ASEN 6060
ADVANCED ASTRODYNAMICS
Computing Periodic Orbit Families, Part 1

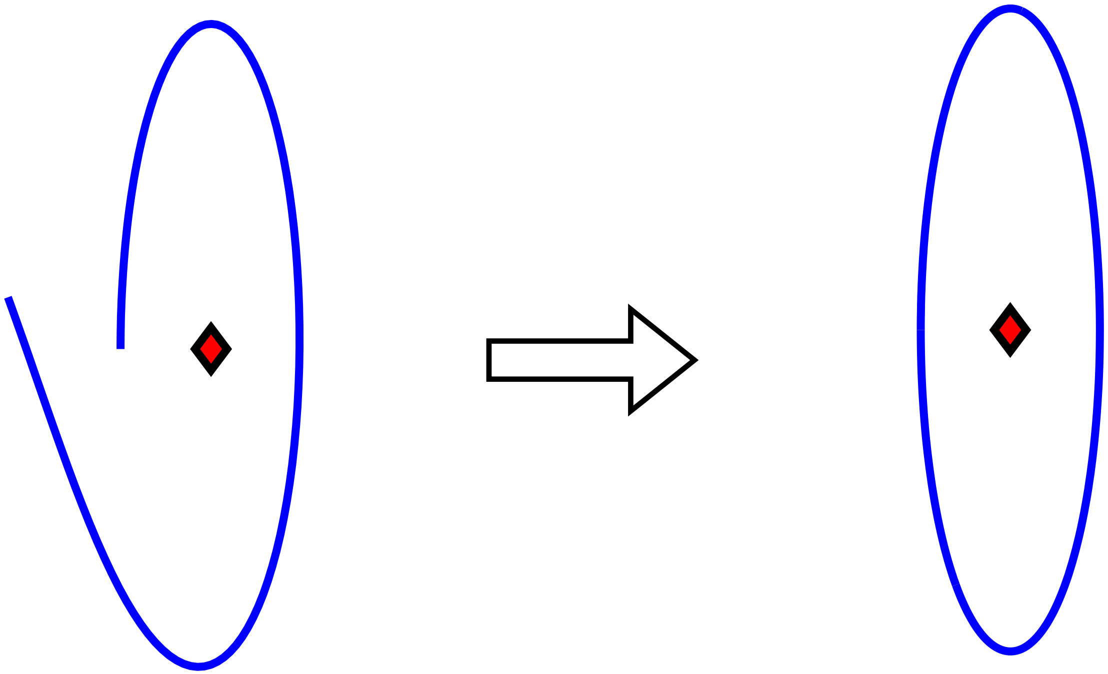
Objectives:

- Define and formulate single-shooting schemes to numerically correct a trajectory to produce a periodic orbit

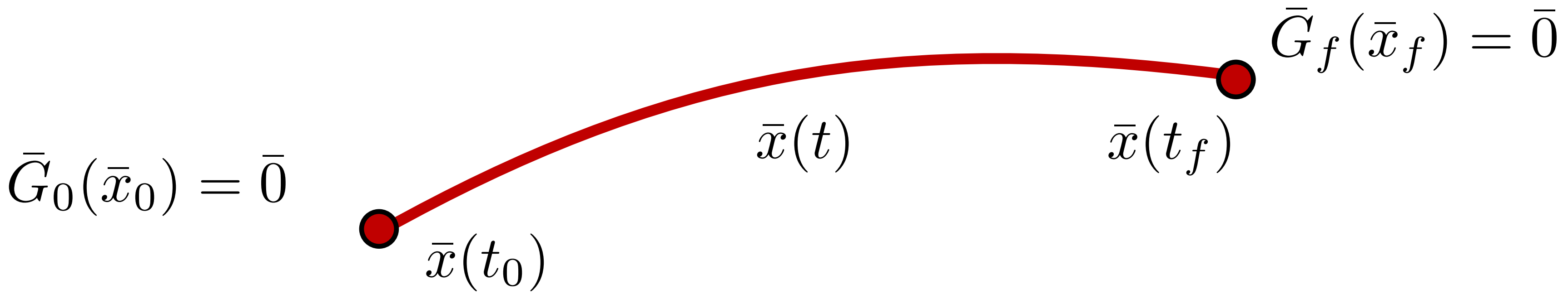
Credit: these notes were developed in collaboration with Dr. Ian Elliott in 2020, and have since been updated

Numerically Computing Periodic Orbits

Goal: Numerically compute a periodic orbit in the CR3BP from a discontinuous/nonperiodic initial guess



Two-Point Boundary Value Problem



- Two-point boundary value problem (TPBVP):

- Shooting method formulates the TPBVP as an initial value problem (IVP):

- Other methods solve this problem too!

Single-Shooting to Correct a Trajectory

- Given an initial guess for an initial condition, iteratively adjust using linearization around current guess until desired trajectory (a solution that satisfies the boundary conditions and other constraints) is recovered to within a specified tolerance
- We will focus on a single-shooting method, implemented using a free variable – constraint vector formulation

Free Variable - Constraint Vector Formulation

Formulate a vectorial root-finding problem to implement the single-shooting scheme:

Free Variable - Constraint Vector Formulation

To perform the iterative updates, define an $(m \times n)$ matrix of partial derivatives of each constraint with respect to each of the free variables:

Can calculate using either:

Free Variable - Constraint Vector Formulation

Assuming at iteration *i*, $\bar{V}_i$ lies sufficiently close to the desired solution $\bar{V}_d$, perform Taylor series expansion in $\bar{F}(\bar{V})$ around $\bar{V}_i$

Retaining only first-order terms:

Because this is a first-order approximation, use this equation to iteratively update $\bar{V}_i$ with the goal of recovering a solution that is sufficiently close to $\bar{V}_d$

Free Variable - Constraint Vector Formulation

Consider an update from $\bar{V}_i$ at iteration i to $\bar{V}_{i+1}$ at iteration *i+1*

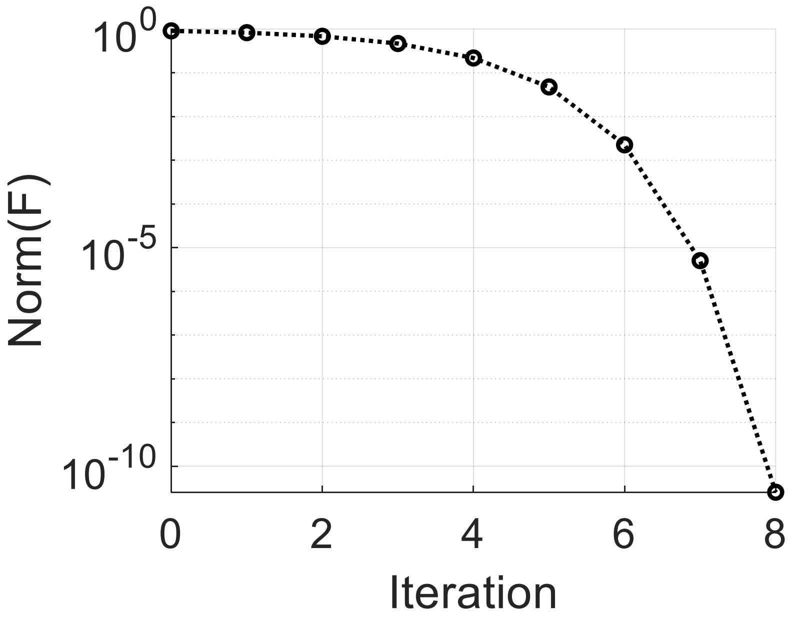
If $m = n$, the update equation (used to update $\bar{V}_i$) is

If $m < n$, the update equation is commonly formed using the minimum-norm solution

