ASEN 6020
Optimal Trajectories
Exam
Spring 2022

Name: Key

E-mail:________________________

Total points = 100
Hand in this copy of the test in addition to any work sheets that you use.

The exam is designed to be taken in a single 90 minute period, and you are allowed up to 180 minutes to complete, scan and upload the exam.
The test is open book and notes, but no internet access except to the course Canvas page.

Write your answer clearly and neatly in the space provided.

If you have any questions as you take the test, feel free to send me a text message question at: 1-720-544-1260. Please only do this between the hours of 8AM and 11PM MT. I will respond as soon as I can.

Scan your exam and any worksheets you used during the exam and upload it to the Canvas site as soon as you are done.

Please sign below to confirm that you have not collaborated with anyone else in taking this test and that you have worked on this exam for, at most, a total of 180 minutes. I will not grade this exam unless it is signed. E-signatures are OK.

Signature: ________________________

Starting Time and Date: ________________________

Ending Time and Date: ________________________

1. (30)
Parametric Optimization:

Minimize the distance between a point in the (x, y) plane at $x = 0$ and $y = 1$ and a cubic equation $y = x^3$. Specifically, assume a cost function

$$J = x^2 + (y-1)^2$$

and a constraint function

$$g(x, y) = y - x^3$$

(a) Write the explicit KKT Necessary Conditions for this problem:

$$L = x^2 + (y-1)^2 + \lambda(y - x^3)$$

$$\boxed{\begin{aligned} \frac{\partial L}{\partial x} &= 2x - 3\lambda x^2 = 0 \\ \frac{\partial L}{\partial y} &= 2(y-1) + \lambda = 0 \\ g &= y - x^3 = 0 \end{aligned}}$$

(b) Show that the point $x = 0$ satisfies all of the necessary conditions ***and*** is a local minimum:

At $x=0$

$L_x = 0$

$g = \boxed{y=0}$

$L_y = -2+\lambda = 0$

$\boxed{\lambda = 2}$

$L^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} > 0$

(c) Note that the value of J, subject to the constraints, equals 1 when $x = 0$ and also equals 1 when $x = 1$. Based on this, what can you say about the global nature of the minimum that occurs at $x = 0$.

When $x=0 \Rightarrow y=0$ and $J = (-1)^2 = 1$

When $x=1 \Rightarrow y=1$ and $J = 1 + (1-1)^2 = 1$

Note, the N.C. cannot be satisfied at $x=1$, thus this is not an extremum. Therefore, there are values of J that are both >1 & <1, meaning that the minimum at $x=0$ is not a global minimum.

See next page...

① $L = J + \lambda g$

ⓐ $L = x^2 + (y-1)^2 + \lambda(y - x^3)$

$$\boxed{\begin{aligned} \frac{\partial L}{\partial x} &= 2x - 3\lambda x^2 = 0 \\ \frac{\partial L}{\partial y} &= 2(y-1) + \lambda = 0 \\ g &= y - x^3 = 0 \end{aligned}}$$

ⓑ Note, $J = x^2 + (x^3 - 1)^2$; expand around $x = 0$

$$J(\delta x) = \overset{=1}{J(0)} + \cancel{\left.\frac{\partial J}{\partial x}\right|_0}^{\,0} \delta x + \frac{1}{2} \left.\frac{\partial^2 J}{\partial x^2}\right|_0 \delta x^2 + \cdots$$

At $x = 0$

$L_x = 0$

$g = \boxed{y = 0}$

$L_y = -2 + \lambda = 0$

$\boxed{\lambda = 2}$

$J_x = 2x + 2(x^3 - 1)3x^2$

$J_{xx} = 2 + 12(x^3 - 1)x + 18x^4 \Big|_{x=0} = \underline{2 > 0}$

ⓒ We can look for an alternate extremum.... $\Rightarrow$

$$x[2-3\lambda x]=0$$

$$2(y-1)+\lambda=0 \quad \lambda = 2-2x^3$$

$$y=x^3$$

$$x[\underbrace{2-3(2-2x^3)x}_{2-6x+6x^4}]=0$$

Alt. candidate is a root of $6x^4-6x+2=0$

$$\boxed{x^4-x+\frac{1}{3}=0}$$

$\Rightarrow$ roots
(A) $x \sim 0.348$ — $y=0.0421, \lambda=1.916$
(B) $x \sim 0.846293\ldots$ — $y=0.606125\ldots$, $\lambda=0.78775\ldots$

$J_A = 1.0386\ldots$

$J_B = 0.8713\ldots < J_0$

2. (30)
Spacecraft Transfers:

Optimal transfer for a sequence of rendezvous points in orbit.

