ASEN 6060
ADVANCED ASTRODYNAMICS
Additional Dynamical Models, Pt 1

Objectives:

- Briefly introduce the following dynamical models:
  - Hill restricted three-body problem
  - Elliptic restricted three-body problem
  - Bicircular four-body problem

For more information: See Szebehely, V., “Theory of Orbits”, Ch. 10

Additional Dynamical Models

- The CR3BP may be a good starting point for approximating the dynamical environment in a multi-body system. However, there are many effects that are not captured:
  - Eccentricity of orbits followed by primaries in a three-body system
  - Gravitational contributions of additional bodies
  - Irregular gravity fields
- These contributions influence the accuracy of an approximation and may significantly influence the structure of the solution space
- In some applications, alternative models may be better suited***
- We will cover three dynamical models:
  - Hill's restricted three-body problem
  - Elliptic restricted three-body problem
  - Bicircular four-body problem

Additional Dynamical Models

- In the autonomous CR3BP: equilibrium points, families of periodic orbits, and families of quasi-periodic orbits
- In a nonautonomous system that is periodic in time with T, may have some significant changes in the solution space
  - Equilibrium points → dynamical substitutes that have period T
  - Periodic orbits: Isolated at resonant ratios of period to T
  - Periodic orbits with nonresonant periods → quasi-periodic orbits with frequency matching periodic term

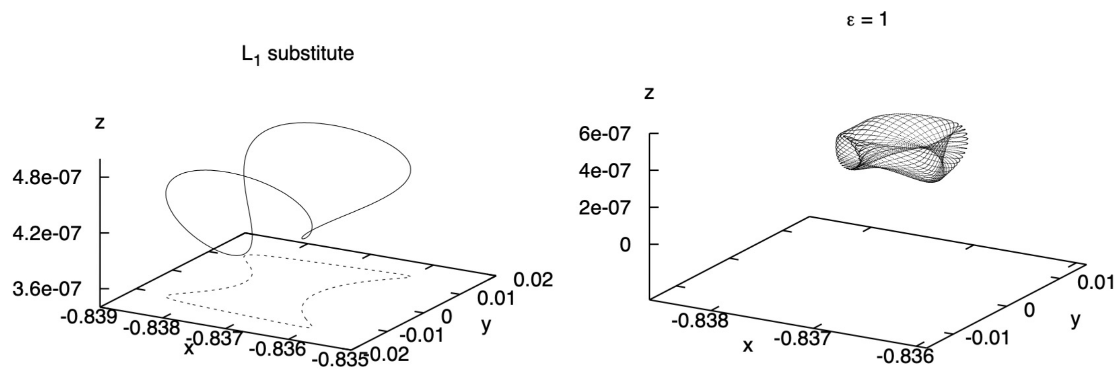


Image credit: G, Gómez, J. Masdemont, J.M. Mondelo, "Libration Point Orbits: A Survey from the Dynamical Point of View"

Figure 1: Dynamical substitutes for the L_1 point in the Earth–Moon system for a time–periodic (left) and a quasi–periodic (right) model.

Circular Restricted Three-Body Problem

Recall equations of motion:

$$\ddot{x} = 2\dot{y} + x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3}$$

$$\ddot{y} = -2\dot{x} + y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}$$

$$\ddot{z} = -\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3}$$

In compact form:

$$\ddot{x} - 2\dot{y} = \frac{\partial U^*}{\partial x} \qquad \ddot{y} + 2\dot{x} = \frac{\partial U^*}{\partial y} \qquad \ddot{z} = \frac{\partial U^*}{\partial z}$$

$$U^* = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

Hill's Restricted Three-Body Problem (HR3BP)

Consider the limit as $\mu \to 0$ and rewrite EOMs to measure *x* from P_2 instead of barycenter to produce:

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\ddot{x} - 2\dot{y} = 3x - \frac{\mu x}{r^3}$$

$$\ddot{y} + 2\dot{x} = -\frac{\mu y}{r^3}$$

$$\ddot{z} = -\frac{\mu z}{r^3}$$

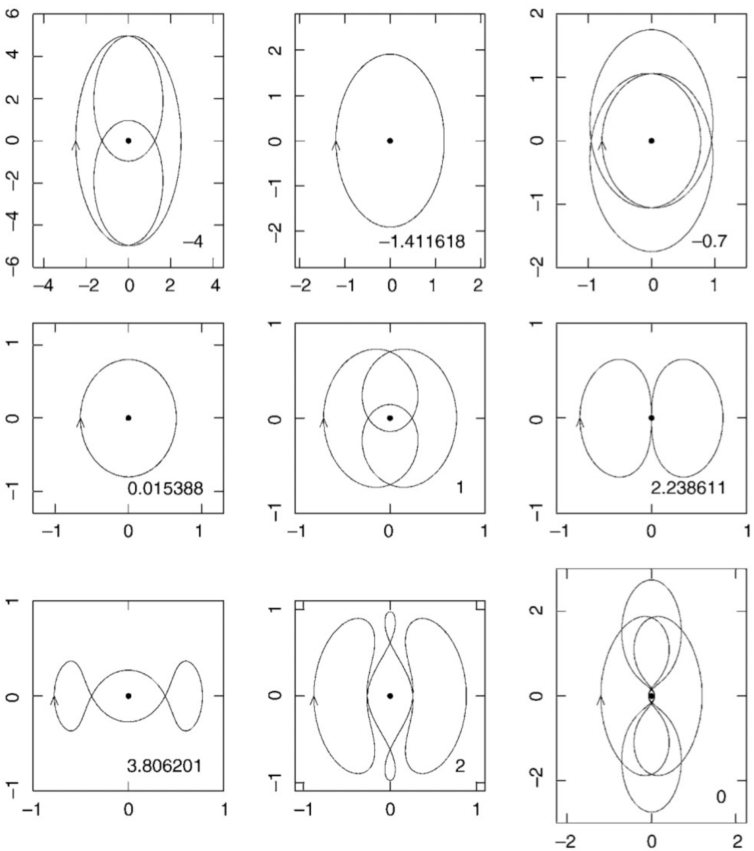
- In the HR3BP, typically focus on motion that remains close to P_2
- Dynamical model is autonomous
- L_1 and L_2 are symmetrically located at Hill radius (distances often scaled with this value):

