{d_C(L_{u_M})}$$

    - Requires knowledge of upper bound on thrust
    - Discontinuous control structure in time
    - Yields impulsive maneuvers if $u_M \longrightarrow \infty$
    - Computationally challenging to solve – *even linear systems present barriers*
  - Minimum Energy: Does not directly relate to propellant usage

$$L_E = \frac{1}{2}\mathbf{u}\cdot\mathbf{u} \qquad \mathbf{u}^* = -\mathbf{p} \qquad \boxed{d_C(L_E)}$$

    - Requires no upper bound
    - Smooth control structure in time
    - Computationally much simpler to solve, *closed form for linear systems*

Bounds on Control Distance

- The minimum energy cost can be used to bound the minimum fuel cost:
  - Applying the Cauchy-Schwarz Inequality we find

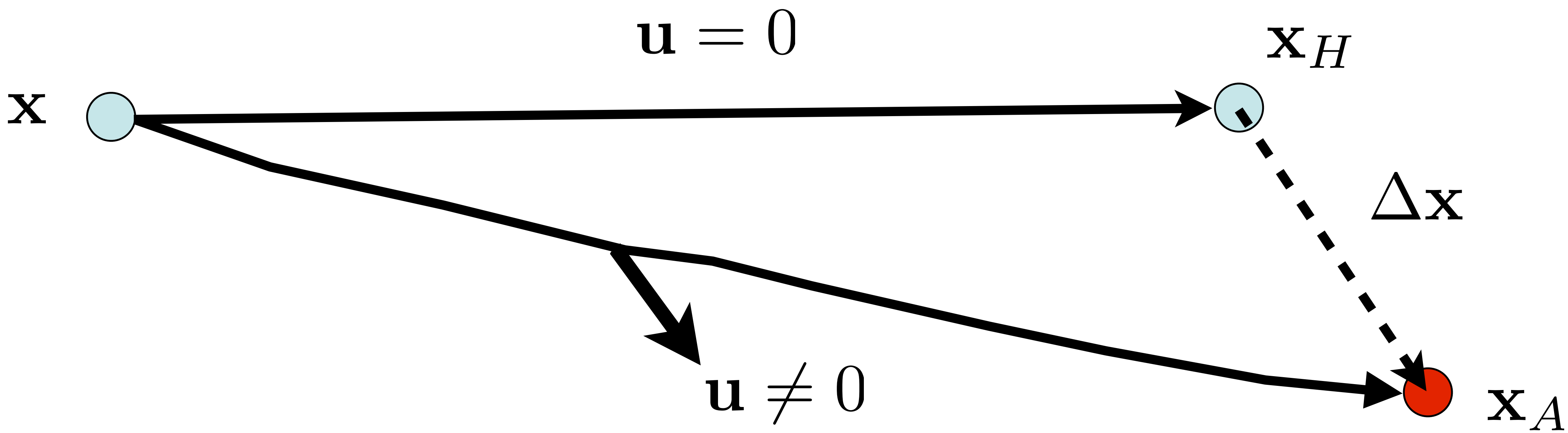
$$\left(\int_0^T \|\mathbf{u}\| d\tau\right)^2 \leq T \int_0^T \mathbf{u} \cdot \mathbf{u} d\tau$$

  - In conjunction with optimal control theory, gives an upper bound on the minimum-fuel control distance
  - Lower bound comes from the optimal impulsive transfer cost

$$d_C(L_\infty) \leq d_C(L_{u_M}) \leq \sqrt{2T d_C(L_E)}$$

  - Enables limits to be placed on fuel usage without intensive optimal control computations

Relatively Simple Computation of Control Distance Ensues



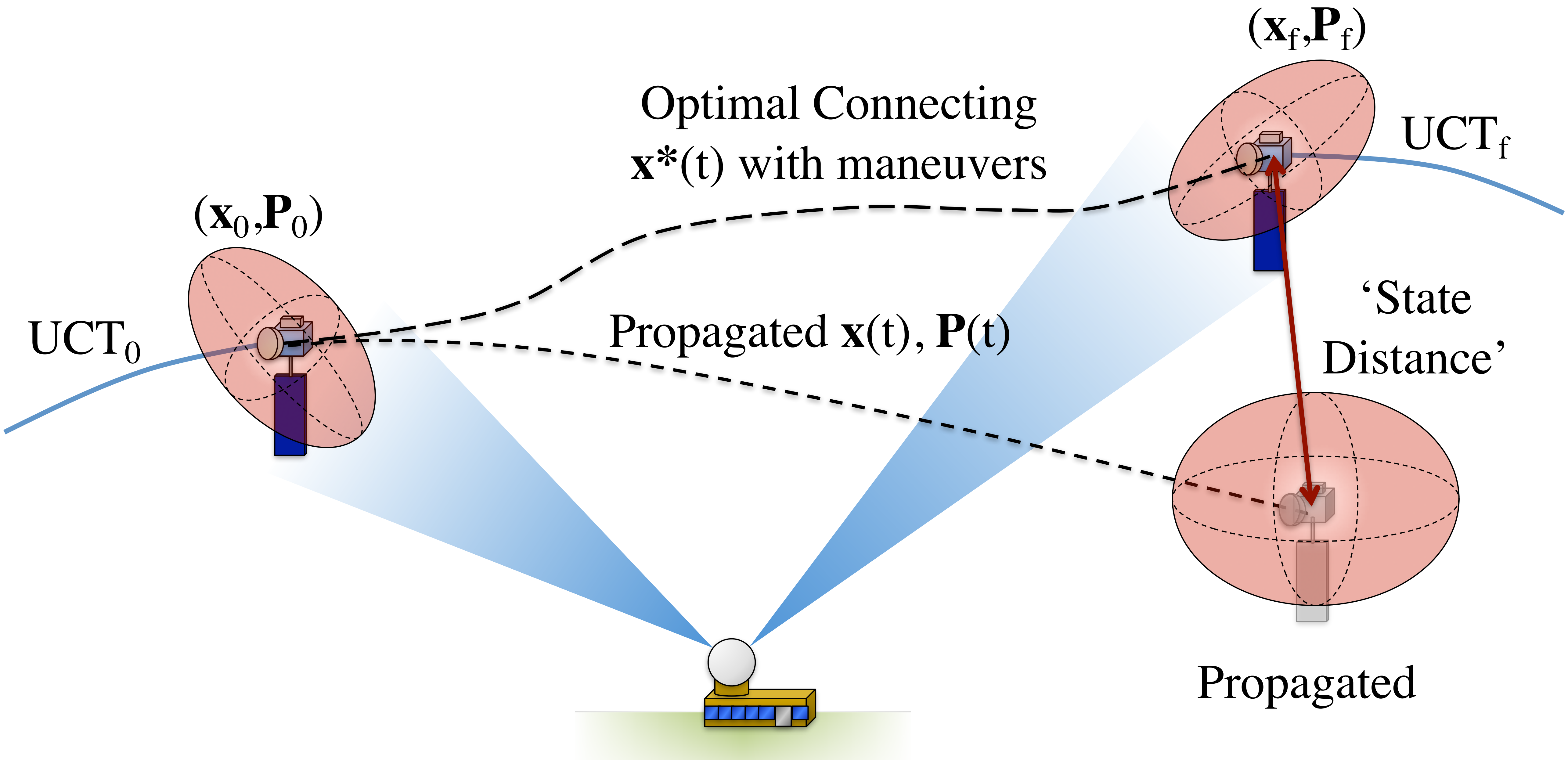
If $\Delta\mathbf{x}$ is "small", can solve explicitly through linearization, if not requires a non-linear solution

