Rigid Body Kinematics

ASEN 5010

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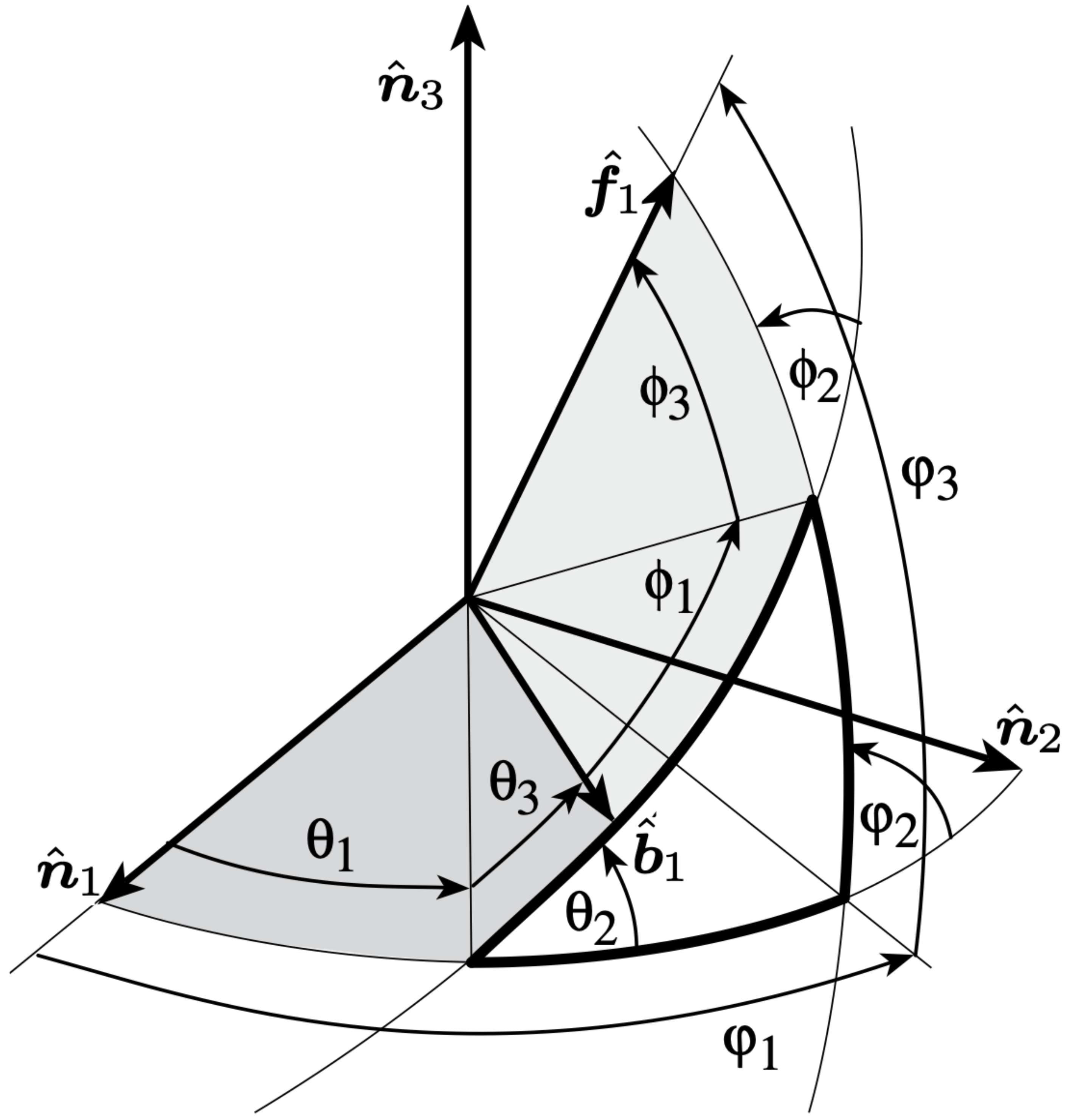
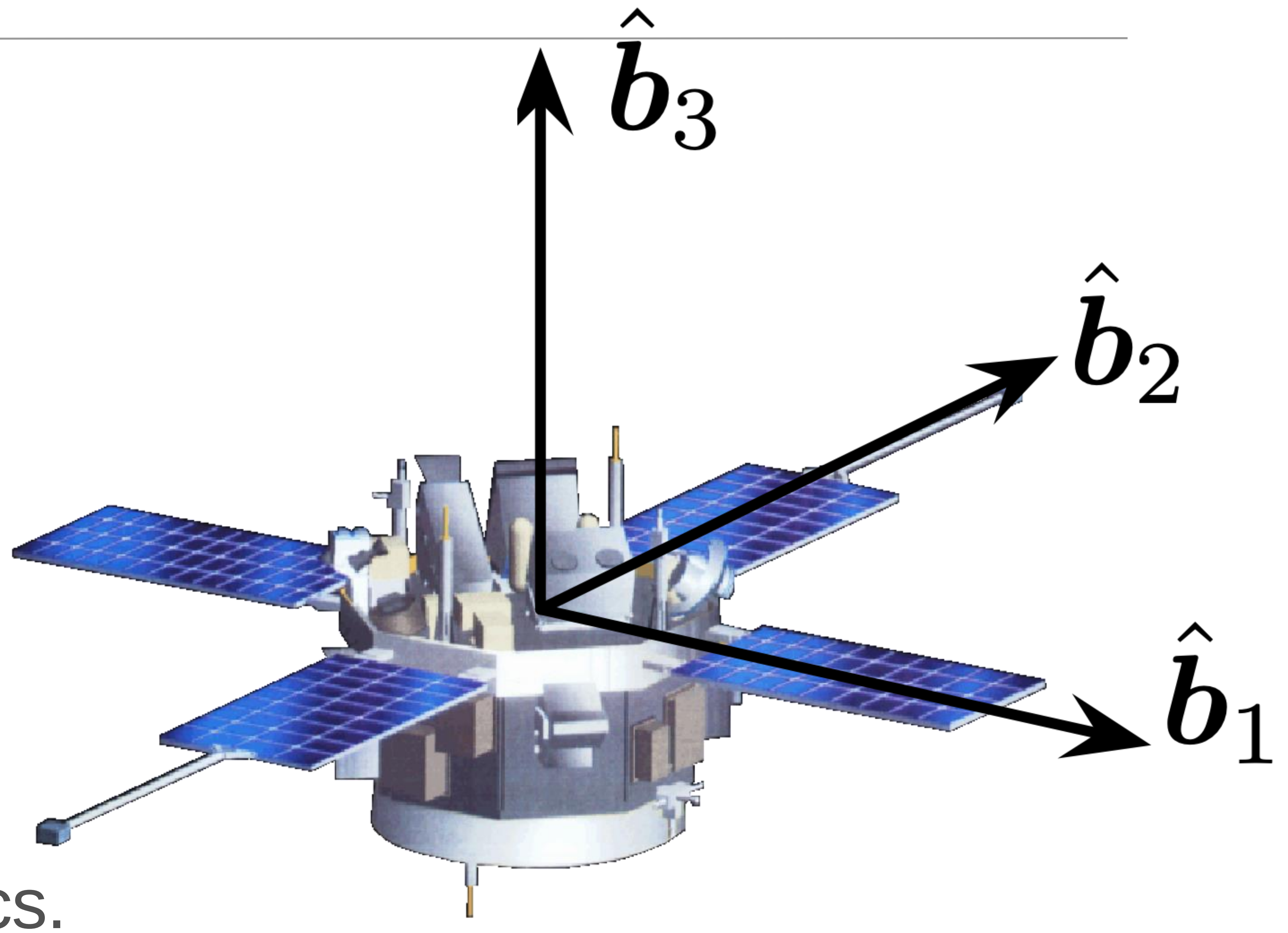
Outline

