# Working with standard optimization packages

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Optimal Trajectories

# Overview

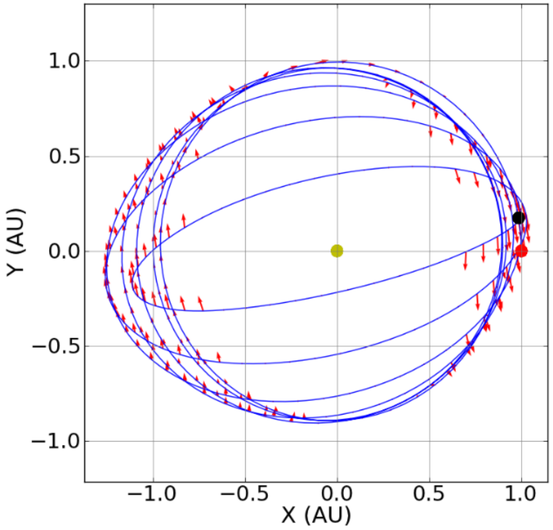
- Personal Background
- Common elements
  - General description
  - Toy problem formulation
  - Scaling
  - Sparsity
  - Local optima
- Optimizer differences
  - SNOPT vs IPOPT vs MATLAB (fmincon, …)
- Advanced example
- Project suggestion

# Personal background

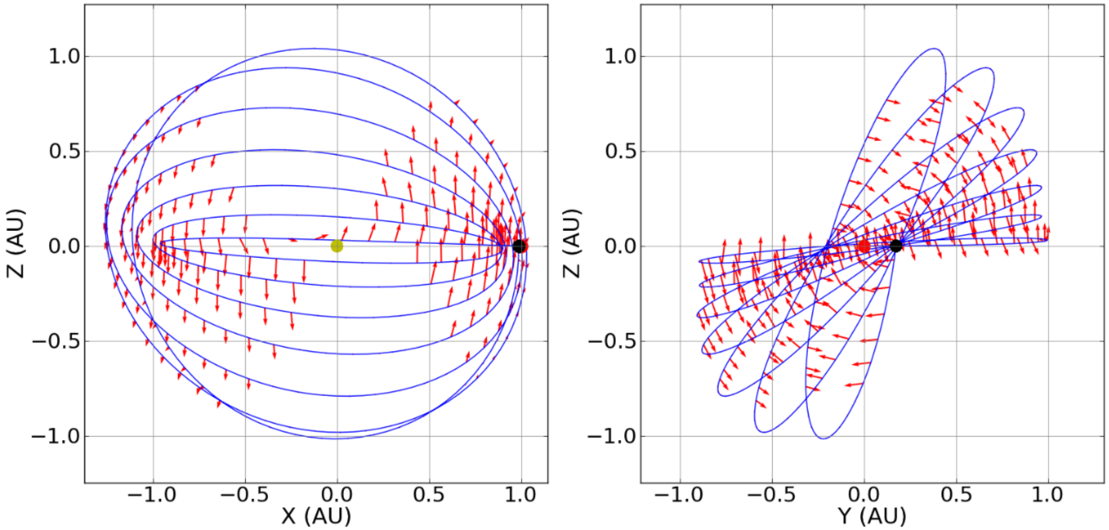
- Spent my master's thesis writing a Sims-Flanagan type low-thrust trajectory optimization tool
  - Essentially a watered down version of JPL's MALTO
  - Interfaced with SNOPT for optimization (same as MALTO)

- Used for optimizing far-side, highly inclined solar observatory trajectories

# Solar observatory




- 7.5 year time of flight
- Orbital period of ~ 1 year
- Final heliocentric inclination of 90 degrees
- C3 = 0, no gravity assists, all solar electric propulsion

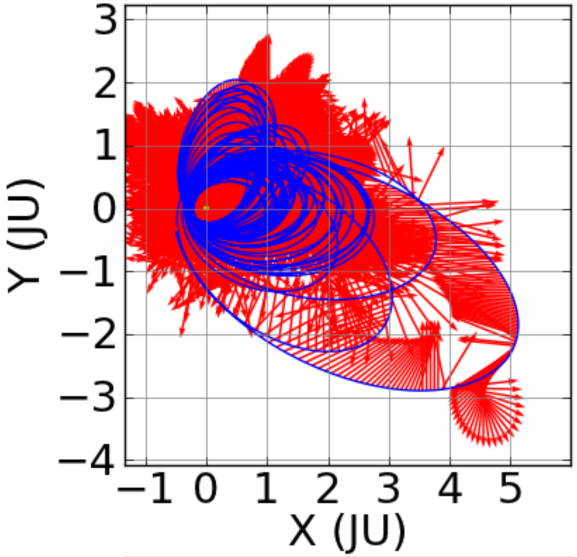


# Personal background

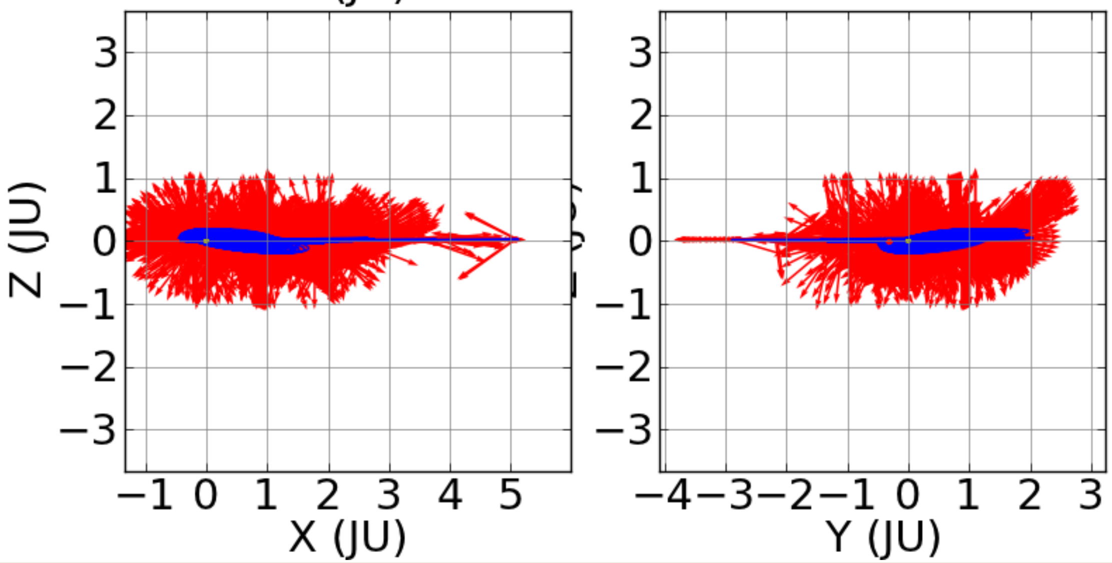
- Realized that even though this code was very specific in its application, it applied very readily to all sorts of problems
  - Human missions to near Earth asteroids
  - Jovian moon tour (GTOC 6)
  - “Warp drive” studies
  - Human fly-by mission of Mars
  - Robotic asteroid tours (GTOC 7)

- Current activities
  - Retired my original low-thrust code
  - Helping out developing a much more powerful implementation of the same method (championed by Stijn de Smet)
  - Working on effective methods for much more complex problems

# Jovian moon tour




- GTOC6
- Scaling gone wrong…
- But it worked, and got us a very nice ranking!



# Mars fly-by mission

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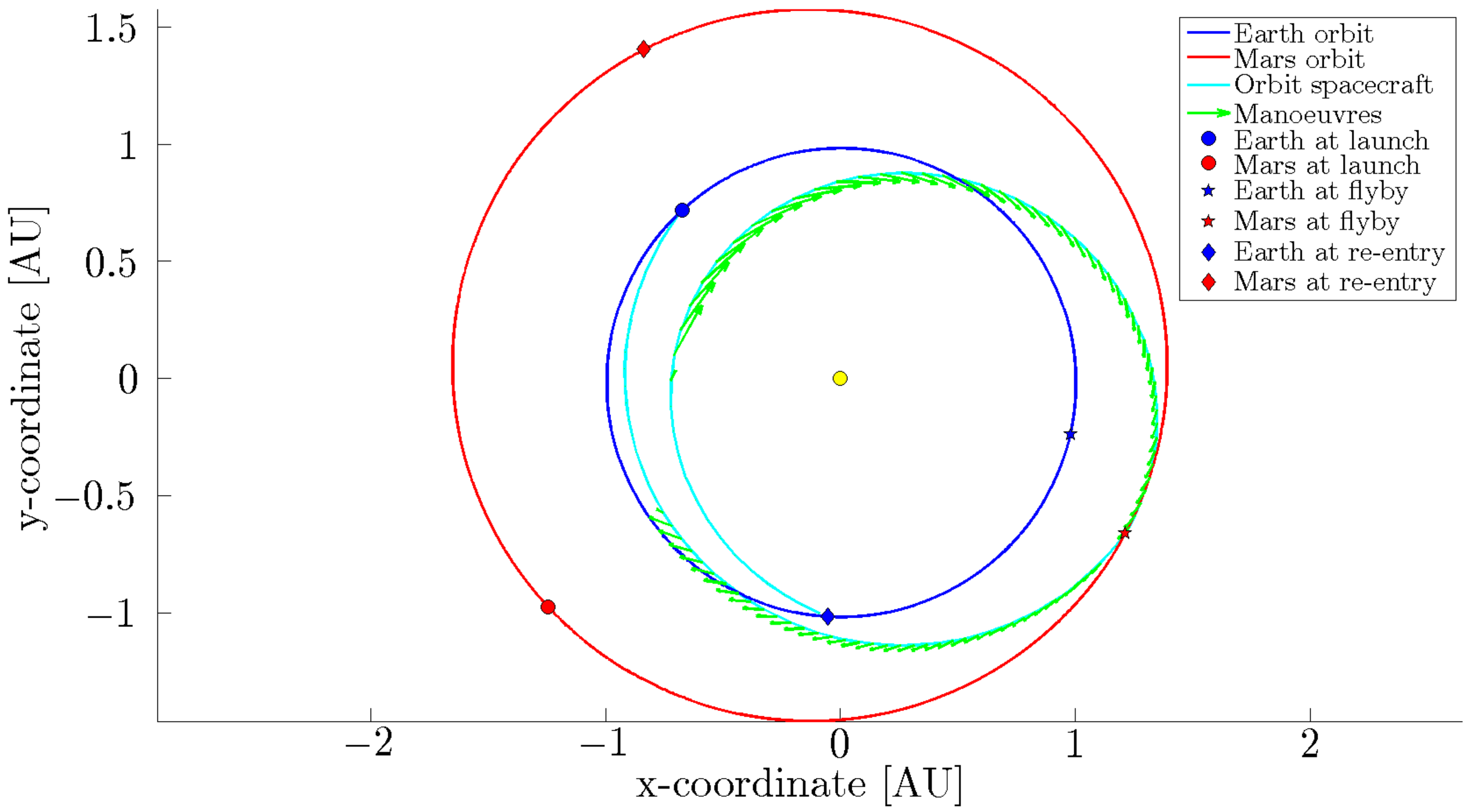

