ASEN 5050
SPACEFLIGHT DYNAMICS
Time Along An Orbit

Objectives:

- Define eccentric anomaly and derive Kepler's equation
- Summarize approaches to solve Kepler's equation
- Derive relationship between true and eccentric anomalies
- Present similar approaches for parabolas and hyperbolas

Introduction

Previously related location along an orbit, via θ^*, to the relative distance between Body 1 and Body 2. However, can also relate these quantities to time. → Will complete for an elliptical orbit

Recall the orbital period, describing the time to complete one revolution:

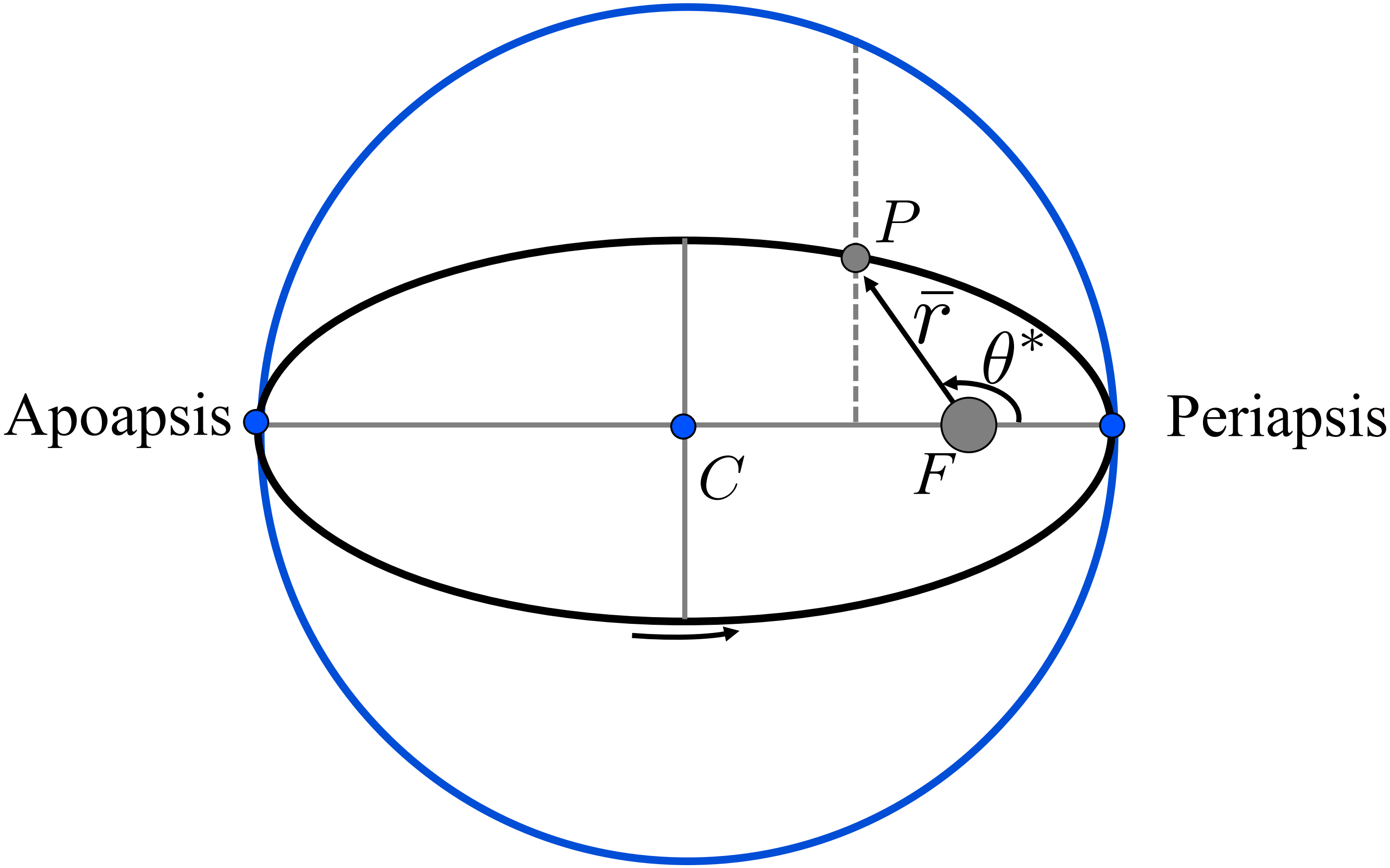
$$P = 2\pi\sqrt{\frac{a^3}{\mu}}$$

By symmetry of an ellipse, time for spacecraft to travel between apoapsis and periapsis is $P/2$ (Not true elsewhere)