$$r_H = \left(\frac{\mu}{3}\right)^{1/3}$$

Hill's Restricted Three-Body Problem (HR3BP)

Symmetry also introduced about *y*-axis

E.g., planar periodic orbits are symmetric about both *x* and *y* axes, and occur in symmetric 'pairs':



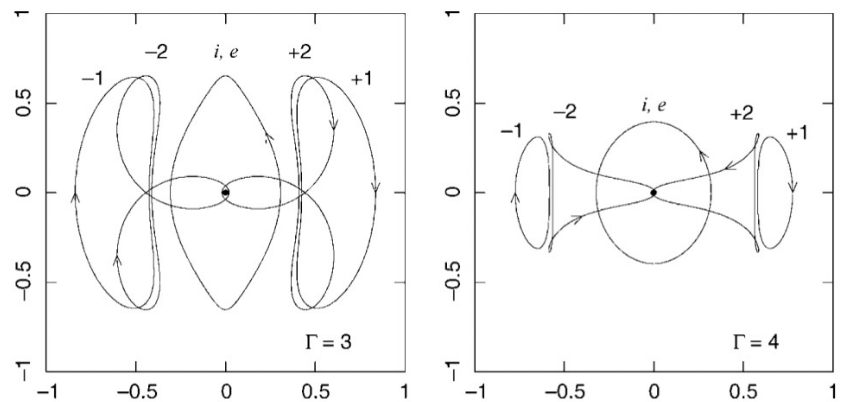


Figure 15. 'Simple' orbits for Γ = 3 and Γ = 4.

Image credit: Hénon, M., "New Families Of Periodic Orbits In Hill's Problem Of Three Bodies" *Celestial Mechanics and Dynamical Astronomy* **85:** 223–246, 2003.

Elliptic Restricted Three-Body Problem

Consider the following assumptions:

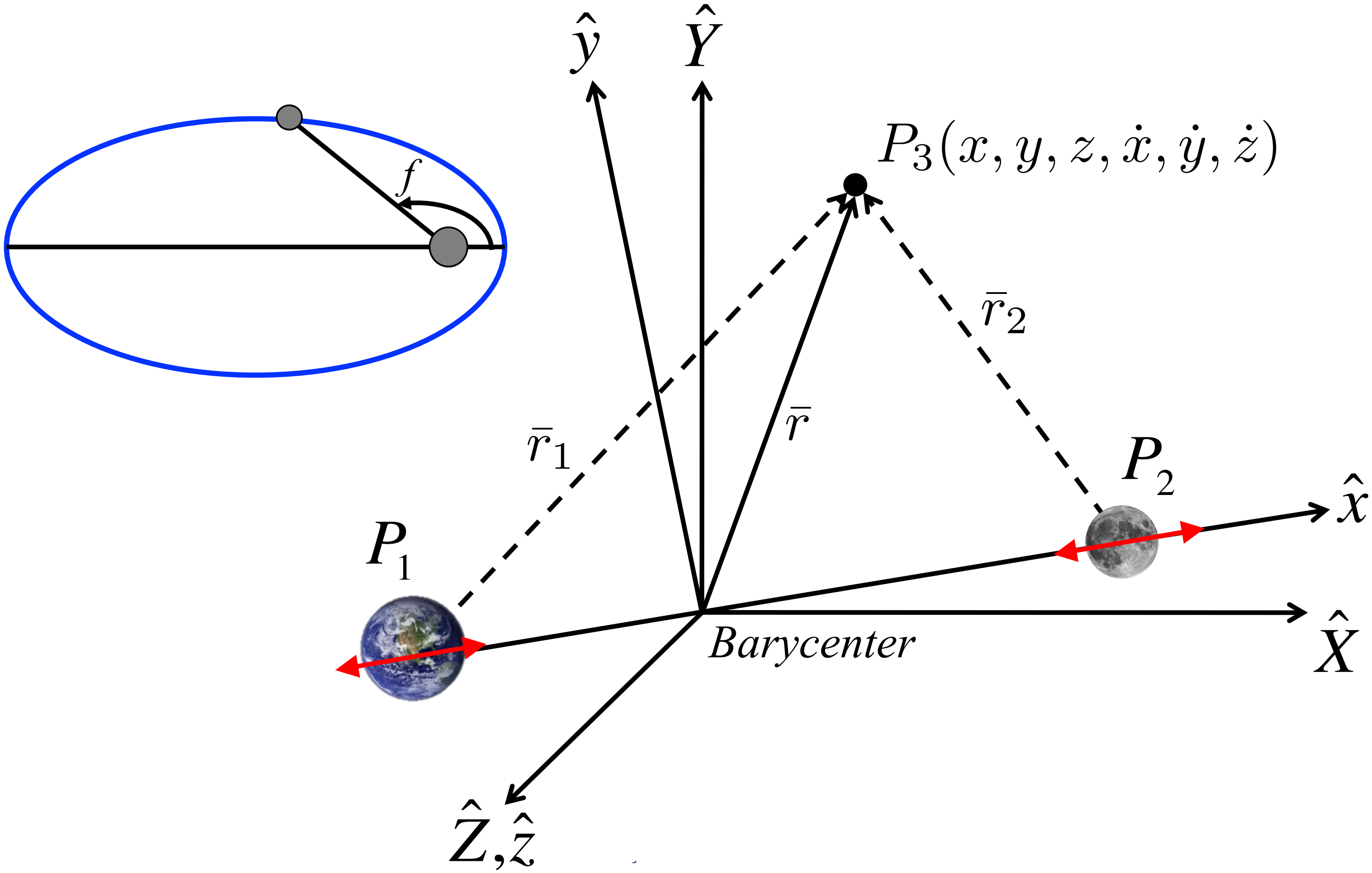
1. Mass of P_3 << Masses of each of P_1, P_2
2. P_3 does not influence the paths of P_1 and P_2, so both primaries travel on conics about their mutual barycenter
3. P_1 and P_2 follow elliptical / non-circular orbits
4. Model each body with same gravity field as point mass (i.e., as a spherically symmetric body) with constant mass