- Direction Cosine Matrix
- Euler Angle Sets
- Principal Rotation Parameters
- Euler Parameters (Quaternions)
- Classical Rodrigues Parameters
- Modified Rodrigues Parameters

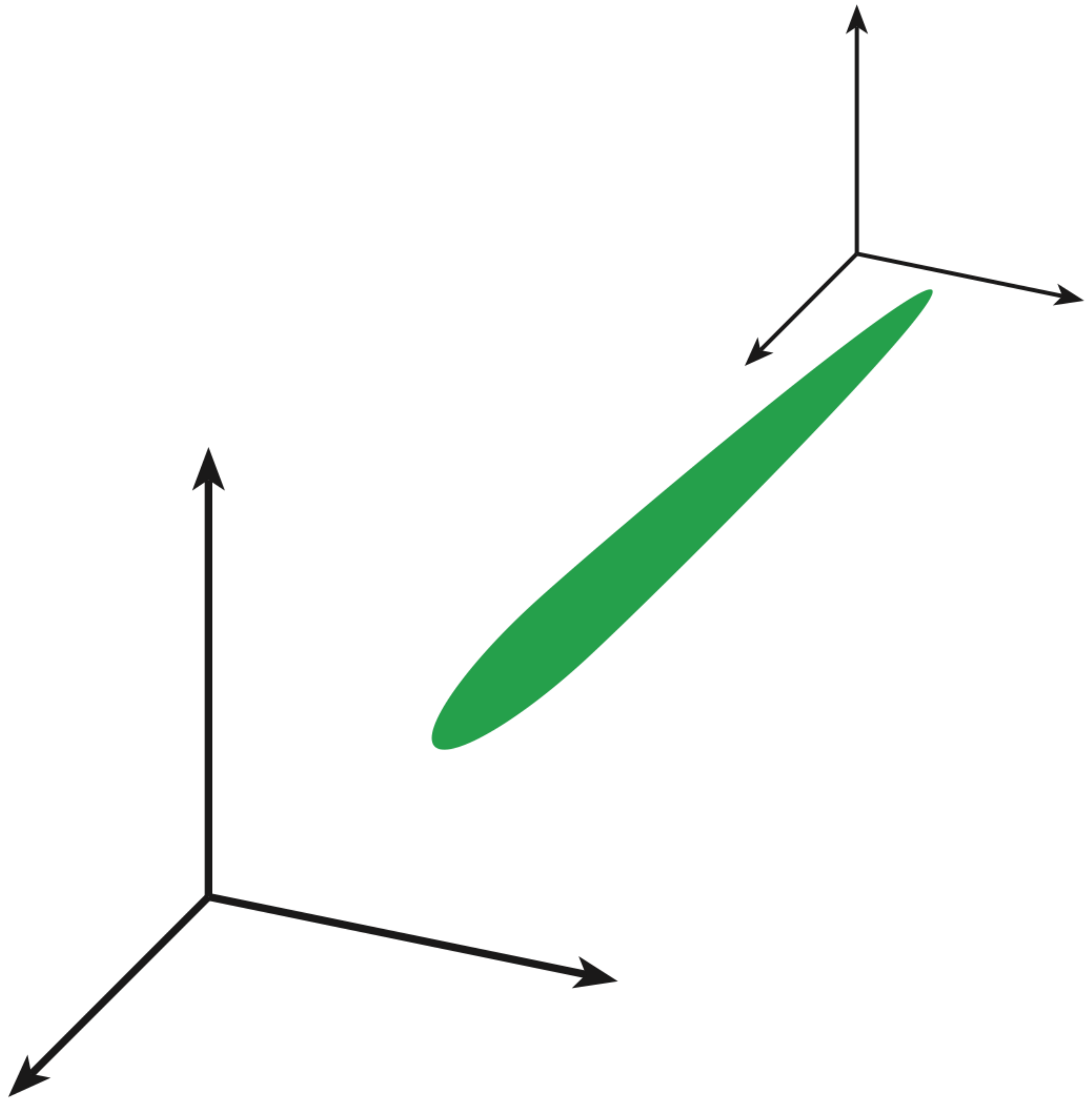


Introduction

- Attitude coordinates are set of coordinates that describe of both a rigid body or a reference frame
- An infinite number of coordinate choices exists, same as with position coordinates
- A good choice in attitude coordinates can greatly simplify the mathematics of the problem solving process
- A bad choice in attitude coordinates can introduce singularities in the attitude description, as well as highly nonlinear mathematics.



Relation to Position Coordinates



Translational errors can grow infinitely large!

Attitude errors can to grow to 180˚!

