Optimal Transfers + Plane Changes

3-Impulse def. untrue

Basic Result

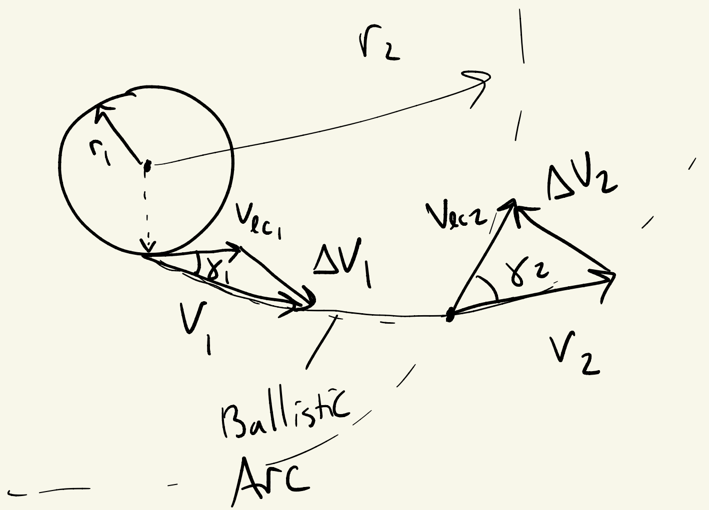
" Hohmann Transfer: Out of all the 2-Impulse Transfers to travel between 2 circular orbits, the H. T. is ΔV optimal." Not true .... if you account for "drag"

also not true.

Fuel cost $\equiv$ "ΔV" cost. For Impulsive maneuvers

$$J = \Delta V_1 + \Delta V_2 = c \ln\left(\frac{m_0}{m_0 - m_1}\right) + c \ln\left(\frac{m_0 - m_1}{m_0 - m_1 - m_2}\right)$$

$$= c \ln\left(\frac{m_0}{m_0 - m_1 - m_2}\right)$$



$$\boxed{\Delta V_1^2 = V_1^2 + V_{ec_1}^2 - 2V_1 V_{ec_1} \cos\gamma_1}$$

$$(\underline{V_1}, \underline{\gamma_1})$$

$$E_1 = \frac{1}{2}V_1^2 - \frac{\mu}{r_1} = \frac{1}{2}V_2^2 - \frac{\mu}{r_2} \Rightarrow$$

$$\boxed{V_2^2 = V_1^2 + 2\mu\left[\frac{1}{r_2} - \frac{1}{r_1}\right]}$$

$$H_1 = r_1 V_1 \cos\gamma_1 = r_2 V_2 \cos\gamma_2 \rightarrow$$

$$\boxed{\cos(\gamma_2) = \frac{r_1 V_1 \cos\gamma_1}{r_2 V_2}}$$

$$\Delta V_2^2 = V_2^2 + V_{ec_2}^2 - 2V_2 V_{ec_2} \cos\gamma_2$$

$$= V_1^2 + 2\mu\left[\frac{1}{r_2} - \frac{1}{r_1}\right] + \frac{\mu}{r_2} - 2\sqrt{\frac{\mu}{r_2}}\, \frac{r_1 V_1}{r_2} \cos\gamma_1$$

$$\boxed{\begin{aligned}\Delta V_1^2 &= V_1^2 + \frac{\mu}{r_1} - 2\sqrt{\frac{\mu}{r_1}}\, V_1 \cos\gamma_1 \\ \Delta V_2^2 &= \frac{3\mu}{r_2} - \frac{2\mu}{r_1} + V_1^2 - 2\sqrt{\frac{\mu}{r_2^3}}\, r_1 V_1 \cos\gamma_1\end{aligned}}$$