$$\mathbf{u}^* = -B^T \Lambda(t_f, t_0) \Delta\mathbf{x}$$

$$d_c = \Delta\mathbf{x} \cdot \Omega(t_f, t_0) \cdot \Delta\mathbf{x}$$

$$\Omega(t_f, t_0) = \frac{1}{2} \int_{t_0}^{t_f} \Lambda^T(\tau, t_0) B B^T \Lambda(\tau, t_0) d\tau$$

Control Distance Object Correlation with Uncertainty



'Control distance' metric is a potentially useful on-orbit object correlation metric

Stochastic State Variables

- How do we account for uncertainty?
  - The "control distance" can be defined as a stochastic quantity:
  - Terminal states are replaced by distributions: $N(\mathbf{x}, P_{xx})$
  - Control distances become stochastic:

$$d_C = d_C(\mathbf{x}_0 + \delta\mathbf{x}_0, t_0, \mathbf{x}_f + \delta\mathbf{x}_f, t_f) \qquad \delta\mathbf{x} \in N(0, P_{xx})$$

  - The control distance can be linearized relative to the mean trajectory at the terminal manifolds:

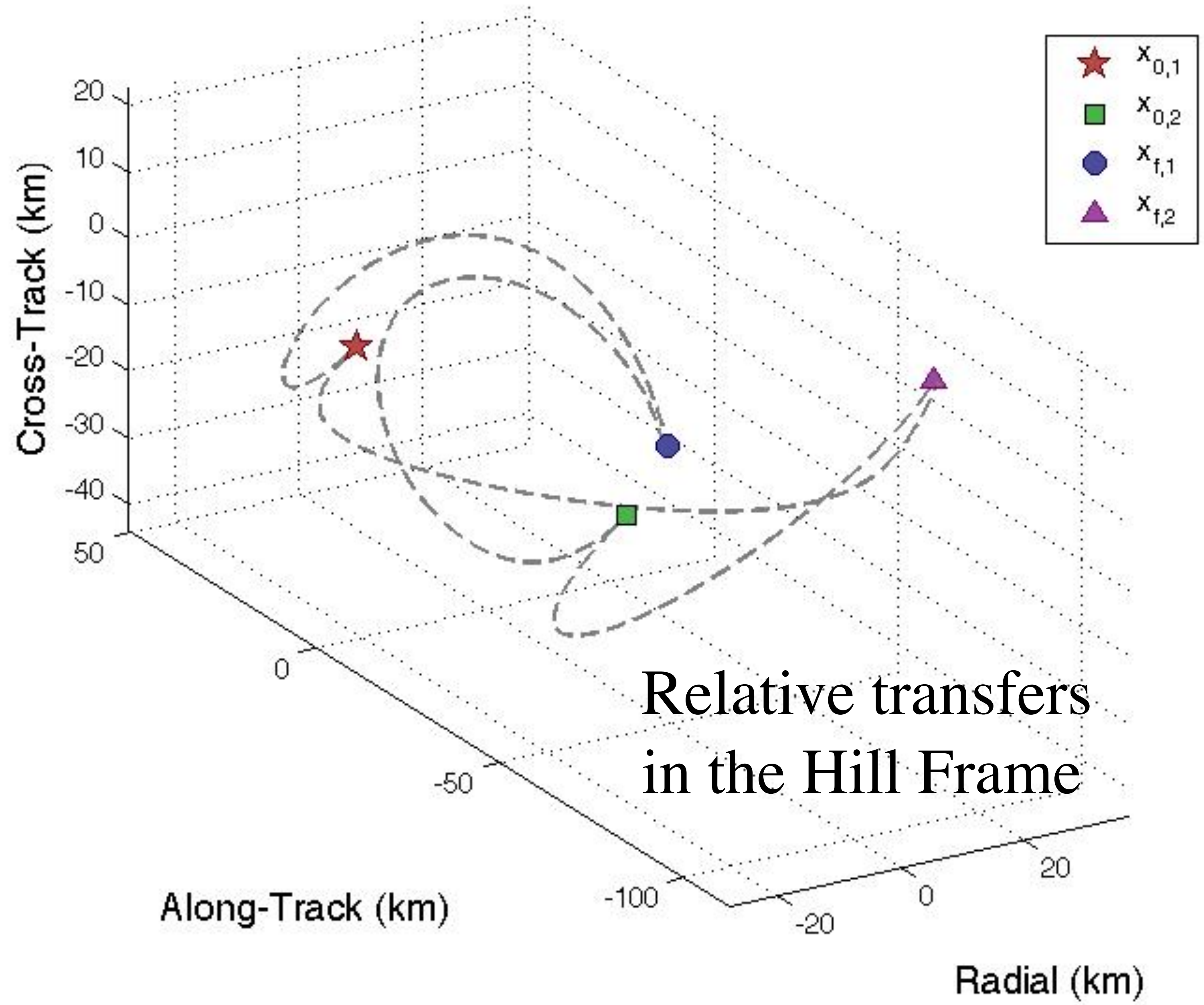
$$d_C = d_C^* + \omega\delta z + \delta z^T \Lambda \delta z \qquad \delta z = [\delta\mathbf{x}_0, \delta\mathbf{x}_f] \qquad \delta z = N\left(0, P\right)$$

