Dynamical Control Problem

$\vec{x} \in \mathbb{R}^n \qquad \dot{\vec{x}} = \vec{f}(\vec{x}, \vec{u}, t)$

$\vec{f} \in \mathbb{R}^n$

$\vec{u} \in \underline{U} \subset \mathbb{R}^m \quad ; \quad \boxed{U \text{ compact}} \Rightarrow$ Infinite Thrust is not allowed.

S/C realignm

$$\vec{x} = \begin{bmatrix} \vec{r} \\ \vec{v} \\ m \end{bmatrix} \qquad \vec{f} = \begin{bmatrix} \vec{v} \\ -\frac{\mu}{r^3}\vec{r} + \frac{\dot{m}}{m}\vec{c}\,\vec{u} \\ f_m(m, \vec{r}, t, u_m) \end{bmatrix}$$

7 dim problem

Our basic problem is to find the control function $\vec{u}(t)$, such that $x_0(t_0) \Rightarrow x_f(t_f)$, where $\vec{x}_f$ & $\vec{x}_0$ are both given.

"Hard Constraint"

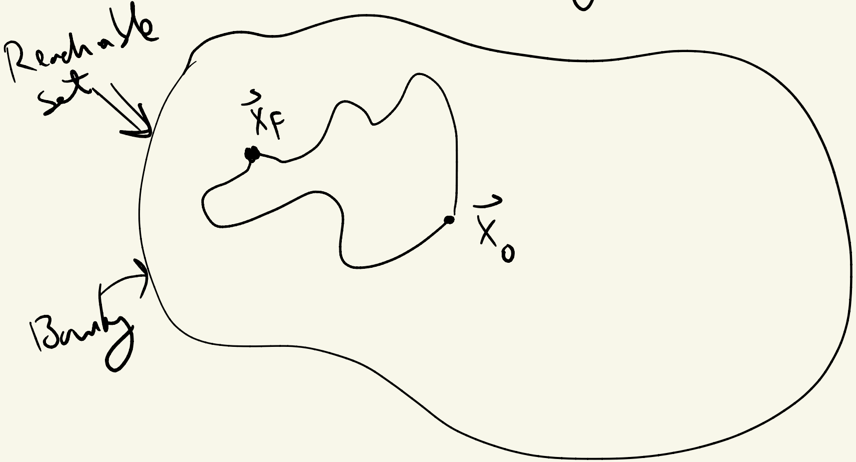
Solution of the EOM (Ord. Diff. Eqns) will be

$$\vec{X}(t) = \varphi(t, \vec{u}(t); \vec{x}_0, t_0) \; ; \quad \text{Solution is defined } \forall \; \vec{u}(t) \subset U$$

System also works backwards in time

$$\vec{X}(t) = \varphi(t; \vec{u}(t); \vec{X}_f, t_f) \quad t < t_f$$

Given $\vec{u} \subset U$, we can generate trajectories starting from an arb. point.



Consider the set of all ~~traj~~ trajectories from $t_0 \to t_f$, $\vec{u} \subset U$

$$R(t; \vec{X}_0, t_0) = \left\{ \vec{X}(t) \;\middle|\; \begin{array}{c} \vec{X}(t) = \varphi(t, \vec{u}; \vec{x}_0, t_0), \\ \vec{u} \subset U \end{array} \right\}$$

Consider a target point $\vec{X}_f$ at time t_f.

If $\vec{X}_f \in R(t_f, t_0, \vec{X}_0)$ the control problem is well posed.

If $\vec{X}_f \notin R$, the BVP is not well posed!

There is no solution $\Rightarrow$ may need to increase t_f, change U.

If $\vec{X}_f \in R(t_f, t_0, \vec{X}_0)$ then $\vec{X}_0 \in R(t_0, t_f, \vec{X}_f) \Rightarrow$ Boundary Value Problem can be solved in either direction.

