Hamilton-Jacobi-Bellman Equation

Recall

$$J = K(\vec{x}_f, t_f) + \int_{t_0}^{t_f} L(\vec{x}, \vec{u}, \tau)\, d\tau$$

$$\dot{\vec{x}} = \vec{F}(\vec{x}, \vec{u}, t)$$

Key Idea

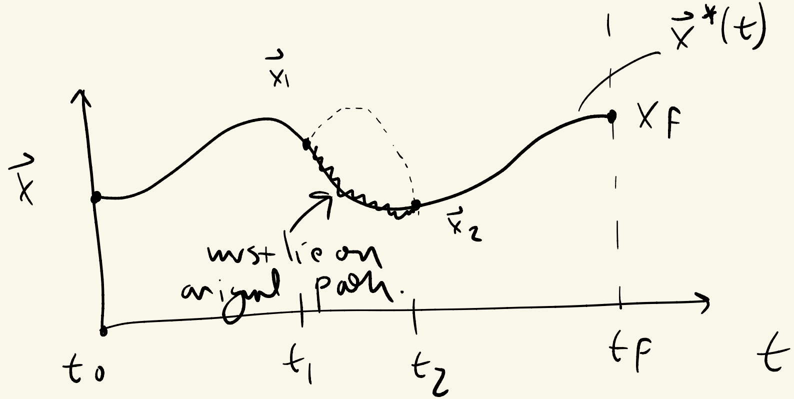
Principle of Optimality

Let $\vec{x}^*(t) = \phi_x(t; \vec{x}_0, t_0, \vec{u}^*)$; $t \in [t_0, t_f]$

be an optimal solution, ie. satisfies NC + SC.

Then the optimal solution to the problem over any sub-interval of time $[t_1, t_2]$; $t_0 \le t_1 < t_2 \le t_f$, denote as $\vec{x}^*_{12}$, is completely contained within the original $\vec{x}^*(t)$.

Another version of the Jacobi non-conjugacy condition.



IF $\exists$ an alternate, lower cost, trajectory portion, $\vec{x}'_{12}$, then that would have been part of the original solution.

Assume a general solution to the O.C.P. $\vec{x}(t)$ + corresponding control $\vec{u}^*(t)$. Consider the cost function starting $t > t_0$

$$J(\vec{x}(t), t, \vec{x}_f, t_f) = K(x_f, t_f) + \int_t^{t_f} L(\vec{x}(\tau), \vec{u}^*(\tau), \tau)\, d\tau$$

Then we have the boundary condition:

$$\boxed{J(\vec{x}(t_f), t_f, \vec{x}_f, t_f) = K(\vec{x}_f, t_f)}$$

Boundary Condition for J.

From the Principle of Optimality the problem is solved from an arbitrary time $t \to t_f$.

Question: What conditions must be satisfied at an earlier time, $t-\Delta t$?

Can say that we don't know $u^*(t-\Delta t)$ ......

$$J(\vec{x}(t-\Delta t), t-\Delta t, \vec{x}_f, t_f) = K + \int_{t-\Delta t}^{t_f} L \, d\tau$$

Expand J:

$$J(\vec{x}(t-\Delta t), t-\Delta t, \vec{x}_f, t_f) = J(\vec{x}(t), t, \vec{x}_f, t_f) + \frac{\partial J}{\partial \vec{x}} \cdot \frac{\partial \vec{x}}{\partial t}(-\Delta t) + \frac{\partial J}{\partial t}(-\Delta t) + \cdots$$

$$K + \int_{t-\Delta t}^{t_f} L \, d\tau = K + \int_{t}^{t_f} L \, d\tau + \underbrace{\int_{t-\Delta t}^{t} L \, d\tau}_{L(t-(t-\Delta t)) = L \Delta t}$$

Balance terms w/ Δt

$$\underbrace{\left[\frac{\partial J}{\partial t} + \frac{\partial J}{\partial \vec{x}} \cdot \vec{F}(\vec{x}, \underline{\underline{u}}, t) + L\left(x(t), \underline{\underline{u}}(t), t\right)\right]}_{\text{Partial Diff. Eq for } J(t,\vec{x}) \text{, with arbitrary } \vec{u}.} \Delta t = 0$$

