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ASEN 5010
Spring 2025
Josh Bhalawat

HW#2

problem ① 3.13 → 3-1-3 rotation → $\{\theta_1, \theta_2, \theta_3\} = \{-30°, 40°, 20°\}$

$$[C] = R_3(\theta_3)\cdot R_1(\theta_2)\cdot R_3(\theta_1) = \begin{bmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_2) & \sin(\theta_2) \\ 0 & -\sin(\theta_2) & \cos(\theta_2) \end{bmatrix}\begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9488 & -0.2429 & 0.2198 \\ 0.0637 & 0.7944 & 0.6040 \\ -0.3214 & -0.5567 & 0.7660 \end{bmatrix}$$

a] $\hat{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \dfrac{1}{2\sin\phi}\begin{bmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{bmatrix}$ $\quad \phi = \cos^{-1}\left(\frac{1}{2}(C_{11}+C_{22}+C_{33}-1)\right) = \cos^{-1}\left(\frac{1}{2}(0.9488+0.7944+0.7660)\right)$

$\phi = 41.18134°$

$$\hat{e} = \frac{1}{2\sin(41.18134°)}\begin{bmatrix} 0.6040 + 0.5567 \\ -0.3214 - 0.2198 \\ -0.2429 - 0.0637 \end{bmatrix} = \boxed{\begin{bmatrix} 0.8814 \\ -0.4110 \\ -0.2329 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}}$$

b] $\boxed{\phi = 41.18134°}$ $\quad \phi' = 2\pi - \phi = \boxed{318.8186° = \phi'}$

c] Using sheppard's method:

$\beta_0^2 = \frac{1}{4}(1 + \mathrm{tr}([C])) = 0.8763$ $\quad$ max $= \beta_0^2 \rightarrow \beta_0 = +\sqrt{\beta_0^2} = 0.9361$

$\beta_1^2 = \frac{1}{4}(1 + 2C_{11} - \mathrm{tr}([C])) = 0.0961$ $\quad \beta_0\beta_1 = \frac{1}{4}(C_{23} - C_{32}) \rightarrow \beta_1 = 0.31$

$\beta_2^2 = \frac{1}{4}(1 + 2C_{22} - \mathrm{tr}([C])) = 0.0209$ $\quad \beta_0\beta_2 = \frac{1}{4}(C_{31} - C_{13}) \rightarrow \beta_2 = -0.1445$

$\beta_3^2 = \frac{1}{4}(1 + 2C_{33} - \mathrm{tr}([C])) = 0.0067$ $\quad \beta_0\beta_3 = \frac{1}{4}(C_{12} - C_{21}) \rightarrow \beta_3 = -0.0819$

$$\boxed{\vec{\beta} = \begin{bmatrix} 0.9361 \\ 0.31 \\ -0.1445 \\ -0.0819 \end{bmatrix}}$$

d] CRP → $q_i = \dfrac{\beta_i}{\beta_0}$ → $q_1 = \beta_1/\beta_0 = \boxed{0.3311 = q_1}$

$q_2 = \beta_2/\beta_0 = \boxed{-0.1544 = q_2}$

$q_3 = \beta_3/\beta_0 = \boxed{-0.0875 = q_3}$

e] MRP → $\sigma_i = \dfrac{\beta_i}{1+\beta_0}$ → $\sigma_1 = \beta_1/(1+\beta_0) = \boxed{0.1601 = \sigma_1}$

$\sigma_2 = \beta_2/(1+\beta_0) = \boxed{-0.0747 = \sigma_2}$

$\sigma_3 = \beta_3/(1+\beta_0) = \boxed{-0.0423 = \sigma_3}$

Problem 2 → 3-15 → $\hat{e}^T\dot{\hat{e}}$ → $\dot{\hat{e}} = \frac{1}{2}\left([\tilde{e}] - \cot\left(\frac{\phi}{2}\right)[\tilde{e}]^2\right)\vec{\omega}$

