ASEN 6060
ADVANCED ASTRODYNAMICS
Exploring the CR3BP

Objectives:

- Discuss properties of dynamical system
- Define fundamental solutions
- Identify important symmetries in the CR3BP
- Derive Jacobi constant
- Define zero velocity surfaces and identify useful trajectory design heuristics

Insights into the CR3BP

Recall the equations of motion:

$$\ddot{x} - 2\dot{y} = \frac{\partial U^*}{\partial x} \qquad \ddot{y} + 2\dot{x} = \frac{\partial U^*}{\partial y} \qquad \ddot{z} = \frac{\partial U^*}{\partial z}$$

where $U^* = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$

$$r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}$$

$$r_2 = \sqrt{(x-1+\mu)^2 + y^2 + z^2}$$

Insights into the CR3BP

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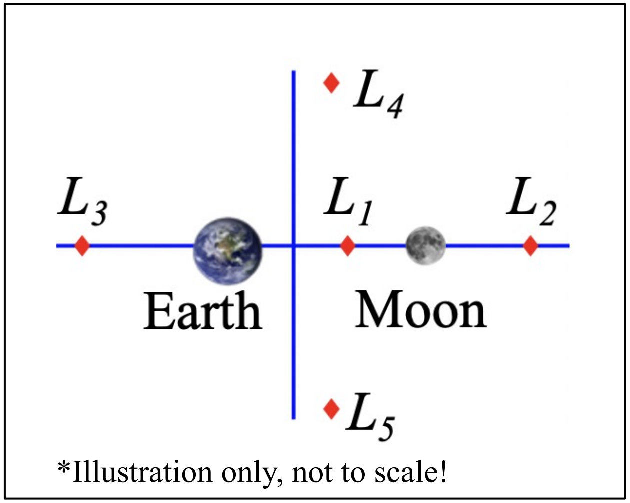
$$\ddot{x} - 2\dot{y} = \frac{\partial U^*}{\partial x} \qquad \ddot{y} + 2\dot{x} = \frac{\partial U^*}{\partial y} \qquad \ddot{z} = \frac{\partial U^*}{\partial z}$$

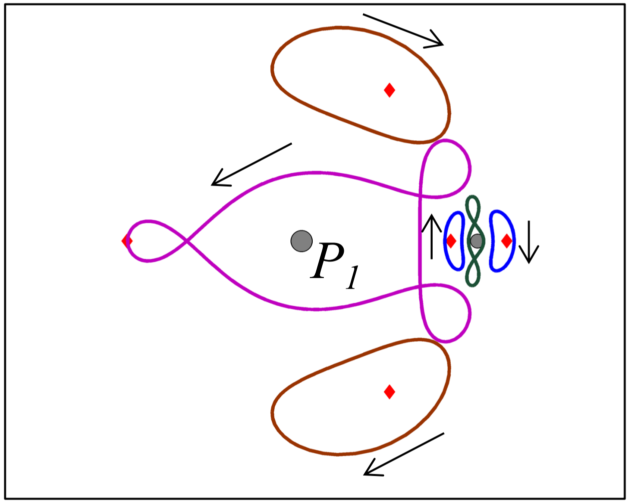
$$U^* = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

Properties of the resulting dynamical system:

