(1)

ASEN 5044
Fall 2024
Jash Bhalavat

$\boxed{\text{Midterm 1}}$

Problem 1 → $(M+m)\ddot{z} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = P$ (1), $l\ddot{\theta} - g\sin\theta = \ddot{z}\cos\theta$ (2), $g = 9.81\ m/s^2$

a] $\ddot{\theta} = \dfrac{\ddot{z}\cos\theta + g\sin\theta}{l}$ ← (3)

Sub. (3) into (1)

$(M+m)\ddot{z} = P - ml\dot{\theta}^2\sin\theta + m\cancel{l}\cos\theta\left(\frac{\ddot{z}}{\cancel{l}}\cos\theta + \frac{g}{\cancel{l}}\sin\theta\right)$

$= P - ml\dot{\theta}^2\sin\theta + m\ddot{z}\cos^2\theta + mg\cos\theta\sin\theta$

$\ddot{z}(M + m - m\cos^2\theta) = P - ml\dot{\theta}^2\sin\theta + mg\sin\theta\cos\theta$ ← $1-\cos^2\theta = \sin^2\theta$

$\ddot{z} = \dfrac{P - ml\dot{\theta}^2\sin\theta + mg\sin\theta\cos\theta}{M + m\sin^2\theta}$ ← (4)

Sub. (4) into (3)

$\ddot{\theta} = \left(\dfrac{P - ml\dot{\theta}^2\sin\theta + mg\sin\theta\cos\theta}{M + m\sin^2\theta}\right)\dfrac{\cos\theta}{l} + \dfrac{g}{l}\sin\theta$

$\ddot{\theta} = \dfrac{\frac{P}{l}\cos\theta - m\dot{\theta}^2\sin\theta\cos\theta + \frac{mg}{l}\sin\theta\cos^2\theta + \frac{g}{l}\sin\theta M + \frac{gm}{l}\sin^3\theta}{M + m\sin^2\theta}$ — (5)

$$x = \begin{bmatrix} z \\ \dot{z} \\ \theta \\ \dot{\theta} \end{bmatrix} \rightarrow \dot{x} = \begin{bmatrix} \dot{z} \\ \ddot{z} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ \dfrac{P - mLx_4^2\sin(x_3) + mg\sin(x_3)\cos(x_3)}{M + m\sin^2(x_3)} \\ x_4 \\ \left(\dfrac{P - mlx_4^2\sin(x_3) + mg\sin(x_3)\cos(x_3)}{M + m\sin^2(x_3)}\right)\dfrac{\cos(x_3)}{l} + \dfrac{g}{l}\sin(x_3) \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$$

$\boxed{\dot{x} = F(x,u)}$ where $u = P$

Let $\dot{z}=0,\ \theta=0,\ \dot{\theta}=0,\ P(t)=0N$ be $\bar{x}, \bar{u}$

$$\dot{x} = F(\bar{x},\bar{u}) = \begin{bmatrix} 0 \\ \dfrac{0 - \cancel{ml(0)^2\sin(0)}^{0} + \cancel{mg\sin(0)\cos(0)}^{0}}{M + \cancel{m\sin^2(0)}^{\to 0}} \\ 0 \\ \left(\dfrac{0 - \cancel{ml(0)^2\sin(0)}^{0} + \cancel{mg\sin(0)\cos(0)}^{0}}{M + \cancel{m\sin^2(0)}^{\to 0}}\right)\dfrac{\cos(0)}{l} + \cancel{\frac{g}{l}\sin(0)}^{\to 0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\because \dot{x} = F(\bar{x},\bar{u}) = 0$, $\bar{x},\bar{u} = x_{eq}, u_{eq}$

$\therefore$ $z=0, \theta=0, \dot{\theta}=0, P(t)=0N$ is an $\boxed{\text{equilibrium state}}$

6] $y = z - l\sin\theta \rightarrow [y] = [x_1 - l\sin(x_3)] = h_1(x,u) = h(x,u)$

Linearize $\rightarrow$

$$\left.\frac{\partial F}{\partial x}\right|_{x_{nom},u_{nom}} = \left.\begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} & \frac{\partial F_1}{\partial x_4} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} & \frac{\partial F_2}{\partial x_4} \\ \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_2} & \frac{\partial F_3}{\partial x_3} & \frac{\partial F_3}{\partial x_4} \\ \frac{\partial F_4}{\partial x_1} & \frac{\partial F_4}{\partial x_2} & \frac{\partial F_4}{\partial x_3} & \frac{\partial F_4}{\partial x_4} \end{bmatrix}\right|_{x_{nom},u_{nom}} \quad x_{nom} = \begin{bmatrix} x_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; u_{nom} = [0]$$

$\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$ — (6)

(4)

$$\frac{\partial F_2}{\partial x_3} = \frac{\partial F_2}{\partial \theta} = \frac{(\bar{M} + m\sin^2\theta)(-ml\dot{\theta}^2\cos\theta + mg(-\sin^2\theta + \cos^2\theta))}{(\bar{M} + m\sin^2\theta)^2} - \frac{(P - ml\dot{\theta}^2\sin\theta + mg\sin\theta\cos\theta)(2m\sin\theta\cos\theta)}{(\bar{M} + m\sin^2\theta)^2}$$

$$\frac{\partial F_2}{\partial \theta} = \frac{(\bar{M} + m\sin^2\theta)[mg(1-\sin^2\theta) - ml\dot{\theta}^2\cos\theta] - (mg\sin\theta\cos\theta - ml\dot{\theta}^2\sin\theta + P)(2m\sin\theta\cos\theta)}{(\bar{M} + m\sin^2\theta)^2}$$

$$\left.\frac{\partial F_2}{\partial \theta}\right|_{x_{nom},u_{nom}} = \frac{\cancel{\bar{M}}[mg] - (0)(0)}{\bar{M}^{\cancel{2}}} = \frac{mg}{\bar{M}}$$

