ASEN 6060
ADVANCED ASTRODYNAMICS
Hyperbolic Invariant Manifolds

Objectives:

- Introduce concept of invariant manifolds
- Define and summarize computation of stable and unstable manifolds associated with equilibrium points
- Define and summarize computation of stable and unstable manifolds associated with periodic orbits

Invariant Manifolds

Invariant manifolds:

- A set of states within the phase space that remain within that set for all time when subject to the flow of the dynamical system

- We will focus on invariant manifolds associated with equilibrium points (eq. pt) and periodic orbits :
  - Stable manifold:
  - Unstable manifold:
  - Center manifold:
- Stable/unstable manifolds are associated with quasi-periodic orbits too but are more complex to compute

Invariant Manifolds

Invariant manifolds play an important role in the CR3BP:

Stability of Equilibrium Points

Recall the EOMs for the CR3BP

$$\ddot{x} - 2\dot{y} = \frac{\partial U^*}{\partial x} \qquad \ddot{y} + 2\dot{x} = \frac{\partial U^*}{\partial y} \qquad \ddot{z} = \frac{\partial U^*}{\partial z}$$

With an equilibrium point located at $\bar{x}_{eq} = [x_{eq}, y_{eq}, z_{eq}, 0, 0, 0]^T$

Linearizing about the equilibrium point produces the following linear system

$$\delta\dot{\bar{x}} = A\delta\bar{x} \qquad \begin{bmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\delta} \\ \ddot{\xi} \\ \ddot{\eta} \\ \ddot{\delta} \end{bmatrix} = \begin{bmatrix} 0_{3\times3} & I_{3\times3} \\ U^*_{XX} & \Omega \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \delta \\ \dot{\xi} \\ \dot{\eta} \\ \dot{\delta} \end{bmatrix}$$

Stability of Equilibrium Points

Calculate 6 eigenvalues of A in the linear system (4 in-plane, 2 out-of-plane)

$$\lambda = a \pm bi$$

$\lambda = a$: Real eigenvalues
If $a > 0$: unstable mode exists
If $a < 0$: stable mode exists

$\lambda = \pm bi$: Imaginary eigenvalues
Oscillatory mode exists

$\lambda = a \pm bi$: Complex eigenvalues
Spiral in/out motion

Stability of Equilibrium Points

Three types of eigenvalues indicate three types of eigenspaces, i.e., subspaces of the linear systems:

- Dimension is determined by number of associated eigenvalues
- In the linear system, a state that starts in a specific subspace remains within that subspace for all time

Stable Manifold Theorem

From Perko, L, 2001, "Differential Equations and Dynamical Systems, 3rd Edition":

- Nonlinear system $\dot{\bar{x}} = \bar{f}(\bar{x})$
- Linear system formed by linearizing around an equilibrium point

$$\dot{\bar{x}} = A\bar{x}$$

- Near an equilibrium point with stable and unstable modes, there are local stable and unstable manifolds $W^S{}_{loc}$ and $W^U{}_{loc}$ that exist in the nonlinear system
- At the equilibrium point, $W^S{}_{loc}$ and $W^U{}_{loc}$ are tangent to E^S and E^U
- $W^S{}_{loc}$ and $W^U{}_{loc}$ possess the same dimensions as E^S and E^U

Center Manifold Theorem

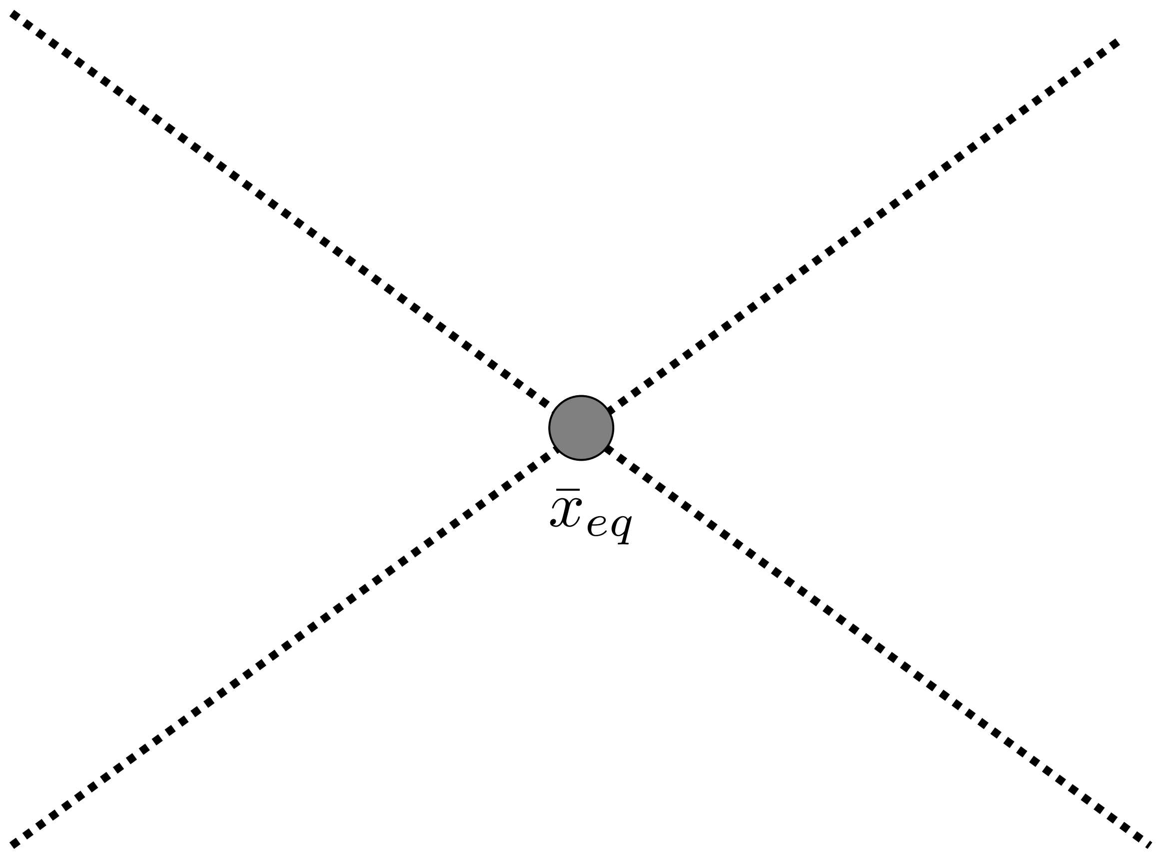
From Perko, L, 2001, “Differential Equations and Dynamical Systems, 3rd Edition”:

