ASEN 5044, Fall 2024

Statistical Estimation for Dynamical Systems

Lecture 05: LTI IVPs Wrap Up; Nonlinear State Space Systems and Linearization

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Announcements

- **Prof. Ahmed out of country next week (SPIE Defense & Security Conference in UK)**
    - **No live classes next Tues 09/17 or Thurs 09/19, BUT pre-recorded Lecture Videos to be posted – WATCH THEM!! (will need them for HW 2 and Quiz)**
    - **Also: TF Aidan will cover Prof. Ahmed's hours next Wed 09/18 4:30-6 pm, AERO N353**
- **HW 1 Due Fri 09/13 at 11:59 pm**
- **Submit to Gradescope (linked via Canvas) –**
    - **All submissions must be legible!!! – zero credit otherwise**
- Advanced Questions: these are optional/extra credit (follow instructions)
- HW 2 will be posted Thurs 09/12, due Fri 09/16.
- Quiz 2 solutions to be posted later today
- Quiz 3: this Friday-Tuesday via Canvas
- **Office hours this week (in person + Zoom):**
    - **Prof. Ahmed: Wed, 4:30-6 pm, AERO N353**
    - **TF Aidan Bagley: Wed 12-1:30 pm, AERO N353**
    - **TF Jiho Lee: Tues 2:30-3:30 pm, AERO N253**
    - **TF Collin Hudson: Mon 1-2 pm, AERO 303**

Overview

Last time: Unforced LTI State Space IVP Solutions and the Matrix Exponential

$$\dot{x}(t) = A x(t), \; x(t_0) = x_0 \longrightarrow x(t) = \underline{\underline{e}}^{A(t-t_0)} \cdot x(t_0)$$

$$e^{A(t-t_0)} \triangleq \sum_{i=0}^{\infty} A^i \frac{(t-t_0)^i}{i!} = I + A(t-t_0) + A^2 \frac{(t-t_0)^2}{2!} + \dots$$

Today:

- Wrap up general LTI SS IVP solutions with forcing inputs
- Choice and Transformation of State Representations
- Nonlinear systems and "standard form" nonlinear state space models
- Linearization and transformation to linear SS models

READ: Chapter 1.6-1.8 in Simon book

General Solution to Forced LTI Matrix-Vector IVPs

- Recall: General LTI state space model with inputs given by

$$\dot{x} = Ax(t) + Bu(t), \qquad x(t_0) = x_0, \quad u(t) \neq 0 \text{ for } t \geq 0$$

$$\text{where } \; x(t) \in \mathbb{R}^n, \; u(t) \in \mathbb{R}^m, \; A \in \mathbb{R}^{n \times n}, \; B \in \mathbb{R}^{n \times m}$$

- If non-zero u(t) and initial condition x(t_0), then general LTI solution is:

$$* \quad \boxed{x(t) = \underbrace{e^{A(t-t_0)} x(t_0)}_{\substack{\text{unforced response} \\ \text{("free response") due to IC's}}} + \underbrace{\int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) d\tau}_{\substack{\text{forced response due to input (0 IC's)} \\ \text{[convolution integral]}}}}$$

"lsim" in Matlab

(note: for LTV systems, general solution is: $x(t) = \underbrace{\Phi(t, t_0)}_{\text{STM}} x(t_0) + \int_{t_0}^{t} \underbrace{\Phi(t, \tau)}_{\text{STM}} B(\tau) u(\tau) d\tau$)

Choice and Transformation of State Representations

- The [A,B,C,D] matrices are not unique for given set of linear ODEs
- Infinitely many possible [A,B,C,D] -- governed by choice of state x
- These choices are all related by invertible similarity transformations
- Example for 1D mass system again: $x = \begin{bmatrix} v \\ d \end{bmatrix}$, $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\dot{x} = Ax$ (unforced)

→ Suppose we want to change basis to get new state vector $\tilde{x} = \begin{bmatrix} d \\ v \end{bmatrix}$

→ clearly, $\tilde{x} = Tx$ where $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ [invertible similarity transform]

→ So $\dot{\tilde{x}} = \frac{d}{dt}(Tx) \underset{\text{(product rule)}}{=} T\dot{x} + \cancel{\frac{d}{dt}(T)}^{\,0}\, x = T(\underbrace{Ax}_{\dot{x}}) \rightarrow \dot{\tilde{x}} = TAx$