Also define the mean motion:

And, in general,

Eccentric Anomaly



E = Eccentric Anomaly
θ^* = True Anomaly

Relating Position to Eccentric Anomaly

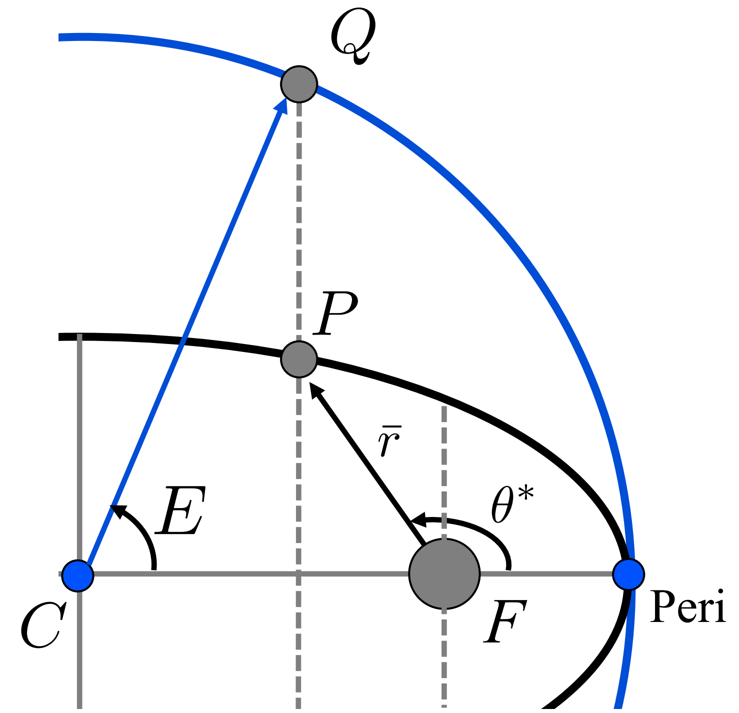
Relate r to E:

From conic equation: $r = \dfrac{p}{1 + e\cos\theta^*}$

$$\cos\theta^* = \frac{p}{re} - \frac{1}{e} \qquad p = a(1 - e^2)$$

Substitute into above equation:

$$\frac{r}{e} = -a\cos E + \frac{a}{e}$$



Relating Position to Time

Differentiate with respect to time $r = a(1 - e\cos(E))$

$$\frac{dr}{dt} = \dot{r} = ae\dot{E}\sin(E)$$

Also differentiate the conic equation:

$$\frac{dr}{dt} = \dot{r} = \frac{d}{dt}\left(\frac{p}{1 + e\cos(\theta^*)}\right) = \frac{pe\dot{\theta}^*\sin(\theta^*)}{(1 + e\cos(\theta^*))^2} = \frac{r^2\dot{\theta}^* e\sin(\theta^*)}{p}$$

Equating these two expressions:

$$\frac{dr}{dt} = ae\dot{E}\sin(E) = \frac{he\sin(\theta^*)}{p}$$

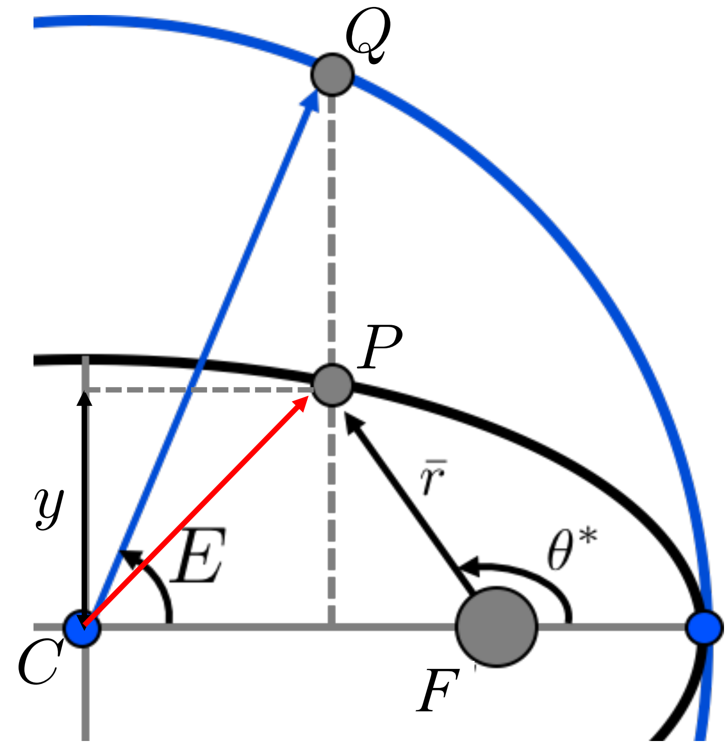
Relating Position to Time

To further manipulate this relationship, from the equation of an ellipse (relative to C!):