Fundamental Solutions

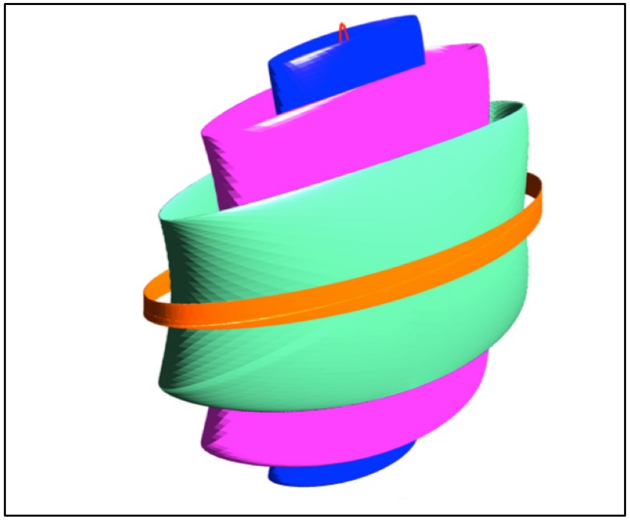
Equilibrium Points

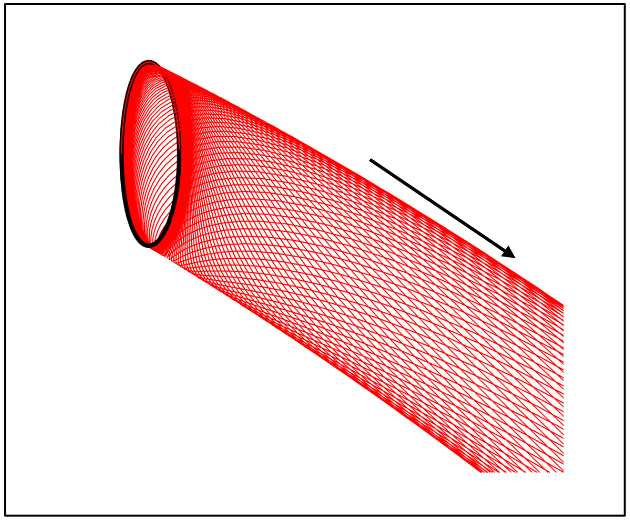




Periodic Orbit

Quasi-Periodic Trajectory tracing out the surface of a torus





Stable/ Unstable Manifolds

Symmetries

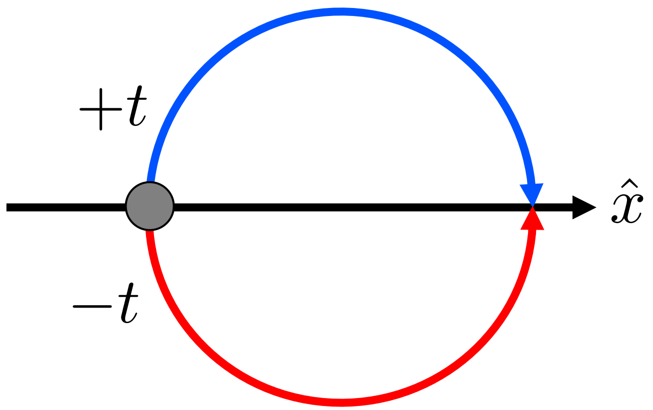
Symmetry #1: $(x, y, z, t) \rightarrow (x, -y, z, -t)$

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = 2\frac{dy}{dt} + x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3}$$

$$\frac{d}{d(-t)}\left(\frac{dx}{d(-t)}\right) = 2\frac{d(-y)}{d(-t)} + x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3}$$

$$\frac{d}{d(-t)}\left(\frac{d(-y)}{d(-t)}\right) = -2(-\dot{x}) + (-y) - \frac{(1-\mu)(-y)}{r_1^3} - \frac{\mu(-y)}{r_2^3}$$

$$\frac{d}{d(-t)}\left(\frac{d(z)}{d(-t)}\right) = -\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3}$$



Symmetries

Symmetry #2: About *xy*-plane

$$(x, y, z, t) \rightarrow (x, y, -z, t)$$

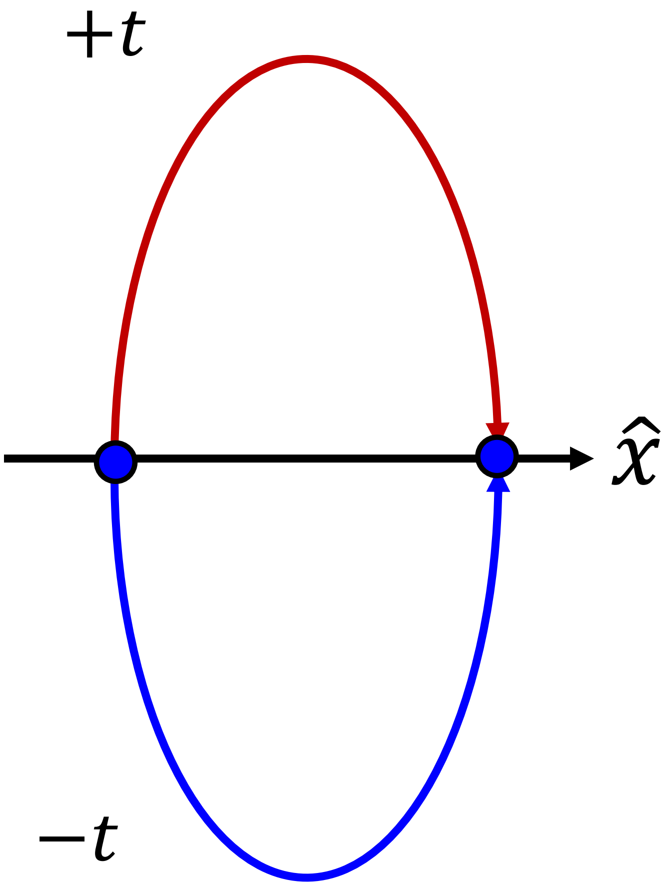
$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = 2\frac{dy}{dt} + x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3}$$