$$\hat{e}^T\dot{\hat{e}} = \frac{1}{2}\left(\hat{e}^T[\tilde{e}] - \cot\left(\frac{\phi}{2}\right)\hat{e}^T[\tilde{e}]^2\right)\vec{\omega}, \quad \text{Let } \hat{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \quad \hat{e}^T = [e_1, e_2, e_3]$$

$$\hat{e}[\tilde{e}] = [e_1 \; e_2 \; e_3]\begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix} = \begin{bmatrix} e_2e_3 - e_2e_3 \\ -e_1e_3 + e_1e_3 \\ e_1e_2 - e_1e_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{e}^T[\tilde{e}]^2 = [e_1 \; e_2 \; e_3]\begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix} = [e_1 \; e_2 \; e_3]\begin{bmatrix} -e_3^2-e_2^2 & e_1e_2 & e_1e_3 \\ e_1e_2 & -e_3^2-e_1^2 & e_2e_3 \\ e_1e_3 & e_2e_3 & -e_2^2-e_1^2 \end{bmatrix}$$

$$= \begin{bmatrix} e_1(-e_3^2-e_2^2) + e_1e_2^2 + e_1e_3^2 \\ e_1^2e_2 + e_2(-e_3^2-e_1^2) + e_2e_3^2 \\ e_1^2e_3 + e_2^2e_3 + e_3(-e_2^2-e_1^2) \end{bmatrix} = \begin{bmatrix} \cancel{-e_1e_3^2} - \cancel{e_1e_2^2} + \cancel{e_1e_2^2} + \cancel{e_1e_3^2} \\ \cancel{e_1^2e_2} - \cancel{e_2e_3^2} - \cancel{e_2e_1^2} + \cancel{e_2e_3^2} \\ \cancel{e_1^2e_3} + \cancel{e_2^2e_3} - \cancel{e_3e_2^2} - \cancel{e_3e_1^2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \hat{e}^T\dot{\hat{e}} = \frac{1}{2}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \cot\left(\frac{\phi}{2}\right)\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)\vec{\omega} = \boxed{0 = \hat{e}^T\dot{\hat{e}}}$$

3-16 → Let $p = [a, b, c]^T$

a) $[\tilde{p}]^2$ (From 3-15) $= \begin{bmatrix} -c^2-b^2 & ab & ac \\ ab & -c^2-a^2 & bc \\ ac & bc & -b^2-a^2 \end{bmatrix}$, $pp^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}[a \; b \; c] = \begin{bmatrix} a^2 & ab & ac \\ ba & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$

$$\therefore [p][p]^T - [I_{3\times3}]p^2 = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix} - \begin{bmatrix} a^2+b^2+c^2 & 0 & 0 \\ 0 & a^2+b^2+c^2 & 0 \\ 0 & 0 & a^2+b^2+c^2 \end{bmatrix} = \begin{bmatrix} -c^2-b^2 & ab & ac \\ ab & -c^2-a^2 & bc \\ ac & bc & -b^2-a^2 \end{bmatrix}$$

$(p^2 = a^2+b^2+c^2)$

$$\therefore \boxed{[\tilde{p}]^2 = [p][p]^2 - [I_{3\times3}]p^2}$$

b) $$[\tilde{p}]^3 = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}\begin{bmatrix} -c^2-b^2 & ab & ac \\ ab & -c^2-a^2 & bc \\ ac & bc & -b^2-a^2 \end{bmatrix} = \begin{bmatrix} -\cancel{abc} + \cancel{abc} & c(c^2+a^2)+b^2c & -bc^2-b^3-a^2b \\ -c^3-b^2c-a^2c & 0 & ac^2+ab^2+a^3 \\ c^2b+b^3+a^2b & -ab^2-ac^2-a^3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & c(a^2+b^2+c^2) & -b(a^2+b^2+c^2) \\ -c(a^2+b^2+c^2) & 0 & a(a^2+b^2+c^2) \\ b(a^2+b^2+c^2) & -a(a^2+b^2+c^2) & 0 \end{bmatrix} = \boxed{-[\tilde{p}]p^2 = [\tilde{p}]^3}$$

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Problem 3 → 3-17 → First, extrapolate from 3-16

