# ASEN 6060
# ADVANCED ASTRODYNAMICS
# Orbital Stability

Objectives:

- Define common approach to assessing orbital stability
- Present an overview of insight gained by assessing orbital stability across a family of periodic orbits
- Define common bifurcations

# *Periodic Orbit Stability*

- Orbital stability describes the behavior of trajectories in the local neighborhood of the periodic orbit
- Lyapunov definition of stability of a reference solution $\bar{x}_R(t)$:

# *Linearizing About a Periodic Orbit*

Linearizing relative to a trajectory produces variational equations

$$\delta\dot{\bar{x}}(t) = \mathbf{A}(t)\delta\bar{x}(t)$$

With the following solution:

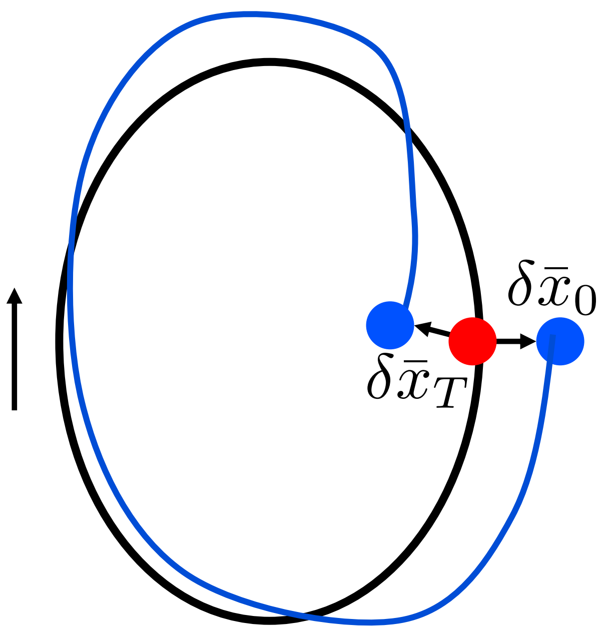
$$\delta\bar{x}(t) = \mathbf{\Phi}(t, t_0)\delta\bar{x}(t_0)$$

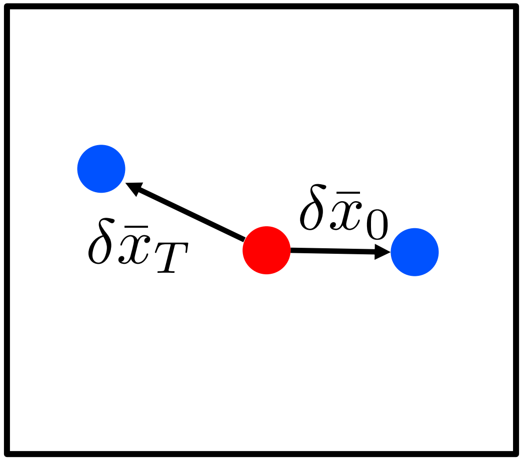
where $\dot{\mathbf{\Phi}}(t, t_0) = \mathbf{A}(t)\mathbf{\Phi}(t, t_0)$ $\mathbf{\Phi}(t_0, t_0) = \mathbf{I}_{6\times 6}$

For a <u>periodic</u> reference path, *A(t)* is time-varying and periodic

# Monodromy Matrix

Monodromy matrix **M** (6x6 matrix):



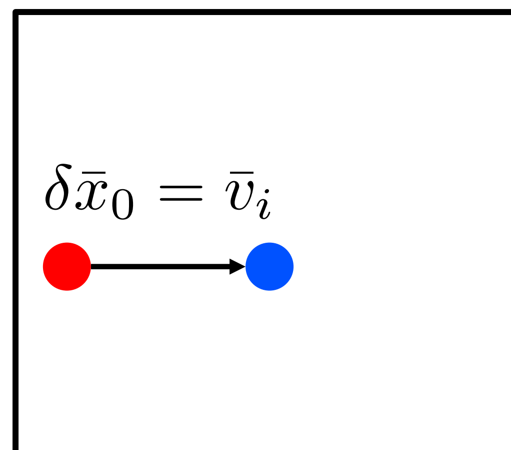



# Eigendecomposition of Monodromy Matrix

If the solution to the linear system is $\delta\bar{x}_T = \mathbf{M}\delta\bar{x}_0$, study types of motion near periodic orbit by decomposing M into eigenvalues and eigenvectors:

where

And recall the definition of an eigenvector:



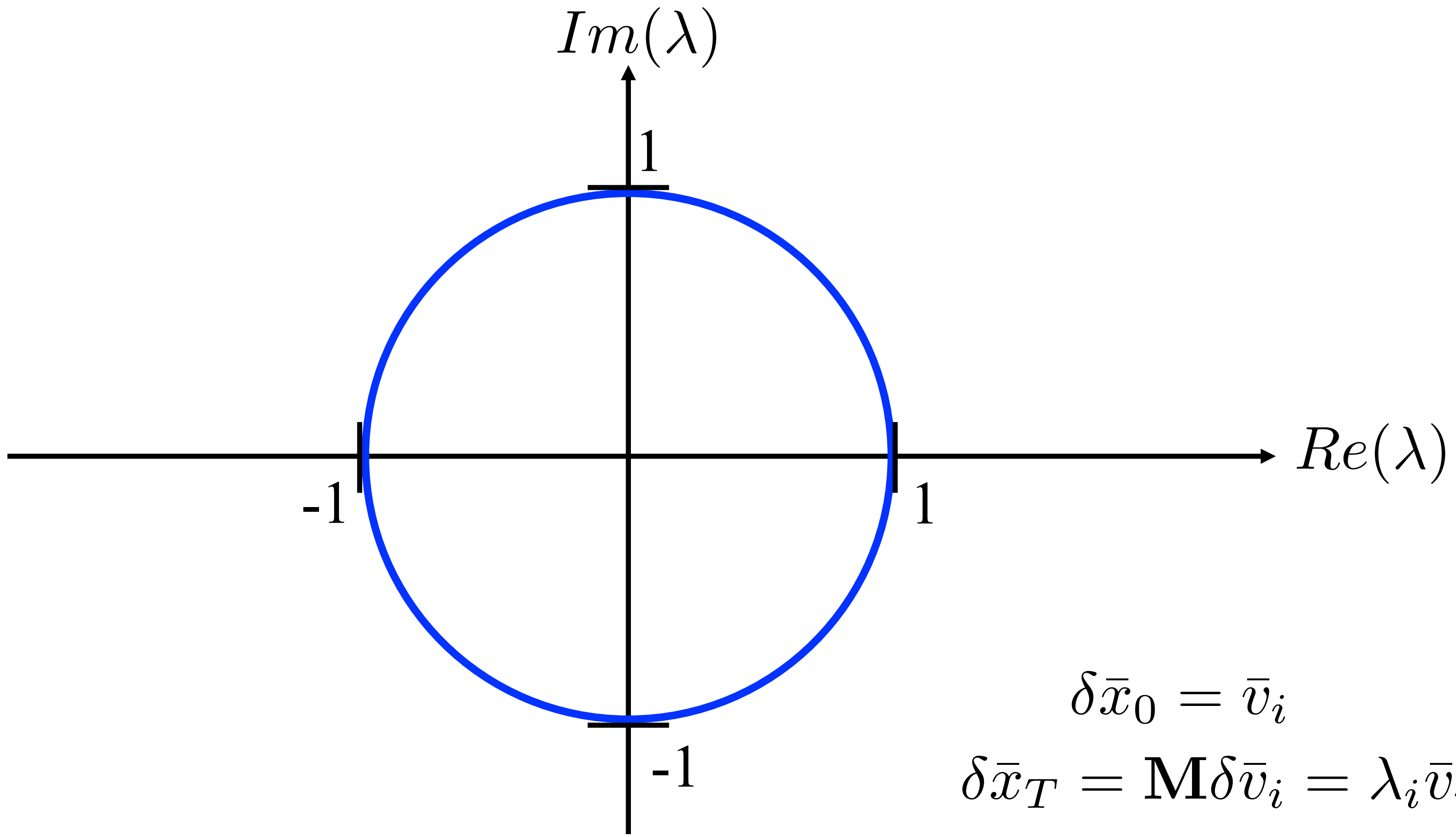
# *Eigendecomposition of Monodromy Matrix*

Decomposing M into eigenvalues and eigenvectors:

$$\mathbf{M} = \mathbf{\Phi}(T,0) \ = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$$