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) = -2\dot{x} + y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}$$

$$\frac{d}{dt}\left(\frac{d(-z)}{dt}\right) = -\frac{(1-\mu)(-z)}{r_1^3} - \frac{\mu(-z)}{r_2^3}$$

Mirror Theorem

- Mirror Theorem in the CR3BP:
  - If a mirror configuration occurs at two distinct epochs, an orbit is periodic
  - For discussion in general *n*-body problems, see: Roy & Ovenden, 1954, "On the Occurrence of Commensurable Mean Motions in the Solar System"



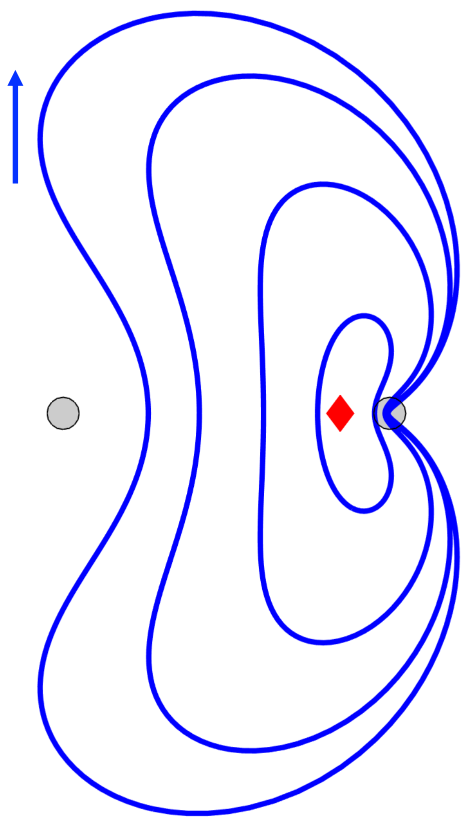
$t \rightarrow -t \qquad y \rightarrow -y$

