ASEN 5044, Fall 2024

Statistical Estimation for Dynamical Systems

Lecture 19:
CT and DT Stochastic Linear Systems

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Announcements

Fri (tomorrow)

- Homework 5: due ~~today~~ on Canvas
- Homework 6 posted today – due in 2 weeks
- ~~Quiz 6 posted due Tues (1 extra day)~~ Quiz 6 Postponed until next week
- Midterm 1: being graded...
- Special Topic Lecture #4 posted tomorrow:
  Intro to Bayesian Estimation Theory
- Quick pitch/info for ASEN 5519: Autonomous Bayesian Reasoning course (Spring 2025)

Last Time

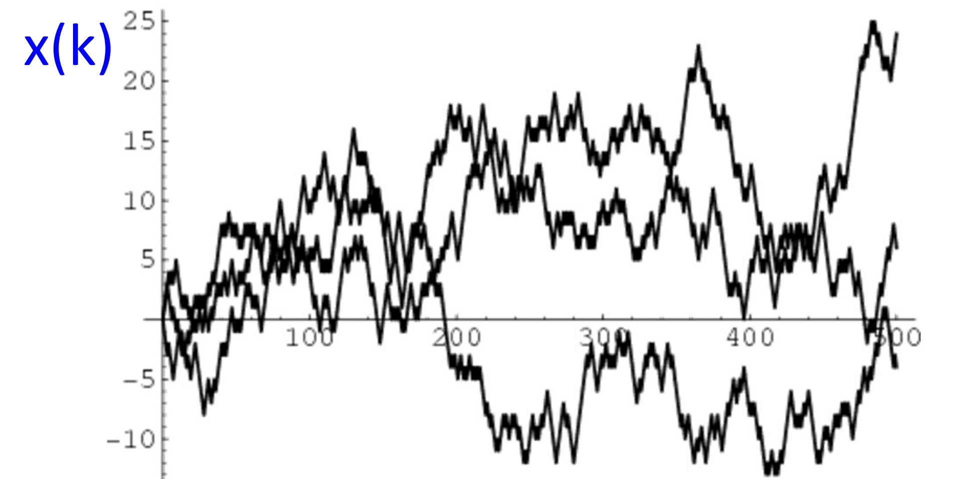
- Stochastic processes
- White noise processes
- DT stochastic processes: white noise sequences, Markov sequences

Example Markov Sequence: DT Random Walk

- Suppose we have a white noise sequence W = {…,w(k),…} with Q(k)=Q, i.e.

let $x(0) = 0$, and $x(k+1) = x(k) + w(k)$

$$\rightarrow \text{clearly } x(k) = \sum_{j=0}^{k-1} w(j) \quad \text{(accumulating noise)}$$



https://mathworld.wolfram.com/RandomWalk1-Dimensional.html

Suppose we want to predict $x(k{+}1)$ given only $\{x(k), w(1), w(2), \cdots, w(k-1)\}$

$\rightarrow$ Then only $x(k)$ needed, since it already has all info from $w(1), w(2), \cdots, w(k-1)$

$\rightarrow$ Therefore: $p(x(k+1)|x(k), w(1), \cdots, w(k-1))) = p(x(k+1)|x(k))$

$\rightarrow X = \{\cdots, x(1), x(2), \cdots, x(k), \cdots\}$ is a *Markov sequence*

(Is this sequence ergodic? Is it stationary?)

