Question 1

Monday, March 7, 2022 12:10

FIRST AND GIVE THE MINIMUM DISTANCE OPTIMAL CONTROL PROBLEM BASED ON CLOSE FOR THE FOLLOWING TERMINAL CONSTRAINTS

ASSUMING THE SAME DYNAMICS FROM PREV CASE: $\dot{y}_1 = \dot{x}^1 + y^2 + u$. THE COST CAME BECAUSE THE FOLLOWS DIAGRAMS:

(a) $y_1 = y + x^2 - y$

WHERE $x(t)$ AND $y(t)$, SEPARATELY DESCRIBE THE SOLUTIONS WHERE $P_1^1(t_f, x(t_f))$, $P_2^1(t_f, x(t_f))$, AND $P_3^1(t_f, x(t_f))$.

Dynamics:

$\dot{x}_1 = \cos\alpha$ $\quad t \geq 0$

$\dot{x}_2 = \sin\alpha$ $\quad x_1 + x_2 = 5$

$\dot{x}_3 = \frac{\sin\alpha}{x_1}$ $\quad$ Terminal constraint for x_1 and x_2 $\quad$ Control

$\dot{x}_1(t_f, y_1) = 0 = \frac{\partial}{\partial x_1} y_1 = 0$ $\rightarrow$ constant, $\alpha(t) = 1$

$\dot{x}_2(t_f, y_2) = 0 = \frac{\partial}{\partial x_2} y_2 = 0$ $\rightarrow$ line of slope α & y-intercept β

FOR THE (FREE) TERMINAL TIME & IDENTIFY THE POINTS (x_1, x_2)

$H = \beta^1 \cdot \dot{x} + \alpha \cos\alpha + \beta_3 \sin\alpha$

$\frac{\partial H}{\partial x_1} = -\beta_1 \cos\alpha + \beta_3 \sin\alpha + 0$ $\qquad \dot{\alpha}\alpha^2 = \frac{\sin\alpha}{x_1} - \frac{\sin\alpha}{x_1^2}$ $\qquad \frac{\partial H}{\partial \alpha} = \frac{\beta_3}{x_1^2}$

$\frac{\sin\alpha}{x_1^2} = \frac{\beta_3}{x_1^2}$

$\dot{\alpha}\alpha^2 = \frac{\beta_3}{x_1^2}$ $\qquad \dot{\alpha}\alpha^2 = \frac{\beta_3}{x_1^2}$ $\qquad \dot{\alpha} = \frac{\beta_3}{x_1^2 \alpha^2}$

$\Rightarrow \frac{\dot{\alpha}\alpha^2}{x_1^2} = \frac{\beta_3}{x_1^2}$ $\Rightarrow \dot{\alpha}^2$ is independent of α^2 or α = constant

EQUATIONS OF MOTION:

$\dot{x}_1 = \frac{\partial H}{\partial \beta_1} = \frac{\beta_3}{x_1^2 \alpha^2}$ $\qquad \dot{\beta}_3 = -\frac{\partial H}{\partial x_1} = \frac{2\beta_3}{x_1^3 \alpha^2}$

$\dot{\beta}_1 = -\frac{\partial H}{\partial x_2} = 0$ $\qquad \dot{\beta}_2 = -\frac{\partial H}{\partial \alpha} = 0$

$\frac{d\alpha}{dx_1} = \frac{\dot{\alpha}}{\dot{x}_1} = \frac{\beta_3}{x_1^2 \alpha^2}$ $\qquad \frac{d\beta_3}{dx_1} = \frac{\dot{\beta}_3}{\dot{x}_1} = \frac{2\beta_3}{x_1^3 \alpha^2}$

$\frac{d\alpha}{d\beta_3} = \frac{1}{2x_1}$ $\qquad \alpha = \frac{1}{2}\ln(\beta_3) + \beta_1$

β_1, β_2 = constant

$\Rightarrow$ find the points (x_1, x_2) & (x_1, y_1). at terminal normals are parallel & point to each other

APPLY TO GIVEN BOUNDARIES

$y_1 = 0 = x_1^2 + x_2^2 + \beta$

$y_2 = 0 = y^2 + x_1^2 \quad \alpha = -y, \beta = 0$

$\Rightarrow \begin{bmatrix} 2x_1 & \alpha \\ 2x_2 & 1 \end{bmatrix} \qquad |G| = 0 = \begin{vmatrix} 2x_1 & -2y_1 \\ 2x_2 & 1 \end{vmatrix}$

$\Rightarrow \boxed{\alpha = \frac{x_2}{x_1}}$

AFTER FROM TRANSVERSALITY CONDITIONS

$\beta_1 = \beta_2 = 0 \qquad \frac{\partial y_1}{\partial x_1} = \frac{\partial y_2}{\partial x_1} = 0 \qquad \frac{\partial y_1}{\partial x_2} = \frac{\partial y_2}{\partial x_2} = 0$

$\beta_3 = \beta_4 \qquad \frac{\partial y_1}{\partial t_f} = \frac{\partial y_2}{\partial t_f} = 0 \qquad \frac{\partial y_1}{\partial \alpha} = \frac{\partial y_2}{\partial \alpha} = 0$

$\Rightarrow -\beta_1 \frac{\partial y_1}{\partial x_1} - \beta_2 \frac{\partial y_2}{\partial x_1} = 0 \qquad \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = 0$

$\Rightarrow -\beta_1 \frac{\partial y_1}{\partial x_2} - \beta_2 \frac{\partial y_2}{\partial x_2} = 0 \qquad \begin{bmatrix} \frac{\partial y_1}{\partial \alpha} & \frac{\partial y_2}{\partial \alpha} \\ \frac{\partial y_1}{\partial t_f} & \frac{\partial y_2}{\partial t_f} \end{bmatrix}$

$|G| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} = 0$ $\qquad$ G = must be singular det(G) = 0

$\frac{\partial y}{\partial x_1} = \frac{\partial y}{\partial x_2} = \frac{\partial y}{\partial \alpha} = \frac{\partial y}{\partial t_f} = 0$ $\qquad$ $|G| = 0$ $\Rightarrow$ $\beta_1 = \beta_2 = 0$