$x \rightarrow x \qquad z \rightarrow z$

Examples of Symmetries

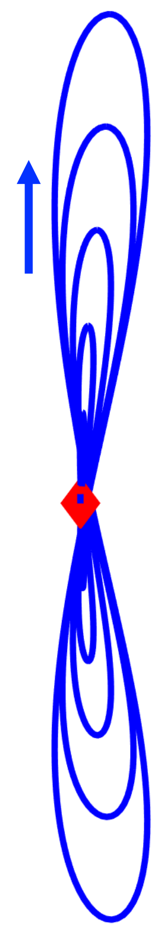
Symmetry #1

L_1 Lyapunov orbits

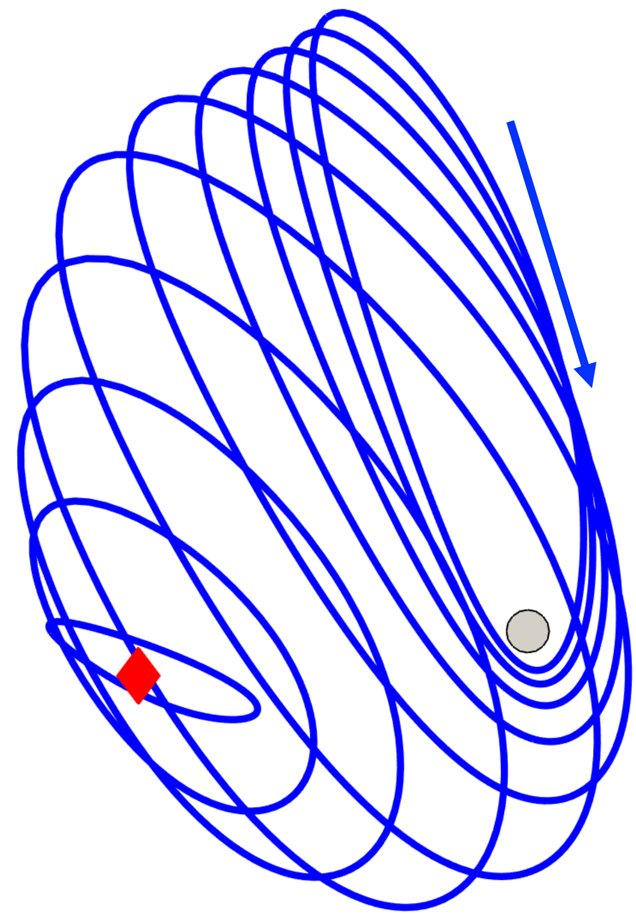


Symmetry #2

L_1 vertical orbits



L_1 halo orbits



Deriving the Jacobi Constant

Derive the Jacobi constant using equations of motion:

$$\ddot{x} - 2\dot{y} = \frac{\partial U^*}{\partial x} \qquad \ddot{y} + 2\dot{x} = \frac{\partial U^*}{\partial y} \qquad \ddot{z} = \frac{\partial U^*}{\partial z}$$

$$U^* = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

Recall the acceleration and velocity vectors:

$$\bar{a} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}$$

$$\bar{v} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$$

Deriving the Jacobi Constant

Take the dot product of acceleration and velocity vectors:

$$\ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z} = \left(2\dot{y} + \frac{\partial U^*}{\partial x}\right)\dot{x} + \left(-2\dot{x} + \frac{\partial U^*}{\partial y}\right)\dot{y} + \left(\frac{\partial U^*}{\partial z}\right)\dot{z}$$

$$\ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z} = \frac{\partial U^*}{\partial x}\dot{x} + \frac{\partial U^*}{\partial y}\dot{y} + \frac{\partial U^*}{\partial z}\dot{z}$$

Deriving the Jacobi Constant

To rewrite the LHS, note

Then

Setting $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$

Zero Velocity Surfaces

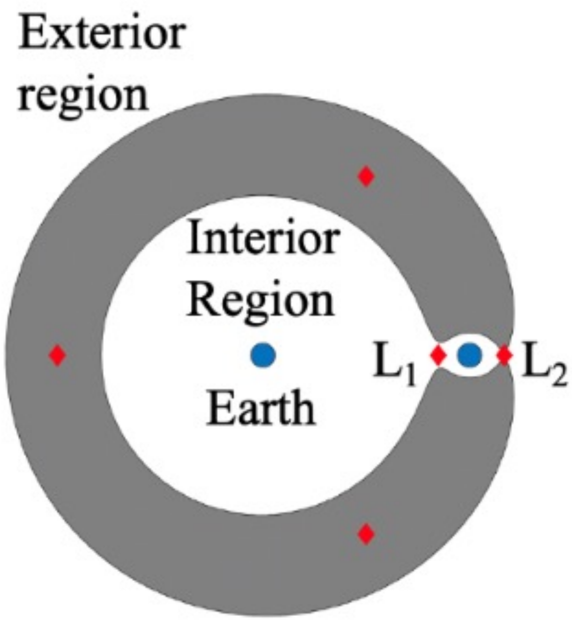
Zero Velocity Surfaces (ZVS) are defined at a single value of the Jacobi constant and composed of infinite states with:

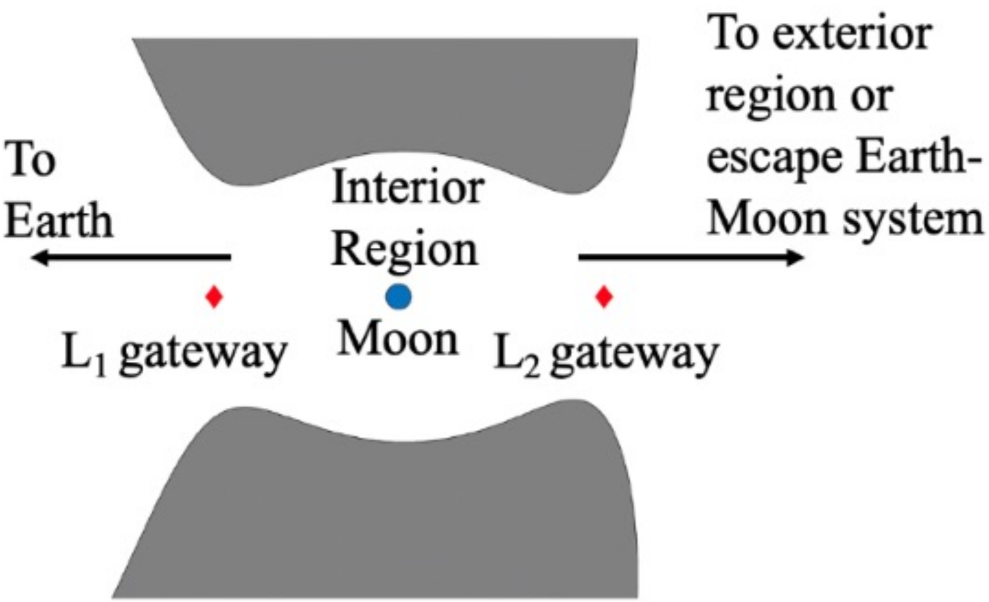
$$\dot{x} = 0, \dot{y} = 0, \dot{z} = 0$$

To calculate ZVS, solve the following relationship for a specified value of Jacobi constant when $v = 0$