4 “Truths” about Attitude Coordinates

- A minimum of **three coordinates** is required to describe the relative angular displacement between two reference frames.
- Any minimal set of three coordinates will contain at least one geometrical orientation where the coordinates are **singular**, namely at least two coordinates are **undefined** or not unique.
- At or near such a geometric singularity, the corresponding **kinematic differential equations are also singular**.
- The geometric singularities and associated numerical difficulties can be avoided altogether through regularization. **Redundant sets of four or more coordinates exist that are universally valid**.

Direction Cosine Matrix

The mother of all attitude parameterizations...

Coordinate Frames

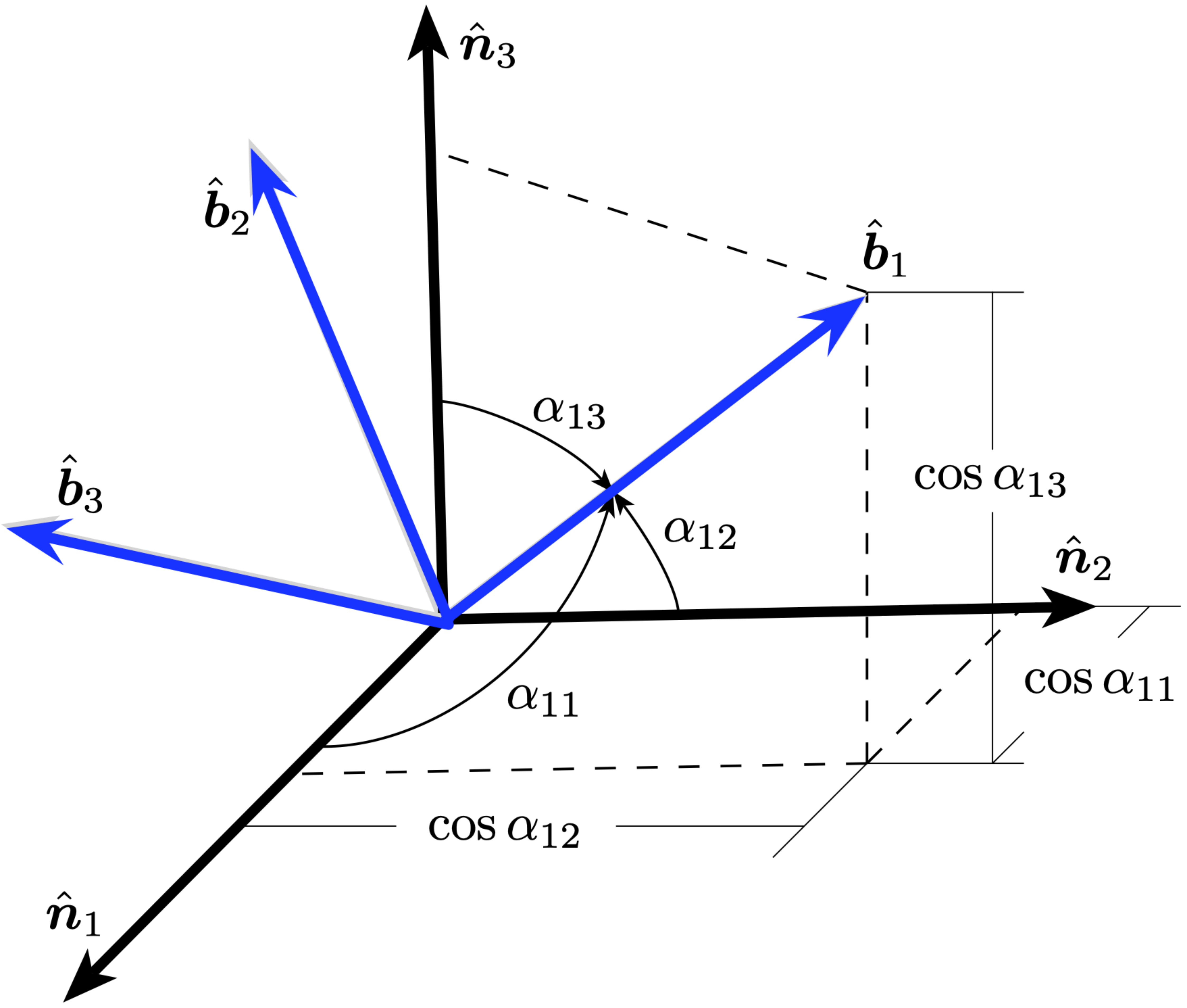
- A vectrix is a matrix of vectors.