$[\tilde{e}]^2 = \hat{e}\hat{e}^T - I[\tilde{e}]^2 \to [\tilde{e}]^n$ when n = even → $[\tilde{e}]^n = (-1)^{\frac{n}{2}+1}\hat{e}\hat{e}^T + (-1)^{\frac{n}{2}} I e^n$, $n \geq 2$

$[\tilde{e}]^3 = -[\tilde{e}]e^2 \to [\tilde{e}]^n$ when n = odd → $[\tilde{e}]^n = (-1)^{\frac{n+1}{2}}[\tilde{e}]e^{(n-1)}$

$$e^{-\phi[\tilde{e}]} = I_{3x3} - \phi[\tilde{e}] + \frac{\phi^2}{2!}[\tilde{e}]^2 - \frac{\phi^3}{3!}[\tilde{e}]^3 + \frac{\phi^4}{4!}[\tilde{e}]^4 - \frac{\phi^5}{5!}[\tilde{e}]^5 + \frac{\phi^6}{6!}[\tilde{e}]^6$$

$$= I_{3x3} - \phi[\tilde{e}] + \frac{\phi^2}{2!}[\hat{e}\hat{e}^T - Ie^2] + \frac{\phi^3}{3!}[\tilde{e}]e^2 + \frac{\phi^4}{4!}[Ie^4 - \hat{e}\hat{e}^T] - \frac{\phi^5}{5}[\tilde{e}]e^4$$
$$+ \frac{\phi^6}{6!}[\hat{e}\hat{e}^T - Ie^6] + \dots$$

Using the fact that $\because e^2 = 1 \to e^n$ (where $n \geq 2$) $\to e^n = 1$

$$= I_{3x3} - \phi[\tilde{e}] + \frac{\phi^2}{2}\hat{e}\hat{e}^T - \frac{\phi^2}{2} + \frac{\phi^3}{3!}[\tilde{e}] + \frac{\phi^4}{4!} - \frac{\phi^4}{4!}\hat{e}\hat{e}^T - \frac{\phi^5}{5!}[\tilde{e}] + \frac{\phi^6}{6!}\hat{e}\hat{e}^T - \frac{\phi^6}{6!} + \dots$$

$$= I_{3x3} - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \dots - \left[\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \dots\right][\tilde{e}] + \left[\frac{\phi^2}{2!} - \frac{\phi^4}{4!} - \frac{\phi^6}{6!}\right]\hat{e}\hat{e}^T$$

Using the fact that → $\cos(x) = 1 - x^2/2! + x^4/4! + \dots$ $\sin x = x - x^3/3! + x^5/5! - \dots$

$$\boxed{e^{-\phi[\tilde{e}]} = \cos(\phi)[I_{3x3}] - \sin(\phi)[\tilde{e}] + (1-\cos(\phi))\hat{e}\hat{e}^T}$$

Problem 4 → 3.22 →

$$[C] = \begin{bmatrix} e_1^2(1-c\phi)+c\phi & e_1e_2(1-c\phi)+e_3s\phi & e_1e_3(1-c\phi)-e_2s\phi \\ e_2e_1(1-c\phi)-e_3s\phi & e_2^2(1-c\phi)+c\phi & e_2e_3(1-c\phi)+e_1s\phi \\ e_3e_1(1-c\phi)+e_2s\phi & e_3e_2(1-c\phi)-e_1s\phi & e_3^2(1-c\phi)+c\phi \end{bmatrix}$$

Using the following eqns → $e_1^2 = \beta_1^2/\sin^2\frac{\phi}{2}$, $e_2^2 = \beta_2^2/\sin^2\frac{\phi}{2}$, $e_3^2 = \beta_3^2/\sin^2\frac{\phi}{2}$, $\beta_0 = \cos\frac{\phi}{2}$

$1-\cos(\phi) = 1-\cos(2\cdot\frac{\phi}{2}) = 1 - 2\cos^2(\phi/2) + 1 = 2(1-\cos^2\frac{\phi}{2}) = 2\sin^2(\phi/2)$ — 1