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Plug in *x* expression, solve for *y*

Then,



Relating Position to Time

Then: $$\frac{dr}{dt} = ae\dot{E}\sin(E) = \frac{he\sin(\theta^*)}{p} = \frac{heb}{rp}\sin(E)$$

Rearranging and noting that $h = \sqrt{\mu p} = \sqrt{\mu a(1-e^2)}$

$$\dot{E} = \frac{hb}{rpa} = \frac{\mu b}{rha} = \frac{\mu a\sqrt{(1-e^2)}}{ra\sqrt{\mu a(1-e^2)}} = \frac{\mu}{r\sqrt{\mu a}} = \frac{1}{r}\sqrt{\frac{\mu}{a}}$$

$$\sqrt{\frac{\mu}{a}} = r\frac{dE}{dt} = a(1-e\cos E)\frac{dE}{dt}$$

From separation of variables:

$$\sqrt{\frac{\mu}{a^3}}dt = (1-e\cos E)dE$$

Relating Position to Time

Integrating from periapsis, where $t=t_p$ and $E=E_0=0$ to a general location along the orbit at t, E:

$$\sqrt{\frac{\mu}{a^3}}(t - t_p) = (E' - e\sin E')\Big|_{E_0=0}^{E}$$

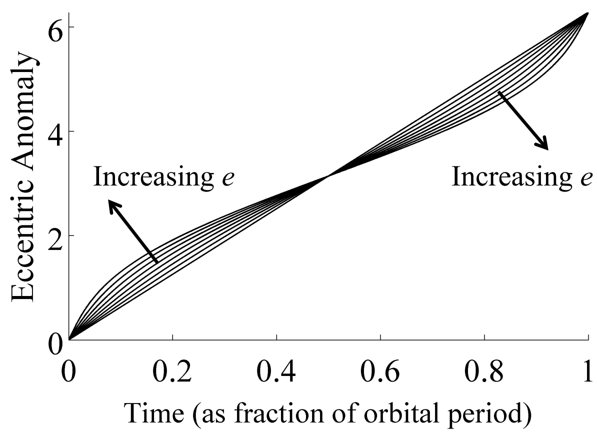
Recalling the mean motion, n, to recover Kepler's equation:

Kepler's Equation

$$M = n(t - t_p) = (E - e\sin(E))$$

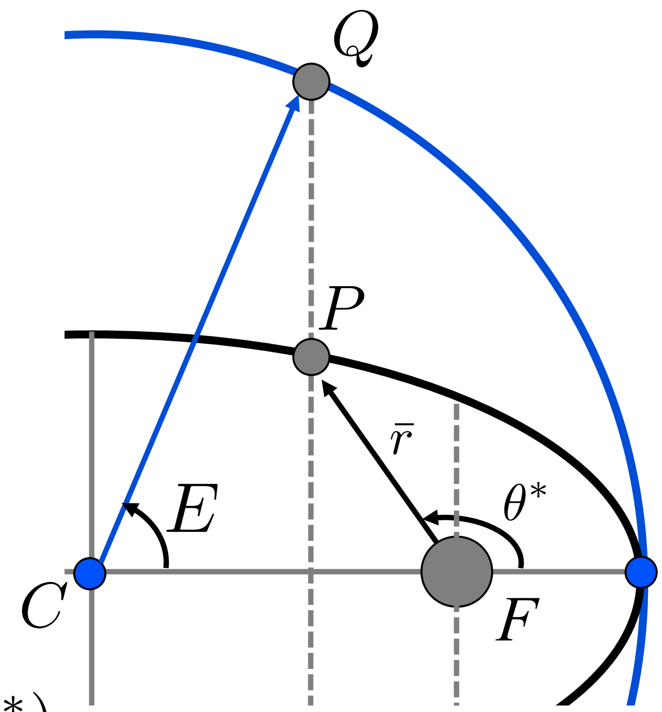
Kepler's equation is a transcendental equation. It cannot be solved analytically to find $E(t\text{-}t_p)$. Instead, we must solve iteratively using a numerical method.

Eccentric Anomaly is not a linear function of time, but the solutions are unique.