But since $\tilde{x} = Tx \rightarrow x = T^{-1}\tilde{x} \longrightarrow \dot{\tilde{x}} = \underbrace{TAT^{-1}}_{\substack{\text{2x2 sq. matrix} \\ \rightarrow \text{ call it } \tilde{A}}}\tilde{x} = \tilde{A}\tilde{x} \Rightarrow \boxed{\dot{\tilde{x}} = \tilde{A}\tilde{x} \text{ where } \tilde{A} = TAT^{-1}}$

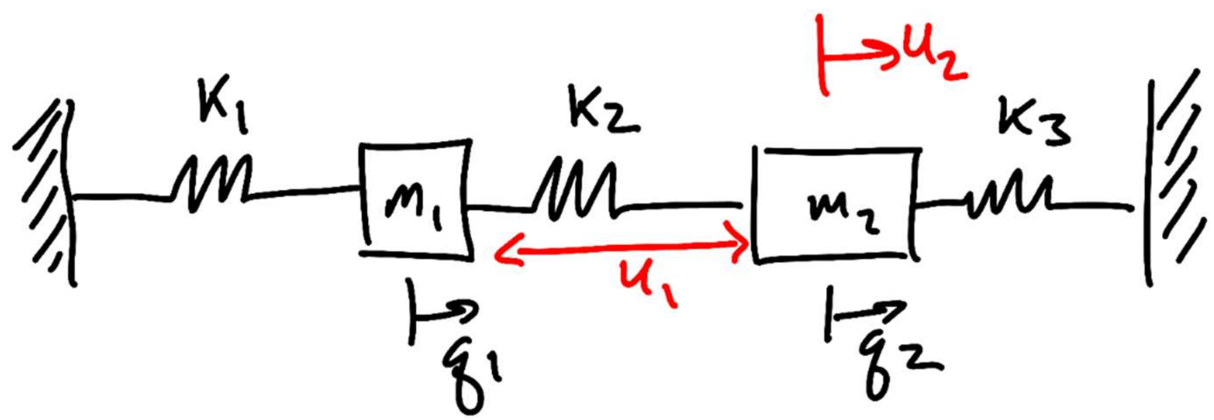
(holds for any similarity transform)

(*) Note: by similar reasoning: can be generalized to $B \rightarrow \tilde{B}$, $C \rightarrow \tilde{C}$, $D \rightarrow \tilde{D}$.

Also: note that y & u Do NOT Transform under T – only the internal state variables in x do!

Linear vs. Nonlinear System Models

- Linear dynamics/ODEs = good approx. for many physical laws, but not all!
- Example: 2 mass / 3 spring system
- Physical springs and actuators always have nonlinear behavior – but sometimes we can ignore these for *a priori*/first principles models in control/estimation