The set of all controls which can take us from $\vec{X}_0 \rightarrow \vec{X}_f$ can be defined as a set

$\boxed{\mathcal{U}(t_0, \vec{X}_0, t_f, \vec{X}_f) \subset U} \Rightarrow \vec{u} \in \mathcal{U} \Rightarrow$ consists of a $\boxed{\vec{u}(t),\ t_0 \le t \le t_f}$

If $\vec{u}_{seq} \in \mathcal{U}$ then $\boxed{\vec{x}_f = \varphi(t_f, \vec{u}_{seq}, t_0, \vec{x}_0)}$

Still ∞ of controls to ~~do~~ this, Process of optimization is to select the "optimal control" one of this set.

$$\boxed{J^* = \min_{u \in \mathcal{U} \subset U} J}$$

<u>Cost Functions</u>: 3 main types.

Problem of Mayer: $J = K(\vec{x}_0, t_0, \vec{x}_f, t_f)$, terminal state function

<u>Performance Index.</u>

Usually we have

$K = m_f$ (Final mass), $(t_f - t_0)$.

Problem of Lagrange:

$$\boxed{J = \int_{t_0}^{t_f} L(\vec{x}(\tau), \vec{u}(\tau), \tau)\, d\tau}$$

Can always be transformed into a Mayer Problem.

Define a new state $\boxed{x_{n+1}(t) = \int_{t_0}^{t} L(\vec{x}, \vec{u}, \tau)\, d\tau}$

$$\boxed{\dot{x}_{n+1} = L(\vec{x}, \vec{u}, t)} \Rightarrow \quad J = x_{n+1}(t_f) = K(t_f)$$

Problem of Bolza

$$J = K(\vec{x}_0, t_0, \vec{x}_f, t_f) + \int_{t_0}^{t_f} L(\vec{x}, \vec{u}, \tau)\, d\tau \Rightarrow$$

Can always be mapped into a Mayer Problem.

Nec. Conditions for Lagrange

Minimize $J = \int_{t_0}^{t_F} L(\vec{x}, \vec{u}(\tau), \tau)\, d\tau$

subject to $\dot{\vec{x}} = \vec{F}(\vec{x}, \vec{u}, t)$, $\vec{u} \subset U \subset \mathbb{R}^m$

such that $\vec{x}_F = \varphi(t_F, \vec{u}; t_0, \vec{x}_0)$

Hard Constraint Problem ... $K = 0$

① How to deal with the differential constraints $\dot{\vec{x}} = \vec{F}$

② " " " " " BVP " $\vec{x}(t_F) = \vec{x}_F$?

② ⇒ Assume they can be satisfied! Becomes the most difficult part of the problem!

① ⇒ Introduce time-varying Lagrange Multipliers

Introduce functionals $\underline{\vec{p}(t)} \in \mathbb{R}^n$ to act as additional degrees of freedom, enforce the constraint

$$\boxed{\dot{\vec{x}} - \vec{F}(\vec{x}, \vec{u}, t) = 0}$$

$$I = \int_{t_0}^{t_F} L(\vec{x}, \vec{u}(\tau), \tau)\, d\tau - \int_{t_0}^{t_F} \underline{\vec{p}(\tau) \cdot \left[\dot{\vec{x}} - F(\vec{x}, \vec{u}, \tau) \right]}\, d\tau$$

$$\boxed{I = \int_{t_0}^{t_F} \left[L(\vec{x}, \vec{u}, \tau) + \vec{p} \cdot \vec{F}(\vec{x}, \vec{u}, \tau) - \vec{p} \cdot \frac{d\vec{x}}{d\tau} \right] d\tau}$$

What is varied? The control $\vec{u}$, under the assumption that its chosen to satisfy the terminal constraints x_f & x_0

Also, the path $\delta\vec{x}$ can be varied, independent of $\delta\vec{u}$ due to the adjoints introduced, $\underline{\vec{p}(t)} \equiv$ adjoints.

Define the "pre-Hamiltonian" function.....

$$\boxed{H(\vec{x},\vec{p},\vec{u},t) = L(\vec{x},\vec{u},t) + \vec{p}\cdot\vec{F}(\vec{x},\vec{u},t)}$$

$$I = \int_{t_0}^{t_F} \left[H(\vec{x},\vec{p},\vec{u},\tau) - \vec{p}\cdot\frac{d\vec{x}}{d\tau} \right] d\tau$$

($H = \vec{p}\cdot\vec{F} - L$ ..... we are using Lawden's notation)

Consider a solution that satisfies the constraints.

