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ASEN 5044
Fall 2024
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HW 5

Problem 1 → Simon 2.10

$$f_{XY}(x,y) = \begin{cases} a e^{-2x} e^{-3y} & x>0,\ y>0 \\ 0 & \text{O'wise} \end{cases}$$

$\lim_{t\to\infty} e^{-t} = 0$

a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a e^{-2x} e^{-3y}\,dx\,dy = 1 = \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2x-3y}\,dx\,dy\right] a$

$= a\int_0^\infty\int_0^\infty e^{-2x-3y}\,dx\,dy \quad [\because f_{xy}(x,y)=0 \text{ when } x\le 0,\ y\le 0]$

$= a\int_0^\infty -\frac{1}{2}(e^{-2x-3y})\Big|_0^\infty dy = a\int_0^\infty +\frac{1}{2}e^{-3y}\,dy = +\frac{a}{2}\left[-\frac{1}{3}e^{-3y}\right]\Big|_0^\infty$

$= +\frac{a}{2}\left[\frac{1}{3}\right] = \frac{a}{6} = 1 \to \boxed{a=6}$

b) $\bar{x} = E[x] = 6\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\cdot e^{-2x-3y}\,dx\,dy = 6\int_0^\infty\int_0^\infty x e^{-2x-3y}\,dx\,dy = 6\int_0^\infty e^{-3y}\int_0^\infty x e^{-2x}\,dx\,dy$

$f=x,\ g'=e^{-2x}dx \to f'=dx,\ g=-\frac{e^{-2x}}{2} \to \quad -\frac{xe^{-2x}}{2}\Big|_0^\infty - \int_0^\infty -\frac{e^{-2x}}{2}dx = \cancel{-\frac{xe^{-2x}}{2}}\Big|_0^{\infty} - \left[+\frac{e^{-2x}}{4}\Big|_0^\infty\right]$

$\bar{x} = 6\int_0^\infty e^{-3y}\left[\frac{1}{4}\right]dy = \frac{6}{4}\left[-\frac{e^{-3y}}{3}\Big|_0^\infty\right] = \boxed{\frac{1}{2} = \bar{x}}$

$\bar{y} = E[y] = 6\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y e^{-2x-3y}\,dx\,dy = 6\int_0^\infty y e^{-3y}\int_0^\infty e^{-2x}\,dx\,dy = 6\int_0^\infty y e^{-3y}\left[-\frac{e^{-2x}}{2}\Big|_0^\infty\right]dx\,dy$

$= 6\int_0^\infty y e^{-3y}\left(\frac{1}{2}\right)dy \to f=y,\ g'=e^{-3y}dy \to f'=dy,\ g=-\frac{e^{-3y}}{3}$

$\bar{y} = 3\left[-\cancel{\frac{ye^{-3y}}{3}}\Big|_0^\infty - \int_0^\infty -\frac{e^{-3y}}{3}dy\right] = 3\left[-\frac{e^{-3y}}{9}\Big|_0^\infty\right] = 3\cdot\frac{1}{9} = \boxed{1/3 = \bar{y}}$

c) $E[x^2] = 6\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 e^{-2x-3y}\,dx\,dy = 6\int_0^\infty e^{-3y}\int_0^\infty x^2 e^{-2x}\,dx\,dy$

from Handy Integrals [on canvas] $\to \int_0^\infty x^n e^{-ax}dx = \frac{n!}{a^{n+1}}$ if n +ve int, $a>0$

$\therefore E[x^2] = 6\int_0^\infty e^{-3y}\left(\frac{2!}{2^3}\right)dy = 6\left(\frac{1}{4}\right)\left[\frac{1}{3}\right] = \boxed{\frac{1}{2} = E[x^2]}$

$E[y^2] = 6\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 e^{-2x-3y}\,dx\,dy = 6\int_0^\infty y^2 e^{-3y}\int_0^\infty e^{-2x}\,dx\,dy = 6\int_0^\infty y^2 e^{-3y}\left(\frac{1}{2}\right)dy$

$= 3\left(\frac{2!}{3^3}\right) = \boxed{\frac{2}{9} = E[y^2]}$

$E[XY] = 6\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^{-2x-3y})xy\,dx\,dy = 6\int_0^\infty y e^{-3x}\int_0^\infty x e^{-2x}\,dx\,dy = 6\int_0^\infty y e^{-3x}\left(\frac{1}{4}\right)dy$