# Personal background



- Spent about 3 years working with SNOPT for a variety of problems & problem formulations
  - And about a month with IPOPT
- Racked up a long list of do's and don'ts over that time
- Hoping to save you all some time by discussing them today
  - Very implementation focused
  - Plenty of published text on the theory behind these tools, easily found through google (or contact me)

# Optimization tools

# Common elements

- At a high level, most optimization libraries are very similar
  - Enormous differences can exist in underlying theory / preferred interface / available options / actual performance
  - Besides those, there are some common features that, if not properly understood, can dominate the effects above
  - The interfaces are ultimately also very similar

- Before we get to some specific differences, its helpful to go over some of these potentially dominant features

# General description

- Parameters (1 x n) $x^L \leq x \leq x^U$
- Cost function (1 x 1) $f(x)$
- Constraints (m x 1) $g^L \leq g \leq g^U$
- Cost derivatives (1 x n) $\nabla f(x) = \frac{\delta f}{dx}$
- Jacobian (m x n) $\nabla g(x) = \frac{\delta g}{dx}$

# General description

- For the purposes of this talk, it is okay to assume that the provided information (constraints, objective, derivatives) are used in a process similar to how a Newton solver is iteratively used to find a root
  - This is obviously an enormous simplification, but there are actually great similarities in these processes

$$X_1 = X_0 - \frac{f(X_0)}{f'(X_0)}$$

- Key take-away:
  - The solver will iteratively improve the optimization state, all the time computing the current value of the objective/constraints, and their derivatives, to determine the state for the next iteration

# General description

- Some solvers also allow the use of second order derivatives
  - Can be a useful feature, but can also be very memory intensive
  - It doesn't really change the aspects discussed today, so will largely be ignored here

- The formulation so far is very generic
  - As expected, smooth functions for the constraints/objective lead to better convergence in this approach
  - Adjoint equations need not be in user formulation
  - Since they are continuous functions, they could be though (and some trajectory optimization tools do this)
  - Typically though, these tools are used for direct optimization

# Toy problem

- To give some concrete examples, we'll work with a toy problem
  - 3 parameters, 2 constraints
  - Designed for instructive/demonstrative value
  - It (probably?) doesn't solve anything useful
  - It should still have a name…

# Toy problem

- To give some concrete examples, we'll work with a toy problem
  - 3 parameters, 2 constraints
  - Designed for instructive/demonstrative value
  - It (probably?) doesn't solve anything useful
  - It should still have a name…

- We will dub this "The POPSICLE-problem"

**The <u>P</u>ointless <u>O</u>ptimization <u>P</u>roblem**
**<u>S</u>o <u>I</u> <u>C</u>an <u>L</u>earn <u>E</u>very-problem**

# POPSICLE-problem formulation

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- Parameters (1 x 3) $$\begin{bmatrix}0\\0\\0\end{bmatrix}^T \le \begin{bmatrix}x_0\\x_1\\x_2\end{bmatrix}^T \le \begin{bmatrix}5\\5\\5\end{bmatrix}^T$$
- Cost function (1 x 1) $$f(x) = x_0^2 + x_1^3 + x_2^2$$
- Constraints (2 x 1) $$\begin{bmatrix}0\\30\end{bmatrix} \le \begin{bmatrix}g_1 = x_0^3 + x_2\\g_2 = x_1^2 + x_2^3\end{bmatrix} \le \begin{bmatrix}2.5\\100\end{bmatrix}$$
- Cost derivatives (1 x 3) $$\nabla f(x) = \begin{bmatrix}2x_0 & 3x_1^2 & 2x_2\end{bmatrix}$$
- Jacobian (2 x 3) $$\nabla g(x) = \begin{bmatrix}3x_0^2 & 0 & 1\\0 & 2x_1 & 3x_2^2\end{bmatrix}$$

# Cost function

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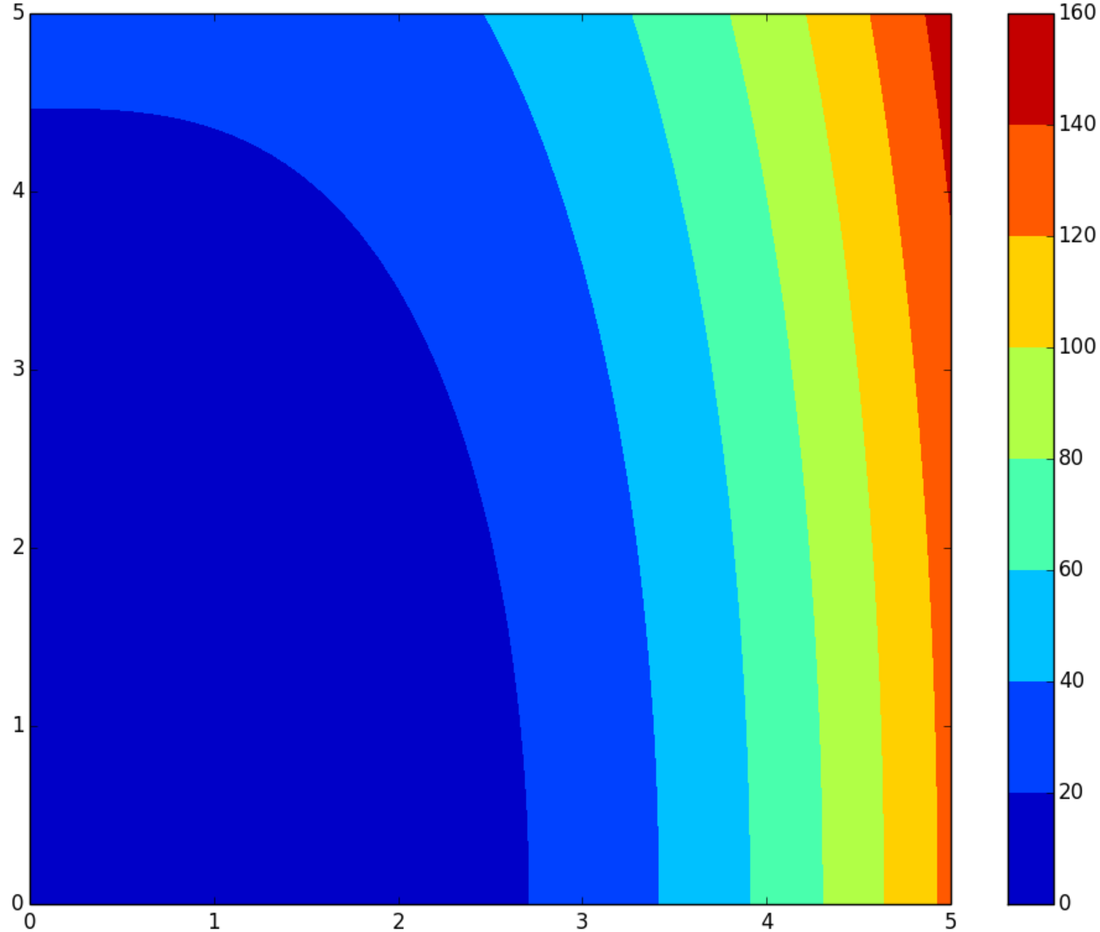
$$x_0 = 0$$
$$f(x) = x_0^2 + x_1^3 + x_2^2$$