Continue iteratively updating free variable vector until

Performance of Newton's Method

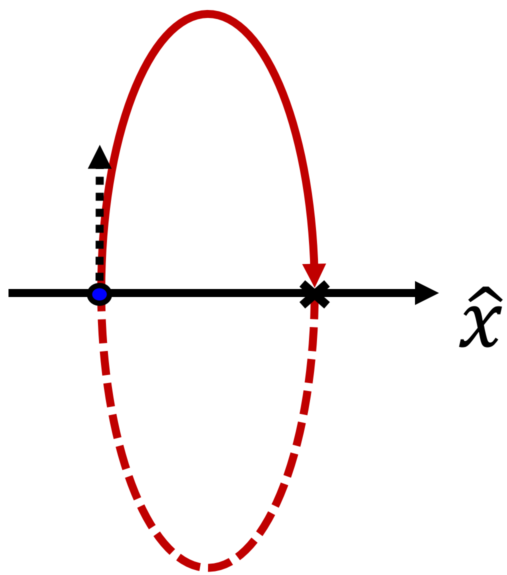
Newton's method converges quadratically to a solution when:



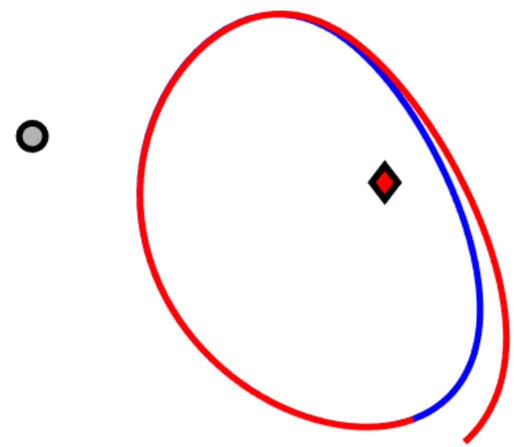
Computing Periodic Orbits via Single Shooting

- Multiple formulations of free variable and constraint vectors for a single shooting method possible
- We will cover a couple of formulations

Variable-Time Mirror Theorem Formulation



General Variable-Time Formulation



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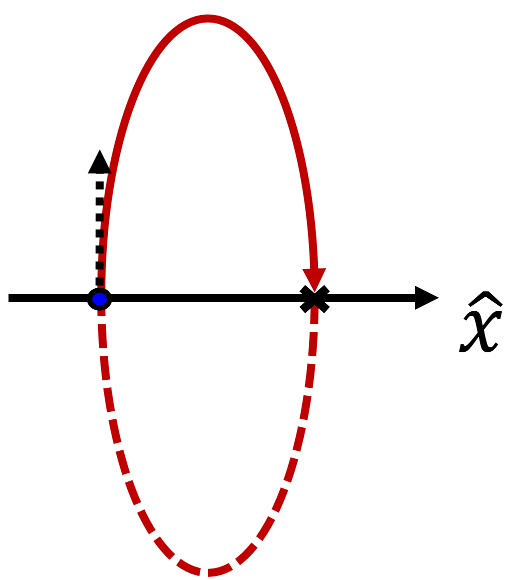
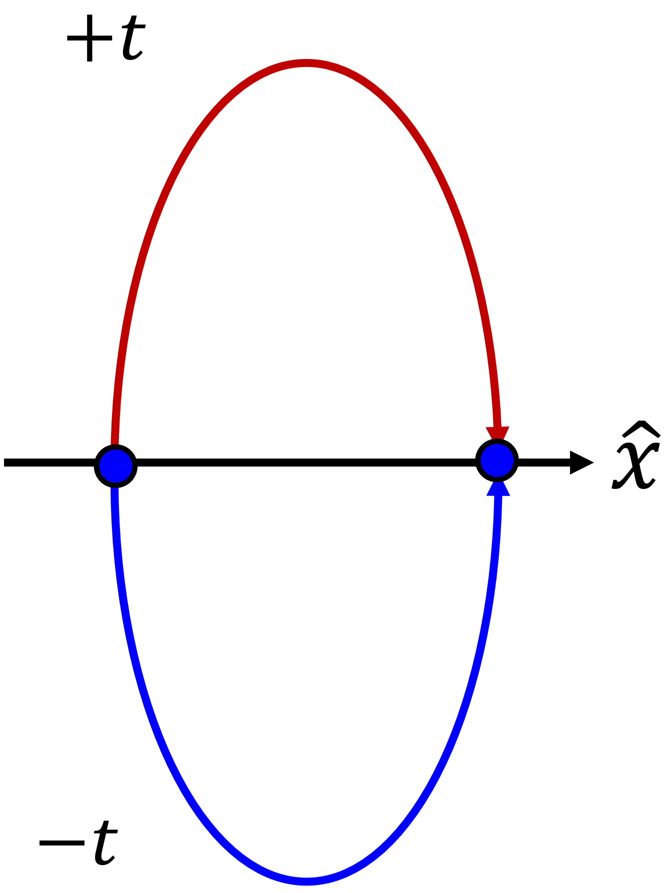


General Variable-Time Formulation

Mirror Theorem

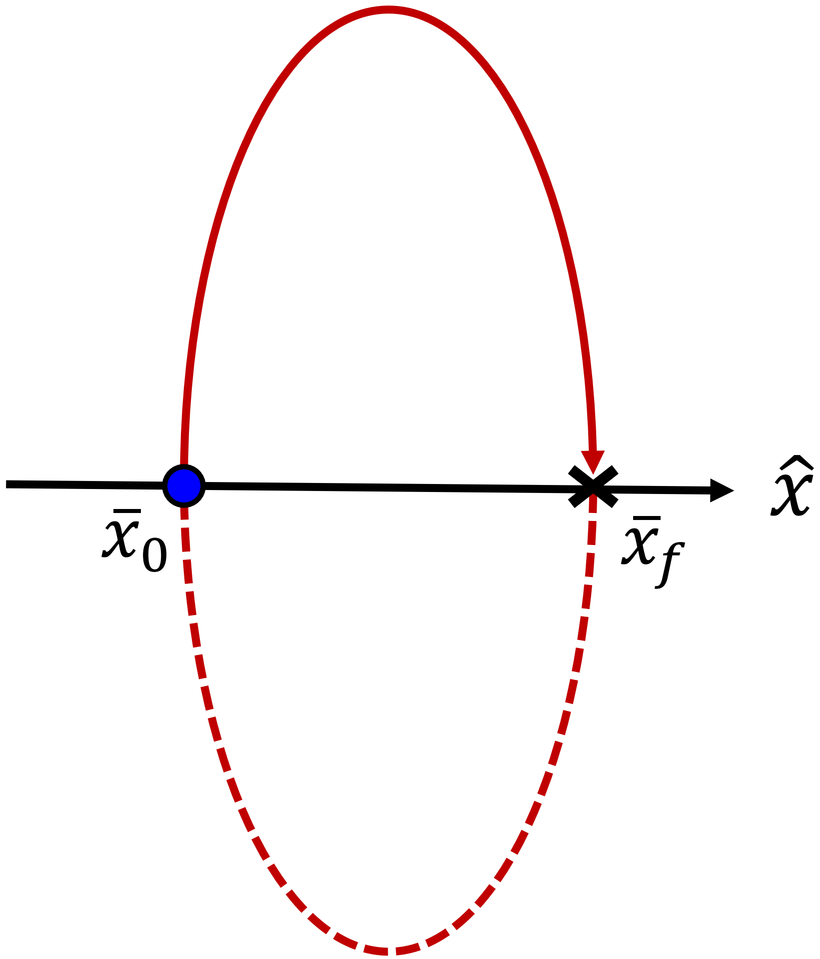
- Mirror Theorem in the CR3BP:
  - If a mirror configuration occurs at two distinct epochs, an orbit is periodic
  - For discussion in general *n*-body problems, see: Roy & Ovenden, 1954, "On the Occurrence of Commensurable Mean Motions in the Solar System"