$$\boxed{J = \sqrt{\Delta V_1^2} + \sqrt{\Delta V_2^2}}$$

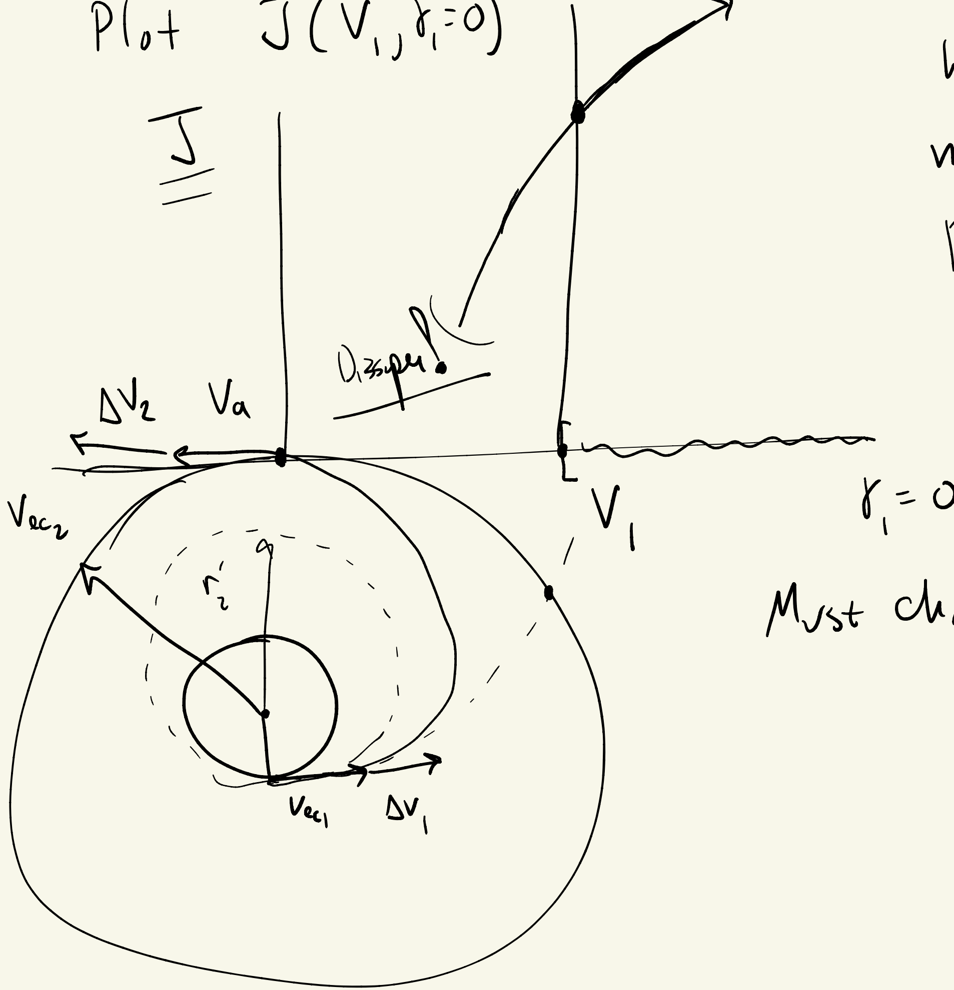
$$J(V_1, \gamma_1) \Rightarrow \boxed{\frac{\partial J}{\partial \gamma_1} = 0 \qquad \frac{\partial J}{\partial V_1} = 0}$$

$$\frac{\partial J}{\partial \gamma_1} = \sqrt{\frac{\mu}{r_1}}\, V_1 \left[\frac{1}{\Delta V_1} + \frac{1}{\Delta V_2}\left(\frac{r_1}{r_2}\right)^{3/2}\right] \sin\gamma_1 = 0 \Rightarrow \boxed{\gamma_1 = 0, \pi} \Rightarrow \boxed{\gamma_1 = 0}$$

$$\frac{\partial J}{\partial V_1} = \frac{1}{2\Delta V_1}\left[2V_1 - 2\sqrt{\frac{\mu}{r_1}}\cos\gamma_1\right] + \frac{1}{2\Delta V_2}\left[2V_1 - 2\sqrt{\frac{\mu}{r_2^3}}\, r_1 \cos\gamma_1\right]\Bigg|_{\gamma_1 = 0}$$

$\Rightarrow$ No valid solution!

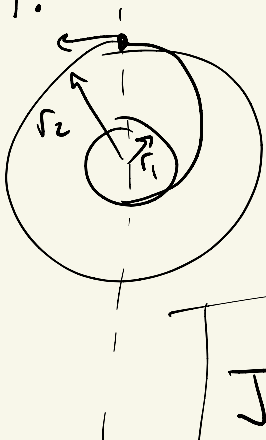
Plot $J(V_1, r_1 = 0)$



What's up? The real solution must satisfy a B.V.P. Boundary Value Problem instead of just "brute force" minimization.