Assume an initially circular orbit about a planet with a gravitational parameter μ and a semi-major axis a. It is desired to transition this orbit into an elliptic orbit of eccentricity e with the same semi-major axis.

This can be performed in at least three ways: a one-impulse maneuver that rotates the velocity vector, a two-impulse transfer first boosting apoapsis and then dropping periapsis, and a two-impulse transfer first dropping periapsis and then boosting apoapsis.

$V_c = \sqrt{\frac{\mu}{a}}$

Recall the expansions $\sqrt{1+ae} = 1 + ae/2 + \ldots$ and $\frac{1}{\sqrt{1+ae}} = 1 - ae/2 + \ldots$ when $e \ll 1$.

(a) The one-impulse transfer will rotate the circular velocity by an angle γ, and it can be shown that after such a maneuver $e = |\sin\gamma|$. State the cost of a one-impulse burn with a target eccentricity of e and the approximate cost if $e \ll 1$.

$\Delta V_1 =$ $2 V_c \sin(\gamma/2)$

ΔV_1 for $e \ll 1 =$ $V_c \cdot e$

(b) State the cost of a two-impulse transfer with a boost to an apoapsis of $a(1+e)$ followed at apoapsis with a drop of periapsis to $a(1-e)$. Again, also evaluate the cost if $e \ll 1$.

$\Delta V_2 =$ $\sqrt{\frac{\mu}{a}}\left[\sqrt{\frac{1+e}{1+\frac{1}{2}e}} - 1 + \frac{1}{\sqrt{1+\frac{1}{2}e}\sqrt{1+e}} - \sqrt{\frac{1-e}{1+e}}\right]$

ΔV_2 for $e \ll 1 =$ $\sqrt{\mu/a}\left(\frac{1}{2}e\right) + \ldots$

(c) State the cost of a two-impulse transfer with a drop to a periapsis of $a(1-e)$ followed at periapsis with a boost of apoapsis to $a(1+e)$. Again, also evaluate the cost if $e \ll 1$.

$\Delta V_3 =$ $\sqrt{\frac{\mu}{a}}\left[1 - \sqrt{\frac{1-e}{1-\frac{1}{2}e}} + \sqrt{\frac{1+e}{1-e}} - \frac{1}{\sqrt{(1-e)(1-\frac{1}{2}e)}}\right]$

ΔV_3 for $e \ll 1 =$ $\sqrt{\mu/a}\left(\frac{1}{2}e\right) + \ldots$

(d) Show that for small e the costs are $\Delta V_2 \sim \Delta V_3 \sim \Delta V_1/2$. ✓

(e) Based on first principles, for transfers when e is not small do you expect ΔV_2 or ΔV_3 to be optimal? Explain in brief:

For earbitrarily large, $\Delta V_2 \sim (\sqrt{2}-1) V_c$ $(Q \to \infty)$ & $\Delta V_3 \sim V_c$ $(q \to 0)$

$\Delta V_2 < \Delta V_3$ in this case, not a proof but a demo...

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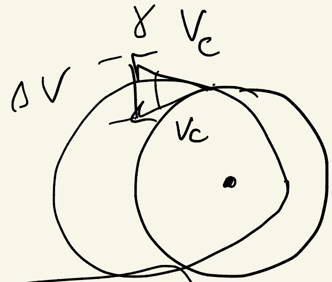
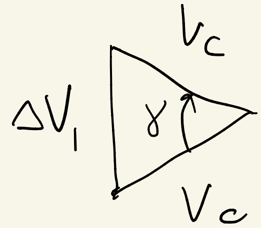
(a)



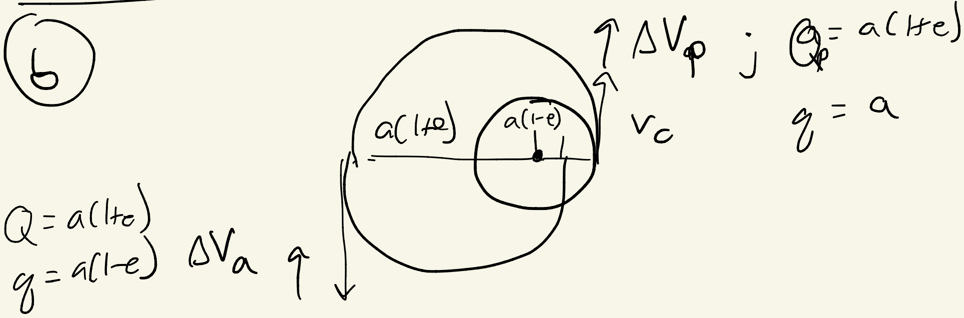


$$\boxed{\Delta V_1 = 2V_c \sin(\gamma/2)}$$

If $e<<1$ & $e=\sin\gamma$, $\gamma<<1$ and $\sin\gamma\sim\gamma$

$$e\sim\gamma \Rightarrow \boxed{\Delta V_1 \sim V_c e}$$

(b)



$$\Delta V_2 = \sqrt{\frac{2\mu}{a+a(1+e)}\frac{(1+e)}{1}} - \sqrt{\frac{\mu}{a}} + \sqrt{\frac{2\mu}{a+a(1+e)}\frac{1}{(1+e)}} - \sqrt{\frac{\mu}{a}\left(\frac{1-e}{1+e}\right)}$$