Today

<u>Today:</u>

- CT and DT linear stochastic systems driven by white noise
- Converting CT white process/measurement noise specs to DT equivalent specs
- Additive white Gaussian noise (AWGN)
- Van Loan’s method for converting CT LTI process noise to DT LTI process noise
- CT to DT white noise transformation example

READ SIMON BOOK, CHAPTER 3.2 - 3.3

General DT Linear Stochastic Systems

- Given CT LTI model (A,B,C,D,Γ):

$\Gamma \in \mathbb{R}^{n \times n_w}$ "process noise to state matrix"

$W \in \mathbb{R}^{n_w \times n_w}$

$$\dot{x} = Ax(t) + Bu(t) + \boxed{\Gamma \tilde{w}(t)}$$

$$y(t) = Cx(t) + Du(t) + \boxed{\tilde{v}(t)}$$

proc. noise PSD intensity

Dirac delta

$\mathbb{R}^{n_w}$ (n_w = # of proc. noise inputs)

$$\tilde{w}(t) = \text{process noise (white: } E[\tilde{w}(t)] = 0,\ E[\tilde{w}(t)\tilde{w}^T(\tau)] = W \cdot \delta(t-\tau))$$

$$\tilde{v}(t) = \text{measurement noise (white: } E[\tilde{v}(t)] = 0,\ E[\tilde{v}(t)\tilde{v}^T(\tau)] = V \cdot \delta(t-\tau))$$

$\in \mathbb{R}^p$

$V \in \mathbb{R}^{p \times p}$ meas. noise PSD intensity

- Want to convert to DT LTI model (F,G,H,M) with ZOH sample time Δt:

$Q \in \mathbb{R}^{n \times n}$

$x \in \mathbb{R}^n \leftarrow$
$$x(k+1) = Fx(k) + Gu(k) + w(k)$$
$\rightarrow w(k) \in \mathbb{R}^n$

$y \in \mathbb{R}^p \leftarrow$
$$y(k) = Hx(k) + Mu(k) + v(k)$$

DT proc. covariance (intensity)

$\in \mathbb{R}^p$

$$w(k) = \text{process noise (white: } E[w(k)] = 0,\ E[w(k)w^T(j)] = Q \cdot \delta(k,j))$$

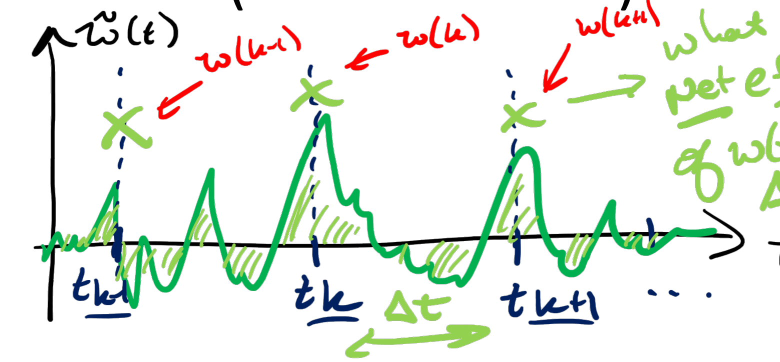
$\in \mathbb{R}^{p \times p}$

$$v(k) = \text{measurement noise (white: } E[v(k)] = 0,\ E[v(k)v^T(j)] = R \cdot \delta(k,j))$$

DT meas. covar. (intensity)

Converting CT to DT Linear Stochastic Systems

- We already know how to get deterministic parts described by (F,G,H,M) matrices (**see Lecture 6**)
- But how to describe random parts w(k) and v(k)? $\leftrightarrow$ $\dot{x} = Ax + Bu + \Gamma\tilde{w}(t)$ $\downarrow$ $x_{k+1} = Fx_k + Gu_k + w_k$
- Start by looking at mean of w(k), i.e. E[w(k)] = ?



what is net effect of $w(t)$ over Δt intervals

Note: over a given Δt sampling window for $t_k \leq t \leq t_{k+1}$:

$$\rightarrow w(k) = w(t_k) = \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}\,\Gamma\,\tilde{w}(\tau)\,d\tau$$ = cumulative effect of CT process noise $\tilde{w}(t)$ on the state x as we move from $t = t_k \rightarrow t = t_{k+1}$

[looks very similar to $\int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)} B\,u(\tau)\,d\tau$ for CT $\rightarrow$ DT control input effect conversion $\rightarrow$ i.e. how we got G matrix w/ ZOH!]