When these assumptions are used, we call the dynamical model the elliptic restricted three-body problem (ER3BP)

Rotating Frames

Pulsating rotating frame

Non-pulsating rotating frame



Deriving EOMs for ER3BP

EOMs for the three-body problem in inertial frame:

$$\ddot{\tilde{\bar{R}}} = -\frac{\tilde{G}\tilde{M}_1}{\tilde{R}_{13}^3}(\tilde{\bar{R}} - \tilde{\bar{R}}_1) - \frac{\tilde{G}\tilde{M}_2}{\tilde{R}_{23}^3}(\tilde{\bar{R}} - \tilde{\bar{R}}_2)$$

Tilde = dimensional quantity

Convert to rotating frame to produce dimensional EOMs:

$$\frac{{}^I d\bar{r}}{dt} = \frac{{}^R d\bar{r}}{dt} + {}^I\bar{\omega}^R \times \bar{r}$$

$${}^I\bar{\omega}^R = \dot{f}\hat{z}$$

$$\frac{{}^I d\bar{v}}{dt} = \frac{{}^R d\bar{v}}{dt} + {}^I\bar{\omega}^R \times \bar{v}$$

Nondimensionalize EOMs

Analyze the *z*-component of the dimensional EOMs in the rotating frame further:

$$\ddot{\tilde{z}} = -\frac{\tilde{G}\tilde{M}_1}{\tilde{r}_{13}^3}(\tilde{z} - \tilde{z}_1) - \frac{\tilde{G}\tilde{M}_2}{\tilde{r}_{23}^3}(\tilde{z} - \tilde{z}_2)$$

Nondimensionalize position coordinates:

$$\tilde{z} = r(f)z$$

Produces coordinates in a pulsating, rotating frame.

Rewrite time derivatives as:

$$\dot{\tilde{z}} = \frac{d\tilde{z}}{dt} = \frac{d(rz)}{dt} = \dot{r}z + r\dot{z}$$

$$\frac{d}{dt}\left(\frac{d\tilde{z}}{dt}\right) = \frac{d}{dt}(\dot{r}z + r\dot{z}) = \ddot{r}z + 2\dot{r}\dot{z} + r\ddot{z}$$

True Anomaly as Independent Variable

Rewrite derivatives of z in terms of true anomaly, f, instead of time:

$$\frac{d}{dt}\left(\frac{d\tilde{z}}{dt}\right) = \ddot{r}z + 2\dot{r}\dot{z} + r\ddot{z}$$

Use notation ()' for derivative with respect to f

$$2\dot{r}\dot{z} = 2\dot{r}\frac{dz}{dt} = 2\dot{r}\frac{dz}{df}\frac{df}{dt} = 2\dot{r}z'\dot{f}$$

$$r\ddot{z} = r\frac{d}{dt}\left(\frac{dz}{dt}\right) = r\left[\dot{f}^2 z'' + z'\ddot{f}\right]$$

Plug back into original equation:

$$\ddot{\tilde{z}} = \ddot{r}z + 2\dot{r}\dot{z} + r\ddot{z} = \ddot{r}z + 2\dot{r}z'\dot{f} + r\left[\dot{f}^2 z'' + z'\ddot{f}\right]$$

True Anomaly as Independent Variable

Recall the derivative of the angular momentum of the primary system in their elliptical orbits in the relative 2BP

$$h = r^2 \dot{f} \qquad \frac{dh}{dt} = 0 = 2r\dot{r}\dot{f} + r^2 \ddot{f}$$

Simplifies the z' terms

$$\ddot{\tilde{z}} = \ddot{r}z + 2\dot{r}\dot{z} + r\ddot{z} = \ddot{r}z + \cancel{2\dot{r}z'\dot{f}} + r\left[\dot{f}^2 z'' + \cancel{z'\ddot{f}}\right]$$

