ASEN 6020 - HW 1
Spring 2025 - Jash Bhalavat

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ASEN 6020
Spring 2020
Jash Bhalavat

HW #1

Problem 1 → $J_{BP} \le J_H \rightarrow (\sqrt{2}-1)\left[1+\frac{1}{\sqrt{r}}\right] \le \sqrt{\frac{2r}{1+r}} - 1 + \frac{1}{\sqrt{r}} - \sqrt{\frac{2}{r(1+r)}}$

Mult by $\sqrt{r}$ → $(\sqrt{2}-1)(\sqrt{r}+1) \le \sqrt{\frac{2r^2}{1+r}} - \sqrt{r} + 1 - \sqrt{\frac{2}{1+r}}$

$$\sqrt{2r} + \sqrt{2} - \cancel{\sqrt{r}} - 1 \le \sqrt{\frac{2}{1+r}}(r-1) - \cancel{\sqrt{r}} + 1$$

$$\sqrt{2}(\sqrt{r}+1) - 2 \le \frac{\sqrt{2}}{\sqrt{1+r}}(r-1) \rightarrow \cancel{\sqrt{2}}(\sqrt{r}+1-\sqrt{2}) \le \cancel{\sqrt{2}}\frac{r-1}{\sqrt{1+r}}$$

1 → $\sqrt{r} - (\sqrt{2}-1) \le \frac{r-1}{\sqrt{1+r}}$ ($\because \sqrt{r} > \sqrt{2}-1$ as $r>1$, square both sides)

let $\boxed{\sqrt{2}-1 = a}$ $\quad (\sqrt{r}-a)^2 \le \frac{(r-1)^2}{1+r}$

multiply by $1+r$ $\quad r - 2a\sqrt{r} + a^2 \le \frac{(r-1)^2}{1+r} \rightarrow (r - 2a\sqrt{r} + a^2)(1+r) \le (r-1)^2$

$$r + \cancel{r^2} - 2a\sqrt{r} - 2ar\sqrt{r} + a^2 + a^2 r \le \cancel{r^2} - 2r + 1$$

square both sides
$(3r+a^2r+a^2) \ge 0$
$(1+2a\sqrt{r}+2ar\sqrt{r}) > 0$

$$3r + a^2 r + a^2 \le 1 + 2a\sqrt{r} + 2ar\sqrt{r}$$

$$9r^2 + 3a^2r^2 + 3a^2r + 3a^2r^2 + a^4r^2 + a^4r^2 + 3a^2r + a^4r + a^4 \le 1 + 2a\sqrt{r} + 2ar\sqrt{r} \ldots$$

$$\ldots + 2a\sqrt{r} + 4a^2r + 4a^2r^2 + 2ar\sqrt{r} + 4a^2r^2 + 4a^2r^3$$

group by r

$$r^3(4a^2) + r^2(4a^2 + 4a^2 - 9 - 3a^2 - 3a^2 - a^4 - a^4) + r(2a\sqrt{r} + 4a^2 + 2a\sqrt{r} - 3a^2 - 3a^2 - a^4)$$

$$+1(1 + 2a\sqrt{r} + 2a\sqrt{r} - a^4) \ge 0$$

Divide by $4a^2$

$$r^3(4a^2) + r^2(2a^2 - a^4 - 9) + r(4a\sqrt{r} - 2a^2 - a^4) + 1(1 + 4a\sqrt{r} - a^4) \ge 0$$

$$r^3 + r^2\left(\frac{1}{2} - \frac{a^2}{4} - \frac{9}{4a^2}\right) + r\left(\frac{\sqrt{r}}{a} - \frac{1}{2} - \frac{a^2}{4}\right) + 1\left(\frac{1}{4a^2} + \frac{\sqrt{r}}{a} - \frac{a^2}{4}\right) \ge 0$$

$$r^3 - r^2(7+4\sqrt{2}) + r(3+4\sqrt{2}) + a - \frac{1}{4a^2} - \frac{a^2}{4} \ge 0$$

$$\boxed{r^3 - r^2(7+4\sqrt{2}) + r(3+4\sqrt{2}) - 1 \ge 0}$$

$\boxed{r = 0.1466,\ 0.5715,\ 11.9388}$ roots < 1 are irrelevant as they violate the core assumption that $r>1$

Now, $r<1$ → start from eqn. 1 → $\sqrt{r} - (\sqrt{2}-1) \le \frac{r-1}{\sqrt{1+r}}$ (let $a = \sqrt{2}-1$)

$$(\sqrt{r}-a)\sqrt{1+r} \le r-1 \rightarrow \sqrt{r^2+r} - a\sqrt{1+r} \le r-1$$

square both sides (>0)

$$\sqrt{r^2+r} + 1 \le r + a\sqrt{1+r} \rightarrow \cancel{r^2}+r + 2\sqrt{r^2+r} + 1 \le \cancel{r^2} + 2a\sqrt{1+r} + a^2(1+r)$$

$$r + 2\sqrt{r(1+r)} + 1 \le 2a\sqrt{1+r} + a^2(1+r)$$

$$r^2 + 2r\sqrt{r^2+r} + r + 2r\sqrt{r^2+r} + 4(r^2+r) + 2\sqrt{r^2+r} + r + 2\sqrt{r^2+r} + 1 \le$$

$$4a^2(1+r) + 4a^3(1+r)^{3/2} + a^4(1+r)^2$$

$$5r^2 + 6r + \sqrt{r^2+r}(2r+2r+2+2) + 1 \le (1+r)(4a^2 + 4a^3\sqrt{r+1} + a^4(1+r))$$

$$5r^2+6r+4\sqrt{r}\,\sqrt[3]{1+r} \le (1+r)(2a+a\sqrt{1+r})^2$$

$$\boxed{5r^2+6r+(1+r)\left(4\sqrt{r}(1+r)^{1/2}-(2a+a\sqrt{1+r})^2\right) \le 0}$$ - solve for 0

This function doesn't have any roots which signifies that when $r<1$, Hohmann transfer is optimal when compared to bi-parabolic.