(4) $\rightarrow$

$$\frac{\partial F_2}{\partial x_4} = \frac{\partial F_2}{\partial \dot{\theta}} = \frac{-2ml\sin\theta\,\dot{\theta}}{\bar{M} + m\sin^2\theta} \rightarrow \left.\frac{\partial F_2}{\partial \dot{\theta}}\right|_{x_{nom},u_{nom}} = \frac{0}{\bar{M}} = 0$$

(5) $\rightarrow$

$$\ddot{\theta} = \frac{\frac{P}{l}\cos\theta - m\dot{\theta}^2\sin\theta\cos\theta + \frac{mg}{l}\sin\theta\cos^2\theta + \frac{\bar{M}g}{l}\sin\theta + \frac{mg}{l}\sin^3\theta}{\bar{M} + m\sin^2\theta}$$

Last 3 Term in numerator of $\ddot{\theta}$

$$\frac{mg}{l}\sin\theta\cos^2\theta + \frac{\bar{M}g}{l}\sin\theta + \frac{mg}{l}\sin^3\theta = (\sin\theta)\cdot\frac{g}{l}[m\cos^2\theta + m\sin^2\theta + \bar{M}]$$

$$= (\sin\theta)\frac{g}{l}[m(\sin^2\theta + \cos^2\theta) + \bar{M}] = [m + \bar{M}]\frac{g}{l}\sin\theta$$

$$\ddot{\theta} = \frac{\frac{P}{l}\cos\theta - m\dot{\theta}^2\sin\theta\cos\theta + [m + \bar{M}]\frac{g}{l}\sin\theta}{\bar{M} + m\sin^2\theta}$$

$$\frac{\partial F_4}{\partial \theta} = \frac{(\bar{M} + m\sin^2\theta)[-\frac{P}{l}\sin\theta - m\dot{\theta}^2(-\sin^2\theta + \cos^2\theta) + (m + \bar{M})\frac{g}{l}\cos\theta]}{(\bar{M} + m\sin^2\theta)^2} - \frac{[\frac{P}{l}\cos\theta - m\dot{\theta}^2\sin\theta\cos\theta + (m+\bar{M})\frac{g}{l}\sin\theta](2m\sin\theta\cos\theta)}{(\bar{M} + m\sin^2\theta)^2}$$

Using (6) $\rightarrow$ $\dfrac{\partial F_4}{\partial \theta} = \dfrac{(\overline{M} + m\sin^2\theta)\left[\left(\frac{g}{l}\right)(\overline{M}+m)\cos\theta - m\dot{\theta}^2(1-2\sin^2\theta) - \frac{P}{l}\sin\theta\right]}{(\overline{M} + m\sin^2\theta)^2}$

$$- \frac{\left[\frac{g}{l}(\overline{M}+m)\sin\theta - m\dot{\theta}^2\sin\theta\cos\theta + \frac{P}{l}\cos\theta\right](2m\sin\theta\cos\theta)}{(\overline{M}+m\sin^2\theta)^2}$$

$$\left.\frac{\partial F_4}{\partial \theta}\right|_{x_{nom},u_{nom}} = \frac{(\overline{M})\left[\frac{g}{l}(\overline{M}+m)\right] - 0}{\overline{M}^2} = \frac{g}{l\overline{M}}(\overline{M}+m)$$

(5) $\rightarrow \dfrac{\partial F_4}{\partial x_4} = \dfrac{\partial F_4}{\partial \dot{\theta}} = \dfrac{-2m\dot{\theta}\sin\theta\cos\theta}{\overline{M}+m\sin^2\theta} \rightarrow \left.\dfrac{\partial F_4}{\partial \dot{\theta}}\right|_{x_{nom},u_{nom}} = \dfrac{0}{\overline{M}} = 0$

$$\left.\frac{\partial F}{\partial x}\right|_{x_{nom},u_{nom}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{mg}{\overline{M}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{l\overline{M}}(\overline{M}+m) & 0 \end{bmatrix} = \overline{A}$$

(4)

$$\left.\frac{\partial F}{\partial u}\right|_{x_{nom},u_{nom}} = \begin{bmatrix} \frac{\partial F_1}{\partial u_1} \\ \frac{\partial F_2}{\partial u_1} \\ \frac{\partial F_3}{\partial u_1} \\ \frac{\partial F_4}{\partial u_1} \end{bmatrix}, \quad \frac{\partial F_2}{\partial u_1} = \frac{\partial F_2}{\partial P} = \frac{1}{\overline{M}+m\sin^2\theta} \rightarrow \left.\frac{\partial F_2}{\partial P}\right|_{x_{nom},u_{nom}} = \frac{1}{\overline{M}}$$

$$\frac{\partial F_4}{\partial u_1} = \frac{\partial F_4}{\partial P} = \frac{\cos\theta}{l(\overline{M}+m\sin^2\theta)} \rightarrow \left.\frac{\partial F_4}{\partial P}\right|_{x_{nom},u_{nom}} = \frac{1}{l\overline{M}}$$

$$= \begin{bmatrix} 0 \\ 1/\overline{M} \\ 0 \\ 1/(l\overline{M}) \end{bmatrix} = \overline{B}$$

$$\left.\frac{\partial h}{\partial x}\right|_{x_{nom},u_{nom}} = \left.\left[\frac{\partial h_1}{\partial x_1} \;\; \frac{\partial h_1}{\partial x_2} \;\; \frac{\partial h_1}{\partial x_3} \;\; \frac{\partial h_1}{\partial x_4}\right]\right|_{x_{nom},u_{nom}}, \quad \frac{\partial h_1}{\partial x_1} = 1, \quad \left.\frac{\partial h_1}{\partial x_1}\right|_{x_{nom},u_{nom}} = 1$$

$$\frac{\partial h_1}{\partial x_3} = -l\cos(x_3) \rightarrow \left.\frac{\partial h_1}{\partial x_3}\right|_{x_{nom},u_{nom}} = -l\cos(0) = -l$$

$$\left.\frac{\partial h}{\partial x}\right|_{x_{nom},u_{nom}} = [1, 0, -l, 0] = \overline{C}$$

$$\left.\frac{\partial h}{\partial u}\right|_{x_{nom},u_{nom}} = \left.\left[\frac{\partial h_1}{\partial u_1}\right]\right|_{x_{nom},u_{nom}} \rightarrow \frac{\partial h_1}{\partial u_1} = \frac{\partial h_1}{\partial P} = 0 \rightarrow \left.\frac{\partial h}{\partial u}\right|_{x_{nom},u_{nom}} = [0] = \overline{D}$$