An equilibrium point possesses an:

- n_C-dimensional center manifold W^C tangent to E^C at eq. pt
- n_S-dimensional stable manifold W^S tangent to E^S at eq. pt
- n_U-dimensional unstable manifold W^U tangent to E^U at eq. pt

- Global stable/unstable manifolds: propagate states from local stable/unstable manifolds backwards/forwards in time
- These manifolds are invariant under the flow of the system

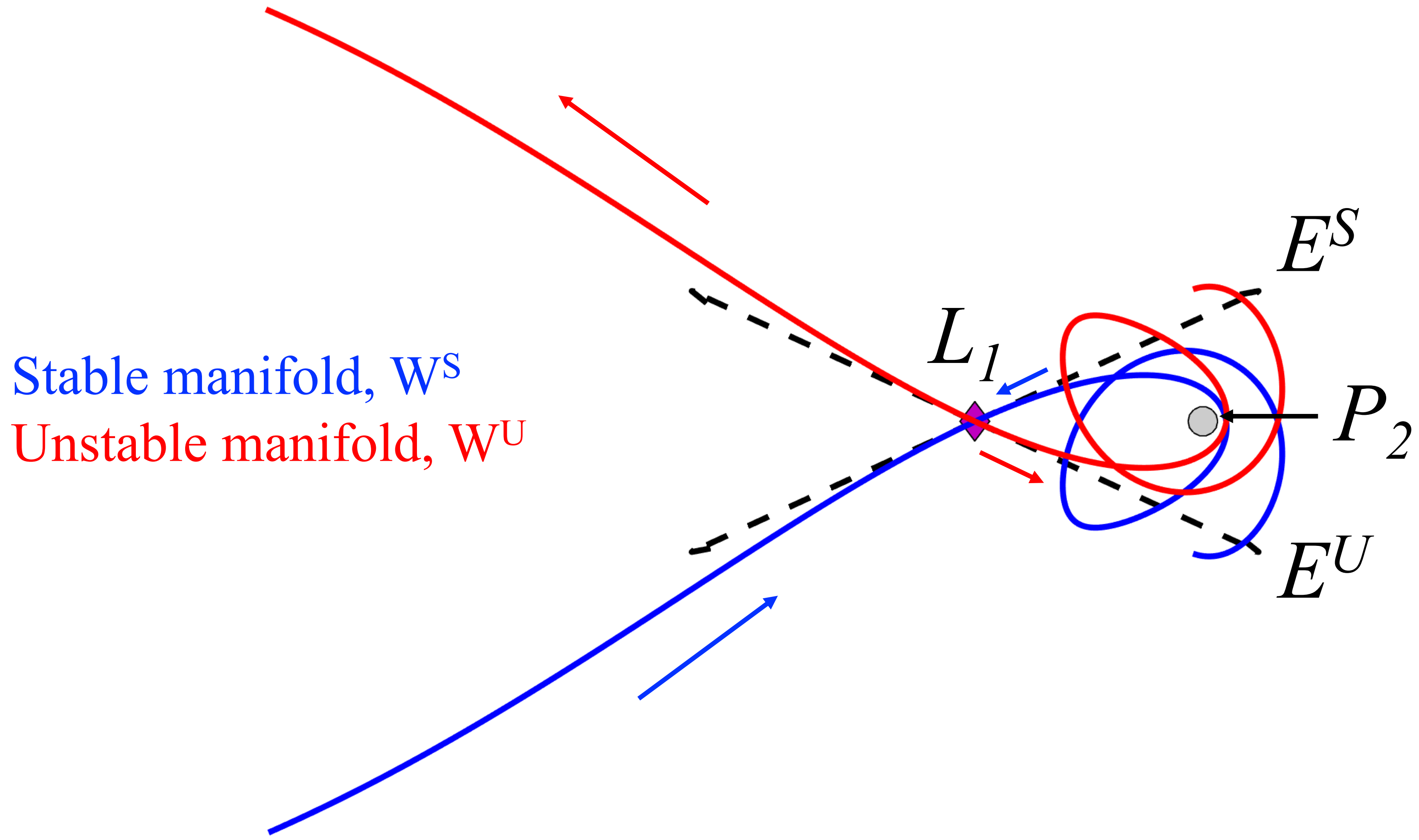
Stable/Unstable Manifolds of Eq. Pts



Global Stable/Unstable Manifolds of Eq. Pts

To calculate a set of states that lie along the global stable/unstable manifolds associated with an equilibrium point:

Global Stable/Unstable Manifolds of EM L_1



Stability of Periodic Orbits

Recall: STM is a linear mapping of the variation relative to a reference trajectory from t_0 to t

Monodromy matrix:

STM propagated for one period along a periodic orbit.

$$\mathbf{M} = \mathbf{\Phi}(T, 0)$$

Stability of a periodic orbit:

- One approach: linearize about the periodic orbit and deduce stability from eigenvalues of the monodromy matrix

Recall:

- There are 6 eigenvalues λ_i of the monodromy matrix
- Eigenvalues of the monodromy matrix occur in reciprocal pairs

Stable Manifold Theorem for Periodic Orbits

Simplified significantly from Perko, L, 2001, "Differential Equations and Dynamical Systems, 3rd Edition":

Consider a periodic orbit described by $\bar{x}_{PO}(t)$ with a period *T*.