$\sin(\phi) = \sin(2\cdot\frac{\phi}{2}) = 2\sin(\frac{\phi}{2})\cos(\frac{\phi}{2}) = -2\sin(\frac{\phi}{2})\cdot\beta_0$ — 2

$\cos(\phi) = \cos(\frac{\phi}{2}\cdot 2) = \cos^2(\frac{\phi}{2}) - \sin^2(\frac{\phi}{2}) = \beta_0^2 - (1)\sin^2(\frac{\phi}{2}) = \beta_0^2 - (e_1^2+e_2^2+e_3^2)\sin^2\frac{\phi}{2}$

$= \beta_0^2 - \beta_1^2 - \beta_2^2 - \beta_3^2$ — 3

Using element wise →

$C_{11} = e_1^2(1-c\phi)+c\phi = \frac{\beta_1^2}{\cancel{\sin^2\phi/2}}\cdot\cancel{2\sin^2(\frac{\phi}{2})} + \beta_0^2 - \cancel{\beta_1^2} - \beta_2^2 - \beta_3^2 = \beta_1^2 + \beta_0^2 - \beta_2^2 - \beta_3^2$ — Using 1, 3

$C_{21} = e_2e_1(1-c\phi) - e_3s\phi = \frac{\beta_1\beta_2}{\cancel{\sin^2\frac{\phi}{2}}}\cdot 2\cancel{\sin^2\frac{\phi}{2}} - \frac{\beta_3}{\cancel{\sin\frac{\phi}{2}}}\cdot 2\cancel{\sin\frac{\phi}{2}}\beta_0 = 2(\beta_1\beta_0 - \beta_0\beta_3)$ — Using 1, 2

$C_{32} = e_3e_1(1-c\phi) + e_2s\phi = \frac{\beta_1\beta_3}{\cancel{\sin^2\frac{\phi}{2}}}\cdot 2\cancel{\sin^2\frac{\phi}{2}} + \frac{\beta_2}{\cancel{\sin\frac{\phi}{2}}}\cdot 2\beta_0\cancel{\sin\frac{\phi}{2}} = 2(\beta_1\beta_3 + \beta_0\beta_2)$ — Using 1, 2

$$C_{12} = l_1 l_2 (1-c\phi) + l_3 s\phi = \frac{\beta_1\beta_2}{\cancel{\sin^2\phi/2}} \cdot 2\cancel{\sin^2\frac{\phi}{2}} + \frac{\beta_3}{\sin\frac{\phi}{2}}\, 2\sin\frac{\phi}{2}\,\beta_0 = 2(\beta_1\beta_2 + \beta_0\beta_3)$$ - Using (1), (2)

$$C_{22} = l_2^2(1-c\phi) + c\phi = \frac{\beta_2^2}{\cancel{\sin^2\phi/2}}\, 2\cancel{\sin^2\frac{\phi}{2}} + \beta_0^2 - \beta_1^2 - \beta_2^2 - \beta_3^2 = \beta_0^2 + \beta_2^2 - \beta_1^2 - \beta_3^2$$ - Using (1), (3)

$$C_{32} = l_3 l_2(1-c\phi) - l_1 s\phi = \frac{\beta_3\beta_2}{\cancel{\sin^2\phi/2}}\, 2\cancel{\sin^2\frac{\phi}{2}} - \frac{\beta_1}{\sin\phi/2}\cdot 2\sin\frac{\phi}{2}\,\beta_0 = 2(\beta_2\beta_3 - \beta_0\beta_1)$$ - Using (1), (2)

$$C_{13} = l_1 l_3(1-c\phi) - l_2 s\phi = \frac{\beta_1\beta_3}{\cancel{\sin^2\phi/2}}\, 2\cancel{\sin^2\frac{\phi}{2}} - \frac{\beta_2}{\sin\phi/2}\, 2\sin\frac{\phi}{2}\,\beta_0 = 2(\beta_1\beta_3 - \beta_0\beta_2)$$ - Using (1), (2)