What must its variations be to satisfy $\underline{\delta I = 0}$.

$$\delta I = \int_{t_0}^{t_F} \left[\frac{\partial H}{\partial \vec{x}} \cdot \delta \vec{x} + \frac{\partial H}{\partial \vec{p}} \cdot \delta \vec{p} + \frac{\partial H}{\partial \vec{u}} \cdot \delta \vec{u} - \dot{\vec{x}} \cdot \delta \vec{p} - \underbrace{\vec{p} \cdot \delta \dot{\vec{x}}} \right] d\tau$$

$$\int_{t_0}^{t_F} \vec{p} \cdot \delta \dot{\vec{x}} \, d\tau = \int_{t_0}^{t_F} \vec{p} \cdot \frac{d}{d\tau}(\delta \vec{x}) \, d\tau = \vec{p} \cdot \delta \vec{x} \Big|_{t_0}^{t_F} - \int_{t_0}^{t_F} \delta \vec{x} \cdot \dot{\vec{p}} \, d\tau$$

$$\delta I = \int_{t_0}^{t_F} \left[\left[\frac{\partial H}{\partial \vec{x}} + \dot{\vec{p}} \right] \cdot \delta \vec{x} + \left[\frac{\partial \vec{H}}{\partial \vec{p}} - \dot{\vec{x}} \right] \cdot \delta \vec{p} + \frac{\partial H}{\partial \vec{u}} \cdot \delta \vec{u} \right] d\tau - p \cdot \delta \vec{x} \Big|_{t_0}^{t_F}$$

For "Hard Constraint" problem $\delta x_0 = \delta x_F = \delta t_0 = \delta t_F = 0$

Note that $\frac{\partial H}{\partial \vec{p}} = \vec{F} = \dot{\vec{x}}$, satisfied by definition, $\delta \vec{p}$ term $\equiv 0$.

$$\delta I = \int_{t_0}^{t_f} \left[\left(\frac{\partial H}{\partial \vec{x}} + \dot{\vec{p}} \right) \cdot \delta \vec{x} + \frac{\partial H}{\partial \vec{u}} \cdot \delta \vec{u} \right] d\tau$$

Introducing $\vec{p}$ allows ~~...~~ $\vec{x}$ & $\vec{u}$ to be varied independently

$$\delta I = 0 \Rightarrow \boxed{\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{x}}} \text{ at each time.}$$

$$\frac{\partial H}{\partial \vec{u}} \cdot \delta \vec{u} \equiv 0 \text{ at each time } \frac{\partial H}{\partial \vec{u}} = 0 \; ; \; \boxed{\delta_{\vec{u}} \vec{H} = 0}$$

These are the Nec. Conditions, when combined.

Process: solve $\frac{\partial H}{\partial \vec{u}}(\vec{x}, \vec{p}, \vec{u}^*, t) = 0 \quad \left(\delta_u \vec{H}(\vec{x}, \vec{p}, \vec{u}^*, t) \right) = 0 \Rightarrow$

$$\boxed{\vec{u}^* = \vec{u}(\vec{x}, \vec{p}, t)}$$ <u>Explicit Guidance Law.</u>

$$\boxed{\vec{u}^* = \underset{\vec{u} \subset U}{\operatorname{argmin}}\ H(\vec{x}, \vec{p}, \vec{u}, t)}$$

$$\dot{\vec{x}} = \vec{f}(\vec{x}, \vec{u}^*(\vec{x}, \vec{p}, t), t) \quad ; \quad \dot{\vec{p}} = -\frac{\partial H}{\partial \vec{x}}\bigg|_{\vec{u}^*} = -\vec{p} \cdot \frac{\partial \vec{f}}{\partial \vec{x}}\bigg|_{u^*}$$

Problem... does our choice of $\vec{u}^*$ guarantee that $\vec{x}(t_f) = \vec{x}_f$

Or.... $\vec{u}^* \subset \mathcal{U}_{x_0 \to x_f}$? No.... depends on $\vec{p}(t)$.

