# Applied Spacecraft Trajectory Optimization: Lecture 2

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ASEN 6020 Guest Lectures

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# Overview

Celestial and Spaceflight Mechanics Laboratory

- Lecture 1: Homotopy Methods and Multi-Objective Optimization
    - Detailed indirect optimization + homotopy example
    - Tips for class project
    - Scalarized methods vs. evolutionary algorithms for MOO
- Lecture 2:
  - Optimization Under Uncertainty
    - Types of uncertainty/stochastics
    - Notation & terminology
    - Techniques
  - Dynamic programming
    - LQR from dynamic programming
    - DDP, HDDP, STT/DDP, SDDP

# Optimization Under Uncertainty

# What is “stochastic optimization”?



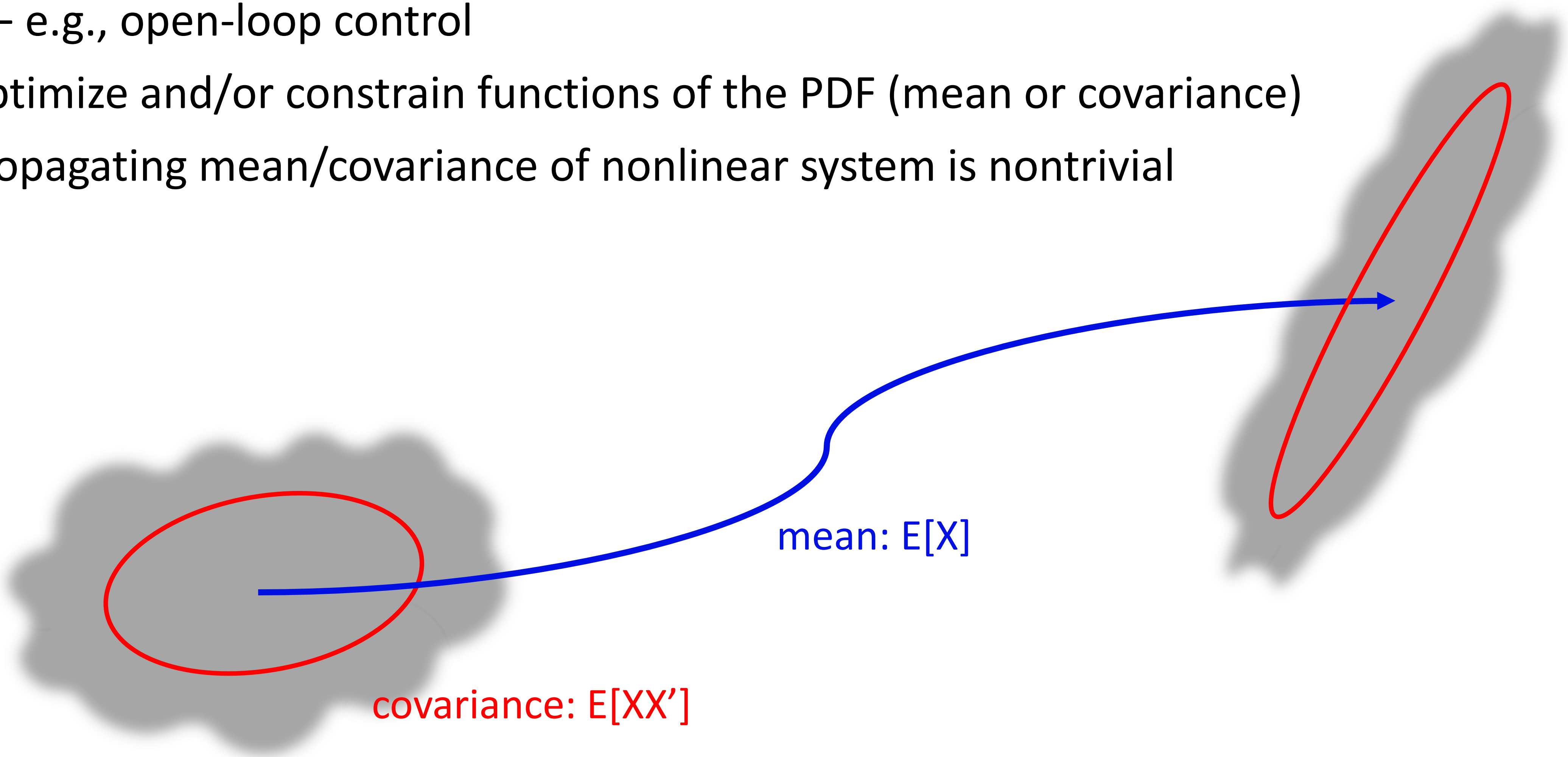
- Stochastics in numerical algorithms:
  - random initial guesses
  - monotonic basin hopping
  - random mutations in genetic algorithms
- Stochastic dynamics:
  - State uncertainty
  - Parameter uncertainty
  - Noise in dynamics

In this lecture

But how do we optimize/constrain something that is unknowable?!

# Probability Density Function (PDF)

- Under certain assumptions, the PDF of a stochastic variable evolves deterministically
  - e.g., open-loop control
- Optimize and/or constrain functions of the PDF (mean or covariance)
- Propagating mean/covariance of nonlinear system is nontrivial



