ASEN 5044, Fall 2024

Statistical Estimation for Dynamical Systems

Lecture 03:
State Space Models for Linear Dynamical Systems

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Announcements

Quiz 1 solutions to be posted on Canvas

HW 1 to be posted this Thurs 09/05, to be due next Thurs 09/12 (via Gradescope)

Office hours: regular days/times starting next week:

- **Prof. Ahmed: Wed 4:30 – 6 pm, AERO N353**
- **TF office hours coming soon!**
- **Zoom link for remote participation: use same link as for lectures (posted on Canvas)**

Overview

- Last time: Quick linear algebra refresher
- **Today:** State Space Models
  - motivation, examples

  →(A,B,C,D) matrix parameters for continuous time (CT) linear dynamical systems

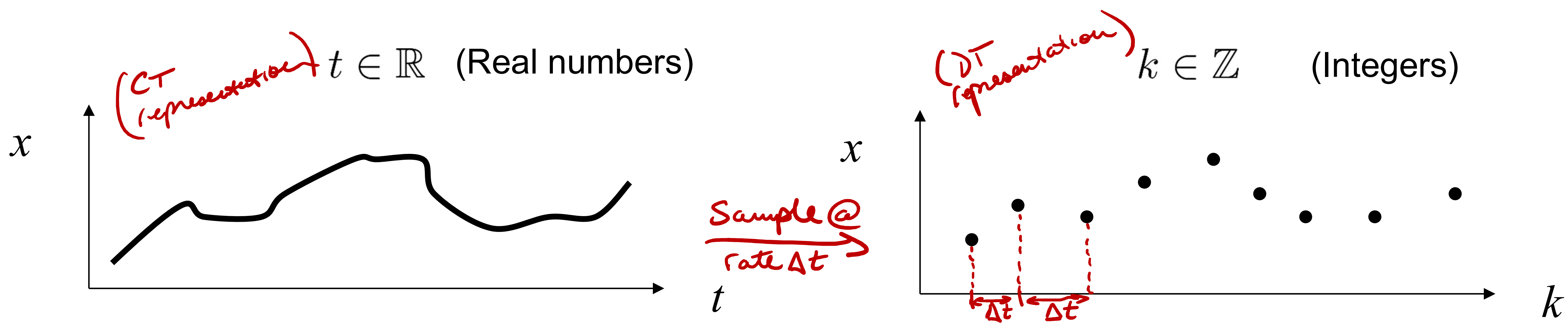
  →(LTV: linear time varying; LTI: linear time invariant)

READ: Chapter 1.2-1.3 in Simon book

Onto Dynamical State Space Systems

- Want to study how **vector quantities change over time**, especially when the **vector elements are related to each other**
  - Vehicle state: position, velocity, attitude, attitude rate,…
  - Physiological state: blood pressure, heart rate, O_2 level…
  - Economic state: GNP, GDP, national debt,…

- Such time-varying variables can define the ***state*** **x** of a system over time
- **Continuous time (CT) systems:** continuous state x(t) depends on **continuous** t time variable
- **Discrete time (DT) systems:** continuous state x(k) depends on **integer** k time variable
- **Often, the state x itself cannot actually be observed – but only some sensed variable y related to x**



The Big Picture

- Goal: analysis, control and estimation of dynamical systems
  - Need to understand behavior over time (so we can influence/change it)
  - Work with mathematical models first…
  - …then go test/implement on real thing (& then go back & fix models if something wrong)

- Interested in (physical) dynamical systems that obey **differential equations**
  - Ex.: scalar **linear ordinary differential equation** (linear ODE):

$$\dot{x}(t) = ax(t) \quad \leftrightarrow \quad x(t) = x(0) \cdot e^{at}$$

Where system is going (d/dt)

Where system is at time t (i.e. system's state)