The introduction of $\vec{p}(t)$ add n additional parameters to be chosen, $\boxed{\vec{p}_0}$, to satisfy $\vec{x}(t_f) = \vec{x}_f$

$$\vec{x}(t) = \varphi(t, \vec{u}^*(\vec{x}, \vec{p}, t); \vec{x}_0, t_0)$$

$$\vec{p}(t) = \Psi(t; \vec{p}_0, \vec{x}_0, t_0)$$

Coupled together.

Solve for $\boxed{\vec{p}_0 \ni \vec{x}(t_f) = \vec{x}_f}$.

Alternate Derivation: Needed since $\frac{\partial H}{\partial \vec{u}} = \vec{0}$

Pontryagin Minimum Principle

$$\min_{\vec{u} \subset U} I = \min_{\vec{u} \subset U} \int_{t_0}^{t_f} \left[H(\vec{x}, \vec{p}, \vec{u}, \tau) - \vec{p} \cdot \dot{\vec{x}} \right] d\tau$$

$$= \int_{t_0}^{t_f} \left[\min_{\vec{u} \subset U} H(\vec{x}, \vec{p}, \vec{u}, \tau) - \vec{p} \cdot \dot{\vec{x}} \right] d\tau$$

$$\boxed{I^* = \int_{t_0}^{t_1} \left[H^*(\vec{x}, \vec{p}, \tau) - \vec{p} \cdot \dot{\vec{x}} \right] d\tau}$$

; Proof is a bit more difficult since

$L_{Ⓐ} = \frac{1}{2}\vec{u}\cdot\vec{u}$; $L_{Ⓑ} = \|\vec{u}\|$

$\vec{F} \in \mathbb{R}^6$; $\vec{F} = \begin{bmatrix} \vec{v} \\ \vec{g}(\vec{r},t) + \vec{u} \end{bmatrix}$

$\vec{x} = \begin{bmatrix} \vec{r} \\ \vec{v} \end{bmatrix}$ $\vec{p} = \begin{bmatrix} \vec{p}_r \\ \vec{p}_v \end{bmatrix}$

$H = L + \vec{p}\cdot\vec{f}$

$$\boxed{H = L + \vec{p}_v \cdot \vec{u} + \cdots}$$

Ⓐ $H = \frac{1}{2}\vec{u}\cdot\vec{u} + \vec{p}_v\cdot\vec{u} + \cdots$

$\frac{\partial H}{\partial \vec{u}} = \vec{u} + \vec{p}_v = \vec{0} \Rightarrow \boxed{\vec{u}^* = -\vec{p}_v}$

Ⓑ $H = \|\vec{u}\| + \vec{p}_v\cdot\vec{u} + \cdots$

$\frac{\partial H}{\partial \vec{u}} = \frac{\vec{u}}{\|\vec{u}\|} + \vec{p}_v = \hat{u} + \vec{p}_v = 0$?

Find $\vec{u}$ such that

$$\boxed{\min_{\vec{u}} \left(\|\vec{u}\| + \vec{p}_v\cdot\vec{u} + \cdots \right)}$$

$$\boxed{\vec{u} = \|\vec{u}\|\,\hat{u}}$$

Choose $\boxed{\hat{u} = -\hat{p}_v}$ $\Rightarrow$ substitute . . . .

$$\min\left(\|\vec{u}\| - |\vec{p}_v| \|\vec{u}\| \right) = \min\left[\|\vec{u}\| \left(1 - |\vec{p}_v|\right) \right] \; ; \; \|\vec{u}\| \leq u_{max} > 0$$

since U is compact.

(a) IF $|\vec{p}_v| < 1$, $\|\vec{u}\| = 0$ $\qquad \vec{p}_v \cdot \|\vec{u}\|(-\hat{p}_v)$

(b) IF $|\vec{p}_v| > 1$, $\|\vec{u}\| = u_{max}$

IF $|\vec{p}_v| = 1$, singular arc $\Rightarrow$ will be studied (much) later.....