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & mg/\bar{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{l\bar{M}}(\bar{M}+m) & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \\ \delta x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/\bar{M} \\ 0 \\ 1/l\bar{M} \end{bmatrix} [\delta u_1]$$

$$\bar{y} = \bar{C}\bar{x} + \bar{D}\bar{u} = \begin{bmatrix} 1 & 0 & -l & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \\ \delta x_4 \end{bmatrix} + [0][\delta u_1]$$

c) $l = 1.85\,m$, $m = 2\,kg$, $\bar{M} = 4\,kg$

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 4.9050 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 7.9541 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 0.25 \\ 0 \\ 0.1351 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 1 & 0 & -1.85 & 0 \end{bmatrix}, \quad \bar{D} = [0]$$

$$\hat{A} = \begin{bmatrix} \bar{A} & \bar{B} \\ 0 & 0 \end{bmatrix} \rightarrow e^{\hat{A}\Delta t} = \underset{\text{(Done in matlab)}}{expm(\hat{A} \cdot 0.05)} = \begin{bmatrix} \bar{F} & \bar{G} \\ 0 & I \end{bmatrix}$$

$$\bar{F} = \begin{bmatrix} 1 & 0.05 & 0.0061 & 1.0299\times10^{-4} \\ 0 & 1 & 0.2461 & 0.0061 \\ 0 & 0 & 1.01 & 0.0502 \\ 0 & 0 & 0.399 & 1.01 \end{bmatrix}, \quad \bar{G} = \begin{bmatrix} 3.1267\times10^{-4} \\ 0.0125 \\ 1.6920\times10^{-4} \\ 0.0068 \end{bmatrix}, \quad \boxed{\bar{H} = \bar{C}}, \quad \boxed{\bar{M} = \bar{D}}$$

$$\therefore \bar{x}_{k+1} = \bar{F}\bar{x}_k + \bar{G}\bar{u}_k, \quad \bar{y}_k = \bar{H}\bar{x}_k + \bar{M}\bar{u}_k$$

$$\boxed{\begin{gathered} \bar{x}_{k+1} = \begin{bmatrix} 1 & 0.05 & 0.0061 & 1.0299\times10^{-4} \\ 0 & 1 & 0.2461 & 0.0061 \\ 0 & 0 & 1.01 & 0.0502 \\ 0 & 0 & 0.399 & 1.01 \end{bmatrix} \begin{bmatrix} \bar{z}_k \\ \dot{\bar{z}}_k \\ \bar{\theta}_k \\ \dot{\bar{\theta}}_k \end{bmatrix} + \begin{bmatrix} 3.1267\times10^{-4} \\ 0.0125 \\ 1.6920\times10^{-4} \\ 0.0068 \end{bmatrix} [P_k] \\ \bar{y}_k = \begin{bmatrix} 1 & 0 & -1.85 & 0 \end{bmatrix} \begin{bmatrix} \bar{z}_k \\ \dot{\bar{z}}_k \\ \bar{\theta}_k \\ \dot{\bar{\theta}}_k \end{bmatrix} + [0][P_k] \end{gathered}}$$

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$\boxed{\text{Midterm 1}}$

Problem 1 → Stability → Get eigenvalues of $\bar{A}$ using matlab.

$eig(\bar{A}) = [0, 0, 2.8203, -2.8203]$

$\because 2.8203 > 0 \rightarrow$ one eigenvalue of $A > 0 \rightarrow \therefore$ $\boxed{\text{The system is } \underline{\text{not}} \text{ asymptotically stable}}$

Observability $\rightarrow O = [\bar{H}, \bar{H}\bar{F}, \bar{H}\bar{F}^2, \bar{H}\bar{F}^3]^T$

Using matlab $\rightarrow O = \begin{bmatrix} 1 & 0 & -1.85 & 0 \\ 1 & 0.05 & -1.8623 & -0.0927 \\ 1 & 0.1 & -1.8994 & -0.1866 \\ 1 & 0.15 & -1.9620 & -0.2831 \end{bmatrix}$ → Using matlab, rank$(O) = 4$

$\boxed{\because \text{Rank}(O) = N, \text{ the system is observable}}$

d] $x(0) = \bar{x}(0)$ $(x_{nom}(0) = 0)$, $u(k) = \bar{u}(k) = -K_c \bar{x}(k)$

$x_k = \bar{F} x_{k-1} + \bar{G} u_{k-1} \rightarrow x_1 = \bar{F} x_0 + \bar{G} u_0 = \bar{F} x_0 + \bar{G}(-K_c) x_0 = x_0(\bar{F} - \bar{G}K_c)$

$y_k = \bar{H} x_k$

$y_1 = \bar{H} x_1 = \bar{H}(\bar{F} x_0 - \bar{G} K_c x_0) = \bar{H} x_0 (\bar{F} - \bar{G} K_c)$

$x_2 = \bar{F} x_1 - \bar{G} K_c x_1 = x_1(\bar{F} - \bar{G} K_c) = x_0 (F - G K_c)^2$

