Traditional view of of opt control:

Best way to get
from A to B

Alternate view

- Are A & B Connected?
- How are they connected?

1) [Are points A & B connected] Object correlation
2) [Are there unmodeled/mismodeled dynamics present] Man Detection
3) [What do the missing dynamics look like] Maneuver Reconstruction
Natural Dynamics Estimation

Object Correlation / Maneuver Detection

- Method for determining likelihood of points being connected given an assumed dynamical model
- Control Distance Metric [Holzinger, Scheeres, Alfriend]

$$d_c = \frac{1}{2}\int_{t_a}^{t_b} \vec{u}(\tau)\, Q^{-1}(\tau)\, \vec{u}(\tau)\, d\tau$$

Given: $t_a, \vec{X}_a, P_a$
$t_b, \vec{X}_b, P_b$
$\dot{\vec{X}} = f(t, \vec{x}, \vec{u})$

Calculate $\vec{u}(t)$ that satisfies boundary conditions and minimizes the metric

Optimal Trajectory (Fall 2014) - Day 2

Notes 2/[illegible]

$$\dot{\vec{x}} = f(t,\vec{x},\vec{u}) = f_n(t,\vec{x}) + B(t)\vec{u}(t)$$

$$\mathcal{H}(t,\vec{x},\vec{u},\vec{p}) = \frac{1}{2}\vec{u}^T Q^{-1}\vec{u} + f\cdot\vec{p}$$
$$= \frac{1}{2}\vec{u}^T Q^{-1}\vec{u} + f_n\cdot\vec{p} + B\vec{u}\cdot\vec{p}$$

Pontryagin: $\hat{u}(t) = \underset{\vec{u}\in\mathcal{U}}{\operatorname{argmin}}\left(\mathcal{H}(t,\vec{x},\vec{u},\vec{p})\right)$

$$\frac{\partial H}{\partial \vec{u}} = \vec{0} \qquad Q^{-1}\vec{u} + B^T\vec{p} = \vec{0}$$

$$\vec{u}(t) = -Q(t)B^T(t)\vec{p}(t)$$

Dynamics: $\dot{\vec{x}} = \frac{\partial\mathcal{H}}{\partial\vec{p}} = f_n(t,\vec{x}) + B(t)\vec{u}(t)$

$$\dot{\vec{p}} = -\frac{\partial\mathcal{H}}{\partial\vec{x}} = -\frac{\partial f_n}{\partial\vec{x}}^T\vec{p}(t)$$

Transversality: No extra Information

Numerical Method: Single Shooter

Nominal trajectory: ~~$t_a, \vec{x}_a; t$~~ $t_a, \vec{x}_a, \vec{p}_a^{(i)}$

$$\vec{x}_b^{(i)} = \phi_x(t_b; t_a, \vec{x}_a, \vec{p}_a^{(i)})$$

$$\vec{x}_b \approx \phi_x(t_b; t_a, \vec{x}_a, \vec{p}_a^{(i)}) + \frac{\partial\phi_x}{\partial\vec{p}_a}\delta\vec{p}_a^{(i)}$$

$$\delta\vec{p}_a^{(i)} = -\Phi_{xp}^{-1}(t_b,t_a)\left(\vec{x}_b^{(i)} - \vec{x}_b\right)$$

$$\vec{p}_a^{(i+1)} = \vec{p}_a^{(i)} + \delta\vec{p}_a^{(i)}$$

Repeat until $|\vec{x}_b^{(i)} - \vec{x}_b|_\infty < \delta$

$\vec{X}_a$ & $\vec{X}_b$ are Gaussian Random Variables

What is the spread in $d_c(a,b)$?

Motion about the trajectory:

$$\delta\vec{x}(t) = \frac{\partial\phi_x}{\partial\vec{x}}\delta\vec{x}_a + \frac{\partial\phi_x}{\partial\vec{p}}\delta\vec{p}_a = \Phi_{xx}(t,t_a)\delta\vec{x}_a + \Phi_{xp}(t,t_a)\delta\vec{p}_a$$

$$\delta\vec{p}(t) = \frac{\partial\phi_p}{\partial\vec{x}}\delta\vec{x}_a + \frac{\partial\phi_p}{\partial\vec{p}}\delta\vec{p}_a = \Phi_{px}(t,t_a)\delta\vec{x}_a + \Phi_{pp}(t,t_a)\delta\vec{p}_a$$

$$\delta\vec{x}_b = \Phi_{xx}(t_b,t_a)\delta\vec{x}_a + \Phi_{xp}(t_b,t_a)\delta\vec{p}_a$$

$$\delta\vec{p}_a = \Phi_{xp}^{-1}(t_b,t_a)\left[\delta\vec{x}_b - \Phi_{xx}(t_b,t_a)\delta\vec{x}_a\right]$$

$$\delta\vec{p}(t) = \left[\Phi_{px}(t,t_a) - \Phi_{pp}(\cancel{t_b},t_a)\Phi_{xp}^{-1}(t_b,t_a)\Phi_{xx}(t_b,t_a)\right]\delta\vec{x}_a$$

$$+ \left[\Phi_{pp}(t,t_a)\Phi_{xp}^{-1}(t_b,t_a)\right]\delta\vec{x}_b$$

$$= \Lambda(t,t_a)\begin{bmatrix}\delta\vec{x}_a\\ \delta\vec{x}_b\end{bmatrix}$$

