ASEN 6060
ADVANCED ASTRODYNAMICS
Equilibrium Points

Objectives:

- Introduce equilibrium points, devise strategy for locating the collinear equilibrium points
- Locate the triangular equilibrium points
- Study stability of equilibrium points via linearization

Locating Equilibrium Points

Recall the equations of motion:

$$\ddot{x} - 2\dot{y} = \frac{\partial U^*}{\partial x} \qquad \ddot{y} + 2\dot{x} = \frac{\partial U^*}{\partial y} \qquad \ddot{z} = \frac{\partial U^*}{\partial z}$$

Where $U^* = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$

$$r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2} \qquad r_2 = \sqrt{(x-1+\mu)^2 + y^2 + z^2}$$

Equilibrium points are stationary solutions in the rotating frame, i.e.,

Locating Equilibrium Points

Plugging these conditions into the EOMs, an equilibrium point is a stationary point in the pseudo-potential function:

Where:

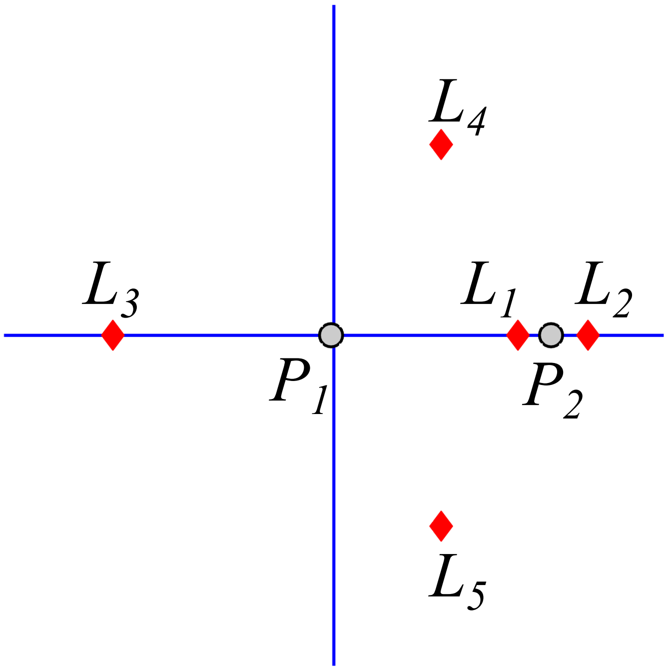
$$\frac{\partial U^*}{\partial x} = x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3}$$

$$\frac{\partial U^*}{\partial y} = y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}$$

$$\frac{\partial U^*}{\partial z} = -\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3}$$

Equilibrium Points in the CR3BP

There are five equilibrium points in the CR3BP:



Locating Equilibrium Points

All five equilibrium points lie in the plane of the primaries

The condition $$\frac{\partial U^*}{\partial z} = -\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3} = 0$$

Locating Collinear Equilibrium Points

One possible solution to the second condition

$$\frac{\partial U^*}{\partial y} = y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3} = 0$$

is $y = 0$, corresponding to collinear equilibrium points L_1, L_2, L_3

For the collinear equilibrium points, the remaining condition is:

$$\frac{\partial U^*}{\partial x} = x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3} = 0$$

Equivalent, in this case, to:

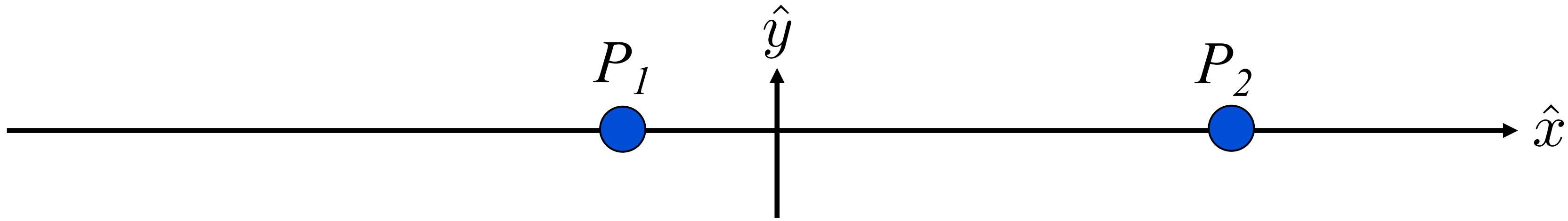
$$x - \frac{(1-\mu)(x+\mu)}{(|x+\mu|)^3} - \frac{\mu(x-1+\mu)}{(|x-1+\mu|)^3} = 0$$