Note, while $\left.\dfrac{\partial H}{\partial \vec{u}}\right|_{\vec{u}^*} \neq 0$, $\boxed{\delta_u H^* = 0}$

(a) IF $|\vec{p}_v| < 1$, $|\vec{u}| \equiv 0$ and $|\overset{0}{\vec{u}} + \delta \vec{u}| \equiv 0 \Rightarrow |\delta \vec{u}| \equiv 0 \Rightarrow \boxed{\delta_u H = 0}$

(b) IF $|\vec{p}_v| > 1$; $|\vec{u}| = u_{max}$ and $|\vec{u} + \delta\vec{u}| = u_{max} \Rightarrow \vec{u} \cdot \delta\vec{u} = 0$

$$\delta_u H = (\hat{u} + \vec{p}_v) \cdot \delta \vec{u} = \cancel{\hat{u} \cdot \delta \vec{u}}^{\,0} + \cancel{\vec{p}_v \cdot \delta \vec{u}}^{\,0} \text{ as } \boxed{\vec{p}_v \parallel -\hat{u}}$$

Necessary Conditions

Given $H(\vec{x},\vec{p},\vec{u},t) = L(\vec{x},\vec{u},t) + \vec{p}\cdot f(\vec{x},\vec{u},t)$

Find $\vec{u}^* = \underset{\vec{u}\subset U}{\operatorname{argmin}}\, H(\vec{x},\vec{p},\vec{u},t) \Rightarrow \boxed{H^*(\vec{x},\vec{p},t) = H(\vec{x},\vec{u}^*(\vec{x},\vec{p},t),t)}$

$$\left(\dot{\vec{x}} = \frac{\partial H^*}{\partial \vec{p}} \quad ; \quad \dot{\vec{p}} = \frac{-\partial H^*}{\partial \vec{x}}\right)$$

$\vec{x}_0$ is specified

$\vec{x}(t_f) = x_f$ is the terminal condition.

$\vec{x}_f = \varphi(t_f; t_0, \vec{x}_0, \vec{p}_0)$ Find $\vec{p}_0$ to satisfy this

$\vec{p}_f = \Psi(t_f; t_0, \vec{x}_0, \vec{p}_0)$ 2-Point Boundary Value Problem.

Very difficult problem to solve.

$\vec{x}_0, \vec{x}_f \in \mathbb{R}^6$; $\vec{p}_0 \in \mathbb{R}^6$

Necessary Conditions w/ general terminal constraints

Now relax the hard constraints.....

1. Allows for a terminal cost function $\boxed{K(\vec{x}_0, t_0, \vec{x}_f, t_f)}$

2. Allows for a more general statement of constraints

$$\vec{g}(\vec{x}_0, t_0, \vec{x}_f, t_f) = \vec{0}, \quad \vec{g} \in \mathbb{R}^{\ell} \; ; \; \boxed{\ell \leq 2n+2}$$

Cost function

$$J = K(\vec{x}_0, t_0, \vec{x}_f, t_f) + \int_{t_0}^{t_f} L(\vec{x}, \vec{u}, \tau)\, d\tau$$

w/ differential constraints

$$I = K + \int_{t_0}^{t_f} [H - \vec{p} \cdot \dot{\vec{x}}]\, d\tau \quad ; \quad H = L + \vec{p} \cdot \vec{F}$$

Terminal Constraints $\Rightarrow$ Introduce addition Lagrange Multipliers

$\vec{\lambda} \in \mathbb{R}^{e}$, giving

$$\boxed{I = K + \int_{t_0}^{t_f} [\vec{H} - \vec{p} \cdot \dot{\vec{x}}] \, d\tau + \vec{\lambda} \cdot \vec{g}}$$

Again, vary $\delta\vec{x}$ & $\delta\vec{u}$ freely, but no longer dictate $\delta x_0 = \delta x_f = \delta t_0 = \delta t_f = 0$