$y_2 = \bar{H} x_2 = \bar{H} x_0 (F - G K_c)^2$

$x_n = x_0 (F - G K_c)^n$, $y_n = H x_0 (F - G K_c)^n$

where $N = 4$

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}}_{\triangleq y} = \underbrace{\begin{bmatrix} H(F - GK_c) \\ H(F - GK_c)^2 \\ \vdots \\ H(F - GK_c)^N \end{bmatrix}}_{\triangleq L} x_0 \rightarrow \hat{x}_0 = (L^T L)^{-1} L^T y = \boxed{\begin{bmatrix} 0.3971 \\ 0.3036 \\ -0.1312 \\ 0.0515 \end{bmatrix} = \bar{x}_0}$$

(using matlab)

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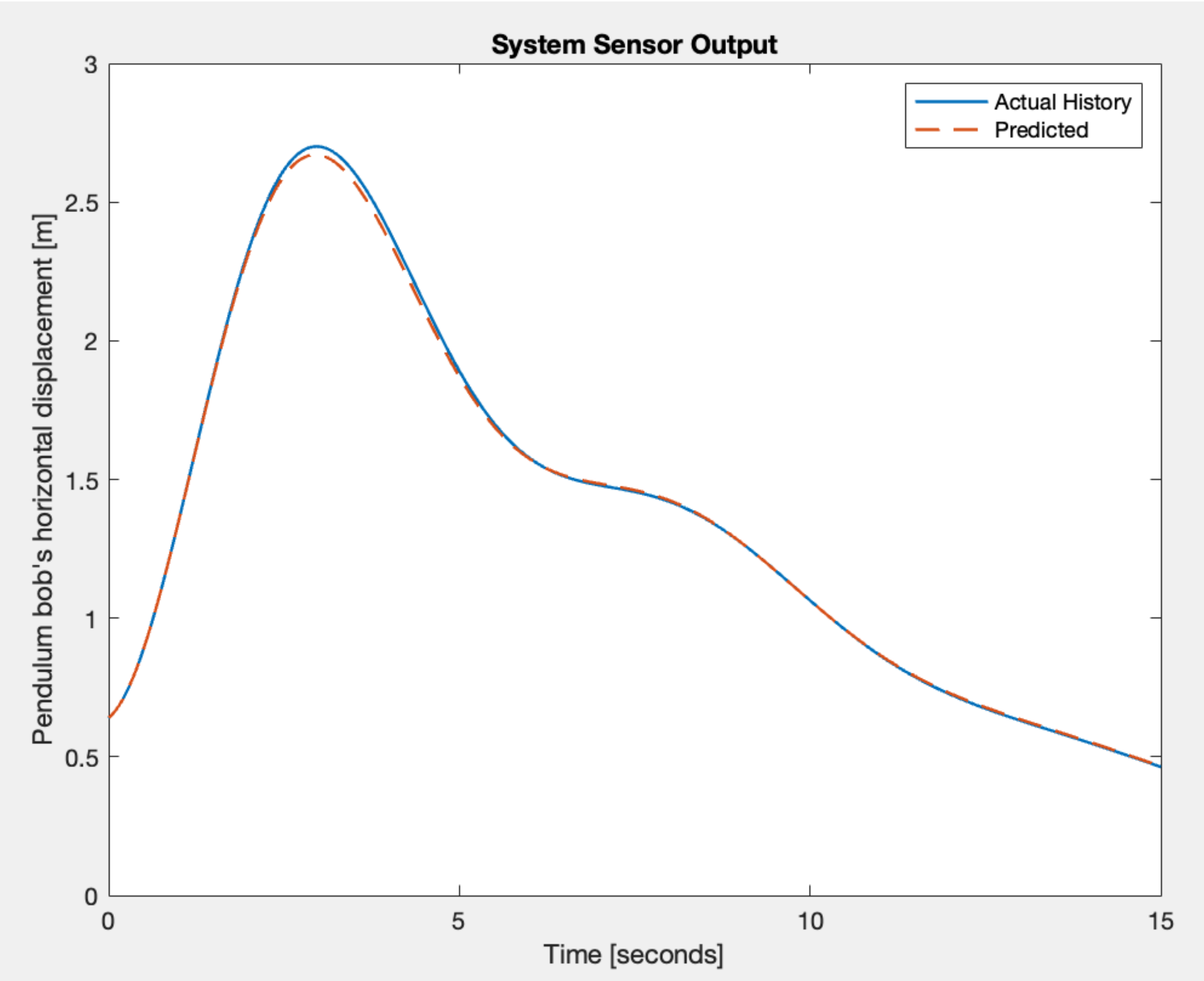
- Problem 1
  - Part e



    - The output predicted by the linearized DT closed-loop model is very close to the actual measurements in ‘yNLhist’. The reason that it does not exactly match the actual measurements is because the non-linear model is very accurate around the nominal state and its error increases the farther the system goes away from the nominal state. Additionally, sensor measurements are also not perfect. So, it may also lead to that small deviation from the predicted model.

- Problem 2

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Problem 2 Outcomes of $R_1 \rightarrow 1,2,3,4,5,6$ | Outcomes of $R_2 \rightarrow 1,2,3,4,5,6$

a] For any R_1 and $R_2 \rightarrow P(R_1) = P(R_2) = \frac{1}{6}$ because the dice are independent and all outcomes are equally probable

$\therefore P(R_1, R_2) = P(R_1) \cdot P(R_2) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

  - Part a
    - The joint probability distribution table would look like this:

P(R1 ∧ R2)	R1 = 1	R1 = 2	R1 = 3	R1 = 4	R1 = 5	R1 = 6
R2 = 1	1/36	1/36	1/36	1/36	1/36	1/36
R2 = 2	1/36	1/36	1/36	1/36	1/36	1/36
R2 = 3	1/36	1/36	1/36	1/36	1/36	1/36
R2 = 4	1/36	1/36	1/36	1/36	1/36	1/36
R2 = 5	1/36	1/36	1/36	1/36	1/36	1/36
R2 = 6	1/36	1/36	1/36	1/36	1/36	1/36