$t \rightarrow -t \qquad y \rightarrow -y$

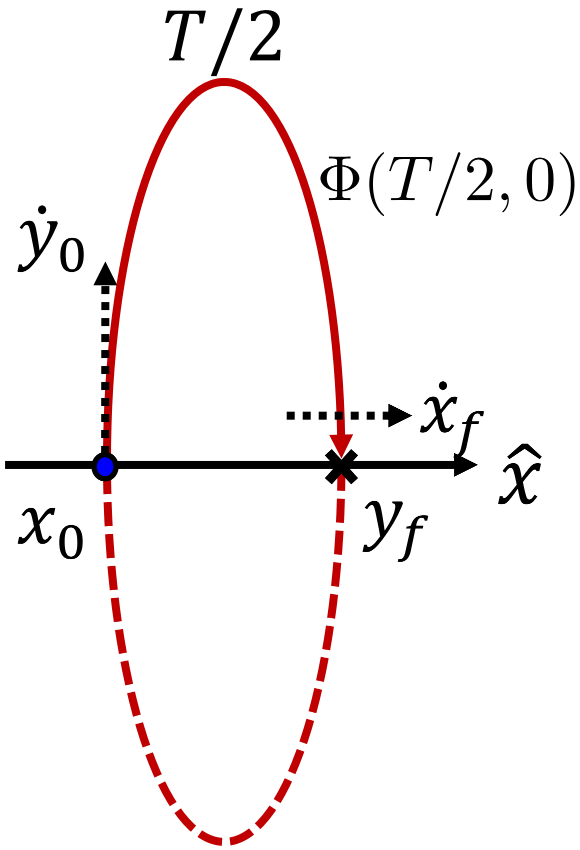
$x \rightarrow x \qquad z \rightarrow z$

Computing Symmetric Planar Periodic Orbits



Variable-Time Mirror Theorem Formulation

Compute the Jacobian as:



Recall: $\Phi_{ij}(t_f, t_0) = \dfrac{\partial x_i(t_f)}{\partial x_j(t_0)}$

Thus:

Update equation:

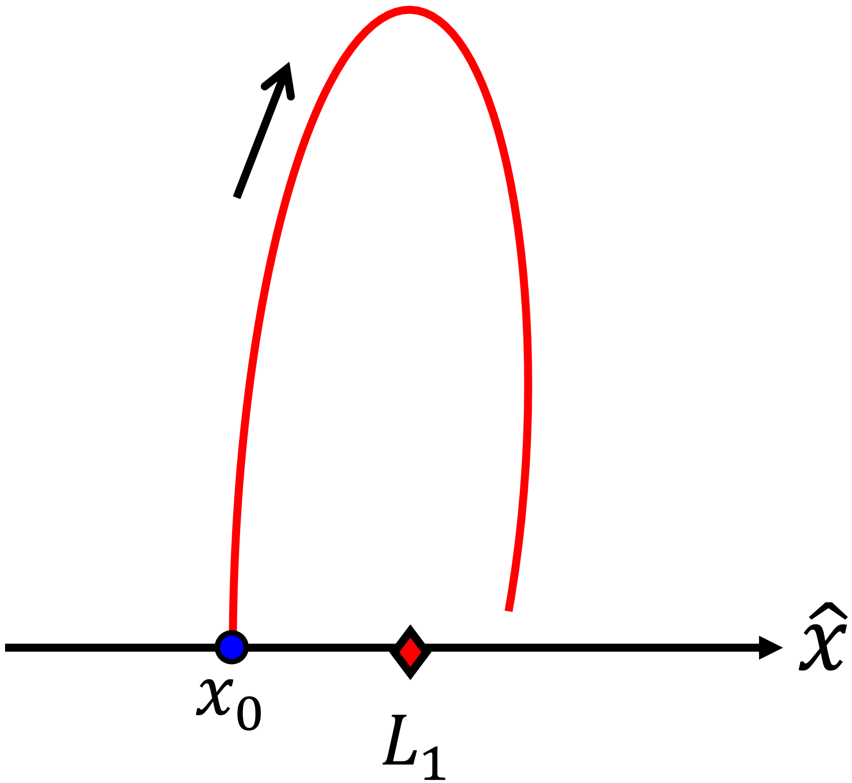
$$\bar{V}_{i+1} = \bar{V}_i - D\bar{F}(\bar{V}_i)^T \left[D\bar{F}(\bar{V}_i) D\bar{F}(\bar{V}_i)^T \right]^{-1} \bar{F}(\bar{V}_i)$$

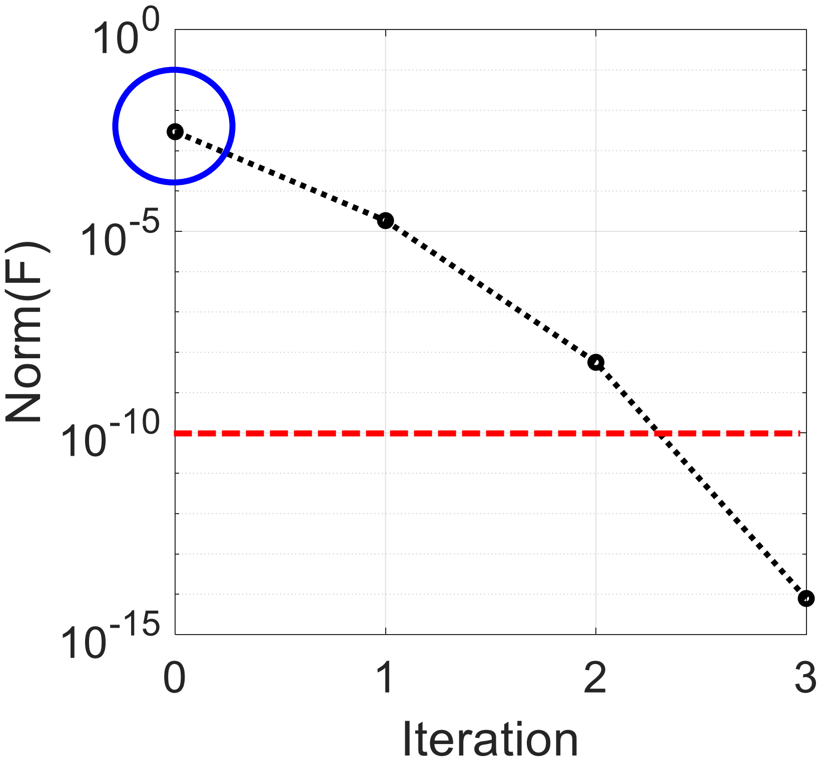
Alternate formulations exist too!

Example: Computing L_1 Lyapunov Orbit

Application of mirror theorem to correct an L_1 Lyapunov orbit

- Initial guess from linearization about L_1
- Tolerance of 1×10^{-10} on constraint vector