Relating Eccentric and True Anomalies

Recall from geometry of ellipse and auxiliary circle:



Substituting the conic equation:

$$a\cos(E) = ae + \frac{a(1-e^2)\cos(\theta^*)}{(1+e\cos(\theta^*))}$$

$$\cos(E) = \frac{e + e^2\cos(\theta^*) + \cos(\theta^*) - e^2\cos(\theta^*)}{(1+e\cos(\theta^*))}$$

$$\cos(E) = \frac{e+\cos(\theta^*)}{1+e\cos(\theta^*)}$$

Relating Eccentric and True Anomalies

Use a trigonometric identity and plug in for $\cos(E)$:

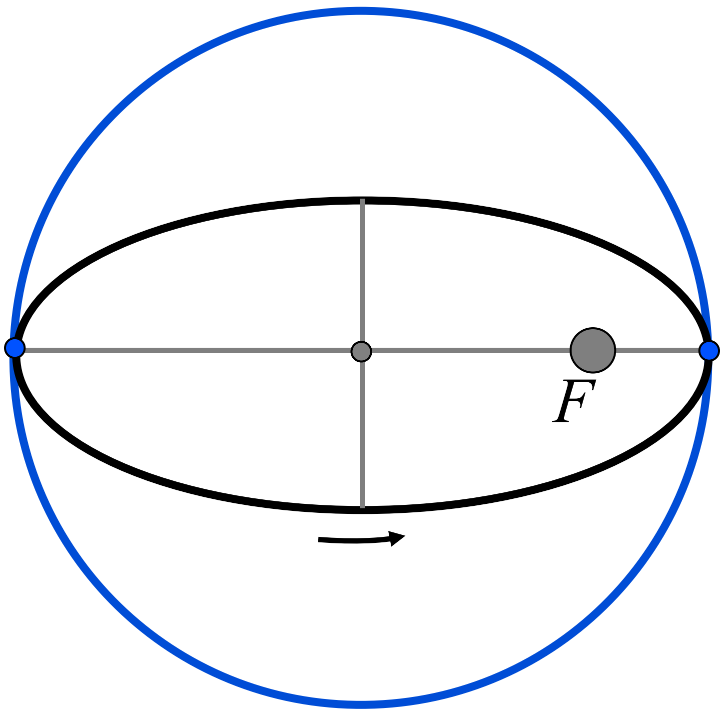
$$\tan^2\left(\frac{E}{2}\right) = \frac{1-\cos(E)}{1+\cos(E)} = \frac{1+e\cos(\theta^*)-e-\cos(\theta^*)}{1+e\cos(\theta^*)+e+\cos(\theta^*)}\frac{(1+e\cos(\theta^*))}{(1+e\cos(\theta^*))}$$

$$\tan^2\left(\frac{E}{2}\right) = \frac{(1-e)}{(1+e)}\tan^2\left(\frac{\theta^*}{2}\right)$$

Then, recovered relationships between E and θ^*:

Calculating Eccentric Anomaly Numerically

$$E = [0, \pi]\text{rad}$$



$$E = [\pi, 2\pi]\text{rad}$$
$$= [-\pi, 0]\text{rad}$$

Iterative scheme using Newton's method, Sec. 2.2.5 of Vallado textbook:

Approximating Eccentric Anomaly

Series solutions exist, but provide only an approximation when truncated to finite number of terms

From Plummer 1918, 'An introductory treatise on dynamical astronomy' [and referenced in Vallado 2022]:

$$E = M + 2\sum_{k=1}^{\infty} \frac{\sin(kM)}{k}\left(J_k(ke)\right)$$

using Bessel functions

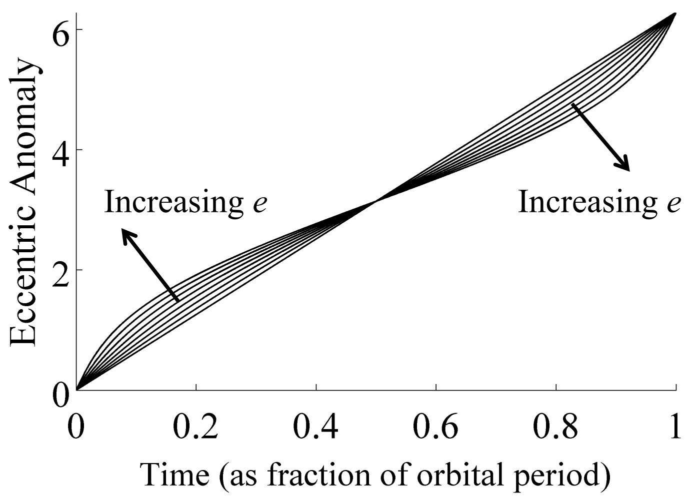
$$J_k(ke) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(k+j)!}\left(\frac{ke}{2}\right)^{k+2j}$$