Expand Control:

$$u(t) = \tilde{u}^*(t) + \delta\vec{u}(t) = -QB^T(t)\left(\tilde{p}(t) + \delta p(t)\right)$$

$$\boxed{\delta\vec{u}(t) = -QB^T(t)\Lambda(t,t_a)\begin{bmatrix}\delta\vec{x}_a\\ \delta\vec{x}_b\end{bmatrix}}$$

$$d_c = \frac{1}{2}\int_{t_a}^{t_b} \left(\tilde{u}(\tau) + \delta\vec{u}(\tau)\right)^T Q^{-1}(\tau)\left(\tilde{u}(\tau) + \delta\vec{u}(\tau)\right) d\tau$$

$$= \frac{1}{2}\int_{t_a}^{t_b} \tilde{u}^T(\tau)\, Q^{-1}(\tau)\, \tilde{u}(\tau)\, d\tau$$

$$+ \underbrace{\left[\int_{t_a}^{t_b} -\tilde{u}^T(\tau) B^T(\tau) \Lambda(\tau, t_a)\, d\tau\right]}_{\vec{\omega}(t_b, t_a)} \delta\vec{z}$$

$$+ \frac{1}{2}\delta\vec{z}^T \underbrace{\left[\int_{t_a}^{t_b} \Lambda^T(\tau, t_a) B(\tau) Q(\tau) B^T(\tau) \Lambda(\tau, t_a)\, d\tau\right]}_{\Omega(t_b, t_a)} \delta\vec{z}$$

$$\boxed{d_c = \tilde{d}_c + \vec{\omega}^T(t_b, t_a)\,\delta\vec{z} + \delta\vec{z}^T\,\Omega(t_b, t_a)\,\delta\vec{z}}$$

$$\mathbb{E}\left[\delta\vec{z}\right] = 0 \qquad \mathbb{E}\left[\delta\vec{z}\,\delta\vec{z}^T\right] = \begin{bmatrix} P_a & 0 \\ 0 & P_b \end{bmatrix} = P_z$$

Mean: $\mu_{d_c} = \tilde{d}_c + \mathrm{Tr}\left[\Omega P_z\right]$

Var: $\sigma_{d_c}^2 = \vec{\omega}^T P_z \vec{\omega} + 2\,\mathrm{Tr}\left[\Omega P_z \Omega P_z\right]$

Hypothesis: A & B are connected on a natural trajectory ($\tilde{u}(t) = 0$)

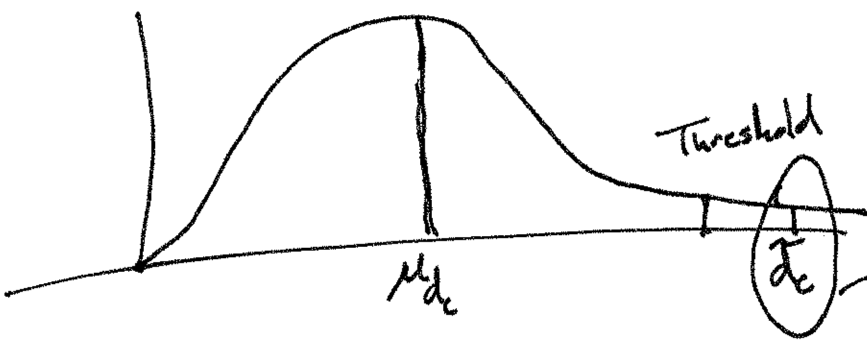
Thus:

$$\mu_{d_c} = Tr\left[\Omega P_z\right]$$

$$\sigma_{d_c}^2 = 2\,Tr\left[\Omega P_z \Omega P_z\right]$$

Model as Gaussian

Is $\tilde{d}_c$ statistically significant?



Detected Maneuver or A & B uncorrelated

Optimal Traj (Fall 2014) - Day 2

Notes 8/17

- Typically not given two states with uncertainties.
  - Generally initial guess $(\bar{X}_{k-1|k-1}, \bar{P}_{k-1|k-1})$ and Measurement $(\vec{Y}_k, R_k)$
  - How to obtain $\hat{X}_k, \hat{P}_{k|k}$? State Estimation

The Optimal Control Based Estimator

Inputs:

A priori

$\bar{X}_{k-1|k-1}, \bar{\bar{P}}_{k-1|k-1}$

t_{k-1}

Measurement

$\vec{Y}_k, R_k, t_k$

$h(t, \vec{x})$

Dynamics

$\dot{\vec{x}} = f_n(t, \vec{x}) + B(t)\vec{u}(t)$

~~$Q(t)\delta$~~

$Q(t)\delta(t-\tau) = \mathbb{E}[\vec{u}(t)\vec{u}(\tau)]$

$$J = K_{k-1}(t_{k-1}, \vec{X}_{k-1}) + K_k(t_k, \vec{X}_k) + \int_{t_{k-1}}^{t_k} L(\tau, \vec{x}, \vec{u})\, d\tau$$

Apriori: $K_{k-1}(\vec{X}_{k-1}) = \frac{1}{2}(\bar{X}_{k-1|k-1} - \vec{X}_{k-1})^T \bar{\bar{P}}_{k-1|k-1}^{-1}(\bar{X}_{k-1|k-1} - \vec{X}_{k-1})$

Measurement: $K_k(\vec{X}_k) = \frac{1}{2}(\vec{Y}_k - h(t_k, \vec{X}_k))^T R_k^{-1}(\vec{Y}_k - h(t_k, \vec{X}_k))$

Dynamics: $L(\vec{u}(t)) = \frac{1}{2}\vec{u}^T(t)\tilde{Q}^{-1}(t)\vec{u}(t)$; $\tilde{Q}(t) = (t_k - t_{k-1})Q(t)$

Differences from Kalman:
- Unpropagated a priori
- term for dynamic uncertainty
- nonlinear
- State estimate at both epochs
- Control estimate

Optimal Traj (Fall 2014) - Day 2 Notes 2/17

- Optimization -

$$\mathcal{H}(t, \vec{x}, \vec{u}, \vec{p}) = h(\vec{u}(t)) + f \cdot \vec{p}(t)$$

Pontryagin: $\hat{u}(t) = -\tilde{Q}(t) B^T(t) \hat{\vec{p}}(t)$

Dynamics: $\dot{\hat{x}} = f_n(t, \hat{x}) + B(t)\hat{u}(t) = f_n(t, \hat{x}) - B(t)\tilde{Q}(t)B^T(t)\hat{p}(t)$

$$\dot{\hat{p}}(t) = -\frac{\partial \mathcal{H}}{\partial x} = -\frac{\partial f_n}{\partial x}^T \hat{p}(t)$$

integrated:
$$\hat{x}_{k|k} = \phi_x(t_k; t_{k-1}, \hat{x}_{k-1|k}, \hat{p}_{k-1|k})$$
$$\hat{p}_{k|k} = \phi_p(t_k; t_{k-1}, \hat{x}_{k-1|k}, \hat{p}_{k-1|k})$$