$$x = [q_1(t), \dot{q}_1(t), q_2(t), \dot{q}_2(t)]^T$$

$$u = [u_1(t), u_2(t)]^T$$

$$y = [q_1(t), q_2(t)]^T$$

u_1 = relative actuator;
u_2 = absolute actuator

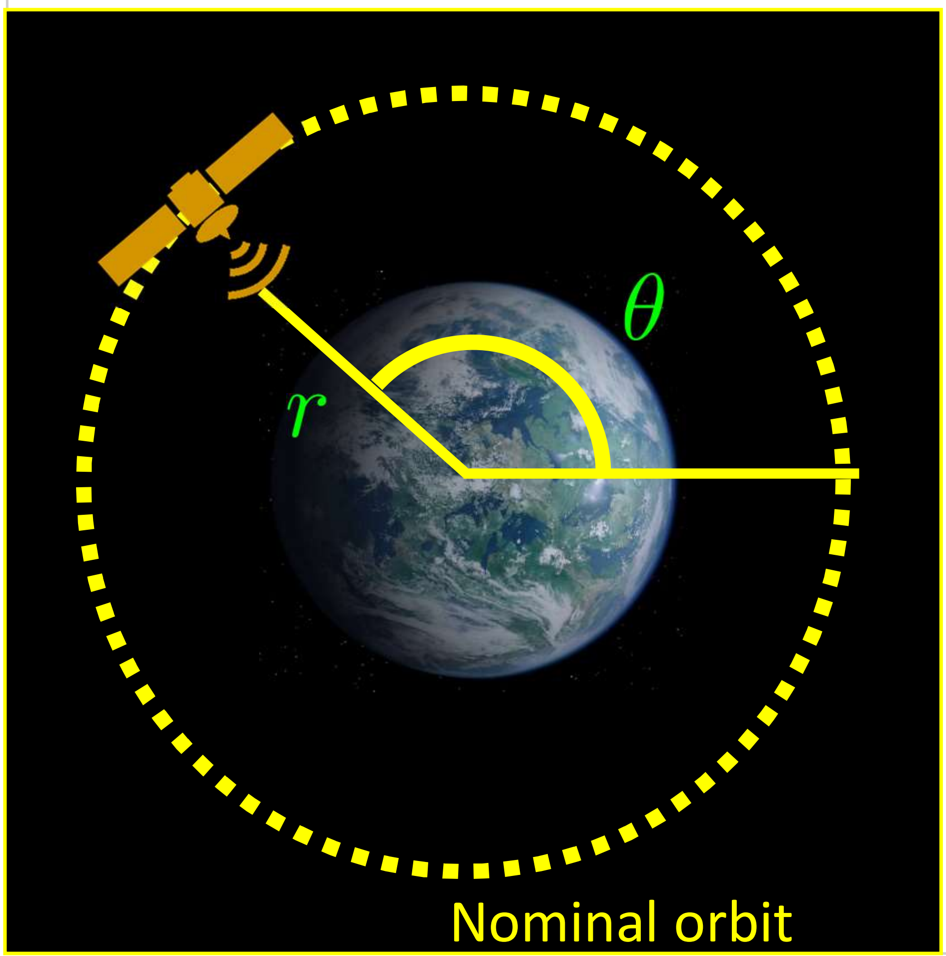
- For $k_1 = k_2 = k_3 = 1$ N/m and $m_1 = m_2 = 1$ kg, can use basic physics to get eqs. of motion and express as LTI SS model:

$$\dot{x} = Ax(t) + Bu(t) \qquad A = \begin{bmatrix} 0 & 1.0 & 0 & 0 \\ -2.0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 \\ 1.0 & 0 & -2.0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$y(t) = Cx(t) + Du(t) \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

What to do about Nonlinear ODEs?

- But most systems have intrinsically nonlinear effects that are not obviously/easily modeled by linear physical relationships
- **What if a priori/first principles give nonlinear (NL) dynamics?**
- Example: equation for orbit plane motion of satellite



$$\ddot{r} - \dot{\theta}^2 r = -\frac{\mu}{r^2} + a_r$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = a_i$$

$a_i,\ a_r$: in-track and radial accelerations
[due to thrust, drag, SRP, grav anomalies, etc.]

$r,\ \theta$: states for spacecraft

NL systems can get very nasty and weird…

- For now, only focus on NL sys with “sufficiently smooth” nonlinearities
  - i.e. derivatives exist for state vars and are bounded

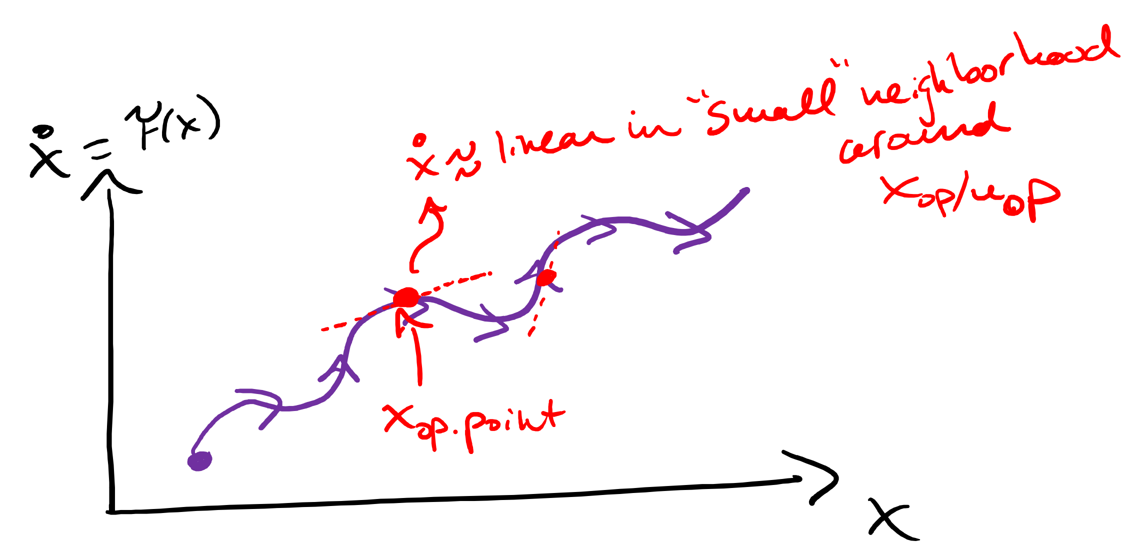
Lipschitz Continuous: for some function f(x), $||f(x_1) - f(x_2)|| \leq c \cdot ||x_1 - x_2||, \ \forall x_1, x_2$ “sufficiently close’
for some constant c (c: Lipschitz constant)