  - Part b

Joint Probability Distribution Table for X and Y						
P(X ∧ Y)	X = 0	X = 1	X = 2	X = 3	X = 4	X = 5
Y = 1	1/36	1/36	1/36	1/36	1/36	1/36

Joint Probability Distribution Table for X and Y						
P(X ∧ Y)	X = 0	X = 1	X = 2	X = 3	X = 4	X = 5
Y = 2	0	1/36	0	0	0	0
Y = 3	0	0	1/36	0	0	0
Y = 4	1/36	0	0	1/36	0	0
Y = 5	0	0	0	0	1/36	0
Y = 6	0	0	0	0	0	1/36
Y = 8	0	1/36	0	0	0	0
Y = 9	0	1/36	0	0	0	0
Y = 16	0	0	2/36	0	0	0
Y = 25	0	0	0	1/36	0	0
Y = 27	1/36	0	0	0	0	0
Y = 32	0	0	0	1/36	0	0
Y = 36	0	0	0	0	1/36	0
Y = 64	0	1/36	0	0	1/36	0
Y = 81	0	1/36	0	0	0	0
Y = 125	0	0	1/36	0	0	0
Y = 216	0	0	0	1/36	0	0
Y = 243	0	0	1/36	0	0	0
Y = 256	1/36	0	0	0	0	0
Y = 625	0	1/36	0	0	0	0
Y = 729	0	0	0	1/36	0	0
Y = 1024	0	1/36	0	0	0	0
Y = 1296	0	0	1/36	0	0	0

Joint Probability Distribution Table for X and Y						
P(X ∧ Y)	X = 0	X = 1	X = 2	X = 3	X = 4	X = 5
Y = 3125	1/36	0	0	0	0	0
Y = 4096	0	0	1/36	0	0	0
Y = 7776	0	1/36	0	0	0	0
Y = 15625	0	1/36	0	0	0	0
Y = 46656	1/36	0	0	0	0	0

- Part c

Marginal Probabilities Table		
Marginal Probability	Description	Value
X = 0	Sum of X = 0 column in part b	1/6
X = 1	Sum of X = 1 column in part b	10/36
X = 2	Sum of X = 2 column in part b	8/36
X = 3	Sum of X = 3 column in part b	1/6
X = 4	Sum of X = 4 column in part b	4/36
X = 5	Sum of X = 5 column in part b	2/36
Y = 1	Sum of Y = 1 row in part b	1/6
Y = 2	Sum of Y = 2 row in part b	1/36
Y = 3	Sum of Y = 3 row in part b	1/36
Y = 4	Sum of Y = 4 row in part b	2/36
Y = 5	Sum of Y = 5 row in part b	1/36

Y = 6	Sum of Y = 6 row in part b	1/36
Y = 8	Sum of Y = 8 row in part b	1/36
Y = 9	Sum of Y = 9 row in part b	1/36
Y = 16	Sum of Y = 16 row in part b	2/36
Y = 25	Sum of Y = 25 row in part b	1/36
Y = 27	Sum of Y = 27 row in part b	1/36
Y = 32	Sum of Y = 32 row in part b	1/36
Y = 36	Sum of Y = 36 row in part b	1/36
Y = 64	Sum of Y = 64 row in part b	2/36
Y = 81	Sum of Y = 81 row in part b	1/36
Y = 125	Sum of Y = 125 row in part b	1/36
Y = 216	Sum of Y = 216 row in part b	1/36
Y = 243	Sum of Y = 243 row in part b	1/36
Y = 256	Sum of Y = 256 row in part b	1/36
Y = 625	Sum of Y = 625 row in part b	1/36
Y = 729	Sum of Y = 729 row in part b	1/36
Y = 1024	Sum of Y = 1024 row in part b	1/36
Y = 1296	Sum of Y = 1296 row in part b	1/36
Y = 3125	Sum of Y = 3125 row in part b	1/36
Y = 4096	Sum of Y = 4096 row in part b	1/36
Y = 7776	Sum of Y = 7776 row in part b	1/36
Y = 15625	Sum of Y = 15625 row in part b	1/36
Y = 46656	Sum of Y = 46656 row in part b	1/36

- Part d
    - X and Y would be independent if P(X = x & Y = y) = P(X=x) * P(Y=y)
        - This is easy to refute
            - P(X = 1 & Y = 1) from table in part b = 1/36 = 0.028
            - P(X = 1) from table in part c = 10/36
            - P(Y = 1) from table in part c = ⅙
            - P(X = 1) * P(Y = 1) = 10/36 * 1/36 = 0.008
            - Since, 0.028 is not equal to 0.008, X and Y are not independent.
    - Using another way to determine independence:
        - If X and Y were independent, P(Y = y | X = x) = P(Y = y)
        - P(Y = 9 | X = 1) = P (X = 1 & Y = 9) / P(X = 1) = 1/36 / 10/36 = 1/10
        - P(Y = 9) = 1/36
        - Since 1/10 is not equal to 1/36, P(Y = 9 | X = 1) is not equal to P(Y = 9) and X and Y are not independent.