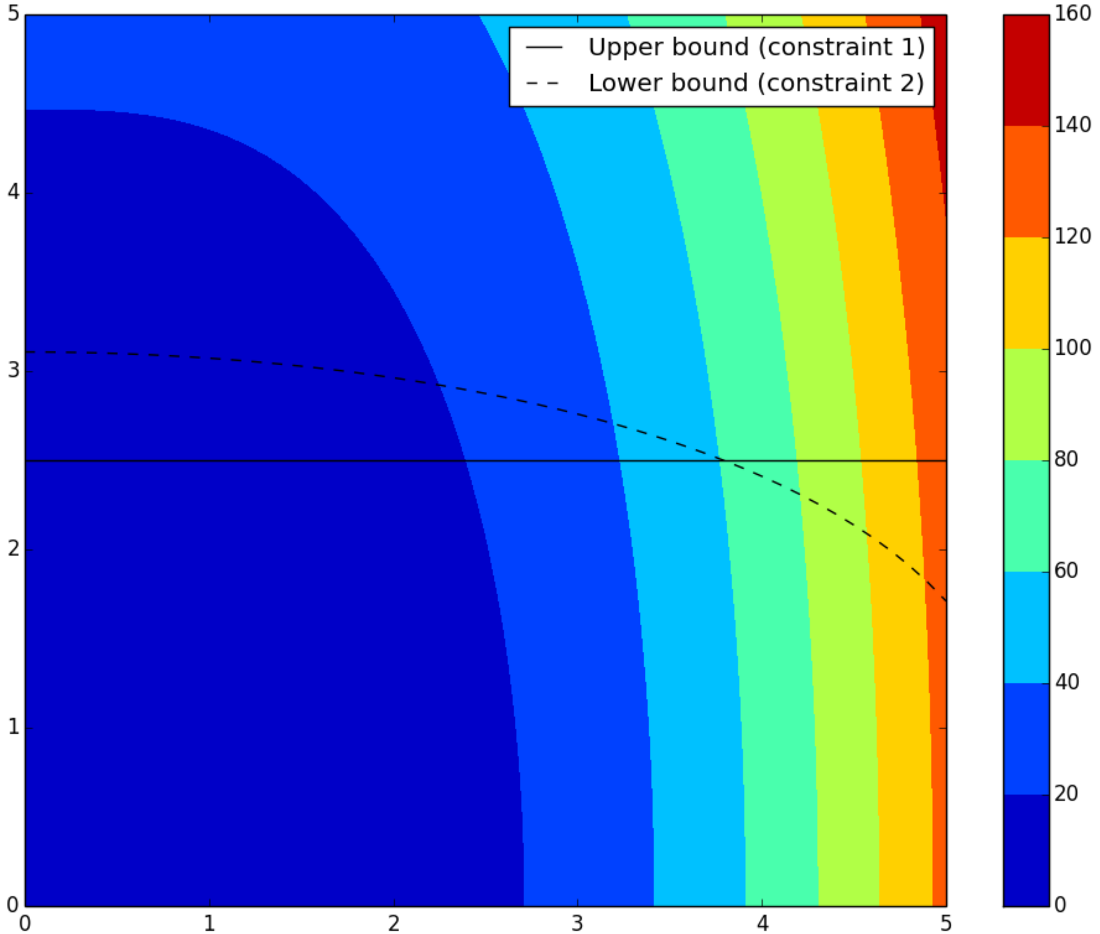


# Cost function & Constraints

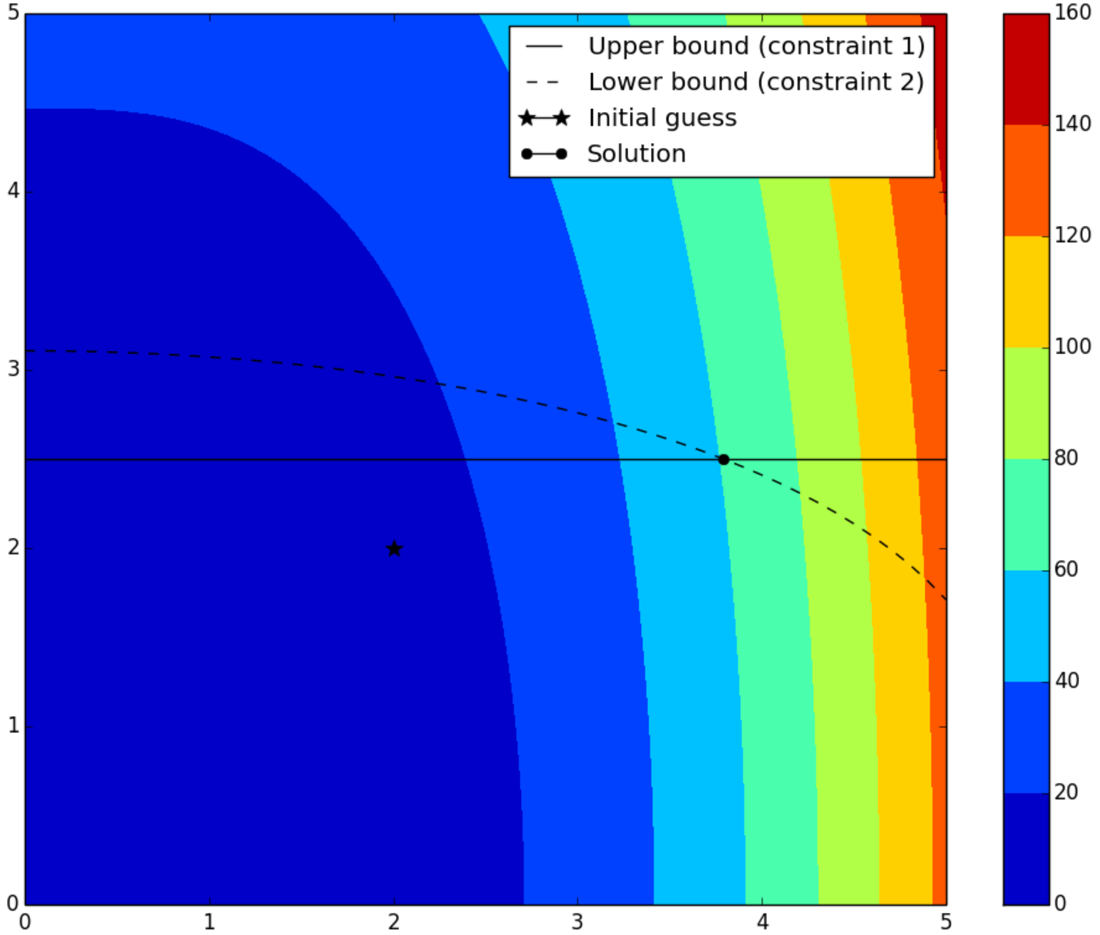
$$x_0 = 0$$

$$\begin{bmatrix} g_1 = x_0^3 + x_2 \\ g_2 = x_1^2 + x_2^3 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 30 \end{bmatrix} \leq \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \leq \begin{bmatrix} 2.5 \\ 100 \end{bmatrix}$$



# Optimal solution



# The importance of scaling!

# POPSICLE-problem formulation

- Parameters (1 x 3)

$$\begin{bmatrix}0\\0\\0\end{bmatrix}^T \le \begin{bmatrix}x_0\\x_1\\x_2\end{bmatrix}^T \le \begin{bmatrix}5\\5\\5\end{bmatrix}^T$$

- Cost function (1 x 1)

$$f(x) = x_0^2 + x_1^3 + x_2^2$$

- Constraints (2 x 1)

$$\begin{bmatrix}0\\30\end{bmatrix} \le \begin{bmatrix}g_1 = x_0^3 + x_2\\g_2 = x_1^2 + x_2^3\end{bmatrix} \le \begin{bmatrix}2.5\\100\end{bmatrix}$$

- Cost derivatives (1 x 3)

$$\nabla f(x) = \begin{bmatrix}2x_0 & 3x_1^2 & 2x_2\end{bmatrix}$$

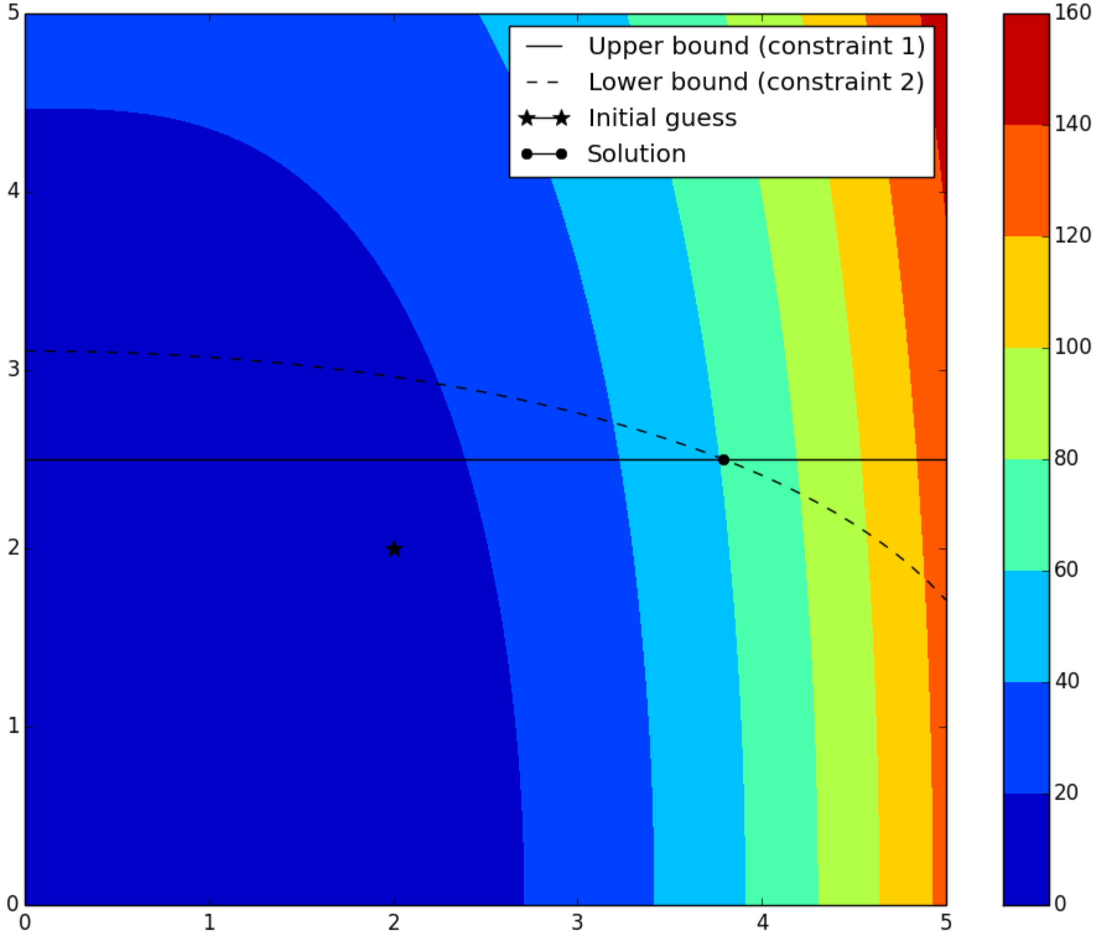
- Jacobian (2 x 3)