$$\bar{d}_C = d_C^* + \mathrm{Trace}\left[\Lambda P\right] \qquad \sigma_{d_C}^2 = \omega^T P \omega + 2\mathrm{Trace}\left[\Lambda P \Lambda P\right]$$

Correlation Using Control Distance

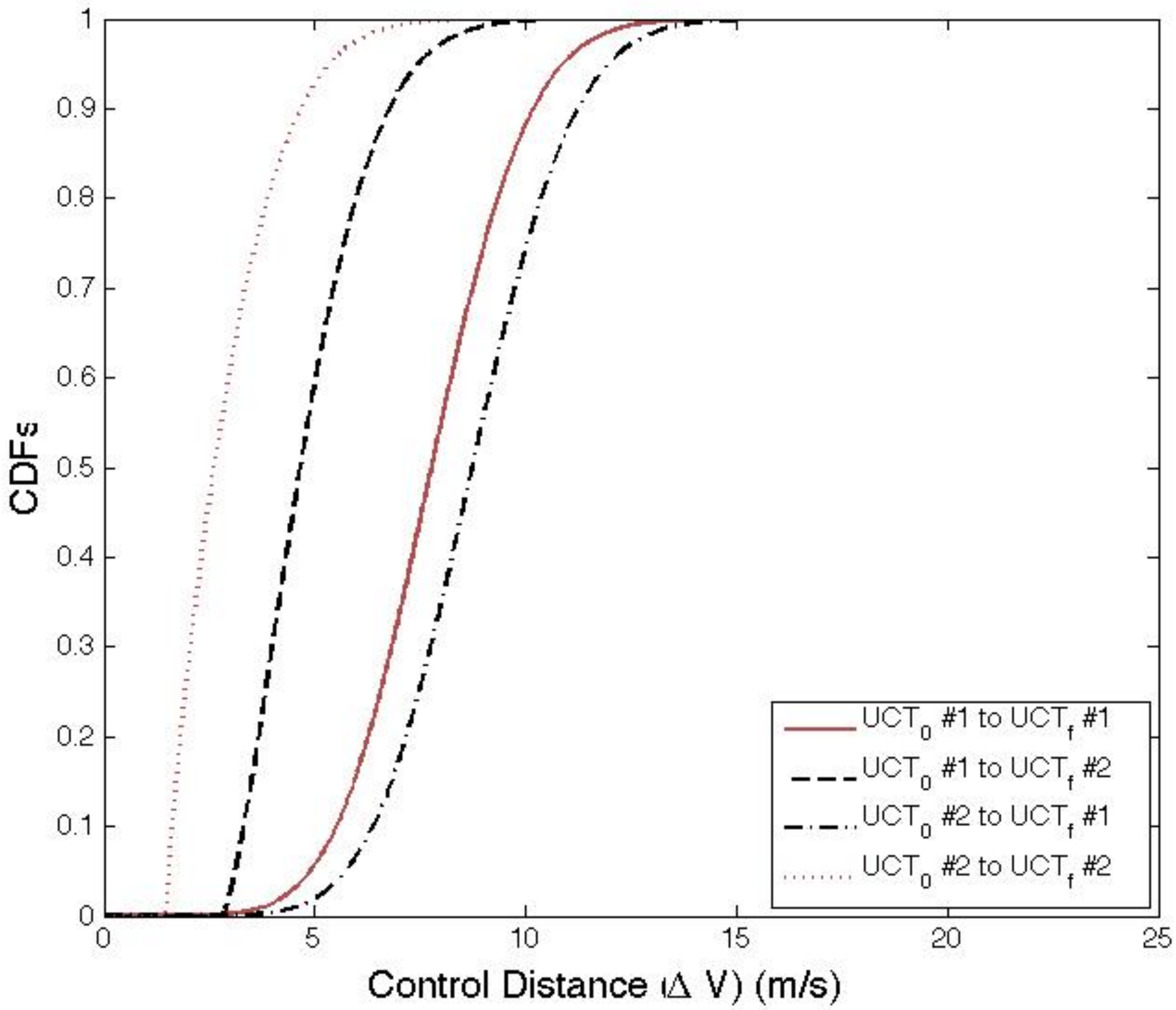
- Two GEO satellites in close formation
  - One satellite performs a 5 m/s maneuver
  - Orbit uncertainties for each satellite are ~ 100 m, 1 m/s
  - Two possible scenarios, can they be compared?