Can take the variations again, but allow for δt_0 & δt_f

$$\delta I = \int_{t_0}^{t_f} \left[\left(\frac{\partial H}{\partial \vec{p}} - \dot{\vec{x}} \right) \cdot \delta \vec{p} + \frac{\partial H}{\partial \vec{x}} \cdot \delta \vec{x} + \frac{\partial H}{\partial \vec{u}} \cdot \delta \vec{u} \right] d\tau \boxed{- \int_{t_0}^{t_f} \vec{p} \cdot \delta \dot{\vec{x}} \, d\tau} \star$$

$$+ \left[H - \vec{p} \cdot \dot{\vec{x}} \right] \delta t \Big|_{\delta t_0}^{\delta t_f} + \left[\frac{\partial K}{\partial \vec{x}_0} \cdot \delta \vec{x}_0 + \frac{\partial K}{\partial \vec{x}_f} \cdot \delta \vec{x}_f + \frac{\partial K}{\partial t_0} \delta t_0 + \frac{\partial K}{\partial t_f} \delta t_f \right]$$

$$\delta \vec{\lambda} \cdot \vec{g} + \vec{\lambda} \cdot \left[\frac{\partial \vec{g}}{\partial \vec{x}_0} \cdot \delta \vec{x}_0 + \frac{\partial \vec{g}}{\partial \vec{x}_f} \cdot \delta \vec{x}_f + \frac{\partial \vec{g}}{\partial t_0} \delta t_0 + \frac{\partial \vec{g}}{\partial t_f} \delta t_f \right] = 0.$$

$$\delta I = \cdots\cdots - \vec{p}_f \left[\delta x_f - \dot{\vec{x}}_f \delta t_f \right] + \vec{p}_0 \left[\delta x_0 - \dot{\vec{x}}_0 \delta t_0 \right]$$

$$H_f \delta t_f - \vec{p} \cdot \dot{\vec{x}}_f \delta t_f - H_0 \delta t_0 + \vec{p} \cdot \dot{\vec{x}}_0 \delta t_0 + \cdots$$

Need to be careful....

what happens to

$$-\int_{t_0}^{t_F} \vec{p}\cdot\delta\dot{\vec{x}}\,d\tau \Rightarrow$$ Integration by parts....

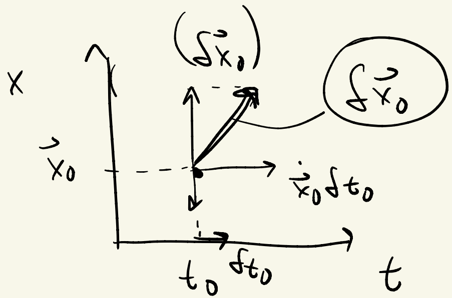
need to accommodate δt_0 & δt_F....

Assume δx is contemporaneous......

$$-\vec{p}\cdot(\delta\vec{x})\Big|_{t_0}^{t_F} + \int_{t_0}^{t_F} \dot{\vec{p}}\cdot\delta\vec{x}\,d\tau$$

The actual variations in δx_0 & δx_F include contem. variations & corrections due to the velocities & δt_0 & δt_F.

$$\underset{\text{Actual}}{\delta\vec{x}_0} = \left(\delta\vec{x}_0\right) + \dot{\vec{x}}_0\,\delta t_0 \qquad \delta\vec{x}_F = (\delta\vec{x})_F + \dot{\vec{x}}_F\,\delta t_F$$



$$-\vec{p}_f \cdot (\delta \vec{x})_f + \vec{p}_0 \cdot (\delta \vec{x})_0 + \int_{t_0}^{t_1} \dot{\vec{p}} \cdot \delta \vec{x} \, d\tau$$

$$-\vec{p}_f \left[\delta \vec{x}_f - \dot{\vec{x}}_f \delta t_f \right]$$

$$+ \vec{p}_0 \left[\delta \vec{x}_0 - \dot{\vec{x}}_0 \delta t_0 \right] + \cdots$$

By def $\quad \dot{\vec{x}} = \frac{\partial H}{\partial \vec{p}} = H_{\vec{p}} \quad , \quad \delta\vec{\lambda}\cdot\vec{g} = 0$

$\delta\vec{x}, \delta\vec{u} \Rightarrow \quad \dot{\vec{p}} = -\frac{\partial H}{\partial \vec{x}} = -H_{\vec{x}} \quad ; \quad \delta_{\vec{u}} H = \frac{\partial H}{\partial \vec{u}}\cdot\delta\vec{u} = 0$