Also know the second time derivative of the distance between two bodies as:

$$\ddot{r} = r\dot{f}^2 - \frac{r^2}{a(1-e^2)}\dot{f}^2$$

True Anomaly as Independent Variable

$$\ddot{\tilde{z}} = -\frac{\tilde{G}\tilde{M}_1}{\tilde{r}_{13}^3}(\tilde{z} - \tilde{z}_1) - \frac{\tilde{G}\tilde{M}_2}{\tilde{r}_{23}^3}(\tilde{z} - \tilde{z}_2)$$

Nondimensionalizing the RHS of the EOMs and plugging in this known quantity:

$$r\dot{f}^2 z'' = -z\left(r\dot{f}^2 - \frac{r^2}{a(1-e^2)}\dot{f}^2\right) - \frac{\tilde{G}\tilde{M}_1(z - z_1)r}{r^3 r_1^3} - \frac{\tilde{G}\tilde{M}_2(z - z_2)r}{r^3 r_2^3}$$

Rearrange:

$$z'' = -z\left(1 - \frac{r}{a(1-e^2)}\right) - \frac{\tilde{G}\tilde{M}_1(z - z_1)}{\dot{f}^2 r^3 r_1^3} - \frac{\tilde{G}\tilde{M}_2(z - z_2)}{\dot{f}^2 r^3 r_2^3}$$

True Anomaly as Independent Variable

From the conic equation: $$\frac{r}{a(1-e^2)} = \frac{1}{(1+e\cos(f))}$$

And note from specific angular momentum that:

$$h = r^2\dot{f} \rightarrow r^4\dot{f}^2 = \tilde{G}(\tilde{M}_1+\tilde{M}_2)a(1-e^2) \qquad \frac{h^2}{\mu_{2BP}} = a(1-e^2)$$

Simplify original expression:

$$z'' = -z\left(1 - \frac{r}{a(1-e^2)}\right) - \frac{\tilde{G}\tilde{M}_1(z-z_1)}{\dot{f}^2 r^3 r_1^3} - \frac{\tilde{G}\tilde{M}_2(z-z_2)}{\dot{f}^2 r^3 r_2^3}$$

To produce:

$$z'' = \frac{1}{1+e\cos f}\left(-ze\cos f - \frac{(1-\mu)(z-z_1)}{r_1^3} - \frac{\mu(z-z_2)}{r_2^3}\right)$$

Nondimensional EOMs in Rotating Frame

The EOMs are then written using true anomaly as an independent variable as:

$$x'' = 2y' + \frac{1}{1 + e\cos f}\left(x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3}\right)$$

$$y'' = -2x' + \frac{1}{1 + e\cos f}\left(y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}\right)$$

$$z'' = \frac{1}{1 + e\cos f}\left(-ze\cos f - \frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3}\right)$$

And note that the independent variable varies with time as:

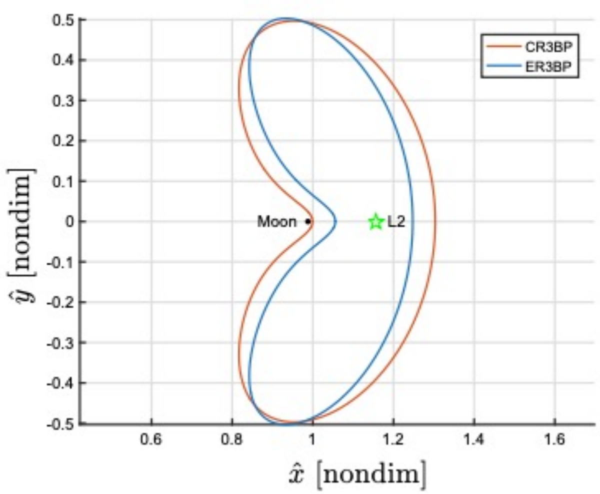
$$\dot{f} = \frac{(1-e^2)^{3/2}}{(1+e\cos(f))^2}$$

Periodic Orbits in ER3BP

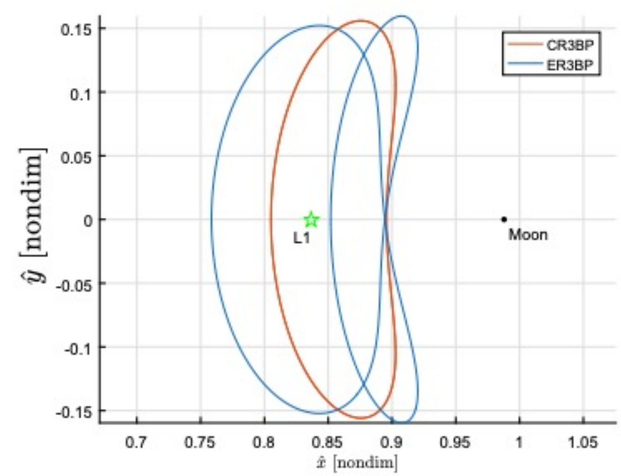
- Isolated periodic orbits exist with periods of $2\pi n$ where n is an integer
  - No longer exist in continuous families
- Periodic orbits from the CR3BP:
  - With a resonant ratio in the period, can be used to calculate periodic orbits in ER3BP (can use continuation in e while constraining the period)
    - Periodic orbits in the ER3BP satisfy the mirror theorem
  - Without this condition on the period satisfied, may be used to compute nearby quasi-periodic orbits in the ER3BP with one frequency matching that of the primary system