- Most cases: want to keep NL sys near operating point/condition
  - Equilibrium: set of x and u such that $\dot{x} = 0$

    (i.e. some $\{x_{eq}, u_{eq}\}$ exists s.t. $\dot{x} = \mathcal{F}(x,u)|_{x_{eq},u_{eq}} = 0$)

  - Nominal trajectory: known valid solutions $x_{nom}(t)$ and $u_{nom}(t)$ to the nonlinear vector ODE $\dot{x} = \mathcal{F}(x,u)$,

    i.e. such that $\dot{x} = \mathcal{F}(x,u)|_{x_{nom}(t),u_{nom}(t)}$



- Can look at dynamics of “small” perturbations near op pt
  - If perturbations small enough, system behaves (almost) linearly!

Linearization of NL ODEs via Multivariable Taylor Series

- Idea: express NL ODEs in (non-linear) standard state vector form → do Taylor expansion near operating point → drop higher order terms (HOTs) → re-arrange to linear SS model

Given set of nonlinear ODES, express in standard nonlinear state space form by picking state vars & stacking into a vector expressed w/ time derivatives as follows:

$$(*)\quad \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \tilde{f}_1(x,u,t) \\ \tilde{f}_2(x,u,t) \\ \vdots \\ \tilde{f}_n(x,u,t) \end{bmatrix} = \tilde{f}(x,u,t) \longrightarrow \text{Stack (vector) of } n \text{ NL ODES}$$

$$(*)\quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} h_1(x,u,t) \\ \vdots \\ h_p(x,u,t) \end{bmatrix} = h(x,u,t) \longrightarrow \text{Stack (vector) of } p \text{ NL algebraic equations}$$

Also, let $u = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix}$ (input vector).

Linearization (cont'd)

- Suppose a nominal solution or operating point is known/given, ie $x_{nom}(t)$ & $u_{nom}(t)$ given

Define slight perturbations from nom. op. point:

perturbation vectors $\begin{cases} \delta x(t) \triangleq x(t) - x_{nom}(t) \longrightarrow x(t) = x_{nom}(t) + \underline{\underline{\delta x(t)}} \\ \delta u(t) \triangleq u(t) - u_{nom}(t) \longrightarrow u(t) = u_{nom}(t) + \underline{\underline{\delta u(t)}} \end{cases}$

→ Plug in expressions for $x(t)$ & $u(t)$ into NL ODEs & do vector Taylor Series expansion:

$$\dot{x}(t) = \tilde{F}(x,u,t) \Longleftrightarrow \dot{x}(t) = \dot{x}_{nom}(t) + \delta\dot{x}(t) \Longleftrightarrow \tilde{F}(x,u,t) = \tilde{F}(x_{nom}+\delta x, u_{nom}+\delta u, t)$$

→ (vector) Taylor Series exp.: $\dot{x}(t) = \tilde{F}(x_{nom}+\delta x, u_{nom}+\delta u, t)$

$$= \tilde{F}(x_{nom}, u_{nom}, t) + \left[\frac{\partial \tilde{F}}{\partial x}\bigg|_{\substack{x_{nom}\\u_{nom}}}\right]\delta x + \left[\frac{\partial \tilde{F}}{\partial u}\bigg|_{\substack{x_{nom}\\u_{nom}}}\right]\delta u + \text{HOTs (higher order terms)}$$

Likewise: $y(t) = y_{nom} + \delta y = h(x_{nom}+\delta x, u_{nom}+\delta u, t)$

$$= h(x_{nom}, u_{nom}, t) + \left[\frac{\partial h}{\partial x}\bigg|_{nom}\right]\delta x + \frac{\partial h}{\partial u}\bigg|_{nom}\delta u + \text{HOTs}$$