# *Eigenvalues of Monodromy Matrix*

$$\lambda = a \pm bi$$



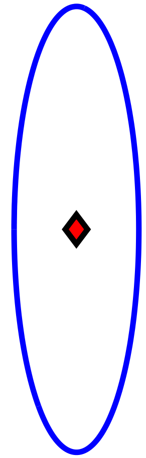

$$\delta \bar{x}_0 = \bar{v}_i$$

$$\delta \bar{x}_T = \mathbf{M} \delta \bar{v}_i = \lambda_i \bar{v}_i$$

# *Eigenvalues of Monodromy Matrix*

One pair of complex eigenvalues on unit circle

$|\lambda| = 1$

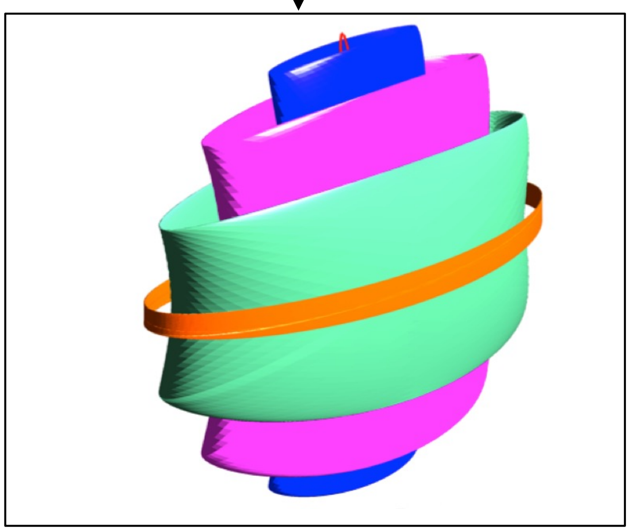


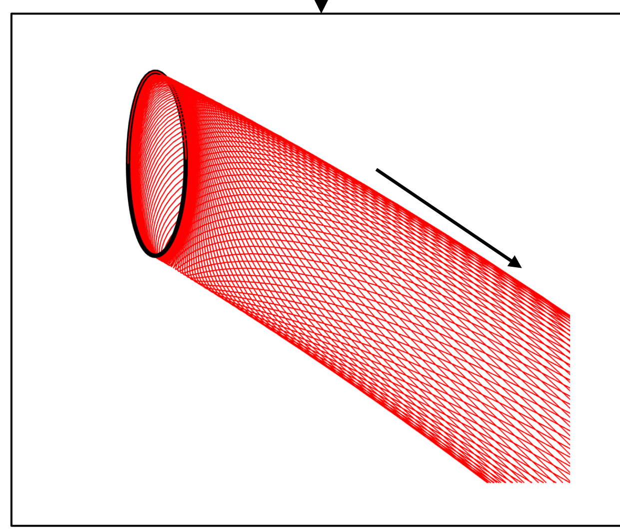
One reciprocal pair of real eigenvalues with

$|\lambda| < 1$

$|\lambda| > 1$

**Quasi-Periodic Trajectory** tracing out the surface of a torus





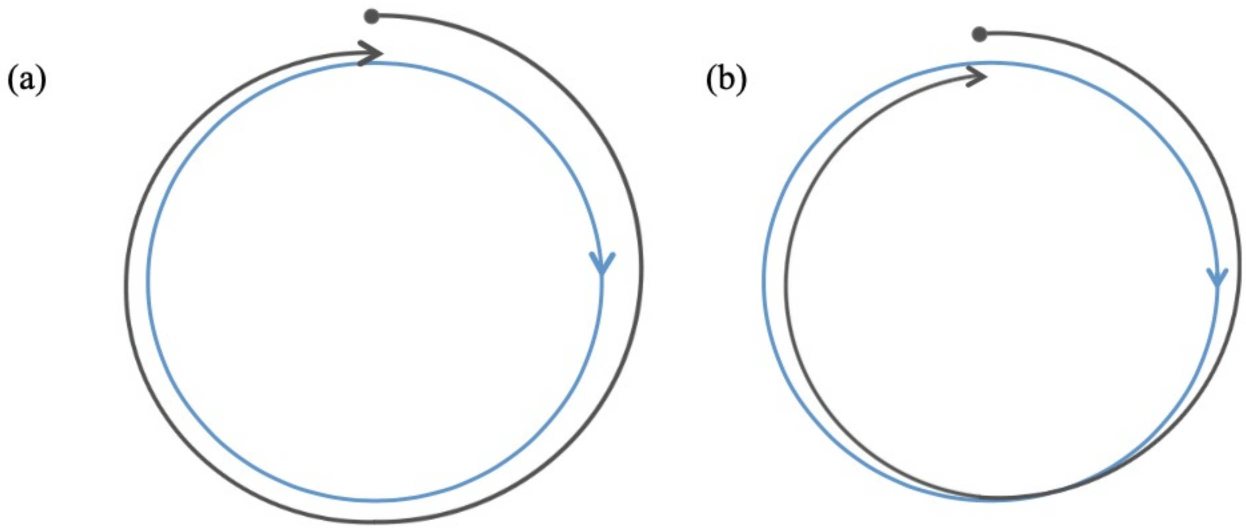
**Stable/ Unstable Manifolds**

# *Eigenvalues of Monodromy Matrix*

Sign of real eigenvalues indicates relative direction of motion and twisting about the orbit when exciting that mode