Periodic Orbits in ER3BP

- Image credits: Ferrari, F., Lavanga, M., “Periodic motion around libration points in the Elliptic Restricted Three-Body Problem,” Nonlinear Dyn (2018) 93:453–462



L_2 Lyapunov orbit with period = 2pi



(a)

L_1 Lyapunov orbit with period = pi

Periodic Orbits in ER3BP

- Image credits: Campagnola, S., Lo, M., Newton, P., “Subregions Of Motion And Elliptic Halo Orbits In The Elliptic Restricted Three-body Problem”, AAS 08-200
- Quote: “For an orbit to be periodic [in the planar ER3BP] it is sufficient that it has two perpendicular crossing with the syzygy-axis, and that the crossings happen at moments when the two primaries are at an apse, (i.e., at maximum or minimum elongation, or apoapsis and periapsis).”

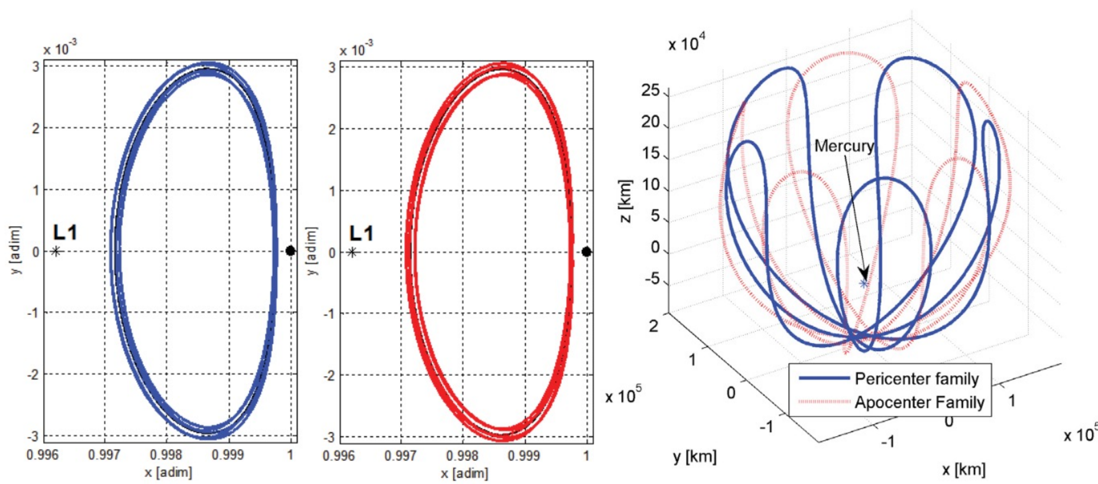


Figure 7 Periapsis and apoapsis elliptic halo orbits in the ER3BP ($e = 0.02$) generated from a 2:5 halo orbit in the CR3BP. The first and second figure from the left show the periapsis and apoapsis halo orbits in the rotating reference frame. The last figure shows both orbits in the inertial reference frame.

Periodic Orbits in ER3BP

- Image credits: Campagnola, S., Lo, M., Newton, P., “Subregions Of Motion And Elliptic Halo Orbits In The Elliptic Restricted Three-body Problem”, AAS 08-200
- Quote: “For an orbit to be periodic [in the planar ER3BP] it is sufficient that it has two perpendicular crossing with the syzygy-axis, and that the crossings happen at moments when the two primaries are at an apse, (i.e., at maximum or minimum elongation, or apoapsis and periapsis).”

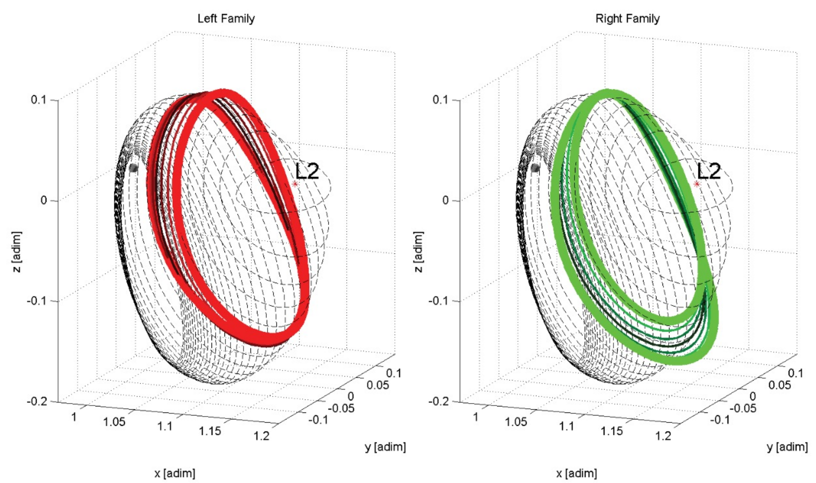
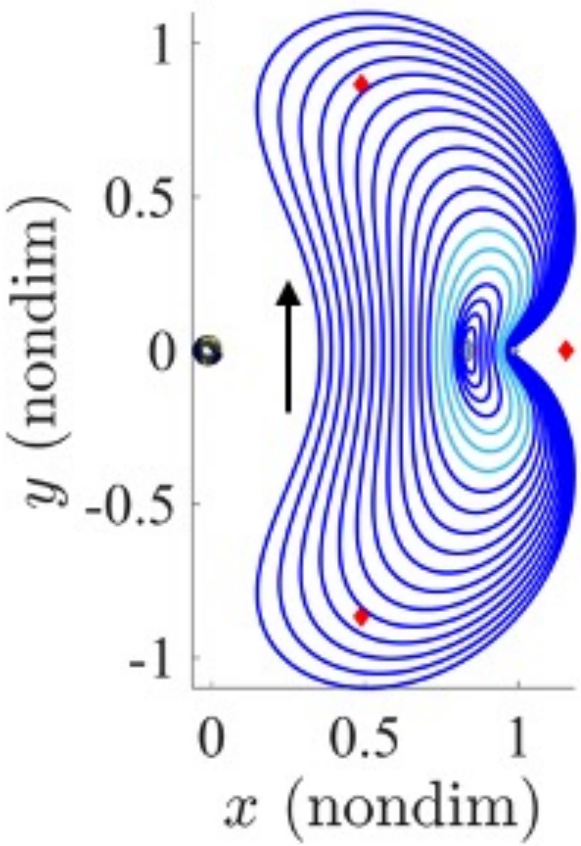