Case	Pairing
1	$X_{0,1}$ -> $X_{f,1}$ $X_{0,2}$ -> $X_{f,2}$
2	$X_{0,1}$ -> $X_{f,2}$ $X_{0,2}$ -> $X_{f,1}$



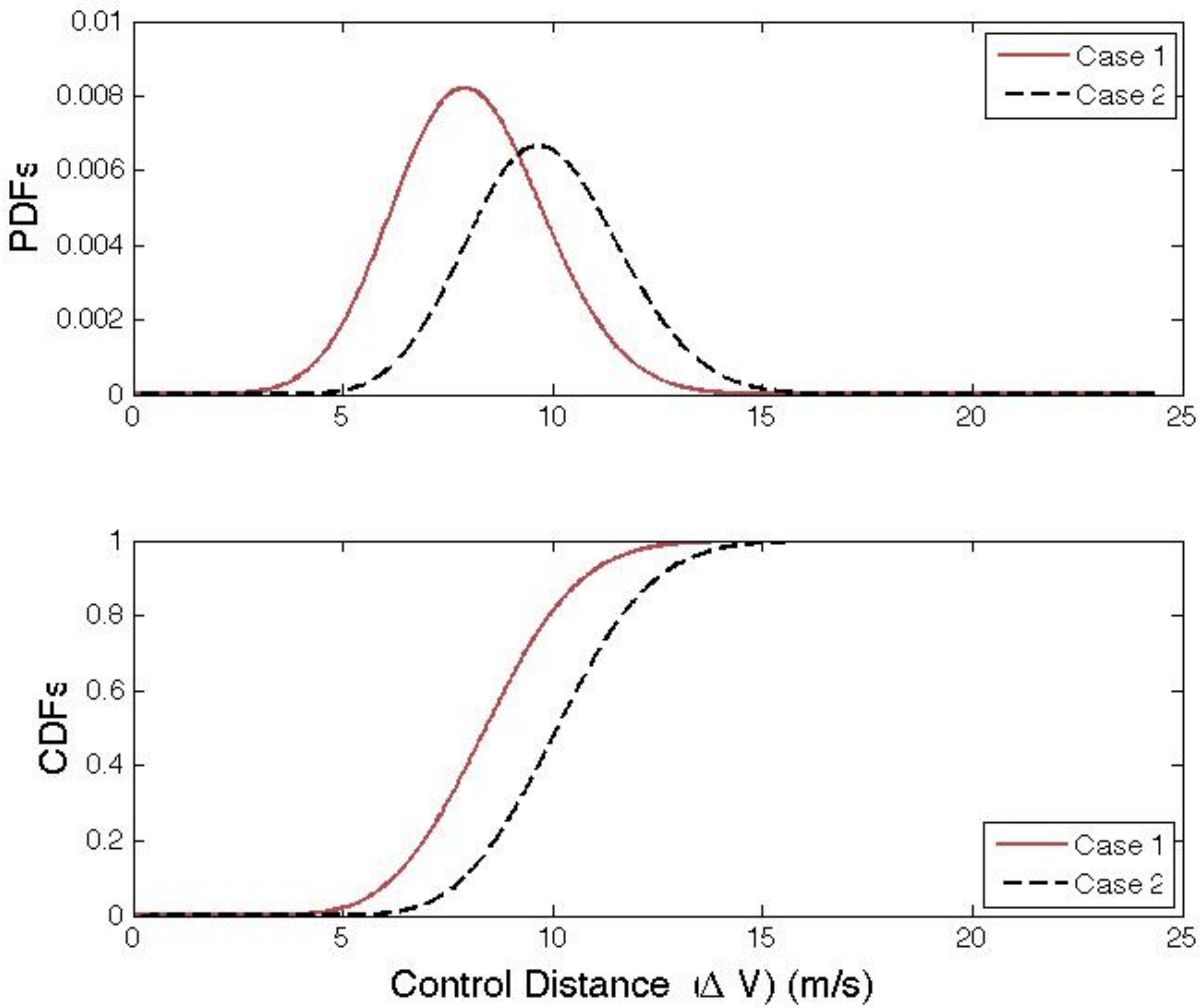
Relative transfers in the Hill Frame

Individual CDFs for each transfer



All are non-zero due to state uncertainties

Deterministic Cost of Each Scenario



Combined Probability Density Functions can be defined for each Case.

To compare, we use Stochastic Dominance, i.e., compare Cumulative Density Functions.

Case 1 is mor… , as its lower … st is alwa… probable th… se 2.