Transversality:

$$\hat{p}_{k-1|k} = \frac{-\partial K_{k-1}}{\partial \vec{x}}^T = +\bar{P}^{-1}_{k-1|k-1}\left(\bar{x}_{k-1|k-1} - \hat{x}_{k-1|k}\right)$$

$$\hat{p}_{k|k} = \frac{\partial K_k}{\partial \vec{x}}^T = -\frac{\partial h}{\partial \vec{x}}^T\Bigg|_{(t_k, \hat{x}_{k|k})} R_k^{-1}\left(\bar{Y}_k - h(t_k, \hat{x}_{k|k})\right)$$

- Nonlinear OCBE -

$$F_1(\hat{x}_{k-1|k}, \hat{x}_{k|k}, \hat{p}_{k-1|k}) = \phi_p(t_k; t_{k-1}, \hat{x}_{k-1|k}, \hat{p}_{k-1|k}) + \frac{\partial h}{\partial \vec{x}}^T\Bigg|_{(t_k, \hat{x}_{k|k})} R_k^{-1}\left(\bar{Y}_k - h(t_k, \hat{x}_{k|k})\right) = \vec{0}$$

$$F_2(\hat{x}_{k-1|k}, \hat{x}_{k|k}, \hat{p}_{k-1|k}) = \hat{p}_{k-1|k} - \bar{P}^{-1}_{k-1|k-1}\left(\bar{x}_{k-1|k-1} - \hat{x}_{k-1|k}\right) = \vec{0}$$

$$F_3(\hat{x}_{k-1|k}, \hat{x}_{k|k}, \hat{p}_{k-1|k}) = \hat{x}_{k|k} - \phi_x(t_k; t_{k-1}, \hat{x}_{k-1|k}, \hat{p}_{k-1|k}) = \vec{0}$$

Optimal Traj (Fall 2014) - Day 2

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- GL-OCBE -

- Nonlinear Result is not implicit
- Want a analytical result
- Linearize!! $\longrightarrow$ $\hat{X}_{k-1}, \hat{P}_{k-1}, \tilde{X}_k = \phi_x(t_k; t_{k-1}, \hat{X}_{k-1}, \hat{P}_{k-1})$

$$\tilde{P}_k = \phi_p(t_k; t_{k-1}, \hat{x}_{k-1}, \hat{p}_{k-1})$$

approach:

$$F_n(\hat{X}_{k-1|k}, \hat{X}_{k|k}, \hat{P}_{k-1|k}) \simeq F_n(\hat{X}_{k-1}, \tilde{X}_k, \hat{P}_{k-1}) + \frac{\partial \tilde{F}_n}{\partial \hat{X}_{k-1|k}} \delta \hat{X}_{k-1|k} + \frac{\partial \tilde{F}_n}{\partial \hat{X}_{k|k}} \delta \hat{X}_{k|k} + \frac{\partial \tilde{F}_n}{\partial \hat{P}_{k-1|k}} \delta \hat{P}_{k-1|k}$$

Definitions:

$$\delta \tilde{y}_k = \bar{Y}_k - h(t_k, \tilde{X}_k)$$

$$\delta \bar{X}_{k-1|k-1} = \bar{X}_{k-1|k-1} - \hat{X}_{k-1}$$

$$\tilde{H}_k = \left.\frac{\partial h}{\partial \tilde{X}_k}\right|_{(t_k, \tilde{X}_k)}$$

$$\Phi(t, t_0) = \begin{bmatrix} \Phi_{xx}(t, t_0) & \Phi_{xp}(t, t_0) \\ \Phi_{px}(t, t_0) & \Phi_{pp}(t, t_0) \end{bmatrix}$$

$$\dot{\Phi}(t, t_0) = \tilde{A}(t) \Phi(t, t_0)$$

$$\tilde{A}(t) = \left.\begin{bmatrix} \frac{\partial f_n}{\partial x} & -B(t)\tilde{Q}(t)B^T(t) \\ \frac{\partial^2}{\partial x^2}(f_n \cdot \vec{p}) & -\frac{\partial f_n}{\partial x}^T \end{bmatrix}\right|_{(t, \tilde{x}, \tilde{p})}$$

Optimal Traj (Fall 2014) - Day 2 Notes 4/17

Results:

$$\left(\tilde{p}_k + \tilde{H}_k^T R_k^{-1} \delta\vec{y}_k\right) + \Phi_{px}\,\delta\hat{x}_{k+1|k} - \tilde{H}_k^T R_k^{-1} \hat{H}_k\,\delta\hat{x}_{k|k} + \Phi_{pp}\,\delta\hat{p}_{k+1|k} = \vec{0} \qquad (1)$$

$$\left(\tilde{p}_{k+1} - \bar{P}_{k+1|k+1}^{-1}\,\delta\bar{x}_{k+1|k+1}\right) + \bar{P}_{k+1|k+1}^{-1}\,\delta\hat{x}_{k+1|k} + \delta\hat{p}_{k+1|k} = \vec{0} \qquad (2)$$

$$-\Phi_{xx}\,\delta\hat{x}_{k+1|k} + \delta\hat{x}_{k|k} - \Phi_{xp}\,\delta\hat{p}_{k+1|k} = \vec{0} \qquad (3)$$

~~Skip to Results:~~

$$\cancel{\delta\hat{x}_{k+1|k} = \left[I + \bar{P}_{k+1|k+1}\left(\Phi_{pp} - \Phi_{px}\bar{P}_{k+1|k+1}\right)^{-1}\Phi_{px}\right]\delta\bar{x}_{k+1|k+1} + L_{k+1}\left(\delta\vec{y}_k - H_k\,\delta\bar{x}_{k+1|k+1}\right)}$$

We need to establish certain properties before rearranging the equations

- BL-OCBE -

Let's ~~assume~~ set $\tilde{p}_{k+1} = \vec{0}$, $\tilde{p}(t) = \phi_p(t; t_{k+1}, \tilde{x}_{k+1}, \tilde{p}_{k+1} = \vec{0}) = \vec{0}$