(*) except: we cannot use ZOH assumption on $\tilde{w}(t)$: b/c it's white & b/c it's noise $\rightarrow$ can't [RV!] control/predict it!)

$\rightarrow$ But: if $\tilde{w}(t)$ is white noise, then it follows:

$$\underset{n\times 1}{E[w(t_k)]} = E\left[\int_{t_k}^{t_{k+1}} \underset{n\times n}{e^{A(t_{k+1}-\tau)}}\,\underset{n\times n_w}{\Gamma}\,\underset{n_w\times 1}{\tilde{w}(\tau)}\,d\tau\right] = \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}\,\Gamma\cdot \underbrace{E[\tilde{w}(\tau)]}_{=0,\text{ b/c it's white noise!}}\,d\tau = 0$$

(Remember $\tilde{w}(t) \in \mathbb{R}^{n_w}$, $w(k) \in \mathbb{R}^{n\times 1}$

$$\longrightarrow \boxed{E[w(t_k)] = E[w(k)] = 0} \quad (*)$$

Converting CT to DT Linear Stochastic Systems

- Now look at covariance of w(k) – need $E[w(k)w(j)^T]$ (b/c $\text{cov}(w_k w_j^T) = E[w_k w_j^T] - E[\cancel{w_k}]\cdot E[\cancel{w_j^T}]$) with $E[w_k] = 0$, $E[w_j^T] = 0$

$$\rightarrow \text{So: } E[w_k w_j^T] = E\left[\int_{t_j}^{t_{j+1}}\int_{t_k}^{t_{k+1}} \underbrace{e^{A(t_{k+1}-\tau_1)} \cdot \Gamma\, \tilde{w}(\tau_1)} \cdot \underbrace{\tilde{w}^T(\tau_2)\, \Gamma^T e^{A^T(t_{j+1}-\tau_2)}}\, d\tau_1\, d\tau_2\right]$$

use linearity of expectation

$$= \int_{t_j}^{t_{j+1}}\int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau_1)} \cdot \underset{n \times n_w}{\Gamma} \cdot \underbrace{E[\tilde{w}(\tau_1)\cdot \tilde{w}^T(\tau_2)]} \cdot \underset{n_w \times n}{\Gamma^T} e^{A^T(t_{j+1}-\tau_2)}\, d\tau_1\, d\tau_2$$

$= \underline{W} \cdot \delta(\tau_1 - \tau_2)$ from def. of autocovr. of white proc. noise

CT PSD intensity ($n_w \times n_w$ matrix) — Dirac

$$= \int_{t_j}^{t_{j+1}}\int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau_1)}\, \Gamma\, \underline{W}\, \Gamma^T e^{A^T(t_{j+1}-\tau_2)} \cdot \delta(\tau_1 - \tau_2)\, d\tau_1\, d\tau_2$$

use sifting property of Dirac delta fcn to get rid of τ_2

$$= \underset{[\text{Kronecker}]}{\delta_{jk}} \cdot \underbrace{\int_{t_k}^{t_{k+1}} \underbrace{\underset{[n \times n]}{e^{A(t_{k+1}-\tau_1)}} \cdot \underset{[n \times n_w]}{\Gamma}\, \underset{[n_w \times n_w]}{\underline{W}}\, \underset{[n_w \times n]}{\Gamma^T}\, \underset{[n \times n]}{e^{A^T(t_{k+1}-\tau_1)}}}_{n \times n}\, d\tau_1}_{= Q \in \mathbb{R}^{n \times n}}$$

$$\Rightarrow \boxed{E[w_k w_j^T] = \delta_{jk} \cdot \underset{\in \mathbb{R}^{n \times n}}{Q}}$$ (makes sense since $w(k), w(j) \in \mathbb{R}^n$)

(*)