- If the monodromy matrix admits k_S stable modes, there exists a (k_S+1)-dimensional stable manifold W^S that is invariant under flow
- If the monodromy matrix admits k_U unstable modes, there exists a (k_U+1)-dimensional unstable manifold W^U that is invariant under the flow
- These stable and unstable manifolds intersect transversally at the periodic orbit

Center Manifold Theorem for Periodic Orbits

Simplified significantly from Perko, L, 2001, “Differential Equations and Dynamical Systems, 3rd Edition”:

If the monodromy matrix admits k_S stable modes, k_U unstable modes, and k_C oscillatory modes, there is a k_C-dimensional center manifold W^C

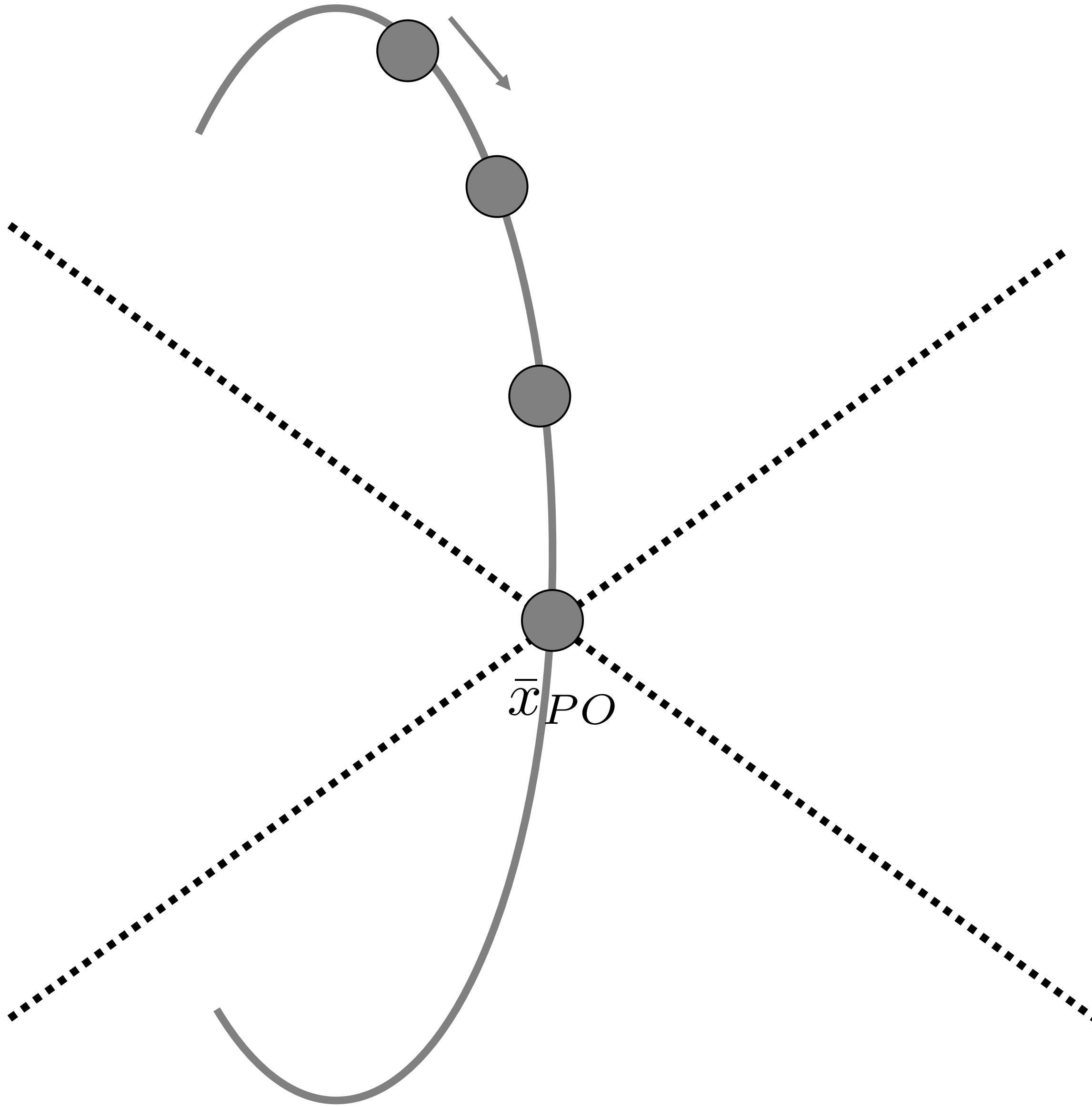
The center manifold W^C is tangent to the center subspace at the periodic orbit.

The stable, unstable and center manifolds intersect transversally at the periodic orbit.