Because: $\dot{\tilde{p}}(t) = -\frac{\partial f_n}{\partial \tilde{x}}^T \tilde{p}(t)$

$$\dot{\Phi}_{px}(t,t_0) = \cancel{-\frac{\partial^2}{\partial x^2}(\vec{f}_n \cdot \vec{p})} \Phi_{xx}(t,t_0) + -\frac{\partial f}{\partial x}^T \Phi_{px}(t,t_0)$$

$$\begin{cases} \Phi_{px}(t,t_0) = \Omega(t,t_0)\,\Phi_{px}(t_0,t_0)\,\Omega(t,t_0) \\ \dot{\Omega}(t,t_0) = -\frac{\partial f_n}{\partial x}^T \Omega(t,t_0) \end{cases}$$

$$\Phi_{px}(t_0,t_0) = 0 \quad \Rightarrow \quad \boxed{\Phi_{px}(t,t_0) = 0 \quad \forall\, t \in \mathbb{R}}$$

Symplectic STM:

$$\Phi_{xx}^T(t,\tau)\,\Phi_{px}(t,\tau) = \Phi_{px}^T(t,\tau)\,\Phi_{xx}(t,\tau)$$

$$\Phi_{pp}^T(t,\tau)\,\Phi_{xp}(t,\tau) = \Phi_{xp}^T(t,\tau)\,\Phi_{pp}(t,\tau)$$

$$\Phi_{xx}^T(t,\tau)\,\Phi_{pp}(t,\tau) - \Phi_{px}^T(t,\tau)\,\Phi_{xp}(t,\tau) = I_{n\times n}$$

Combine Properties:

a) $\Phi_{px} = 0$

$\Phi_{xx}^T \Phi_{pp} = I \rightarrow \quad \Phi_{pp}(\Phi_{xx}^T \Phi_{pp})\Phi_{pp}^{-1} = \Phi_{pp}(I)\Phi_{pp}^{-1}$

b) $\Phi_{pp}\Phi_{xx}^T = I$

$\Phi_{pp}^T \Phi_{xp} = \Phi_{xp}^T \Phi_{pp} \rightarrow \quad \Phi_{xx}(\Phi_{pp}^T \Phi_{xp})\Phi_{xx}^T = \Phi_{xx}(\Phi_{xp}^T \Phi_{pp})\Phi_{xx}^T$

c) $\Phi_{xp}\Phi_{xx}^T = \Phi_{xx}\Phi_{xp}^T$

Sub in Results:

$$\tilde{H}_k^T R_k^{-1} \delta\vec{y}_k - \tilde{H}_k^T R_k^{-1} \tilde{H}_k \delta\hat{x}_{k|k} + \Phi_{pp}\delta\hat{p}_{k-1|k} = \vec{0} \qquad (1)$$

$$-\bar{P}_{k-1|k-1}^{-1}\delta\bar{x}_{k-1|k-1} + \bar{P}_{k-1|k-1}^{-1}\delta\hat{x}_{k-1|k} + \delta\hat{p}_{k-1|k} = \vec{0} \qquad (2)$$

$$-\Phi_{xx}\delta\hat{x}_{k-1|k} + \delta\hat{x}_{k|k} - \Phi_{xp}\delta\hat{p}_{k-1|k} = \vec{0} \qquad (3)$$

Rearrange:

$$\delta\hat{x}_{k|k} = \Phi_{xx}\delta\hat{x}_{k-1|k} + \Phi_{xp}\delta\hat{p}_{k-1|k} \qquad (4)$$

$$\delta\hat{p}_{k-1|k} = -\bar{P}_{k-1|k-1}^{-1}\delta\hat{x}_{k-1|k} + \bar{P}_{k-1|k-1}^{-1}\delta\bar{x}_{k-1|k-1} \qquad (5)$$

Sub (5) into (4)

$$\delta\hat{x}_{k|k} = (\Phi_{xx} - \Phi_{xp}\bar{P}_{k-1|k-1}^{-1})\delta\hat{x}_{k-1|k} + \Phi_{xp}\bar{P}_{k-1|k-1}^{-1}\delta\bar{x}_{k-1|k-1} \qquad (6)$$

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Sub (6) & (5) into (1)

$$\hat{H}_k^T R_k^{-1} \delta\vec{y}_k - \hat{H}_k^T R_k^{-1} \hat{H}_k \Phi_{xp} \bar{P}_{k-1|k-1}^{-1} \delta\bar{x}_{k-1|k-1} + \Phi_{pp} \bar{P}_{k-1|k-1}^{-1} \delta\bar{x}_{k-1|k-1}$$
$$= \left[\hat{H}_k^T R_k^{-1} \hat{H}_k \left(\Phi_{xx} - \Phi_{xp} \bar{P}_{k-1|k-1}^{-1}\right) + \Phi_{pp} \bar{P}_{k-1|k-1}^{-1}\right] \delta\hat{x}_{k-1|k}$$

Rearrange:

$$\delta\hat{x}_{k-1|k} = \left[\hat{H}_k^T R_k^{-1} \hat{H}_k \left(\Phi_{xx} - \Phi_{xp} \bar{P}_{k-1|k-1}^{-1}\right) + \Phi_{pp} \bar{P}_{k-1|k-1}^{-1}\right]^{-1} \tag{7}$$
$$\times \left[\hat{H}_k^T R_k^{-1} \delta\vec{y}_k + \left(\Phi_{pp} - \hat{H}_k^T R_k^{-1} \hat{H}_k \Phi_{xp}\right) \bar{P}_{k-1|k-1}^{-1} \delta\bar{x}_{k-1|k-1}\right]$$

Sub (7) into (6):

$$\delta\hat{x}_{k|k} = M_k \delta\bar{x}_{k-1|k-1} + L_k \delta\vec{y}_k$$

$$M_k = \left[\left(\Phi_{xx} - \Phi_{xp} \bar{P}_{k-1|k-1}^{-1}\right)\left[\hat{H}_k^T R_k^{-1} \hat{H}_k \left(\Phi_{xx} - \Phi_{xp} \bar{P}_{k-1|k-1}^{-1}\right) + \Phi_{pp} \bar{P}_{k-1|k-1}^{-1}\right]^{-1} \right.$$
$$\left. \times \left(\Phi_{pp} - \hat{H}_k^T R_k^{-1} \hat{H}_k \Phi_{xp}\right) + \Phi_{xp}\right] \bar{P}_{k-1|k-1}^{-1}$$

$$L_k = \left(\Phi_{xx} - \Phi_{xp} \bar{P}_{k-1|k-1}^{-1}\right)\left[\Phi_{pp} \bar{P}_{k-1|k-1}^{-1} + \hat{H}_k^T R_k^{-1} \hat{H}_k \left(\Phi_{xx} - \Phi_{xp} \bar{P}_{k-1|k-1}^{-1}\right)\right]^{-1} \hat{H}_k^T R_k^{-1}$$