Note: we will <u>not</u> use this approach in this class as it can suffer from lower accuracy depending on truncation order, orbit, and location. But it can provide a useful initial guess for an iterative scheme

Calculating Eccentric Anomaly Numerically

What is a good initial guess E_0? $M = n(t - t_p) = (E - e\sin(E))$

Common approach:
$E_0 = M$



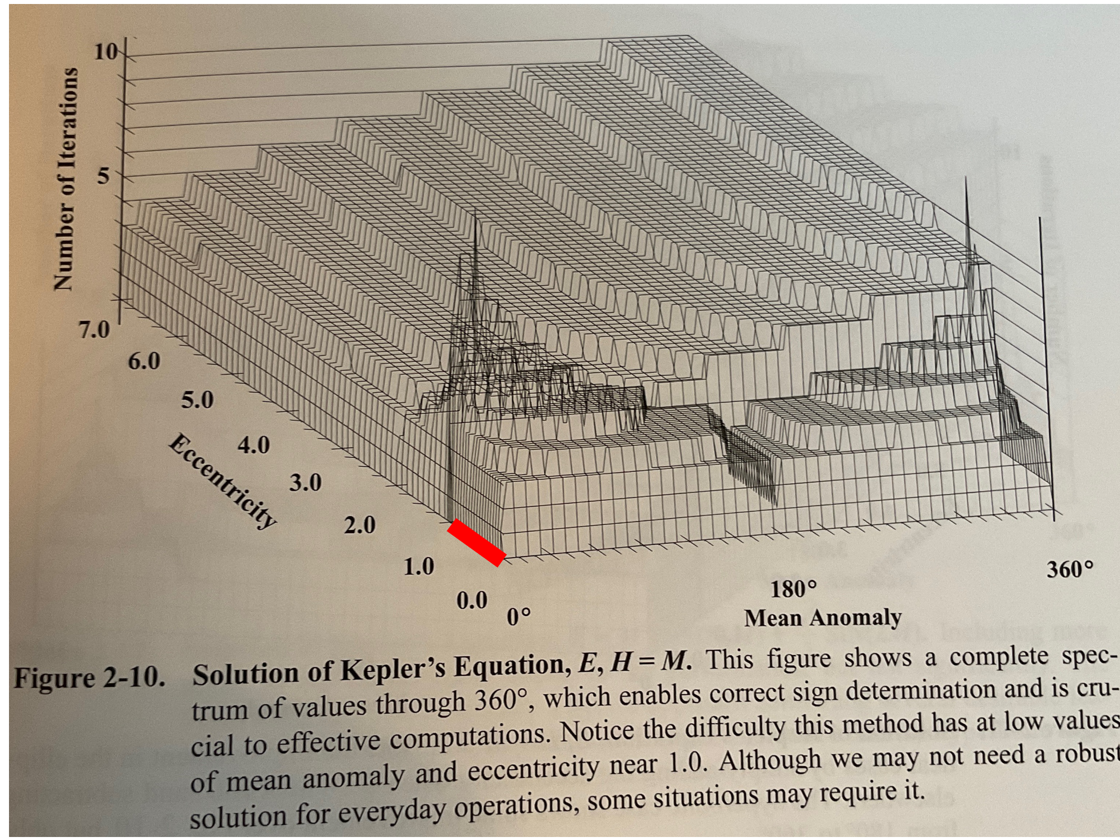


Image credit: Vallado, "Fundamentals of Astrodynamics, 5th edition", 2022

Calculating Eccentric Anomaly Numerically

What is a good initial guess E_0?