$\Rightarrow$ find the points (x_1, y_1) & (x_2, y_2). at terminal normals are parallel & point to each other

AFTER FROM TRANSVERSALITY CONDITIONS

$\beta_1 = \beta_2 = 0 \qquad \frac{\partial y_1}{\partial x_1} = \frac{\partial y_2}{\partial x_1} = 0 \qquad \frac{\partial y_1}{\partial x_2} = \frac{\partial y_2}{\partial x_2} = 0$

$\beta_3 = \beta_4 \qquad \frac{\partial y_1}{\partial t_f} = \frac{\partial y_2}{\partial t_f} = 0 \qquad \frac{\partial y_1}{\partial \alpha} = \frac{\partial y_2}{\partial \alpha} = 0$

$\Rightarrow -\beta_1 \frac{\partial y_1}{\partial x_1} - \beta_2 \frac{\partial y_2}{\partial x_1} = 0 \qquad \Rightarrow -\beta_1(2x_1) = \beta_2 y^2$

$\Rightarrow -\beta_1 \frac{\partial y_1}{\partial x_2} - \beta_2 \frac{\partial y_2}{\partial x_2} = 0 \qquad \Rightarrow -\beta_1(2x_2) = \beta_2 y^2$

and $\frac{\partial y_1}{\partial \alpha} = \frac{\partial y_2}{\partial \alpha} = \frac{-2\beta_2 y}{\partial x_1 \partial \alpha} = \frac{\partial y_2}{\partial \alpha} = 2 \Rightarrow \boxed{\text{slope } \alpha^2 = \frac{1}{2}}$ $\rightarrow$ the infinite level of the real numbers?

$\frac{\partial y_1}{\partial t_f} = \frac{\partial y_2}{\partial t_f} = 0$

Dynamics

$\dot{x}_1 = \frac{1}{\sqrt{\beta_3^2 + \beta_4^2}} = 1$

$\Rightarrow \beta_3^2 + \beta_4^2 = 1 \qquad \Rightarrow \sqrt{\beta_3^2 + \beta_4^2} = 1 \qquad \Rightarrow \beta_3 \in \mathbb{R}, \beta_4 \in \mathbb{R}$

$\beta_3^2 + \beta_4^2 = 1 \qquad \Rightarrow \beta_3 \in \mathbb{R}, \beta_4 \in \mathbb{R}$ indefinite

$\dot{x}_1 = \frac{\beta_3}{\sqrt{\beta_3^2 + \beta_4^2}} = \frac{\beta_3}{\sqrt{\beta_3^2 + 1}}$ $\qquad \dot{x}_2 = \frac{\beta_4}{\sqrt{\beta_3^2 + \beta_4^2}} = \frac{\beta_4}{\sqrt{\beta_3^2 + 1}}$

$\Rightarrow \frac{\dot{x}_1}{\dot{x}_2} = \frac{\beta_3}{\beta_4} = \frac{\beta_3}{\sqrt{\beta_3^2 + 1}} \qquad \Rightarrow \frac{\dot{x}_1}{\dot{x}_2} = \frac{\beta_3}{\sqrt{\beta_3^2 + 1}}$

$\Rightarrow \frac{\dot{x}_1}{\dot{x}_2} = \frac{\beta_3}{\sqrt{\beta_3^2 + 1}} = \frac{\beta_3}{\sqrt{1 + \beta_3^2}}$

$\Rightarrow \frac{\dot{x}_1}{\dot{x}_2

Question 2

Monday, March 14, 2022 11:58

RECALL THE CONSTANT GRAVITY, PRESCRIBED THRUST PROBLEM:

$$\dot{x} = u$$
$$\dot{y} = v$$
$$\dot{u} = f(t)\cos\theta$$
$$\dot{v} = f(t)\sin\theta - g$$

WHERE THE THRUST f IS A PRESCRIBED FUNCTION OF TIME AND θ IS THE CONTROL ANGLE. ASSUMING THE LAGRANGE COST FUNCTION $L \equiv 0$, STATE THE OPTIMAL CONTROL HAMILTONIAN AND SOLVE FOR THE GENERAL FORM OF THE OPTIMAL CONTROL LAW, AS SEEN IN CLASS. SHOW THAT THE TIME RATE OF CHANGE OF THE CONTROL ANGLE θ IS MONOTONIC, i.e. THAT IT ONLY INCREASES OR DECREASES AS A FUNCTION OF TIME.

$$H = L + \vec{p}\cdot\vec{f}$$

$$= 0 + \begin{bmatrix} p_x \\ p_y \\ p_u \\ p_v \end{bmatrix} \cdot \begin{bmatrix} u \\ v \\ f(t)\cos\theta \\ f(t)\sin\theta - g \end{bmatrix}$$

$$H = p_x \cdot u + p_y v + p_u f(t)\cos\theta + p_v(f(t)\sin\theta - g)$$

$$\frac{\partial H}{\partial \theta} = -p_u f(t)\sin\theta + p_v f(t)\cos\theta = 0$$

if $f(t) \neq 0$ [no thrust]

$$\frac{p_v}{p_u} = \frac{\sin\theta}{\cos\theta} \Rightarrow \boxed{\tan\theta^* = \frac{p_v}{p_u}}$$

$$\cos\theta^* = \frac{p_u}{\sqrt{p_v^2+p_u^2}} \qquad \sin\theta^* = \frac{p_v}{\sqrt{p_u^2+p_v^2}}$$

$$H^* = p_x u + p_y v + \underline{f(t)(p_u\cos\theta^* + p_v\sin\theta^*)} - p_v g$$

$$f(t)\left[p_u \frac{p_u}{\sqrt{p_v^2+p_u^2}} + p_v \frac{p_v}{\sqrt{p_u^2+p_v^2}}\right]$$

$$f(t)\left[\frac{p_u^2+p_v^2}{\sqrt{p_u^2+p_v^2}}\right] = f(t)\sqrt{p_v^2+p_u^2}$$

$$\boxed{H^* = p_x u + p_y v + f(t)\sqrt{p_u^2+p_v^2} - p_v g}$$

↳ OPTIMAL CONTROL HAMILTONIAN

STATE EQUATIONS

$$\dot{x} = \frac{\partial H^*}{\partial p_x} = u \qquad \dot{u} = \frac{\partial H^*}{\partial p_u} = \frac{f p_u}{\sqrt{p_u^2+p_v^2}}$$

$$\dot{y} = \frac{\partial H^*}{\partial p_y} = v \qquad \dot{v} = \frac{\partial H^*}{\partial p_v} = \frac{f p_v}{\sqrt{p_u^2+p_v^2}} - g$$