Example: $|a| < 1$

$$\lambda = a > 0 \qquad\qquad \lambda = a < 0$$




$$\delta \bar{x}_0 = \bar{v}_i$$

$$\delta \bar{x}_T = \mathbf{M} \delta \bar{v}_i = \lambda_i \bar{v}_i$$

Image credit: Bosanac 2016

# *Stability Index*

Tracking the eigenvalues of the monodromy matrix for members of a periodic orbit family supplies insight into the stability of those members, and the identification of any structural changes in the local neighborhood of those orbits

Challenging to visualize the nontrivial eigenvalues in complex plane

Often use stability indices:

2 stability indices for periodic orbit (don't calculate for +1,+1 pair)

There are also other (less intuitive) ways to represent stability information succinctly!

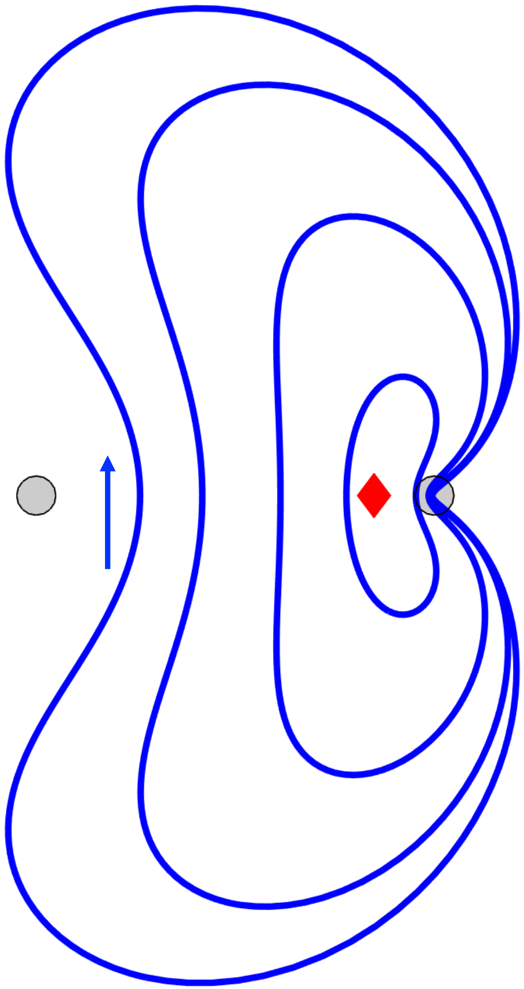
# Stability Index

Stability indices:

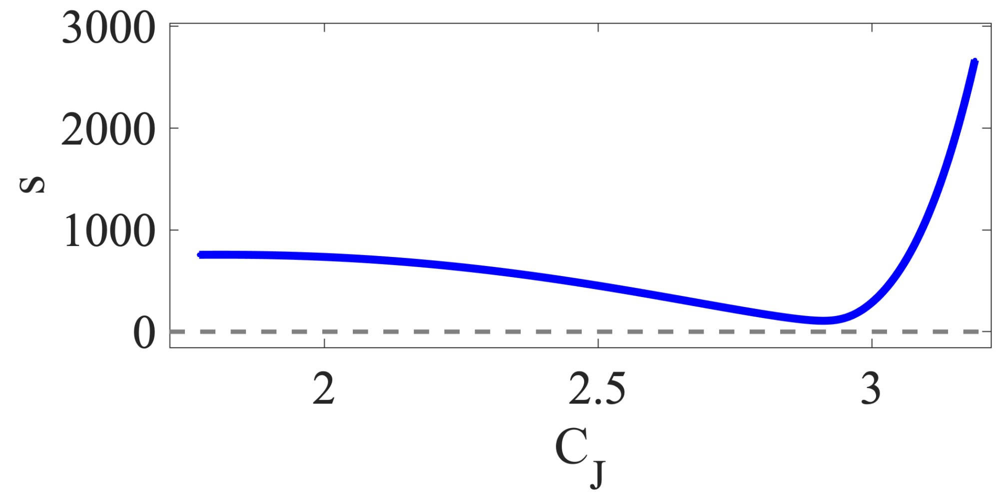
$$s_1 = \lambda_1 + \lambda_2 \qquad\qquad s_2 = \lambda_3 + \lambda_4$$

Observations:

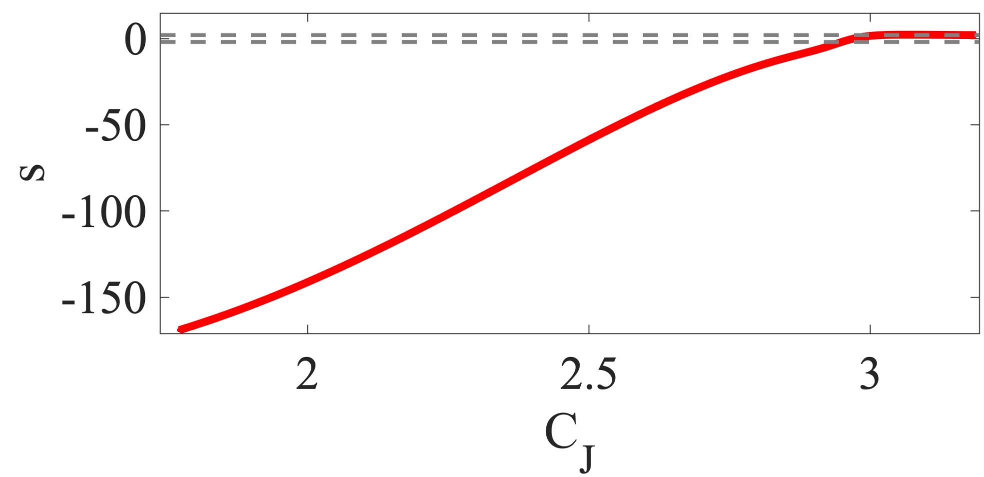
# Example: $L_1$ Lyapunov Orbits



Stability index for in-plane modes




Stability index for out-of-plane modes



# *Local Bifurcations*

Recall: Periodic orbits exist in a family that may be described by a continuously varying natural parameter

A local bifurcation occurs if a change in the natural parameter results in a change in the qualitative behavior of the trajectories in the vicinity of a periodic orbit

A variety of bifurcations exist and may produce a change in the stability of periodic orbits along the family, formation of a new family/ies of periodic orbits, or termination of the current family

- Selected examples discussed in these notes.

(Interesting reading: Ch. 4 of Perko, 2000, "Differential Equations and Dynamical Systems" 3rd edition, See link on Canvas page to free download via CU libraries)

# *Local Bifurcations*

Nontrivial eigenvalues pass through critical values +1,+1, $s = 2$

Three well-known types:

- *Saddle-node/fold bifurcation*: stability change, no new/intersecting families
- *Pitchfork bifurcation*: as stability of periodic orbits change along family, two additional families of similar period are formed with same stability as members of original family before bifurcation
- *Transcritical bifurcation*: family of stable orbits intersects family of unstable orbits and the two families exchange stability characteristics at the intersection

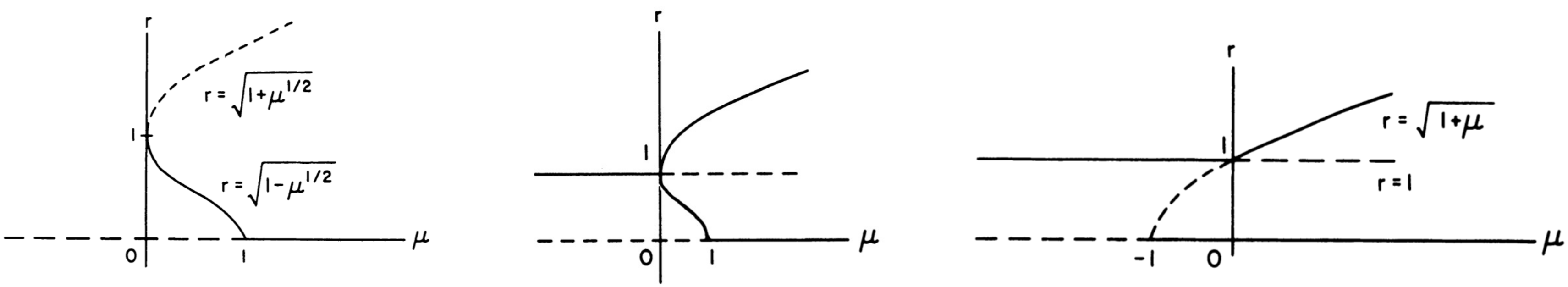



Image credits: Perko, 2001

# *Period-Multiplying Bifurcation*

A family of period-$mq$ orbits emerges from a family of period-$q$ orbit (with orbit period $T$)