$= 6\left(\frac{1}{9}\right)\left(\frac{1}{4}\right) = \boxed{\frac{1}{6} = E[XY]}$

d) $[x\ y]^T \to R_{X,Y} = E\left[\begin{bmatrix}x\\y\end{bmatrix}[x\ y]\right] = \begin{bmatrix} E[x^2] & E[xy] \\ E[yx] & E[y^2]\end{bmatrix}$

From part c $\to E[x^2]=\frac{1}{2},\ E[y^2]=\frac{2}{9},\ E[XY]=E[YX]=\frac{1}{6} \to \boxed{R_{X,Y} = \begin{bmatrix} 1/2 & 1/6 \\ 1/6 & 2/9\end{bmatrix}}$

e) $\sigma_x^2 = \mathrm{var}(x) = E[(x-\bar{x})^2] = E[(x-\tfrac{1}{2})^2] = E[x^2] - E[x]^2 = \frac{1}{2} - \frac{1}{4} = \boxed{1/4 = \sigma_x^2}$

$\sigma_y^2 = \mathrm{var}(y) = E[(y-\bar{y})^2] = E[(y-\tfrac{1}{3})^2] = E[y^2] - E[y]^2 = \frac{2}{9} - \frac{1}{9} = \boxed{1/9 = \sigma_y^2}$

$\mathrm{cov}(x,y) = E[xy] - \bar{x}\bar{y} = \frac{1}{6} - \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \boxed{0 = \mathrm{cov}(x,y)}$

f) $C_{xy} = E\left[\begin{bmatrix} x-\bar{x} \\ y-\bar{y} \end{bmatrix}[(x-\bar{x}), (y-\bar{y})]\right]$

$= \begin{bmatrix} E[(x-\bar{x})^2] & E[(x-\bar{x})(y-\bar{y})] \\ E[(x-\bar{x})(y-\bar{y})] & E[(y-\bar{y})^2] \end{bmatrix}$

$cov_{x,y} = E[(x-\bar{x})(y-\bar{y})]$
$= E[(y-\bar{y})(x-\bar{x})]$

From Part e $\rightarrow$ $\boxed{C_{xy} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/9 \end{bmatrix}}$

From Part e $\rightarrow$ g) $corr(x,y) = \dfrac{cov(x,y)}{\sqrt{var(x)} \cdot \sqrt{var(y)}} = \dfrac{0}{\sqrt{\frac{1}{4}} \cdot \sqrt{\frac{1}{9}}} = \boxed{0 = corr(x,y)}$

Problem 2 $\rightarrow$ $y = cx + de$, $x \sim N(\bar{x}, \sigma_x^2)$, $e \sim N(0, \sigma_e^2)$, $x \perp\!\!\!\perp e$, $E[y] = c\bar{x}$

a) $cov(x,y) = E[(x-\bar{x})(y-\bar{y})] = E[(x-\bar{x})(cx+de-\bar{y})]$

$= E[x(cx+de) - x\bar{y} - \bar{x}(cx+de) + \bar{x}\bar{y}]$

$= E[cx^2 + dex - cx\bar{x} - cx\bar{x} - de\bar{x} + c\bar{x}^2]$

$= E[c(x^2 - 2x\bar{x} + \bar{x}^2) + dex - de\bar{x}] = E[c(x-\bar{x})^2 + \cancel{dex}^{0} - \cancel{de\bar{x}}^{0}]$

$= cE[(x-\bar{x})^2] = c[E[x^2] - \bar{x}^2]$

From lecture 15, slide 17 $\rightarrow$ $E[xy] = cE[x^2]$

$cov(x,y) = cE[x^2] - \bar{x} \cdot c\bar{x} = \boxed{E[xy] - \bar{x}\bar{y} = cov(x,y)}$

b) $var(y) = E[(y-\bar{y})^2] = E[y^2 - 2y\bar{y} + \bar{y}^2] = E[(cx+de)^2 - 2(cx+de)c\bar{x} + c^2\bar{x}^2]$

$= E[c^2x^2 + 2cxde + d^2e^2 - 2c^2x\bar{x} - 2cde\bar{x} + c^2\bar{x}^2]$

$= E[c^2(x^2 - 2x\bar{x} + \bar{x}^2) + 2cde(x-\bar{x}) + d^2e^2]$

$= E[c^2(x-\bar{x})^2] + E[2cde(x-\bar{x})] + E[d^2(e-\bar{e})^2]$ (where $\bar{e} = 0$)