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Rearrange L_k:

$$L_k = \left[\Phi_{pp}\bar{P}^{-1}_{k-1|k-1}\left(\Phi_{xx} - \Phi_{xp}\bar{P}^{-1}_{k-1|k-1}\right)^{-1} + \hat{H}_k^T R_k^{-1}\hat{H}_k\right]^{-1}\hat{H}_k^T R_k^{-1}$$

$$A = \Phi_{pp}\bar{P}^{-1}_{k-1|k-1}\left(\Phi_{xx} - \Phi_{xp}\bar{P}^{-1}_{k-1|k-1}\right)^{-1} = \left[\Phi_{xx}\bar{P}_{k-1|k-1}\Phi_{xx}^T - \Phi_{xp}\Phi_{xx}^T\right]^{-1}$$

$$= \bar{P}_{k|k-1}$$

$$B = \hat{H}_k^T R_k^{-1}$$

$$C = \hat{H}_k$$

$$L_k = (A + BC)^{-1}B$$

Apply Schur Identity

$$= \left[A^{-1} - A^{-1}B(I + CA^{-1}B)^{-1}CA^{-1}\right]B$$

$$= A^{-1}B\left[I - (I + CA^{-1}B)^{-1}CA^{-1}B\right]$$

$$= A^{-1}B(I + CA^{-1}B)^{-1}\left[I + CA^{-1}B - CA^{-1}B\right]$$

$$= A^{-1}\hat{H}_k^T R_k^{-1}\left(I + \hat{H}_k A^{-1}\hat{H}_k^T R_k^{-1}\right)^{-1}$$

$$= A^{-1}\hat{H}_k^T\left(R_k + \hat{H}_k A^{-1}\hat{H}_k^T\right)^{-1}$$

$$\boxed{L_k = \bar{P}_{k|k-1}\hat{H}_k^T\left(R_k + \hat{H}_k\bar{P}_{k|k-1}\hat{H}_k^T\right)^{-1}}$$

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Rearrange M_k:

$$M_k = \Phi_{xp}\bar{P}_{k-1|k-1}^{-1} + (\Phi_{xx} - \Phi_{xp}\bar{P}_{k-1|k-1}^{-1})\left[\Phi_{pp}\bar{P}_{k-1|k-1}^{-1} + \hat{H}_k^T R_k^{-1}\hat{H}_k(\Phi_{xx} - \Phi_{xp}\bar{P}_{k-1|k-1}^{-1})\right]^{-1}\Phi_{pp}\bar{P}_{k-1|k-1}^{-1}$$

$$- \underbrace{\left[(\Phi_{xx} - \Phi_{xp}\bar{P}_{k-1|k-1}^{-1})\left[\Phi_{pp}\bar{P}_{k-1|k-1}^{-1} + \hat{H}_k^T R_k^{-1}\hat{H}_k(\Phi_{xx} - \Phi_{xp}\bar{P}_{k-1|k-1}^{-1})\right]^{-1}\hat{H}_k^T R_k^{-1}\right]}_{L_k}\hat{H}_k\Phi_{xp}\bar{P}_{k-1|k-1}^{-1}$$

$$= \Phi_{xp}\bar{P}_{k-1|k-1}^{-1} + \left[I + (\Phi_{xx} - \Phi_{xp}\bar{P}_{k-1|k-1}^{-1})(\Phi_{pp}\bar{P}_{k-1|k-1}^{-1})^{-1}(\hat{H}_k^T R_k^{-1}\hat{H}_k)\right]^{-1}(\Phi_{xx} - \Phi_{xp}\bar{P}_{k-1|k-1}^{-1})$$

$$- L_k\hat{H}_k\Phi_{xp}\bar{P}_{k-1|k-1}^{-1}$$

Define:

$A = \Phi_{pp}\bar{P}_{k-1|k-1}^{-1}$

$B = \hat{H}_k^T R_k^{-1}\hat{H}_k$

$C = \Phi_{xx} - \Phi_{xp}\bar{P}_{k-1|k-1}^{-1}$

$$M_k = \Phi_{xp}\bar{P}_{k-1|k-1}^{-1} + (I + CA^{-1}B)^{-1}C - L_k\hat{H}_k\Phi_{xp}\bar{P}_{k-1|k-1}^{-1}$$

$$M_k = \left[I - (I + CA^{-1}B)^{-1}\right]\Phi_{xp}\bar{P}_{k-1|k-1}^{-1} + (I + CA^{-1}B)^{-1}\Phi_{xx}$$

$$- L_k\hat{H}_k\Phi_{xp}\bar{P}_{k-1|k-1}^{-1}$$

$$= (I + CA^{-1}B)^{-1}(CA^{-1}B)\Phi_{xp}\bar{P}_{k-1|k-1}^{-1} + (I + CA^{-1}B)^{-1}\Phi_{xx}$$

$$- L_k\hat{H}_k\Phi_{xp}\bar{P}_{k-1|k-1}^{-1}$$

$$= C(A + BC)^{-1}B\Phi_{xp}\bar{P}_{k-1|k-1}^{-1} + (I + CA^{-1}B)^{-1}\Phi_{xx} - L_k\hat{H}_k\Phi_{xp}\bar{P}_{k-1|k-1}^{-1}$$

$$= \cancel{\left[L_k\hat{H}_k\Phi_{xp}\bar{P}_{k-1|k-1}^{-1} - L_k\hat{H}_k\Phi_{xp}\bar{P}_{k-1|k-1}^{-1}\right]} + (I + CA^{-1}B)^{-1}\Phi_{xx}$$

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$$M_k = (I + CA^{-1}B)^{-1}\,\Phi_{xx}$$
$$= (I + CA^{-1}B)^{-1}\left[I + (CA^{-1}B - CA^{-1}B)\right]\Phi_{xx}$$
$$= \left[I - (I + CA^{-1}B)^{-1}CA^{-1}B\right]\Phi_{xx}$$
$$= \left[I - C(A+BC)^{-1}B\right]\Phi_{xx}$$
$$= (I - L_k \tilde{H}_k)\,\Phi_{xx}$$