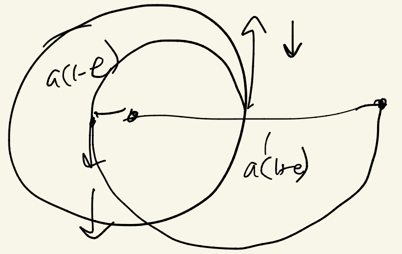
$$= \sqrt{\frac{\mu}{a}}\left[\sqrt{\frac{1+e}{1+\frac{1}{2}e}} - 1 + \sqrt{\frac{1}{(1+\frac{1}{2}e)(1+e)}} - \sqrt{\frac{1-e}{1+e}}\right]$$

for $e<<1$

$$\Delta V_2 = \sqrt{\frac{\mu}{a}}\left[\left(1+\frac{1}{2}e\right)\left(1-\frac{1}{4}e\right) - 1 + \left(1-\frac{1}{4}e\ldots\right)\left(1-\frac{1}{2}e\ldots\right) - \left(1-\frac{1}{2}e\right)\left(1-\frac{1}{2}e\right)\right]$$

$$= \sqrt{\frac{\mu}{a}}\left[\cancel{1} + \frac{1}{4}e + \cdots - \cancel{1} + \underbrace{\cancel{1} - \frac{3}{4}e \cdots - \cancel{1} + e + \cdots}_{\frac{1}{4}e}\right] = \sqrt{\frac{\mu}{a}}\,\frac{1}{2}e$$

2c



$$\Delta V_a = \sqrt{\frac{\mu}{a}} - \sqrt{\frac{2\mu}{a+a(1-e)}(1-e)}$$

$$\Delta V_p = \sqrt{\frac{\mu}{a}\left(\frac{1+e}{1-e}\right)} - \sqrt{\frac{2\mu}{a+a(1-e)}\,\frac{1}{1-e}}$$

$$\Delta V_3 = \sqrt{\frac{\mu}{a}}\left[1 - \sqrt{\frac{1-e}{1-\frac{1}{2}e}} + \sqrt{\frac{1+e}{1-e}} - \frac{1}{\sqrt{(1-e)(1-\frac{1}{2}e)}}\right]$$

low ecc

$$\Delta V_3 = \sqrt{\frac{\mu}{a}}\left[1 - \left(1-\tfrac{1}{2}e\ldots\right)\left(1+\tfrac{1}{4}e\ldots\right) + \left(1+\tfrac{1}{2}e\ldots\right)\left(1+\tfrac{1}{2}e+\ldots\right) - \left(1+\tfrac{1}{2}e\ldots\right)\left(1+\tfrac{1}{4}e\ldots\right)\right]$$

$$= \sqrt{\frac{\mu}{a}}\left[1 - \left(1-\tfrac{1}{2}e+\tfrac{1}{4}e\ldots\right) + 1 + \tfrac{1}{2}e+\tfrac{1}{2}e+\ldots - \left(1+\tfrac{1}{2}e+\tfrac{1}{4}e+\ldots\right)\right]$$

$$= \sqrt{\frac{\mu}{a}}\left[\tfrac{1}{4}e + \tfrac{1}{4}e + \ldots\right] = \frac{1}{2}\sqrt{\frac{\mu}{a}}\,e$$

(e) Cost to set $q=0 \Rightarrow \Delta V = \sqrt{\frac{\mu}{a}}$

" " " $Q=\infty \Rightarrow \Delta V = (\sqrt{2}-1)\sqrt{\frac{\mu}{a}} < \sqrt{\frac{\mu}{a}}$

ΔV_2

3. (40)

Assume a simple 1-DOF dynamical system:

$$\dot{r} = v$$
$$\dot{v} = u$$

where $r, v \in \mathbb{R}$ and $|u| \le 1$ is the control. The optimal cost function to be minimized is:

$$J = \int_0^{t_f} |u| d\tau$$

There is a constraint on the state and time at the initial epoch:

$$r(t_o) = r_o$$
$$v(t_o) = v_o$$
$$t_o = 0$$

The final time t_f and final states r_f and v_f are <u>completely free</u>.