Must choose V_1 s.t. $r = r_2$

H.T.



$r_2 \geq r_1$

$$\boxed{\begin{aligned}\Delta V_1 &= \sqrt{\frac{2\mu}{r_1+r_2}\frac{r_2}{r_1}} - \sqrt{\frac{\mu}{r_1}} \\ \Delta V_2 &= \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu}{r_1+r_2}\frac{r_1}{r_2}}\end{aligned}}$$

$(r_2 \geq r_1)$ $\Delta V_1 \geq 0$

≥ 0

$\gamma_1 = \gamma_2 = 0$

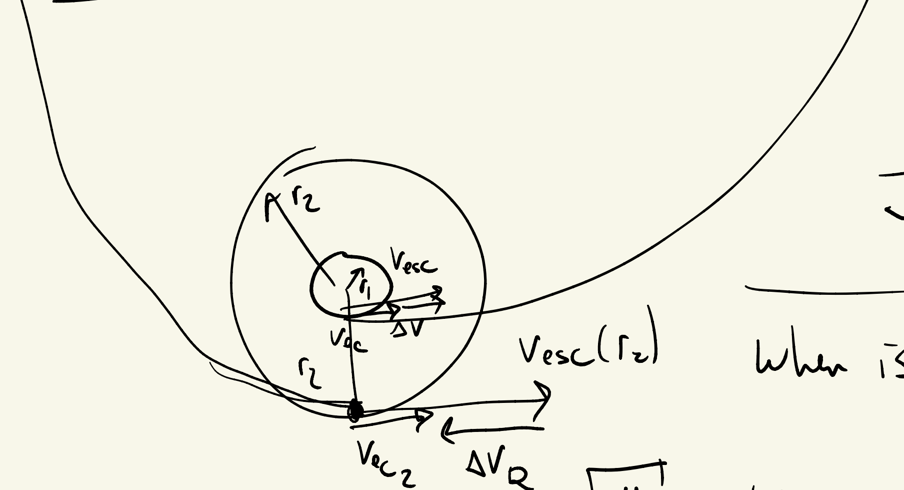
$$\boxed{J_H(r_1, r_2) = \sqrt{\frac{2\mu}{r_1+r_2}\frac{r_2}{r_1}} - \sqrt{\frac{\mu}{r_1}} + \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{2\mu}{r_1+r_2}\frac{r_1}{r_2}}}$$

Is this always optimal? No

IF r_2/r_1 is large enough, other options exist.

3-Impulse, Parabolic Transfer, Aero-assist

Parabolic Transfer $\quad r_2 \gg r_1$



$$J_p = \overbrace{\sqrt{\frac{2\mu}{r_1}} - \sqrt{\frac{\mu}{r_1}}}^{\Delta V_1} + \overbrace{\sqrt{\frac{2\mu}{r_2}} - \sqrt{\frac{\mu}{r_2}}}^{\Delta V_2} \geq 0$$

When is $\boxed{J_p \leq J_H}$

Scale both the costs by $\sqrt{\frac{\mu}{r_1}}$, define ratio $\boxed{\mathcal{R} = \frac{r_2}{r_1} \geq 1}$

$$\boxed{\tilde{J}_H(\mathcal{R}) = \sqrt{\frac{2\mathcal{R}}{1+\mathcal{R}}} - 1 + \frac{1}{\sqrt{\mathcal{R}}} - \sqrt{\frac{2}{\mathcal{R}(1+\mathcal{R})}}}$$

$$\boxed{\tilde{J}_p(\mathcal{R}) = \sqrt{2} - 1 + \frac{1}{\sqrt{\mathcal{R}}}\left(\sqrt{2}-1\right)}$$

When is $\tilde{J}_p \leq \tilde{J}_H$?

$$(\sqrt{2}-1)\left[1+\frac{1}{\sqrt{r}}\right] \leq \sqrt{\frac{2}{r(1+r)}}\,[r-1] - \frac{1}{\sqrt{r}}(\sqrt{r}-1) \qquad r \geq 1$$

What conditions are placed on r.