Recall $r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}$ $r_2 = \sqrt{(x-1+\mu)^2 + y^2 + z^2}$

Locating Collinear Equilibrium Points

$$x - \frac{(1-\mu)(x+\mu)}{(|x+\mu|)^3} - \frac{\mu(x-1+\mu)}{(|x-1+\mu|)^3} = 0$$

There are three regions on the x-axis where a solution may lie:



Root-finding problem to solve *f(x)*=0, using 3 initial guesses

Locating Triangular Equilibrium Points

Another solution to the second condition

$$\frac{\partial U^*}{\partial y} = y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3} = 0$$

exists with $y \neq 0$, locating the triangular equilibrium points L_4, L_5

The values of (x,y) at L_4, L_5 simultaneously satisfy:

$$\frac{\partial U^*}{\partial x} = x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3} = 0$$

$$\frac{\partial U^*}{\partial y} = y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3} = 0$$

Locating Triangular Equilibrium Points

Recall
$$U^* = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

The problem of finding stationary points in U^* was posed as:

$$0 = \frac{\partial U^*}{\partial x} = \frac{\partial U^*}{\partial y}$$

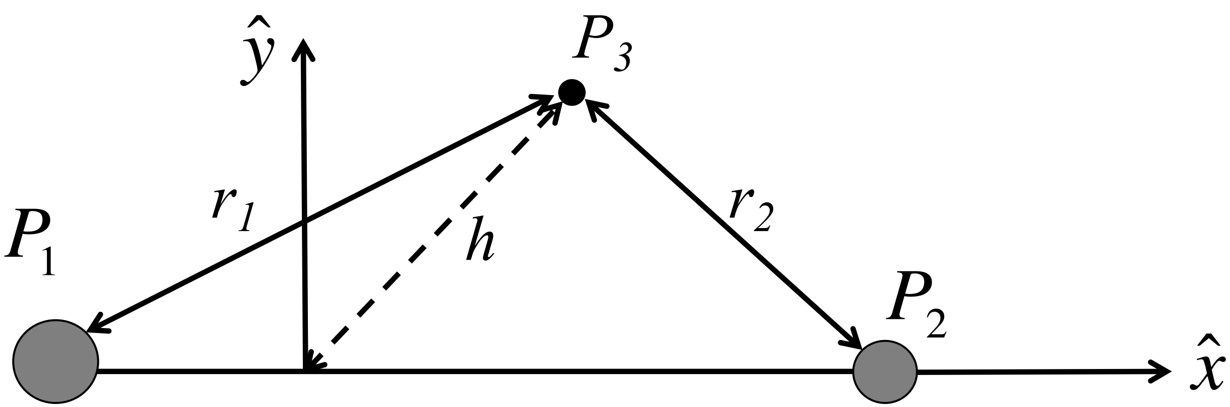
Since:
$$\frac{\partial U^*}{\partial x} = 0 = \frac{\partial U^*}{\partial r_1}\frac{\partial r_1}{\partial x} + \frac{\partial U^*}{\partial r_2}\frac{\partial r_2}{\partial x}$$

This is equivalent to solving the following conditions:

Easier to solve for L_4, L_5 by rewriting U^* in r_1, r_2

Locating Triangular Equilibrium Points

To rewrite U^* as a function of r_1, r_2, note:



Known quantities at significant locations:

Locating Triangular Equilibrium Points

Solve for a, b, c by using known quantities $h^2 = x^2 + y^2$

At location A):

$$h^2 = ar_1^2 + br_2^2 + c$$

$$(1-\mu)^2 = a + c \longrightarrow c = (1-\mu)^2 - a$$

At location B):

$$h^2 = ar_1^2 + br_2^2 + c$$

$$\mu^2 = b + c \longrightarrow c = \mu^2 - b$$

Combine these two results and simplify:

$$a = 1 - 2\mu + b$$

Locating Triangular Equilibrium Points

At location C),:

$$h^2 = ar_1^2 + br_2^2 + c$$

$$0 = a\mu^2 + b(1-\mu)^2 + c$$

Plug in for *a*(*b*) and *c*(*b*) to solve for b: $b = \mu$

Plug this value for *b* back into *a*(*b*) and *c*(*b*) to produce:

$$h^2 = x^2 + y^2 = ar_1^2 + br_2^2 + c = (1-\mu)r_1^2 + \mu r_2^2 + \mu(\mu - 1)$$

$$x^2 + y^2 = (1-\mu)r_1^2 + \mu r_2^2 + \mu(\mu - 1)$$

Locating Triangular Equilibrium Points

$$U^* = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

Use the previous result to rewrite pseudo-potential function in r_1, r_2

$$U^* = \frac{1}{2}\big((1-\mu)r_1^2 + \mu r_2^2 + \mu(\mu-1)\big) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

Then, solve

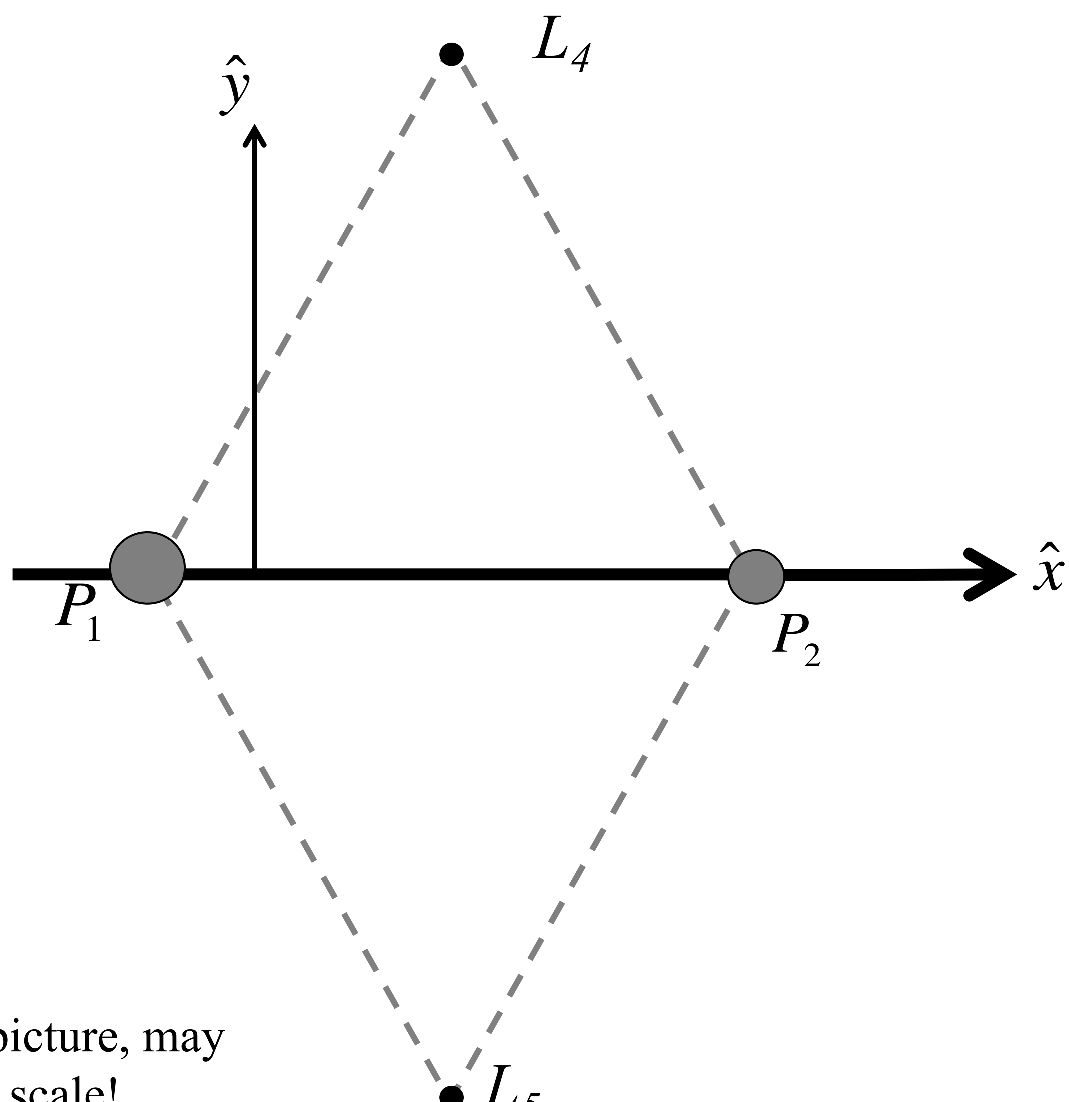
$$0 = \frac{\partial U^*}{\partial r_1} = \frac{\partial U^*}{\partial r_2}$$