$$\nabla g(x) = \begin{bmatrix}3x_0^2 & 0 & 1\\0 & 2x_1 & 3x_2^2\end{bmatrix}$$

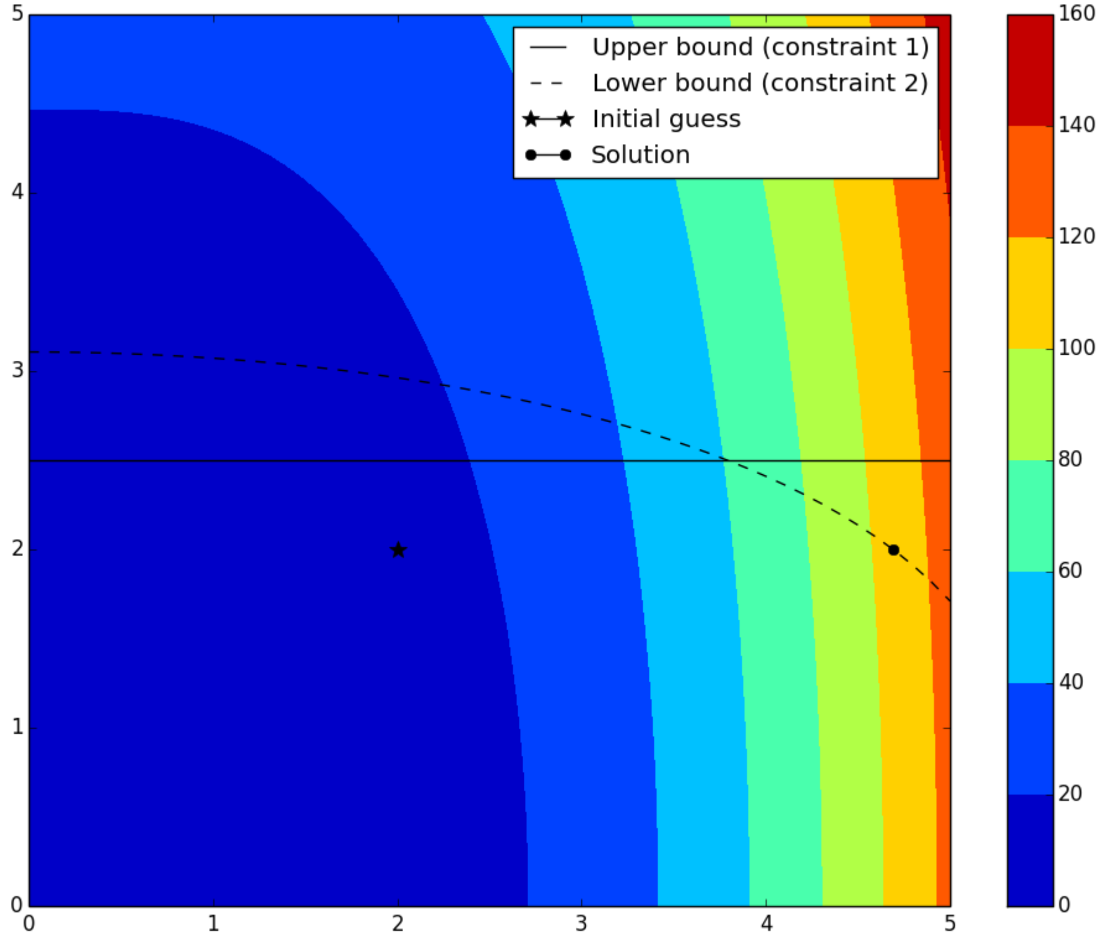
# Scaling example

- Parameters (1 x 3) $$\begin{bmatrix}0\\0\\0\end{bmatrix}^T \leq \begin{bmatrix}x_0\\x_1\\x_2\end{bmatrix}^T \leq \begin{bmatrix}5\\5\cdot 10^7\\5\end{bmatrix}^T$$
- Cost function (1 x 1) $$f(x) = x_0^2 + (x_1 \cdot 10^{-7})^3 + x_2^2$$
- Constraints (2 x 1) $$\begin{bmatrix}0\\30\end{bmatrix} \leq \begin{bmatrix}g_1 = x_0^3 + x_2\\g_2 = (x_1 \cdot 10^{-7})^2 + x_2^3\end{bmatrix} \leq \begin{bmatrix}2.5\\100\end{bmatrix}$$
- Cost derivatives (1 x 3) $$\nabla f(x) = \begin{bmatrix}2x_0 & 3{x_1}^2 \cdot (10^{-7})^3 & 2x_2\end{bmatrix}$$
- Jacobian (2 x 3) $$\nabla g(x) = \begin{bmatrix}3x_0^2 & 0 & 1\\0 & 2x_1 \cdot (10^{-7})^2 & 3x_2^2\end{bmatrix}$$

# Optimal solution



# Poorly scaled solution



# Scaling

- Clearly, this arbitrary scale factor of $10^{-7}$ is absurd, right?
  - Sure, except… $\frac{V_{Earth}}{AU} \approx \frac{30\ km/s}{150 \cdot 10^6\ km} = 2 \cdot 10^{-7}$
- **Scaling. Is. Everything.** It can (or rather; It will)…
  - Make a feasible solution appear infeasible
  - Make an infeasible solution appear feasible
  - Change runtimes by orders of magnitude
  - Determine whether or not the problem converges at all
  - Give you enormous headaches
- Scale appropriately!
  - **Everything** should be in the [0,1] domain (or [-0.5,0.5])
  - That means parameters, constraints, and objective function
  - You can deviate to weight certain aspects, **but tread carefully!**

# Sparsity

- Most optimization problems have many derivatives that are always zero → Sparsity

- An important way to reduce memory usage
  - Which can also lead to improved runtimes

- Easy to show implementation with the POPSICLE-problem
  - Harder to show importance

- Will follow the sparse POPSICLE-problem with a real example

# Sparsity: IPOPT format

- Actual Jacobian

$$\nabla g(x) = \begin{bmatrix} 3x_0^2 & 0 & 1 \\ 0 & 2x_1 & 3x_2^2 \end{bmatrix}$$

# Sparsity: IPOPT format

- Actual Jacobian

$$\nabla g(x) = \begin{bmatrix} 3x_0^2 & 0 & 1 \\ 0 & 2x_1 & 3x_2^2 \end{bmatrix}$$

- Input of dense (full) Jacobian

$$\begin{aligned} \nabla g(x) &= [3x_0^2 \quad 0 \quad 1 \quad 0 \quad 2x_1 \quad 3x_2^2] \\ iRow &= [0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1] \\ jCol &= [0 \quad 1 \quad 2 \quad 0 \quad 1 \quad 2] \end{aligned}$$

# Sparsity: IPOPT format

- Actual Jacobian

$$\nabla g(x) = \begin{bmatrix} 3x_0^2 & 0 & 1 \\ 0 & 2x_1 & 3x_2^2 \end{bmatrix}$$

- Input of dense (full) Jacobian

$$\nabla g(x) = [3x_0^2 \quad 0 \quad 1 \quad 0 \quad 2x_1 \quad 3x_2^2]$$
$$iRow = [0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1]$$
$$jCol = [0 \quad 1 \quad 2 \quad 0 \quad 1 \quad 2]$$

- Input of sparse Jacobian

$$\nabla g(x) = [3x_0^2 \quad 1 \quad 2x_1 \quad 3x_2^2]$$
$$iRow = [0 \quad 0 \quad 1 \quad 1]$$
$$jCol = [0 \quad 2 \quad 1 \quad 2]$$

# Sparsity

- Example from my own research
- Low-thrust trajectory split into segments
  - Each segment contains a number of parameters & constraints
  - Sparsity grows (almost) linearly with problem size

| Segments | 100 | 1000 | 10 000 |
|---|---|---|---|
| Non-zero Jacobian elements (%) | 1.8 | 0.18 | 0.018 |
| Full Jacobian (megabyte) | 7.1 | 700 | 70 000 |
| Sparse Jacobian (megabyte) | 0.1 | 1.3 | 12.6 |

- Sparsity matters!

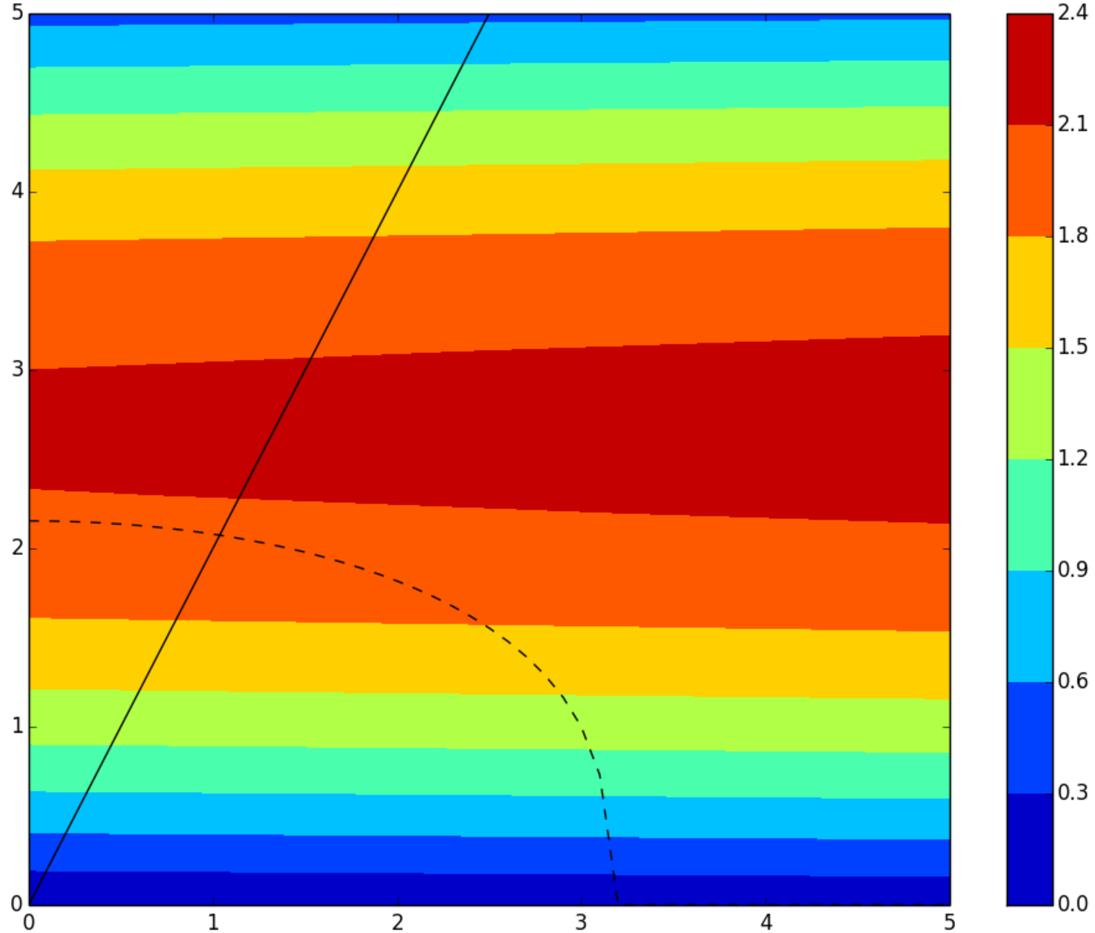
# Local optima

# POPSICLE-2

- Parameters (1 x 3) $$\begin{bmatrix}0\\0\\0\end{bmatrix}^T \leq \begin{bmatrix}x_0\\x_1\\x_2\end{bmatrix}^T \leq \begin{bmatrix}5\\5\\5\end{bmatrix}^T$$
- Cost function (1 x 1) $$f(x) = x_0 + 0.01x_1 + x_2(1.6 - 0.3x_2)$$
- Constraints (2 x 1) $$\begin{bmatrix}-10\\10\end{bmatrix} \leq \begin{bmatrix}g_1 = -2x_1 + x_2\\ g_2 = x_1^2 + x_2^3\end{bmatrix} \leq \begin{bmatrix}0\\100\end{bmatrix}$$
- Cost derivatives (1 x 3) $$\nabla f(x) = \begin{bmatrix}1 & 0.01 & 1.6 - 0.6x_2\end{bmatrix}$$
- Jacobian (2 x 3) $$\nabla g(x) = \begin{bmatrix}0 & -2 & 1\\ 0 & 2x_1 & 3x_2^2\end{bmatrix}$$