Figure 8 Left and Right elliptic halo orbits in the ER3BP ($0 < e < 0.3$, $\mu \approx 0.0123$)**. The dash lines are the L2 halo orbits in the CR3BP. The bold solid line is the elliptic halo for** $e = 0.3$**. On the Left: ‘Left’ family that bifurcates from the L2 halo in the pulsating reference frame. On the Right: ‘Right’ family that bifurcates from the L2 halo in the pulsating reference frame.**

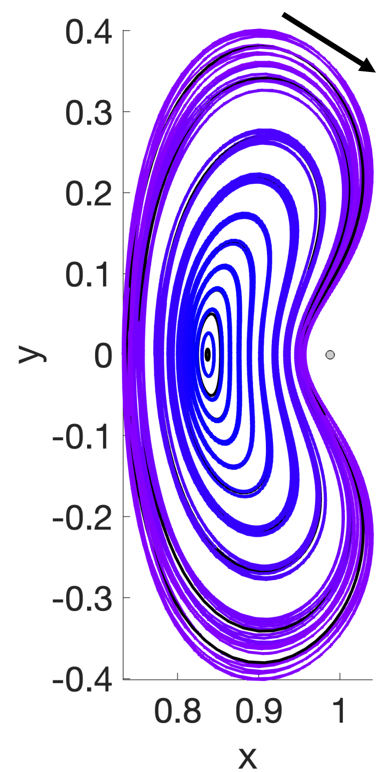
Bounded Motions in ER3BP

Motions resembling L_1 Lyapunov orbit family from CR3BP

CR3BP



ER3BP, e = 0.0549



Bicircular Four-Body Problem

Consider the Sun-Earth-Moon bicircular four-body problem (BC4BP)