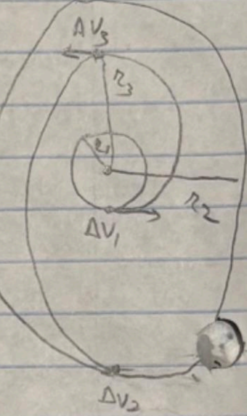
Problem 2

Problem 2 → When $1<l<r$ for bi-elliptic transfers →

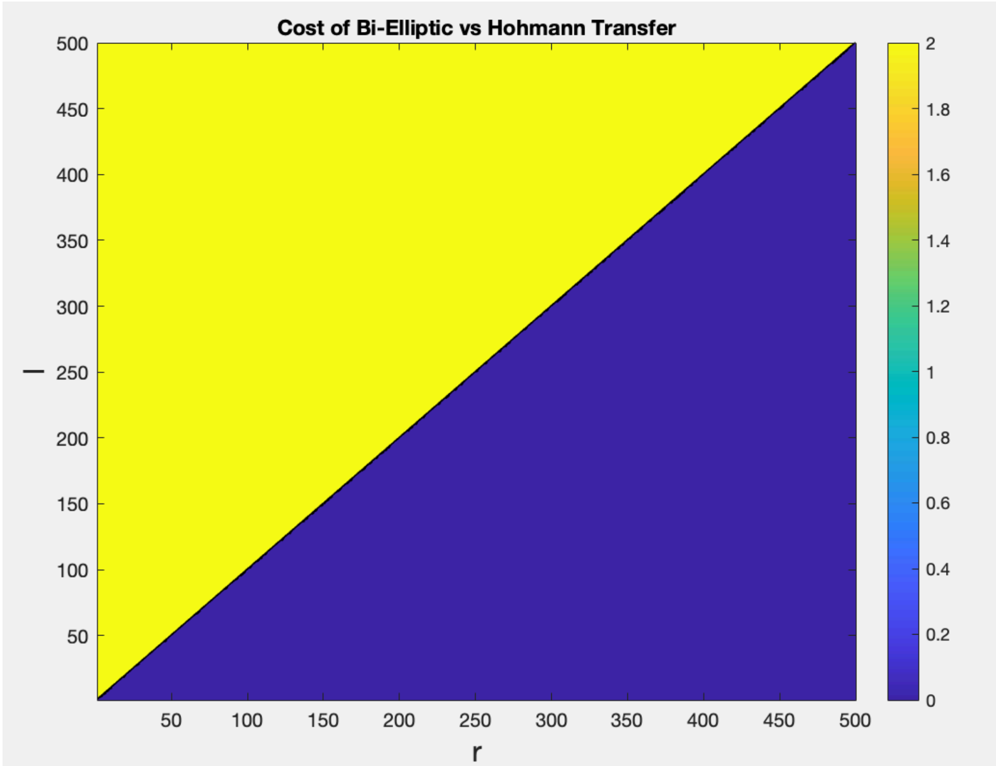
$$J_{BE} = \underbrace{\sqrt{\frac{2\mu}{r_1+r_3}\cdot\frac{r_3}{r_1}} - \sqrt{\frac{\mu}{r_1}}}_{\Delta V_1} + \underbrace{\sqrt{\frac{2\mu}{r_2+r_3}\cdot\frac{r_2}{r_3}} - \sqrt{\frac{2\mu}{r_1+r_3}\cdot\frac{r_1}{r_3}}}_{\Delta V_3} + \underbrace{\sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu}{r_3+r_2}\cdot\frac{r_3}{r_2}}}_{\Delta V_2}$$

$$\boxed{\tilde{J}_{BE} = \sqrt{\frac{2l}{1+l}} - 1 + \sqrt{\frac{2r}{l(l+r)}} - \sqrt{\frac{2}{l(l+1)}} + \frac{1}{\sqrt{r}} - \sqrt{\frac{2l}{r(l+r)}}}$$

comparing with $\tilde{J}_H(r) = \sqrt{\frac{2r}{1+r}} - 1 + \frac{1}{\sqrt{r}} - \sqrt{\frac{2}{r(1+r)}}$ numerically



The contour plot below shows the cost of bi-elliptic vs Hohmann transfers for 1<l<r. The yellow region below is inaccessible (for this problem) because l > r. The blue region Hohmann transfers cost less than Bi-Elliptic transfers. This is different from when l > r because at no point here is the Bi-Elliptic transfer the optimal option.