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<u>Problem 3</u> $\rightarrow$ $p(x) = \begin{cases} K(2-x^6) & -1 \le x \le 1 \\ 0 & \end{cases}$

a] $\int_{-\infty}^{\infty} p(x)\,dx = 1$ ← From probability axioms

$\int_{-\infty}^{\infty} p(x)\,dx = \cancelto{0}{\int_{-\infty}^{-1} 0\,dx} + \int_{-1}^{1} K(2-x^6)\,dx + \cancelto{0}{\int_{1}^{\infty} 0\,dx} = 1$

$\int_{-1}^{1} 2K - Kx^6\,dx = 2Kx - \frac{K}{7}x^7 \Big|_{-1}^{1}$

$= [2K - K/7] - [-2K + K/7] = 4K - 2K/7 = K(4 - 2/7) = \frac{26}{7}K = 1$

$\therefore \boxed{K = 7/26}$

$E[x] = \int_{-\infty}^{\infty} p(x)\cdot x\,dx = \cancelto{0}{\int_{-\infty}^{-1} 0\cdot x\,dx} + \int_{-1}^{1} \frac{7}{26}x(2-x^6)\,dx + \cancelto{0}{\int_{1}^{\infty} 0\cdot x\,dx}$

$= \int_{-1}^{1} \frac{14x}{26} - \frac{7x^7}{26}\,dx = \frac{7x^2}{26} - \frac{7x^8}{208}\Big|_{-1}^{1}$

$= \cancel{\left[\frac{7}{26} - \frac{7}{208}\right]} - \cancel{\left[\frac{7}{26} - \frac{7}{208}\right]} = \boxed{0 = E[x]} = \mu_x$

$var(x) = E[(x - \cancelto{0}{\mu_x})^2] = E[x^2] = \int_{-\infty}^{\infty} p(x)\cdot x^2\,dx$

$= \cancelto{0}{\int_{-\infty}^{-1} 0\cdot x^2\,dx} + \int_{-1}^{1} \frac{7}{26}x^2(2-x^6)\,dx + \cancelto{0}{\int_{1}^{\infty} 0\cdot x^2\,dx}$

$= \int_{-1}^{1} \frac{14x^2}{26} - \frac{7x^8}{26}\,dx = \frac{14x^3}{78} - \frac{7x^9}{234}\Big|_{-1}^{1}$

$= \left[\frac{14}{78} - \frac{7}{234}\right] - \left[-\frac{14}{78} + \frac{7}{234}\right] = \frac{28}{78} - \frac{14}{234} = \boxed{\frac{70}{234} = 0.299145 = var(x)}$

b] $P_x(\xi) = \int_{-\infty}^{\xi} p(x)\,dx$

If $\xi < -1 \longrightarrow p(x) = 0 \rightarrow \int_{-\infty}^{\xi} 0\,dx = \boxed{0} = P_x(\xi)$

If $-1 \le \xi \le 1 \longrightarrow P_x(\xi) = \int_{-1}^{\xi} \frac{7}{26}(2-x^6)\,dx = \int_{-1}^{\xi} \frac{14}{26} - \frac{7x^6}{26}\,dx$

$= \frac{14x}{26} - \frac{x^7}{26}\Big|_{-1}^{\xi} = \left[\frac{14\xi}{26} - \frac{\xi^7}{26}\right] - \left[-\frac{14}{26} + \frac{1}{26}\right]$

$P_x(\xi) = \boxed{\frac{14\xi - \xi^7}{26} + \frac{1}{2}}$

If $\zeta > 1 \rightarrow P_x(\zeta) = \int_{-1}^{1} \frac{7}{26}(2-x^6)\,dx + \int_{1}^{\infty} \underbrace{0\,dx}_{0} = \int_{-1}^{1} \frac{14}{26} - \frac{7x^6}{26}\,dx$

$$= \frac{14}{26}x - \frac{x^7}{26}\Big|_{-1}^{1} = \left[\frac{14}{26} - \frac{1}{26}\right] - \left[-\frac{14}{26} + \frac{1}{26}\right] = \frac{13}{26} + \frac{13}{26} = \boxed{1}$$

$$\therefore \boxed{P_x(\zeta) = \begin{cases} 0 & \zeta < -1 \\ \frac{14\zeta - \zeta^7}{26} + \frac{1}{2} & -1 \leq \zeta \leq 1 \\ 1 & \zeta > 1 \end{cases}}$$

c] $P(|X| < 0.65) = P(-0.65 < x < 0.65)$

By definition, $P_x(0.65) = P(-\infty < X \leq 0.65)$

$\therefore P(-0.65 < x < 0.65) = P_x(0.65) - p(0.65) - P_x(-0.65)$

$$P_x(0.65) = \frac{14(0.65) - (0.65)^7}{26} + \frac{1}{2} = 0.848114$$

$$p(0.65) = \frac{7}{26}(2 - 0.65^6) = 0.518156$$

$$P_x(-0.65) = \frac{14(-0.65) - (-0.65)^7}{26} + \frac{1}{2} = 0.151885$$

$P(|X| < 0.65) = 0.848114 - 0.518156 - 0.151885 = \boxed{0.178073 = P(|X| < 0.65)}$

Problem 4 → a] Typical hop begins with assumption that $P(H=1)=0.5$

$\therefore$ Probability axiom says that $P(H=0)=0.5$

$A^* = \max\left(E_h[U(H,A)]\right)$

calculate E_H for $A=0,1$

Get $U(H,A)$ values from table 1

$$E_H[U(H,A=0)] = \sum_h U(H=h, A=0)\cdot P(H=h) = U(H=0,A=0)P(H=0) + U(H=1,A=0)P(H=1)$$
$$= (0)(0.5) + (-1000)(0.5) = -500$$

$$E_H[U(H,A=1)] = \sum_h U(H=h, A=1)\cdot P(H=h) = U(H=0,A=1)P(H=0) + U(H=1,A=1)P(H=1)$$
$$= (-10)(0.5) + (100)(0.5) = 45$$