# Problem space

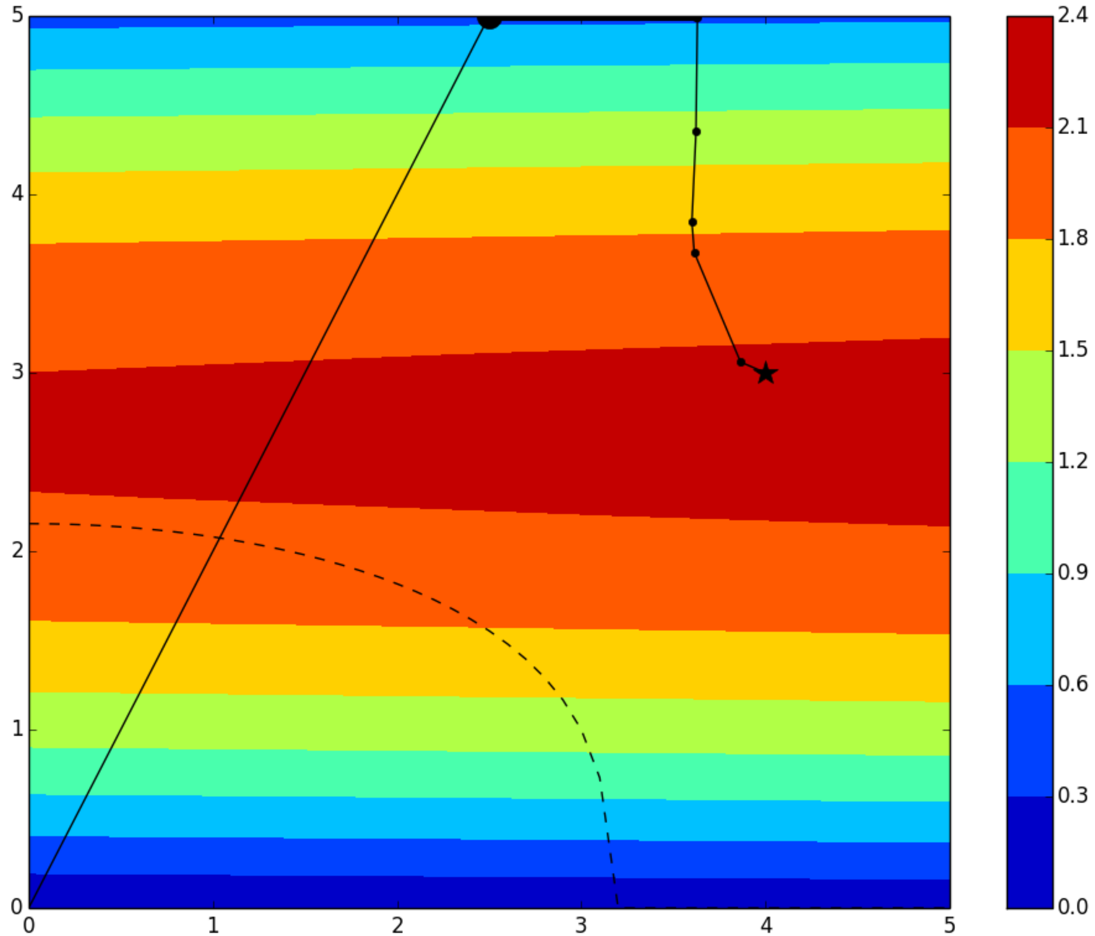
$$f(x) = x_0 + 0.01x_1 + x_2(1.6 - 0.3x_2)$$



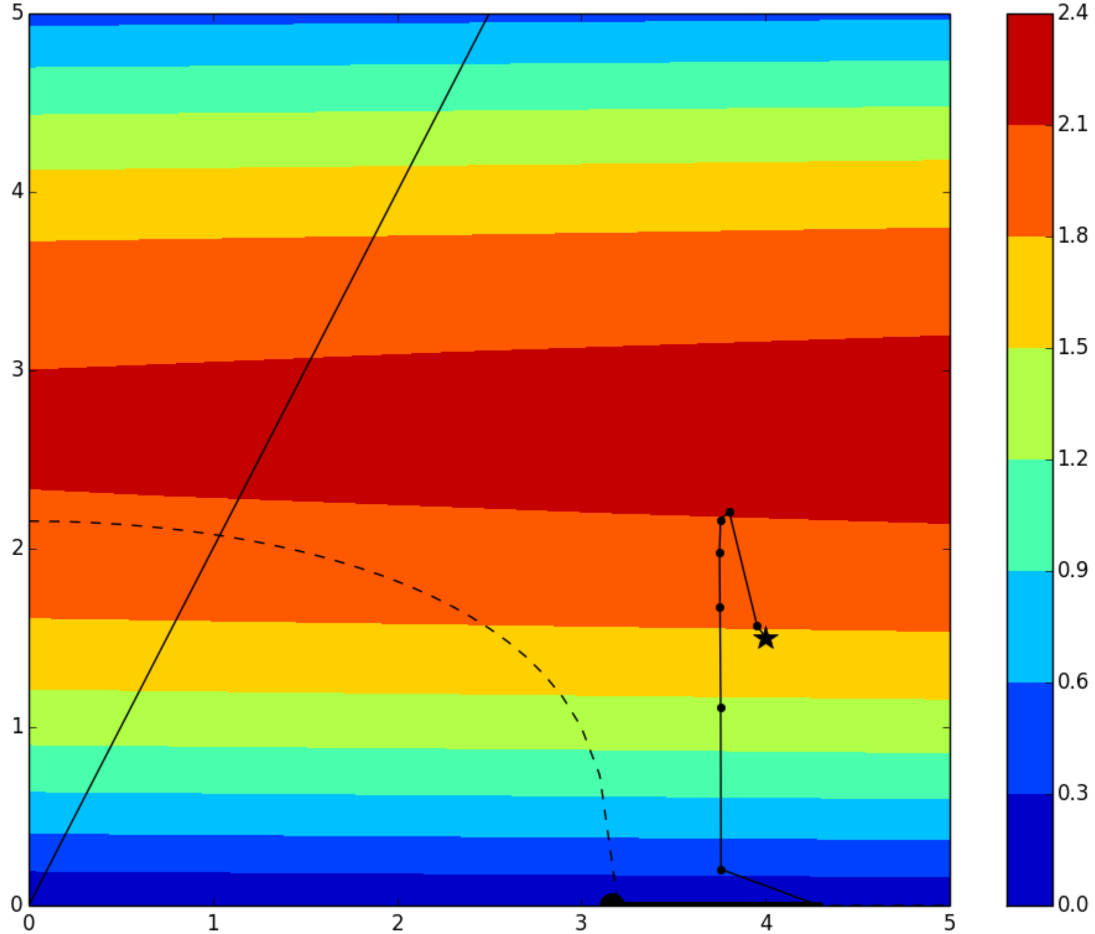

$$\begin{bmatrix} g_1 = -2x_1 + x_2 \\ g_2 = x_1^2 + x_2^3 \end{bmatrix}$$

$$\begin{bmatrix} -10 \\ 10 \end{bmatrix} \leq \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 100 \end{bmatrix}$$

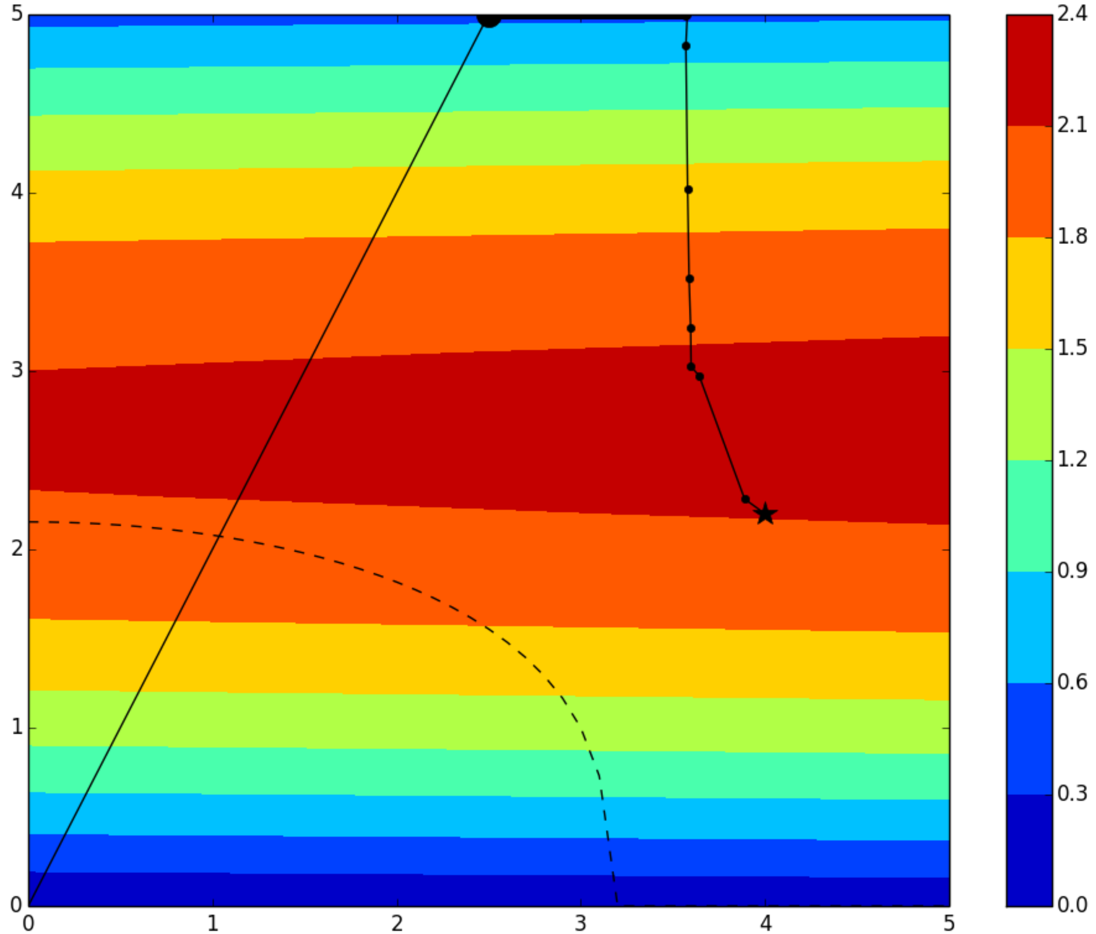
# Local optimum



# Optimal solution



# Crossing to local optimum



# Code examples

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(Switch to code)

# Optimizer overview

- SNOPT
- IPOPT
- MATLAB (fmincon, …)
- Many others exist
  - GPU-based
  - …

# SNOPT

- SNOPT (Sparse Nonlinear OPTimizer)
  - Written in Fortran 77 (Fortran 2003 version in development)
  - Interfaces exist for C, C++, Fortran, MATLAB, Python, …
  - CU has an academic license to a fairly recent version
  - Sparsity notation a little more cumbersome than IPOPT
  - No parallel solving, and only first order derivatives
  - Performance generally excellent
  - Has become somewhat of an industry standard in optimization
  - Assumes/prefers you install on Linux…
  - Website: http://ccom.ucsd.edu/~optimizers/

# IPOPT

- IPOPT (Interior Point OPTimizer)
  - Written in C++
  - Interfaces exist for C, C++, Fortran, MATLAB, Python, …
  - Completely functional as open source distribution
  - Additional capability available through (academic) licenses
  - Very modular in design, essentially a wrapper for several solvers
  - Support for parallel solving & second order derivatives
  - Sparsity notation extremely straightforward
  - Seems to perform somewhat-to-significantly better than SNOPT
  - Very widely used (and seems to be growing)
  - EXCELLENT documentation & community support
  - Assumes/prefers you install on Linux…
  - Website: http://www.coin-or.org/Ipopt/

# fmincon



- fmincon (MATLAB)
  - Easy to get access to (it's in MATLAB…done)
  - MATLAB probably also has other solvers
  - No personal experience
  - Similarly in its setup/functionality to SNOPT/IPOPT
  - It does support sparse matrices
  - Might be perfect for a project, probably not for research
  - Website: http://www.mathworks.com/help/optim/ug/fmincon.html