# Uncertainty in Space Mission Design

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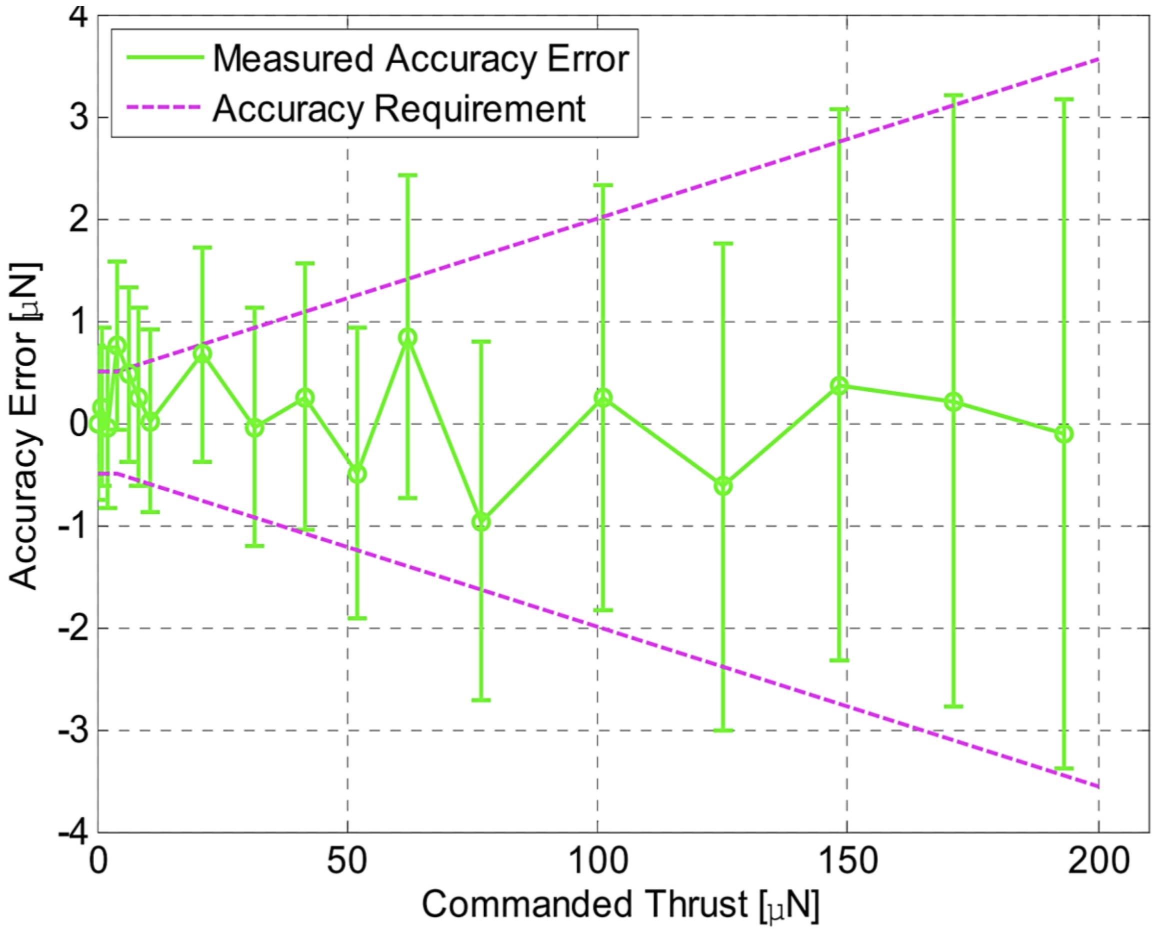
- Navigation uncertainty (open-loop control)
  - Assume some initial PDF
- Navigation uncertainty (closed-loop control)
  - Assume covariance updates at pre-scheduled times
  - Include estimator dynamics
- Parameter uncertainty
  - Add uncertain parameter as a “state” with $\dot{\mu} = 0$
- Maneuver execution errors
- Missed thrust events (MTEs)
- Dynamical modeling errors
  - Additive noise/process noise

# Control-Dependent Noise

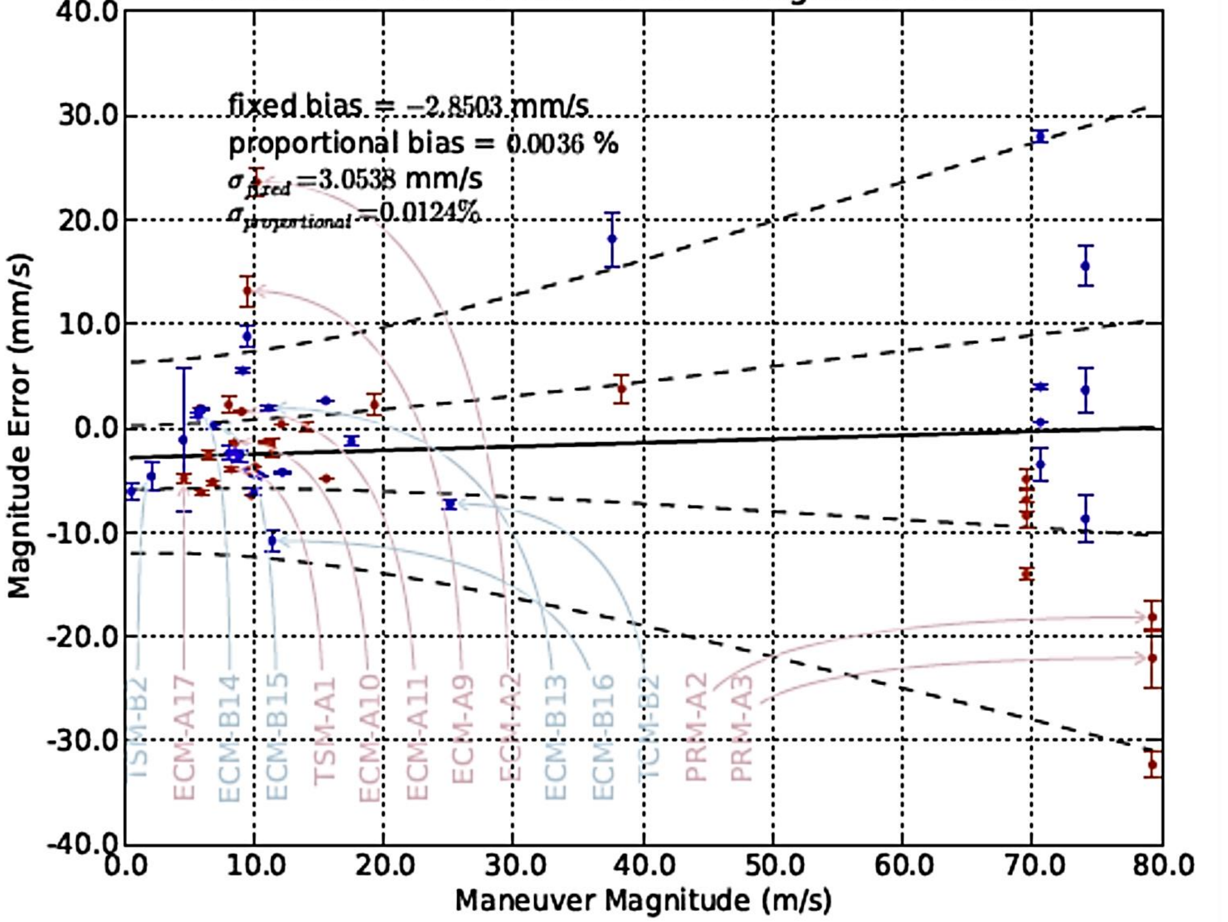
– Electric propulsion: Control-linear noise (Nicolini 2009)

– Chemical propulsion, turn-burn-turn: Gates model

LISA Pathfinder Thrust Accuracy Error (Nicolini)




Combined GRAIL-A and GRAIL-B Magnitude Errors (Goodson)



# Gates Model: Error Sources

**Shutoff error**

– in the direction of $\Delta v$, std. dev. proportional to $\Delta v$

**Residual error**

– in the direction of $\Delta v$, fixed std. dev.