Example: period-doubling bifurcation

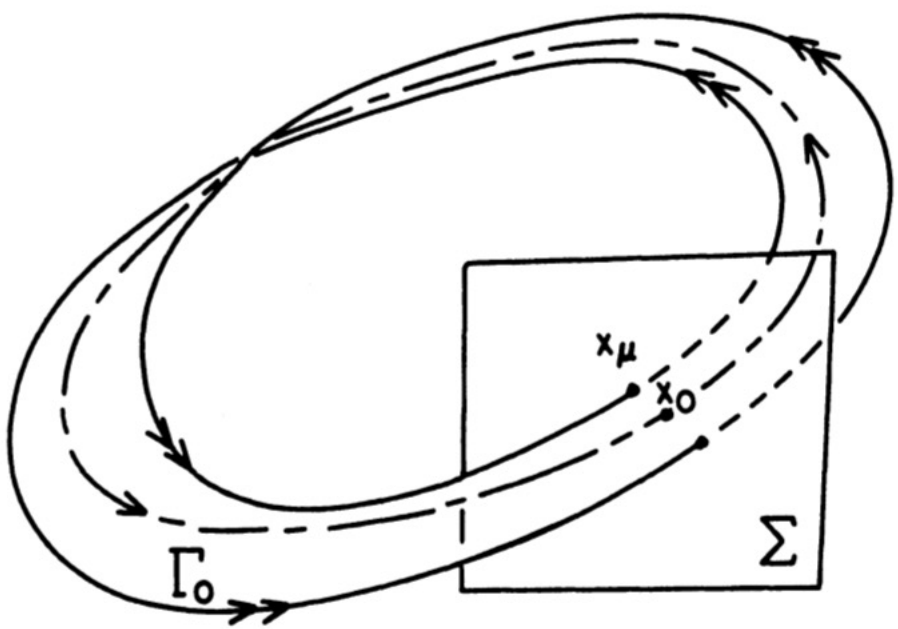



Image credit: Perko, 2000, "Differential Equations and Dynamical Systems" 3rd edition

# Period-Multiplying Bifurcation

Express monodromy matrix of period-*mq* orbit, $\mathbf{\Phi}_{mq}(mT,0)$, in terms of monodromy matrix of period-*q* orbit, $\mathbf{\Phi}_q(T,0)$, at the intersection of the two families

$$\mathbf{\Phi}_{mq}(mT,0) = \mathbf{\Phi}_q(mT,(m-1)T)\mathbf{\Phi}_q((m-1)T,(m-2)T)...\mathbf{\Phi}_q(2T,T)\mathbf{\Phi}_q(T,0)$$

$$\mathbf{\Phi}_{mq}(mT,0) = [\mathbf{\Phi}_q(T,0)]^m$$

The eigenvalues of the period-*mq* orbit are related to the eigenvalues of the period-*q* orbit

$$\lambda_{mq} = \lambda_q^m$$

# *Period-Multiplying Bifurcation*

At the intersection of the two families when the period-*mq* orbits form, there is one nontrivial pair with:

- $\lambda_{mq} = 1, 1$
- $\lambda_q$ are the roots of unity
- Detect the bifurcation in the original period-*q* family when the eigenvalues equal:

and when the stability index equals:

# *Period-Doubling Bifurcation*

Limiting case is the period-doubling bifurcation when:

$$\lambda_i, \lambda_j = -1, -1$$

and is accompanied by a change in stability

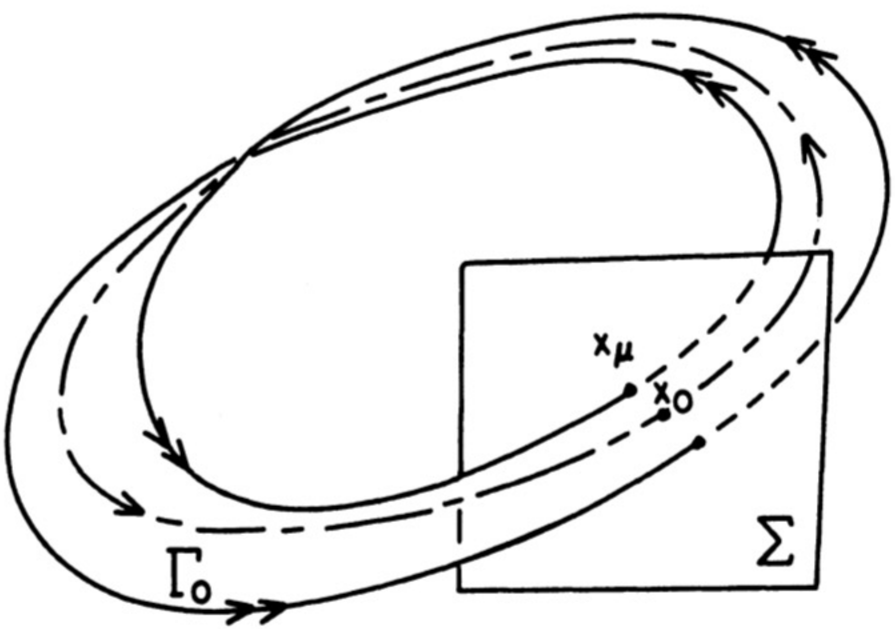
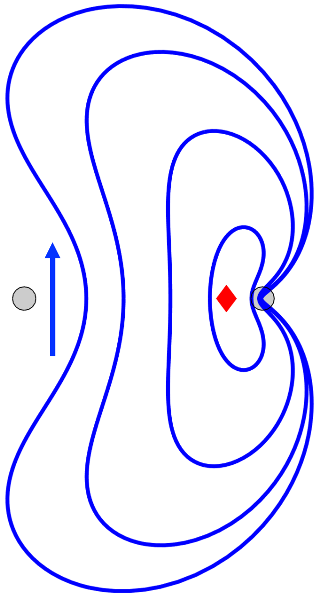
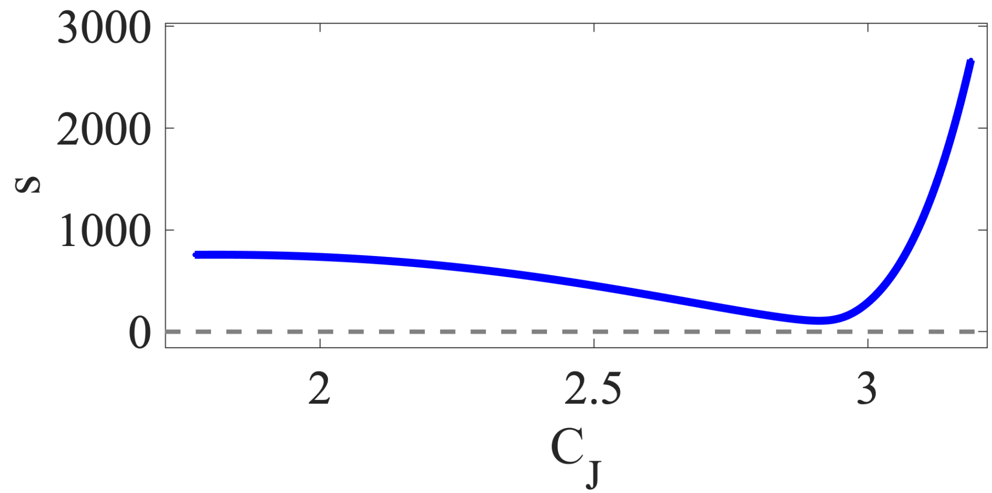



Image credit: Perko, 2000, "Differential Equations and Dynamical Systems" 3rd edition