ADJOINT EQUATIONS

$$\dot{p}_x = -\frac{\partial H^*}{\partial x} = 0 \qquad \dot{p}_y = -\frac{\partial H^*}{\partial y} = 0 \rightarrow p_x \,\&\, p_y \text{ are constant}$$

$$\dot{p}_u = -\frac{\partial H^*}{\partial u} = -p_x = -p_{xo} \qquad \dot{p}_v = -\frac{\partial H^*}{\partial v} = -p_y = -p_{yo}$$

p_u & p_v have linear relationship to time

$$\boxed{p_u(t) = p_{uo} - p_{xo}t \;;\; p_v(t) = p_{vo} - p_{yo}t}$$

Substitute the above expressions into $\tan\theta^* = \frac{p_v}{p_u}$

$$\boxed{\tan\theta^* = \frac{p_{vo} - p_{yo}t}{p_{uo} - p_{xo}t}}$$ → GENERAL FORM OF OPTIMAL CONTROL LAW

SHOW $\dot{\theta}(t)$ IS MONOTONIC

$$\frac{d}{dt}(\arctan(x)) = \frac{1}{1+x^2}$$

$$\dot{\theta}^* = \frac{d}{dt}\left[\arctan\left(\frac{p_{vo} - p_{yo}t}{p_{uo} - p_{xo}t}\right)\right]$$

$$\dot{\theta}^* = \frac{1}{1+\left(\frac{p_{vo} - p_{yo}t}{p_{uo} - p_{xo}t}\right)^2} \cdot \frac{d}{dt}\left[\frac{p_{vo} - p_{yo}t}{p_{uo} - p_{xo}t}\right]$$

$$= \frac{1}{1+\left(\frac{p_{vo} - p_{yo}t}{p_{uo} - p_{xo}t}\right)^2}\left(\frac{(p_{uo} - \cancel{p_{xo}t})(-p_{yo}) - (p_{vo} - \cancel{p_{yo}t})(-p_{xo})}{(p_{uo} - p_{xo}t)^2}\right)$$

$$= \frac{\overbrace{p_{vo}p_{xo} - p_{uo}p_{yo}}^{\text{CONSTANT}}}{\underbrace{(p_{uo} - p_{xo}t)^2}_{>0} + \underbrace{(p_{vo} - p_{yo}t)^2}_{>0}}$$

$$\boxed{\dot{\theta}^* = \frac{p_{vo}p_{xo} - p_{uo}p_{yo}}{(p_{uo} - p_{xo}t)^2 + (p_{vo} - p_{yo}t)^2}}$$

- $p_{vo}, p_{yo}, p_{uo}, p_{xo}$ → constants
- denominator > 0

sgn $(\dot{\theta}^*)$ CONSTANT DEPENDING ON SIGNS/VALUES OF CONSTANTS IN THE NUMERATOR ⇒ θ IS MONOTONIC

Question 3

Monday, March 14, 2022 14:08

CONSIDER THE OPTIMAL CONTROL OF A TIME-INVARIANT, LINEAR DYNAMICAL SYSTEM:

$$\dot{\bar{x}} = A\bar{x} + B\bar{u}$$

$n \times 1$ $n \times n$ $n \times m$ $m \times 1$

WHERE $\bar{x} \in \mathbb{R}^n$, $\bar{u} \in \mathbb{R}^m$ WITH $n \leq m$, $A \in \mathbb{R}^{n \times n}$ IS A CONSTANT MATRIX, AND $B \in \mathbb{R}^{n \times m}$ IS A CONSTANT MATRIX THE COST FUNCTION IS:

$$J = \underbrace{\frac{1}{2}\bar{x}_f^T S \bar{x}_f}_{K} + \int_{t_0}^{t_f} \underbrace{\frac{1}{2}\left[\bar{x}^T Q \bar{x} + \bar{u}^T R \bar{u}\right]}_{L} d\tau$$

$$\int_{t_0}^{t_f} L \, d\tau$$

WHERE $S \in \mathbb{R}^{n \times n}$ IS A SYMMETRIC MATRIX THAT MAY BE SINGULAR, $Q \in \mathbb{R}^{n \times n}$ IS A SYMMETRIC, NON-SINGULAR MATRIX. THE END TIME IS CONSTRAINED TO EQUAL t_f, BUT THE END STATE IS NOT CONSTRAINED. THE INITIAL STATE IS FULLY CONSTRAINED AND EQUAL TO $\bar{x}_0$ AT TIME $t = 0$.

(a) WRITE OUT THE NECESSARY CONDITIONS IN THEIR ENTIRETY, SOLVING FOR THE OPTIMAL CONTROL LAW, THE DYNAMICAL EQUATIONS, AND THE TRANSVERSALITY CONDITIONS. RECALL THAT A TIME INVARIANT LINEAR DYNAMICAL SYSTEM'S SOLUTION CAN BE WRITTEN AS:

$$\dot{\bar{y}} = M\bar{y}$$
$$\bar{y}(t) = e^{Mt}\bar{y}_0$$

WHERE e^M DENOTES THE EXPONENTIAL FUNCTION FOR A MATRIX A AND $\bar{y}_0$ IS THE INITIAL CONDITION.