Cost of Bi-Elliptic vs Hohmann Transfer
r
l

Problem 3

Problem 3 →

$$\bar{J}_H = \frac{1}{\sqrt{\alpha_1}} - \sqrt{\frac{2\alpha_2}{(\alpha_1+\alpha_2)\alpha_1}} + \sqrt{\frac{2\alpha_1}{(\alpha_1+\alpha_2)\alpha_2}} - \frac{1}{\sqrt{\alpha_2}}, \quad \bar{J}_{AE} = \frac{1}{\sqrt{\alpha_1}} - \sqrt{\frac{2}{\alpha_1(1+\alpha_1)}} + \frac{1}{\sqrt{\alpha_2}} - \sqrt{\frac{2}{\alpha_2(1+\alpha_2)}}$$

$$g_I = J_H - J_{AE}$$

$$= \cancel{\frac{1}{\sqrt{\alpha_1}}} - \sqrt{\frac{2\alpha_2}{\alpha_1(\alpha_1+\alpha_2)}} + \sqrt{\frac{2\alpha_1}{\alpha_2(\alpha_1+\alpha_2)}} - \frac{1}{\sqrt{\alpha_2}} - \cancel{\frac{1}{\sqrt{\alpha_1}}} + \sqrt{\frac{2}{\alpha_1(1+\alpha_1)}} - \frac{1}{\sqrt{\alpha_2}} + \sqrt{\frac{2}{\alpha_2(1+\alpha_2)}}$$

$$\boxed{g_I = \sqrt{\frac{2}{\alpha_1(1+\alpha_1)}} - \sqrt{\frac{2\alpha_2}{\alpha_1(\alpha_1+\alpha_2)}} + \sqrt{\frac{2\alpha_1}{\alpha_2(\alpha_1+\alpha_2)}} + \sqrt{\frac{2}{\alpha_2(1+\alpha_2)}} - \frac{2}{\sqrt{\alpha_2}}}$$

$$g_I(1+a\epsilon, 1+\epsilon) = 0$$

Taylor series expansion $\rightarrow g_I(1+a\epsilon, 1+\epsilon) = g_I\big|_{\epsilon=0} + \frac{\partial g_I}{\partial \epsilon}\Big|_{\epsilon=0}\epsilon + \dots = 0$

$$g_I\big|_{\epsilon=0} = \cancel{\sqrt{\frac{2}{1(1+1)}}} - \cancel{\sqrt{\frac{2(1)}{1(1+1)}}} + \cancel{\sqrt{\frac{2(1)}{1(1+1)}}} + \cancel{\sqrt{\frac{2}{1(1+1)}}} - \cancel{\frac{2}{\sqrt{1}}} = 0$$

Take partial derivatives term by term for simplification

$$\frac{\partial}{\partial \epsilon}\left(\sqrt{\frac{2}{(1+a\epsilon)(1+1+a\epsilon)}}\right) = -\frac{a(2a\epsilon+3)}{\sqrt{2}(a\epsilon+1)^2(a\epsilon+2)^2\sqrt{\frac{1}{(a\epsilon+1)(a\epsilon+2)}}}\Bigg|_{\epsilon=0} = \frac{-3a}{\sqrt{2}(4)\cdot\frac{1}{\sqrt{2}}} = -\frac{3a}{4}$$

$$\frac{\partial}{\partial \epsilon}\left(\sqrt{\frac{2(1+\epsilon)}{(1+a\epsilon)(2+a\epsilon+\epsilon)}}\right) = -\frac{(a^2+a)\epsilon^2 + (2a^2+2a)\epsilon + 3a-1}{\sqrt{2}(a\epsilon+1)^2[(a+1)\epsilon+2]^2\sqrt{\frac{\epsilon+1}{(a\epsilon+1)(a\epsilon+\epsilon+2)}}}\Bigg|_{\epsilon=0} = \frac{3a-1}{\sqrt{2}(4)\frac{1}{\sqrt{2}}} = -\frac{3a-1}{4}$$

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$$\frac{\partial}{\partial \epsilon}\left(\sqrt{\frac{2(1+a\epsilon)}{(1+\epsilon)(2+a\epsilon+\epsilon)}}\right) = -\frac{(a+1)\epsilon(a\epsilon+2) - a+3}{\sqrt{2}(\epsilon+1)^2[(a+1)\epsilon+2]^2\sqrt{\frac{a\epsilon+1}{(\epsilon+1)[(a+1)\epsilon+2]}}}\Bigg|_{\epsilon=0} = -\frac{3-a}{\sqrt{2}(4)\frac{1}{\sqrt{2}}} = -\frac{3-a}{4}$$

$$\frac{\partial}{\partial \epsilon}\left(\sqrt{\frac{2}{(1+\epsilon)(1+1+\epsilon)}}\right) = -\frac{2\epsilon+3}{\sqrt{2}(\epsilon+1)^2(\epsilon+2)^2\sqrt{\frac{1}{(\epsilon+1)(\epsilon+2)}}}\Bigg|_{\epsilon=0} = -\frac{3}{\sqrt{2}(4)\frac{1}{\sqrt{2}}} = -\frac{3}{4}$$

$$\frac{\partial}{\partial \epsilon}\left(\frac{2}{\sqrt{1+\epsilon}}\right) = -\frac{1}{(\epsilon+1)^{3/2}}\Bigg|_{\epsilon=0} = -1$$

Bringing them all together $\rightarrow \frac{\partial g_I}{\partial \epsilon}\Big|_{\epsilon=0} = -\frac{3a}{4} + \frac{3a-1}{4} - \frac{3-a}{4} - \frac{3}{4} + 1 = 0 \rightarrow \boxed{a=3}$

$$\alpha_1 = 1+3\epsilon, \ \alpha_2 = 1+\epsilon \rightarrow \epsilon = \frac{\alpha_1-1}{3} = \alpha_2 - 1 \rightarrow \alpha_2 = \frac{\alpha_1}{3} - \frac{1}{3} + 1 = \boxed{\frac{\alpha_1}{3} + \frac{2}{3} = \alpha_2}$$

Problem 4

Assuming l = 10 (as the optimal plane change cost is at this extremity).