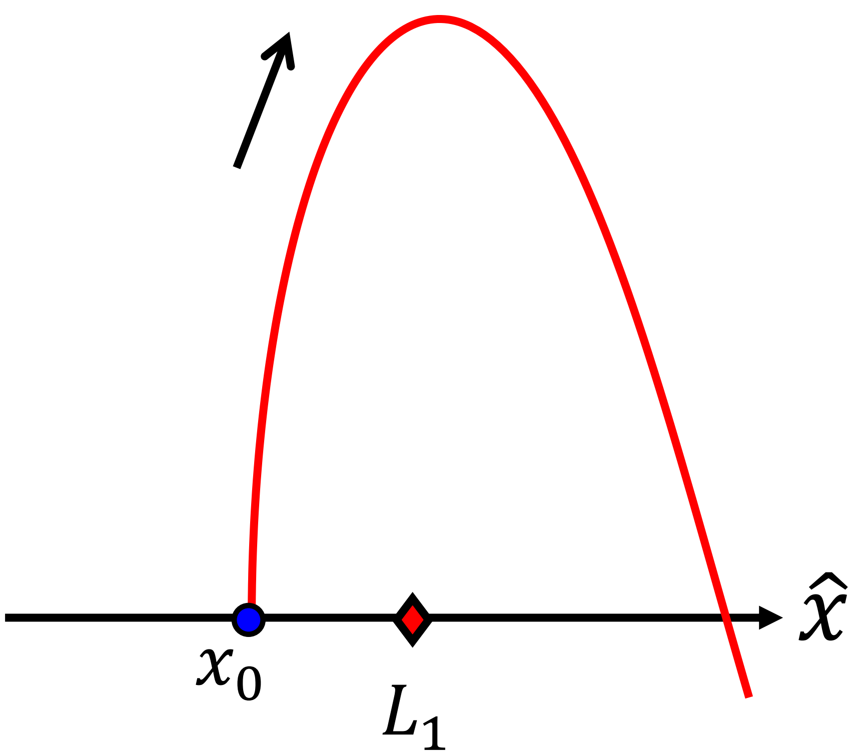


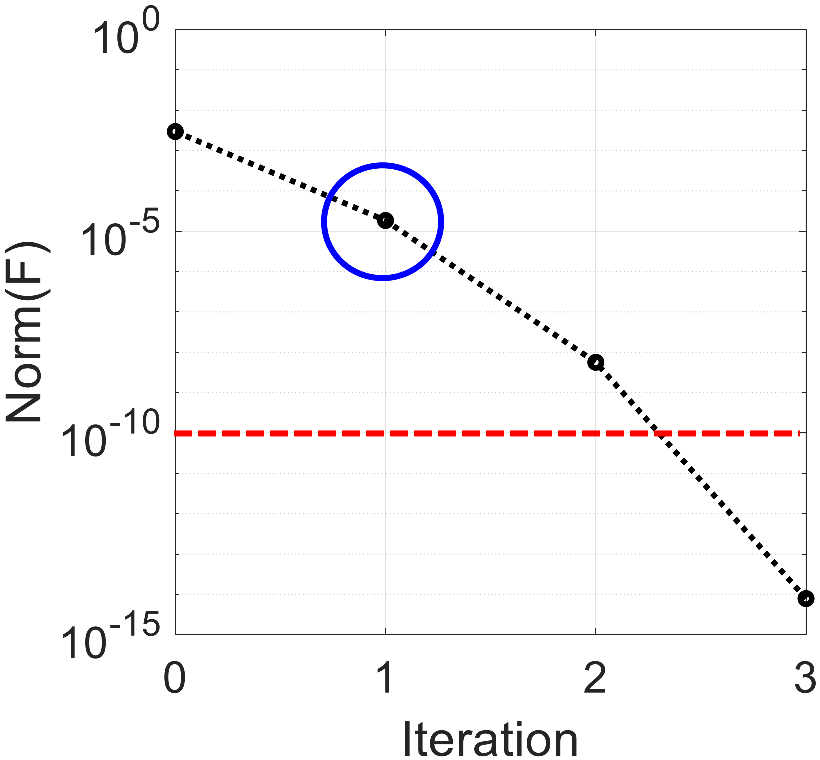


Initial guess

Example: Computing L_1 Lyapunov Orbit

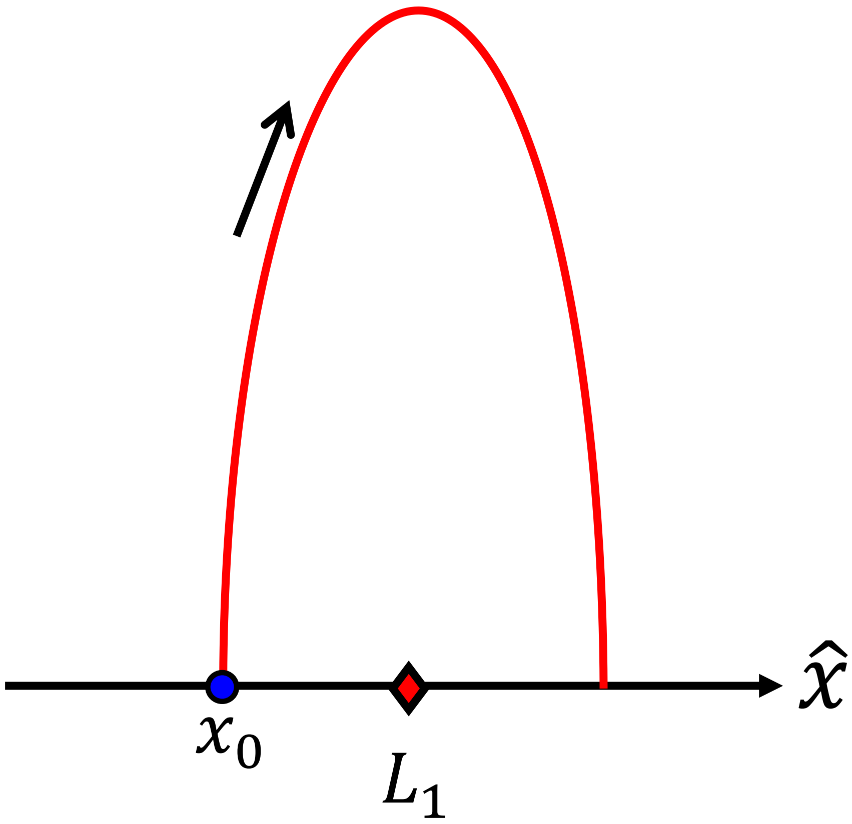
Application of mirror theorem to correct an L_1 Lyapunov orbit