Linearization (cont'd)

- Partial derivative matrices = **Jacobians** w.r.t. x and u

$$\left[\frac{\partial \mathcal{F}}{\partial x}\right]\Big|_{x_{nom},u_{nom}} = \begin{bmatrix} \frac{\partial \mathcal{F}_1}{\partial x_1} & \frac{\partial \mathcal{F}_1}{\partial x_2} & \cdots & \frac{\partial \mathcal{F}_1}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial \mathcal{F}_n}{\partial x_1} & \frac{\partial \mathcal{F}_n}{\partial x_2} & \cdots & \frac{\partial \mathcal{F}_n}{\partial x_n} \end{bmatrix}\Big|_{x_{nom},u_{nom}} \rightarrow \boxed{n \times n \text{ matrix}}$$

$$\left[\frac{\partial \mathcal{F}}{\partial u}\right]\Big|_{x_{nom},u_{nom}} = \begin{bmatrix} \frac{\partial \mathcal{F}_1}{\partial u_1} & \frac{\partial \mathcal{F}_1}{\partial u_2} & \cdots & \frac{\partial \mathcal{F}_1}{\partial u_m} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial \mathcal{F}_n}{\partial u_1} & \frac{\partial \mathcal{F}_n}{\partial u_2} & \cdots & \frac{\partial \mathcal{F}_n}{\partial u_m} \end{bmatrix}\Big|_{x_{nom},u_{nom}} \rightarrow \boxed{n \times m}$$

(similarly for $\underbrace{\frac{\partial h}{\partial x}\Big|_{nom}}_{[p \times n]}$ and $\underbrace{\frac{\partial h}{\partial u}\Big|_{nom}}_{[p \times m]}$)

For small enough δx, δu, can neglect HOTs

- Get linearized eqs **for dynamics of perturbations** δx, δy w.r.t. δu "near" nominal op point:

$$(i)\quad \dot{x}(t) = \cancel{\dot{x}_{nom}(t)} + \underbrace{\delta\dot{x}}_{\text{same!}} \approx \cancel{\mathcal{F}(x_{nom}, u_{nom}, t)} + \left.\frac{\partial \mathcal{F}}{\partial x}\right|_{nom}\delta x + \left.\frac{\partial \mathcal{F}}{\partial u}\right|_{nom}\delta u + \cancel{HOTS}$$

$$(ii)\quad y(t) = \cancel{y_{nom}(t)} + \underbrace{\delta y}_{\text{same!}} \approx \cancel{h(x_{nom}, u_{nom}, t)} + \left.\frac{\partial h}{\partial x}\right|_{nom}\delta x + \left.\frac{\partial h}{\partial u}\right|_{nom}\delta u + \cancel{HOTS}$$

→ left with

$$\boxed{\begin{aligned} \delta\dot{x} &= \left[\left.\frac{\partial \mathcal{F}}{\partial x}\right|_{nom}\right]\delta x + \left[\left.\frac{\partial \mathcal{F}}{\partial u}\right|_{nom}\right]\delta u \\ \delta y &= \left[\left.\frac{\partial h}{\partial x}\right|_{nom}\right]\delta x + \left[\left.\frac{\partial h}{\partial u}\right|_{nom}\right]\delta u \end{aligned}}$$

Perturbation Dynamics

valid only if δx & δu stay small!

Let

$\bar{x} = \delta x$

$\bar{y} = \delta y$

$\bar{u} = \delta u$

$\bar{A}|_{nom} = \left.\frac{\partial \mathcal{F}}{\partial x}\right|_{nom}$

$\bar{B}|_{nom} = \left.\frac{\partial \mathcal{F}}{\partial u}\right|_{nom}$

$\bar{C}|_{nom} = \left.\frac{\partial h}{\partial x}\right|_{nom}$

$\bar{D}|_{nom} = \left.\frac{\partial h}{\partial u}\right|_{nom}$

$$\boxed{\begin{aligned} \dot{\bar{x}} &= \bar{A}|_{nom}\bar{x} + \bar{B}|_{nom}\bar{u} \\ \bar{y} &= \bar{C}|_{nom}\bar{x} + \bar{D}|_{nom}\bar{u} \end{aligned}}$$

Looks like SS [A, B, C, D] matrices for linear system

But now depends on x_{nom} & u_{nom}!