$$\{\hat{\boldsymbol{n}}\} \equiv \begin{bmatrix} \hat{\boldsymbol{n}}_1 \\ \hat{\boldsymbol{n}}_2 \\ \hat{\boldsymbol{n}}_3 \end{bmatrix}$$

$$\{\hat{\boldsymbol{b}}\} \equiv \begin{bmatrix} \hat{\boldsymbol{b}}_1 \\ \hat{\boldsymbol{b}}_2 \\ \hat{\boldsymbol{b}}_3 \end{bmatrix}$$

$${}^{N}\underline{r} = 3\hat{n}_1 + 4\hat{n}_2 + \hat{n}_3$$

$$[3, 4, 1] \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix}$$



Coordinate Frames

$$\begin{Bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix}$$

DCM

$1^2+0^2+0^2 = 1$

$$\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix}$$

Frame base vectors are related through:

$$\hat{\boldsymbol{b}}_1 = \cos\alpha_{11}\hat{\boldsymbol{n}}_1 + \cos\alpha_{12}\hat{\boldsymbol{n}}_2 + \cos\alpha_{13}\hat{\boldsymbol{n}}_3$$
$$\hat{\boldsymbol{b}}_2 = \cos\alpha_{21}\hat{\boldsymbol{n}}_1 + \cos\alpha_{22}\hat{\boldsymbol{n}}_2 + \cos\alpha_{23}\hat{\boldsymbol{n}}_3$$
$$\hat{\boldsymbol{b}}_3 = \cos\alpha_{31}\hat{\boldsymbol{n}}_1 + \cos\alpha_{32}\hat{\boldsymbol{n}}_2 + \cos\alpha_{33}\hat{\boldsymbol{n}}_3$$

