Minimum Time / Max Range / Reachable Sets

To date, our space traj. problems have the form:

$$\vec{X} = \begin{bmatrix} \vec{r} \\ \vec{v} \end{bmatrix} \qquad \dot{\vec{X}} = \begin{bmatrix} \dot{\vec{r}} \\ \dot{\vec{v}} \end{bmatrix} = \underbrace{\begin{bmatrix} \vec{v} \\ \vec{G}(\vec{r}) \end{bmatrix}}_{\vec{F}(\vec{x})} + \begin{bmatrix} 0 \\ I \end{bmatrix} \vec{u} \; ; \text{ with } L(\vec{u}) = \begin{cases} \frac{1}{2}\vec{u}\cdot\vec{u} \\ \|\vec{u}\| \end{cases}$$

Then

$$H = \vec{P}_r \cdot \vec{v} + \vec{P}_v \cdot \vec{G}(\vec{r}) + \vec{P}_v \cdot \vec{u} + L(\vec{u})$$

$$H = \vec{P}\cdot\vec{F} + \begin{cases} \frac{1}{2}\vec{u}\cdot\vec{u} + \vec{P}_v\cdot\vec{u} \\ \|\vec{u}\|\left(1 + \vec{P}_v\cdot\hat{u}\right) \end{cases} \Rightarrow \vec{u}^* = \begin{cases} \vec{u} = -\vec{P}_v \\ \hat{u} = -\hat{P}_v \\ \|u\| = 0 \quad (1-P_v) > 0 \\ \|u\| = u_{max} \quad (1-P_v) < 0 \\ \|u\| \in (0, u_{max}) \quad P_v \equiv 1 \end{cases}$$

For $\|\vec{u}\|$ - Singular Control arcs can exist for $\|u\|$ if $P_v = 1$, $\dot{P}_v = 0$, etc.

- Periods of thrusting + coasting for $\|u\|$.

For $\frac{1}{2}\vec{u}\cdot\vec{u}$, the control is always on but is variable, $\boxed{\vec{u} = -\vec{P}_v}$

IF at some point $\vec{P}_v = \vec{0}$ & $\dot{\vec{P}}_v = \vec{0}$, $\Rightarrow \vec{P}_r = \vec{0}$ + the thrust is "off" for all time

$$\dot{\vec{P}}_r = -\vec{P}_v \cdot G_{\vec{r}} \quad ; \quad \dot{\vec{P}}_v = -\vec{P}_r \Rightarrow \vec{P} \equiv \vec{0}$$

Let's consider the case when $L \equiv 0$, we have a Mayer Problem with $\boxed{J = K(\vec{X}_f, t_f)}$ (assume $\vec{X}_0, t_0$ are constrained).

Then $H = \vec{p} \cdot \left(\vec{F}(\vec{x}) + B\vec{u} \right) = \vec{p} \cdot \vec{F} + \vec{p}_v \cdot \vec{u}$

$$H = \vec{p}_v \cdot \vec{u} + \vec{p} \cdot F$$

Assume $\|u\| \le \|u\|_{max}$

$K(\vec{x}_F, t_F) \Rightarrow$ 2 possible classes $\Bigg\{$ ~~[illegible]~~ problems, t_F (min time) or x_F (max state)

Only appers in Trns. Conds ....

Optimal Control will be the same for both

$$\vec{u}^* \arg\min_{u \in \mathcal{U}} H = \begin{cases} \text{Chose } \hat{u} = -\hat{p}_v \\ \text{" } \|u\| = \|u\|_{max} \end{cases} = -\hat{p}_v \|u\|_{max}$$

$$\boxed{H^* = \vec{p} \cdot \vec{F} - p_v \|u\|_{max}} \Rightarrow \text{Dynamics}$$

$$\dot{\vec{X}} = \frac{\partial H^{*}}{\partial \vec{P}} = \vec{F} - \underbrace{B \cdot \hat{P}_V \|u\|_{max}} \quad ; B = \begin{bmatrix} 0 \\ I \end{bmatrix} \quad \left(\frac{\partial P_V}{\partial \vec{P}_V} = \hat{P}_V \right)$$

$$\dot{\vec{P}} = \begin{bmatrix} \dot{\vec{P}}_r \\ \dot{\vec{P}}_V \end{bmatrix} = \frac{-\partial H^{*}}{\partial \vec{X}} = \begin{bmatrix} -\vec{P}_V \cdot \frac{\partial \vec{G}}{\partial \vec{r}} \\ -\vec{P}_r \end{bmatrix}$$

; Can the problem be singular,

$P_V = 0$; $H_u = 0$; $\hat{P}_V$ undefined

$\dot{P}_V = \hat{P}_V \cdot (-\vec{P}_r) = 0$; $\vec{P}_r = 0$

$$\left. \begin{aligned} H_{\vec{u}} &= \vec{P}_V = \vec{0} \\ \dot{\vec{P}}_V &= -\vec{P}_r = \vec{0} \end{aligned} \right\} \vec{P} \equiv 0.$$

Ill posed problem.