$K = \frac{1}{2}\bar{x}_f^T S \bar{x}_f$

$L = \frac{1}{2}\left[\bar{x}^T Q \bar{x} + \bar{u}^T R \bar{u}\right]$

Equality constraints:

$$\left.\begin{aligned} g_{te} &= t(t_f) - t_f = 0 \\ g_{t0} &= t(t_0) - t_0 = 0 \\ \bar{g}_0 &= \bar{x}(t_0) - \bar{x}_0 = \bar{0} \end{aligned}\right\} \bar{g} = \bar{0}$$

PRE-HAMILTONIAN

USING DENOMINATOR LAYOUT

$$H = L + \bar{p} \cdot \bar{f}$$

$$H = \frac{1}{2}\left[\bar{x}^T Q \bar{x} + \bar{u}^T R \bar{u}\right] + \bar{p} \cdot (A\bar{x} + B\bar{u})$$

$$\frac{\partial H}{\partial \bar{u}} = \frac{1}{2}\left[2R\bar{u}\right] + B^T\bar{p} = 0$$

R IS SYMMETRIC

$$R\bar{u} = -B^T\bar{p}$$

$$\boxed{\bar{u}^* = -R^{-1}B^T\bar{p}} \rightarrow \text{OPTIMAL CONTROL LAW}$$

OPTIMAL CONTROL HAMILTONIAN

$$H^* = \frac{1}{2}\left[\bar{x}^T Q \bar{x} + \left[R^{-1}B^T\bar{p}\right]^T \cancel{R}\left[\cancel{R^{-1}}B^T\bar{p}\right]\right] + \bar{p} \cdot (A\bar{x} - R^{-1}B^T\bar{p})$$

$$= \frac{1}{2}\bar{x}^T Q \bar{x} + \frac{1}{2}\bar{p}^T B \underbrace{R^{-1}}_{\text{sym}} B^T \bar{p} + \bar{p}^T A\bar{x} - \bar{p}^T R^{-1} B^T \bar{p}$$

$$\boxed{H^* = \frac{1}{2}\bar{x}^T Q \bar{x} - \frac{1}{2}\bar{p}^T B R^{-1} B^T \bar{p} + \bar{p}^T A \bar{x}} \rightarrow \text{OPTIMAL CONTROL HAMILTONIAN}$$

DYNAMICAL EQUATIONS

$$\frac{\partial \bar{x}}{\partial t} = \frac{\partial H^*}{\partial \bar{p}}$$

$B^T R^{-1} B^T$

$B R^{-1} B^T$

$$\dot{\bar{x}} = -\frac{1}{2}\left[BR^{-1}B^T + (BR^{-1}B^T)^T\right]\bar{p} + A\bar{x}$$

$$= -\frac{1}{2}\left[2BR^{-1}B^T\right]\bar{p} + A\bar{x}$$

$$\boxed{\dot{\bar{x}} = -BR^{-1}B^T\bar{p} + A\bar{x}}$$

ADJOINT EQUATIONS

$$\frac{\partial \bar{p}}{\partial t} = -\frac{\partial H^*}{\partial \bar{x}}$$

Q: sym

$$\boxed{\dot{\bar{p}} = -Q\bar{x} - A^T\bar{p}}$$

$$\dot{\bar{y}} = \begin{bmatrix} \dot{\bar{x}} \\ \dot{\bar{p}} \end{bmatrix} = \overbrace{\begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}}^{M} \bar{y}$$

$$\bar{y}(t) = e^{Mt}\bar{y}_0 \rightarrow \bar{y}_0 = \begin{bmatrix} \bar{x}_0 \\ \bar{p}_0 \end{bmatrix}$$

TRANSVERSALITY CONDITIONS

1) $\bar{p}_0 = -\frac{\partial K}{\partial \bar{x}_0} - \bar{\lambda} \cdot \frac{\partial \bar{g}}{\partial \bar{x}_0}$

2) $\bar{p}_f = \frac{\partial K}{\partial \bar{x}_f} + \bar{\lambda} \cdot \frac{\partial \bar{g}}{\partial \bar{x}_f}$

3) $H_0 = \frac{\partial K}{\partial t_0} + \bar{\lambda} \cdot \frac{\partial \bar{g}}{\partial t_0}$

$H_f = -\frac{\partial K}{\partial t_f} - \bar{\lambda} \cdot \frac{\partial \bar{g}}{\partial t_f}$

1) $\bar{p}_0 = -\cancel{\frac{\partial K}{\partial \bar{x}_0}}^{0} - \bar{\lambda} \cdot \frac{\partial \bar{g}}{\partial \bar{x}_0} = -\bar{\lambda}_0 \cdot (-1) = \bar{\lambda}_0$

$$\boxed{\bar{p}_0 = \bar{\lambda}_0}$$

2) $\bar{p}_f = \frac{\partial K}{\partial \bar{x}_f} + \bar{\lambda} \cdot \frac{\partial \bar{g}}{\partial \bar{x}_f} = S\bar{x}_f \Rightarrow \boxed{\bar{p}_f = S\bar{x}_f}$

3) $H_0 = \cancel{\frac{\partial K}{\partial t_0}}^{0} + \bar{\lambda} \cdot \frac{\partial \bar{g}}{\partial t_0} = -\bar{\lambda}_{t_0} \Rightarrow \boxed{H_0 = -\bar{\lambda}_{t_0}}$

4) $H_f = -\cancel{\frac{\partial K}{\partial t_f}}^{0} - \bar{\lambda} \cdot \frac{\partial \bar{g}}{\partial t_f} = -\bar{\lambda}_{tf} \cdot (-1) = \bar{\lambda}_{tf} \Rightarrow \boxed{H_f = \bar{\lambda}_{tf}}$

(b) SOLVE THE PROBLEM FOR THE FOLLOWING CASE WITH $n=2$ & $m=1$.