$$M = n(t - t_p) = (E - e\sin(E))$$

Alternative approaches

$E_0 = M +/- e$

$$E_0 = M + e\sin(M) + \frac{e^2}{2}\sin(2M)$$

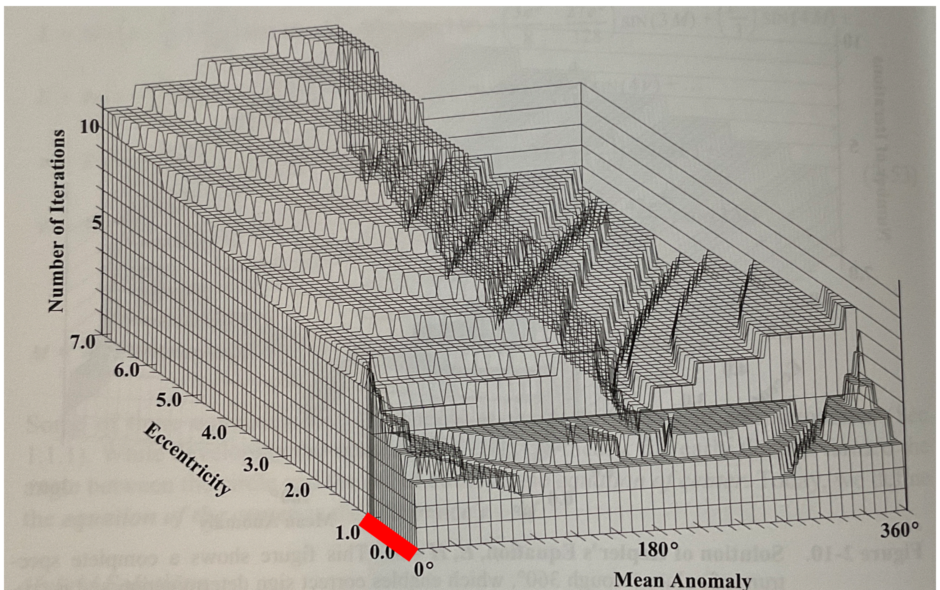


Figure 2-11. Solution of Kepler's Equation, *E*, *H* = *M* ± *e*. Note the improvement in the elliptical cases by simply adding the eccentricity when $0° < M < 180°$ and subtracting elsewhere. The hyperbolic case shows some improvement over Fig. 2-10, but only from 180° to 360°.

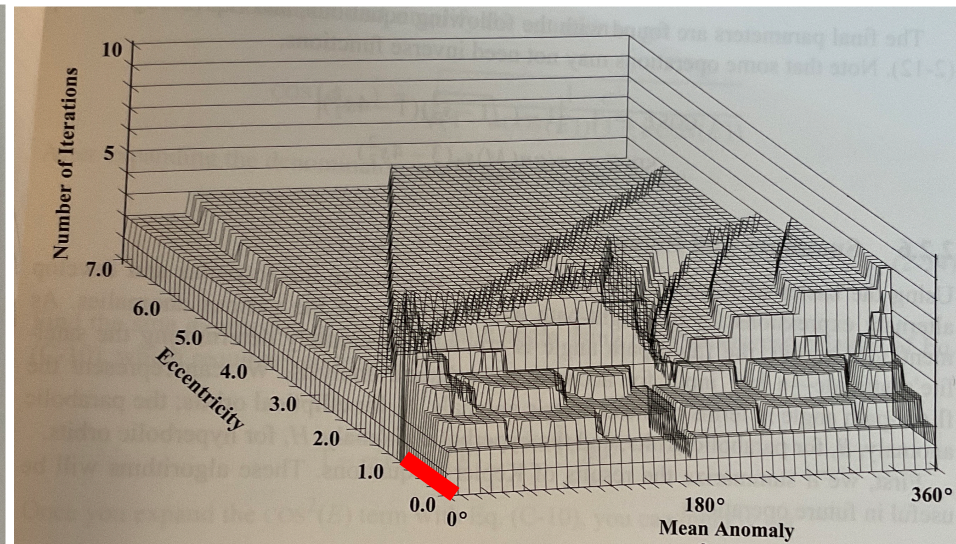


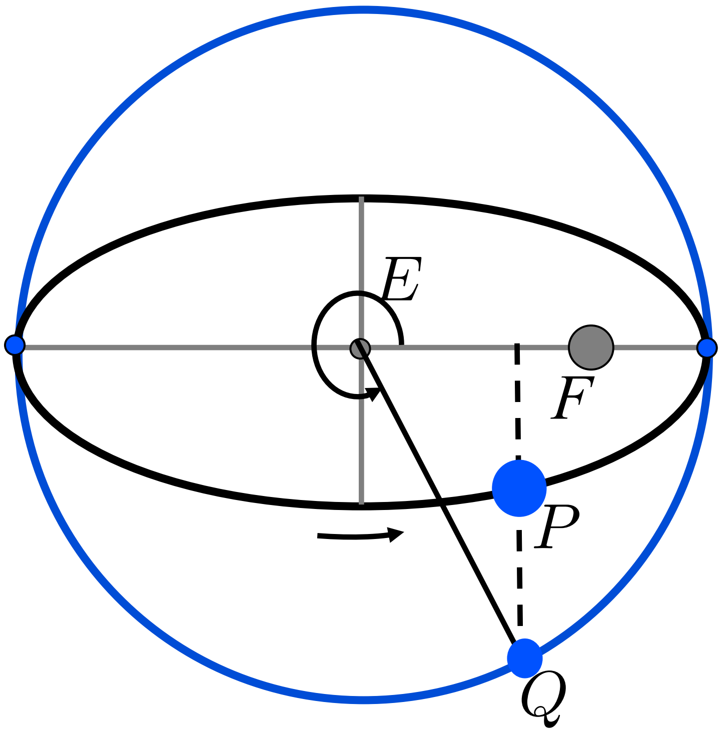
Figure 2-12. Solution of Kepler's Equation, $E = M + e\mathrm{SIN}(M) + \frac{e^2}{2}\mathrm{SIN}(2M)$. Including more terms in the initial guess improves the performance, but not significantly. The hyperbolic solution is actually a hybrid approach combining several desirable features of solution from the linearity of the problem. Notice that the hyperbolic solution actually improves as eccentricity increases.