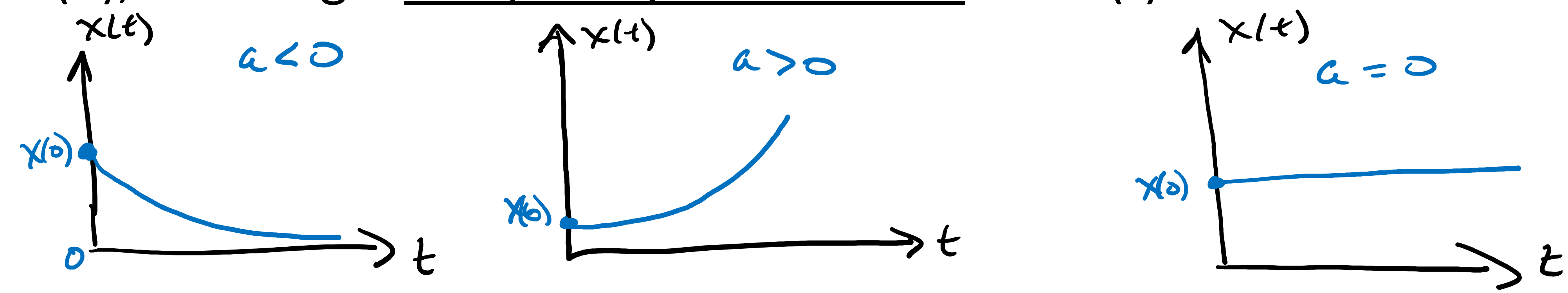
Where system started

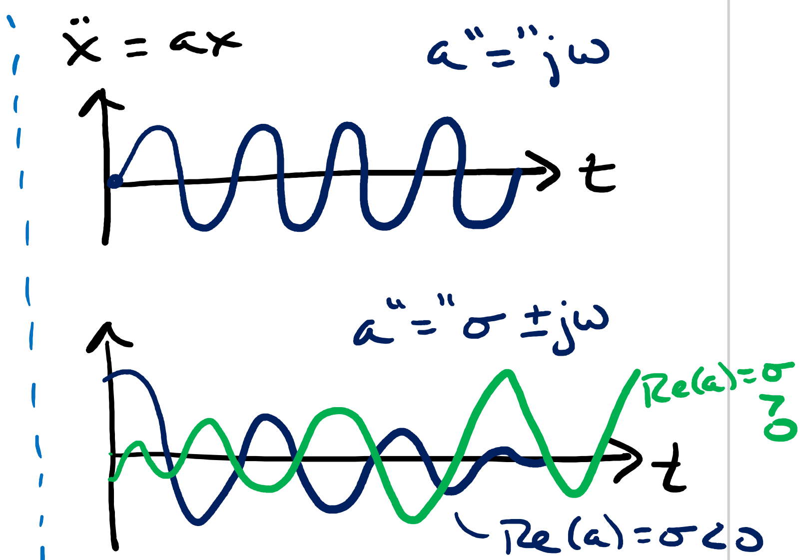
How system behaves from time 0 to time t

Refresher/Motivation: $\dot{x}(t) = ax(t) \iff x(t) = e^{at}x(0)$

- Given *a*, knowing x(0) <u>completely determines</u> x(t) **in future**
- Given x(0), knowing *a* <u>completely determines</u> how x(t) **behaves**

$\dot{x} = ax$





How to generalize these insights for (ie from) scalar linear systems/ODEs to more complicated systems with vector variables & (linear) vector-matrix ODEs?

e.g.

$$\left.\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, x_3, \ldots, x_n) \\ \dot{x}_2 &= f_2(x_1, x_2, x_3, \ldots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, x_3, \ldots, x_n) \end{aligned}\right\}$$

behavior of x_1 depends on behavior & initial conditions of $x_1, x_2, x_3, \ldots, x_n$

→ Likewise for $x_2, x_3, \ldots x_n$ as fxns of time (all coupled)

→ simultaneous sol'ns for all ODEs required!!

State Space Models

- Idea of "state variable" (state vector):
  - Completely summarize information about condition of a dynamical system

  (i.e. summarizes results of past events leading to present)
  - Sufficient to **completely and uniquely** describe system at all future times (given an input + dynamics model)

  (i.e. the state is such that "knowing 'it' now is enough to tell you all about 'it' later")

Look @ scalar linear ODE again:

$$\dot{x}(t) = a x(t) \; (+ \underbrace{b u(t)}_{\text{external forcing input}}) \iff x(t) = \underbrace{e^{at} x(0)}_{\substack{\text{"free response"} \\ \text{(response to ICs)}}} \; + \; \underbrace{\left(+ \int_0^t e^{a\tau} b u(\tau) d\tau \right)}_{\text{"forced response": convolution integral}}$$