DETERMINE OPTIMAL CONTROL LAW $\bar{u}^*$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix};\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix};\ S = I_{2\times2};\ Q = 0_{2\times2};\ R = 1;\ \bar{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\dot{\bar{y}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & \overset{2\times1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \cdot \overset{1\times2}{\begin{bmatrix} 0 & 1 \end{bmatrix}} \\ 0 & 0 & 1 \\ 0 & 0 & 1 \cdot 0 \cdot 0 \\ 0 & 0 & 1 \cdot -1 \cdot 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{bmatrix}$$

$$\dot{\bar{y}} = \overbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}}^{M} \bar{y}$$

$$\bar{y} = e^{Mt}\bar{y}_0$$

USING MATLAB SYMBOLIC TOOLBOX & expm(Mt)

$$e^{Mt} = \begin{bmatrix} 1 & t & \frac{1}{6}t^3 & -\frac{1}{2}t^2 \\ 0 & 1 & \frac{1}{2}t^2 & -t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -t & 1 \end{bmatrix}$$

APPLY AT FINAL CONDITIONS

$$\underbrace{\begin{bmatrix} \bar{x}_f \\ \bar{p}_f \end{bmatrix}}_{\bar{x}_f} = \underbrace{\begin{bmatrix} 1 & t_f & t_f^3/6 & -t_f^2/2 \\ 0 & 1 & t_f^2/2 & -t_f \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -t_f & 1 \end{bmatrix}}_{\Phi} \begin{bmatrix} 1 \\ 1 \\ \bar{p}_0 \end{bmatrix} \Bigg\} \bar{x}_0 \qquad \Phi = \begin{bmatrix} \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_4 \end{bmatrix}$$

$$\underbrace{\bar{p}_f}_{S\bar{x}_f} = \underbrace{\begin{bmatrix} 1 & 0 \\ -t_f & 1 \end{bmatrix}}_{S} \bar{p}_0 = \bar{x}_f$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_f = x_f$$

TWO EXPRESSIONS FOR $\bar{x}_f \rightarrow$ SOLVE FOR $\bar{p}_0 \rightarrow$ OPTIMAL CONTROL LAW ONLY RELIES ON $\bar{p}$ & KNOW DYNAMICS GIVEN $\bar{p}_0$. ALSO FROM $\bar{y} = e^{Mt}\bar{y}_0$ CAN COMPUTE $\bar{x}_f$ THUS FULLY SOLVING THE OPTIMAL CONTROL PROBLEM

$$\Phi_1\bar{x}_0 + \Phi_2\bar{p}_0 = \cancel{\Phi_3\bar{x}_0}^{0} + \Phi_4\bar{p}_0$$

$$\left[\Phi_4 - \Phi_2\right]\bar{p}_0 = \Phi_1\bar{x}_0 \Rightarrow \boxed{\bar{p}_0 = \overbrace{\left[\Phi_4 - \Phi_2\right]^{-1}\Phi_1}^{\Phi(t_f, t_0)}\bar{x}_0}$$

$$\boxed{\begin{bmatrix} \bar{x}(t) \\ \bar{p}(t) \end{bmatrix} = \Phi(t, t_0)\begin{bmatrix} \bar{x}_0 \\ \bar{p}_0 \end{bmatrix}}$$

$$\bar{u}^* = -R^{-1}B^T\bar{p} = \overset{1\times1}{-1}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = -p_2$$

$$\boxed{\bar{u}^* = -p_2(t) = \begin{bmatrix} -t & 1 \end{bmatrix}\bar{p}_0}$$

Question 4

Friday, March 11, 2022 12:20

HERE WE ANALYZE A 1-DIMENSIONAL DYNAMICAL SYSTEM'S OPTIMAL CONTROL. ASSUME A COST FUNCTION AND DYNAMICAL SYSTEM:

$$J = \int_0^T |u| \, dt$$
$$\dot{x} = -x + u$$
$$|u| \le 1$$

WHERE $x \in \mathbb{R}$ AND WITH SPECIFIED INITIAL CONDITION $x(0) = 1$ AND FINAL CONDITION $x(T) = 0$. NOTE THAT THE INITIAL TIME IS SET TO $t = 0$.

(a) WRITE OUT THE PRE-HAMILTONIAN FOR THIS SYSTEM:

$$\begin{aligned} H(x,p,u) &= L + p \cdot \vec{f} \\ &= |u| + p \cdot (-x + u) \\ &\quad \hookrightarrow u \cdot \text{sgn}(u) \\ &= u \cdot \text{sgn}(u) + pu - px \\ \boxed{H(x,p,u)} &\boxed{= -px + u(p + \text{sgn}(u))} \end{aligned}$$

(b) USING THE MINIMUM PRINCIPLE, FIND THE OPTIMAL CONTROL LAW FOR THIS SYSTEM:

$$u^* = \underset{u}{\text{argmin}} \; H(u,p,x) = \underset{u}{\text{argmin}} \left[-px + u(p + \text{sgn}(u)) \right]$$