Image credits: Vallado, "Fundamentals of Astrodynamics, 5th edition", 2022

Calculating Time Past Periapsis

Assume a, e known

$$M = n(t - t_p) = (E - e\sin(E))$$



a)

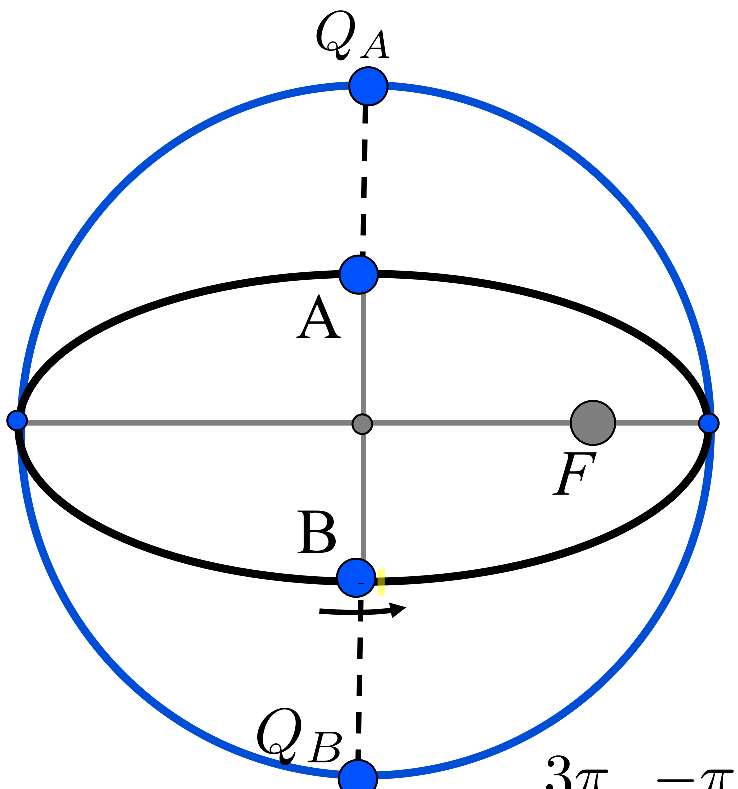
b)

Calculating Time Between Two Locations

Assume a, e known

$$T = 2\pi/n$$

$$E_A = \frac{\pi}{2} \text{ rad}$$



$$E_B = \frac{3\pi}{2}, \frac{-\pi}{2} \text{ rad}$$

Time for spacecraft to travel from A to B

Recall: $t - t_p = \dfrac{T}{2\pi}(E - e\sin(E))$

Use:

Time for spacecraft to travel from B to A

Use:

Calculating Time Between Two Locations

Assume *a, e* known

General expression when less than one revolution performed:

$$t_{AB} = \frac{1}{n}\left[(E_B - e\sin(E_B)) - (E_A - e\sin(E_A))\right]$$

General expression when *k* revolutions of orbit performed:

$$t_{AB} = \frac{1}{n}\left[2\pi k + (E_B - e\sin(E_B)) - (E_A - e\sin(E_A))\right]$$

