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ASEN 5010
Spring 2025
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$\boxed{\text{HW \#5}}$

Problem 1 → 8.3

$\ddot{\theta} + 2\xi\omega_n\dot{\theta} + \omega_n^2\sin\theta = 0$

a] $\theta_0 = 0$, $\vec{\theta} = [\theta, \dot{\theta}]^T$

$$\dot{\vec{\theta}} = \begin{bmatrix}\dot{\theta}\\ \ddot{\theta}\end{bmatrix} = \vec{f}(\vec{\theta}) = \begin{bmatrix}\dot{\theta}\\ -2\xi\omega_n\dot{\theta} - \omega_n^2\sin\theta\end{bmatrix}, \quad [A] = \frac{\partial \vec{f}}{\partial \vec{\theta}} = \begin{bmatrix}\partial f_1/\partial\theta & \partial f_1/\partial\dot{\theta}\\ \partial f_2/\partial\theta & \partial f_2/\partial\dot{\theta}\end{bmatrix}$$

$$[A] = \begin{bmatrix}0 & 1\\ -\omega_n^2\cos\theta & -2\xi\omega_n\end{bmatrix} \rightarrow \delta\dot{\vec{\theta}} \simeq \begin{bmatrix}0 & 1\\ -\omega_n^2\cos\theta & -2\xi\omega_n\end{bmatrix}\delta\vec{\theta}$$

Reference → $\theta_0 = 0$ → $\dot{\vec{\theta}} \simeq [A]\vec{\theta} + \cancel{[B]\vec{u}}^{\,\to 0} = \boxed{\begin{bmatrix}0 & 1\\ -\omega_n^2 & -2\xi\omega_n\end{bmatrix}\vec{\theta} = \dot{\vec{\theta}}}$

b] $\xi = 0 \rightarrow \dot{\vec{\theta}} = \begin{bmatrix}0 & 1\\ -\omega_n^2 & 0\end{bmatrix}\vec{\theta} = [A]\vec{\theta}$

Calculate eigenvalues → $|A - \lambda I| = \left|\begin{bmatrix}0 & 1\\ -\omega_n^2 & 0\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right| = \begin{vmatrix}-\lambda & 1\\ -\omega_n^2 & -\lambda\end{vmatrix} = \lambda^2 + \omega_n^2 = 0$

$\lambda^2 = -\omega_n^2 \rightarrow \lambda = \pm\sqrt{-\omega_n^2}$, $\omega_n^2 > 0 \rightarrow \lambda$ is purely imaginary ∴ linearized system is marginally stable ∴ small motions about θ_0 are marginally stable

c] $V(\vec{\theta}) = \vec{\theta}^T[P]\vec{\theta}$ ← candidate Lyapunov function

$\dot{V}(\vec{\theta}) = \dot{\vec{\theta}}^T[P]\vec{\theta} + \vec{\theta}^T[P]\dot{\vec{\theta}} \qquad (\dot{\vec{\theta}} = [A]\vec{\theta})$

$= ([A]\vec{\theta})^T[P]\vec{\theta} + \vec{\theta}^T[P][A]\vec{\theta} = \vec{\theta}^T([A]^T[P] + [P][A])\vec{\theta} = -\vec{\theta}^T[R]\vec{\theta}$

$$\begin{bmatrix}0 & -\omega_n^2\\ 1 & 0\end{bmatrix}\begin{bmatrix}P_{11} & P_{12}\\ P_{12} & P_{22}\end{bmatrix} + \begin{bmatrix}P_{11} & P_{12}\\ P_{12} & P_{22}\end{bmatrix}\begin{bmatrix}0 & 1\\ -\omega_n^2 & 0\end{bmatrix} = -\begin{bmatrix}R_{11} & R_{12}\\ R_{12} & R_{22}\end{bmatrix}$$

$$\begin{bmatrix}-P_{12}\omega_n^2 & -P_{22}\omega_n^2\\ P_{11} & P_{12}\end{bmatrix} + \begin{bmatrix}-P_{12}\omega_n^2 & P_{11}\\ -P_{22}\omega_n^2 & P_{12}\end{bmatrix} = -\begin{bmatrix}R_{11} & R_{12}\\ R_{12} & R_{22}\end{bmatrix}$$

$\boxed{R_{11} = +2P_{12}\omega_n^2,\ R_{12} = P_{22}\omega_n^2 - P_{11},\ R_{22} = -2P_{12}}$

$R = \begin{bmatrix}2P_{12}\omega_n^2 & P_{22}\omega_n^2 - P_{11}\\ P_{22}\omega_n^2 - P_{11} & -2P_{12}\end{bmatrix} \rightarrow |R| = -4P_{12}^2\omega_n^2 - (P_{22}\omega_n^2 - P_{11})^2$