For the Mayer Problems the control is "always on" and at its max limit.

Some example cost functions:

$K = \underline{\underline{t_f}}$, min time problem

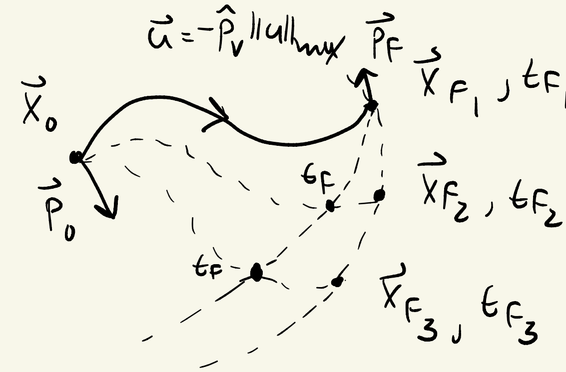
Requires some non-trivial terminal manifold constraint $\boxed{g(\vec{X}_f) = \vec{0}}$ $\Rightarrow$ $\vec{v}_f$ = free ; $\boxed{\vec{P}_v = 0}$

$\vec{r}_f = \vec{R}$

$$\vec{P}_f = \cancel{\frac{\partial K}{\partial \vec{X}_f}}^{\,0} + \vec{\lambda} \cdot \frac{\partial \vec{g}}{\partial \vec{X}_f} = \vec{\lambda}_f \cdot \frac{\partial \vec{g}}{\partial \vec{X}_f} = \text{value of } \vec{P}_f \text{ is "arbitrary" \& needs to be solved for.}$$

$$H_f^* = \frac{-\partial K}{\partial t_f} - \cancel{\vec{\lambda} \cdot \frac{\partial \vec{g}}{\partial t_f}} = -1$$

$$H_f^* = \boxed{\vec{P}_f \cdot \vec{F}(\vec{X}_f) - P_{v_f} \|u\|_{max} = -1}$$



Max. State Cost Functions

$$S = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

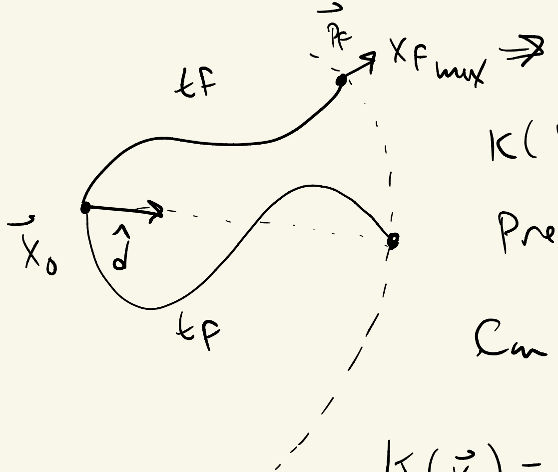
$$K = \begin{cases} -\frac{1}{2} x_f^T x_f = -\frac{1}{2} x_f^T S x_f \\ -\vec{x}_f \cdot \hat{d}, \quad \hat{d} \in \mathbb{R}^6 \text{ pointing direction in phase space.} \\ -\|\vec{x}_f\| \end{cases}$$

Optimal Control $\quad \vec{u}^* = -\hat{p}_v \|u\|_{max}$

Assume $\vec{x}_0$, t_0, t_f specified

$$H_f^* = -\frac{\partial K}{\partial t_f}^{\,0} - \vec{\lambda} \cdot \frac{\partial g}{\partial t_f} = \text{unspecified.}$$

$$\vec{p}_f = \frac{\partial K}{\partial \vec{x}_f} + \vec{\lambda} \cdot \cancel{\frac{\partial g}{\partial \vec{x}_f}} \to 0 \quad \Rightarrow \quad \vec{p}_f = \begin{cases} \left.\begin{array}{l} -\vec{x}_f \\ -S\vec{x}_f \end{array}\right\} \begin{array}{l} -\frac{1}{2}\vec{x}_f^T \vec{x}_f \\ -\frac{1}{2}\vec{x}_f^T S \vec{x}_f \end{array} \\ -\hat{d}; \text{ we specify } \hat{d}. \\ -\hat{x}_f \end{cases}$$



$\Rightarrow K(x_f) = \frac{1}{2} x_f^T x_f$

$K(\vec{x}_f)$

Presupposes a specific form for $K(\vec{x}_f)$

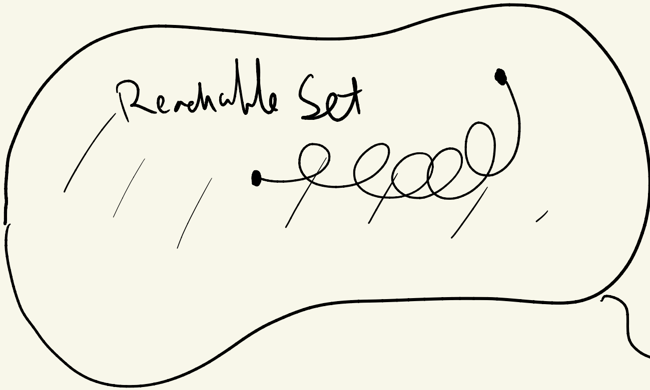
Can get rid of an analytic form for K

$K(\vec{x}_f) = 0 \Rightarrow$ where is J?

