Problem 1, 2, 3a

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ASEN 5044
Fall 2024
Josh Bhalavat

HW 4

Problem 1 → Total cards = 52.

Ideal → Ace of any suit & any face card / 10 of any suit

Event A → P(Ace of any suit) = $4/52 = P(A)$

Event B → Total of face cards + 10s = $(3)(4) + (1)(4) = 16 \rightarrow P(B) = 16/52$

Assuming an event occurs, the total # of cards decreases to 51.

option 1 → A happens first, then B → $P(A) \cdot P(B) = \frac{4}{52} \cdot \frac{16}{51} = 0.024133 = 2.4133\%$

option 2 → B happens first, then A → $P(B) \cdot P(A) = \frac{16}{52} \cdot \frac{4}{51} = 0.024133 = 2.4133\%$

Ideal combination is option 1 or option 2

∴ Probability of ideal combination = P(option 1) + P(option 2) $\boxed{= \frac{128}{2652} = 4.8265\%}$

Problem 2 → 2.3

$$f_X(x) = \begin{cases} ax(1-x) & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

A valid probability density function must always have → $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

∵ $f_X(x) \neq 0$ for $x \in [0,1] \rightarrow \int_0^1 f_X(x)\,dx = 1$

$\int_0^1 f_X(x)\,dx = \int_0^1 ax - ax^2\,dx = \frac{1}{2}ax^2 - \frac{1}{3}ax^3 \Big|_0^1 = \frac{a}{2} - \frac{a}{3} = \frac{a}{6} = 1 \rightarrow \boxed{a = 6}$

Problem 3 → 2.5

$$f_X(x) = \begin{cases} ae^{-ax} & x > 0 \\ 0 & x \leq 0 \end{cases}, \quad a \geq 0$$

a] cumulative distribution function: $F(-\infty \leq X \leq \xi) = \int_{-\infty}^{\xi} f_X(x)\,dx = c(\xi)$

for $x \leq 0$ $\quad F(x) = \int_{-\infty}^{0} 0\,dx = 0$

for $x > 0$ $\quad F(x) = \underbrace{\int_{-\infty}^{0} 0\,dx}_{0} + \int_0^x ae^{-at}\,dt = a\int_0^x e^{-at}\,dt = a\left[-\frac{1}{a}e^{-at}\right]_0^x$

$= a\left[-\frac{1}{a}e^{-ax} + \frac{1}{a}\right] = -e^{-ax} + 1$

$$\boxed{F(x) = \begin{cases} -e^{-ax} + 1 & x > 0 \\ 0 & x \leq 0 \end{cases}}$$

Problem 3b, 3c, 3d, 3e

b] $\bar{x} = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^{\infty} x a e^{-ax}\,dx = a\int_0^{\infty} x e^{-ax}\,dx$

Let $f = x$, $g' = e^{-ax}dx \rightarrow f' = dx$, $g = -\frac{1}{a}e^{-ax} \rightarrow \int fg' = fg - \int f'g$

$a\int_0^{\infty} x e^{-ax} = a\left[x\left(-\frac{1}{a}e^{-ax}\right) - \int_0^{\infty} -\frac{1}{a}e^{-ax}dx\right] = a\left[-\frac{x}{a}e^{-ax}\Big|_0^{\infty} + \frac{1}{a}\int_0^{\infty} e^{-ax}\,dx\right]$

$= a\left[-\frac{x}{a}e^{-ax}\Big|_0^{\infty} + \frac{1}{a}\left(-\frac{1}{a}e^{-ax}\right)\Big|_0^{\infty}\right] = -xe^{-ax}\Big|_0^{\infty} - \frac{1}{a}\left[\left(\cancel{\lim_{x\to\infty} e^{-ax}}\right)^{0} - e^0\right]$

$= -xe^{-ax}\Big|_0^{\infty} + \frac{1}{a} = \cancel{\lim_{x\to\infty}\left(-xe^{-ax}\right)}^{0} - 0 + \frac{1}{a}$