$E_H[U(H,A=0)] < E_H[U(H,A=1)] \rightarrow \therefore$ $\boxed{A^*=1 \text{ is selected}}$

b] $L=0$, using Bayes' Rule

Priori → $P(H=1)=0.5$, $P(H=0)=0.5$ ($\because P(H=0)=1-P(H=1)$)

$$P(H=1|L=0) = \frac{P(H=1)\cdot P(L=0|H=1)}{\sum_h P(H=h)\cdot P(L=0|H=h)} = \frac{P(H=1)\cdot P(L=0|H=1)}{P(H=0)\cdot P(L=0|H=0)+P(H=1)\cdot P(L=0|H=1)}$$
$$= \frac{(0.5)(0.05)}{(0.5)(0.92)+(0.5)(0.05)} = \boxed{0.0515464 = P(H=1|L=0)}$$

$$P(H=0|L=0) = \frac{P(H=0)\cdot P(L=0|H=0)}{\sum_h P(H=h)\cdot P(L=0|H=h)} = \frac{P(H=0)\cdot P(L=0|H=0)}{P(H=0)\cdot P(L=0|H=0)+P(H=1)P(L=0|H=1)}$$
$$= \frac{(0.5)(0.92)}{(0.5)(0.92)+(0.5)(0.05)} = \boxed{0.948454 = P(H=0|L=0)}$$

(also $1-P(H=1|L=0)$)

$$E_H[U(H,A=0)] = \sum_h U(H=h,A=0)\cdot P(H=h) = U(H=0,A=0)P(H=0) + U(H=1,A=0)P(H=1)$$
$$= \cancel{(0)(0.948450)}^{0} + (-1000)(0.0515464) = -51.5464$$

$$E_H[U(H,A=1)] = \sum_h U(H=h,A=1)\cdot P(H=h) = U(H=0,A=1)P(H=0) + U(H=1,A=1)P(H=1)$$
$$= (-10)(0.948454) + (100)(0.0515464) = -4.3298$$

$E_H[U(H,A=0)] < E_H[U(H,A=1)] \rightarrow \therefore$ $\boxed{A^*=1 \text{ is selected}}$

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Fall 2024
Jash Bhalavat

Midterm 1

Problem 4 → c) i] $L_1=0, L_2=0$, $\because P(L_1,L_2|H)=P(L_1|H)\,P(L_2|H)$ it allows us to consecutively use H_{n-1} to calculate $P(H_n|L_n)$

For first assumption, $P(H_0=1)=\frac{1}{2}$, $P(H_0=0)=1/2$

From part b → $P(H_1=1|L_1=0) = 0.0515464$, $P(H_1=0|L_1=0) = 0.948454$

Use these to calculate $P(H_2=0)$ and $P(H_2=1)$

$$P(H_2=0|L_2=0) = \frac{P(H_1=0)\cdot P(L=0|H=0)}{\Sigma_h P(H_1=h)\cdot P(L=0|H=h)} = \frac{P(H_1=0)\,P(L=0|H=0)}{P(H_1=0)\,P(L=0|H=0)+P(H_1=1)\,P(L=0|H=1)}$$

$$= \frac{(0.948454)(0.92)}{(0.948454)(0.92)+(0.0515464)(0.05)} = 0.997055$$

$$P(H_2=1|L_2=0) = \frac{P(H_1=1)\,P(L=0|H=1)}{\Sigma_h P(H_1=h)\,P(L=0|H=h)} = \frac{P(H_1=1)\,P(L=0|H=1)}{P(H_1=0)P(L=0|H=0)+P(H_1=1)P(L=0|H=1)}$$

$$= \frac{(0.0515464)(0.05)}{(0.948454)(0.92)+(0.0515464)(0.05)} = 0.00294499$$

$$E_H[U(H,A=0)] = \Sigma_h U(H=h,A=0)\,P(H_2=h) = U(H=0,A=0)\,P(H_2=0) + U(H=1,A=0)\,P(H_2=1)$$

$$= \cancel{(0)(0.997055)}^{0} + (-1000)(0.00294499) = 2.9449$$

$$E_H[U(H,A=1)] = \Sigma_h U(H=h,A=1)\,P(H_2=h) = U(H=0,A=1)\,P(H_2=0) + U(H=1,A=1)\,P(H_2=1)$$

$$= (-10)(0.997055) + (100)(0.00294499) = -9.6760$$

$E_H[U(H;A=0)] > E_H[U(H,A=1)] \rightarrow \therefore$ $\boxed{A^*=0 \text{ is selected}}$

ii] $L_1=1, L_2=0 \rightarrow P(H_0=0)=\frac{1}{2}$, $P(H_0=1)=1/2$

$$P(H_1=0|L_1=1) = \frac{P(H_0=0)\cdot P(L=1|H=0)}{\Sigma_h P(H_0=h)\cdot P(L=1|H=h)} = \frac{P(H_0=0)\cdot P(L=1|H=0)}{P(H_0=0)P(L=1|H=0)+P(H_0=1)\,P(L=1|H=1)}$$

$$= \frac{(0.5)(0.08)}{(0.5)(0.08)+(0.5)(0.95)} = 0.0776699$$

$P(H_1=1|L_1=1) = 1 - P(H_1=0|L=1) = 0.922330$

Use H_1 probabilities to calculate $P(H_2=0)$ & $P(H_2=1)$ given $L_2=0$

$$P(H_2=0|L_2=0) = \frac{P(H_1=0)\cdot P(L=0|H=0)}{\sum_h P(H_1=h)\cdot P(L=0|H=h)} = \frac{P(H_1=0)\,P(L=0|H=0)}{P(H_1=0)P(L=0|H=0)+P(H_1=1)P(L=0|H=1)}$$

$$= \frac{(0.0776699)(0.92)}{(0.0776699)(0.92)+(0.922330)(0.05)} = 0.607762$$

$$P(H_2=1|L_2=0) = \frac{P(H_1=1)P(L=0|H=1)}{\sum_h P(H_1=h)P(L=0|H=h)} = \frac{P(H_1=1)\,P(L=0|H=1)}{P(H_1=0)P(L=0|H=0)+P(H_1=1)P(L=0|H=1)}$$

$$= \frac{(0.922330)(0.05)}{(0.0776699)(0.92)+(0.922330)(0.05)} = 0.392238$$

$$E_H[U(H,A=0)] = \sum_h U(H=h,A=0)\,P(H_2=h) = U(H=0,A=0)P(H_2=0)+U(H=1,A=0)P(H_2=1)$$