Recall $\boxed{H = \vec{p} \cdot \vec{F} + L} = H(\vec{x}, \vec{p}, \vec{u}, t)$

$$\boxed{\frac{\partial J}{\partial t} + H\left(\vec{x}, \frac{\partial J}{\partial \vec{x}}, \vec{u}, t\right) = 0}$$

Partial Differential Equation for $J(\vec{x}, t)$ for arbitrary control $\vec{u}$.

From the Nec. Conds, if J is to take on an optimal value at $J(t-\Delta t)$, then H must satisfy

$$\boxed{H^*\left(\vec{x}, \frac{\partial J}{\partial \vec{x}}, u^*, t\right) = \min_{\vec{u} \subset \mathcal{U}} H\left(\vec{x}, \frac{\partial J}{\partial \vec{x}}, \vec{u}, t\right)}$$

This yields the $\boxed{\text{HJB}}$ Equation

$$\boxed{\begin{aligned} &\frac{\partial J}{\partial t} + \min_{\vec{u} \in \mathcal{U}} H\left(\vec{x}, \frac{\partial J}{\partial \vec{x}}, \vec{u}, t\right) = 0 \\ &J(x(t_f), t_f) = K(\vec{x}_f, t_f) \end{aligned}}$$

P.D.E $J(\vec{x}, t)$

Thm: Let $J(\vec{x}, t)$ be a solution to the HJB Equation w/ Boundary Condition and be C^1 on $\mathbb{R}^n \times [t_0, t_f]$.

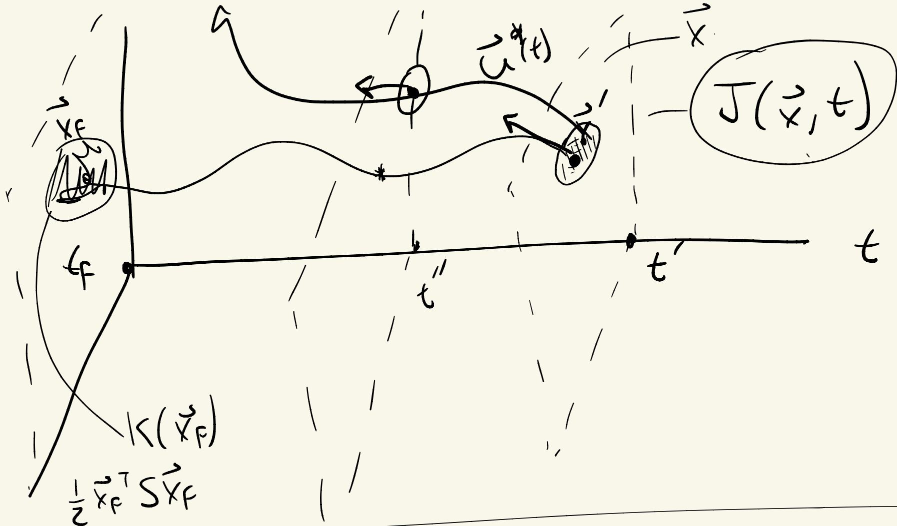
- Let $\vec{u}^*(\vec{x}, t) = \arg\min_{\vec{u} \in \mathcal{U}} H\left(\vec{x}, \frac{\partial J}{\partial \vec{x}}, \vec{u}, t\right)$
- Let $\dot{\vec{x}} = \vec{F}(\vec{x}, \vec{u}^*, t)$ have a solution $\vec{x}(t) = \varphi_x(t; \vec{x}_0, t_0)$

Then $\vec{x}(t)$ & $\vec{u}^*(t)$ are a solution of the optimal control problem and the minimum cost is

$$J(\vec{x}, t, \vec{x}_f, t_f) \text{ for arbitrary } \underline{\vec{x}, t \le t_f}$$