is our independent var

$p < -1$: u_{max}

$-1 < p < 0$: 0 } $|p| \le 1$: $u^* = 0$

$0 < p < 1$: 0

$p > 1$: $-u_{max}$

$|p| > 1$: $-u_{max} \cdot \text{sgn}(p)$

Look here some cases

$p = 1 + \epsilon$

$$(1+\epsilon)u + u \cdot \text{sgn}(u)$$

$u = +1 \to 1 - \epsilon + 1 \to 2 - \epsilon \quad +$

$u = -1 \to \epsilon - 1 = (-1) \quad \epsilon \quad +$

$u = 0 \to 0 \quad 0$

$\Rightarrow p > 1 \quad u^* = -1$

$p = 1 + \epsilon$

$$(1+\epsilon)u + u \cdot \text{sgn}(u)$$

$u = +1 \quad 1 + \epsilon + 1 = 2 + \epsilon \to +$

$u = -1 \quad -\epsilon - 1 + 1 \to -\epsilon \to -$

$u = 0 \to 0 \quad \to +$

$\Rightarrow p > 1 \quad u^* = -1$

$p = -0.5$

$$u(-0.5 + \text{sgn}(u))$$

$u = 0 \to 0$

$u = -x \quad -u(-0.5 - 1) \to +$

$u = x \quad x(-0.5 + 1) \to +$

$p = -1.5$

$$u(-1.5 + \text{sgn}(u))$$

$u = 0 \to 0$

$u = -x \quad -u(-1.5 - 1) \to +$

$u = x \quad x(-1.5 + 1) \to -$

$p = 0.5$

$$u(0.5 + \text{sgn}(u))$$

$u = 0 \to 0$

$u = -x \quad -u(0.5 - 1) \to +$

$u = x \quad x(0.5 + 1) \to +$

$p = 1.5$

$$u(1.5 + \text{sgn}(u))$$

$u = 0 \to 0$

$u = -x \quad -u(1.5 - 1) \to -$

$u = x \quad x(1.5 + 1) \to +$

$$\boxed{u^* = \begin{cases} 0 & |p| \le 1 \\ -u_{max} \, \text{sgn}(p) & |p| > 1 \end{cases}}$$

$u_{max} = 1$

(c) SUBSTITUTE THIS BACK INTO THE HAMILTONIAN TO FIND THE OPTIMAL CONTROL HAMILTONIAN, & THE ASSOCIATED EQUATIONS OF MOTION FOR THE STATES & ADJOINTS.

$$\boxed{H^*(x,p) = \begin{cases} -px & |p| \le 1 \quad \to u^* = 0 \\ -px - |p|+1 & |p| > 1 \quad \to u^* = -\text{sgn}(p) \end{cases}}$$

$$\dot{p} = -\frac{\partial H^*}{\partial x}$$

$$\boxed{\dot{p} = p}$$

$$\dot{x} = \frac{\partial H^*}{\partial p} = \begin{cases} -x & |p| \le 1 \\ -x - \frac{p}{|p|} & |p| > 1 \end{cases} \quad \begin{cases} p > 1 & -x - 1 \\ p < -1 & -x + 1 \end{cases}$$

$\hookrightarrow \pm 1 \to \text{sgn}(p)$

(d) THE ABOVE DIFFERENTIAL EQUATIONS FOR x & p CAN BE SOLVED IN CLOSED FORM. WRITE OUT THE EXPLICIT SOLUTIONS FOR THESE EQUATIONS FOR THE DIFFERENT PHASES OF THE OPTIMAL CONTROL LAW FOR THE GIVEN INITIAL CONDITION FOR x AND LEAVING THE INITIAL CONDITION FOR THE ADJOINT AS p_0.

THE FOLLOWING RESULT FROM CALCULUS MAY BE HELPFUL

$$\int_{x(t_0)}^{x(t)} \frac{dx}{ax+b} = \frac{1}{a} \ln(ax+b) \Big|_{t_0}^{t}$$

$x(0) = 1 \quad x(T) = 0$

$$\dot{p} = -\frac{\partial H^*}{\partial x} = p$$

$\Rightarrow \boxed{p(t) = p_0 e^t}$

$\hookrightarrow$ $p(t)$ CANNOT CHANGE SIGNS

$$\dot{x} = \frac{\partial H^*}{\partial p}$$

$|p| \le 1 \quad \dot{x} = -x \to \frac{dx}{dt} = -x$

$$-\int \frac{dx}{x} = \int dt$$
$$-\ln(x) = t + c$$
$$x = Ce^{-t} \qquad \begin{matrix} x(0) = 1 \\ \Rightarrow C = 1 \end{matrix}$$

$$\boxed{x = e^{-t}}$$

$|p| > 1 \quad \dot{x} = -x - \text{sgn}(p)$

$$\frac{dx}{dt} = -(x + \text{sgn}(p))$$
$$-\int \frac{dx}{x + \text{sgn}(p)} = \int dt$$
$$-\ln(x + \text{sgn}(p)) \Big|_0^t = t$$
$$-\ln(x_t + \text{sgn}(p)) + \ln(1 + \text{sgn}(p_0)) = t$$
$$\ln(x + \text{sgn}(p)) = -t + \ln(1 + \text{sgn}(p))$$
$$x(t) = \underbrace{(1 + \text{sgn}(p))}_{C} e^{-t} - \text{sgn}(p)$$

$\text{sgn}(p) = 1 \quad C = 2$

$\text{sgn}(p) = -1 \quad C = 0 \quad \Rightarrow \quad \boxed{x(t) = 1} \quad p < -1$

$$\boxed{x(t) = 2e^{-t} - 1} \quad p > 1$$

WITH GIVEN CONTROL, AUTOMATICALLY CONSTANT BECAUSE $x(t) = 0$

CASE 1	$\lvert p \rvert \le 1$	$x(t) = e^{-t}$	$u^* = 0$	COAST
CASE 2	$p > 1$	$x(t) = 2e^{-t} - 1$	$u^* = -1$	THRUST
CASE 3	$p < -1$	$x(t) = 1$	$u^* = 1$	$\begin{matrix} x = -x + u \\ = -1 + 1 \\ = 0 \end{matrix}$

(e) FIRST ASSUME THE FINAL TIME T IS "FREE". FIND THE OPTIMAL CONTROL & COST, SATISFYING THE TRANSVERSALITY CONDITIONS & THE PROBLEM STATEMENT.

$x(T) = 0$

$$x(T) = e^{-t_s} + 2e^{-(T-t_s)} - 1 = 0$$

Consider letting $T \to \infty$

$$0 = e^{-t_s} + 2\cancel{e^{-\infty}} - 1$$
$$-e^{-t_s} = -1$$

$\Rightarrow t_s \to 0$ since no switching occurs - always in coast case

$$\boxed{\begin{aligned} u^* &= 0 \\ J &= \int_0^\infty |u^*| \, dt = 0 \\ T &= \infty \end{aligned}}$$

(f) NEXT ASSUME THAT THE FINAL TIME IS SPECIFIED AS $T = \ln 2$. FIND THE OPTIMAL CONTROL & COST, SATISFYING THE TRANSVERSALITY CONDITIONS & THE PROBLEM STATEMENT.

$$x(T) = e^{-t_s} + 2e^{-(T-t_s)} - 1 = 0$$
$$x(\ln 2) = e^{-t_s} + 2e^{-(\ln 2 - t_s)} - 1$$

BY INSPECTION $t_s = 0 \Rightarrow$ NO SWITCHING OCCURS $\Rightarrow$ ALWAYS IN CASE 2 $\Rightarrow$ THRUST

$$u^* = -1$$
$$J = \int_0^{\ln 2} |-1| \, dt = \ln 2$$

$$\boxed{u^* = -1 \quad ; \quad J = \ln 2}$$

(g) FINALLY, ASSUME THAT THE FINAL TIME IS SPECIFIED AT A VALUE OF T SUCH THAT $T > \ln 2$. FIND THE OPTIMAL CONTROL & COST, SATISFYING THE TRANSVERSALITY CONDITIONS & THE PROBLEM STATEMENT.