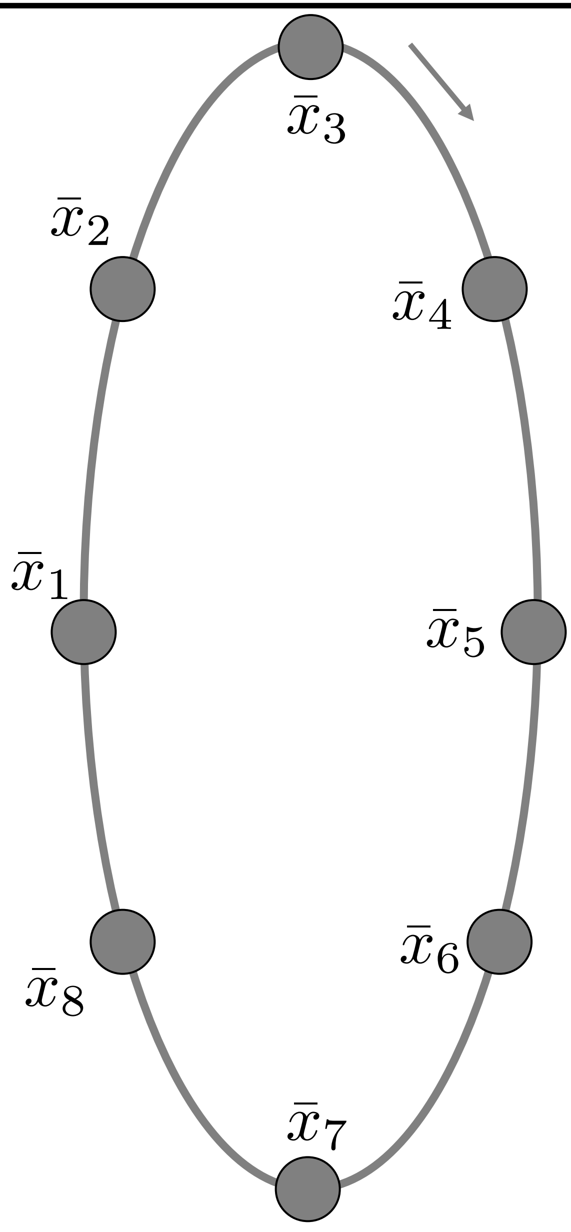
Eigenspaces for Periodic Orbits

Three types of eigenvalues of a monodromy matrix $\mathbf{M} = \mathbf{\Phi}(T,0)$ indicate three types of eigenspaces:

Stable/Unstable Manifolds of Periodic Orbits



Eigenvalues of the Monodromy Matrix



Tip 1:

Proof:

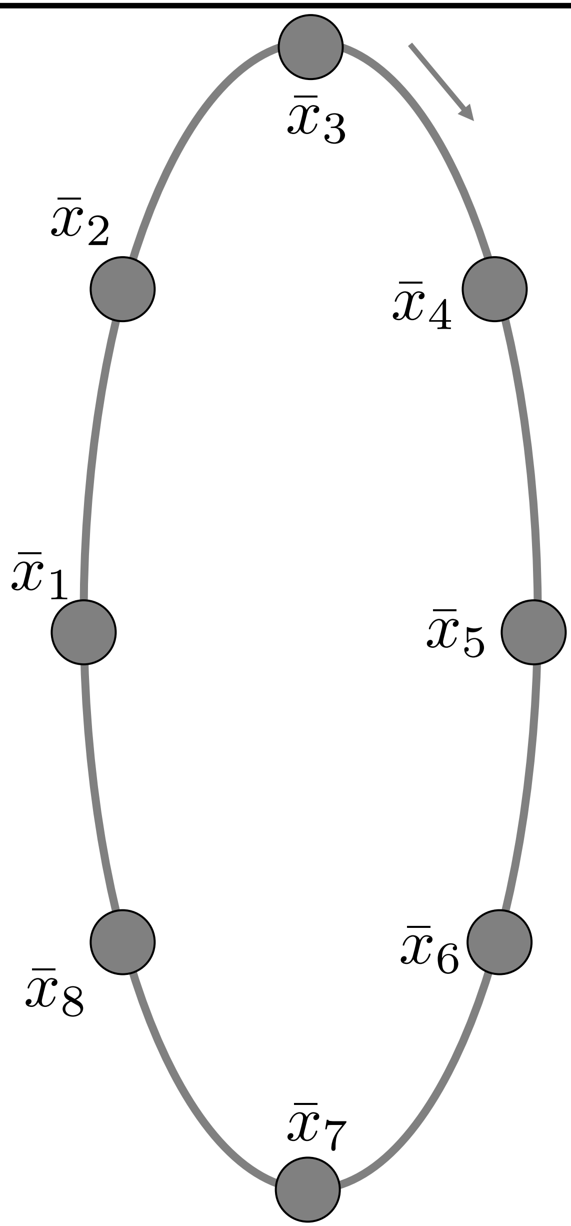
$$\Phi(t_j + T, t_j) = \Phi(t_j + T, t_1 + T)\Phi(t_1 + T, t_j)$$

Note 1: $\Phi(t_j + T, t_1 + T) = \Phi(t_j, t_1)$

Note 2: $\Phi(t_1 + T, t_1) = \Phi(t_1 + T, t_j)\Phi(t_j, t_1)$

$$\Phi(t_j + T, t_j) = \Phi(t_j, t_1)\Phi(t_1 + T, t_1)\Phi(t_j, t_1)^{-1}$$

Eigenvectors of the Monodromy Matrix



Tip 2:

Proof:

Eigenvectors at times t_1 and t_j satisfy

$$\Phi(t_j + T, t_j)\bar{v}(t_j) = \lambda\bar{v}(t_j)$$

$$\Phi(t_1 + T, t_1)\bar{v}(t_1) = \lambda\bar{v}(t_1)$$

At t_1, multiply by STM from t_1 to t_j

$$\Phi(t_j, t_1)\Phi(t_1 + T, t_1)\bar{v}(t_1) = \lambda\Phi(t_j, t_1)\bar{v}(t_1)$$

Plug in relationship from previous slide

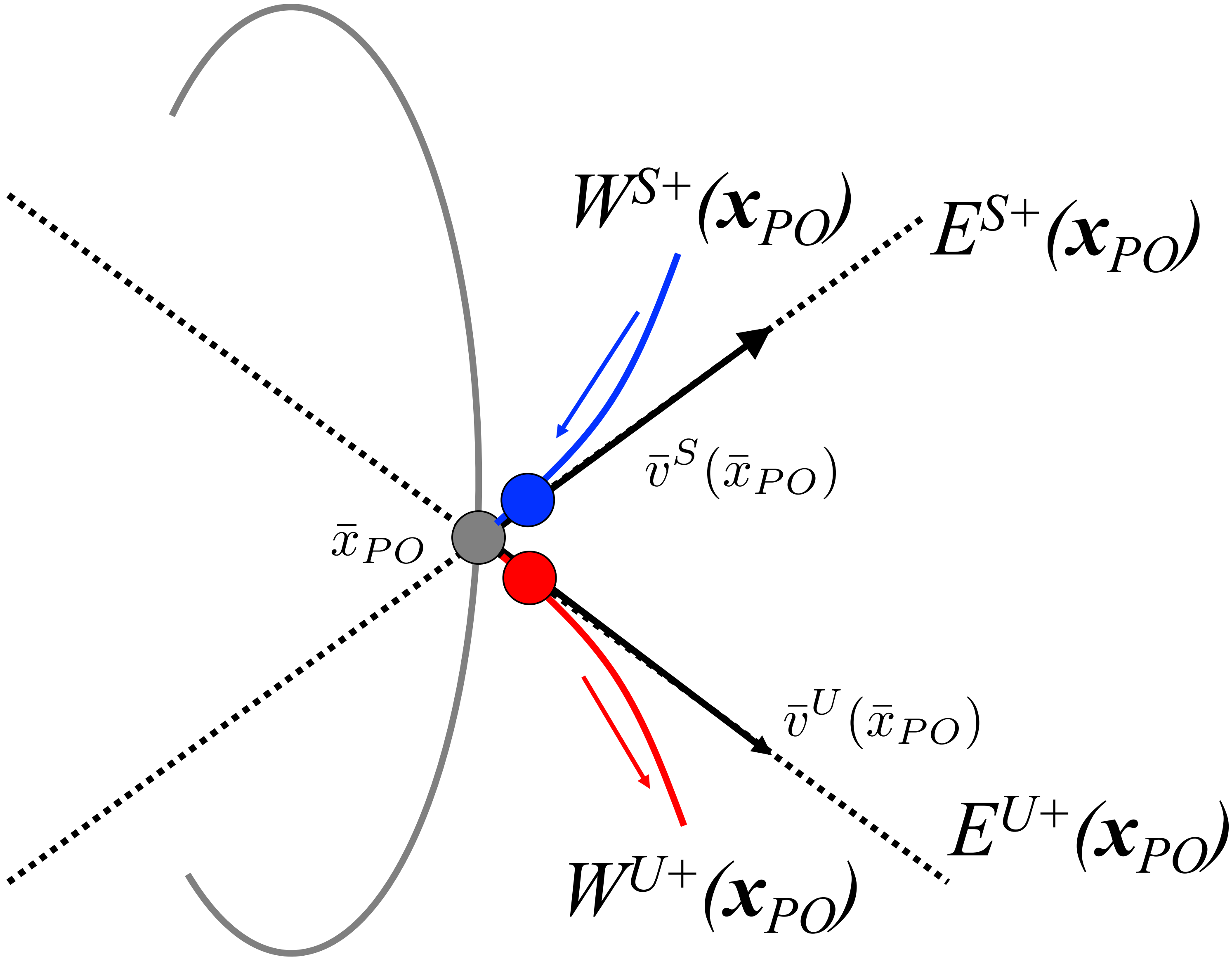
$$\Phi(t_j + T, t_j)\Phi(t_j, t_1)\bar{v}(t_1) = \lambda\Phi(t_j, t_1)\bar{v}(t_1)$$