Example (please read on your own!):
2nd order NL ODE (no input/unforced)

$$\ddot{z} + (1+z)\dot{z} - 2z + 0.5z^3 = 0 \;\; (\leftrightarrow \ddot{z} = 2z - 0.5z^3 - (1+z)\dot{z})$$

Step 1: define state variables and put into standard NL SS form:

$$\begin{aligned} x_1 &= z \\ x_2 &= \dot{z} \end{aligned} \quad \rightarrow \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \rightarrow \quad \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 2x_1 - 0.5x_1^3 - (1+x_1)x_2 \end{bmatrix} = \begin{bmatrix} \dot{z} \\ \ddot{z} \end{bmatrix}$$

$$\rightarrow \mathcal{F}(x) = \begin{bmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 2x_1 - 0.5x_1^3 - (1+x_1)x_2 \end{bmatrix} = \dot{x} \quad \text{(now in standard NL SS form)}$$

$$\text{Define output} \quad y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{(already linear – no linearization needed!)}$$

$\rightarrow$ Suppose we now linearize dynamics around *equilibrium points*...

Example (please read on your own!):
2nd order NL ODE (cont'd)

Step 2: look for eq. points to use as x_{nom} op. point:

Equilibrium points: solutions of $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} |_{x_1,x_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{F}_1(x) \\ \mathcal{F}_2(x) \end{bmatrix} |_{x_1,x_2} = \begin{bmatrix} x_2 \\ 2x_1 - 0.5x_1^3 - (1+x_1)x_2 \end{bmatrix}$

(i.e. $\dot{x} = \mathcal{F}[x] = 0$)

$\rightarrow$ So, we must have: $x_2 = 0$

and $2x_1 - 0.5x_1^3 - (1+x_1)x_2 = 0$

$\rightarrow$ Solve for the roots of the 2nd equation (since $x_2 = 0$ is known via first eq.)

$\rightarrow$ Get 3 equilibrium points:

$$\left.\begin{aligned} x_{eq,1} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \left(= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \\ x_{eq,2} &= \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ x_{eq,3} &= \begin{bmatrix} -2 \\ 0 \end{bmatrix} \end{aligned}\right\}$$

Now we take Jacobians of NL ODE with respect to x and evaluate at these different x_{nom} operating points

Example (please read on your own!): 2nd order NL ODE (cont'd)

Step 3: Find Jacobians at x_{nom} points

$$\left[\frac{\partial \mathcal{F}}{\partial x}\right]\Big|_{x_{nom}} = \begin{bmatrix} \frac{\partial \mathcal{F}_1}{\partial x_1} & \frac{\partial \mathcal{F}_1}{\partial x_2} \\ \frac{\partial \mathcal{F}_2}{\partial x_1} & \frac{\partial \mathcal{F}_2}{\partial x_2} \end{bmatrix}\Big|_{x_{nom}} = \begin{bmatrix} 0 & 1 \\ (2 - \frac{3}{2}x_1^2 - x_2) & -(1 + x_1) \end{bmatrix}\Big|_{x_{nom}}$$

$\rightarrow$ for $x_{eq,1}$: $\bar{A}_{1,nom} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$

$x_{eq,2}$: $\bar{A}_{2,nom} = \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix}$

$x_{eq,3}$: $\bar{A}_{3,nom} = \begin{bmatrix} 0 & 1 \\ -4 & 1 \end{bmatrix}$

3 different linearized ODEs

$\dot{\bar{x}} = \bar{A}_i \bar{x}$ for $\bar{x} = \delta x$

for different eq. points $i = 1, 2, 3$

Example (please read on your own!): 2^{nd} order NL ODE (cont'd)

Step 4: Put into LTI SS form: what is the state vector?

$$\bar{x} = \delta x = \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} \quad = \text{perturbation state vector} \quad (\neq \text{total state vector } x(t)!!)$$

$\rightarrow$ What is actual total state of NL system at any time (w.r.t. op. pt.)?

$$x(t) = x_{nom}(t) + \delta x(t)$$
$$= \begin{bmatrix} x_{nom,1}(t) \\ x_{nom,2}(t) \end{bmatrix} + \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$$
$$= \begin{bmatrix} z_{nom,i}(t) \\ \dot{z}_{nom,i}(t) \end{bmatrix} + \begin{bmatrix} \delta z \\ \delta \dot{z} \end{bmatrix}$$

(for eq. pt. i in example)