Leaves terms w/ $\delta\vec{x}_0$, $\delta\vec{x}_f$, δt_0 & δt_f $\Rightarrow$ Balance

$$\left[-\vec{p}_f + K_{\vec{x}_f} + \vec{\lambda}\cdot\vec{g}_{\vec{x}_f}\right]\cdot\delta\vec{x}_f + \left[\vec{p}_0 + K_{\vec{x}_0} + \vec{\lambda}\cdot\vec{g}_{\vec{x}_0}\right]\cdot\delta\vec{x}_0$$

$$+\left[H_f + K_{t_f} + \vec{\lambda}\cdot\vec{g}_{t_f}\right]\delta t_f + \left[-H_0 + K_{t_0} + \vec{\lambda}\cdot\vec{g}_{t_0}\right]\delta t_0 = 0$$

Transversality Conditions:

$$\left.\begin{aligned} \vec{p}_0 &= -\frac{\partial K}{\partial \vec{x}_0} - \vec{\lambda}\cdot\frac{\partial \vec{g}}{\partial \vec{x}_0} & H_0 &= \frac{\partial K}{\partial t_0} + \vec{\lambda}\cdot\frac{\partial \vec{g}}{\partial t_0} \\ \vec{p}_f &= \frac{\partial K}{\partial t_f} + \vec{\lambda}\cdot\frac{\partial \vec{g}}{\partial \vec{x}_f} & H_f &= -\frac{\partial K}{\partial t_f} - \vec{\lambda}\cdot\frac{\partial \vec{g}}{\partial t_f} \end{aligned}\right\}$$

We know K & $\vec{g}$

We don't know $\vec{\lambda}$

Full set of Nec. Conditions

$$J = K + \int_{t_0}^{t_f} L \, d\tau \quad , \quad \vec{g}(\vec{x}_0, t_0, \vec{x}_f, t_f) = \vec{0} \in \mathbb{R}^l , \quad \dot{\vec{x}} = \vec{f}(\vec{x}, t, \vec{u})$$

$$\vec{u} \subset U$$

$$H(\vec{x}, \vec{p}, \vec{u}, t) = L + \vec{p} \cdot \vec{f} \quad ; \quad \vec{u}^*(\vec{x}, \vec{p}, t) = \underset{\vec{u} \subset U}{\arg\min}\, H(\vec{x}, \vec{p}, \vec{u}, t)$$

2n IC's: $\vec{x}_0, \vec{p}_0$
l Lagr. Mult: $\vec{\lambda}$ consts
2 times t_0 & t_f

$$\boxed{H^*(\vec{x}, \vec{p}, t)} = H(\vec{x}, \vec{p}, \vec{u}^*(\vec{x}, \vec{p}), t)$$

EOM:

$$\frac{d\vec{x}}{dt} = \frac{\partial H^*}{\partial \vec{p}}$$

$$\frac{d\vec{p}}{dt} = -\frac{\partial H^*}{\partial \vec{x}}$$

Solve these Diff Eqs

while satisfying $\vec{g} = \vec{0}$

and the Transversality Conditions

2n+2 Trans. Conds
l consts

$$\vec{p}_0 = -\frac{\partial K}{\partial \vec{x}_0} - \vec{\lambda} \cdot \frac{\partial \vec{g}}{\partial \vec{x}_0} \qquad H_0 = \frac{\partial K}{\partial t_0} + \vec{\lambda} \cdot \frac{\partial \vec{g}}{\partial t_0}$$

$$\vec{p}_f = \frac{\partial K}{\partial \vec{x}_f} + \vec{\lambda} \cdot \frac{\partial \vec{g}}{\partial \vec{x}_f} \qquad H_f = -\frac{\partial K}{\partial t_f} - \vec{\lambda} \cdot \frac{\partial \vec{g}}{\partial t_f}$$

The conditions + Diff Eqs are all balanced.