Thus

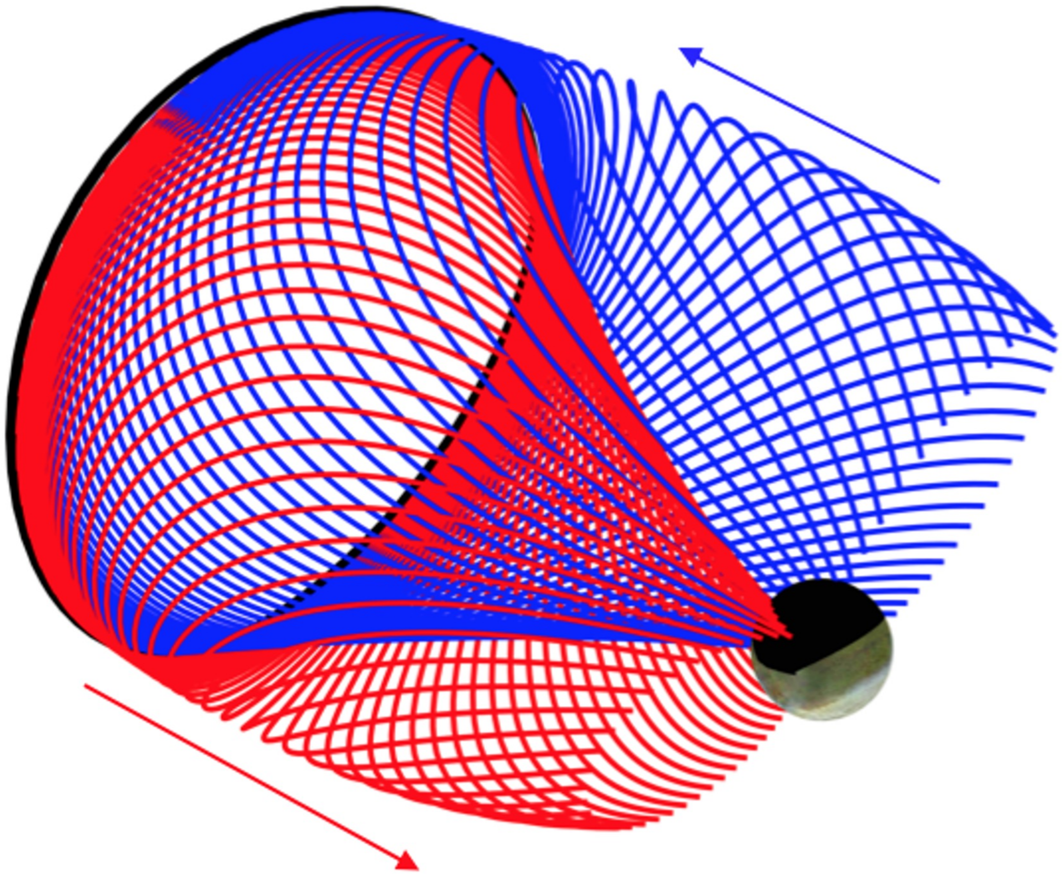
Stable/Unstable Manifolds of Periodic Orbits

To calculate a set of states that lie along the stable/unstable manifolds of a periodic orbit:

Stable/Unstable Manifolds of Periodic Orbits

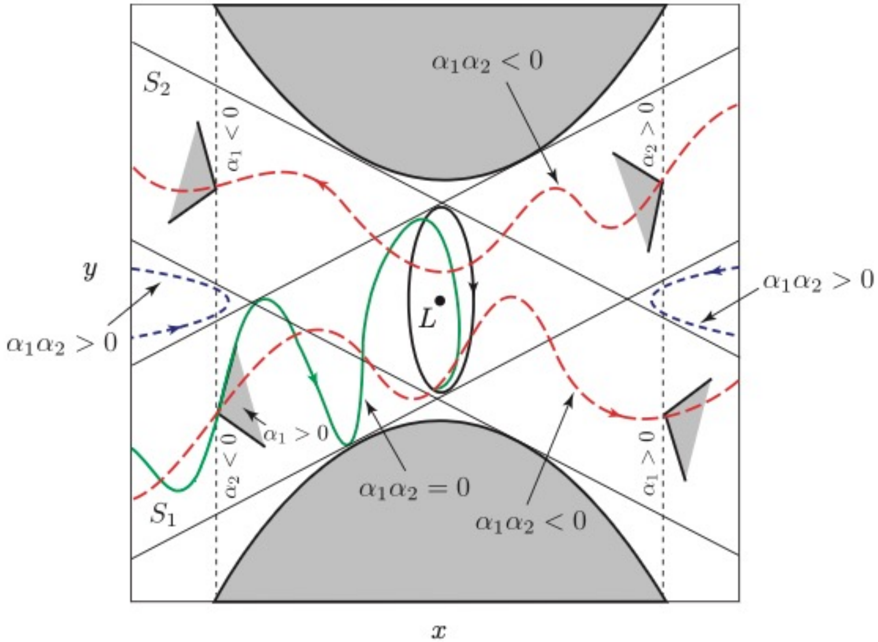


Stable/Unstable Manifolds of Periodic Orbits

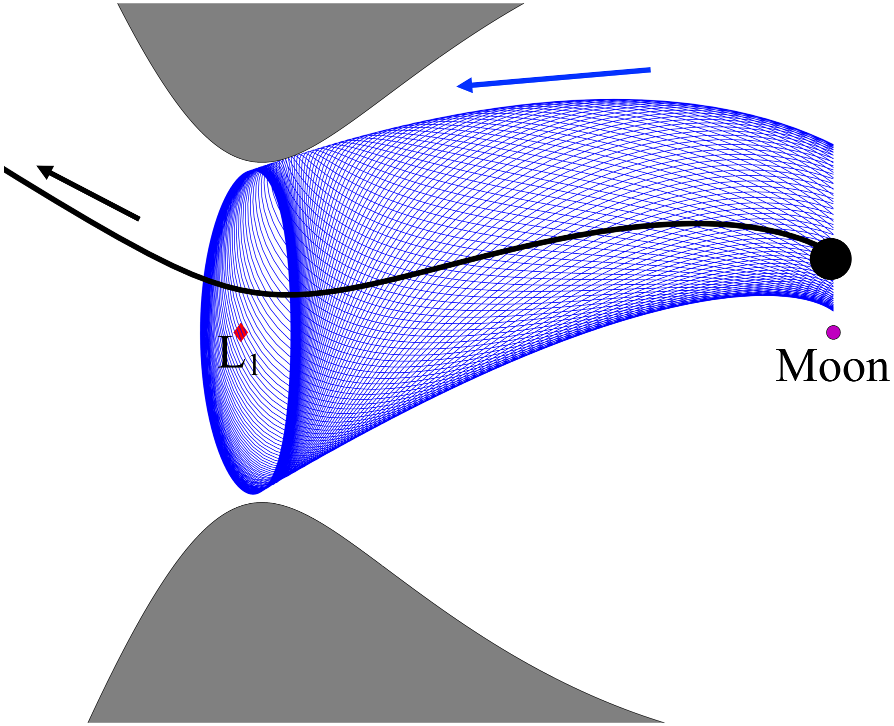


Passing Through the Gateways

Hyperbolic invariant manifolds associated with periodic and quasi-periodic orbits near the collinear libration points govern trajectories passing through the gateways