$\boxed{\bar{x} = \frac{1}{a}}$

c] $E[x^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = a\int_0^{\infty} x^2 e^{-ax}\,dx$

Let $f = x^2$, $g' = e^{-ax}dx \rightarrow f' = 2x\,dx$, $g = -\frac{1}{a}e^{-ax} \rightarrow \int fg' = fg - \int f'g$

$E[x^2] = a\left[x^2\left(-\frac{1}{a}e^{-ax}\right)\Big|_0^{\infty} + \frac{1}{a}\int_0^{\infty} 2x e^{-ax}\,dx\right]$

$= \cancel{-\frac{x^2}{a}e^{-ax}\Big|_0^{\infty}}^{0} + \frac{2}{a}\int_0^{\infty} x e^{-ax}\,dx$

Let $f = x$, $g' = e^{-ax}dx \rightarrow f' = dx$, $g = -\frac{1}{a}e^{-ax} \rightarrow \int fg' = fg - \int f'g$

$E[x^2] = \frac{2}{a}\left[\cancel{x\left(-\frac{1}{a}e^{-ax}\right)\Big|_0^{\infty}}^{0} + \frac{1}{a}\int_0^{\infty} e^{-ax}\,dx\right]$

$= \frac{2}{a}\left[-\frac{1}{a}\left(e^{-ax}\right)\Big|_0^{\infty}\right] = \frac{2}{a}\left[\cancel{\lim_{x\to\infty} -\frac{1}{a}e^{-ax}}^{0} + \frac{1}{a}e^0\right] = \boxed{\frac{2}{a^2} = E[x^2]}$

d] $var(x) = E[(x-\mu_x)^2] = \sigma_x^2 = \int_{-\infty}^{\infty}(x-\mu_x)^2 f_X(x)\,dx$

$= \int_0^{\infty}\left(x - \frac{1}{a}\right)^2 a e^{-ax}\,dx = \int_0^{\infty}\left(x^2 - \frac{2x}{a} + \frac{1}{a^2}\right)a e^{-ax}\,dx$

$= \int_0^{\infty} x^2 a e^{-ax}\,dx - \int_0^{\infty}\frac{2x}{a}a e^{-ax}\,dx + \frac{1}{a^2}\int_0^{\infty} \cancel{a}e^{-ax}\,dx$

from part c $\rightarrow \int_0^{\infty} x^2 a e^{-ax}\,dx = \frac{2}{a^2}$

From Part b $\rightarrow \int_0^{\infty} x a e^{-ax}\,dx = \frac{1}{a}$

$\therefore \frac{2}{a}\int_0^{\infty} x a e^{-ax} = \frac{2}{a}\left(\frac{1}{a}\right) = \frac{2}{a^2}$

$\int_0^{\infty} e^{-ax}\,dx = -\frac{1}{a}e^{-ax}\Big|_0^{\infty} = \cancel{\lim_{x\to\infty} -\frac{1}{a}e^{-ax}}^{0} - \left(-\frac{1}{a}e^0\right) = \frac{1}{a} \rightarrow \frac{1}{a}\int_0^{\infty} e^{-ax} = \frac{1}{a^2}$