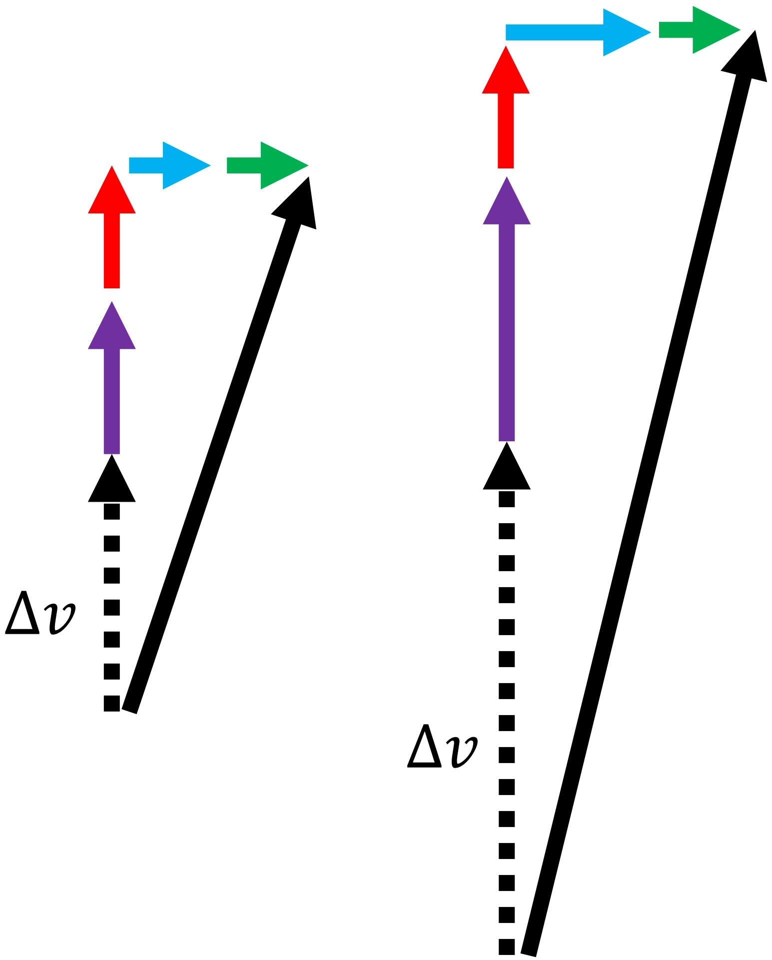
**Pointing error**

– perpendicular to $\Delta v$, std. dev. proportional to$\Delta v$

**Autopilot error**

– perpendicular to $\Delta v$, fixed std. dev

All independent, zero-mean




$$P = (\sigma_s^2 - \sigma_r^2)E[\Delta v \Delta v^T] + (\sigma_r^2 - \sigma_a^2)E\left[\frac{\Delta v \Delta v^T}{\|\Delta v\|^2}\right] + \left(E[\Delta v^T \Delta v]\sigma_p^2 + \sigma_a^2\right)I$$

# Missed Thrust Events (MTEs)



- Spacecraft safe mode or component failure
- Unknowns:
  - # of MTEs
  - initial time of an MTE
  - duration of an MTE
- Some possible approaches:
  - enforce coast arcs to ensure recovery is possible
  - missed thrust fraction (Rubinsztejn 2021):

$$\left.\begin{array}{l} N \text{ is the fraction available thrust} \\ N = 1 \text{ during nominal performance} \\ N = 0 \text{ during MTE} \end{array}\right\} \text{ replace } N \text{ with } E[N]$$

- **See guest lectures by Dr. CK Venigalla**

# Stochastic Differential Equations (SDE)

$$\frac{dX_t}{dt} = f(t, X_t) + b(t, X_t)\underbrace{\xi_t}_{\text{Gaussian white noise}}$$

– Two main types of stochastic calculus: Itô and Statonovich

– Itô SDE: $$dX_t = f(t, X_t)dt + b(t, X_t)\underbrace{dW_t}_{\text{Wiener process (i.e., Brownian motion)}}$$

– Additive noise: $b(t)$

– Multiplicative noise: $b(t, X_t)$

# Wiener Process (Brownian Motion)

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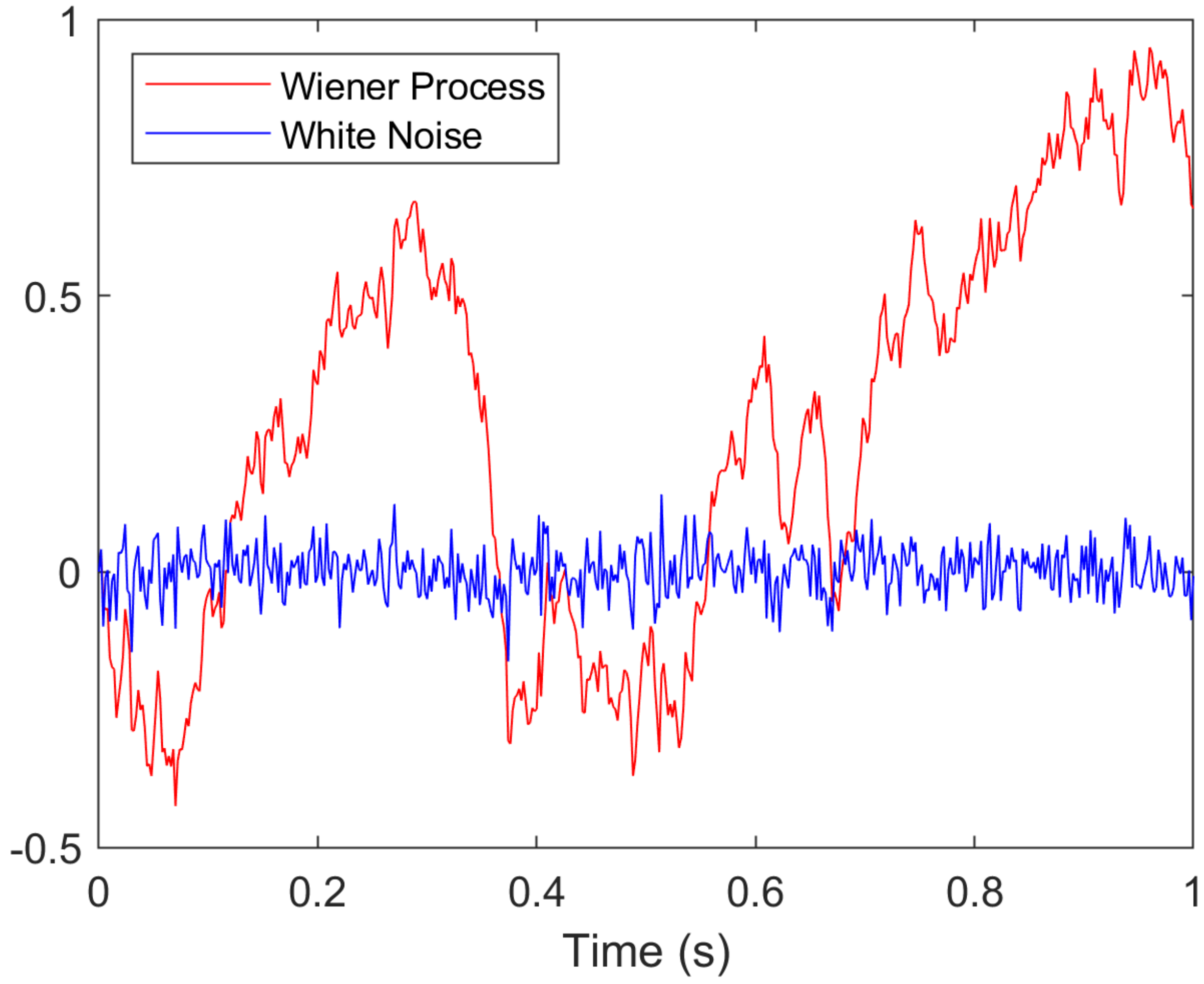
- Continuous-time stochastic process
- Integral of Gaussian white noise