— Similar process for $\delta\hat{x}_{k-1|k}$ & $\delta\hat{p}_{k-1|k}$ —

$$\left[\begin{aligned}
\delta\hat{x}_{k-1|k} &= \delta\bar{x}_{k-1|k-1} + L_{k-1}(\delta\bar{y}_k - \tilde{H}_k\,\delta\bar{x}_{k|k-1}) \\
\delta\hat{x}_{k|k} &= \delta\bar{x}_{k|k-1} + L_k(\delta\vec{y}_k - \tilde{H}_k\,\delta\bar{x}_{k|k-1}) \\
\delta\hat{p}_{k-1|k} &= -\bar{P}^{-1}_{k-1|k-1} L_{k-1}(\delta\vec{y}_k - \tilde{H}_k\,\delta\bar{x}_{k|k-1})
\end{aligned}\right]$$

~ Estimates

$$\left[\begin{aligned}
\delta\bar{x}_{k|k-1} &= \Phi_{xx}\,\delta\bar{x}_{k-1|k-1} \\
\bar{P}_{k|k-1} &= \Phi_{xx}\bar{P}_{k-1|k-1}\Phi_{xx}^T - \Phi_{xp}\Phi_{xx}^T
\end{aligned}\right]$$

~ Propagated

$$\left[\begin{aligned}
L_k &= \bar{P}_{k|k-1}\hat{H}_k^T(R_k + \hat{H}_k\bar{P}_{k|k-1}\hat{H}_k^T)^{-1} \\
L_{k-1} &= \bar{P}_{k-1|k-1}\Phi_{xx}^T\hat{H}_k^T(R_k + \hat{H}_k\bar{P}_{k|k-1}\hat{H}_k^T)^{-1}
\end{aligned}\right]$$

~ Gains

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— Uncertainties —

Propagated:

$$\mathbb{E}\left[\left(\delta\bar{x}_{k|k-1} - \delta\bar{x}^{**}_{k}\right)\left(\delta\bar{x}_{k|k-1} - \delta\bar{x}^{*}_{k}\right)^T\right] = \bar{P}_{k|k-1}$$

- Look at dynamical equation

$$\dot{\bar{P}}(t) = \frac{d}{dt}\left(\Phi_{xx}\bar{P}_{k-1|k-1}\Phi_{xx}^T - \Phi_{xp}\Phi_{xx}^T\right)$$

$$= \dot{\Phi}_{xx}\bar{P}_{k-1|k-1}\Phi_{xx}^T + \Phi_{xx}\bar{P}_{k-1|k-1}\dot{\Phi}_{xx}^T - \dot{\Phi}_{xp}\Phi_{xx}^T - \Phi_{xp}\dot{\Phi}_{xx}^T$$

$$\dot{\Phi}_{xx} = \frac{\partial f_n}{\partial x}\Phi_{xx} \qquad \dot{\Phi}_{xp} = \frac{\partial f_n}{\partial x}\Phi_{xp} - B(t)\hat{Q}(t)B(t)\Phi_{pp}$$

$$= \frac{\partial f_n}{\partial x}\Phi_{xx}\bar{P}_{k-1|k-1}\Phi_{xx}^T + \Phi_{xx}\bar{P}_{k-1|k-1}\Phi_{xx}^T\frac{\partial f_n}{\partial x}^T - \frac{\partial f_n}{\partial x}\Phi_{xp}\Phi_{xx}^T + B(t)\hat{Q}(t)B(t)\Phi_{pp}\Phi_{xx}^T - \Phi_{xp}\Phi_{xx}^T\frac{\partial f_n}{\partial x}^T$$

$$= \frac{\partial f_n}{\partial x}\left(\Phi_{xx}\bar{P}_{k-1|k-1}\Phi_{xx}^T - \Phi_{xp}\Phi_{xx}^T\right) + \left(\Phi_{xx}\bar{P}_{k-1|k-1}\Phi_{xx}^T - \Phi_{xp}\Phi_{xx}^T\right)\frac{\partial f_n}{\partial x}^T + B(t)\hat{Q}(t)B(t)\left(\Phi_{pp}\Phi_{xx}^T\right)$$

$$\boxed{\dot{\bar{P}}(t) = \frac{\partial f_n}{\partial x}\bar{P}(t) + \bar{P}(t)\frac{\partial f_n}{\partial x}^T + B(t)\hat{Q}(t)B(t)}$$

~ Same as Process Noise

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$$\cancel{\mathbb{E}\left[(\delta\bar{x}_{k|k-1} - \delta\vec{x}_k^*)(\delta\bar{x}_{k-1|k-1} - \delta\vec{x}_{k-1}^*)^T\right] = \Phi_{xx}\bar{P}_{k-1|k-1}}$$

Stochastic :

$$\cancel{\bar{P}_{k|k}} =$$

$$\delta\bar{x}_{k|k-1} = \delta\vec{x}_k^* + \vec{\eta}_k \qquad \mathbb{E}[\vec{\eta}_k] = 0 \qquad \mathbb{E}[\vec{\eta}_k\vec{\eta}_k^T] = \bar{P}_{k|k-1}$$

$$\delta\bar{x}_{k-1|k-1} = \delta\vec{x}_{k-1}^* + \vec{\eta}_{k-1} \qquad \mathbb{E}[\vec{\eta}_{k-1}] = 0 \qquad \mathbb{E}[\vec{\eta}_{k-1}\vec{\eta}_{k-1}^T] = \bar{P}_{k|k-1}$$

$$\mathbb{E}[\vec{\eta}_k\vec{\eta}_{k-1}^T] = \Phi_{xx}\bar{P}_{k-1|k-1}$$

$$\delta\vec{y}_k = \tilde{H}_k\delta\vec{x}_k^* + \vec{\epsilon}_k \qquad \mathbb{E}[\vec{\epsilon}_k] = 0 \qquad \mathbb{E}[\vec{\epsilon}_k\vec{\epsilon}_k^T] = R_k$$