# Finite differencing

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- Instead of the user defining the Jacobian, some optimizers offer the functionality of finite differencing the objective & constraint functions
  - Generally worthless, since it comes at tremendous computational cost and is less accurate than analytical derivatives
  - It does allow for very quick implementations of problem formulations, so there is some value during prototyping

- Similarly, most optimizers can check the user-provided derivatives through finite differencing
  - This is **invaluable**, since it allows for (comparatively) easy identification of scaling errors, indexing errors, and 100 other errors you didn't know were even possible…

# The joys of compilation

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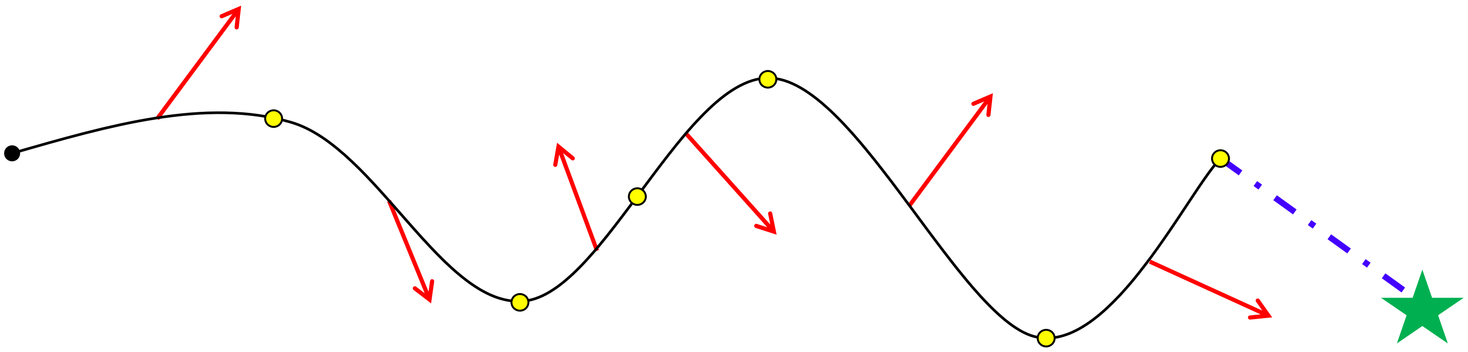
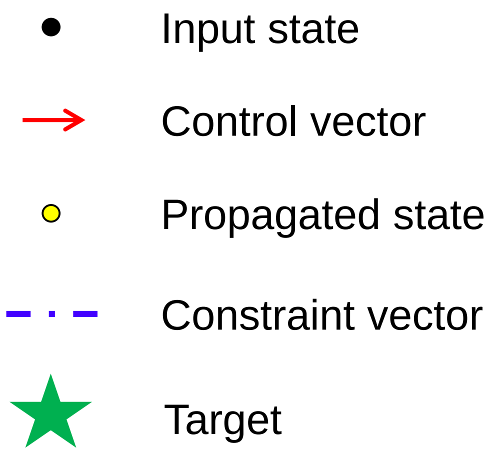
- It can be a chore to get SNOPT/IPOPT to compile
  - Especially if you are not extremely familiar with Fortran/C++
- Be prepared: don't plan to do this when you are highly pressed for time
- Doing this in Linux will typically save a lot of work
- I'd be happy to assist, but my abilities to do so are entirely experience based and include large gaps of knowledge
  - Like you, I'm not a computer scientist, and just want to get it running
- Just for this reason, it might be preferable to work with fmincon for this class

# Research example

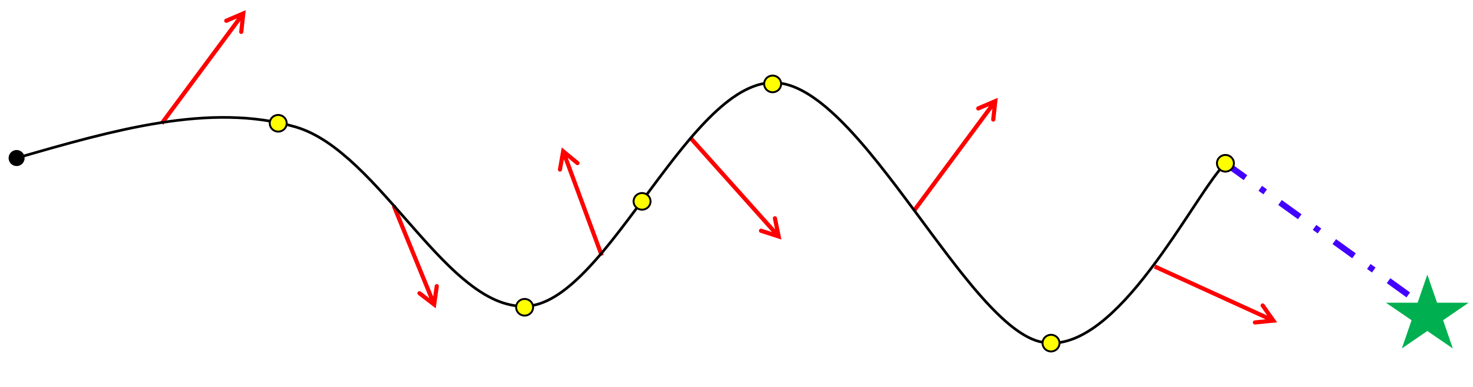
- Low-thrust trajectory optimization
- Direct transcription through Gauss-Lobatto collocation
- Some appealing aspects…
  - Very direct control over problem
  - Extremely sparse Jacobian
  - Embarrassingly parallel formulation
- Has proven to be very effective with large and/or complex problems

# Single shooting





# Single shooting



Input state

Control vector

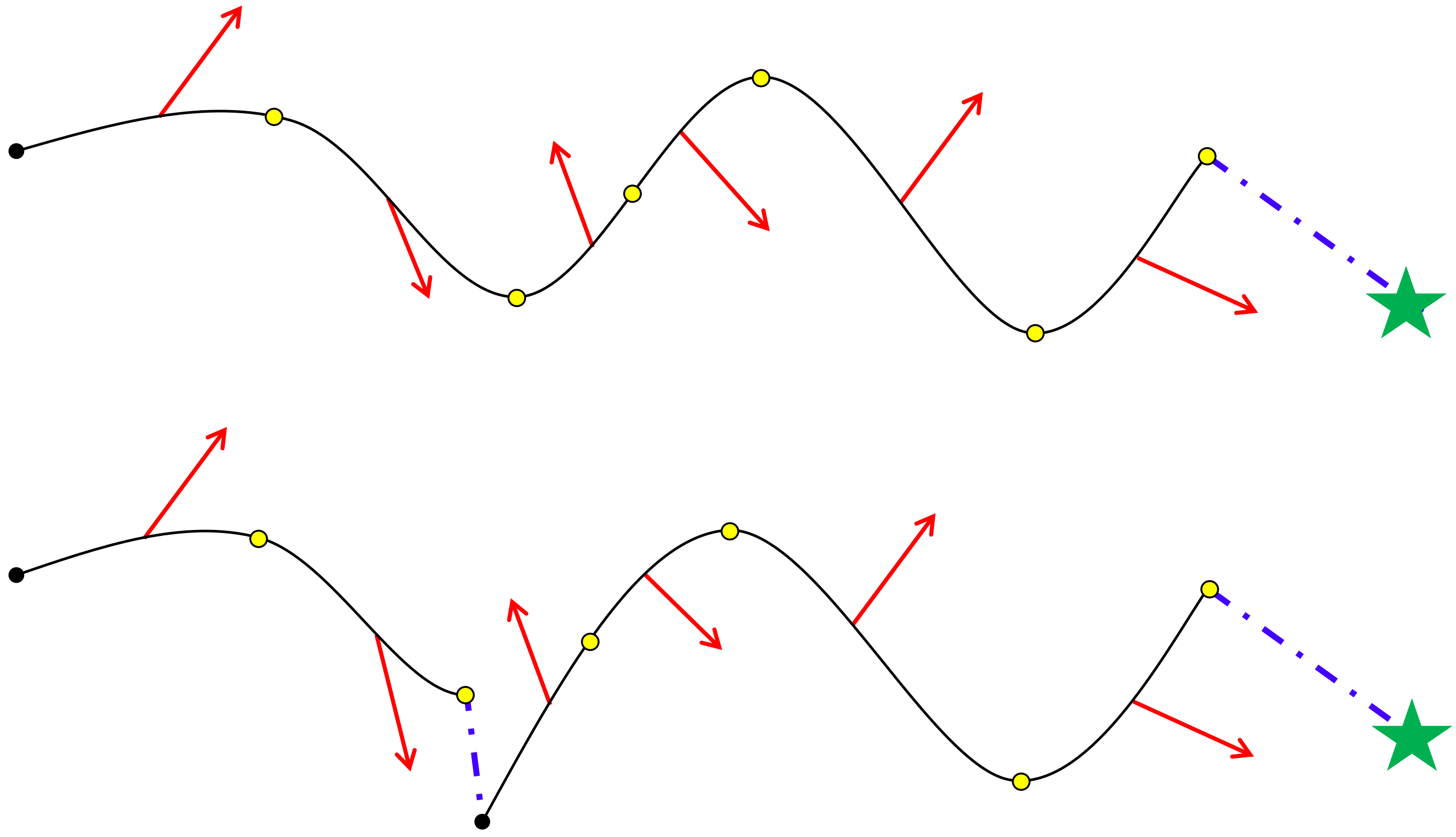
→ **<u>This is all you directly control</u>**