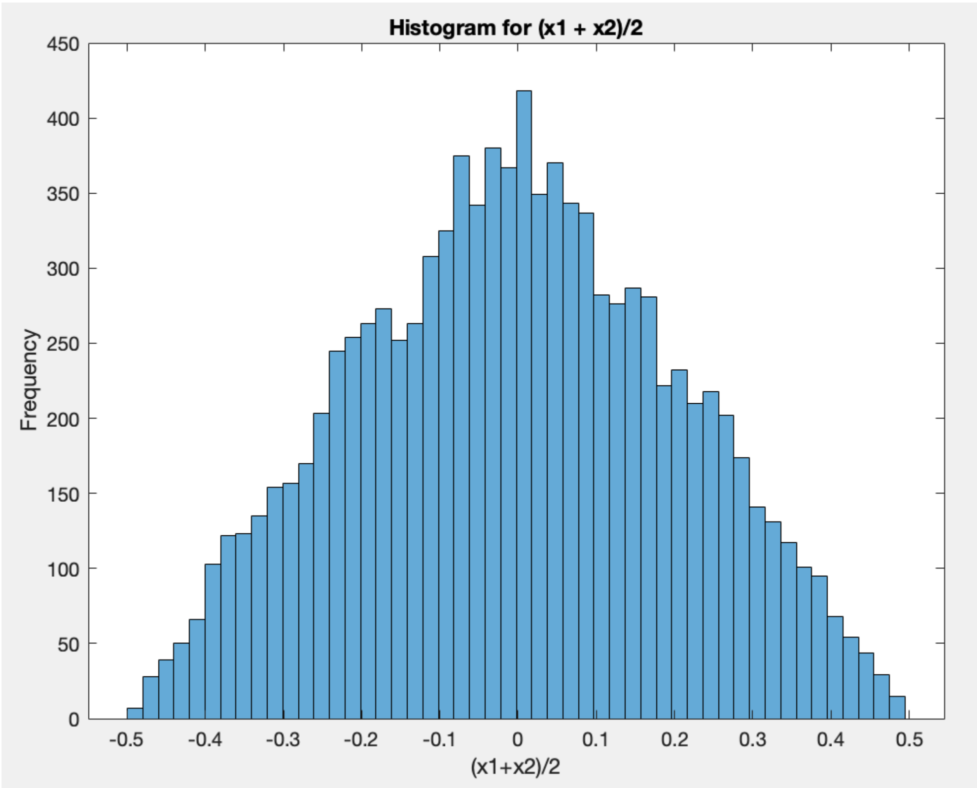
$var(x) = \cancel{\frac{2}{a^2}} - \cancel{\frac{2}{a^2}} + \frac{1}{a^2} = \boxed{\frac{1}{a^2} = var(x)}$

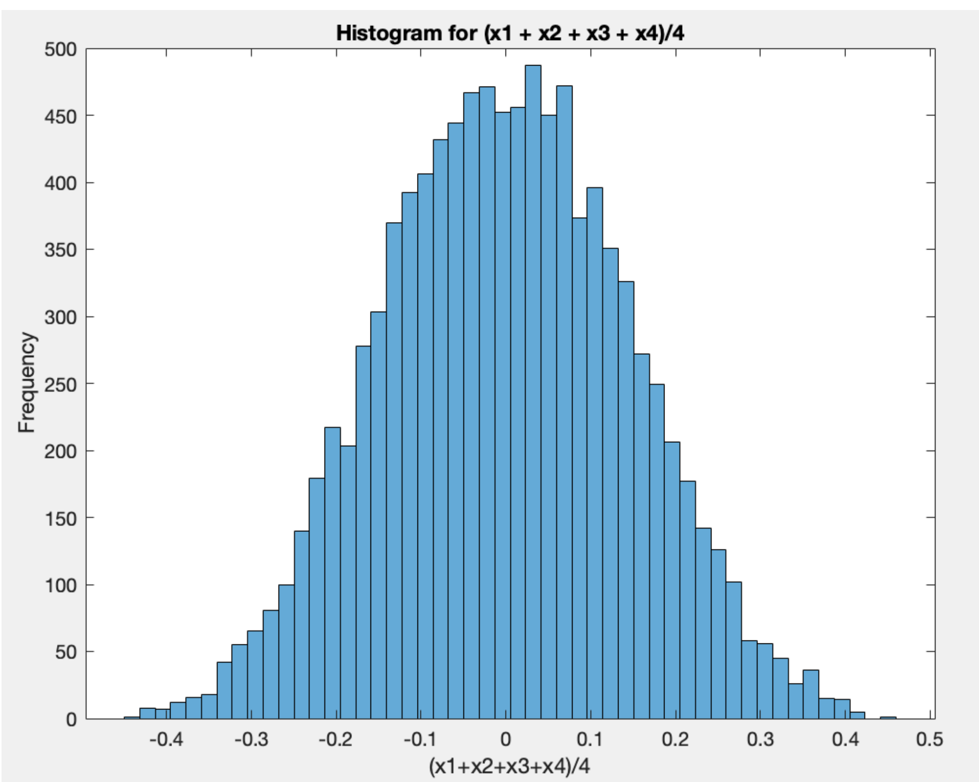
e] $\sigma_x = \sqrt{\frac{1}{a^2}} = \pm\frac{1}{a} \rightarrow \mu_x \pm \sigma_x = [0, 2/a]$