$$W_0 = 0$$

$$\mathrm{E}[W_t] = 0$$

$$\mathrm{Var}(W_t) = t$$



# Itô's Lemma

$$dX_t = f(t, X_t)dt + b(t, X_t)dW_t$$

e.g., $d(X_t X_t^T)$

$P = E[X_t X_t^T]$

– How to derive differential equations for functions of $X_t$?

– Apply Itô's Lemma to each term: standard chain rule + Itô correction term ($C$)

$$d(x_t y_t) = x_t dy_t + y_t dx_t + C(X_t, b)$$

# Linear Covariance Propagation



– Stochastic linear system with additive noise:

$$dX_t = A(t)X_t dt + B(t)u(t)dt + G(t)dW_t$$

– Use Itô's Lemma to compute $\mathrm{E}[d(X_t X_t^T)]$ :

$$\dot{P} = A(t)P + PA(t)^T + G(t)G(t)^T$$

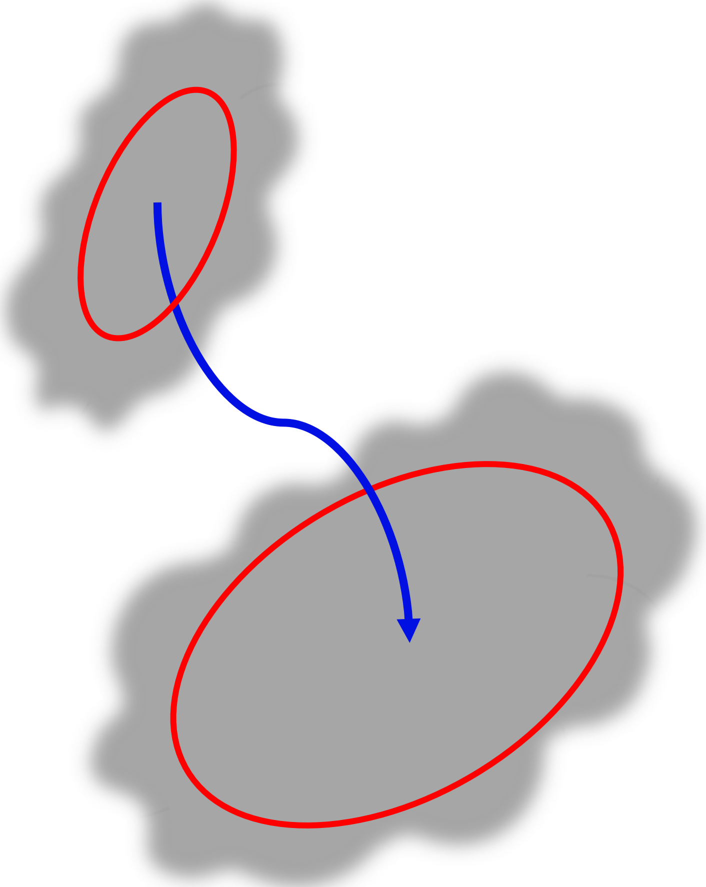
$$P(t) = \Phi(t, t_0)P_0\Phi(t, t_0)^T + \int_{t_0}^{t} \Phi(t, \tau)G(\tau)G(t)^T\Phi(t, \tau)^T d\tau$$

# Fokker-Planck/Forward Kolmogorov

- Time evolution of PDF
- Result of Itô's Lemma

$$dx(t) = f[x(t), t]\,dt + G[x(t), t]\,d\beta(t)$$

$$\frac{\partial p(\boldsymbol{x}, t)}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial \boldsymbol{x}_i}[p(\boldsymbol{x}, t)\boldsymbol{f}_i(\boldsymbol{x}, t)] + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial \boldsymbol{x}_i \partial \boldsymbol{x}_j}\{p(\boldsymbol{x}, t)[\boldsymbol{G}(\boldsymbol{x}, t)\boldsymbol{Q}(t)\boldsymbol{G}^T(\boldsymbol{x}, t)]_{ij}\}$$



- Can find some solutions to Fokker-Planck eqn.
  - e.g., when there are no noise terms (Park 2006)

# Nonlinear Uncertainty Propagation

Some options:

– Propagate higher-order moments of state PDF using state transition tensors (does not accommodate noise!)

– Unscented transform

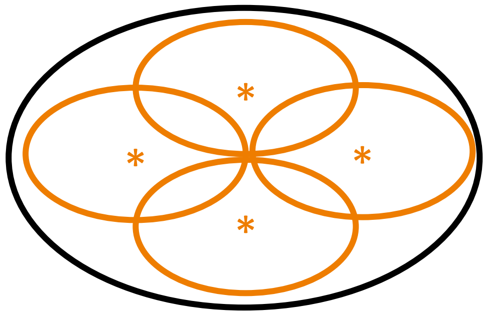
– Gaussian mixture models

– Polynomial chaos expansion

Things to consider:

– Computation time

– Computing derivatives in gradient-based methods



# Optimization Under Uncertainty Techniques

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- Sequential Optimization and Reliability Assessment (SORA) (Ren 2015)
  - iterate between deterministic optimization and robustness evaluation
- Linear covariance propagation (Zimmer 2005, Jenson 2020)
  - minimize covariance using linear covariance propagation & indirect methods
- Desensitized Optimal Control (Seywald 2019)
  - introduce additional state variables that capture sensitivity to dynamical perturbations; can include feedback controls
- Polynomial chaos + pseudospectral methods (Xiong 2015)
- Unscented optimal; controlling unscented transform (Ross 2014)
- Stochastic chance constraints (Oguri 2019)
- Stochastic differential dynamic programming (Ozaki 2018)
  - Use unscented transform to propagate mean state in DDP
- *Many* more…