Recall Hard Constraint Problem

$$\vec{g} = \begin{bmatrix} \vec{x}(t_0) - \vec{x}_0 \\ \vec{x}(t_f) - \vec{x}_f \\ t|_{t_f} - t_f \\ t|_{t_0} - t_0 \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{0} \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{2n+2}$$

$; \; K \equiv 0$

$$\frac{\partial \vec{g}}{\partial (\vec{x}_0, \vec{x}_f)} = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right]$$

$$\frac{\partial \vec{g}}{\partial (t_0, t_f)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \; ; \; \vec{\lambda} \in \mathbb{R}^{2n+2}$$

$$\vec{\lambda} = \begin{bmatrix} \vec{\lambda}_0 \\ \vec{\lambda}_f \\ \lambda_{t_0} \\ \lambda_{t_f} \end{bmatrix}$$

$$\boxed{\begin{aligned} \vec{p}_0 &= -\vec{\lambda}_0 & \qquad H_0 &= \lambda_{t_0} \\ \vec{p}_f &= \vec{\lambda}_f & \qquad H_f &= -\lambda_{t_f} \end{aligned}}$$

Special Cases

Free Variables & Time

- A state is called "free" if it does not appear in either K or $\vec{g}$. This yields a strong simplification.

Let $x_{\hat{i}}$ be a free variable at both end points.

$$P_{0_{\hat{i}}} = -\frac{\partial K}{\partial x_{0_{\hat{i}}}} - \vec{\lambda} \cdot \frac{\partial \vec{g}}{\partial x_{0_{\hat{i}}}} \equiv 0$$; Similar for $P_{f_{\hat{i}}} = 0$.

If its specified at one point but free at the other.....

$$P_{0_i} \equiv -\frac{\partial K}{\partial x_{0_i}} - \vec{\lambda} \cdot \frac{\partial \vec{g}}{\partial x_{0_i}}$$

$$P_{f_i} = 0$$

If time is not specified in K or $\vec{g}$, i.e., free then $H_0^* = H_f^* = 0$, does not mean that $H^* \equiv 0$; If specified at one end...

$$H_0^* = \lambda_{t_0}, \quad H_f^* = 0$$

Conditions for the Hamiltonian to be constant.

$H^*(\vec{x},\vec{p},t)$; $\quad \frac{d}{dt}H^* = \frac{\partial H^*}{\partial \vec{p}}\cdot\frac{d\vec{p}}{dt} + \frac{\partial H^*}{\partial \vec{x}}\cdot\frac{d\vec{x}}{dt} + \frac{\partial H}{\partial t}$

From the EOM $\quad \dot{\vec{p}} = -\frac{\partial H^*}{\partial \vec{x}}$ & $\dot{\vec{x}} = \frac{\partial H^*}{\partial \vec{p}}$

$$\frac{d}{dt}H^* = \cancel{-\frac{\partial H^*}{\partial \vec{p}}\cdot\frac{\partial H^*}{\partial \vec{x}}} + \cancel{\frac{\partial H^*}{\partial \vec{x}}\cdot\frac{\partial H^*}{\partial \vec{p}}} + \frac{\partial H^*}{\partial t} = \quad \boxed{\frac{dH^*}{dt} = \frac{\partial H^*}{\partial t}}$$

<u>If $\vec{f}(\vec{x})$ & $L(\vec{x},\vec{u})$</u> are time invariant, then $\frac{\partial H^*}{\partial t} = 0$

and $\boxed{H^*(\vec{x},\vec{p}) = \text{constant.}}$

Ignorable Coordinates:

If a state variable $x_{\hat{i}}$ does not appear in the dynamics or cost function (i.e., does not appear in $H^*(\vec{x}, \vec{p}, t)$), then its conjugate adjoint, $p_{\hat{i}}$, is a constant and x_i can be solved for after the full problem is solved.

$$\dot{p}_{\hat{i}} = -\frac{\partial H^*}{\partial x_{\hat{i}}} = 0 \Rightarrow \boxed{p_{\hat{i}} = \text{Constant}} \; ; \quad \dot{x}_i = \frac{\partial H^*}{\partial p_{\hat{i}}}$$

Need to solve $\dot{\vec{x}} = \frac{\partial H}{\partial \vec{p}}$ or $\Rightarrow \underline{\vec{p}(\vec{x}, \dot{\vec{x}})}$

But we need $\left|\frac{\partial^2 H}{\partial \vec{p}^2}\right| \neq 0$, but $\left|\frac{\partial^2 H^*}{\partial \vec{p}^2}\right| = 0$ we can restate as a Lagrangian System ....