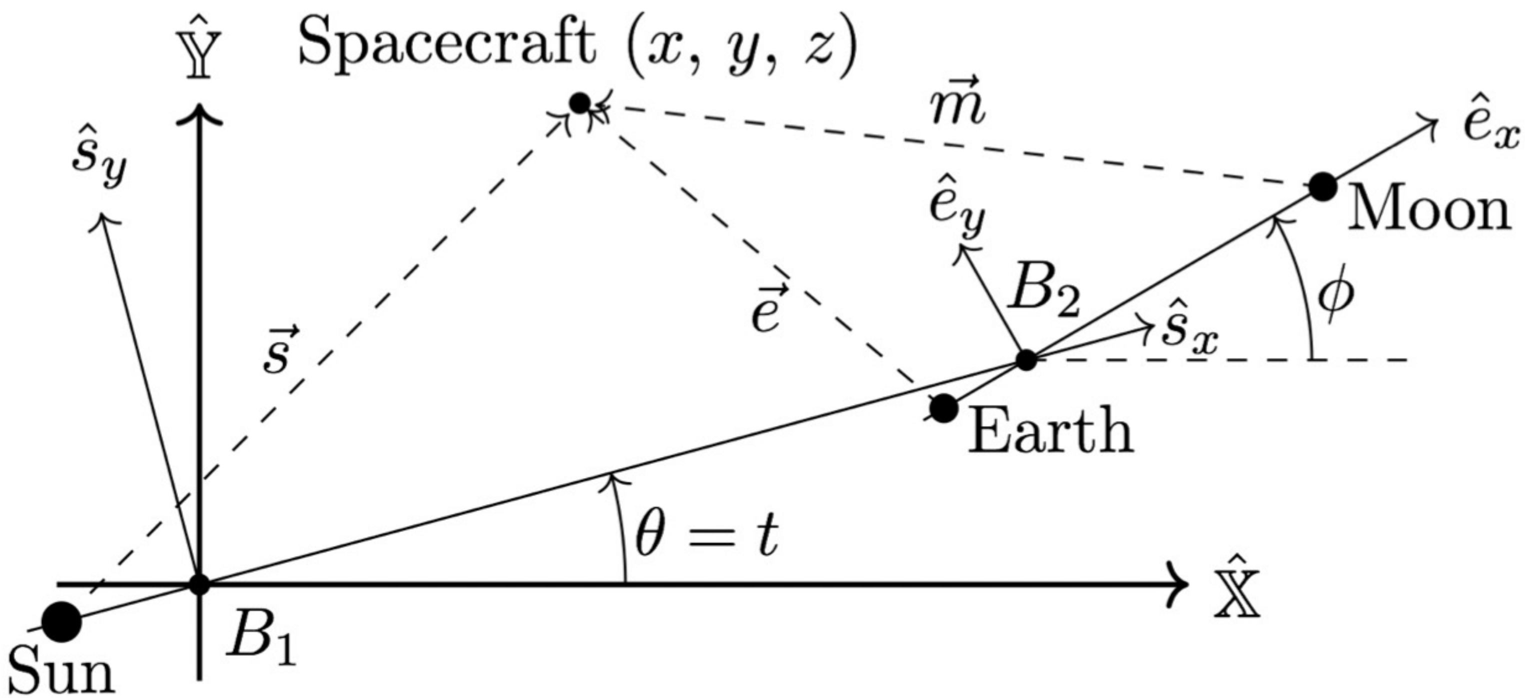


Image credit: Folta, D; Bosanac, N; Cox, A; Howell, K.C., 'The Lunar Icecube Mission Design: Construction Of Feasible Transfer Trajectories With A Constrained Departure,' 26th AAS/AIAA Space Flight Mechanics Meeting, 2016

Bicircular Four-Body Problem

Equations of motion:

$$\ddot{x} - 2\dot{y} = \frac{\partial U_4^*}{\partial x} \qquad \ddot{y} + 2\dot{x} = \frac{\partial U_4^*}{\partial y} \qquad \ddot{z} = \frac{\partial U_4^*}{\partial z}$$

Where: $$U_4^* = 0.5(x^2 + y^2) + \frac{1-\mu}{s} + \frac{\mu - \nu}{e} + \frac{\nu}{m}$$

μ = Non-dimensional mass of Earth-Moon system

ν = Non-dimensional mass of Moon

This is a non-autonomous dynamical model

Bicircular Four-Body Problem

Consider the Earth-Moon-Sun bicircular four-body problem (BC4BP)

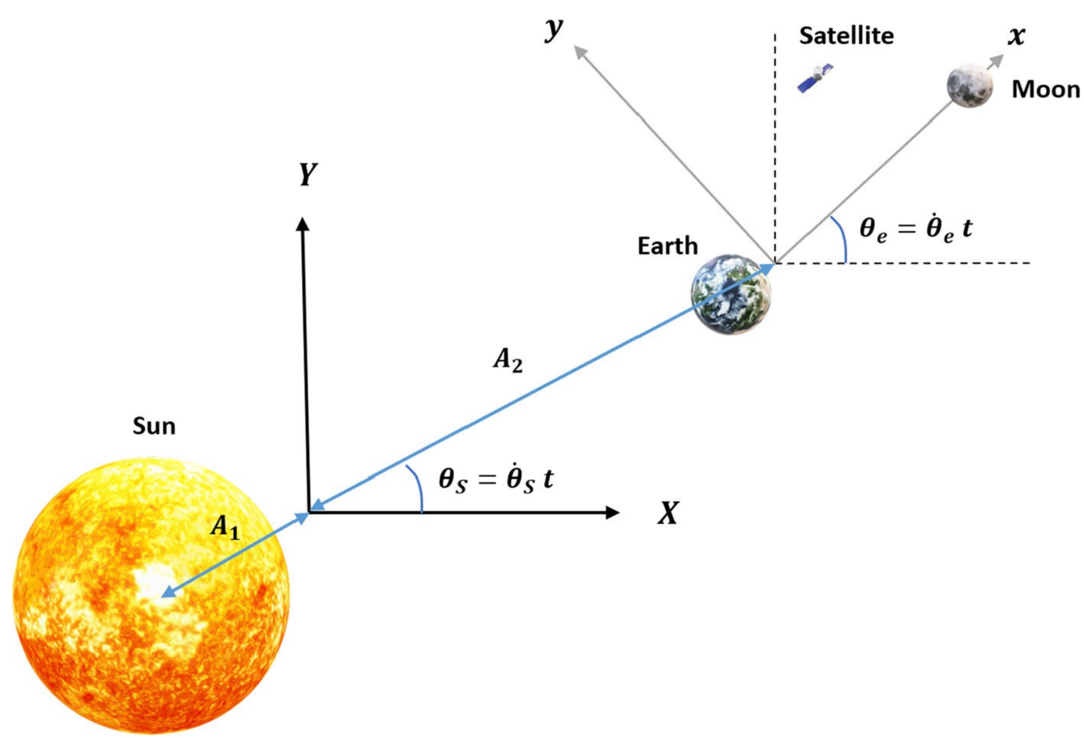


Image credit: Wilmer, A; Bettinger, B, Lagrangian dynamics and the discovery of cislunar periodic orbits, Nonlinear Dynamics, 2023

Bicircular Four-Body Problem

Equations of motion:

$$\ddot{x} = x + 2\dot{y} - \frac{(1-\mu)(x+\mu)}{r^3_{\text{sat}/e}} - \frac{\mu(x-1+\mu)}{r^3_{\text{sat}/m}} - \frac{\mu_S(x-x_S)}{r^3_{\text{sat}/S}} + A_2\dot{\theta}_S^2\cos(\theta_S - \theta_e)$$

$$\ddot{y} = y - 2\dot{x} - \frac{(1-\mu)y}{r^3_{\text{sat}/e}} - \frac{\mu y}{r^3_{\text{sat}/m}} - \frac{\mu_S(y-y_S)}{r^3_{\text{sat}/S}} + A_2\dot{\theta}_S^2\sin(\theta_S - \theta_e)$$

$$\ddot{z} = -\frac{(1-\mu)z}{r^3_{sat/e}} - \frac{\mu z}{r^3_{sat/m}} - \frac{\mu_S z}{r^3_{sat/S}}$$