# References


- D. Nicolini et al., "Direct thrust and thrust noise measurements on the LISA pathfinder field emission thruster," Proc. 31st Int. Elect. Propul. Conf., 2009
- T. D. Goodson, "Execution-Error Modeling and Analysis of the GRAIL Spacecraft Pair," AAS 13-268.
- C. R. Gates, "Technical Report No. 32-504: A Simplified Model of Midcourse Maneuver Execution Errors," Tech. rep., Jet Propulsion Laboratory, Oct. 1963.
- A. Rubinsztejn, C. G. Sandel, R. Sood, F. E. Laipert, "Designing trajectories resilient to missed thrust events using expected thrust fraction," Aerospace Science and Technology, 2021, DOI: 10.1016/j.ast.2021.106780
- R. S. Park, D. J. Scheeres, "Nonlinear Mapping of Gaussian Statistics: Theory and Applications to Spacecraft Trajectory Design," Journal of Guidance, Control, and Dynamics, 2006, DOI: 10.2514/1.20177
- Y. Ren, J. Shan, "Reliability-Based Soft Landing Trajectory Optimization near Asteroid with Uncertain Gravitational Field," Journal of Guidance, Control, and Dynamics, 2015, DOI: 10.2514/1.G000903
- Zimmer, "Reducing Spacecraft State Uncertainty Through Indirect Trajectory Optimization." PhD thesis, Univ. of Texas, Austin, TX, U.S.A, 2005.
- E. L. Jenson, D. J. Scheeres, "Multi-Objective Optimization of Covariance and Energy for Asteroid Transfers," Journal of Guidance, Control, and Dynamics, 2021, DOI: 10.2514/1.G005609
- H. Seywald, K. Seywald, "Desensitized Optimal Control," AIAA SciTech Forum, 2019, DOI: 10.2514/6.2019-0651
- F. Xiong, Y. Xiong, B. Xue, "Trajectory Optimization under Uncertainty based on Polynomial Chaos Expansion," AIAA SciTech Forum, 2015, DOI: 10.2514/6.2015-1761
- I. M. Ross, R. J. Proulx, and M. Karpenko, "Unscented Optimal Control for Orbital and Proximity Operations in an Uncertain Environment: A New Zermelo Problem," AIAA/AAS Astrodynamics Specialist Conference, DOI: 10.2514/6.2014-4423
- K. Oguri, J. McMahon, "Risk-aware Trajectory Design with Continuous Thrust: Primer Vector Theory Approach," AAS/AIAA Astrodynamics Specialist Conference, 2019.
- N. Ozaki, S. Campagnola, R. Funase, and C. H. Yam. Stochastic Differential Dynamic Programming with Unscented Transform for Low-Thrust Trajectory Design. Journal of Guidance, Control, and Dynamics, 2018.

# Dynamic Programming

# Bellman’s Principle of Optimality



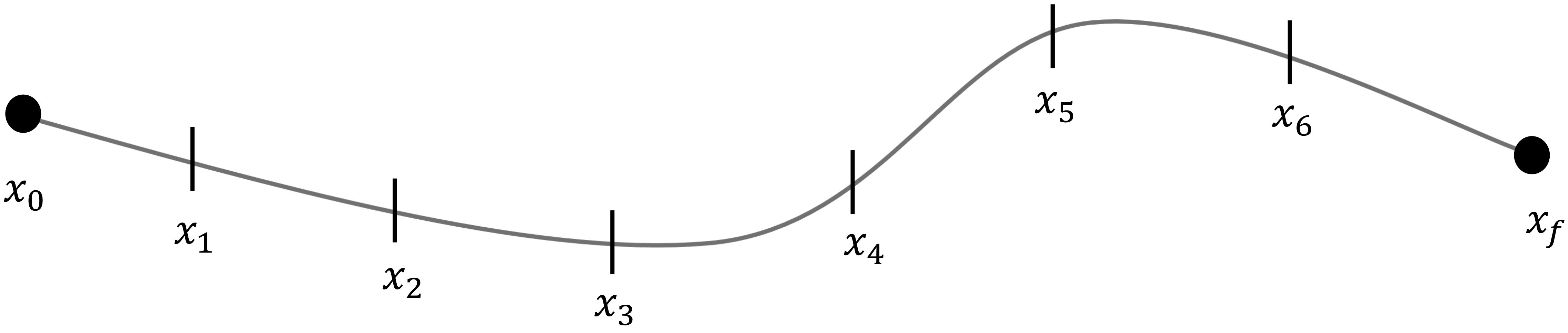
“An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.”
(Bellman 1957)

???

# Bellman's Principle of Optimality

"An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."
(Bellman 1957)

$$x_{k+1} = F(x_k, u_k)$$



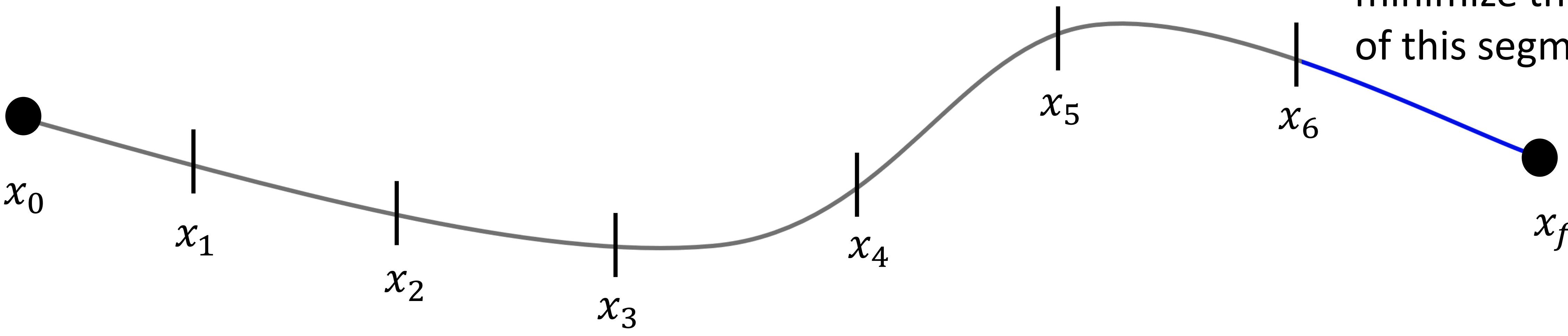
# Bellman’s Principle of Optimality



“An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.”
(Bellman 1957)