The cost function for the 3 plane change maneuvers (divided into 3 plane changes - 2 at periapsis and 1 at apoapsis) is given below along with the restricted bi-elliptic maneuver (where the plane change occurs at the apoapsis):

Problem 4 → $\Delta i = 90^\circ$, $\ell \leq 10$, $3\times\ \Delta V$

$$\Delta V_1 = \sqrt{\frac{\mu}{r_1} + \frac{2\mu}{r_1+r_2}\cdot\frac{r_2}{r_1} - 2\sqrt{\frac{\mu}{r_1}\cdot\frac{2\mu}{r_1+r_2}\cdot\frac{r_2}{r_1}}\cos(\Delta i_1)}$$

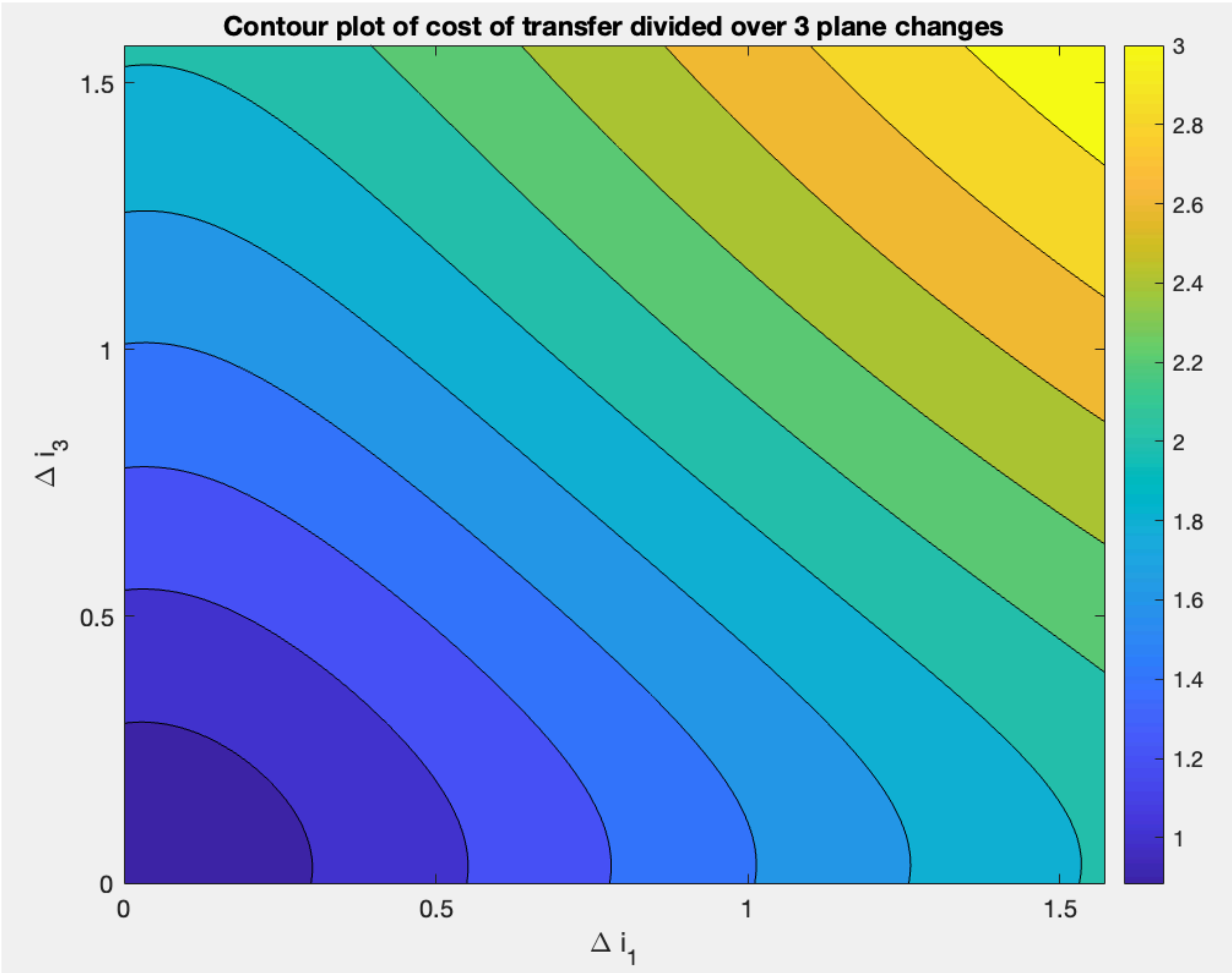
$$\Delta V_2 = 2\sqrt{\frac{2\mu}{r_1+r_2}\cdot\frac{r_1}{r_2}}\sin(90^\circ - \Delta i_1 - \Delta i_3), \quad \Delta V_3 = \sqrt{\frac{\mu}{r_1} + \frac{2\mu}{r_1+r_2}\cdot\frac{r_2}{r_1} - 2\sqrt{\frac{\mu}{r_1}\cdot\frac{r_2}{r_1}\cdot\frac{2\mu}{r_1+r_2}}\cos(\Delta i_3)}$$

$\Delta V = \Delta V_1 + \Delta V_2 + \Delta V_3$ – Divide by $\sqrt{\frac{\mu}{r_1}}$ and $r = r_2/r_1 = \ell$

