Singular Control: $\Rightarrow$ Nec. Conditions for S.C.

Recall that if $|H_{uu}| = 0 \Rightarrow$ lose suff. conditions.

We also lose the ability to uniquely solve for $\vec{u}$, or some component of it.

Can we still satisfy NC?

★ Consider the case of a <u>singular control</u>

$$H = \cdots\cdots \underbrace{\varphi_s(\vec{x}, \vec{p}, t)}_{\text{Switching Function}} \cdot \underbrace{u}_{\text{Control}} \quad ; \quad \frac{\partial H}{\partial u} = H_u = \varphi_s$$

If $H_u = 0$ over a finite interval $\left(\varphi_s = 0\right)$.

then $\dot{H}_u = 0$, $\ddot{H}_u = 0$, etc ....

$H_u = 0$

$$\dot{H}_u = H_{ux}\dot{x} + H_{up}\dot{p} = H_{ux}H_p - H_{up}H_x = 0$$

$$\ddot{H}_u = H_{uxx}H_p\dot{x} + H_{uxp}H_p\dot{p} + H_{ux}H_{px}\dot{x} + H_{ux}H_{pp}\dot{p} - H_{up} \cdots\cdots$$

Repeat until $\vec{u}$ explicitly appears in the Equations.

Provides a constraint for solving for $\vec{u}$ to satisfy the condition $\boxed{H_u = 0}$

Deeper analysis leads to a nec. condition for this singular control to be an extremal solution.

Generalized Legendre - Clebsch Condition | N.C. for optimality

$$(-1)^m \left(\frac{\partial}{\partial u}\right)\left(\frac{d^{2m}}{dt^{2m}}\right) H_u > 0 \quad (u + x \text{ scalar})$$

where $2m$ is the first order at which u appears explicitly.

Note: $\frac{\partial}{\partial u}\left(\frac{d^l}{dt^l}\right) H_u = 0 \quad l \text{ is odd}$

$$(-1)^{2n} \left(\frac{\partial}{\partial u}\right)\left(\frac{d^{2n}}{dt^{2n}}\right) H_u \geq 0$$

Simple Example

$t \in [0, 2]$

$$J = x^2(2) + \int_0^2 x^2 \, dt \quad ; \quad \dot{x} = u \qquad |u| \le 1$$

or $u \in [-1, 1]$

$x(0) = 1$

$$\boxed{H = x^2 + p \cdot u} \qquad u^* = \begin{cases} 1 & p < 0 \\ -1 & p > 0 \\ ? & p = 0 \end{cases}$$

$$\boxed{\dot{p} = -2x \quad ; \quad \dot{x} = u^*}$$

Transversality: $P(0) = P_0$; $P(2) = \frac{\partial K}{\partial X} = 2X(2)$

$$\boxed{P(2) = 2X(2)}$$

H is constant, but end. at times are not free.

Suppose we have $\boxed{p = 0}$, what should the singular control be?

$\frac{\partial H}{\partial u} = H_u = p$;

If $p=0$ then $\boxed{\dot{p} = -2x = 0}$ $\Rightarrow$ $\boxed{x = 0 \text{ along a singular arc.}}$

$\ddot{p} = -2\dot{x} = \boxed{-2u = 0}$ we need $\boxed{u = 0}$

If not, solution leaves the "singular manifold"

Solution Process $\Rightarrow$ Need specify P_0

- If $P_0 < 0$, $u^* = +1$ $\quad \dot{x} > 0 \quad \dot{p} < 0$
  $\quad x > 0 \quad P < 0$ $\Bigg\}$ $p(z) \neq z\,x(z)$

- If $P_0 = 0$; we cannot stay on the sing. manifold as $x > 0$
  $P \downarrow$

- IF $P_0 > 0$ $u = -1$ $\Rightarrow$ $\boxed{\begin{aligned} x(t) &= 1 - t \\ p(t) &= p_0 + t(t-2) \end{aligned}}$

$x > 0 \downarrow$ at $t = 1$ $\boxed{\begin{aligned} x(1) &= 0 \\ P(1) &= P_0 - 1 \end{aligned}}$

IF $P_0 = 1$, get $P(1) = 0$ + possible singular control.

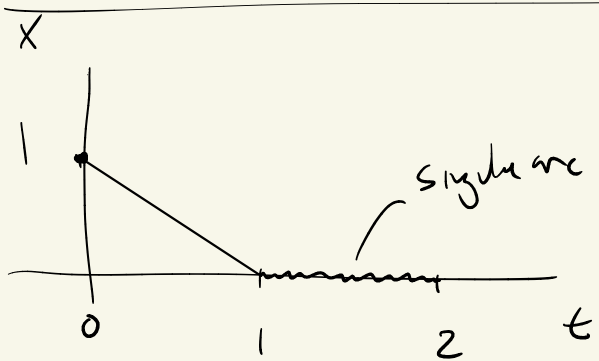
Alternative is $P_0 < 1$ or $P_0 > 1$ $\Rightarrow$ Can show that both give us inconsistent solutions.