For R to be Positive definite, $|R| > 0$

∴ $\underbrace{-4P_{12}^2\omega_n^2}_{-ve} > \underbrace{(P_{22}\omega_n^2 - P_{11})^2}_{+ve}$ → so, $|R|$ is not > 0 → $[R]$ is not Positive definite

$\boxed{\text{theorem 8.6 does not hold}}$, ∵ $[A]$ is not stable, it makes sense that theorem 8.6 does not hold

This means that $\dot{V} = -\vec{x}^T[R]\vec{x}$ is not strictly negative definite $\therefore$ the system cannot be guaranteed to be Lyapunov stable or asymptotically stable.

d) $A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix}$, calculate evals $\rightarrow |A-\lambda I| = 0 \rightarrow \begin{vmatrix} -\lambda & 1 \\ -\omega_n^2 & -2\xi\omega_n - \lambda \end{vmatrix} = (-\lambda)(-2\xi\omega_n - \lambda) + \omega_n^2 = 0$

$\lambda^2 + 2\lambda\xi\omega_n + \omega_n^2 = 0 \rightarrow \lambda = \dfrac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2} = \dfrac{-2\xi\omega_n \pm 2\omega_n\sqrt{\xi^2 - 1}}{2}$

$2\omega_n\sqrt{\xi^2-1}$ will be imaginary. $-2\xi\omega_n$ will be $< 0 \rightarrow \therefore Re(\lambda) < 0$

$\therefore [A]$ is strictly stable ($\because Re(\lambda) < 0$)

e) $[A]^T[P] + [P][A] = -[R] \rightarrow \begin{bmatrix} 0 & -\omega_n^2 \\ 1 & -2\xi\omega_n \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix}$

$= \begin{bmatrix} -P_{12}\omega_n^2 & -P_{22}\omega_n^2 \\ P_{11} - 2P_{12}\xi\omega_n & P_{12} - 2P_{22}\xi\omega_n \end{bmatrix} + \begin{bmatrix} -P_{12}\omega_n^2 & P_{11} - 2P_{12}\xi\omega_n \\ -P_{22}\omega_n^2 & P_{12} - 2P_{22}\xi\omega_n \end{bmatrix} = -\begin{bmatrix} R_{11} & R_{12} \\ R_{12} & R_{22} \end{bmatrix}$

$\boxed{R_{11} = 2P_{12}\omega_n^2, \quad R_{12} = P_{22}\omega_n^2 + 2P_{12}\xi\omega_n - P_{11}, \quad R_{22} = 4P_{22}\xi\omega_n - 2P_{12}}$

f) $R = \begin{bmatrix} R_{11} & 0 \\ 0 & R_{22} \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix} \rightarrow R_{12} = 0 = P_{11} = P_{22}\omega_n^2 + 2P_{12}\xi\omega_n$

Evals of $[R] \rightarrow R_{11}, R_{22} \rightarrow R_{11} > 0, R_{22} > 0$ ($\because R$ is pos. def)

$R_{11} > 0 \rightarrow 2P_{12}\omega_n^2 > 0 \rightarrow P_{12} > 0$

$R_{22} > 0 \rightarrow 4P_{22}\xi\omega_n > 2P_{12} \rightarrow 2P_{22}\xi\omega_n > P_{12} \rightarrow P_{12} > 0 \rightarrow \therefore \underline{P_{22} > 0}$ ($\because \omega_n > 0$ because natural frequency is +ve, $\xi > 0$)

$P = \begin{bmatrix} P_{22}\omega_n^2 + 2P_{12}\xi\omega_n & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$

$|P| = P_{22}^2\omega_n^2 + \underbrace{2P_{12}P_{22}\xi\omega_n - P_{12}^2}$

$\because 2P_{22}\xi\omega_n > P_{12} \rightarrow 2P_{12}P_{22}\xi\omega_n > P_{12}^2 \rightarrow \therefore \boxed{2P_{12}P_{22}\omega_n\xi > P_{12}^2}$

And, $P_{22}^2\omega_n^2 > 0$ (square of 2 terms)

$\therefore \boxed{|P| > 0} \rightarrow$ P is positive definite

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$$\boxed{{}^B\vec{\omega}^T\sigma = \sigma^T\,{}^B\vec{\omega}}$$