E.g.: Planar trajectory starting within stable manifold of L_1 Lyapunov orbit



Left image credit: Koon, Lo, Marsden, Ross, 2011, “Dynamical Systems, the Three-Body Problem and Mission Design”

Stable/Unstable Manifolds

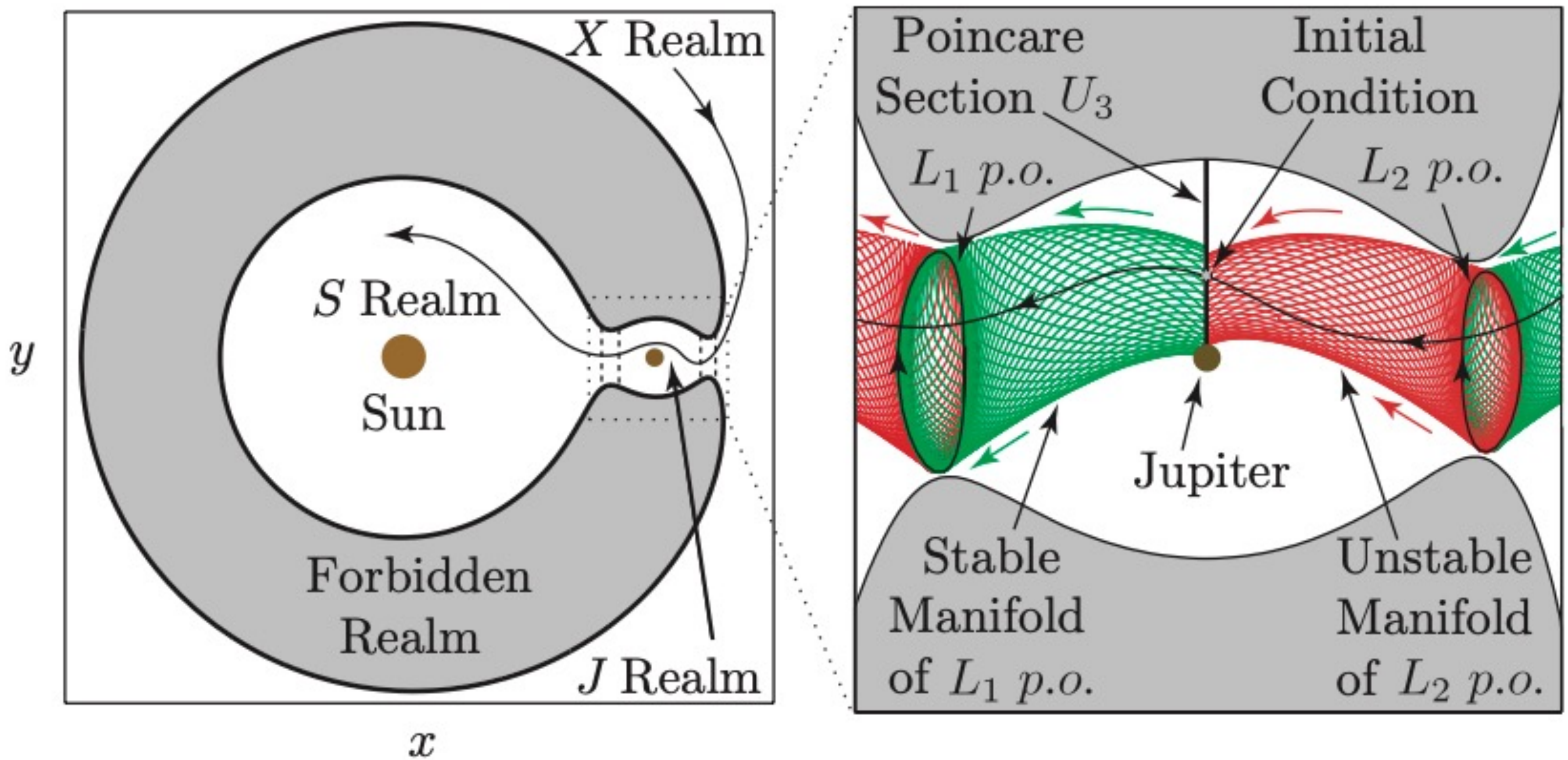


Image credit: Koon, Lo, Marsden, Ross, 2011, “Dynamical Systems, the Three-Body Problem and Mission Design”