$\because x \perp\!\!\!\perp e$ $= c^2\sigma_x^2 + 2cd\,\cancel{E[e]}^{0}\,E[x-\bar{x}] + d^2\sigma_e^2 = \boxed{c^2\sigma_x^2 + d^2\sigma_e^2 = var(y)}$

Problem 3 $\rightarrow$ $y = \ln(x)$, $x > 0 \rightarrow g(x) = \ln(x) \rightarrow y = \ln(x)$

$h(y) = e^y \rightarrow h'(y) = e^y \rightarrow |h'(y)| = |e^y|$

a) $P(x) = U[a,b] = \frac{1}{b-a}$ [when $a \le x \le b$]

$P_y(y) = P_x(e^y) \cdot |e^y| \rightarrow a \le x \le b \rightarrow a \le e^y \le b \rightarrow \ln(a) \le y \le \ln(b)$

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$P_y(y) = \frac{1}{\ln(b)-\ln(a)} \cdot |e^y| = U[\ln(a), \ln(b)]|e^y|$ when $\ln(a) \le y \le \ln(b)$

$\boxed{P(y) = U[\ln(a), \ln(b)] \cdot |e^y|}$

b) $P(\frac{1}{x}) = U[c, d]$ — $0 < c \le x^{-1} \le d \rightarrow P(1/x) = 1/(d-c)$

$0 < c \le \frac{1}{x} \le d \rightarrow 0 < \frac{1}{d} \le x \le \frac{1}{c}$

$\therefore P(x) = U[\frac{1}{d}, \frac{1}{c}] \rightarrow P(x) = \begin{cases} \frac{1}{1/c - 1/d} & 0 < \frac{1}{d} \le x \le \frac{1}{c} \\ 0 & \text{o'wise} \end{cases}$

$P_y(y) = P_x(e^y)|e^y| \rightarrow \frac{1}{d} \le x \le \frac{1}{c} \rightarrow \frac{1}{d} \le e^y \le \frac{1}{c} \rightarrow \ln(\frac{1}{d}) \le y \le \ln(\frac{1}{c})$

$P_y(y) = \frac{1}{\ln(1/c) - \ln(1/d)} \cdot |e^y| = U[\ln(\frac{1}{d}), \ln(\frac{1}{c})]|e^y|$ when $\ln(\frac{1}{d}) \le y \le \ln(\frac{1}{c})$

$\boxed{P(y) = \begin{cases} \frac{|e^y|}{\ln(1/c) - \ln(1/d)} & \ln(1/d) \le y \le \ln(1/c) \\ 0 & \text{o'wise} \end{cases} = U[\ln(\frac{1}{d}), \ln(1/c)]|e^y|}$

c) $P(y) = U[l, m] = \frac{1}{m-l}$ when $l \le y \le m$

$y = \ln(x) \rightarrow e^y = x \rightarrow g(y) = e^y$, inverse of $g(y) \rightarrow h(x) = \ln(x)$

$h'(x) = \frac{1}{x} \rightarrow |h'(x)| = |\frac{1}{x}| = \frac{1}{|x|}$

$P_x(x) = P_y(\ln(x)) \cdot \frac{1}{|x|} \rightarrow l \le y \le m \rightarrow l \le \ln(x) \le m \rightarrow e^l \le x \le e^m$

$P_x(x) = \frac{1}{e^m - e^l} \cdot \frac{1}{|x|} = U[e^l, e^m] \cdot \frac{1}{|x|}$

$\boxed{P(x) = \begin{cases} \frac{1}{e^m - e^l} \cdot \frac{1}{|x|} & e^l \le x \le e^m \\ 0 & \text{o'wise} \end{cases} = U[e^l, e^m] \cdot \frac{1}{|x|}}$

d) $P(y) = \mathcal{N}(\mu_y, \sigma_y^2) = \frac{1}{\sqrt{2\pi}\,\sigma_y} \exp(-\frac{1}{2\sigma_y^2}(y-\mu_y)^2)$

$P_x(x) = P_y(\ln(x)) \cdot \frac{1}{|x|} = \boxed{\frac{1}{|x|} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_y} \exp\left[\frac{-1}{2\sigma_y^2}(\ln(x) - \mu_y)^2\right] = P(x)}$