$c\left(\frac{2}{a}\right) = -e^{-a(2/a)} + 1 = -e^{-2} + 1 = \boxed{0.864664 = c\left(\frac{1}{a}\right)}$

This is the probability that an exponentially distributed RV takes on a value within one standard deviation of the mean.

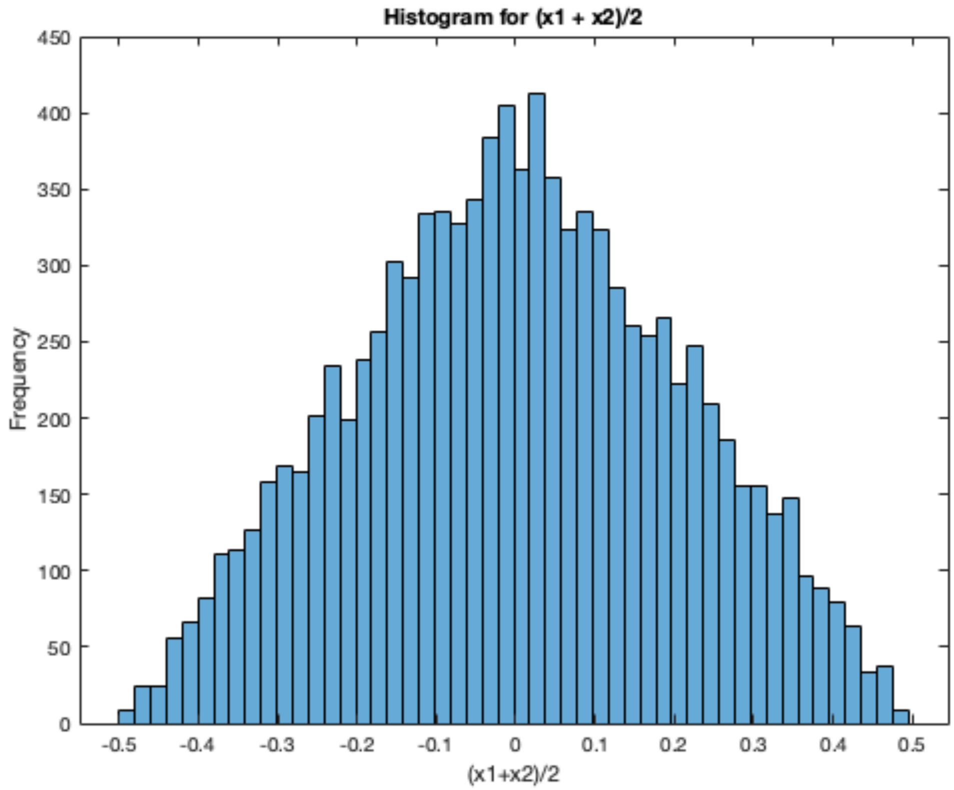
Problem 4 - Simon Problem 2.16

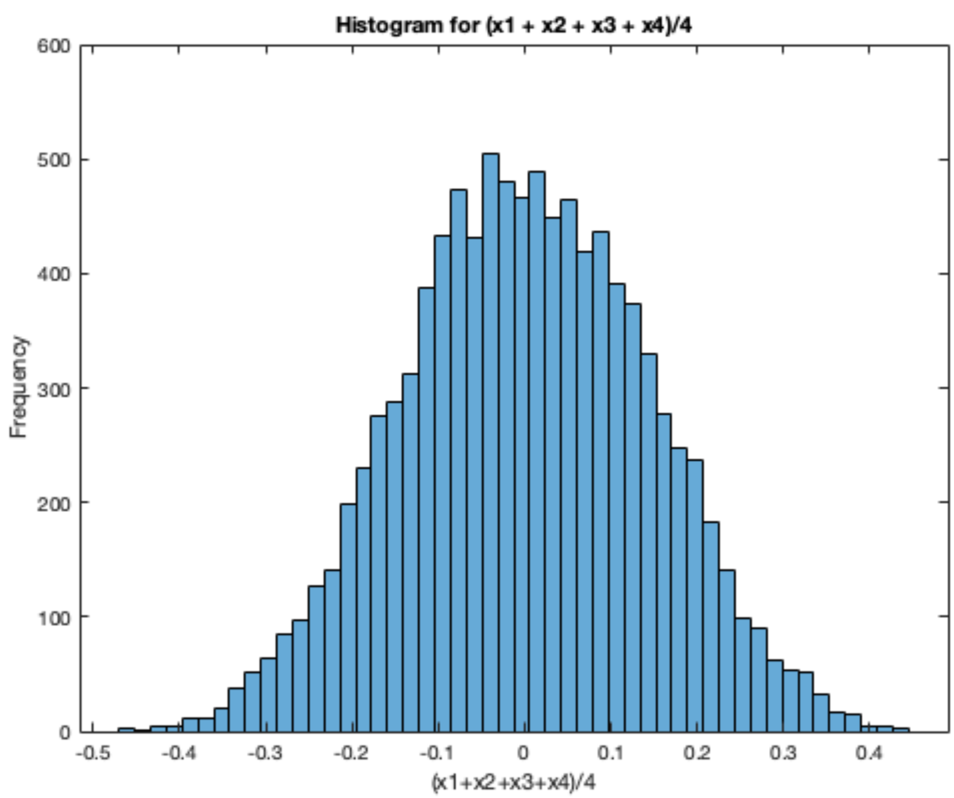




The histogram for (x1 + x2)/2 indicates that the variance for the distribution is higher than the distribution for (x1 + x2 + x3 + x4)/4. This can be explained by the higher frequency around the mean (which is 0) and lower frequency around the mean. The second distribution, (x1 + x2 + x3 + x4)/4 also resembles a normal distribution more than the first distribution, (x1 + x2)/2.

```
clear; clc; close all;

% Problem 4
% Textbook - 2.16
n = 10000;

lower_bound = -1/2;
upper_bound = 1/2;

x1 = lower_bound + (upper_bound - lower_bound).*rand(n, 1);
x2 = lower_bound + (upper_bound - lower_bound).*rand(n, 1);
x3 = lower_bound + (upper_bound - lower_bound).*rand(n, 1);
x4 = lower_bound + (upper_bound - lower_bound).*rand(n, 1);

samples12 = (x1 + x2)/2;
samples1234 = (x1 + x2 + x3 + x4)/4;

nbins = 50;

figure()
histogram(samples12, nbins)
xlabel("(x1+x2)/2")
ylabel("Frequency")
title("Histogram for (x1 + x2)/2")

figure()
histogram(samples1234, nbins)
xlabel("(x1+x2+x3+x4)/4")
ylabel("Frequency")
title("Histogram for (x1 + x2 + x3 + x4)/4")
```


Histogram for (x1 + x2)/2
Frequency
(x1+x2)/2
450
400
350
300
250
200
150
100
50
0
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5


Histogram for (x1 + x2 + x3 + x4)/4
Frequency
(x1+x2+x3+x4)/4
600
500
400
300
200
100
0
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4