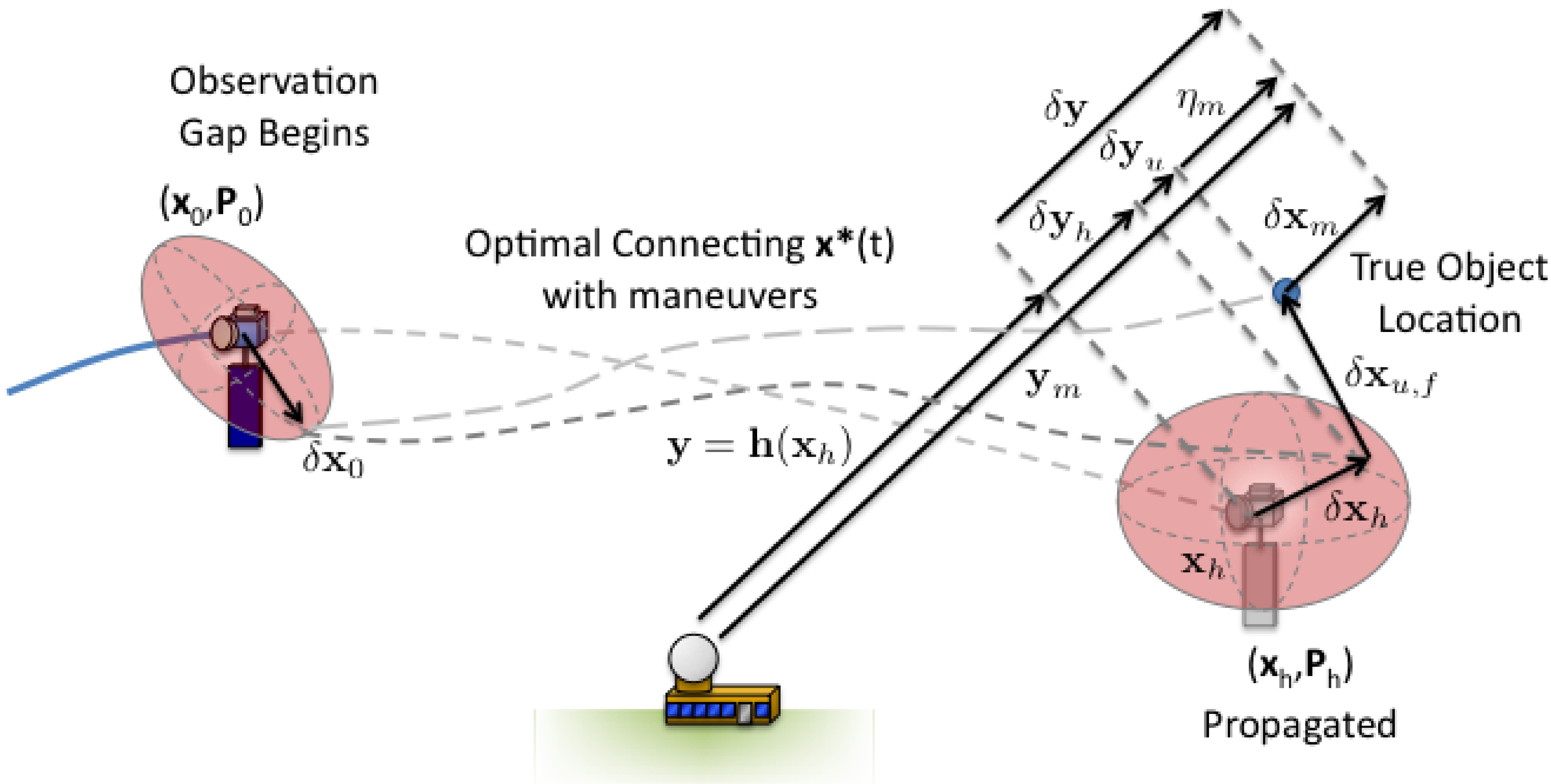
Assumes Principle II

Maneuver Detection & Model Improvement

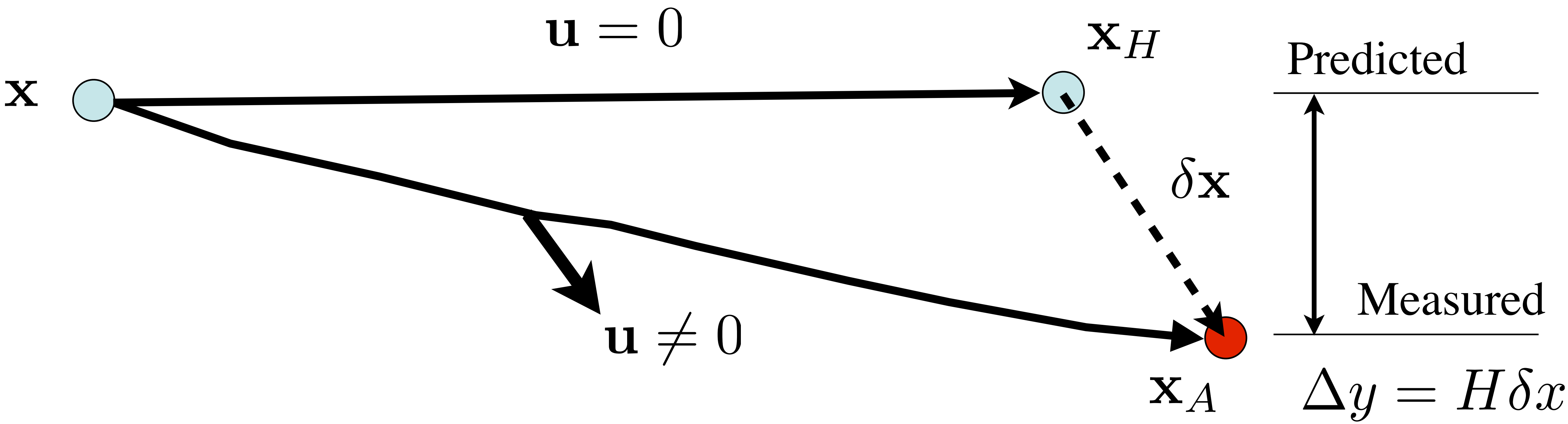
D.P. Lubey and D.J. Scheeres. “Identifying and Quantifying Mis-Modeled Dynamics via Optimal Control Problem Distance Metrics,” paper presented at the AIAA/AAS Astrodynamics Specialist Meeting, Minneapolis, August 2012.

D.P. Lubey and D.J. Scheeres. “An Optimal Control Based Estimator for Maneuver Detection,” paper presented at the AAS/AIAA Astrodynamics Specialist Meeting, Hilton Head, August 2013.