Propagated state

Constraint vector

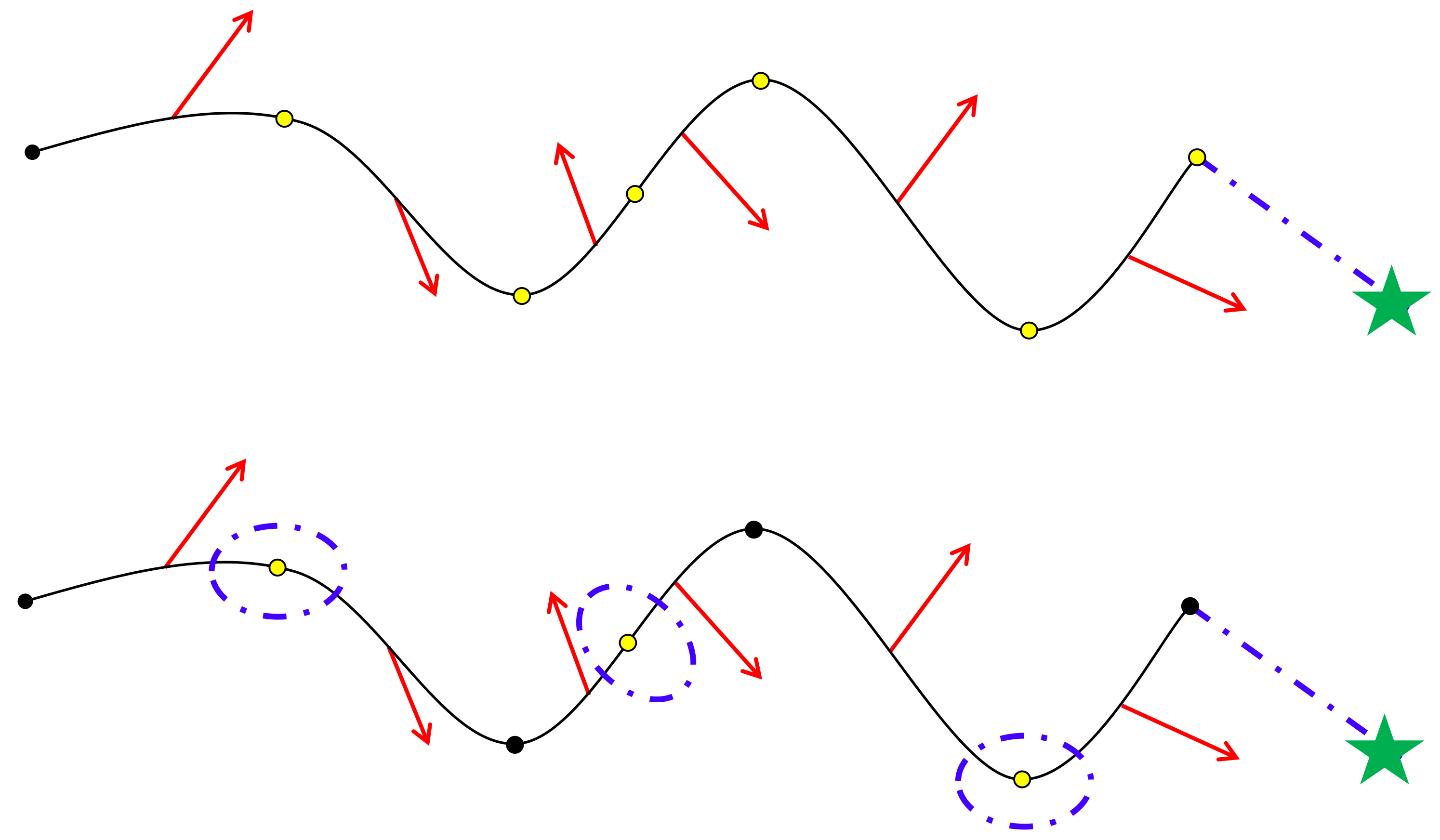
Target

# Multiple shooting



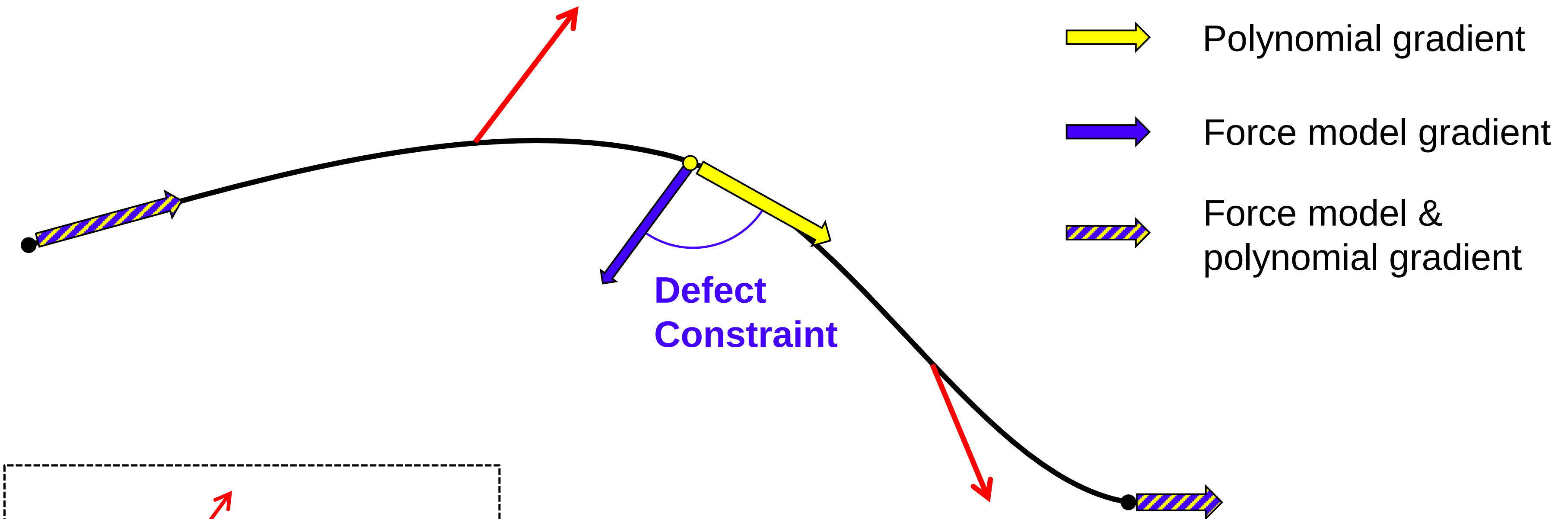
● Input state → Control vector ○ Propagated state — · — Constraint vector ★ Target

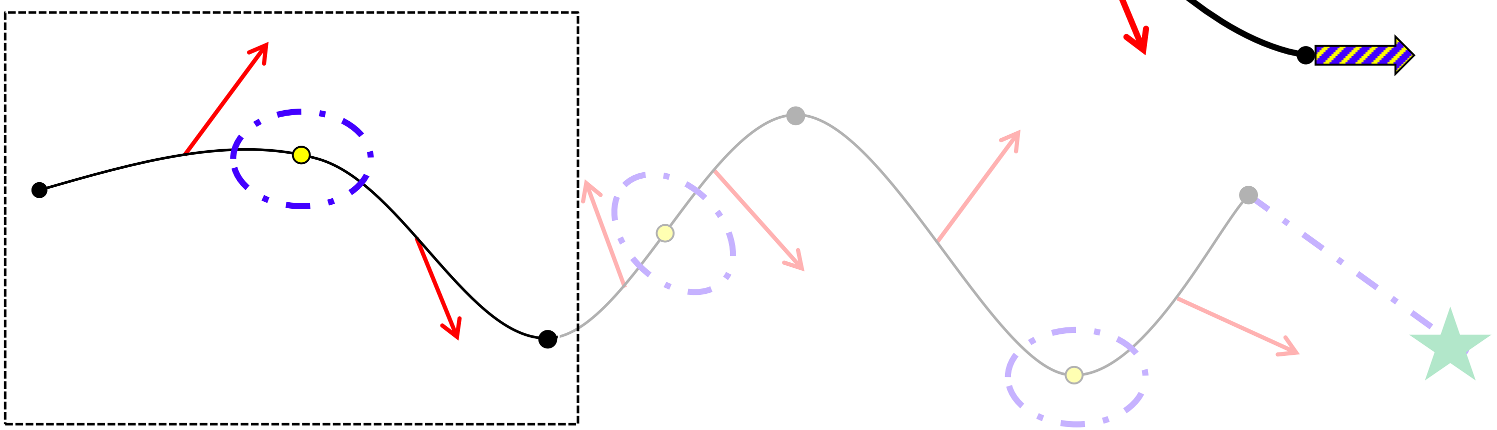
# Collocation



● Input state → Control vector ○ Propagated state — · — Constraint vector ★ Target

# Defect constraints



Input state
Control vector
Propagated state
Constraint vector
Target

# Example problem

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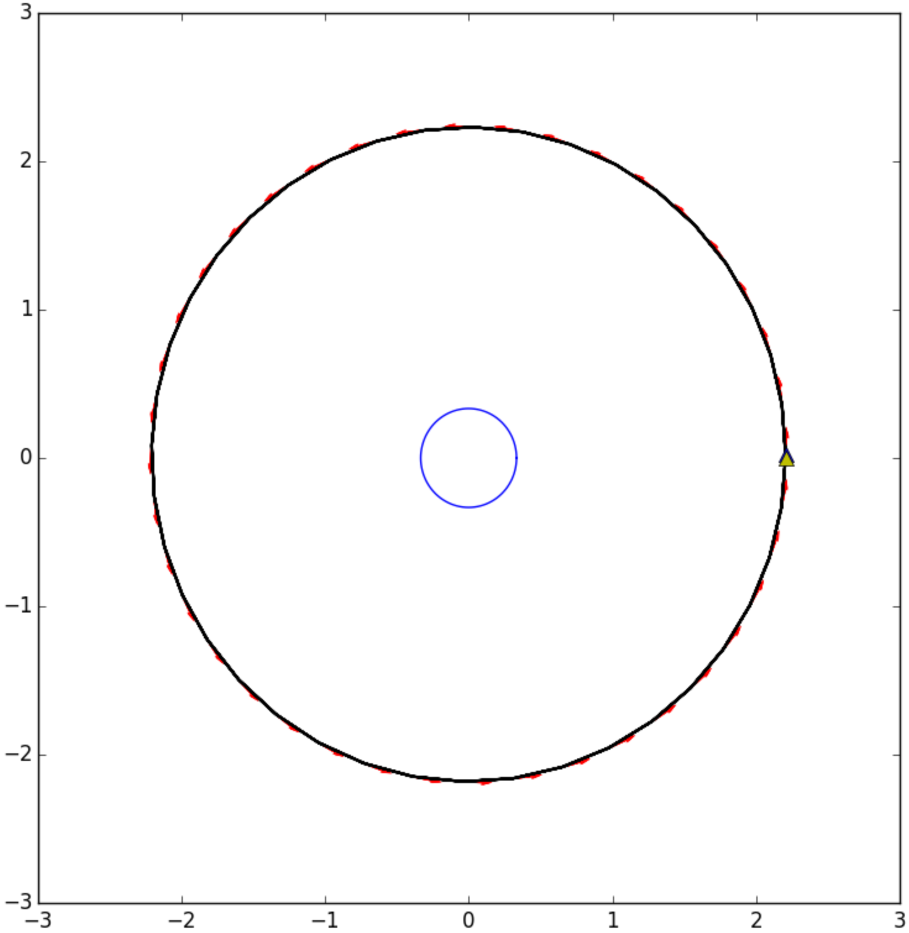
- Outward spiral from GEO
  - ~5 days flight time
  - Two-body motion
  - 3000 kg initial mass
  - 50 kW constant power (60% jet efficiency)
  - 2000 s specific impulse