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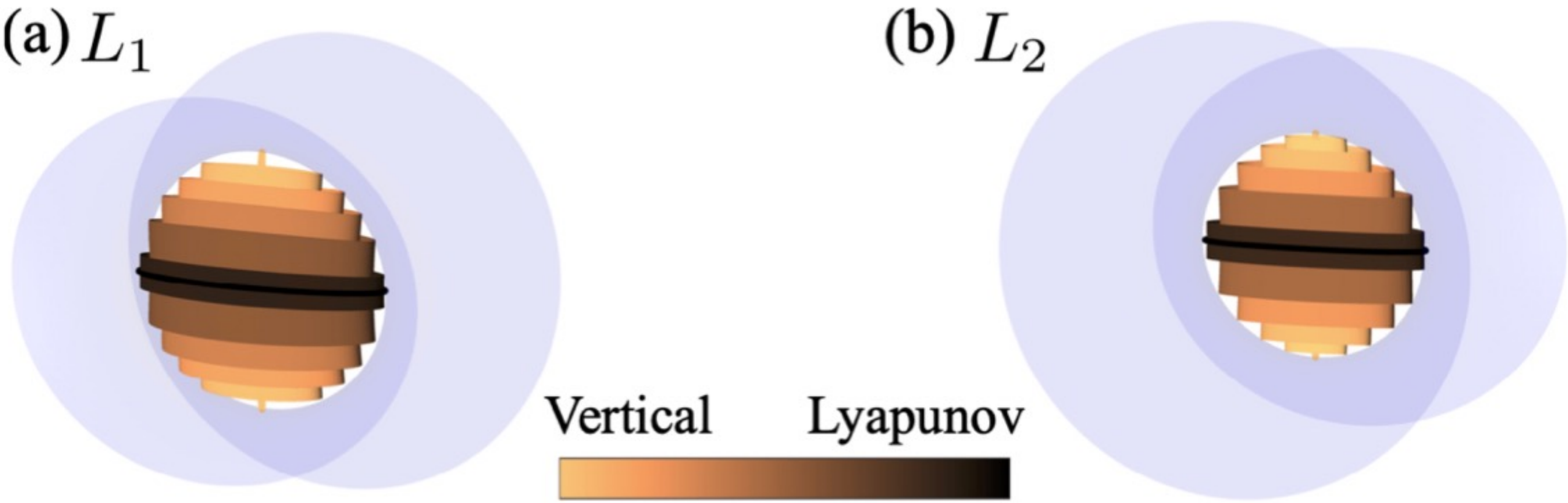

Image credit: Bonasera and Bosanac, 2022, "Applying Data Mining Techniques to Higher-Dimensional Poincaré Maps in the Circular Restricted Three-Body Problem", Celestial Mechanics and Dynamical Astronomy

Passing Through the Gateways

Use stable/unstable manifolds to understand natural flow for trajectory design

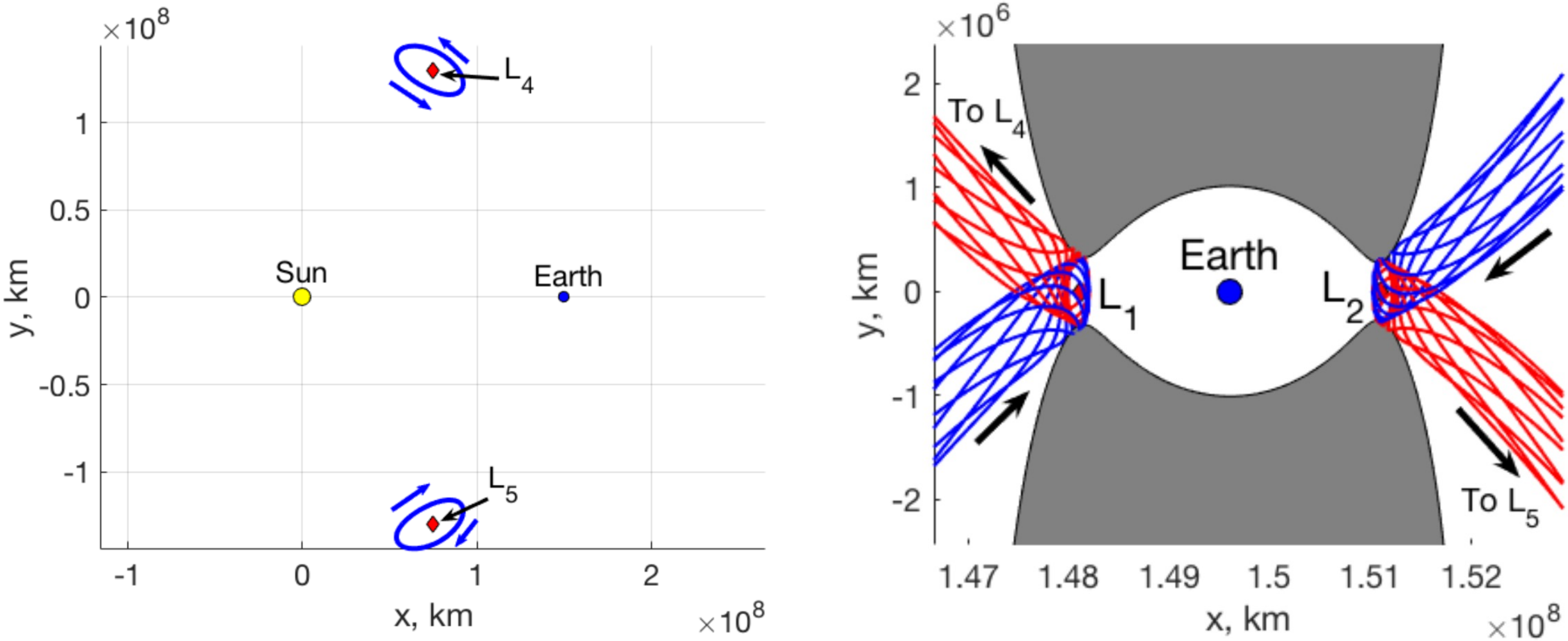

Image credit: Sullivan, Elliott, Bosanac, Alibay, Stuart, 2019, "Exploring the Low-Thrust Trajectory Design Space for SmallSat Missions to the Sun-Earth Triangular Equilibrium Points"