$x_f = F(x_6, u_6)$
get control policy $u_6(x_6)$ to minimize the cost of this segment

$$x_{k+1} = F(x_k, u_k)$$



# Bellman’s Principle of Optimality

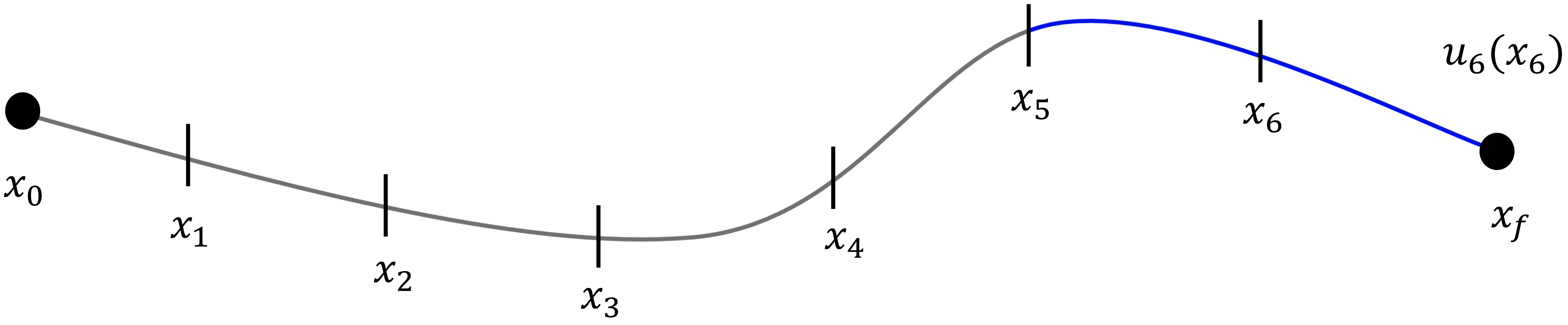


“An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.” (Bellman 1957)

$x_6 = F(x_5, u_5)$

get $u_5(x_5)$ to minimize cost over final two segments

$x_{k+1} = F(x_k, u_k)$



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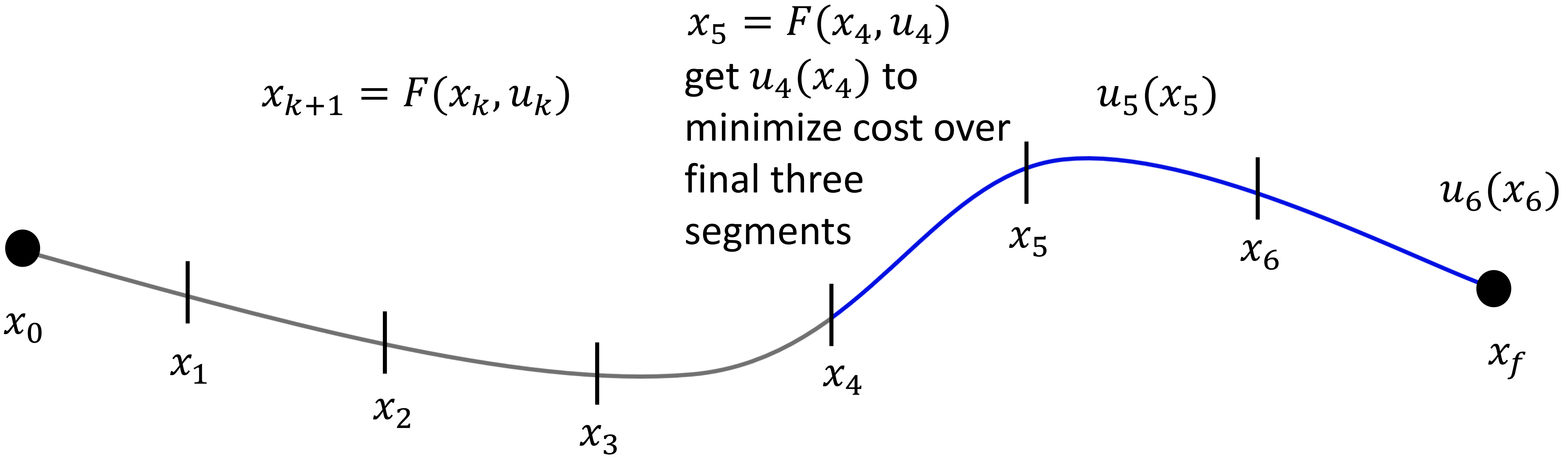


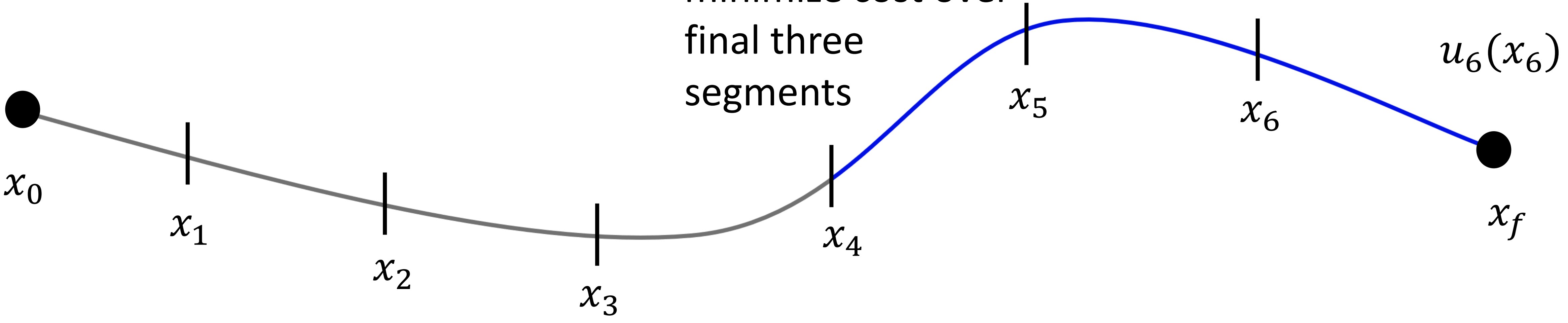
# Bellman’s Principle of Optimality

“An … d
initi … al
poli…
(Bellman 1957)

solving recursive sub-problems backwards in time

$x_{k+1} = F(x_k, u_k)$

$x_5 = F(x_4, u_4)$
get $u_4(x_4)$ to minimize cost over final three segments



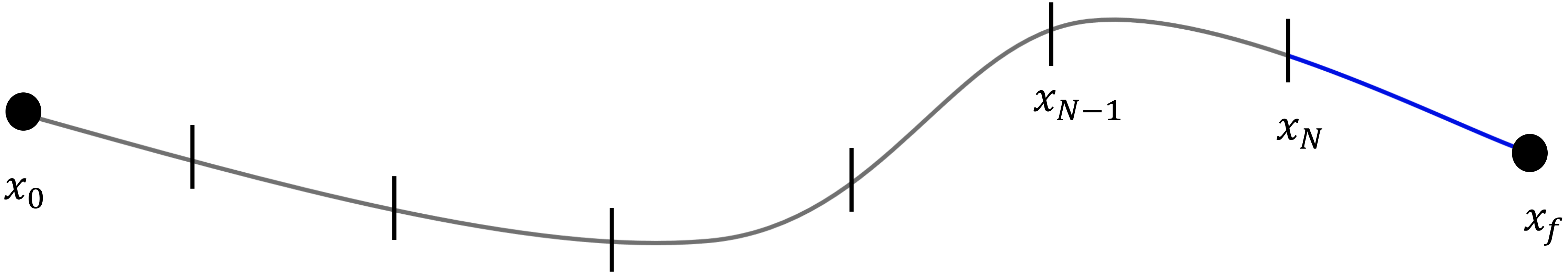
# Bellman Equation

- Cost function: $J = \sum_{k=0}^{N} C_k(x_k, u_k) + \psi(x_f)$

- Value function, cost-to-go from $x_N$ to $x_f$: $V_N(x_N, u_N) = C_N + \psi(x_f)$