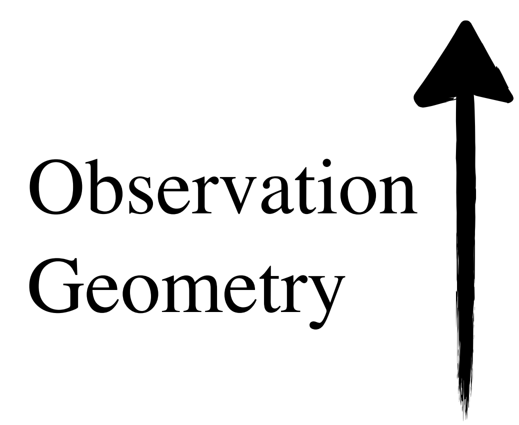
Maneuver Detection



Maneuver Detection



If the satellite has maneuvered, it will initially show up as a deviation from the predicted measurement residual.

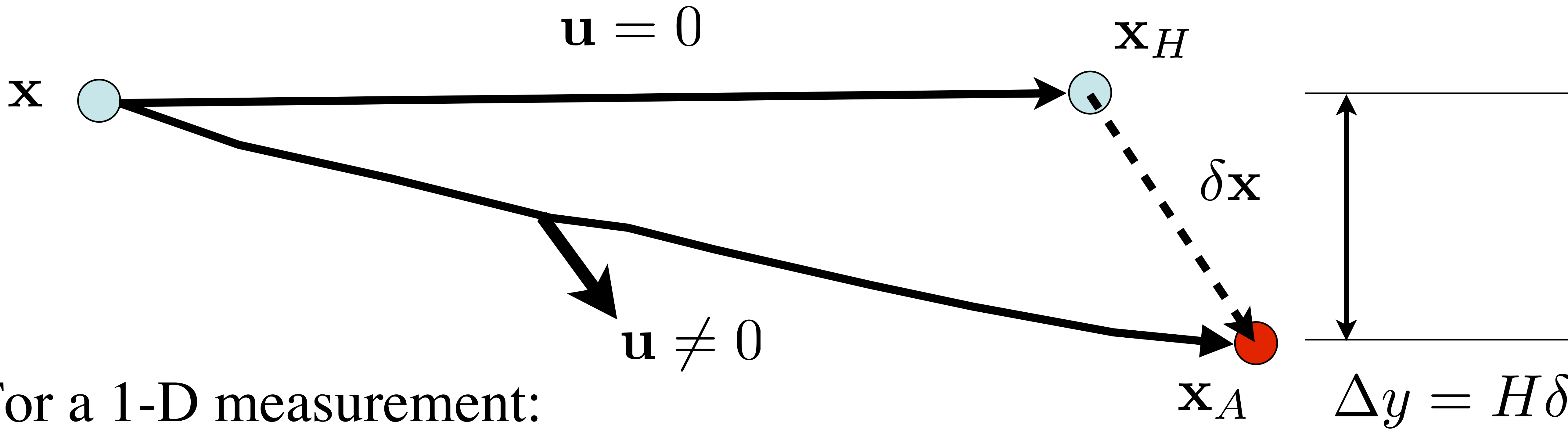


Maneuver Detection

- Given an estimated state, we propagate its mean and covariance to the next measurement epoch
  – Measurements are generically "singular" in that they only sense sub-manifolds of the actual satellite state:

$$\Delta y = H \cdot \delta\mathbf{x} \qquad H \in \mathbb{R}^{s\times n} \qquad \Delta y \in \mathbb{R}^{s}$$

- In filtering theory, following a measurement we define the residual, expected noise from state uncertainty, and expected noise from measurement:
  – These can be jointly transformed into expected control distances
  – Optimal control theory provides simple ways to measure control effort to a sub-manifold through the transversality conditions
  – Only $s < n$ conditions to satisfy – lowering control distance cost.



For a 1-D measurement:

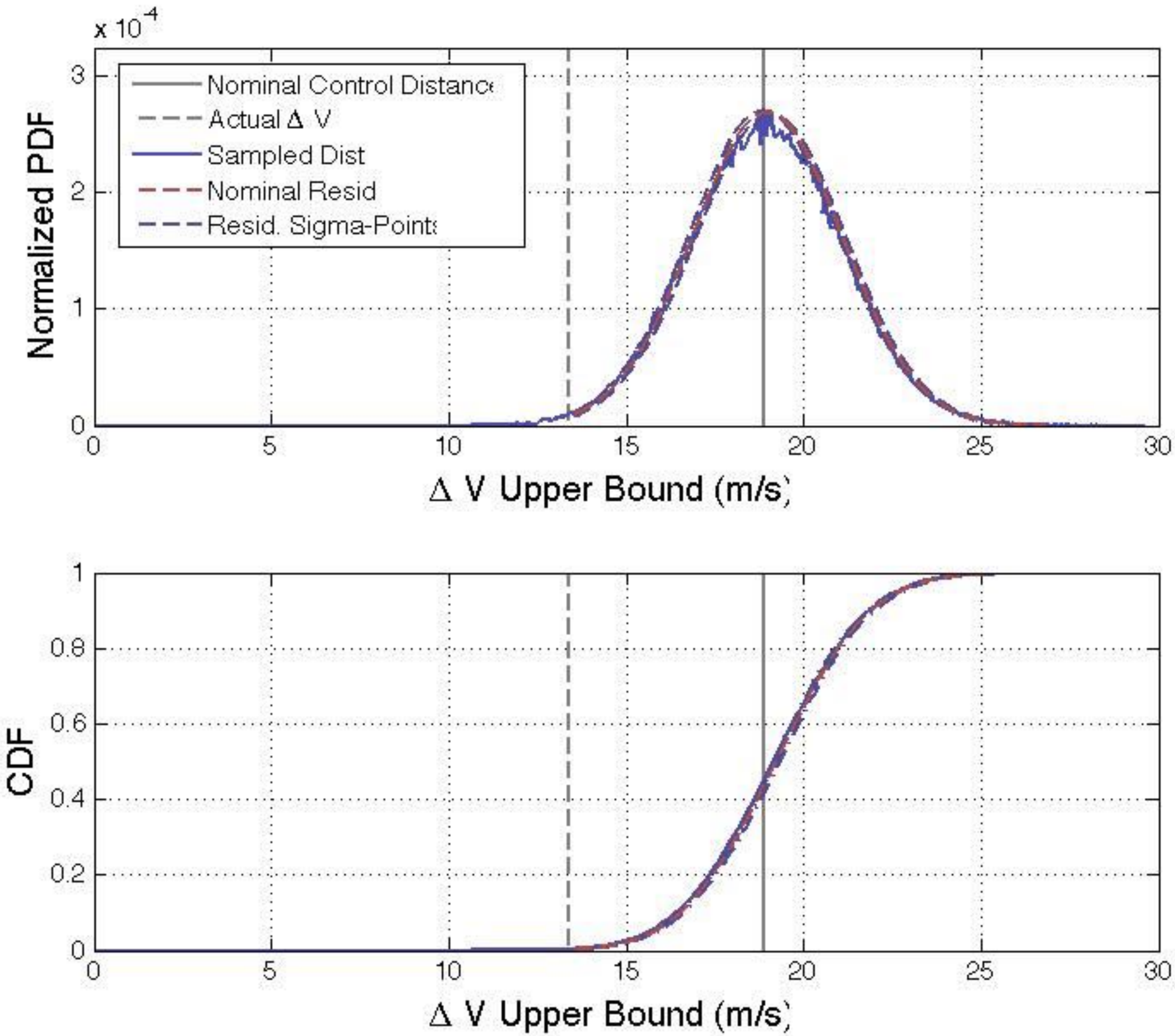
$$\Delta y = H\delta x$$

$$\mathbf{u}^* = \frac{-1}{H\Lambda(t_f, t_0)H^T} B^T H^T \Delta y$$

$$d_c = \Delta y \cdot \Omega_H(t_f, t_0) \cdot \Delta y$$

$$\Omega_H(t_f, t_0) = \frac{1}{2\left(H\Lambda(t_f, t_0)H^T\right)^2} \int_{t_0}^{t_f} HBB^T H^T d\tau$$

Example GEO maneuver detection



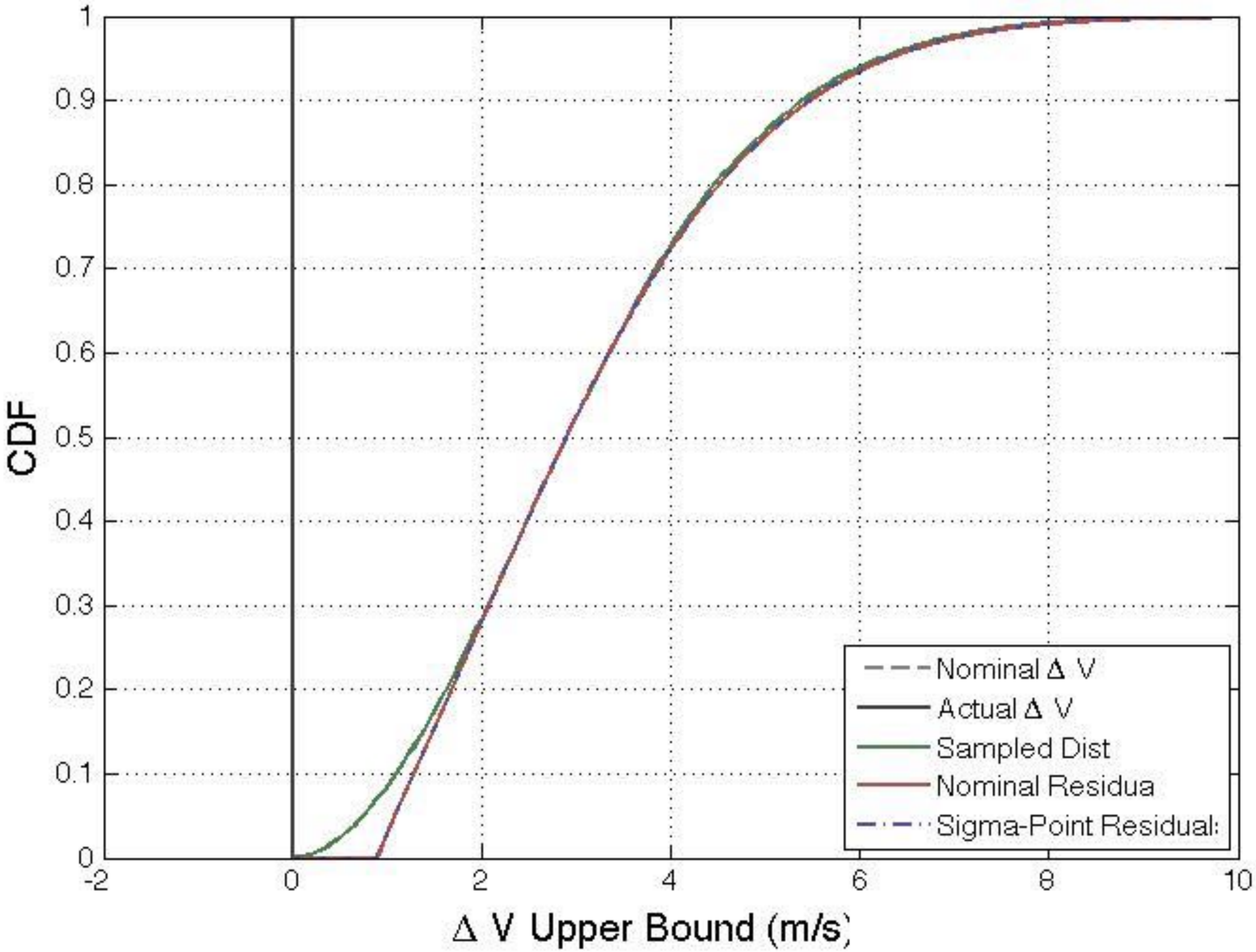
GEO Satellite:

13.4 m/s out-of-plane maneuver

Angle observation 15 hrs post-maneuver

Similar orbit state uncertainties as prior

Non-Maneuver Computation Case



Allows "visible" maneuvers to be automatically bounded

Defines threshold of what maneuvers can be seen as a function of measurements and state knowledge

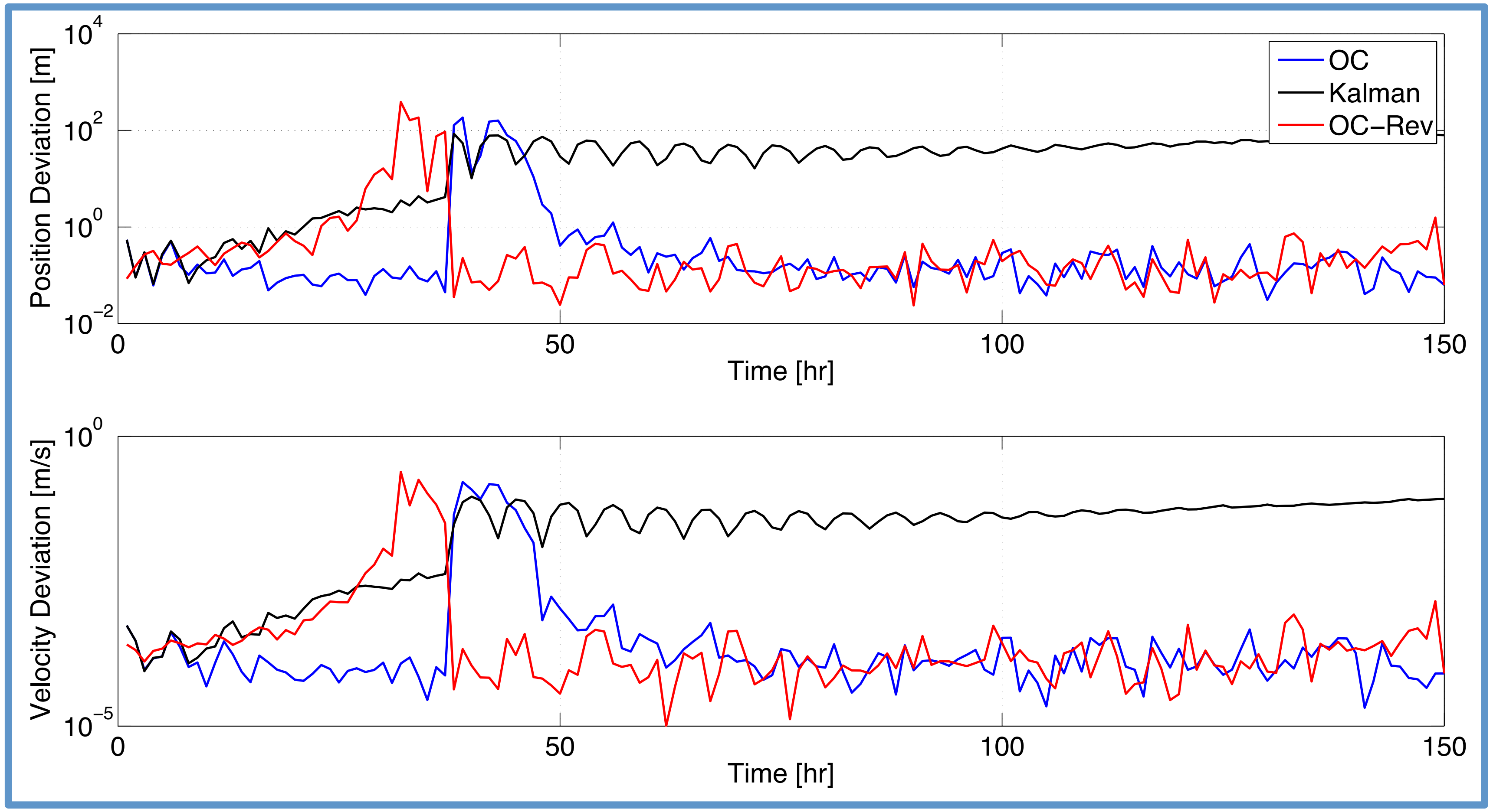
Simulations & Results

- Setup
  - LEO Spacecraft
  - Dynamics
    - Two Body Gravity
    - J2
    - Atmospheric Drag
    - Solar Radiation Pressure
- Observations
  - One observation each hour for 150 hours
    - Observed from 1 of 3 GEO spacecraft
    - Range and Range Rate
    - Error of 0.1 m and 1 mm/s for measurements, respectively
- Simulation
  - Drag Mismodeled (5% overestimate)
  - Impulse (0.1 m/s in cross-track direction at ~38 hours)

Simulations & Results

- Kalman Tracking gradually diverges due to drag mismodeling, then large divergence after impulse
- Tracking is best for true level of uncertainty, σ_Q = 1E-08 m/s^2
- Inflation of covariance displayed in uncertainty plot
- Bias in control indicates drag mismodeling
- Impulse is identified through forward-reverse technique, and reconstructed with finite controls
  - Data run through estimator forward, then re run in reverse to mitigate divergence from impulse
- Maneuver detection results indicate when true disturbance magnitude is selected

Deviation from True Trajectory in Position and Velocity



Simulations & Results

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Uncertainty in Position and Velocity