Expectations:

$$\mathbb{E}[\delta\hat{x}_{k-1|k}] = \mathbb{E}[\delta\bar{x}_{k-1|k-1}] + L_{k-1}\left(\mathbb{E}(\delta\vec{y}_k - \tilde{H}_k\delta\bar{x}_{k|k-1})\right)$$

$$= \delta\vec{x}_{k-1}^* + L_{k-1}\left(\tilde{H}_k\delta\vec{x}_k^* - \tilde{H}_k\delta\vec{x}_k^*\right)$$

$$= \delta\vec{x}_{k-1}^* \qquad \underline{\underline{\text{unbiased}}}$$

$$\mathbb{E}[\delta\hat{x}_{k|k}] = \delta\vec{x}_k^*$$

$$\mathbb{E}[\delta\hat{p}_{k-1|k}] = 0$$

Uncertainties:

$$\hat{P}_{k-1|k} = \mathbb{E}\left[\left(\delta\hat{x}_{k-1|k} - \delta\vec{x}^{*}_{k-1}\right)\left(\delta\hat{x}_{k-1|k} - \delta\vec{x}^{*}_{k-1}\right)^{T}\right]$$

$$= \mathbb{E}\left[\left(\vec{\eta}_{k-1} + L_{k-1}\left(\vec{\epsilon}_{k} - \hat{H}_{k}\vec{\eta}_{k}\right)\right)\left(\vec{\eta}_{k-1} + L_{k-1}\left(\vec{\epsilon}_{k} - \hat{H}_{k}\vec{\eta}_{k}\right)\right)^{T}\right]$$

$$= \mathbb{E}\left[\vec{\eta}_{k-1}\vec{\eta}_{k-1}^{T}\right] + L_{k-1}\mathbb{E}\left[\vec{\epsilon}_{k}\vec{\epsilon}_{k}^{T} + \hat{H}_{k}\vec{\eta}_{k}\vec{\eta}_{k}^{T}\hat{H}_{k}^{T}\right]L_{k-1}^{T}$$
$$- \mathbb{E}\left[\vec{\eta}^{*}_{k-1}\vec{\eta}_{k}^{T}\right]\hat{H}_{k}^{T}L_{k-1}^{T} - L_{k-1}\hat{H}_{k}\mathbb{E}\left[\vec{\eta}_{k}\vec{\eta}_{k-1}^{T}\right]$$

$$= \bar{P}_{k-1|k-1} + L_{k-1}\left(R_{k} + \hat{H}_{k}\bar{P}_{k|k-1}\hat{H}_{k}^{T}\right)L_{k-1}^{T}$$
$$- \bar{P}_{k-1|k-1}\Phi_{xx}^{T}\hat{H}_{k}^{T}L_{k-1}^{T} - L_{k-1}\hat{H}_{k}\Phi_{xx}\bar{P}_{k-1|k-1}$$

$$= \bar{P}_{k-1|k-1} + L_{k-1}\left(R_{k} + \hat{H}_{k}\bar{P}_{k|k-1}\hat{H}_{k}^{T}\right)L_{k-1}^{T}$$
$$- \bar{P}_{k-1|k-1}\Phi_{xx}^{T}\hat{H}_{k}^{T}\left(R_{k} + \hat{H}_{k}\bar{P}_{k|k-1}\hat{H}_{k}^{T}\right)^{-1}\left(R_{k} + \hat{H}_{k}\bar{P}_{k|k-1}\hat{H}_{k}^{T}\right)L_{k-1}^{T}$$
$$- L_{k-1}\left(R_{k} + \hat{H}_{k}\bar{P}_{k|k-1}\hat{H}_{k}^{T}\right)\left(R_{k} + \hat{H}_{k}\bar{P}_{k|k-1}\hat{H}_{k}^{T}\right)^{-1}\hat{H}_{k}\Phi_{xx}\bar{P}_{k-1|k-1}$$

$$\boxed{\begin{aligned}\hat{P}_{k-1|k} &= \bar{P}_{k-1|k-1} - L_{k-1}\left(R_{k} + \hat{H}_{k}\bar{P}_{k|k-1}\hat{H}_{k}^{T}\right)L_{k-1}^{T} \\ \hat{P}_{k|k} &= \left(I - L_{k}\hat{H}_{k}\right)\bar{P}_{k|k-1}\left(I - L_{k}\hat{H}_{k}\right)^{T} + L_{k}R_{k}L_{k}^{T}\end{aligned}}$$

Equivalence to Kalman:

- $\bar{P}_{k|k-1}$ → same as Process Noise
- L_k → same a Kalman Gain
- $\delta\hat{x}_{k|k}$ → Equivalent to Kalman
- $\hat{P}_{k|k}$ → Equivalent to Kalman
- $\delta\hat{x}_{k-1|k}$ → Equivalent to Smoothed Kalman
- $\hat{P}_{k-1|k}$ → Equivalent to Smoothed Kalman

Summary: BL-OCBE is generalization of Kalman

- Equivalent estimate & uncertainty
- Automatic smoother [Additional full smoothing Algorithm]
- Automatic dynamic uncertainty

Plus: $\hat{u}(t) = -\tilde{Q}(t) B^T(t) \Phi_{pp}(t, t_{k-1}) \delta\hat{p}_{k-1|k}$

Analyze for:
- maneuver detection
- Maneuver Characterization
- Maneuver reconstruction
- Natural Dynamics Estimation