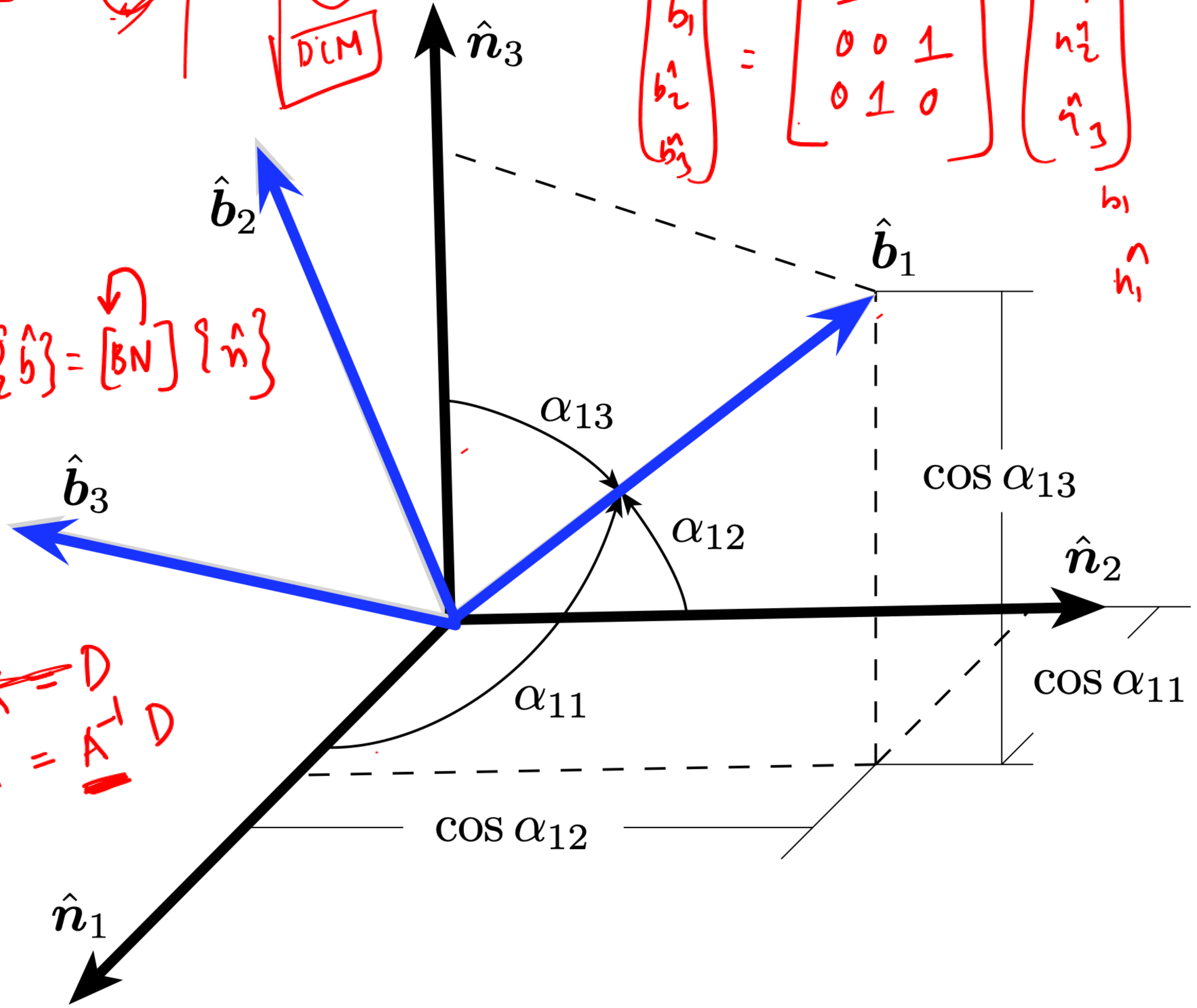
$$\boxed{\{\hat{\boldsymbol{b}}\}} = \underbrace{\begin{bmatrix} \cos\alpha_{11} & \cos\alpha_{12} & \cos\alpha_{13} \\ \cos\alpha_{21} & \cos\alpha_{22} & \cos\alpha_{23} \\ \cos\alpha_{31} & \cos\alpha_{32} & \cos\alpha_{33} \end{bmatrix}}_{(3\times3)}^{[C]} \boxed{\{\hat{\boldsymbol{n}}\}} = [C]\{\hat{\boldsymbol{n}}\}$$

$\{\hat{b}\} = [BN]\{\hat{n}\}$

Note that: $C_{ij} = \cos(\angle\, \hat{\boldsymbol{b}}_i, \hat{\boldsymbol{n}}_j) = \hat{\boldsymbol{b}}_i \cdot \hat{\boldsymbol{n}}_j$

$AX = D$

$X = A^{-1} D$



b_1
$\hat{n}_1$

Analogously, we can find:

$$\{\hat{\boldsymbol{n}}\} = \begin{bmatrix} \cos\alpha_{11} & \cos\alpha_{21} & \cos\alpha_{31} \\ \cos\alpha_{12} & \cos\alpha_{22} & \cos\alpha_{32} \\ \cos\alpha_{13} & \cos\alpha_{23} & \cos\alpha_{33} \end{bmatrix} \{\hat{\boldsymbol{b}}\} = \boxed{[C]^T}\{\hat{\boldsymbol{b}}\}$$

Matrix Inverse

$\{\hat{n}\} = [C]^T \{\hat{b}\}$

$\{\hat{b}\} = [C]\{\hat{n}\}$

Combining these two results, we find

$$\{\hat{\boldsymbol{b}}\} = \boxed{[C][C]^T}\{\hat{\boldsymbol{b}}\} \quad \Rightarrow \quad [C][C]^T = [I_{3\times3}]$$

$$\{\hat{\boldsymbol{n}}\} = [C]^T[C]\{\hat{\boldsymbol{n}}\} \quad \Rightarrow \quad [C]^T[C] = [I_{3\times3}]$$

Therefore, the inverse of a direction cosine matrix is simply the transpose operation.