(a) Write out the pre-Hamiltonian for this system:

$H(r, v, p_r, p_v, u) =$ $H = |u| + u P_v + P_r V = |u|(1 + P_v \, \text{sgn}(u)) + P_r V$

(b) Using the minimum principle, find the optimal control law for this system:

$u^* = \text{argmin}_u H(r, v, p_r, p_v, u) =$ ______

$$u^* = \begin{cases} 0 & \text{if } |P_v| < 1 \\ -1 & \text{if } P_v > 1 \\ +1 & \text{if } P_v < -1 \\ \text{nb.} & \text{if } |P_v| = 1 \end{cases}$$

(c) Substitute this back into the Hamiltonian to find the optimal control Hamiltonian, and the associated equations of motion for the states and adjoints:

$H^*(r, v, p_r, p_v) =$ ______

$$\boxed{H^* = \begin{cases} P_r V & \text{if } |P_v| \le 1 \\ (1 - P_v) + P_r V & \text{if } P_v > 1 \\ (1 + P_v) + P_r V & \text{if } P_v < -1 \end{cases}}$$

$\dot{r} = \frac{\partial H}{\partial P_r} = V$ for all cases

$\dot{v} = \frac{\partial H}{\partial P_v} = 0 \ (|P_v| \le 1), \ -1 \ (P_v > 1), \ 1 \ (P_v < -1)$

$\dot{p_r} = \frac{-\partial H}{\partial r} = 0$

$\dot{p_v} = -\frac{\partial H}{\partial V} = P_r$

(d) Evaluate the Transversality Conditions to find the constraints on the initial and final values of the adjoints and Hamiltonian:

$p_{ro} =$ $-\lambda_{r_0}$

$p_{vo} =$ $-\lambda_{v_0}$

$H_o =$ λ_{t_0}

$p_{rf} =$ 0

$p_{vf} =$ 0

$H_f =$ 0

(e) Based on the above conditions and dynamics, what is the solution to these necessary conditions. Explain your answer.

We find $P_r = \text{const} = 0$

$P_v = \text{const} = 0 \Rightarrow u^* = 0.$

Thus, optimal control is to do nothing! This makes sense as there are no target states for the system, yet it wants to minimize J.

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ⓐ $H = u p_v + |u| + p_r V = |u|\left(1 + \mathrm{sgn}(u) p_v\right) + p_r V$

ⓑ Min principle[#1] is that $\mathrm{sg}(u) = -\mathrm{sgn}(p_v)$,

thus $H = |u|\left(1 - \mathrm{sgn}(p_v)\cdot p_v\right) + p_r V$

$= |u|\left(1 - |p_v|\right) + p_r V$

Next, if $(1 - |p_v|) > 0 \Rightarrow$ set $|u| = 0$

$(1 - |p_v|) < 0 \Rightarrow$ set $|u| = |u_{max}| = 1$

Thus

$$\boxed{u^* = \begin{cases} 0 & \text{if } |p_v| < 1 \\ -1 & \text{if } p_v > 1 \\ +1 & \text{if } p_v < -1 \\ \text{arb.} & \text{if } |p_v| = 1 \end{cases}}$$

ⓒ $$\boxed{H^* = \begin{cases} p_r V & \text{if } |p_v| \leq 1 \\ (1 - p_v) + p_r V & \text{if } p_v > 1 \\ (1 + p_v) + p_r V & \text{if } p_v < -1 \end{cases}}$$

(d) Trans. Conditions.

r_f, v_f & t_f free

r_0, v_0, t_0 fixed

$$g = \begin{bmatrix} r(t_0) \\ v(t_0) \\ t_0 \end{bmatrix} - \begin{bmatrix} r_0 \\ v_0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} P_{r_0} &= -\lambda_{r_0} \\ P_{v_0} &= -\lambda_{v_0} \\ H_0 &= \lambda_{t_0} \end{aligned}\right\} \text{Constants to be found.}$$

$P_{r_f} = P_{v_f} = H_f = 0 \Rightarrow$ free

(e) Solve system.... $\dot{P}_r = 0 \Rightarrow P_r =$ Constant

but ~~from~~ above, then $P_r = P_{r_f} = 0 = -\lambda_{r_0}$

Then if $\dot{P}_v = 0$, $P_v =$ Const, and $P_v = P_{v_f} = 0 = -\lambda_{v_0}$

Thus if $P_v = 0$, then $u^*(P_v = 0) = 0$!

Then $\dot{V} = 0 \Rightarrow V(t) = V_0$

$\dot{r} = V \Rightarrow r(t) = r_0 + V_0 \cdot t$

Optimal control to minimize L is to do nothing, since there are no targets we need to hit!
A bit of a trick question, but which the N.C. give a precise solution to.