$$C_{23} = l_2 l_3(1-c\phi) + l_1 s\phi = \frac{\beta_2\beta_3}{\cancel{\sin^2\phi/2}}\, 2\cancel{\sin^2\frac{\phi}{2}} + \frac{\beta_1}{\sin\phi/2}\, 2\sin\frac{\phi}{2}\,\beta_0 = 2(\beta_2\beta_3 + \beta_0\beta_1)$$ - Using (1), (2)

$$C_{33} = l_3^2(1-c\phi) + c\phi = \frac{\beta_3^2}{\cancel{\sin^2\phi/2}}\, 2\cancel{\sin^2\frac{\phi}{2}} + \beta_0^2 - \beta_1^2 - \beta_2^2 - \cancel{\beta_3^2} = \beta_0^2 + \beta_3^2 - \beta_1^2 - \beta_2^2$$ - Using (1), (3)

$$\therefore \boxed{[C] = \begin{bmatrix} \beta_0^2+\beta_1^2-\beta_2^2-\beta_3^2 & 2(\beta_1\beta_2+\beta_0\beta_3) & 2(\beta_1\beta_3-\beta_0\beta_2) \\ 2(\beta_1\beta_2-\beta_0\beta_3) & \beta_0^2+\beta_2^2-\beta_1^2-\beta_3^2 & 2(\beta_2\beta_3+\beta_0\beta_1) \\ 2(\beta_1\beta_3+\beta_0\beta_2) & 2(\beta_2\beta_3-\beta_0\beta_1) & \beta_0^2+\beta_3^2-\beta_1^2-\beta_2^2 \end{bmatrix}}$$

Problem 5 → start with $[\dot{C}] = -[\tilde{\omega}][C]$

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$$[\dot{C}] = -\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = -\begin{bmatrix} -\omega_3C_{21}+\omega_2C_{31} & -\omega_3C_{22}+\omega_2C_{32} & -\omega_3C_{23}+\omega_2C_{33} \\ \omega_3C_{11}-\omega_1C_{31} & \omega_3C_{12}-\omega_1C_{32} & \omega_3C_{13}-\omega_1C_{33} \\ -\omega_2C_{11}+\omega_1C_{21} & -\omega_2C_{12}+\omega_1C_{22} & -\omega_2C_{13}+\omega_1C_{23} \end{bmatrix}$$

Shepperd's method → $\beta_1^2 = \frac{1}{4}(1+2C_{11}-C_{11}-C_{22}-C_{33}) = \frac{1}{4}(1+C_{11}-C_{22}-C_{33})$

$\beta_1 = \pm\frac{1}{2}\sqrt{1+C_{11}-C_{22}-C_{33}}$, $\quad \dot{\beta}_1 = \frac{1}{2}\cdot\frac{1}{2}(1+C_{11}-C_{22}-C_{33})^{-1/2}(\dot{C}_{11}-\dot{C}_{22}-\dot{C}_{33})$

Get $\dot{C}_{11}, \dot{C}_{22}, \dot{C}_{33}$ from $[\dot{C}]$ matrix

$$\dot{\beta}_1 = \frac{1}{4}\left[+\omega_3C_{21}-\omega_2C_{31}+\omega_3C_{12}-\omega_1C_{32}-\omega_2C_{13}+\omega_1C_{23}\right]\cdot\frac{1}{2\beta_1}$$
$$= \frac{1}{8\beta_1}\left[\omega_3(C_{21}+C_{12})-\omega_2(C_{31}+C_{13})-\omega_1(C_{32}-C_{23})\right]$$

From shepperd's method → $\beta_0\beta_1 = \frac{C_{23}-C_{32}}{4}$, $\beta_1\beta_2 = \frac{C_{12}+C_{21}}{4}$, $\beta_1\beta_3 = \frac{C_{31}+C_{13}}{4}$

$$\boxed{\dot{\beta}_1 = \frac{1}{2}\left[\omega_3\beta_2-\omega_2\beta_3+\omega_1\beta_0\right]}$$