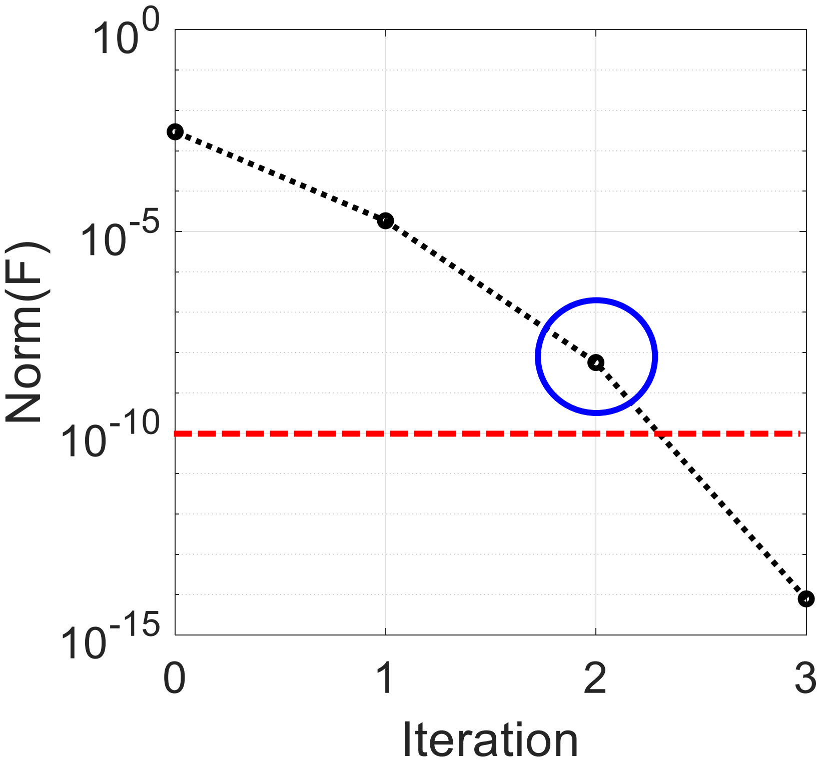




Example: Computing L_1 Lyapunov Orbit

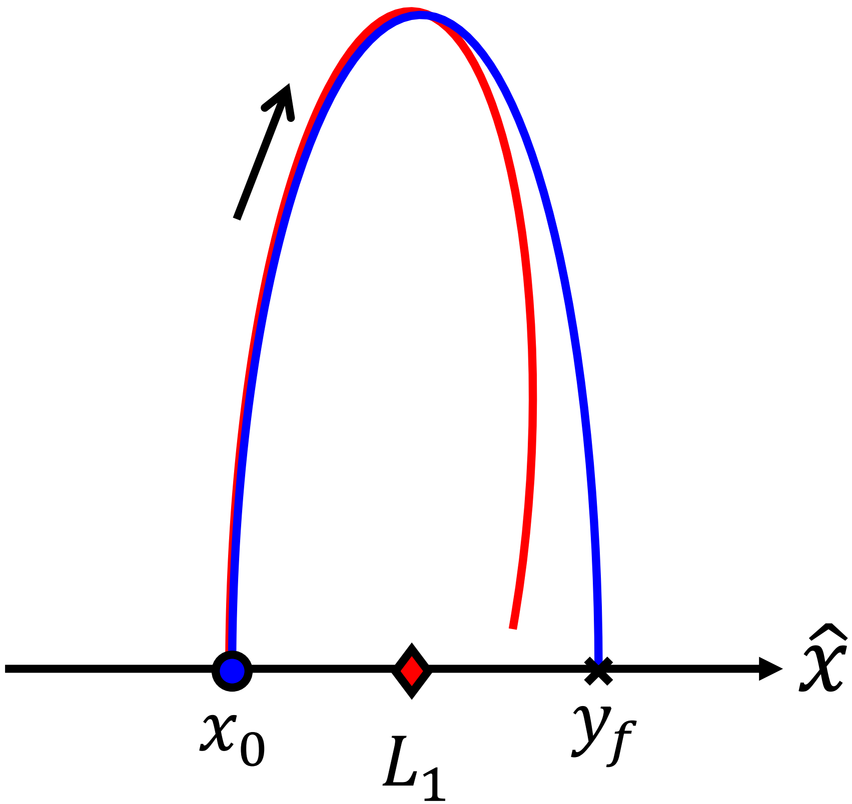
Application of mirror theorem to correct an L_1 Lyapunov orbit

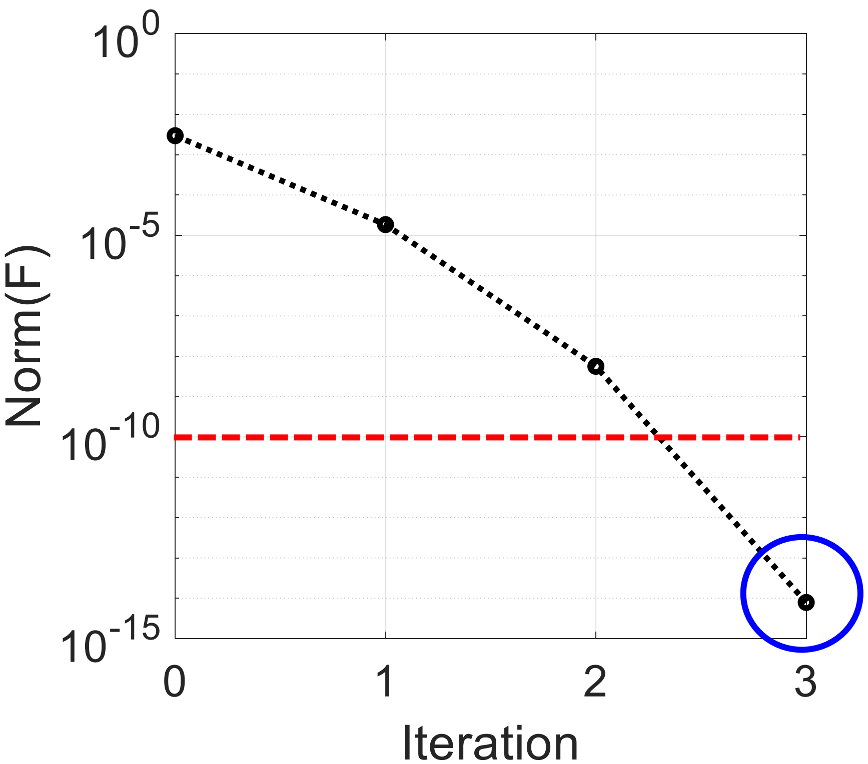




Example: Computing L_1 Lyapunov Orbit

Application of mirror theorem to correct an L_1 Lyapunov orbit

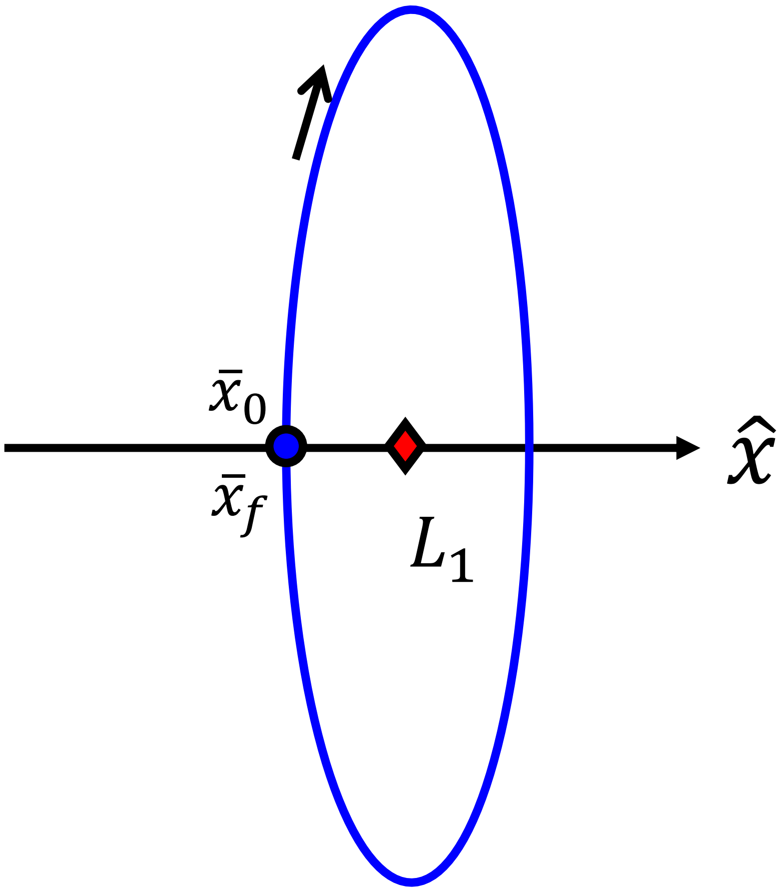




Initial guess Recovered solution

Example: Computing L_1 Lyapunov Orbit

L_1 Lyapunov orbit
propagated for a complete period



Check for periodicity by
comparing initial and final state

$$\bar{x}_0 - \bar{x}_f = \begin{bmatrix} -1.15\times10^{-12} \\ 5.06\times10^{-13} \\ 0 \\ -3.36\times10^{-12} \\ 1.52\times10^{-12} \\ 0 \end{bmatrix}$$