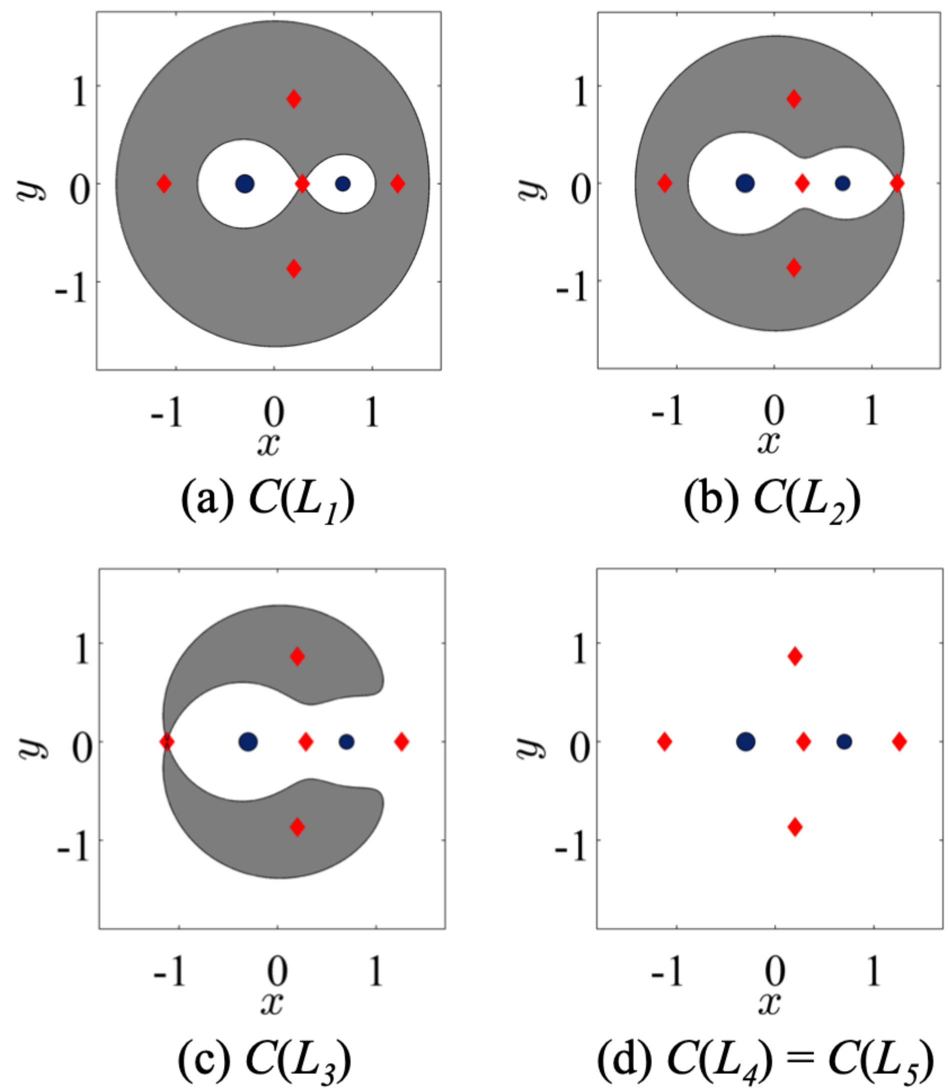
$$C = 2U^* - v^2 = 2U^*$$

Useful Terminology





Zero Velocity Curves



(a) $C(L_1)$

(b) $C(L_2)$

(c) $C(L_3)$

(d) $C(L_4) = C(L_5)$

Image credit:
Bosanac, 2016

Zero Velocity Surfaces

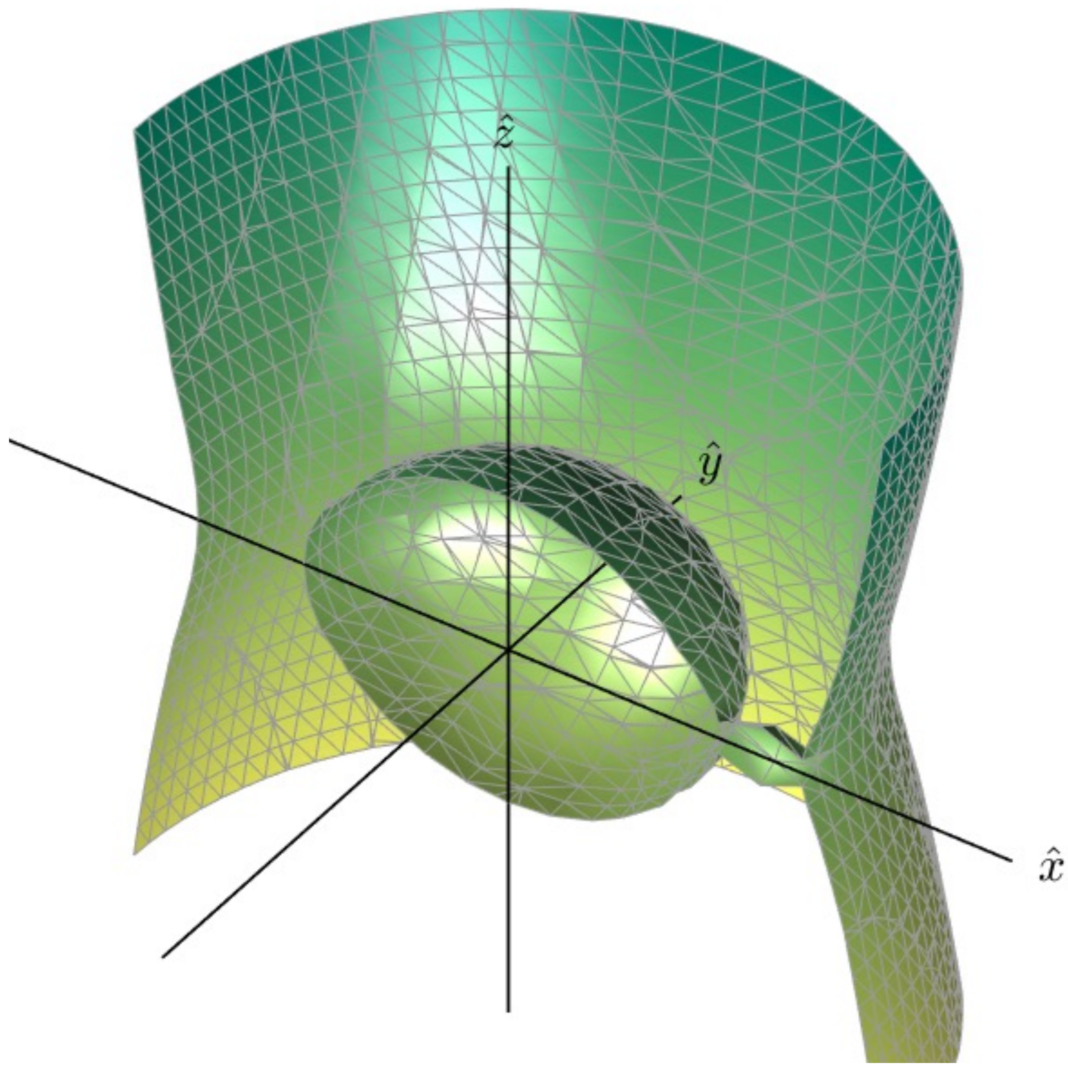


Image credit:
Ian Elliott