similarly,

$\beta_2 = \pm\frac{1}{2}\sqrt{1+C_{22}-C_{11}-C_{33}} \rightarrow \dot{\beta}_2 = \frac{1}{2}\cdot\frac{1}{2}(1+C_{22}-C_{11}-C_{33})^{-1/2}(\dot{C}_{22}-\dot{C}_{11}-\dot{C}_{33})$

$$= \frac{1}{4}\left[-\omega_3C_{12}+\omega_1C_{32}-\omega_3C_{21}+\omega_2C_{31}-\omega_2C_{13}+\omega_1C_{23}\right]\cdot\frac{1}{2\beta_2}$$
$$= \frac{1}{8\beta_2}\left[-\omega_3(C_{12}+C_{21})+\omega_1(C_{32}+C_{23})+\omega_2(C_{31}-C_{13})\right]$$

From shepperd's method → $\beta_0\beta_2 = \frac{C_{31}-C_{13}}{4}$, $\beta_1\beta_2 = \frac{C_{12}+C_{21}}{4}$, $\beta_2\beta_3 = \frac{C_{23}+C_{32}}{4}$

$$\boxed{\dot{\beta}_2 = \frac{1}{2}\left[-\omega_3\beta_1+\omega_1\beta_2+\omega_2\beta_0\right]}$$

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$$\beta_3 = \pm\frac{1}{2}\sqrt{1+C_{33}-C_{11}-C_{22}} \rightarrow \dot{\beta}_3 = \frac{1}{2}\cdot\frac{1}{2}(1+C_{33}-C_{11}-C_{22})^{-1/2}\cdot(\dot{C}_{33}-\dot{C}_{11}-\dot{C}_{22})$$

$$= \frac{1}{4}\left[w_2C_{13}-w_1C_{23}-w_3C_{21}+w_2C_{31}+w_3C_{12}-w_1C_{32}\right]$$

$$= \frac{1}{8\beta_3}\left[w_2(C_{13}+C_{31})-w_1(C_{23}+C_{32})-w_3(C_{21}-C_{12})\right]$$

from sheppard's method $\rightarrow \beta_0\beta_3 = \frac{C_{12}-C_{21}}{4},\ \beta_1\beta_3 = \frac{C_{31}+C_{13}}{4},\ \beta_2\beta_3 = \frac{C_{23}+C_{32}}{4}$

$$\boxed{\dot{\beta}_3 = \frac{1}{2}\left[w_2\beta_1 - w_1\beta_2 + w_3\beta_0\right]}$$

Problem 6 $\rightarrow$

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$$\bar{q}_i = \frac{\beta_i}{\beta_0},\ i=1,2,3, \rightarrow \text{Verify} \rightarrow \beta_0 = \frac{1}{\sqrt{1+\bar{q}^T\bar{q}}},\quad \beta_i = \frac{q_i}{\sqrt{1+\bar{q}^T\bar{q}}}\quad i=1,2,3$$

$$\bar{q}^T\bar{q} = [q_1\ q_2\ q_3]\begin{bmatrix}q_1\\q_2\\q_3\end{bmatrix} = q_1^2+q_2^2+q_3^2 = \frac{\beta_1^2+\beta_2^2+\beta_3^2}{\beta_0^2} = \frac{1-\beta_0^2}{\beta_0^2}$$

$$\beta_0 = \frac{1}{\sqrt{1+\frac{1-\beta_0^2}{\beta_0^2}}} = \frac{1}{\sqrt{\frac{\cancel{\beta_0^2}+1-\cancel{\beta_0^2}}{\beta_0^2}}} = \frac{1}{\sqrt{\frac{1}{\beta_0^2}}} = \frac{1}{\frac{1}{\beta_0}} = \boxed{\beta_0 = \beta_0}$$

$$\beta_i = \frac{q_i}{\sqrt{1+\frac{1-\beta_0^2}{\beta_0^2}}} = \frac{q_i}{\sqrt{\frac{\beta_0^2+1-\beta_0^2}{\beta_0^2}}} = \frac{q_i}{\frac{1}{\beta_0}} = q_i\beta_0 = \frac{\beta_i}{\cancel{\beta_0}}\cdot\cancel{\beta_0} = \boxed{\beta_i = \beta_i}$$