IF $P_0 > 1$ after $t > 1$ $x \downarrow$ $x < 0$

$\dot{p} > 0$, $P > 0$ $\quad p(2) \neq 2x(2)$

$P_0 < 1$ $\Rightarrow$ at $t = 2$ $P < 0$ $x > 0$

Must choose the singular arc.



$$p_0 = 1$$

$$u^* = \begin{cases} -1 & t \in [0, 1] \\ 0 & t \in [1, 2] \end{cases}$$

$$x(2) = 2\, p(2)$$

$$\downarrow \qquad \downarrow$$

$$0 = 0 = 0$$

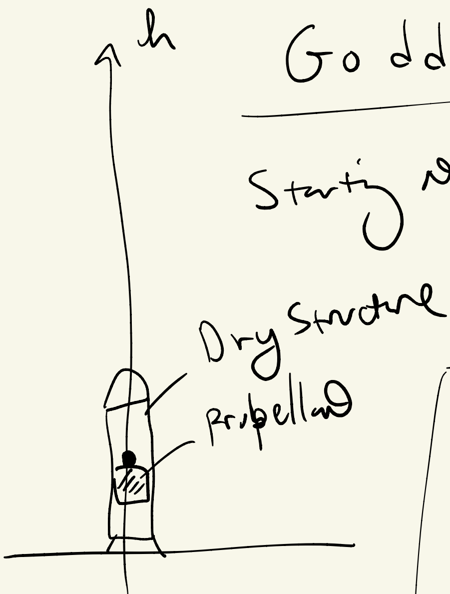
We can check the L-C conditions

$$(-1)^1 \cdot \frac{\partial}{\partial u}\left(\frac{d^2}{dt^2} H_u\right) = -\frac{\partial}{\partial u}\left(-2u\right) = (-)(-2) = 2 > 0$$

An extremal!

"Classical" Singular Arc Problem is the

Goddard Rocket Problem



$T \in [0, T_{max}]$

Starting w/ $h(0) = V(0) = t_0 = 0$, $m(0) = M$

Maximize altitude, $V(t_f) = \text{free}$

$t_f = \text{free}$

$\boxed{m(t_f) = M_F}$

$$\boxed{\begin{aligned} \dot{h} &= V \\ \dot{V} &= \frac{1}{m}(T-D) - g \\ \dot{m} &= -\frac{T}{c} \end{aligned}}$$

$$\boxed{K = -h(t_f)}$$

$$\boxed{H = p_h V + \frac{p_V}{m}(T-D) - p_V g - \frac{p_m T}{c}} \Rightarrow H = \left[\frac{p_V}{m} - \frac{p_m}{c}\right] T + \cdots\cdots$$

$$T^* = \begin{cases} 0 & \frac{p_v}{m} - \frac{p_m}{c} > 0 \\ T_{max} & \frac{p_v}{m} - \frac{p_m}{c} < 0 \\ (0, T_{max}) & \frac{p_v}{m} - \frac{p_m}{c} = 0 \text{ over a finite interval.} \end{cases}$$

$$\dot{p}_h = 0 \qquad \dot{p}_V = -p_h + \frac{p_V}{m}\frac{\partial D}{\partial V} \qquad \dot{p}_m = \frac{p_V}{m^2}(T - D)$$

Trans. Conditions t_f free $\quad H_f = 0 = H$

$$p_m = p_{m_f}, \quad p_h = -1, \quad p_{v_f} = 0$$

What conditions lead to a singular arc?

$$H_T = \left(\frac{p_v}{m} - \frac{p_m}{c}\right) = 0 \;;\; \dot{H}_T = 0, \; \ddot{H}_T = 0, \ldots$$

$$\frac{d}{dt}H_T = \frac{\overset{0}{\dot{P}_v}}{m} - \frac{P_v}{m^2}\dot{m} - \frac{\dot{P}_m}{c} = -\frac{P_h}{m} + \frac{P_v}{m^2}\frac{\partial D}{\partial V} + \frac{P_v}{m^2 c} D = 0$$

$$\dot{H}_T = \frac{1}{\cancel{m}}\left\{ \frac{P_v}{m}\left[D_v + \frac{D}{c} \right] - P_h \right\} = 0 \Rightarrow$$

Constraint to satisfy if the arc is singular.

$$\ddot{H}_T = \frac{\overset{0}{\dot{P}_v}}{m}\left(D_v + \frac{D}{c}\right) - \frac{P_v}{m^2}\left(D_v + \frac{D}{c}\right)\dot{m} + \frac{P_v}{m}\left(D_{vv} + \frac{D_v}{c}\right)\dot{V} - \cancel{\dot{P}_h}^{\,0} = 0$$

Are a function of T.

$$\frac{P_v}{mc}\left[\frac{D}{c} + 2D_v + cD_{vv} \right] T = \frac{D}{c}\left[P_h + \frac{P_v}{m}\left(cD_{vv} + D_v + \frac{cD_{vv}g}{D} + \frac{D_v g}{D} \right) \right]$$

$$\boxed{T_{sing} = F(m, D, V, h, P_v, P_h, P_m)}$$

Nec. Cond. for a singular arc extremal:

$$-\frac{\partial}{\partial T}\ddot{H}_u = \boxed{-\frac{P_v}{mc}\Big[\underbrace{2D_v + \frac{D}{c} + c\,D_{vv}}_{\geq 0}\Big] \geq 0}$$

— N.C. for the singular arc

Expect $P_v < 0$;

IF singular control is implemented, at some point we will revert back to $T \sim T_{max}$.

$$T_{sing} \sim \frac{D P_h m}{P_v} + \cdots \quad ; \; P_h = -1 \; , \quad P_v < 0 \; , \; P_{vf} = 0$$

as $t \to t_f$ $P_v \to 0^-$

$T_{sing} \to \infty$ as $t \to t_f$

$T_{sing} \to T_{max}$