Problem 2 →

$$V = 2K\ln(1+\sigma^T\sigma) \qquad \sigma^2 = \sigma^T\sigma$$

$$\dot{V}(\sigma) = \frac{\partial V}{\partial \sigma}\cdot\dot{\sigma} = \frac{2K}{1+\sigma^2}\cdot(2\sigma^T)\cdot\dot{\sigma} = \frac{4K\sigma^T}{1+\sigma^2}\cdot\dot{\sigma}$$

$$\dot{\sigma} = \frac{1}{4}\left[(1-\sigma^2)I_{3\times3} + 2[\tilde{\sigma}] + 2\sigma\sigma^T\right]{}^B\vec{\omega}$$

$$\dot{V}(\sigma) = \frac{\cancel{4}K\sigma^T}{1+\sigma^2}\cdot\frac{1}{\cancel{4}}\left[(1-\sigma^2)I_{3\times3} + 2[\tilde{\sigma}] + 2\sigma\sigma^T\right]{}^B\vec{\omega}$$

$$= \frac{K}{1+\sigma^2}\cdot\left[\sigma^T\,{}^B\omega - \sigma^T\sigma^2\,{}^B\vec{\omega} + \underbrace{\cancel{2\sigma^T[\tilde{\sigma}]\,{}^B\vec{\omega}}}_{\to 0} + 2\sigma^T\sigma\sigma^T\,{}^B\vec{\omega}\right]$$

$\sigma^T\sigma^2\,{}^B\vec{\omega}$ is the same as $\sigma^T\sigma\sigma^T\,{}^B\vec{\omega} \rightarrow \therefore -\sigma^T\sigma^2\,{}^B\vec{\omega} + 2\sigma^T\sigma\sigma^T\,{}^B\vec{\omega} = \sigma^T\sigma^2\,{}^B\vec{\omega}$

$[\tilde{\sigma}]\,{}^B\omega = \sigma \times {}^B\vec{\omega} \rightarrow$ Resulting vector is $\perp$ to σ, ${}^B\vec{\omega} \rightarrow \therefore \sigma^T[\tilde{\sigma}]\,{}^B\vec{\omega} = \vec{0}$

$$\dot{V}(\sigma) = \frac{K}{1+\sigma^2}\left[\sigma^T\,{}^B\vec{\omega} + \sigma^T\sigma^2\,{}^B\vec{\omega}\right] = \frac{K}{\cancel{1+\sigma^2}}(\sigma^T\,{}^B\vec{\omega})\left[\cancel{1+\sigma^2}\right] = K\sigma^T\,{}^B\vec{\omega} = \boxed{{}^B\vec{\omega}^T(K\sigma) = \dot{V}}$$

Alternate derivation →

$$\dot{V}(\sigma) = \frac{2K}{1+\sigma^2}(\sigma^T\dot{\sigma} + \dot{\sigma}^T\sigma) = \frac{2K}{1+\sigma^2}\Big[\sigma^T\cdot\frac{1}{4}\left((1-\sigma^2)I_{3\times3} + 2[\tilde{\sigma}] + 2\sigma\sigma^T\right){}^B\vec{\omega}$$
$$+ {}^B\vec{\omega}^T\left(\frac{1}{4}\left((1-\sigma^2)I_{3\times3} - 2[\tilde{\sigma}] + 2\sigma^T\sigma\right)\right)\sigma\Big]$$

$$= \frac{2K}{1+\sigma^2}\cdot\frac{1}{4}\Big[\sigma^T(1-\sigma^2)\,{}^B\vec{\omega} + \cancel{\sigma^T\cdot 2[\tilde{\sigma}]\,{}^B\omega} + 2\sigma^T\sigma\sigma^T\,{}^B\vec{\omega} + {}^B\vec{\omega}^T(1-\sigma^2)\sigma - \cancel{2\,{}^B\vec{\omega}^T[\tilde{\sigma}]\sigma} + 2\,{}^B\vec{\omega}^T\sigma\sigma^T\sigma\Big]$$

$$= \frac{K}{2(1+\sigma^2)}\left[2\,{}^B\vec{\omega}^T(1-\sigma^2)\sigma + 4\,{}^B\vec{\omega}^T\sigma^2\sigma\right] = \frac{K}{\cancel{2}(1+\sigma^2)}\left[\cancel{2}\,{}^B\vec{\omega}^T\sigma - \cancel{2}\,{}^B\vec{\omega}^T\sigma^2\sigma + \cancel{4}\,{}^B\vec{\omega}^T\sigma^2\sigma\right]$$

$$= \frac{K}{1+\sigma^2}\left[{}^B\vec{\omega}^T\sigma + {}^B\vec{\omega}^T\sigma^2\sigma\right] = \frac{K}{\cancel{1+\sigma^2}}\left[\vec{\omega}^T\sigma\left(\cancel{1+\sigma^2}\right)\right]$$

$$= K\vec{\omega}^T\sigma = \boxed{\vec{\omega}^T(K\sigma) = \dot{V}(\sigma)}$$