$$= \cancelto{0}{(0)(0.607762)} + (-1000)(0.392238) = -392.238$$

$$E_H[U(H,A=1)] = \sum_h U(H=h,A=1)\,P(H_2=h) = U(H=0,A=1)P(H_2=0)+U(H=1,A=1)P(H_2=1)$$

$$= (-10)(0.607762)+(100)(0.392238)$$

$$= 33.1461$$

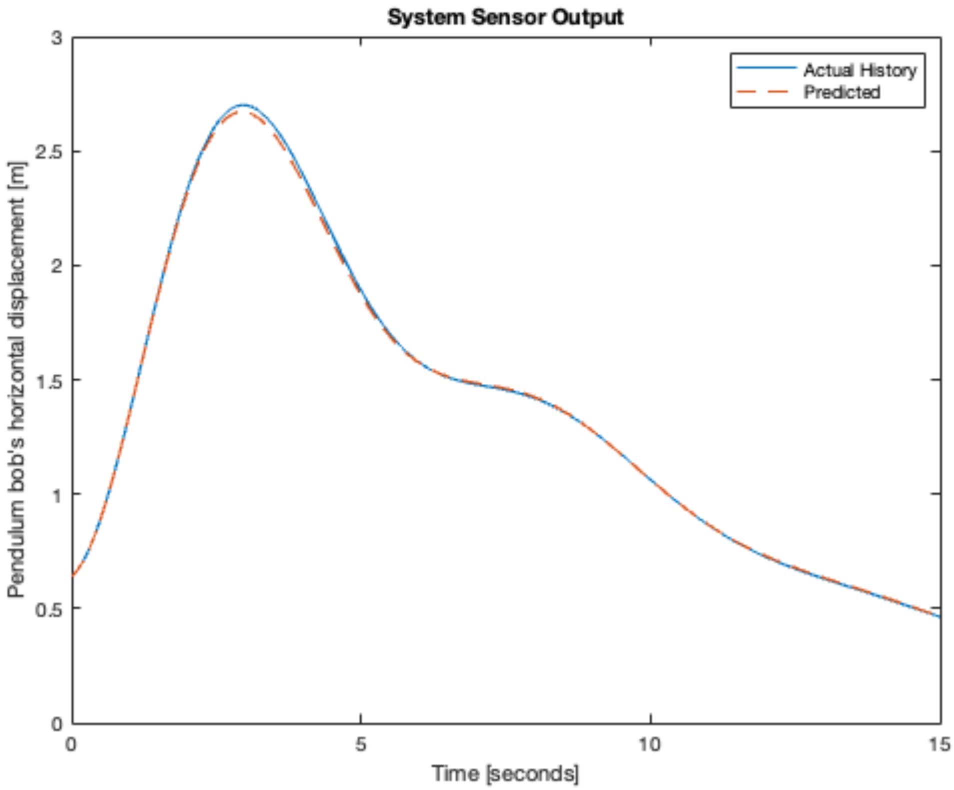
$$E_H[U(H,A=0)] < E_H[U(H,A=1)] \rightarrow \therefore \boxed{A^*=1 \text{ is selected}}$$

```
clear; clc; close all;

% ASEN 5044
% Midterm 1, 10/10/2024
% Author - Jash Bhalavat
```

Problem 1

```
% Given constants
g = 9.81;
l = 1.85;
m = 2;
M = 4;

% Load data and assign variables
data = load("midterm1problem1data.mat");
Kc = data.Kc;
thist = data.thist;
yNLhist = data.yNLhist;

% Part c
delta_t = 0.05;

% CT LTI matrices as computed in part c
A_bar = [0 1 0 0; 0 0 m*g/M 0; 0 0 0 1; 0 0 (g/(l*M))*(M + m) 0];
B_bar = [0; 1/M; 0; 1/(l*M)];
C_bar = [1 0 -l 0];
D_bar = [0];

% Construct A_hat matrix
size_A = size(A_bar);
size_B = size(B_bar);
zeros_A_hat = zeros(size_B(2), size_A(1) + size_B(2));
A_hat = [A_bar, B_bar; zeros_A_hat];

% Compute matrix exponential of A_hat matrix
matrix_exponential_A_hat = expm(A_hat * delta_t);

% Get F, G, H, M matrices
F = matrix_exponential_A_hat(1:4, 1:4);
G = matrix_exponential_A_hat(1:4, end);
H = C_bar;
M = D_bar;

% Stability
eigenvalues_F = eig(A_bar);

% Observability
O = [H; H*F; H*F*F; H*F*F*F];
observ = rank(O);
```

```
% Part d
% Construct L and y matrices as computed in part d
L = [H*(F - G*Kc); H*(F - G*Kc)^2; H*(F - G*Kc)^3; H*(F - G*Kc)^4];
y = [yNLhist(2); yNLhist(3); yNLhist(4); yNLhist(5)];

% Compute x_hat_0
x_hat_0 = inv(L' * L) * L' * y;

% Using x_hat_0, get first u_0 and y_0
u_0 = -Kc * x_hat_0;
y_0 = H * x_hat_0;

% Assign first approximations to predicted vectors
x_predicted(:,1) = x_hat_0;
u_predicted(1) = u_0;
y_predicted(1) = y_0;

% Go through time history and calculate predicted vectors using CT model
for i = 2:length(thist)
    x_predicted(:,i) = F * x_predicted(:,i-1) + G * u_predicted(i-1);
    u_predicted(i) = -Kc * x_predicted(:,i);
    y_predicted(i) = H * x_predicted(:,i);
end

% Plot figure
figure()
plot(thist, yNLhist, 'LineWidth',1.2)
hold on
plot(thist, y_predicted, '--', 'LineWidth',1.2)
legend("Actual History", "Predicted")
title("System Sensor Output")
xlabel("Time [seconds]")
ylabel("Pendulum bob's horizontal displacement [m]")
```



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