- Optimize final segment: $V_N^*(x_N) = \min_{u_N} V_N(x_N, u_N)$

- Control policy: $u_N^* = f(x_N)$



# Bellman Equation

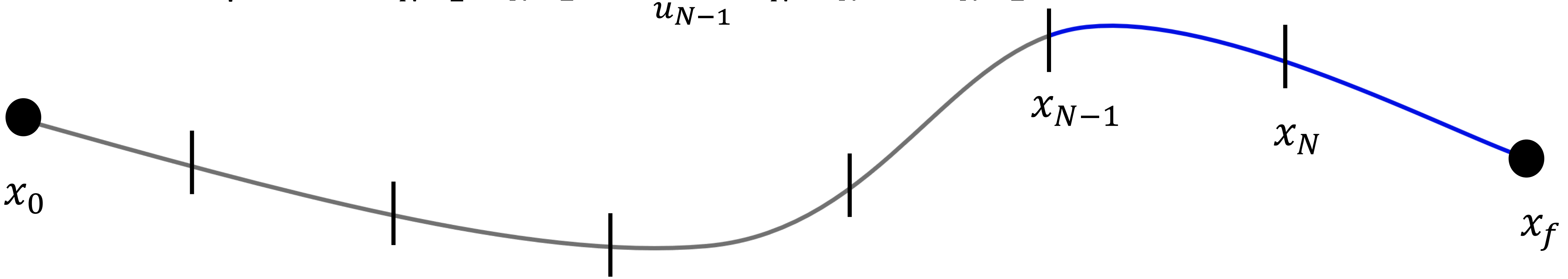
– Cost function: $J = \sum_{k=0}^{N} C_k(x_k, u_k) + \psi(x_f)$

– Value function, cost-to-go from $x_N$ to $x_f$: $V_N(x_N, u_N) = C_N + \psi(x_f)$

– Optimize final segment: $V_N^*(x_N) = \min_{u_N} V_N(x_N, u_N)$

– Control policy: $u_N^* = f(x_N)$

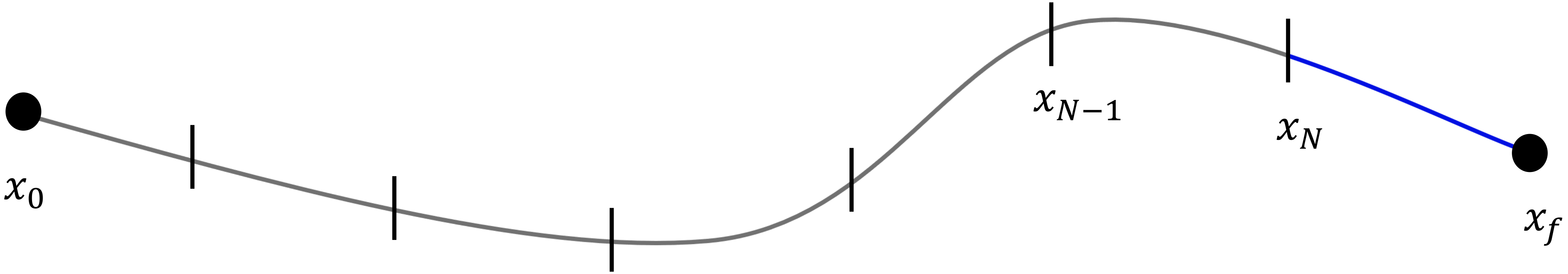
– Bellman Equation: $V_{N-1}^*(x_{N-1}) = \min_{u_{N-1}} [V_N^*(x_N) + C_{N-1}]$



# Dynamic Programming: Finite- Horizon LQR

- Dynamics: $x_{k+1} = Ax_k + Bu_k$
- Cost function: $J = x_f^T Q x_f + \sum_{k=0}^{N} x_k^T Q x_k + u_k^T R u_k$
- Value function, final segment: $V_N = x_f^T Q x_f + x_N^T Q x_N + u_N^T R u_N$

Replace with
$P_f = Q$



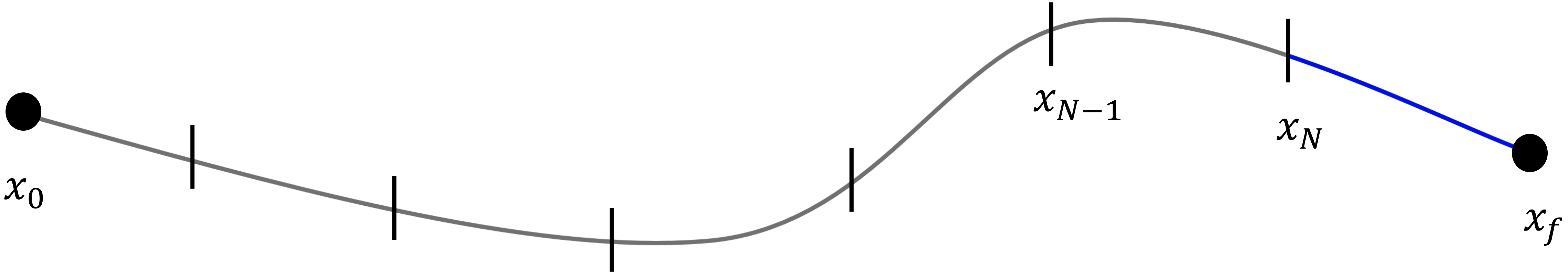
# Dynamic Programming: Finite- Horizon LQR



– Value function, final segment: $V_N = x_f^T P_f x_f + x_N^T Q x_N + u_N^T R u_N$

– Sub in $x_f = Ax_N + Bu_N$

$$V_N = x_N^T (A^T P_f A + Q) x_N + u_N^T (B^T P_f B + R) u_N + 2u_N^T B^T P_f A x_N$$



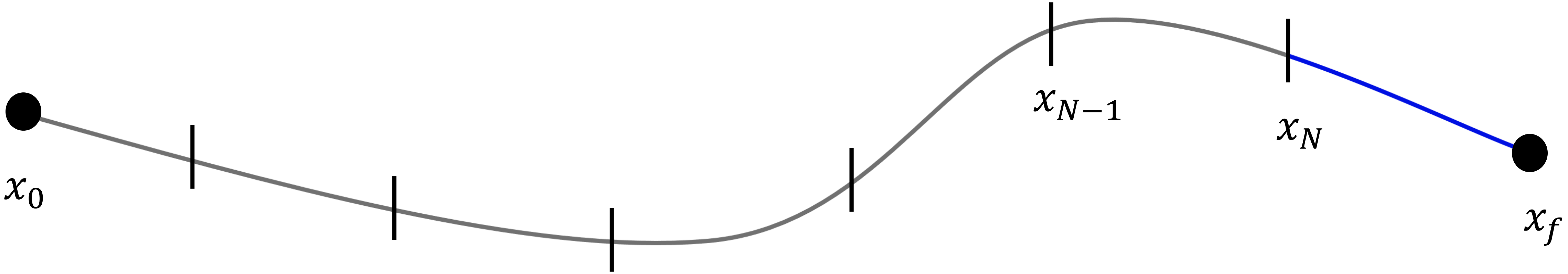
# Dynamic Programming: Finite- Horizon LQR