FOR $T > \ln 2$ A COMBINATION OF CASE 1 (COAST) & CASE 2 (THRUST) WILL OCCUR. SWITCHING OCCURS ONLY ONCE SINCE P IS MONOTONICALLY INCREASING.

TO DETERMINE CONTROL SCHEME SOLVE FOR CONTINUITY BETWEEN THE TWO SCHEMES

$u^* = 0 \quad x(t) = e^{-t_s} \qquad x(t_s) = x(t_s)$

$u^* = -1 \quad x(t_s) = Ce^{-t_s} - 1 \qquad e^{-t_s} = Ce^{-t_s} - 1$

$\Rightarrow C = e^{t_s} + 1$

$$\boxed{\begin{aligned} x(t) &= \begin{cases} e^{-t} & 0 < t \le t_s \\ (e^{t_s}+1)e^{-t} - 1 & t_s \le t < T \end{cases} \\ u^* &= 0 \quad 0 < t \le t_s \\ u^* &= -1 \quad t_s < t_s \le T \quad T > \ln 2 \end{aligned}}$$

SOLVE FOR SWITCHING TIME

$$x(T) = 0 = (e^{t_s}+1)e^{-T} - 1$$
$$e^{t_s} = e^T - 1$$
$$\Rightarrow \boxed{t_s = \ln(e^T - 1)}$$

$$J^* = \int_0^{t_s} 0 \, dt + \int_{t_s}^{T} |-1| \, dt$$

$$\boxed{J^* = T - t_s}$$

$\hookrightarrow J^* = T - \ln(e^T - 1)$

Question 5

Monday, March 14, 2022 16:13

For the following 1-DOF problem (i.e. a 2-dimensional dynamical system), state the full set of necessary conditions & solve them, to the extent that you can, for an arbitrary initial state at time $t=0$, x_0, and v_0. This problem finds the minimum time solution from an initial position & speed to the origin with no speed.

$$\ddot{x} = u$$
$$|u| \le 1$$
$$x_f = 0$$
$$\dot{x}_f = 0$$
$$L \equiv 0$$
$$K = t_f$$

Pre-Hamiltonian

$$H = \vec{f}^T \vec{p} \cdot \vec{f} \qquad \vec{f} = \begin{bmatrix} v \\ u \end{bmatrix} \qquad \vec{p} = \begin{bmatrix} p_x \\ p_v \end{bmatrix}$$

$\hookrightarrow \vec{x} = \begin{bmatrix} x \\ v \end{bmatrix}$

$$H = p_x v + p_v u$$

$$\frac{\partial H}{\partial u} = p_v = 0 \rightarrow \text{this method goes nowhere...}$$

Pontryagin Minimum Principle

$$u^* = \underset{u}{\operatorname{argmin}}\ H = \underset{u}{\operatorname{argmin}} \left[p_x v + p_v u \right]$$
$$\Rightarrow u^* = -\underbrace{u_{max}}_{=1} \operatorname{sign}(p_v) = -\frac{p_v}{|p_v|}$$
$$= -\operatorname{sign}(p_v)$$

$$\boxed{u^* = -\operatorname{sign}(p_v) = -\frac{p_v}{|p_v|}}$$

Optimal Control Hamiltonian

$$H^* = p_x v - \frac{p_v \cdot p_v}{|p_v|} = p_x v - |p_v|$$

$$\boxed{H^* = p_x v - |p_v|}$$

State Dynamics

$$\dot{\vec{x}} = \frac{\partial H}{\partial \vec{p}} = \begin{bmatrix} v \\ u^* \end{bmatrix}$$

Adjoint Dynamics

$$\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{x}} = \begin{bmatrix} 0 \\ -p_x \end{bmatrix} \rightarrow \begin{matrix} \dot{p}_x = 0 \rightarrow p_x \text{ is constant} \\ \dot{p}_v = -p_x \end{matrix} \quad \boxed{p_{vo} = p_{xo}}$$

$$\frac{dp_v}{dt} = -p_x \rightarrow \int dp_v = -\int p_x\, dt$$
$$\boxed{p_v = -p_x t + p_{vo}}$$

$\rightarrow$ p_v is linear, which means $p_v = 0$ at most once & switching can only occur 0 times or 1 time.

Transversality Conditions

$$K = t_f \quad ; \quad \vec{g} = \begin{cases} r(t_f) - r_f \\ v(t_f) - v_f \\ t(0) - t_0 \end{cases}$$

r_0 & v_0 are not constrained.