μ_S = Non-dimensional mass of the Sun

x_S, y_S = Position of the Sun WRT E-M barycenter

Coordinates are WRT Earth-Moon rotating reference frame

$$\theta_S = \dot{\theta}_S t = \sqrt{\frac{\mu_S + 1}{(A_1 + A_2)^3}}\,t$$

$$\theta_e = \dot{\theta}_e t = (1)t$$

$$r^2_{sat/e} = (x+\mu)^2 + y^2 + z^2$$

$$r^2_{sat/m} = \left(x - (1-\mu)\right)^2 + y^2 + z^2$$

$$r^2_{sat/S} = (x - x_S)^2 + (y - y_S)^2 + z^2$$

Bicircular Four-Body Problem

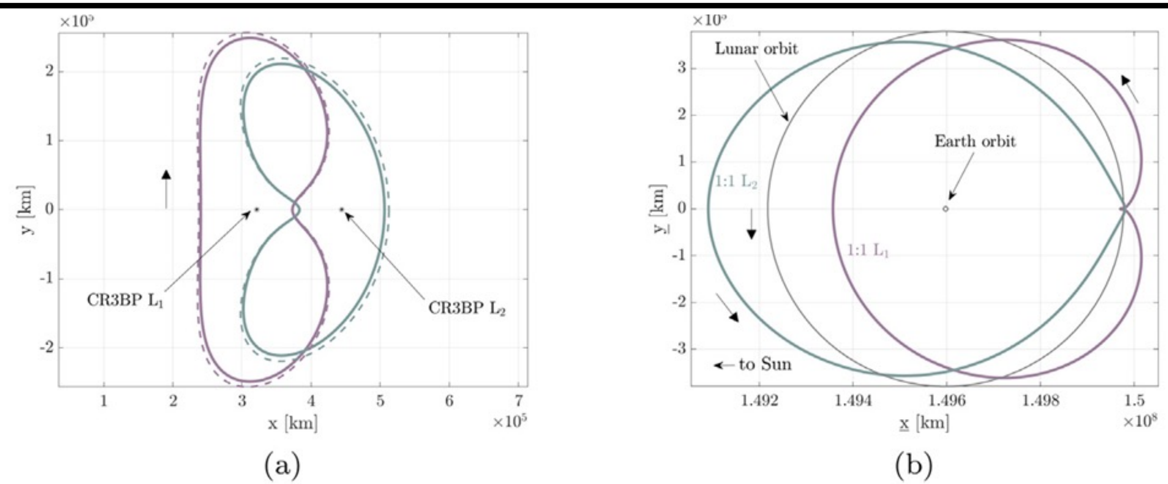


Fig. 11. 1:1 L_1 and L_2 Lyapunov orbits in the BCR4BP (solid) and in the CR3BP (dashed), as seen in the Earth-Moon rotating frame (a). 1:1 L_1 and L_2 Lyapunov orbits in the BCR4BP, as observed in the Sun-B_1 rotating frame. The Moon and Earth orbits are indicated for reference.

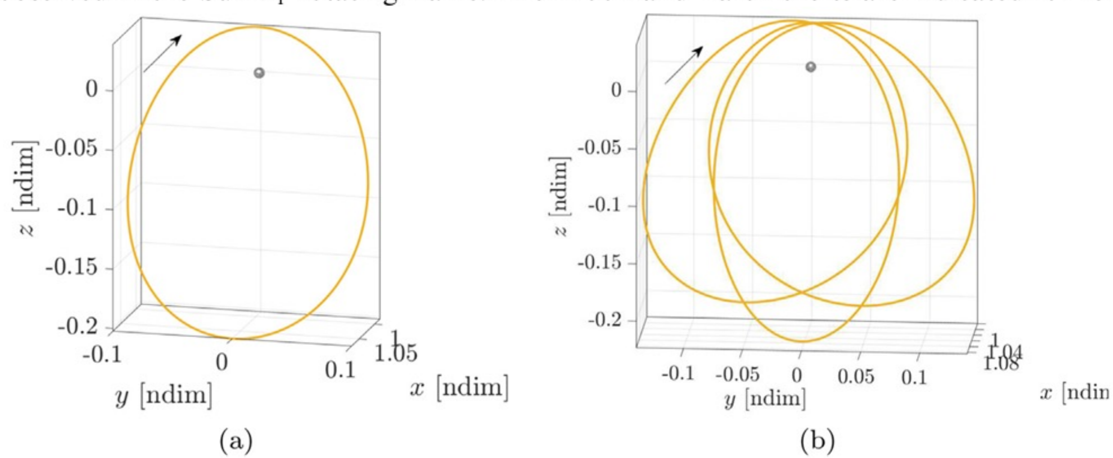


Fig. 12. CR3BP 3 : 1 NRHO (a) and its T_{syn}-periodic BCR4BP counterpart (b).

Image credit: Boudad, K.Z., Howell, K.C., Davis, D.C., “Dynamics of synodic resonant near rectilinear halo orbits in the bicircular four-body problem,” Advances in Space Research, 2020