$$[C]^{-1} = [C]^T$$

Orthogonal matrix

Orthonormal

Coordinate Frame Transformation

- Let a vector have its components taken in the body frame *B* or the inertial frame *N*:

$$\boldsymbol{v} = v_{b_1}\hat{\boldsymbol{b}}_1 + v_{b_2}\hat{\boldsymbol{b}}_2 + v_{b_3}\hat{\boldsymbol{b}}_3 = \{v_b\}^T\{\hat{\boldsymbol{b}}\}$$

$$\boldsymbol{v} = {}^{B}\boldsymbol{v} = {}^{N}\boldsymbol{v}$$

- we can now rearrange the vector expression as

$$\boldsymbol{v} = v_{n_1}\hat{\boldsymbol{n}}_1 + v_{n_2}\hat{\boldsymbol{n}}_2 + v_{n_3}\hat{\boldsymbol{n}}_3 = \{v_n\}^T\{\hat{\boldsymbol{n}}\}$$

- Equating components, we find that the two vector component sets must be related through

$$\boldsymbol{v} = \{v_n\}^T\{\hat{\boldsymbol{n}}\} = \{v_n\}^T[C]^T\{\hat{\boldsymbol{b}}\} = \{v_b\}^T\{\hat{\boldsymbol{b}}\}$$

$$\{\hat{n}\} = [C]^T\{\hat{b}\}$$

- From here on, we will make use of the short-hand notation:

$$\boldsymbol{v}_b = [C]\boldsymbol{v}_n \qquad \boldsymbol{v}_n = [C]^T\boldsymbol{v}_b$$

$${}^{B}\underline{v} = [C]\,{}^{N}\underline{v}$$

$${}^{\mathcal{B}}\boldsymbol{v} \equiv \boldsymbol{v}_b \qquad {}^{\mathcal{N}}\boldsymbol{v} \equiv \boldsymbol{v}_n$$

$$\boxed{{}^{B}\underline{v} = [BN]\,{}^{N}\underline{v}}$$

DCM Determinant

- Let's find the determinant of the [*C*] by first evaluating

$$\det\left(CC^T\right) = \det\left([I_{3\times 3}]\right) = 1$$

- Since [*C*] is a square matrix, we find that

$$\det(C)\,\det(C^T) = 1$$

- Because det([*C*]) is the same as det([*C*]^T), this is further reduced to

$$\left(\det(C)\right)^2 = 1 \iff \boxed{\det(C) = \pm 1}$$

- Note that this is true for any orthogonal matrix.

- For a proper rotation matrix with right-handed coordinate system, then $\boxed{\det(C) = +1.}$

$$\begin{bmatrix} \alpha_1 \\ \beta_1 \\ \gamma_2 \end{bmatrix} + \begin{bmatrix} \alpha_2' \\ \beta_1' \\ \gamma_2' \end{bmatrix} =$$

$\det(x) = +1$

Adding DCM's

$[RN]\ \cancel{[\,]}\ [RB]$

$\boxed{BN = [RB]^T [RN]}$

$\omega_{B/N} = -\omega_{R/B} + \omega_{R/N}$

- Assume three coordinate frames given: $\mathcal{N} : \{\hat{\boldsymbol{n}}\}$ $\mathcal{B} : \{\hat{\boldsymbol{b}}\}$ $\mathcal{R} : \{\hat{\boldsymbol{r}}\}$
- Let *N* and *B* frame orientation be related through $\boxed{\{\hat{\boldsymbol{b}}\} = [C]\{\hat{\boldsymbol{n}}\}}$
- Let *R* and *B* frame orientation be related through $\{\hat{\boldsymbol{r}}\} = [C']\{\hat{\boldsymbol{b}}\}$

$\{\hat{b}\} = [BN]\{\hat{n}\}$

$\{\hat{r}\} = [RB]\{\hat{b}\}$

- Then the *R* and *N* frame orientation are directly related through $\{\hat{\boldsymbol{r}}\} = [C'][C]\{\hat{\boldsymbol{n}}\} = [C'']\{\hat{\boldsymbol{n}}\}$
- Let us introduce the two-letter DCM notation [*NB*] as mapping from *B* to *N* frame, then the DCM addition is

$$\boxed{[RN] = [RB][BN]}$$

$\{\hat{r}\} = [RN]\{\hat{n}\}$

$\{\hat{r}\} = [RB][BN]\{\hat{n}\}$

Kinematic Differential Equation

- What does this mean??
  - kinematic ➠ position description

what is $$\boxed{[\dot{C}] = \frac{\mathrm{d}}{\mathrm{d}t}[C]}$$

  - differential equation ➠ time rate equation
- How does the [*C*] direction cosine matrix evolve over time. The rotation rate of a rigid body is expressed through the body angular velocity vector:

$$\boxed{\boldsymbol{\omega} = \omega_1 \hat{\boldsymbol{b}}_1 + \omega_2 \hat{\boldsymbol{b}}_2 + \omega_3 \hat{\boldsymbol{b}}_3}$$