Computing Periodic Orbits via Single Shooting

- Multiple formulations of free variable and constraint vectors for a single shooting method possible
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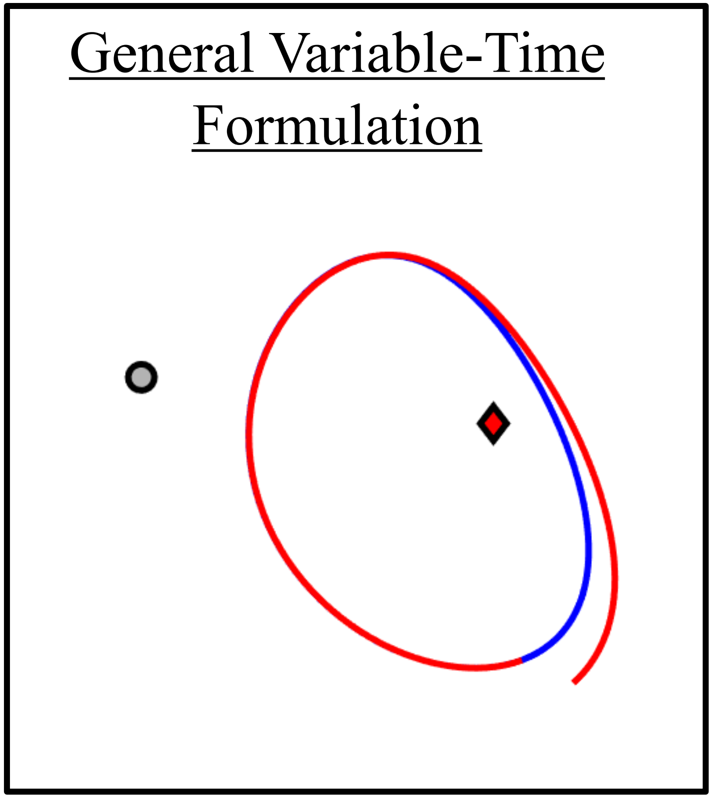


General Variable-Time Formulation

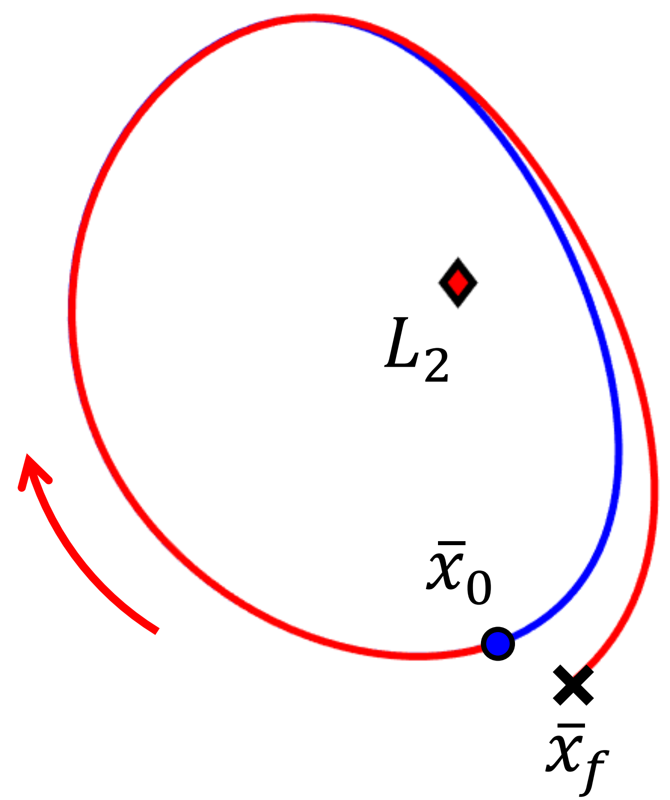
Consider alternative approach that does not use mirror theorem.

Example: compute a spatial periodic orbit in the CR3BP

Define free variable vector as

Define constraint vector as

L_2 southern halo orbit

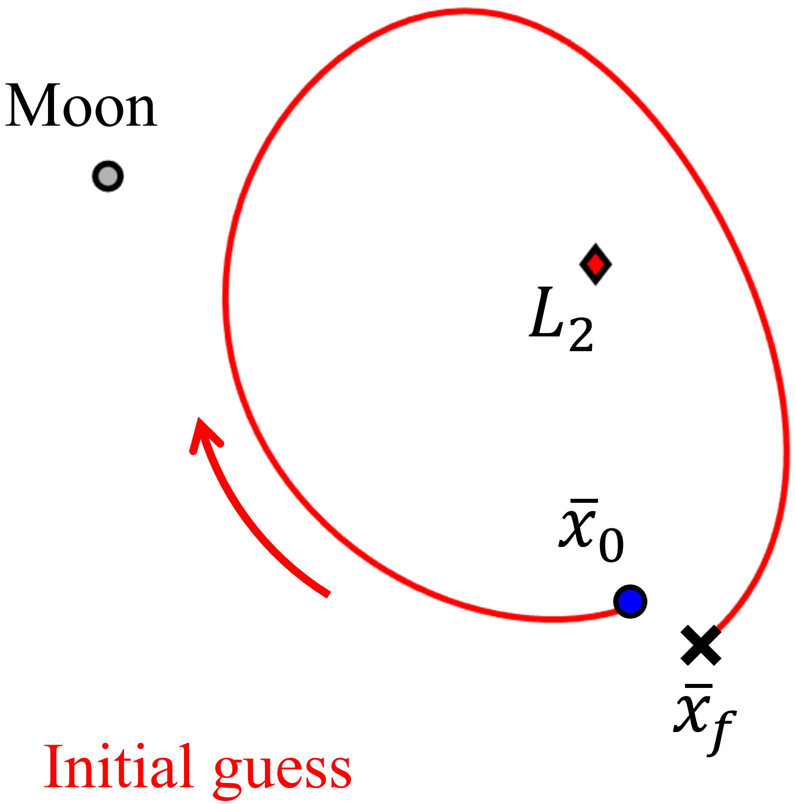


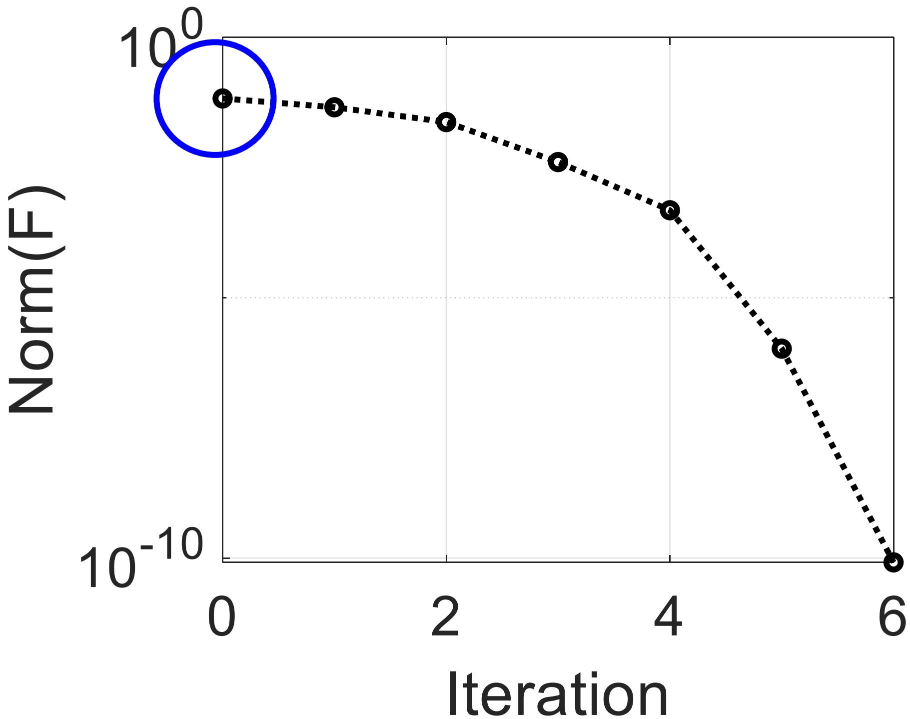
General Variable-Time Formulation

The Jacobian is:

Example: L_2 Southern Halo Orbit

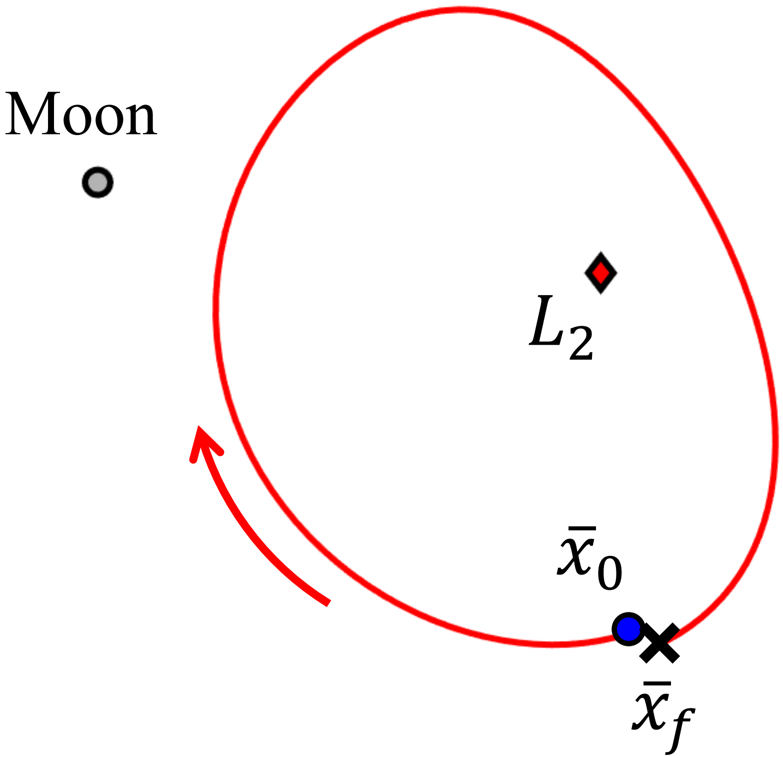
Application of general single shooting formulation to compute an L_2 southern halo orbit

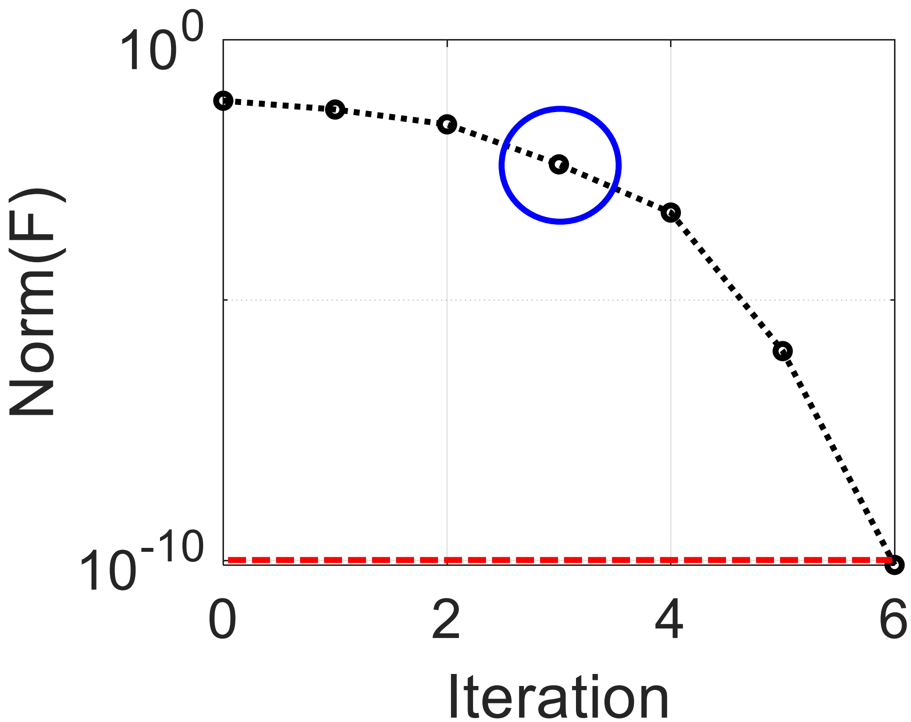




Example: L_2 Southern Halo Orbit

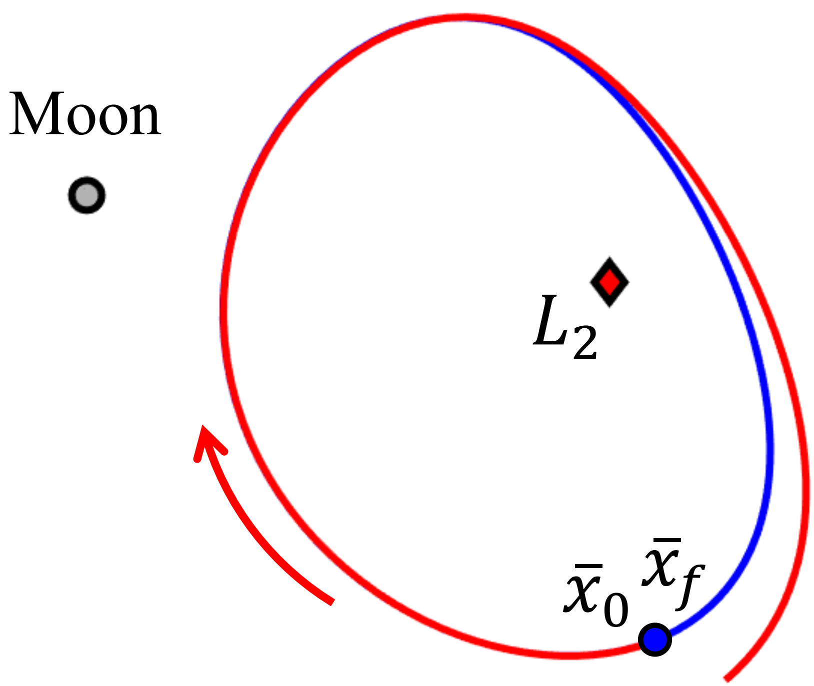
Application of general single shooting formulation to correct an L_2 southern halo orbit



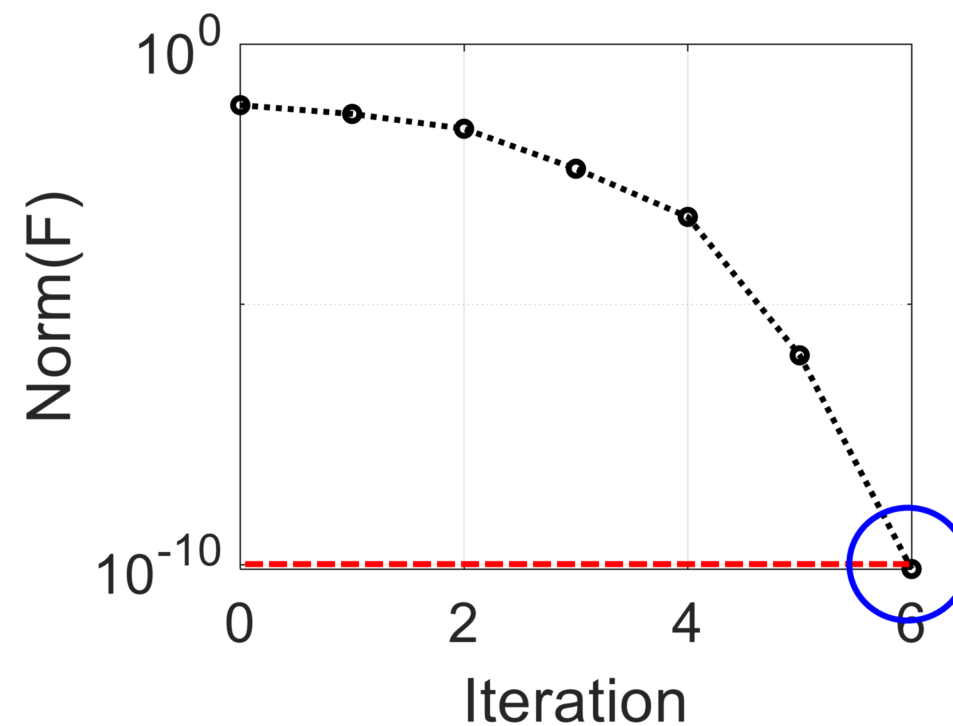


Example: L_2 Southern Halo Orbit

Application of general single shooting formulation to correct an L_2 southern halo orbit