1. Mult. by $\sqrt{r}$

$$(\sqrt{2}-1)\left[\sqrt{r}+1\right] \leq \sqrt{\frac{2}{1+r}}\,(r-1) - (\sqrt{r}-1)$$

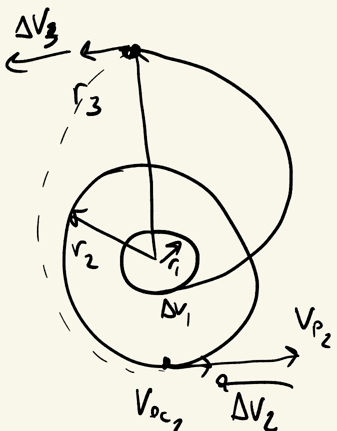
$$\sqrt{r}-(\sqrt{2}-1) \leq \frac{r-1}{\sqrt{r+1}} \Rightarrow \sqrt{r} > \sqrt{2}-1 \text{ as } r \geq 1$$

2. Square both sides $\Rightarrow$ $r^3-(7+4\sqrt{2})r^2+(3+4\sqrt{2})r-1 \geq 0$

Square again . . . . $r = 0.1466\ldots, 0.5715\ldots, 11.9388$

$$\boxed{J_P \le J_H \quad \text{when} \quad \eta \geq 11.9388\ldots}$$

Bi-Parabolic transfer a limiting version of the $\underline{\text{Bi-Elliptic Transfer}}$



$$J_{BE} = \underbrace{\sqrt{\frac{2\mu}{(r_1+r_3)}\frac{r_3}{r_1}} - \sqrt{\frac{\mu}{r_1}}}_{\Delta V_1} + \underbrace{\sqrt{\frac{2\mu}{r_2+r_3}\frac{r_2}{r_3}} - \sqrt{\frac{2\mu}{r_1+r_3}\frac{r_1}{r_3}}}_{\Delta V_3}$$

$$+ \underbrace{\sqrt{\frac{2\mu\, r_3}{(r_3+r_2)\, r_2}} - \sqrt{\frac{\mu}{r_2}}}_{\Delta V_2}$$

Let $\eta = \frac{r_2}{r_1}$

$\ell = \frac{r_3}{r_1} \geq \eta$

$$\boxed{\tilde{J}_{BE}(\eta, \ell) = \sqrt{\frac{2\ell}{1+\ell}} - 1 + \sqrt{\frac{2\eta}{\ell(\ell+\eta)}} - \sqrt{\frac{2}{\ell(\ell+1)}} + \sqrt{\frac{2\ell}{\eta(\ell+\eta)}} - \frac{1}{\sqrt{\eta}}}$$

$$\boxed{\lim_{\ell\to\infty} \tilde{J}_{BE}(\eta,\ell) = \sqrt{2} - 1 + \frac{1}{\sqrt{\eta}}\left(\sqrt{2}-1\right) = \tilde{J}_P} \quad \boxed{\tilde{J}_{BE}(\eta,\eta) = \tilde{J}_H}$$

When is $\boxed{\tilde{J}_{BE}(r, \ell) \leq \tilde{J}_H(r)}$; Recall that $J_{BE}(r, r) = J_H(r)$

When might a small increase in ℓ give a lower cost?

$$\tilde{J}_{BE}(r, r(1+\epsilon)) \leq \tilde{J}_H(r) \qquad \ell = r(1+\epsilon) \; ; \text{ if } \epsilon << 1$$

we can expand the $\tilde{J}_{BE}(r, r(1+\epsilon)) = \tilde{J}_{BE}\Big|_{\epsilon=0} + \frac{d\tilde{J}_{BE}}{d\epsilon}\Big|_{\epsilon=0} \cdot \epsilon + \cdots\cdots$

Use binomial Thm:

$$(1+\epsilon)^{\pm n} = 1 \pm n\epsilon + \mathcal{O}(\epsilon^2) \; ; \; \sqrt{1+\epsilon} = (1+\epsilon)^{1/2} = 1 + \frac{1}{2}\epsilon + \mathcal{O}(\epsilon^2)$$

$\Downarrow$ Expand

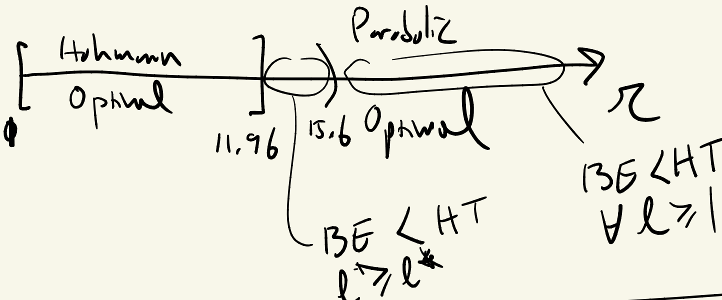
$$\left(r^3 - 15r^2 - 9r - 1 \geq 0\right) \Rightarrow \boxed{r \geq 15.5817\ldots}$$