$r = \ell$

$$\boxed{\Delta\tilde{V} = \sqrt{1 + \frac{2r}{1+r} - 2\sqrt{\frac{2r}{1+r}}\cos(\Delta i_1)} + \sqrt{1 + \frac{2r}{1+r} - 2\sqrt{\frac{2r}{1+r}}\cos(\Delta i_3)} + 2\sqrt{\frac{2}{r(1+r)}}\sin\left(\frac{90 - \Delta i_1 - \Delta i_3}{2}\right)}$$

$$\Delta\tilde{V}_{IB} = 2\left[\sqrt{\frac{2r}{1+r}} - 1 + \sqrt{\frac{2}{r(1+r)}}\sin\left(\frac{90^\circ}{2}\right)\right] \Rightarrow @\ \ell = 10 \rightarrow \Delta\tilde{V}_{IB} = 0.8875$$



A contour plot of the cost function is also provided above.

As expected, when the periapsis plane changes are small, the cost of the transfer is low. The lowest cost of the maneuver is 0.8851 which occurs when $\Delta i_1 = 1.4414°$ & $\Delta i_3 = 1.4414°$. Compared to the single apoapsis maneuver cost (0.8875), this is lower. That's why it's more optimal compared to the single apoapsis maneuver.

Problem 5

Problem5 → Eqn 13 → $\Delta V_{tot} = \sqrt{V_{1t}^2 + V_0^2 - 2V_{1t}V_0\cos(\eta\Delta i)} + \sqrt{V_f^2 + V_{2t}^2 - 2V_{2t}V_f\cos((1-\eta)\Delta i)}$

$$= \sqrt{\frac{2\mu r_2}{(r_1+r_2)r_1} + \frac{\mu}{r_1} - 2\sqrt{\frac{\mu^2}{r_1^2}\cdot\frac{2r_2}{r_1+r_2}}\cos(\eta\Delta i)} + \sqrt{\frac{2\mu}{r_1+r_2}\cdot\frac{r_1}{r_2} - \frac{2\mu r_1}{r_2(r_1+r_2)}\cos((1-\eta)\Delta i)}$$

Divide by v_0

$\lambda = \frac{r_2}{r_1} \geq 1$

$= \lambda$

$$\boxed{\Delta V_{tot} = \sqrt{\frac{2\lambda}{1+\lambda}+1-2\sqrt{\frac{2\lambda}{\lambda+1}}\cos(\eta\Delta i)} + \sqrt{\frac{4}{\lambda(1+\lambda)} - \frac{2}{\lambda(1+\lambda)}\cos((1-\eta)\Delta i)}}$$

$$\boxed{\Delta i = |i_f - i_0|}$$

$$F = \frac{\Delta i\, v_0 v_{1t}\sin(\eta^*\Delta i)}{\Delta V_1} - \frac{\Delta i\, v_f v_{2t}\sin((1-\eta^*)\Delta i)}{\Delta V_2} = 0$$

$$= \frac{\Delta i\sqrt{\frac{\mu}{r_1}\cdot\frac{2\mu}{r_1+r_2}\cdot\frac{r_2}{r_1}}\sin(\eta^*\Delta i)}{\Delta V_1} - \frac{\Delta i\left(\frac{2\mu}{r_1+r_2}\cdot\frac{r_1}{r_2}\right)\sin((1-\eta^*)\Delta i)}{\Delta V_2} = 0$$