Initial guess
Corrected solution



Modified Constraint Formulation

- Sometimes, the presented formulation can exhibit numerical sensitivities, particularly during continuation (next class!).
- In that case, can use the implicit constraint on a natural trajectory (i.e., the arc generated in single shooting) that ‘conserves’ the Jacobi constant
- Approach:

Modified Constraint Formulation

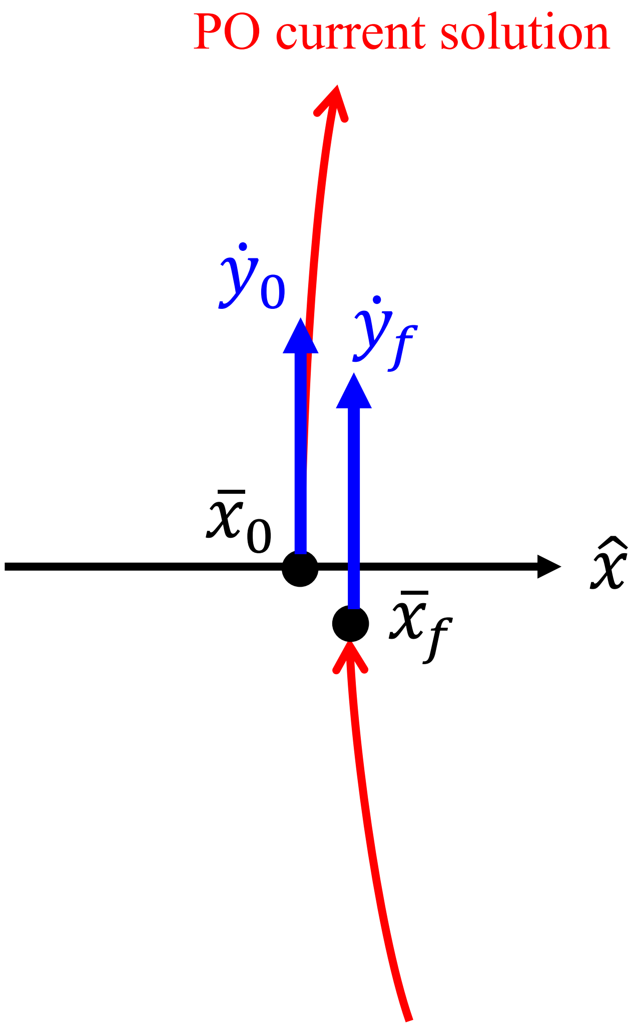
- Because $C_0 = C_f$ for a natural trajectory:

$$2U_0^* - \dot{x}_0^2 - \dot{y}_0^2 - \dot{z}_0^2 = 2U_f^* - \dot{x}_f^2 - \dot{y}_f^2 - \dot{z}_f^2$$

- If continuity constraints are applied to the $x, y, z, \dot{x}, \dot{z}$ components of the state vector, this relationship tells us that along the natural periodic orbit, the following must be true:

Modified Constraint Formulation

- Consider when initial conditions lie on the $\hat{x}$ axis, i.e., $y_0 = 0$
- $\dot{y}_0$ is most likely to possess the same sign in this case throughout corrections and continuation, while the magnitude is constrained by 'conservation' of Jacobi constant along a natural trajectory
- Words of warning:



Modified Constraint Formulation

Remove continuity constraint on $\dot{y}$ and replace with constraint to place initial condition on $\hat{x}$ axis, i.e., $y_0 = 0$. (Update constraint if the periodic orbit does not pass through the *x*-axis)

Before: General constraint formulation:

$$\bar{F}(\bar{V}) = \begin{bmatrix} x_f - x_0 \\ y_f - y_0 \\ z_f - z_0 \\ \dot{x}_f - \dot{x}_0 \\ \dot{y}_f - \dot{y}_0 \\ \dot{z}_f - \dot{z}_0 \end{bmatrix}$$

After: Modified constraint formulation:

$$\bar{F}(\bar{V}) = \begin{bmatrix} x_f - x_0 \\ y_f - y_0 \\ z_f - z_0 \\ \dot{x}_f - \dot{x}_0 \\ \dot{z}_f - \dot{z}_0 \\ y_0 \end{bmatrix}$$

Extensions

When single-shooting methods fail, can use multiple-shooting methods (we will cover later in the course!) to compute periodic orbits

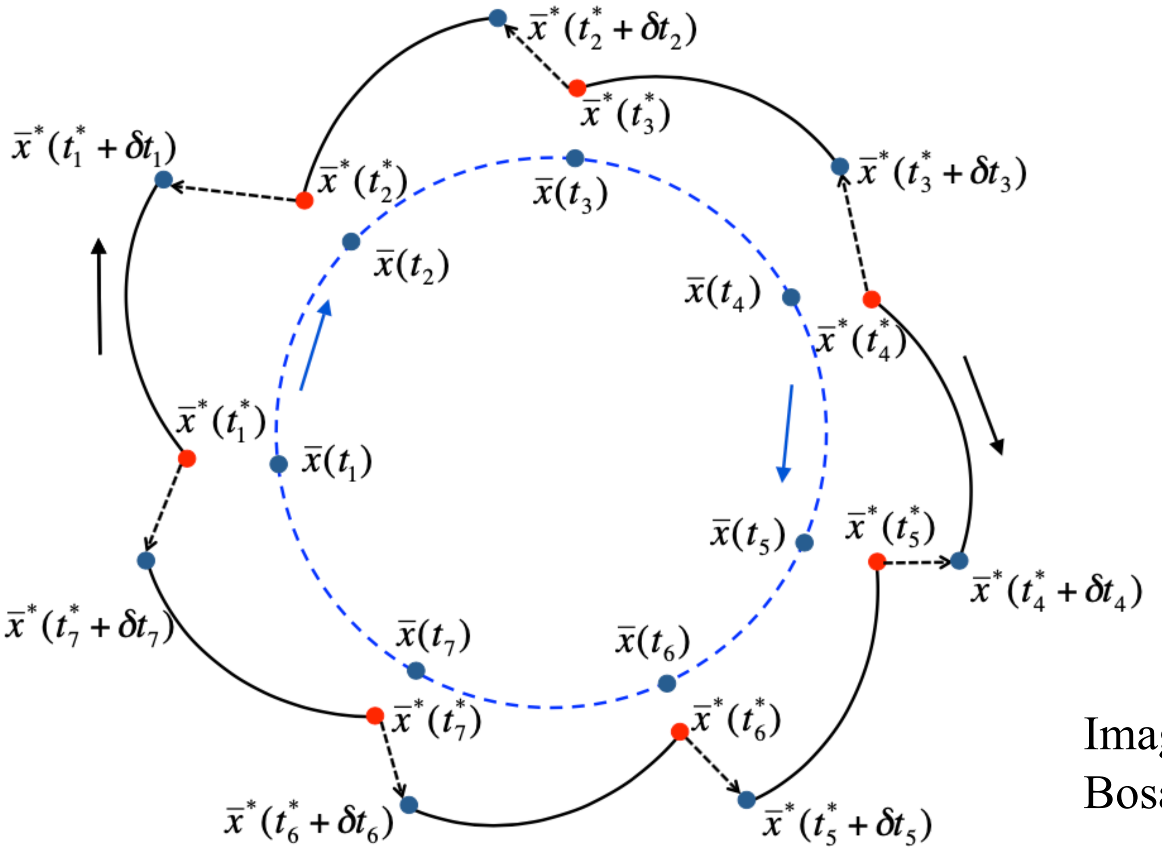


Image credit: Bosanac 2016