where

$$\frac{\partial U^*}{\partial r_1} = (1-\mu)r_1 - \frac{1-\mu}{r_1^2}$$

$$\frac{\partial U^*}{\partial r_2} = \mu r_2 - \frac{\mu}{r_2^2}$$

Locating Triangular Equilibrium Points



Note: conceptual picture, may not be perfectly to scale!

Equilibrium Points in the CR3BP

$$C(L_1) > C(L_2) > C(L_3) > C(L_4) = C(L_5)$$

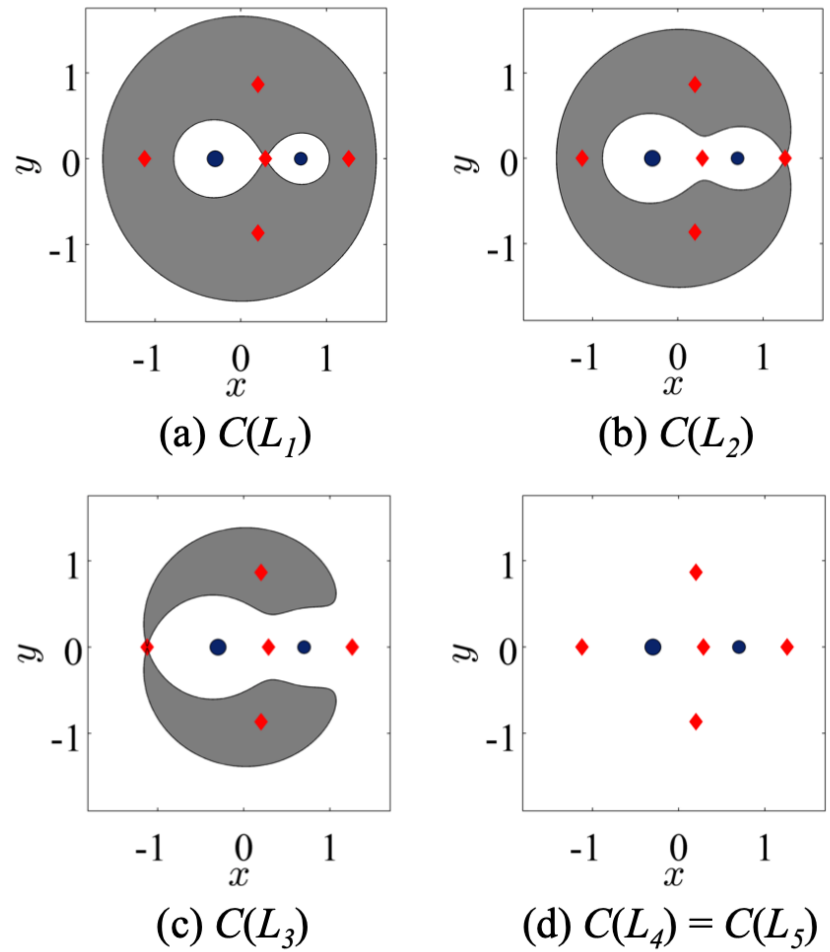


Image credit:
Bosanac, 2016

Assessing Stability of Equilibrium Points

Consider the ODEs for our nonlinear system $\dot{\bar{x}} = \bar{f}(\bar{x})$

The equilibrium points $\bar{x}_{eq}$ are the solutions where $\bar{f}(\bar{x}_{eq}) = \bar{0}$

Linearize about each equilibrium point to write linear system

$$\delta\dot{\bar{x}} = A\delta\bar{x}$$

Eigenvalues of constant matrix A reveal stability information in linear system. In some cases, can extrapolate to characterize stability in nonlinear system.