- Objective: maximize orbital energy

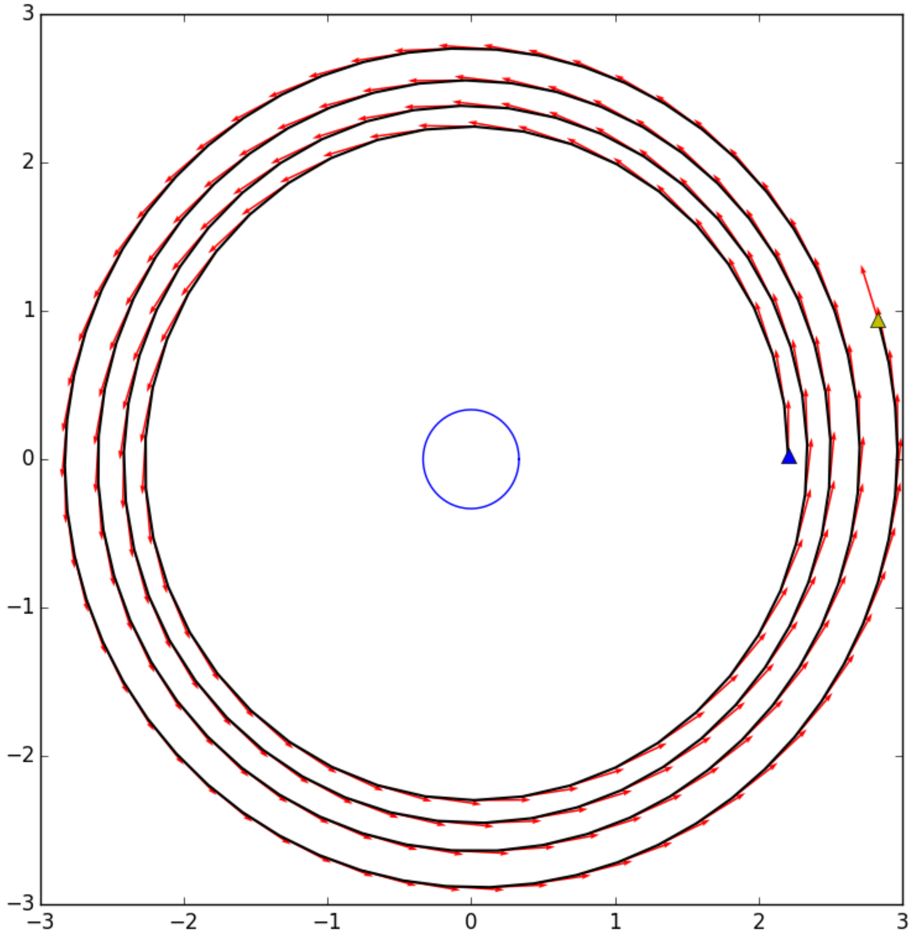
# Example problem



## Initial guess




## Solution



# Convergence behavior



(Video)

# Project suggestion

# Project suggestion

- It is actually quite manageable to write a simple but reasonably powerful low-thrust tool
  - The final product can be about 100-150 lines of code, plus some 50-100 lines to initialize the problem
  - Took me about an hour to write it from scratch (where I count several years of experience with these tools as "scratch")

- What do you need?
  - A ready-to-go optimizer that can compute derivatives through finite differencing (fmincon?)
  - A propagator for two-body motion (MATLAB's rk4?)
  - Perhaps start by building the POPSICLE-problem to get the interface down, after which all you need to do is change the contents of the function

# Project suggestion

- Suggested problem
  - 180 degree Earth-Mars transfer
  - 10 trajectory segments, single shooter
  - 2000 kg initial mass
  - 30 kW of constant power
  - Fixed specific impulse (but can be optimized with the trajectory)
  - Flight time of 250 days

- Variables
  - Hardcode the initial & final state, including the initial mass
  - That leaves 1 variable for power, 1 variable for the specific impulse, and three thrust components for 10 segments: 32 variables

- Constraints
  - 3 for the final position, 3 for the final velocity, and 10 thrust constraints for a total of 16 constraints

# Project suggestion

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- Propagation
  - Divide total flight time (250 days) by 10
  - Apply the 3 thrust components of the current segment as an impulsive Delta-V to the initial state (just set all of these to 10% of max thrust for the initial guess)
  - Store the thrust magnitude for the correct thrust constraint
  - Propagate for the 25 days of the current segment
  - Repeat, with as initial state the final state of the last 25 day propagation
  - At the end of the 10 segments, compute the error with respect to the target state (Mars) and store these as the appropriate constraint values

- Cost function
  - Take the mass at the end of the propagation, either maximize its positive value or minimize its negative value (most tools minimize by default)

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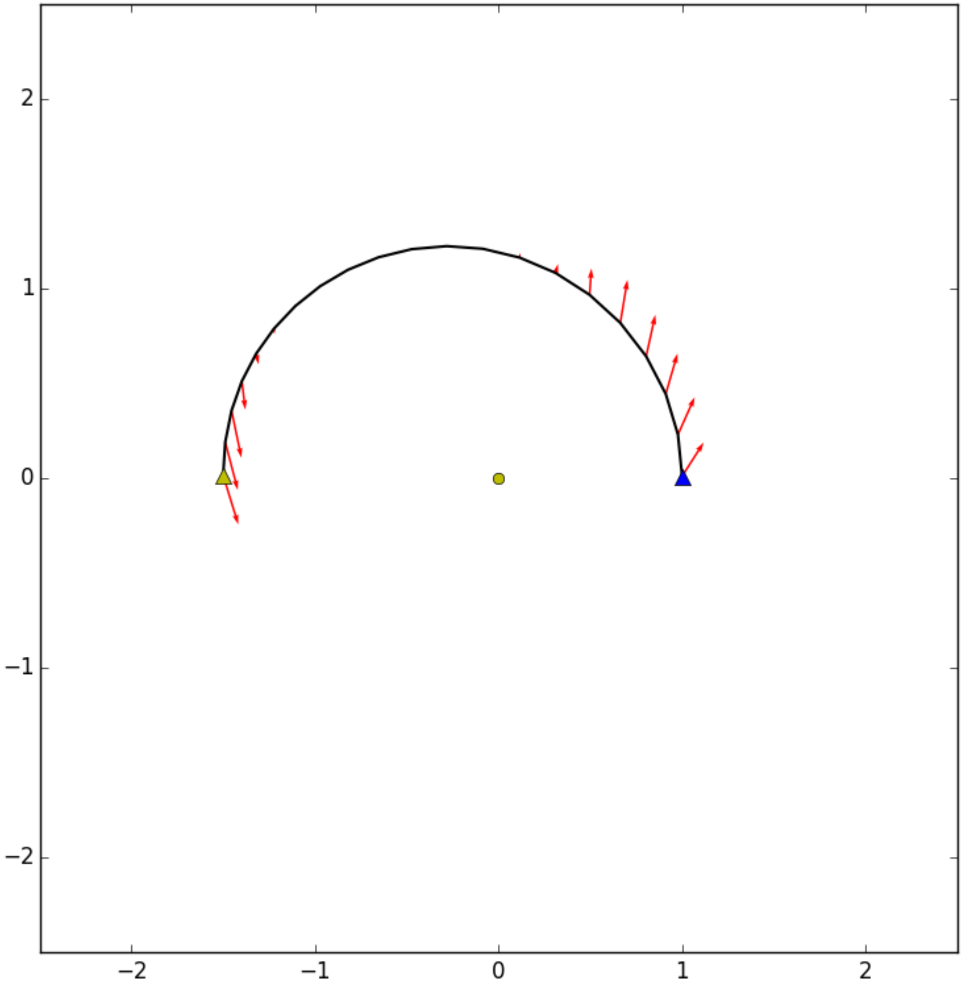
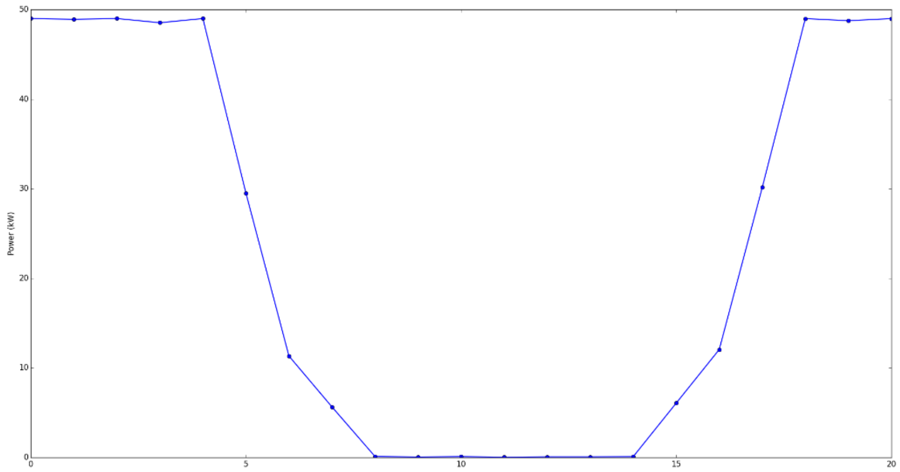
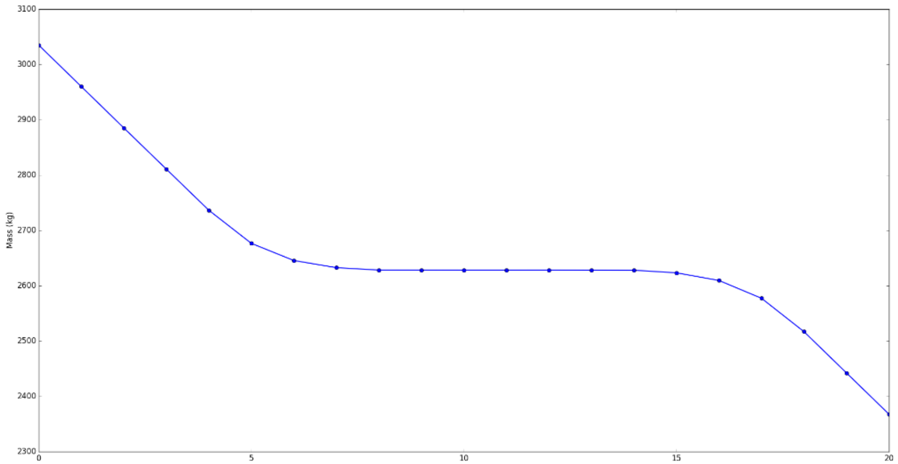
# Resulting transfer



# Project suggestion



- Perhaps a bit much to take in right now, but a very manageable exercise

- What you'll get out of it…
  - Familiarity with these types of tools
  - Insight into the effects of scaling
  - Optimizer will independently introduce 2 thrust arcs and 1 coast arc
  - Optimizer will independently optimize the value of the specific impulse and/or spacecraft power if you open up those variables
  - Once you've done this once in such a controlled formulation, you'll be very comfortable using these tools for much more advanced problems (you could actually evolve this simple example into a very powerful low-thrust optimization tool…)

# Conclusion

- High-level overview of the key aspects & features of optimization packages
- Hopefully, this is a useful starting point for building your own IPOPT/SNOPT/fmincon/... implementations
  - Try out the POPSICLE-problem!
  - Slides & supporting material will be online
- Contact me if you need a hand with these packages and/or the problem suggestion!
  - jon.herman@colorado.edu