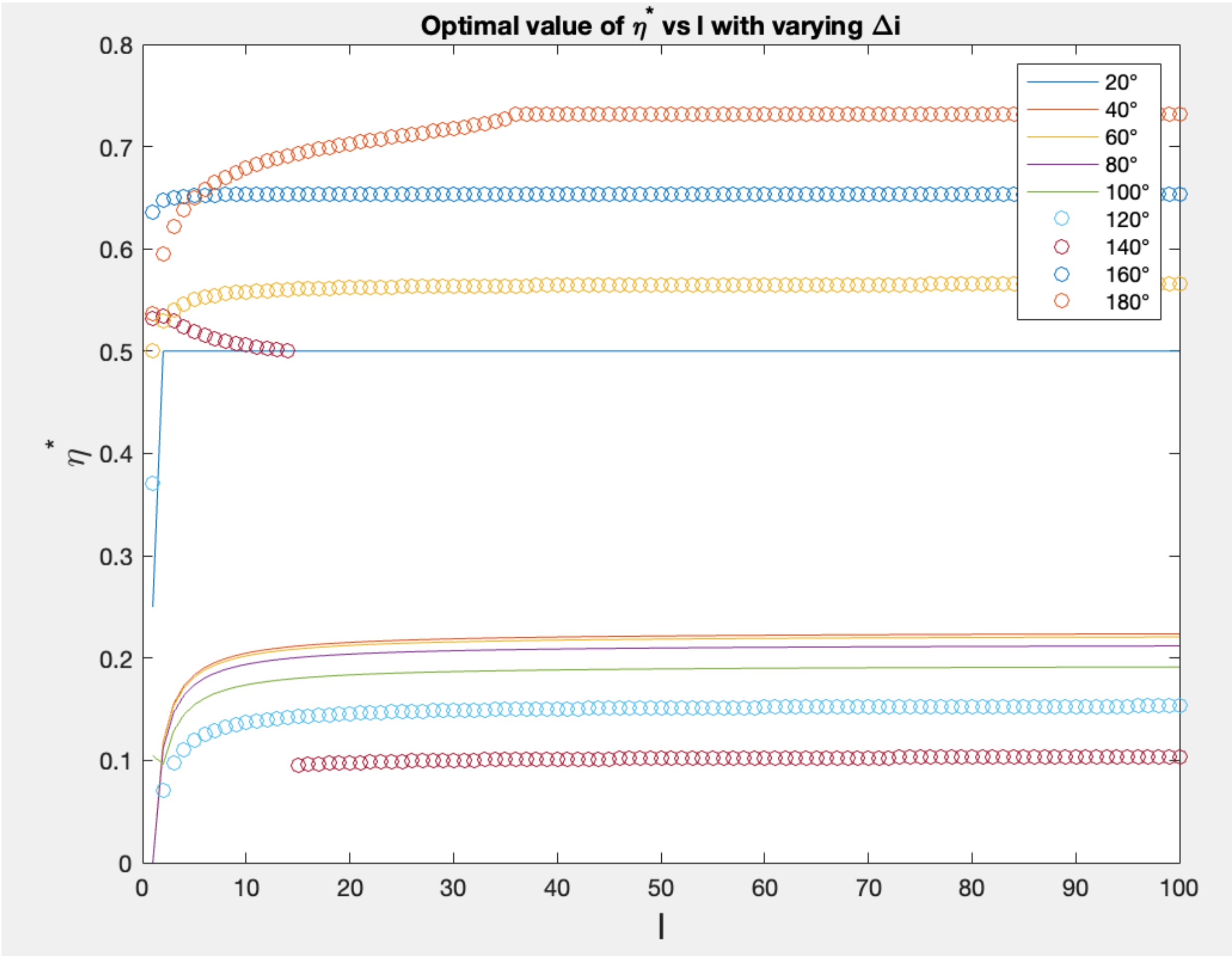
Multiply by $\frac{v_0}{v_0^2}$ →

$$\frac{\Delta i\,\cancel{\mu}\sqrt{\frac{2r_2}{r_1^3(r_1+r_2)}}\sin(\eta^*\Delta i)}{\sqrt{\frac{2\lambda}{1+\lambda}+1-\sqrt{\frac{2\lambda}{1+\lambda}}\cos(\eta^*\Delta i)}} - \frac{\Delta i\,\cancel{\mu}\left(\frac{2r_1}{r_2(r_1+r_2)}\right)\sin((1-\eta^*)\Delta i)}{\sqrt{\frac{4}{\lambda(1+\lambda)} - \frac{2}{\lambda(1+\lambda)}\cos[(1-\eta^*)\Delta i]}} = 0$$

$$\boxed{F = \frac{\Delta i\sqrt{\frac{2\lambda}{1+\lambda}}\sin(\eta^*\Delta i)}{\sqrt{\frac{2\lambda}{1+\lambda}+1-\sqrt{\frac{2\lambda}{1+\lambda}}\cos(\eta^*\Delta i)}} - \frac{\Delta i\left(\frac{2}{\lambda(1+\lambda)}\right)\sin((1-\eta^*)\Delta i)}{\sqrt{\frac{4}{\lambda(1+\lambda)} - \frac{2}{\lambda(1+\lambda)}\cos[(1-\eta^*)\Delta i]}} = 0}$$

Matlab’s fsolve() function is used to numerically compute the roots of the cost function.

The plot below shows the optima value η* for varying change in inclination (Δi). As seen, η* settles down as l increases for a particular change in inclination. The η* plots are not continuous as Matlab’s fsolve() function tries to numerically solve the roots for the cost function.



The optimal η* settles down as I increases. As seen, it also clearly depends on the Δi. When the parameters are known, this method can be used to find the optimal factor for the dog-leg maneuver.

Problem 6

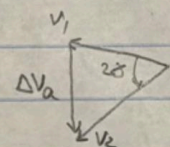
Problem 6 → $\omega = 0^\circ$

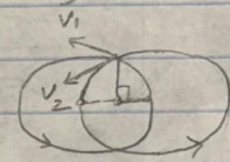
a) @ $\nu = 90^\circ$, $r(\nu) = p$, $V(\nu) = \sqrt{\frac{\mu}{p}}\sqrt{1+e^2}$, $\tan\gamma = e \rightarrow \gamma = \tan^{-1}(e)$

$V_1 = \sqrt{\frac{\mu}{p}}\sqrt{1+e^2}$, $V_2 = \sqrt{\frac{\mu}{p}}\sqrt{1+e^2}$

$\Delta V_a = \sqrt{V_1^2 + V_2^2 - 2V_1V_2\cos(2\gamma)}$

$= \sqrt{\frac{2\mu}{p}(1+e^2) - 2\frac{\mu}{p}(1+e^2)\cos(2\tan^{-1}(e))}$





$\cos(2\theta) = \frac{1-\tan^2(\theta)}{1+\tan^2(\theta)} \longrightarrow \cos(2\tan^{-1}(e)) = \frac{1-\tan^2(\mathrm{atan}(e))}{1+\tan^2(\mathrm{atan}(e))} = \frac{1-e^2}{1+e^2}$

$= \sqrt{\frac{2\mu}{p}(1+e^2)\left(1 - \frac{1-e^2}{1+e^2}\right)} = \sqrt{\frac{2\mu}{p}\cancel{(1+e^2)}\left(\frac{\cancel{1}+e^2\cancel{-1}+e^2}{\cancel{1+e^2}}\right)} = \sqrt{\frac{4\mu}{p}e^2} = \boxed{2\sqrt{\frac{\mu}{p}}e = \Delta V_a}$