Linearizing About Equilibrium Points

Recall the EOMs for the CR3BP

$$\ddot{x} - 2\dot{y} = \frac{\partial U^*}{\partial x} \qquad \ddot{y} + 2\dot{x} = \frac{\partial U^*}{\partial y} \qquad \ddot{z} = \frac{\partial U^*}{\partial z}$$

And setting $\bar{x}_{eq} = (x_e, y_e, z_e)$

Perturb each equilibrium point:

Linearizing About Equilibrium Points

Eg: Linearize in two dimensions x,y about equilibrium point at (a,b)

$$f(x,y) \approx f(a,b) + \left.\frac{\partial f(x,y)}{\partial x}\right|_{a,b}(x-a) + \left.\frac{\partial f(x,y)}{\partial y}\right|_{a,b}(y-b)$$

Linearizing the EOMs for the CR3BP about an equilibrium point to produce following variational equations:

$$\ddot{\xi} - 2\dot{\eta} = U^*_{xx}\big|_{\bar{x}_{eq}}\xi + U^*_{xy}\big|_{\bar{x}_{eq}}\eta + U^*_{xz}\big|_{\bar{x}_{eq}}\delta$$

$$\ddot{\eta} + 2\dot{\xi} = U^*_{yx}\big|_{\bar{x}_{eq}}\xi + U^*_{yy}\big|_{\bar{x}_{eq}}\eta + U^*_{yz}\big|_{\bar{x}_{eq}}\delta$$

$$\ddot{\delta} = U^*_{zx}\big|_{\bar{x}_{eq}}\xi + U^*_{zy}\big|_{\bar{x}_{eq}}\eta + U^*_{zz}\big|_{\bar{x}_{eq}}\delta$$

Linearizing About Equilibrium Points

$$U_{xx}^* = 1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} + \frac{3(1-\mu)(x+\mu)^2}{r_1^5} + \frac{3\mu(x-1+\mu)^2}{r_2^5}$$

$$U_{xy}^* = \frac{3(1-\mu)(x+\mu)y}{r_1^5} + \frac{3\mu(x-1+\mu)y}{r_2^5} = U_{yx}^*$$

$$U_{xz}^* = \frac{3(1-\mu)(x+\mu)z}{r_1^5} + \frac{3\mu(x-1+\mu)z}{r_2^5} = U_{zx}^*$$

$$U_{yy}^* = 1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} + \frac{3(1-\mu)y^2}{r_1^5} + \frac{3\mu y^2}{r_2^5}$$

$$U_{yz}^* = +\frac{3(1-\mu)yz}{r_1^5} + \frac{3\mu yz}{r_2^5} = U_{zy}^*$$

$$U_{zz}^* = -\frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} + \frac{3(1-\mu)z^2}{r_1^5} + \frac{3\mu z^2}{r_2^5}$$

Variational Equations

At the equilibrium points, z=0 → $U^*_{xz} = U^*_{zx} = U^*_{yz} = U^*_{zy} = 0$

The variational equations about the equilibrium points become:

$$\ddot{\xi} - 2\dot{\eta} = U^*_{xx}\Big|_{\bar{x}_{eq}} \xi + U^*_{xy}\Big|_{\bar{x}_{eq}} \eta$$

$$\ddot{\eta} + 2\dot{\xi} = U^*_{yx}\Big|_{\bar{x}_{eq}} \xi + U^*_{yy}\Big|_{\bar{x}_{eq}} \eta$$

$$\ddot{\delta} = U^*_{zz}\Big|_{\bar{x}_{eq}} \delta$$

Out-of-Plane Modes

Focus on $\ddot{\delta} = U^*_{zz}\Big|_{\bar{x}_{eq}} \delta$

At eq. pt $U^*_{zz}\Big|_{\bar{x}_{eq}} = -\dfrac{1-\mu}{r_1^3} - \dfrac{\mu}{r_2^3} + \dfrac{3(1-\mu)z^2}{r_1^5} + \dfrac{3\mu z^2}{r_2^5} < 0$