$\hookrightarrow$ corresponding adjoints equal some unknown lagrange mult.

$$H_f = -\frac{\partial K}{\partial t_f} - \vec{\lambda}^T \cancel{\frac{\partial \vec{g}}{\partial t_f}}^{\,0} = -1 \qquad \boxed{H_f = -1}$$

$$\frac{d}{dt} H^* = 0 \rightarrow H_0 = H_f \qquad H_f = -1 = -\lambda_{t_0}$$
$$\boxed{H_0 = H_f = -1} \qquad \lambda_{t_0} = 1$$

Sub into H^*

$$p_{xo} = p_{vf}$$

$$p_{xo} v_0 + p_{vo} u^* = p_{xo} \cancel{v_f}^{\,0} + p_{vf} u^*_f$$
$$p_{xo} v_0 + p_{vo} u^* = p_{vf} u^*$$

$$H_f = -1 = \cancel{p_{xo} v_f} + p_{vf} u^*_f$$
$$p_{vf} u^*_f = -1$$
$$(-p_{xo} t_f + p_{vo}) u^*_f = -1 \rightarrow \underline{p_{vf} = -p_{xo} t_f + p_{vo}}$$

$$H_0 = -1 = p_{xo} v_0 + p_{vo} u^*_0 \rightarrow \boxed{p_{xo} = \frac{-p_{vo} u^*_0 - 1}{v_0}} \rightarrow \frac{|p_{vo}| - 1}{v_0}$$

$$p_{vo} u^*_0 = -|p_{vo}|$$
$$p_{vf} u^*_f = -|p_{vf}|$$

$$H_0 = H_f$$
$$p_{xo} v_0 + p_{vo} u^*_0 = p_{xo} v_f + p_{vf} u^*_f$$
$$p_{xo} v_0 - |p_{vo}| = -|p_{vf}|$$
$$p_{xo} v_0 = |p_{vo}| - |p_{vf}|$$

$$p_{xo} v_0 = |p_{vo}| - |-p_{xo} t_f + p_{vo}|$$

$$\frac{|p_{vo}| - 1}{v_0} v_0 = |p_{vo}| - \left| \frac{|-p_{vo}|}{v_0} t_f + p_{vo} \right|$$

$$\boxed{1 = \left| \frac{1 - |p_{vo}|}{v_0} t_f + p_{vo} \right|}$$

$\rightarrow$ the solution to takes the form of "bang-bang" control, where where the control acts at its maximum value towards the origin & then at some point the control switches to its minimum to achieve $v_f = 0$. This is reflected in the value of u^* that was determined to be $u^* = -u_{max} \cdot \operatorname{sign}(p_v)$.

There are 4 solution cases dependent on r_0 & v_0.

① $u^* = 1 \quad \forall t$

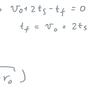
② $u^* = -1 \quad \forall t$

③ $u^* = \begin{cases} 1, & t < t_s \\ -1, & t > t_s \end{cases}$

④ $u^* = \begin{cases} -1, & t < t_s \\ 1, & t > t_s \end{cases}$

Case 1 $u^* = 1 \quad \forall t$

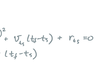
$$x(t_f) = t_f^2/2 + v_0 t_f + r_0 = 0$$
$$v(t_f) = t_f + v_0 = 0 \rightarrow \underline{t_f = -v_0} \rightarrow v_0 = 0 \text{, since } t_f > 0$$
$$x(t_f) = (-v_0)^2/2 - v_0^2 + r_0 = 0$$
$$= r_0 - v_0^2/2 = 0$$
$$\boxed{r_0 = v_0^2/2 > 0}$$



* Special case w/ $r_0 = \frac{v_0^2}{2}$, $v_0 < 0 \rightarrow J = t_f = |v_0|$

Case 2 $u^* = -1 \quad \forall t$

$$v(t_f) = 0 = -t_f + v_0 \quad \underline{t_f = v_0 > 0}$$
$$x(t_f) = -t_f^2/2 + v_0 t_f + r_0 = 0$$
$$-v_0^2/2 + v_0^2 + r_0 = 0$$
$$\boxed{r_0 = -v_0^2/2 < 0}$$



* Similar special case to ①

$r_0 = -\frac{v_0^2}{2}$, $v_0 > 0 \rightarrow J = t_f = |v_0|$

Switching Cases

Case 3 $u^* = \begin{cases} 1, & t < t_s \\ -1, & t > t_s \end{cases} \Rightarrow r_0 < 0$

@ t_s: $r(t_s) = t_s^2/2 + v_0 t_s + r_0$, $v(t_s) = v_0 + t_s$

@ t_f: $r(t_f) = \frac{-(t_f - t_s)^2}{2} + v_0 t_s (t_f - t_s) + r_0 = 0$, $v(t_f) = v(t_s) - (t_f - t_s)$

Sub. values for $r(t_s)$ & $v(t_s)$ & solve system of eqns

$$r(t_f) = \frac{-(t_f - t_s)^2}{2} + (v_0 + t_s)(t_f - t_s) + \left(\frac{t_s^2}{2} + v_0 t_s + r_0 \right) = 0$$

$$v(t_f) = v_0 + t_s - (t_f - t_s) = 0 \rightarrow v_0 + 2t_s - t_f = 0$$
$$t_f = v_0 + 2t_s$$

Solve for t_s, t_f

$$\underline{t_s = -v_0 \pm \sqrt{\tfrac{1}{2} v_0^2 - r_0}}$$
$$t_f = v_0 + 2\left(-v_0 \pm \sqrt{\tfrac{1}{2} v_0^2 - r_0}\right)$$
$$= -v_0 \pm 2\sqrt{\tfrac{1}{2} v_0^2 - r_0}$$
$$\underline{J = t_f = -v_0 \pm 2\sqrt{\tfrac{1}{2} v_0^2 - r_0}}$$

Case 4 $u^* = \begin{cases} -1 & t < t_s \\ 1 & t > t_s \end{cases}$

Similar to above

@ t_s: $r(t_s) = -t_s^2/2 + v_0 t_s + r_0$, $v(t_s) = v_0 - t_s$

@ t_f: $r(t_f) = \frac{(t_f - t_s)^2}{2} + v_0 (t_f - t_s) + r_0 = 0$, $v(t_f) = v(t_s) - (t_f - t_s)$

Set up system & solve

$$\underline{t_s = v_0 \pm \sqrt{\tfrac{1}{2} v_0^2 - r_0}}, \quad \underline{J = t_f = v_0 \pm 2\sqrt{\tfrac{1}{2} v_0^2 - r_0}}$$

Choose $\pm$ in t_s & t_f expressions for ③ & ⑤ s.t. $t_s, t_f > 0$