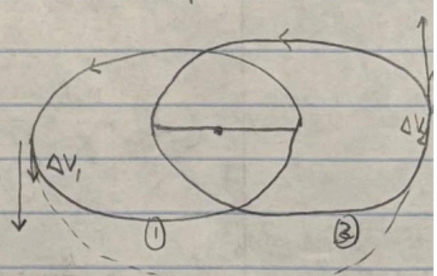
b) @ Apoapsis → $\nu = 180^\circ$, $r(\nu) = \frac{p}{1-e}$, $V(\nu) = \sqrt{\frac{\mu}{p}}\sqrt{1-2e+e^2} = (1-e)\sqrt{\frac{\mu}{p}}$, $\gamma = 0^\circ$

@ Apoapsis → $V_1 = \sqrt{\frac{\mu}{p}}(1-e)$,

$V_{1t} = V_{circ}$ @ $r(\nu) = \frac{p}{1-e} \rightarrow V_{circ} = V_{1t} = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{\mu}{p}}\sqrt{1-e}$

$\Delta V_1 = V_{1t} - V_1 = \sqrt{\frac{\mu}{p}}(\sqrt{1-e} - (1-e))$

$\Delta V_1 = \Delta V_2 \rightarrow \therefore \Delta V_b = 2\Delta V_1 = \boxed{2\sqrt{\frac{\mu}{p}}(\sqrt{1-e} - (1-e)) = \Delta V_b}$



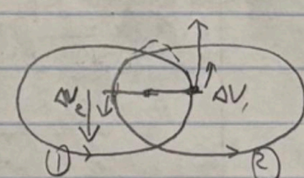
c) @ Periapsis → $\nu = 0^\circ$, $r(\nu) = \frac{p}{1+e}$, $V(\nu) = \sqrt{\frac{\mu}{p}}\sqrt{1+e}$, $\gamma = 0^\circ$

$V_1 = \sqrt{\frac{\mu}{p}}(1+e)$

$V_{1t} = V_{circ}$ @ $r(\nu) = \frac{p}{1+e} \rightarrow V_{circ} = V_{1t} = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{\mu}{p}}\sqrt{1+e}$

$\Delta V_1 = V_1 - V_{1t} = \sqrt{\frac{\mu}{p}}[(1+e) - \sqrt{1+e}]$

$\Delta V_1 = \Delta V_2 \rightarrow \therefore \Delta V_c = 2\Delta V_1 = \boxed{2\sqrt{\frac{\mu}{p}}[(1+e) - \sqrt{1+e}] = \Delta V_c}$



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d) $\Delta V_b \le \Delta V_a \rightarrow \sqrt{1-e} - 1 \cancel{+e} \le \cancel{e} \rightarrow \sqrt{1-e} \le 1$ $(\sqrt{1-e} \ge 0) \rightarrow$ square both sides

$1-e \le 1 \rightarrow e \ge 0$ — $\boxed{\Delta V_b \text{ is always better than } \Delta V_a}$ $(0 \le e \le 1)$

$\Delta V_b \le \Delta V_c \rightarrow \sqrt{1-e} - 1 + \cancel{e} \le 1 + \cancel{e} - \sqrt{1+e} \rightarrow \sqrt{1-e} \le -\sqrt{1+e} + 2$ (Both sides ≥ 0)

$\cancel{1} - e \le 4 - 4\sqrt{1+e} + \cancel{1} + e \rightarrow 2\sqrt{1+e} \le e + 2 \rightarrow$ Square $\rightarrow 4(1+e) \le e^2 + 4e + 4$

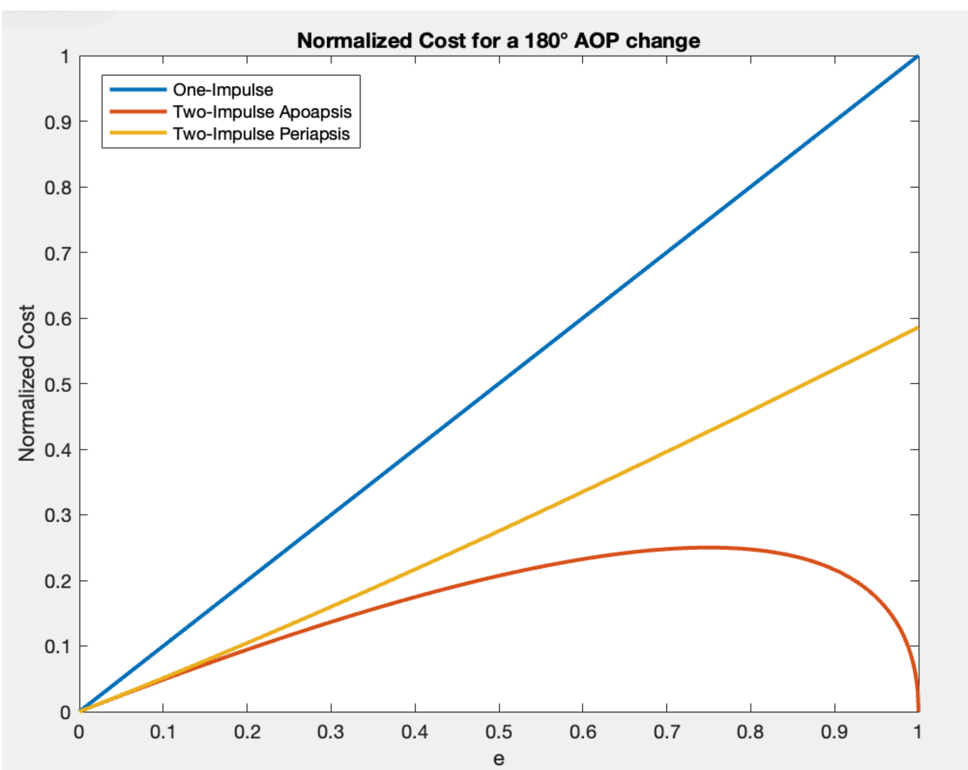
$\cancel{4} + \cancel{4e} \le e^2 + \cancel{4e} + \cancel{4} \rightarrow 0 \le e^2$ & $\because 0 \le e \le 1 \rightarrow e^2 \ge 0 \rightarrow \therefore$ $\boxed{\Delta V_b \text{ is always better than } \Delta V_a}$