Hyperbolic Orbits

Analogous relationships exist with hyperbolas, but new definitions

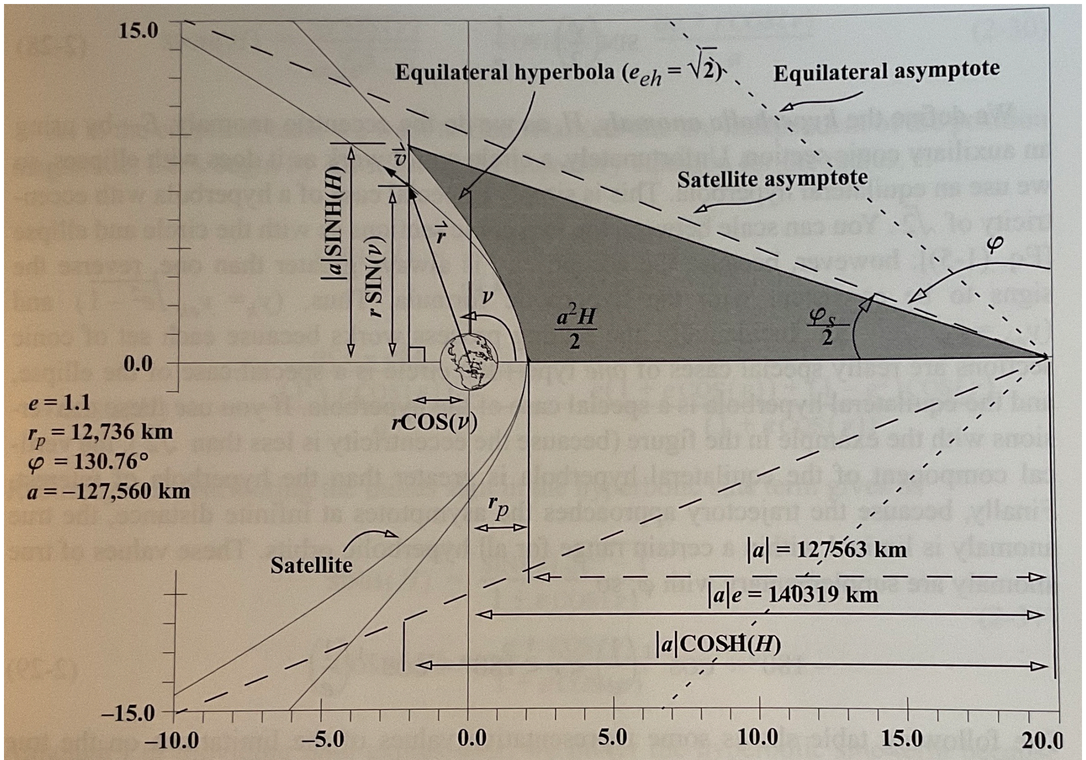


Figure 2-4. Hyperbolic Anomaly. The hyperbolic anomaly, H, is actually a measure of area, shown as the shaded region in the figure. Notice we use an equilateral hyperbola to determine H, just as we use a circle for elliptical orbits. Semimajor axis quantities are shown as absolute values to keep distances positive. The satellite and equilateral hyperbola have the same semimajor axis value. Thus, the focus for the equilateral hyperbola does not coincide with that of the satellite and its $r_p = 52811$ km. Note that the turning angle extends past the page to the satellite asymptote.

Equilateral hyperbola:

$$e = \sqrt{2}$$

$$r_p = a(1 - e)$$

H = Hyperbolic Anomaly

Image credit: Vallado, "Fundamentals of Astrodynamics, 5th edition", 2022

Hyperbolic Orbits

H, Hyperbolic anomaly similar to area quantity.

$$M_h = \sqrt{\frac{\mu}{|a|^3}}(t - t_p) = e \sinh(H) - H$$

Like Kepler's equation, cannot solve for H analytically.

Then, relationships between H and θ^* for hyperbolic orbits:

$\theta^* \rightarrow H$: $$\tanh\left(\frac{H}{2}\right) = \sqrt{\frac{(e-1)}{(e+1)}} \tan\left(\frac{\theta^*}{2}\right)$$

$H \rightarrow \theta^*$: $$\tan\left(\frac{\theta^*}{2}\right) = \sqrt{\frac{(e+1)}{(e-1)}} \tanh\left(\frac{H}{2}\right)$$

Relate r and H: $r = a(1 - e\cosh(H))$

Parabolic Orbits

Since $e = 1$ for a parabola, can find one relationship between time and true anomaly → Barker's equation

$$\sqrt{\frac{\mu}{p^3}}(t - t_p) = M_p = \frac{1}{6}\tan^3\left(\frac{\theta^*}{2}\right) + \frac{1}{2}\tan\left(\frac{\theta^*}{2}\right)$$

This expression for time as a function of true anomaly can be solved iteratively or inverted to find an analytical solution:

$$\tan\left(\frac{\theta^*}{2}\right) = \left(3M_p + \sqrt{(3M_p)^2 + 1}\right)^{1/3} - \left(3M_p + \sqrt{(3M_p)^2 + 1}\right)^{-1/3}$$

Can also write similar solutions in other forms.