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Maneuver Detection:

$$d_c = \frac{1}{2}\int_{t_{k-1}}^{t_k} \vec{u}^T(\tau)\,\tilde{Q}^{-1}(\tau)\,\vec{u}(\tau)\,d\tau$$

$$= \frac{1}{2}(\delta\vec{y}_k - \tilde{H}_k\delta\bar{x}_{k|k-1})^T L_{k-1}^T \bar{P}_{k-1|k-1}^{-1}\left[\int_{t_{k-1}}^{t_k} \Phi_{pp}^T(\tau,t_{k-1})\,B(\tau)\,\tilde{Q}(\tau)\,B^T(\tau)\,\Phi_{pp}(\tau,t_{k-1})\,d\tau\right]$$

$$\times \bar{P}_{k-1|k-1}^{-1} L_{k-1}(\delta\vec{y}_k - \tilde{H}_k\delta\bar{x}_{k|k-1})$$

$$\Phi_{pp}^T(\tau,t_{k-1}) = \Phi_{xx}^{-1}(\tau,t_{k-1}) = \Phi_{xx}(t_{k-1},\tau) = \Phi_{xx}(t_{k-1},t_k)\,\Phi_{xx}(t_k,\tau)$$

$$= \Phi_{xx}^{-1}(t_k,t_{k-1})\,\Phi_{xx}(t_k,\tau) = \Phi_{pp}^T(t_k,t_{k-1})\,\Phi_{xx}(t_k,\tau)$$

$$d_c = (\delta\vec{y}_k - \tilde{H}_k\delta\bar{x}_{k|k-1})^T D_k (\delta\vec{y}_k - \tilde{H}_k\delta\bar{x}_{k|k-1})$$

$$D_k = \frac{1}{2}L_{k-1}^T\bar{P}_{k-1|k-1}^{-1}\Phi_{pp}^T\underbrace{\left[\int_{t_{k-1}}^{t_k}\Phi_{xx}(t_k,\tau)\,B(\tau)\,\tilde{Q}(\tau)\,B^T(\tau)\,\Phi_{xx}^T(t_k,\tau)\,d\tau\right]}_{-\Phi_{xp}\Phi_{xx}^T}\Phi_{pp}\bar{P}_{k-1|k-1}^{-1}L_{k-1}$$

$$= -\frac{1}{2}L_{k-1}^T\bar{P}_{k-1|k-1}^{-1}\Phi_{pp}^T\Phi_{xp}\Phi_{xx}^T\Phi_{pp}\bar{P}_{k-1|k-1}^{-1}L_{k-1}$$

$$D_k = -\frac{1}{2}L_{k-1}^T\bar{P}_{k-1|k-1}^{-1}\Phi_{pp}^T\Phi_{xp}\bar{P}_{k-1}^{-1}L_{k-1}$$

Mean:

$$\mu_{d_c} = \mathbb{E}[d_c] = \text{Trace}\left[D_k\left(R_k + \hat{H}_k \bar{P}_{k|k-1} \hat{H}_k^T\right)\right]$$

$$= \underbrace{\text{Trace}\left[D_k R_k\right]}_{\text{Measurement}} + \underbrace{\text{Trace}\left[D_k \hat{H}_k \bar{P}_{k|k-1} \hat{H}_k^T\right]}_{\text{State \& Dynamic}}$$

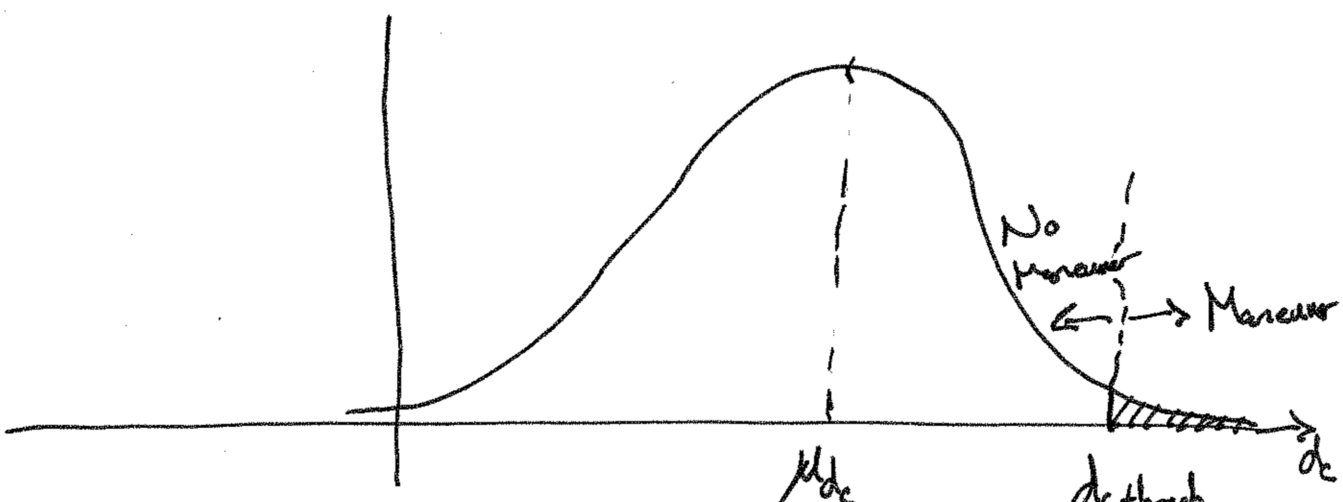
Variance:

$$\sigma_{d_c}^2 = \mathbb{E}\left[(d_c - \mu_{d_c})^2\right]$$

$$= 2\,\text{Trace}\left[D_k\left(R_k + \hat{H}_k \bar{P}_{k|k-1} \hat{H}_k^T\right) D_k\left(R_k + \hat{H}_k \bar{P}_{k|k-1} \hat{H}_k^T\right)\right].$$

Assume Gaussian Distribution:

Threshold @ $z^* = \dfrac{d_{c,thresh} - \mu_{d_c}}{\sigma_{d_c}}$



- Adjust $\tilde{Q}(t)$ until no maneuver is detected.

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- Reconstruct with $\hat{U}(t)$ when proper $\tilde{Q}(t)$ chosen
- Characterize with $Q(t)$ → tells order of mag of magnitude
  & d_c → similar to $|\Delta\vec{v}|$
- Natural Dynamics Estimation - Lubey & Scheeres
  JGCD, Vol. 37, N. 5
  (show Results if time allows)

Summary:

1) More than one way to look at Optimal Control
   - Control, Estimation, Dynamics Estimation, Object Correlation
2) Control Distance Metric
   - Object Correlation: Are A & B obs of the same object?
   - Maneuver Detection: Are there unknown dynamics connecting A & B?
   - Stochastic Process
3) Optimal Control Based Estimator
   - Generalization of Kalman: Auto Process Noise, Auto Smoother
     Equiv to Kalman @ t_e
   - Extra Information: $\hat{U}(t)$
     - Man Detect/Characterization/Reconstruction
     - Nat Dyn Est
   - Nonlinear form allows for more accurate estimation