$\Delta V_c \le \Delta V_a \rightarrow 1 + \cancel{e} - \sqrt{1+e} \le \cancel{e} \rightarrow 1 \le \sqrt{1+e} \rightarrow 1 \le 1+e \rightarrow e \ge 0$

$\because e$ is always ≥ 0 when $0 \le e \le 1 \rightarrow$ $\boxed{\Delta V_c \text{ is always better than } \Delta V_a}$

Finally – $\boxed{\Delta V_b \le \Delta V_c \le \Delta V_a}$

As seen, two-impulse periapsis is more optimal than two-impulse apoapsis and one-impulse maneuvers for this case. To further illustrate that, the normalized cost is computed at discrete e steps for all three maneuvers and it matches the analytical solution - two-impulse apoapsis is most optimal followed by two-impulse periapsis and followed by the one-impulse maneuver:



Problem 7

Problem 7 → 1-impulse → $\Delta V_a = V_\infty - V_{lc}$

2-impulse $(l<r)$ → $\Delta V_b = \Delta V_1 + \Delta V_2 = \sqrt{\frac{\mu}{r}} - \sqrt{\frac{2\mu}{r+l}\cdot\frac{l}{r}} + V_\infty - \sqrt{\frac{2\mu}{r+l}\cdot\frac{r}{l}}$

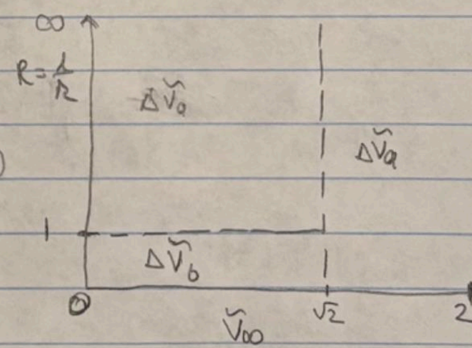
3-impulse $(l>r)$ → $\Delta V_c = \Delta V_1 + \Delta V_2 = \sqrt{\frac{2\mu}{r+l}\cdot\frac{l}{r}} - \sqrt{\frac{\mu}{r}} + V_\infty - \sqrt{\frac{2\mu}{r+l}\cdot\frac{r}{l}}$

let $R = l/r$, normalize by $V_{lc} = \sqrt{\frac{\mu}{r}}$

$\Delta\tilde{V}_a = \tilde{V}_\infty - 1$

$0 < R < 1$: $\Delta\tilde{V}_b = 1 - \sqrt{\frac{2R}{1+R}} + \tilde{V}_\infty - \sqrt{\frac{2}{R(1+R)}}$

$R > 1$: $\Delta\tilde{V}_c = \sqrt{\frac{2R}{1+R}} - 1 + \tilde{V}_\infty - \sqrt{\frac{2}{R(1+R)}}$



$\Delta\tilde{V}_a \le \Delta\tilde{V}_b \rightarrow \cancel{\tilde{V}_\infty} - 1 \le 1 - \sqrt{\frac{2R}{1+R}} + \cancel{\tilde{V}_\infty} - \sqrt{\frac{2}{R(1+R)}}$

$\sqrt{\frac{2R}{1+R}} + \sqrt{\frac{2}{R(1+R)}} \le 2$

$\frac{2R}{1+R} + 2\sqrt{\frac{4\cdot R^2}{R(1+R)^2}} + \frac{2}{R(1+R)} \le 2$

$\frac{2R^2+2}{R(1+R)} + \frac{4\cdot R}{(1+R)}\cdot\frac{1}{\sqrt{R}} \le 2$

$\rightarrow \boxed{R \ge \sqrt{2} \leftarrow \Delta\tilde{V}_a \le \Delta\tilde{V}_b}$

$\Delta\tilde{V}_a \le \Delta\tilde{V}_c \rightarrow \cancel{\tilde{V}_\infty}\cancel{-1} \le \sqrt{\frac{2R}{1+R}} \cancel{-1} + \cancel{\tilde{V}_\infty} - \sqrt{\frac{2}{R(1+R)}}$

$\boxed{\Delta\tilde{V}_a \text{ always} \le \Delta\tilde{V}_c}$

$\Delta\tilde{V}_b \le \Delta\tilde{V}_c \rightarrow 1 - \sqrt{\frac{2R}{1+R}} + \cancel{\tilde{V}_\infty} - \cancel{\sqrt{\frac{2}{R(1+R)}}} \le \sqrt{\frac{2R}{1+R}} - 1 + \cancel{\tilde{V}_\infty} - \cancel{\sqrt{\frac{2}{(R+1)R}}}$

$\sqrt{\frac{2R}{1+R}} \ge 1$ – always → $\therefore \boxed{\Delta\tilde{V}_b \le \Delta\tilde{V}_c}$