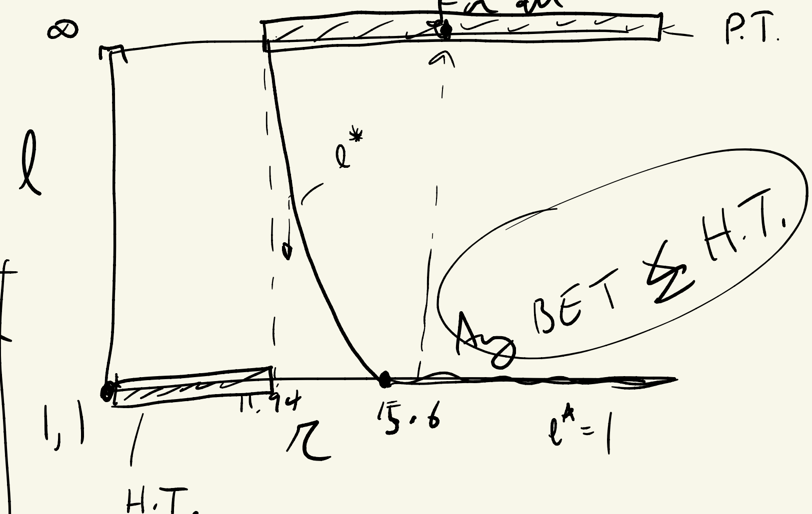
Note: $\boxed{\tilde{J}_{BE}(r, l) > \tilde{J}_p(r)}$

If $r \geq 11.94$ $\quad \tilde{J}_p \leq \tilde{J}_H$

$r \geq 15.6$ $\quad \tilde{J}_{BE} \leq \tilde{J}_H$ for any $l \geq r$

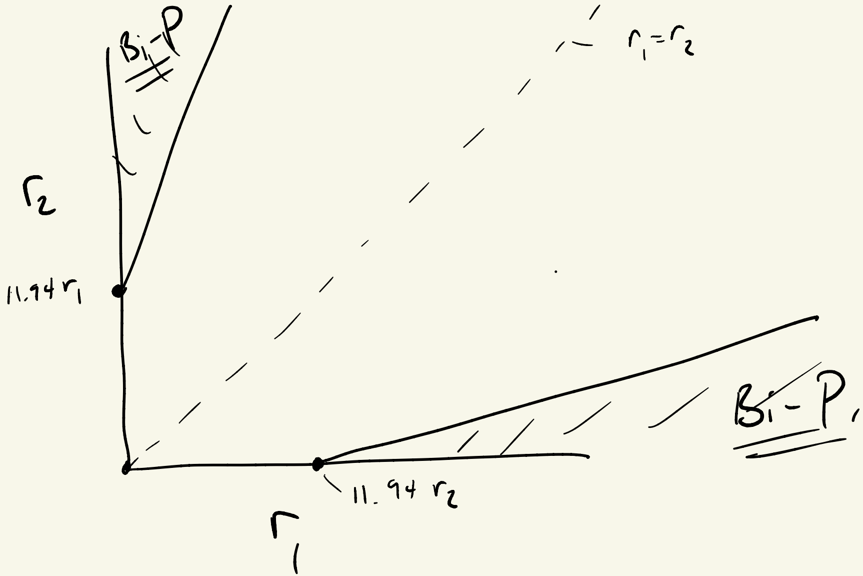
When $11.94 \leq r \leq 15.6$ there exists an $l^*(r)$ s.t. $\tilde{J}_{BE}(r, l^*) \leq \tilde{J}_H(r)$ for all $l \geq l^*$





Global Optimal results is not close to the H.T.

$$\frac{\partial J_{BE}}{\partial l} < 0 \quad \text{when } l \geq l^*$$



Bi-P
$r_1 = r_2$
r_2
$11.94\, r_1$
Bi-P
$11.94\, r_2$
r_1