Rewrite out-of-plane variational equation as:

If

In-Plane Modes

Focus on in-plane variational equations

$$\ddot{\xi} - 2\dot{\eta} = U^*_{xx}\Big|_{\bar{x}_{eq}}\xi + U^*_{xy}\Big|_{\bar{x}_{eq}}\eta \qquad \ddot{\eta} + 2\dot{\xi} = U^*_{yx}\Big|_{\bar{x}_{eq}}\xi + U^*_{yy}\Big|_{\bar{x}_{eq}}\eta$$

In matrix form:

$$\begin{bmatrix}\dot{\xi}\\ \dot{\eta}\\ \ddot{\xi}\\ \ddot{\eta}\end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ U^*_{xx}\Big|_{\bar{x}_{eq}} & U^*_{xy}\Big|_{\bar{x}_{eq}} & 0 & 2\\ U^*_{yx}\Big|_{\bar{x}_{eq}} & U^*_{yy}\Big|_{\bar{x}_{eq}} & -2 & 0\end{bmatrix}\begin{bmatrix}\xi\\ \eta\\ \dot{\xi}\\ \dot{\eta}\end{bmatrix}$$

$$\begin{bmatrix}\dot{\xi}\\ \dot{\eta}\\ \ddot{\xi}\\ \ddot{\eta}\end{bmatrix} = \begin{bmatrix} 0_{2\text{x}2} & I_{2\times 2}\\ U^*_{XX} & \Omega\end{bmatrix}\begin{bmatrix}\xi\\ \eta\\ \dot{\xi}\\ \dot{\eta}\end{bmatrix} \longrightarrow \delta\dot{\bar{x}}_{2D} = A_{2D}\delta\bar{x}_{2D}$$

In-Plane Modes

Solutions to this system of in-plane variational equations take the following form:

$$\xi = \sum_{i=1}^{4} A_i e^{\lambda_i t} \qquad \eta = \sum_{i=1}^{4} B_i e^{\lambda_i t}$$

To calculate eigenvalues, calculate characteristic equation for $[A_{2D}]$

$$|A_{2D} - \lambda I| = 0$$

In-Plane Modes

Setting $\Lambda = \lambda^2$, can rewrite characteristic equation as:

Applying quadratic equation: (dropping $\bar{x}_{eq}$ subscript for conciseness)

Two solutions: Λ_1, Λ_2 → Four eigenvalues λ_i

Stability of Equilibrium Points

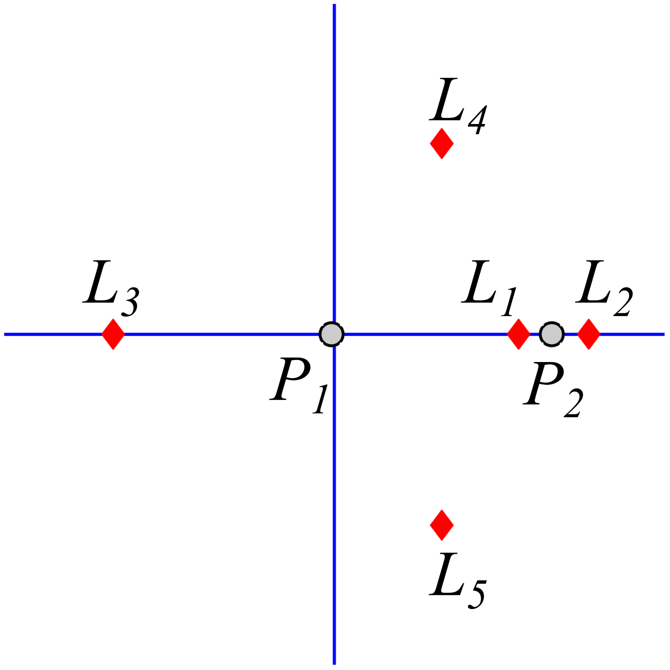
In linearized system, 6 eigenvalues
(2 out-of-plane, 4 in-plane)

$$\lambda = a \pm bi$$

$$\lambda = a$$

$$\lambda = \pm bi$$

$$\lambda = a \pm bi$$



Stability of Equilibrium Points

In some cases, stability in linear system offers prediction for nonlinear system:

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