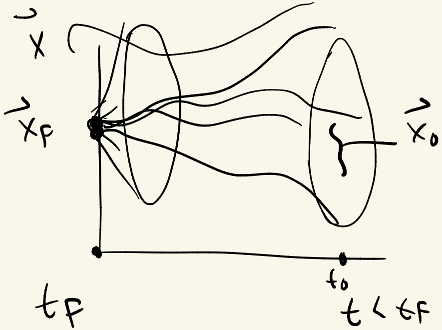
$$\vec{u}^*(\vec{x}, t) = \underset{\vec{u} \subset U}{\operatorname{argmin}} \vec{H}\left(\vec{x}, \frac{\partial J}{\partial \vec{x}}, \vec{u}, t\right) \equiv \text{the optimal feedback control law.}$$

Given a solution to the HJB over the domain $[\mathbb{R}^n, t]$ that satisfies the Boundary Condition, we can generate the control law which optimally drives us to the final state as an <u>arbitrary</u> function of state and time.



; $\left.\dfrac{\partial J}{\partial \vec{x}}\right|_{t', \vec{x}'} \Rightarrow H \Rightarrow \vec{u}^*(t)$

If solution is a "point" (hard constraint problem), $\vec{u}^*(t, \vec{x})$ causes all trajectories to converge to the final state, $\vec{x}_f$, and $K \equiv 0$ so that $\boxed{J(\vec{x}_f, t_f) = 0}$. Note, $\left.\dfrac{\partial J}{\partial \vec{x}}\right|_{t_f} \neq 0$ in general.



$\boxed{J(\vec{x}, t)}$

What about terminal constraints?

$\vec{g}(\vec{x}_0, t_0) = 0$ easily dealt with.

$\vec{g}(\vec{x}_f, t_f) = \vec{0} \Rightarrow$ Can eliminate + substitute into $K(\vec{x}_f, t_f)$, the terminal B.C. $\Big\}$ ~~Not easy....~~

How does the HJB solution compare to the Nec. Conds?

We can show that, along a particular trajectory,

$$\vec{p}_f = -\frac{\partial J}{\partial \vec{x}_f}\bigg|_{t_f} \;;\quad \boxed{\vec{p}_0 = \frac{\partial J}{\partial \vec{x}_0}\bigg|_{t_0}}$$

Gives us the solution to the 2PBVP.

$J(\vec{x},t) \equiv$ Value Function

Compare the Value Function with Hamilton's Principal Function

Recall the Action Integral

$$I = \int_{t_0}^{t_f} L\, d\tau = \int_{t_0}^{t_f} \left[H^* - \vec{p}\cdot\dot{\vec{x}} \right] d\tau \;;\quad H^*(\vec{x},\vec{p},t) = L + \vec{p}\cdot\vec{F} = \min_{\vec{u}\subset U} H(\vec{x},\vec{p},\vec{u},t)$$

$$\delta I = 0 \;\Rightarrow\; \boxed{\dot{\vec{x}} = \frac{\partial H^*}{\partial \vec{p}} \;,\quad \dot{\vec{p}} = -\frac{\partial H^*}{\partial \vec{x}}}$$

Action I evaluated along an extremal $\equiv$ Hamilton's Principal Function

$W(\vec{x}_0, t_0, \vec{x}_F, t_F)$ + satisfies Hamilton-Jacobi Eqn.

$$\left.\begin{aligned} &\frac{\partial W}{\partial t_0} - H\left(\vec{x}_0, -\frac{\partial W}{\partial \vec{x}_0}, t_0\right) = 0 \\ &\frac{\partial W}{\partial t_F} + H\left(\vec{x}_F, \frac{\partial W}{\partial \vec{x}_F}, t_F\right) = 0 \end{aligned}\right\} \qquad \vec{P}_F = \frac{\partial W}{\partial \vec{x}_F} \qquad \vec{P}_0 = -\frac{\partial W}{\partial \vec{x}_0}$$

Value Function $J = -W$, $W \equiv$ Hamilton's Principal Function.

Creates/Maintains a fundamental connection between Optimal Control + Hamiltonian Dynamics.