We can combine these ideas to ask: what are the set of <u>maximal states</u> that we can reach in a given time?

What are the set of <u>all</u> states that we can reach " " " ?

Given t_F, u_{max}



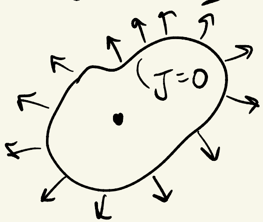
Maximal states that can be reached. Achieved by thrusting over the entire t_F interval.

$J(\vec{x}, t_F)$

Abstracts away the specific form of $K(x)$, and we can use

$J(\vec{x}_F, t_F) = 0$ as the terminal set for a given t_F.

This means that initially, at $t_F = t_0$, $\vec{x}_F = \vec{x}_0$ & $J(\vec{x}_0, t_0) = 0$



$\equiv$ In this forward view the HJB equation is a wave propagation equation $\equiv$ P.D.E.

Initiate the wave front at $\vec{x}_0, t_0$ to cover all of phase space.

HJB

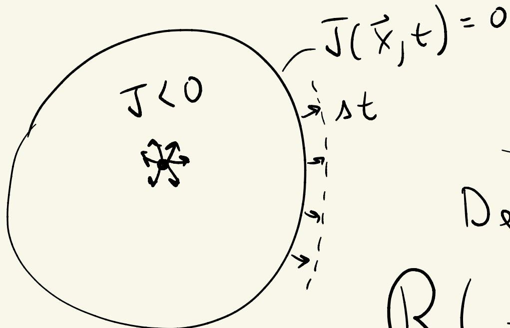
$$\frac{dJ(\vec{x},t)}{dt} = \frac{\partial J}{\partial t} + \min_{u} H\left(\vec{x}, \frac{\partial J}{\partial \vec{x}}, \vec{u}, t\right) = 0$$

$$\vec{x} = \begin{bmatrix} \vec{r} \\ \vec{v} \end{bmatrix}$$

$$\boxed{\frac{\partial J}{\partial t} + \frac{\partial J}{\partial \vec{x}} \cdot \vec{F}(\vec{x}) - \underbrace{\left| \frac{\partial J}{\partial \vec{v}} \right|}_{\text{non-linearity.}} u_{max} = 0}$$

Even if $F(x) = A \cdot X$

$$J(\vec{x}_F, t_F) = 0$$



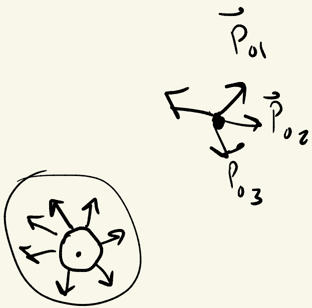
How do we initiate the gradient $\left.\frac{\partial J}{\partial \vec{x}}\right|_{t_0}$

Definition of Reachable Set

$$R(t; V(\vec{x}_0, t_0)) = \left\{ \vec{x} \,\middle|\, V(\vec{x}, t) \leq 0 \right\}$$

where $V(\vec{x}, t)$ satisfies the HJB equation evaluated at the terminal condition $V(\vec{x}, t) = 0$

Procedure for solving the HJB for the Reachable Set is difficult.



In general, we can sample $\vec{P}_0 = \frac{\partial J}{\partial \vec{X}_0}$

Can integrate each $\vec{P}_{0_i}$ over t_F following the Opt. Control Law. When found, provide "local" minimum time solutions to the final state.

- Can sample all possible $\hat{P}_0$ + find nec. initial magnitude to satisfy Trans. Conditions. Each $\hat{P}_{0_i}$ has an $\vec{X}_{F_i}$ on the boundary of the set, reached in minimum time. (papers w/ Patel, Nathersan)
- Can define an initial manifold $\vec{g}(\vec{X}_0) = 0$ + evaluate $V(\vec{X}_0, t_0)$ on this manifold. Then $\vec{P}_0 = \frac{\partial V}{\partial \vec{X}_0}$, is well defined.

$g(\vec{x}_0) = \frac{1}{2} X^T E^{-1} X - 1 = 0.$ (Holzinger)

- Can solve the max state problem, sweeping through all $\hat{\lambda}$ directions + solving over a timespan t_f. (Natherson)

- Can solve the HJB (semi)-analytically. (Park)

- Numerical CFD-type solutions, "Viscosity" Solutions.

- Reachable + Controllable Set

Reachable Set: Starting on a specified manifold, what are all the final states that can be reached in a given time?

Controllable Set: Ending on a specified manifold, what are all the initial states that can reach it in a given time.
(Like the Optimal Feedback Control)

Only difference is the time flow.