– Optimize final segment: $\dfrac{\partial V_N}{\partial u_N} = 0 = 2(B^T P_f B + R)u_N + 2B^T P_f A x_N$

– Optimal control policy: $u_N^* = -(B^T P_f B + R)^{-1} B^T P_f A x_N$

– Optimal value function: $V_N^*(x_N) = x_N^T P_N x_N$

– Recursive update for $P$: $P_N = A^T \underline{P_f} A + Q - A^T \underline{P_f} B(B^T \underline{P_f} B + R)^{-1} B^T \underline{P_f} A$



# Dynamic Programming: Finite- Horizon LQR

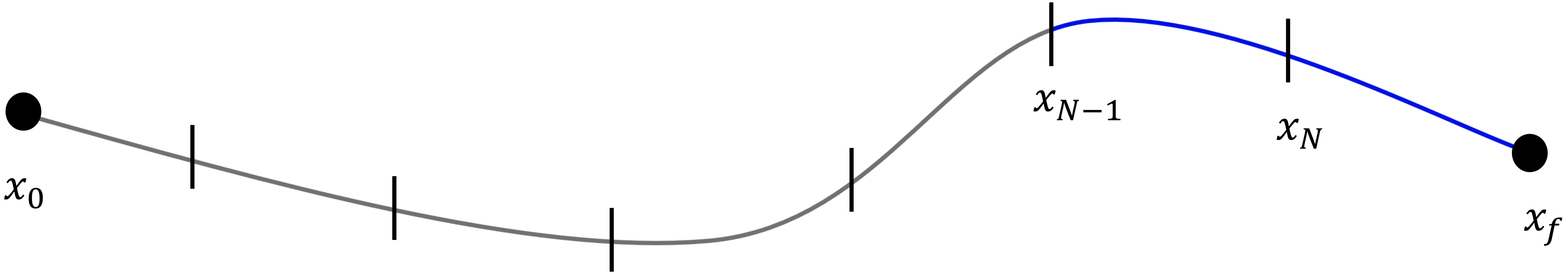
– Where we started: $$V_N = x_f^T P_f x_f + x_N^T Q x_N + u_N^T R u_N$$

– Optimal value function: $$V_N^*(x_N) = x_N^T P_N x_N$$

Looks familiar!

– From Bellman Eqn:

$$V_{N-1}^*(x_{N-1}) = \min_{u_{N-1}} [x_N^T P_N x_N + x_{N-1}^T Q x_{N-1} + u_{N-1}^T R u_{N-1}]$$



# Dynamic Programming: Finite- Horizon LQR

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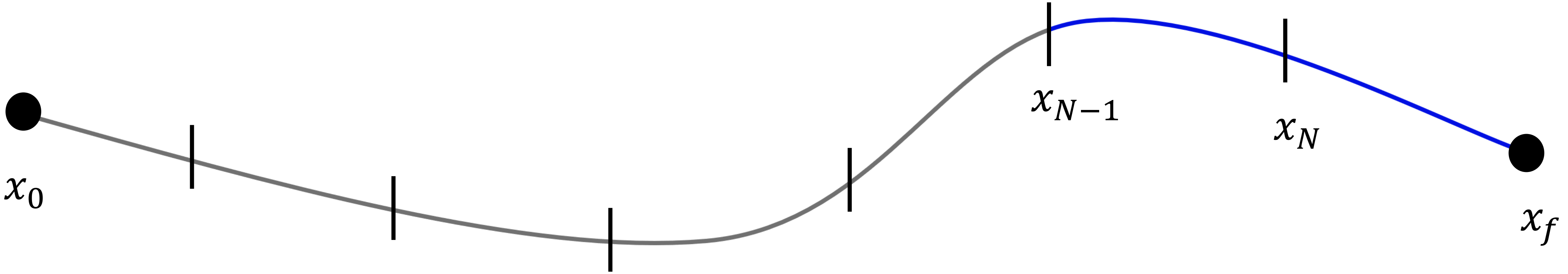
– For every time step:

$$u_k^* = -(B^T P_{k+1} B + R)^{-1} B^T P_{k+1} A x_k$$

– Compute all $P$ recursively, backwards in time; initialize with $P_f = Q$

$$P_k = A^T P_{k+1} A + Q - A^T P_{k+1} B (B^T P_{k+1} B + R)^{-1} B^T P_{k+1} A$$



# Differential Dynamic Programming

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- Differential Dynamic Programming (DDP):
  - Second-order gradient-based method
  - Begin with initial guess for control & parameters
  - You will optimize *deviations* from your initial guess, i.e., $\delta u$
  - Perform **forward sweep** (compute trajectory) with $\delta u = 0$
  - Evaluate optimality conditions
  - In **backward sweep**, apply Bellman's Principle of optimality to get optimal control update $\delta u^*$
  - Approximations made for dynamics and cost function; method is inexact & iteration necessary
  - (see Aziz 2018)
- Hybrid DDP (HDDP) (Lantoine 2012)
- State Transition Tensor DDP (STT/DDP) (Boone 2020)
- Stochastic DDP using unscented transform (SDDP) (Ozaki 2018)

# References


- R. E. Bellman. Dynamic Programming. Princeton University Press, Princeton, NJ, 1957.
- J. D. Aziz, "Low-Thrust Many-Revolution Trajectory Optimization," PhD Dissertation, 2018.
- G. Lantoine and R. P. Russell. A Hybrid Differential Dynamic Programming Algorithm for Constrained Optimal Control Problems. Part 1: Theory. Journal of Optimization Theory and Applications, 2012.
- G. Lantoine and R. P. Russell. A Hybrid Differential Dynamic Programming Algorithm for Constrained Optimal Control Problems. Part 2: Application. Journal of Optimization Theory and Applications, 2012.
- S. Boone, J. McMahon, "Rapid Local Trajectory Optimization Using Higher-Order State Transition Tensors and Differential Dynamic Programming," 2020 AAS/AIAA Astrodynamics Specialist Conference.
- N. Ozaki, S. Campagnola, R. Funase, and C. H. Yam. Stochastic Differential Dynamic Programming with Unscented Transform for Low-Thrust Trajectory Design. Journal of Guidance, Control, and Dynamics, 2018.

# Questions?

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Thanks for watching!