- This vector determines how a body will rotate, and thus also how the DCM describing the orientation will evolve.

Kinematic Differential Equation

- Let's study how the body frame vectors will evolve over time as seen by the inertial frame. To do so, we differentiate the vectrix of body frame orientation vectors.

$$\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{b}}_i = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{b}}_i + \underline{\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \hat{\boldsymbol{b}}_i}$$

$\boxed{\underline{x} \times 0}$

- Let us introduce the matrix cross-product operator: $[\tilde{\boldsymbol{x}}] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$ where $\boldsymbol{x} \times \boldsymbol{y} \equiv [\tilde{\boldsymbol{x}}]\boldsymbol{y}$ and $[\tilde{\boldsymbol{x}}]^T = -[\tilde{\boldsymbol{x}}]$

- The body frame vectrix differential equation is then simply

$$\boxed{\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\{\hat{\boldsymbol{b}}\} = -[\tilde{\boldsymbol{\omega}}]\{\hat{\boldsymbol{b}}\}}$$

$[\tilde{x}]^T = -[\tilde{x}]$ → Skew symmetric

$[X]^T = [X]$ Symmetric

Kinematic Differential Equation

- Next take the inertial derivative of $\boxed{\{\hat{\boldsymbol{b}}\} = [C]\{\hat{\boldsymbol{n}}\}}$

$$\frac{{}^N\mathrm{d}}{\mathrm{d}t}\{\hat{\boldsymbol{b}}\} = \frac{{}^N\mathrm{d}}{\mathrm{d}t}\left([C]\{\hat{\boldsymbol{n}}\}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left([C]\right)\{\hat{\boldsymbol{n}}\} + [C]\cancelto{0}{\frac{{}^N\mathrm{d}}{\mathrm{d}t}\left(\{\hat{\boldsymbol{n}}\}\right)} = [\dot{C}]\{\hat{\boldsymbol{n}}\}$$

$$\frac{{}^N\mathrm{d}}{\mathrm{d}t}\{\hat{\boldsymbol{b}}\} = -[\tilde{\boldsymbol{\omega}}]\{\hat{\boldsymbol{b}}\} = \boxed{-[\tilde{\boldsymbol{\omega}}][C]\{\hat{\boldsymbol{n}}\} = [\dot{C}]\{\hat{\boldsymbol{n}}\}}$$

- This leads to

$$\left([\dot{C}] + [\tilde{\boldsymbol{\omega}}][C]\right)\{\hat{\boldsymbol{n}}\} = 0$$

$\{\hat{n}\}$

- Since this must be true for any *N* frame orientation, we find

$$\boxed{[\dot{C}] = -[\tilde{\boldsymbol{\omega}}][C]}$$

$\checkmark\checkmark$ $\frac{d}{dt}[BN] = -[\tilde{\omega}_{B/N}][BN]$

B frame coordinatization only

Kinematic Differential Equation

- An interesting fact is that this matrix differential equation holds for *any NxN orthogonal matrix*!

$$\frac{\mathrm{d}}{\mathrm{d}t}\left([C][C]^T\right) = [\dot{C}][C]^T + [C][\dot{C}]^T = 0$$

using the differential equation $[\dot{C}] = -[\tilde{\boldsymbol{\omega}}][C]$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left([C][C]^T\right) = -[\tilde{\boldsymbol{\omega}}][C][C]^T - [C][C]^T[\tilde{\boldsymbol{\omega}}]^T$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left([C][C]^T\right) = -[\tilde{\boldsymbol{\omega}}] + [\tilde{\boldsymbol{\omega}}] = 0$$