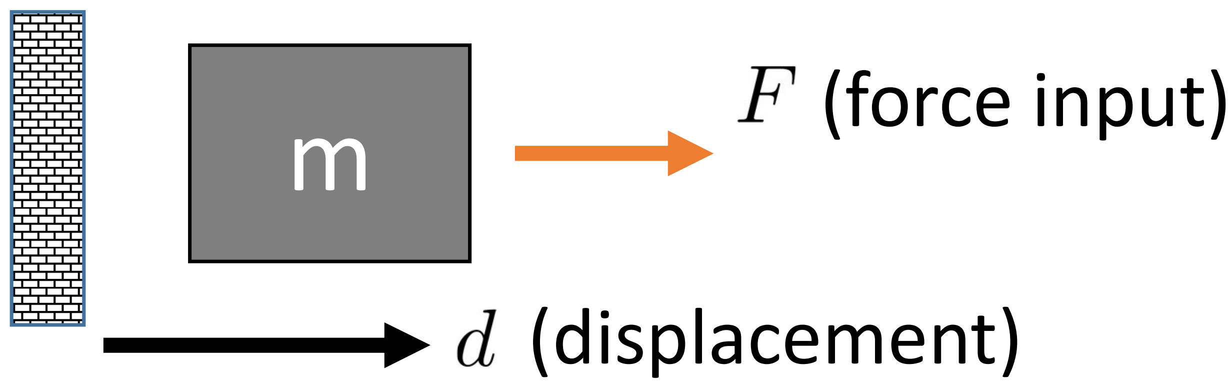
if $t_0 \neq 0$ (initial time):

$$x(t-t_0) = e^{a(t-t_0)} x(t_0) \left(+ \int_{t_0}^{t} b e^{a(\tau - t_0)} u(\tau) \, d\tau \right)$$

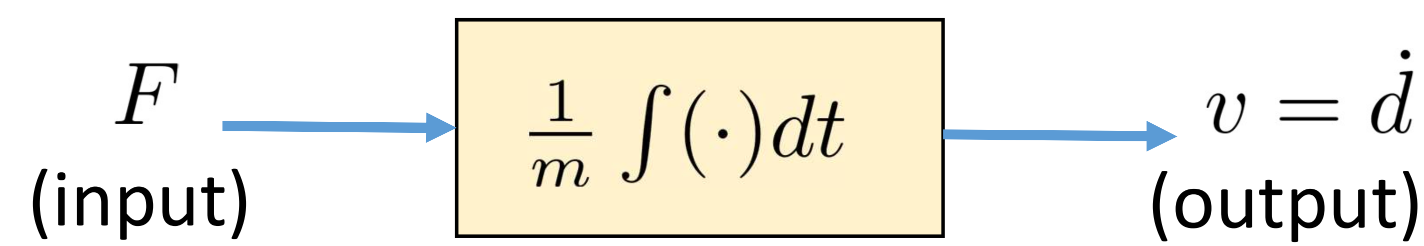
"state variable": (*)
Minimal set of initial conds/ICs required to describe unique solution to ODE

how to extend idea to more complicated systems w/ more variables?

Example #1: Mass with 1 Deg of Freedom



Time Domain Block Diagram



$\rightarrow$ want velocity [output] $v(t)$ as a fxn of time and external force $F(t)$ (input)

$F = m\dot{v}$ (physical law) $\leftrightarrow$ our ODE model $\rightarrow \dot{v} = \frac{F(t)}{m} \left(+ \underset{a}{0} \cdot v(t) \right)$

$\rightarrow$ 1st Fundamental Theorem of Calculus: $\int_{t_0}^{t} \dot{v}(\tau)\,d\tau = [v(t) - v(t_0)]$

$\rightarrow$ re-arrange & solve for $v(t) = v(t_0) + \frac{1}{m}\int_{t_0}^{t} F(\tau)\,d\tau$

but this looks very much like: $x(t) = e^{a(t-t_0)}x(t_0) + \int_{t_0}^{t} b e^{a(\tau-t_0)} u(\tau)\,d\tau$

$\frac{1}{m} \leftrightarrow b$
$F(t) \leftrightarrow u(t)$
$0 \leftrightarrow a$
$v(t) \leftrightarrow x(t)$

$\rightarrow$ So: State var. is $x(t) = v(t)$ [1D state var]:
For some known $F(t)$ over $[t_0, t]$, the output $v(t)$ is different whenever $v(t_0)$ is different

Example #2: Same Mass, Slightly Different Model

Block Diagram:

F (input) $\longrightarrow \boxed{\cdots} \longrightarrow$ d (output)

$$F = m\dot{v} \text{ (ODE physical law)}$$

Different output:

Now want displacement $d(t)$ as fxn of time and $F(t)$ (input)

Easy to Show:

$$v(t) = \frac{1}{m}\int_{t_0}^{t} F(\tau)\,d\tau + \underline{\underline{v(t_0)}}$$

$$\& \quad d(t) = \int_{t_0}^{t} v(\tau)\,d\tau + \underline{\underline{d(t_0)}}$$

Now Need 2 IC's, hence the state vector is $x(t) = \begin{bmatrix} v(t) \\ d(t) \end{bmatrix}$

→ If we don't keep track of $v(t)$ & $d(t)$ in our $x(t)$, then we cannot uniquely specify output $d(t)$ for given $F(t)$ input! [i.e. we must know both IC's for $d(t)$ to be predicted]

→ System has memory: Need to integrate $F(t)$ to get $v(t)$ & integrate $v(t)$ to get $d(t)$!

Re-arrange Dynamics to Reflect States (want d vs. f)

- Let $x = \begin{bmatrix} v \\ d \end{bmatrix} \triangleq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ & $u = F(t)$ (input) $\xrightarrow[\text{elementwise}]{d/dt\ \&\ x}$ $\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{v} \\ \dot{d} \end{bmatrix} = ??$

(state vector $\in \mathbb{R}^2$)

But recall: $v(t) = v(t_0) + \frac{1}{m}\int_{t_0}^{t} F(\tau)\, d\tau$

$d(t) = d(t_0) + \int_{t_0}^{t} v(\tau)\, d\tau$

take d/dt

$\dot{v}(t) = \frac{d}{dt} v(t) = \frac{1}{m} F(t)$

$\dot{d}(t) = \frac{d}{dt} d(t) = v(t)$

(Note: this step follows from the Second Fundamental Theorem of Calculus -- this is different from the application of the First Fund Theorem Application from slide 8 since we are differentiating a definite integral, rather than integrating a derivative... intuitively, the result here follows from the fact that we are saying v(t) is a constant plus a Riemann sum (definite integral) over infinitesimal F(\tau) inputs multiplied by 1/m and d\tau -- by taking the time derivative, with respect to variable t (which is the upper limit of the integral), we are saying the slope of the function v at t is given by F(t) times 1/m -- the act of differentiation -- taking the limit of difference as the interval of the difference goes to zero -- gets rid of the rest of the integral for us, which is exactly what the second fundamental theorem of calculus says happens)

$\rightarrow$ So: $\boxed{\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{v} \\ \dot{d} \end{bmatrix} = \begin{bmatrix} \frac{1}{m} F(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{m} u(t) \\ x_1 \end{bmatrix}}$

where "sensed output" that we actually see is $\boxed{y(t) = d(t) = x_2}$

Re-arrange Dynamics to Reflect States (want d vs. f)

Since everything here is **linear**, can rewrite all this in **general matrix-vector ODE form**:

$$\dot{x} = \begin{bmatrix} \dot{v}(t) \\ \dot{d}(t) \end{bmatrix} = \begin{bmatrix} 0\cdot v(t) + 0\cdot d(t) + \frac{1}{m}f(t) \\ 1\cdot v(t) + 0\cdot d(t) + 0\cdot f(t) \end{bmatrix}$$

(n=2, # states)

(where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} v \\ d \end{bmatrix}$
$\rightarrow \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{v} \\ \dot{d} \end{bmatrix}$)

$$= \begin{bmatrix} 0\cdot x_1 + 0\cdot x_2 + \frac{1}{m}u(t) \\ 1\cdot x_1 + 0\cdot x_2 + 0\cdot u(t) \end{bmatrix}$$

$$\rightarrow \boxed{\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{n\times n = 2\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix}}_{\substack{n\times m \\ = 2\times 1}} u(t), \quad \text{w/ IC } x(0) = \begin{bmatrix} v(0) \\ d(0) \end{bmatrix}}$$

"A" "B" (m=1 input)

how states track "internal memory" of dyn. sys.

where output $y(t) = d(t) = x_2$ is given by

gives "snapshot" of system via some operation on some subset of states

$$y(t) = 0\cdot x_1 + 1\cdot x_2 + 0\cdot u(t)$$
$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\,u(t)$$

"C" "D"

Linear State space model