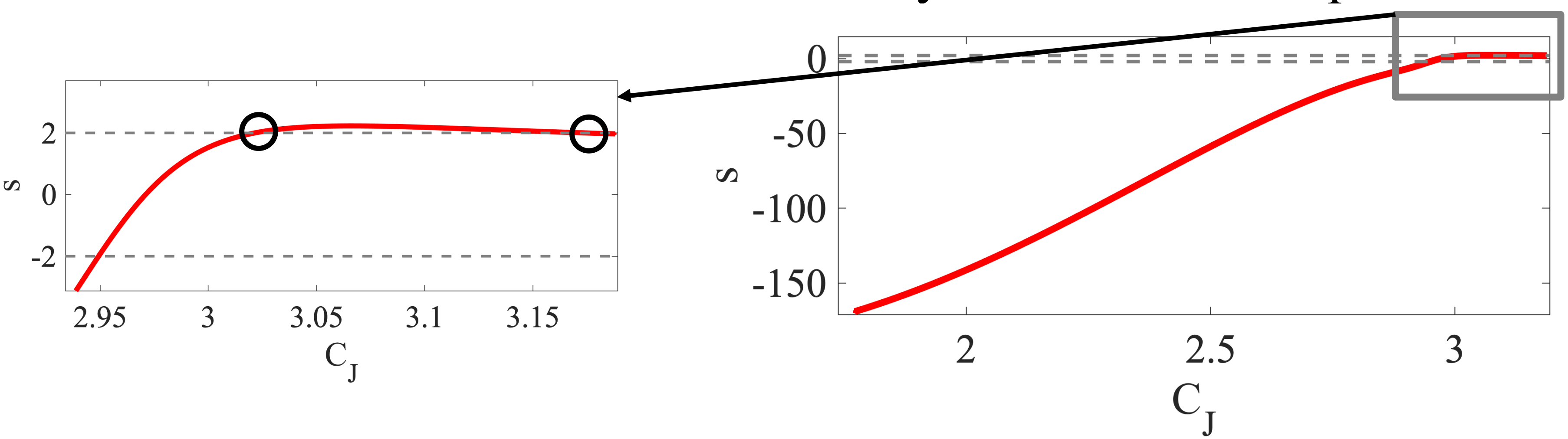
# Example: $L_1$ Lyapunov Orbits



Stability index for in-plane modes




Stability index for out-of-plane modes



# *Periodic Orbit Families near $L_1$*

Bifurcations along $L_1$ Lyapunov orbit family correspond to intersections with $L_1$ halo and axial orbit families

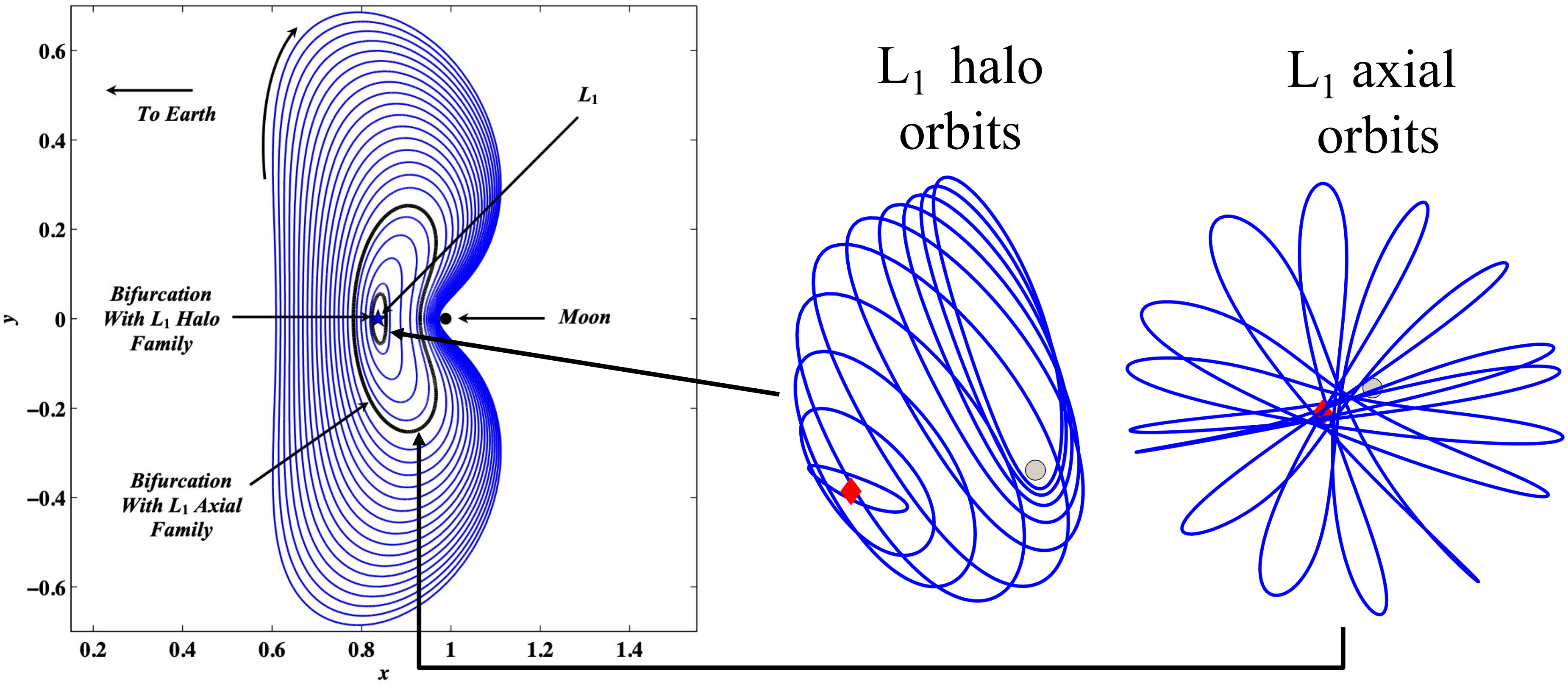



Figure 3.5 The $L_1$ Lyapunov Family

Left image credit: Grebow, D. 2006, "GENERATING PERIODIC ORBITS IN THE CIRCULAR RESTRICTED THREE- BODY PROBLEM WITH APPLICATIONS TO LUNAR SOUTH POLE COVERAGE", MS Thesis

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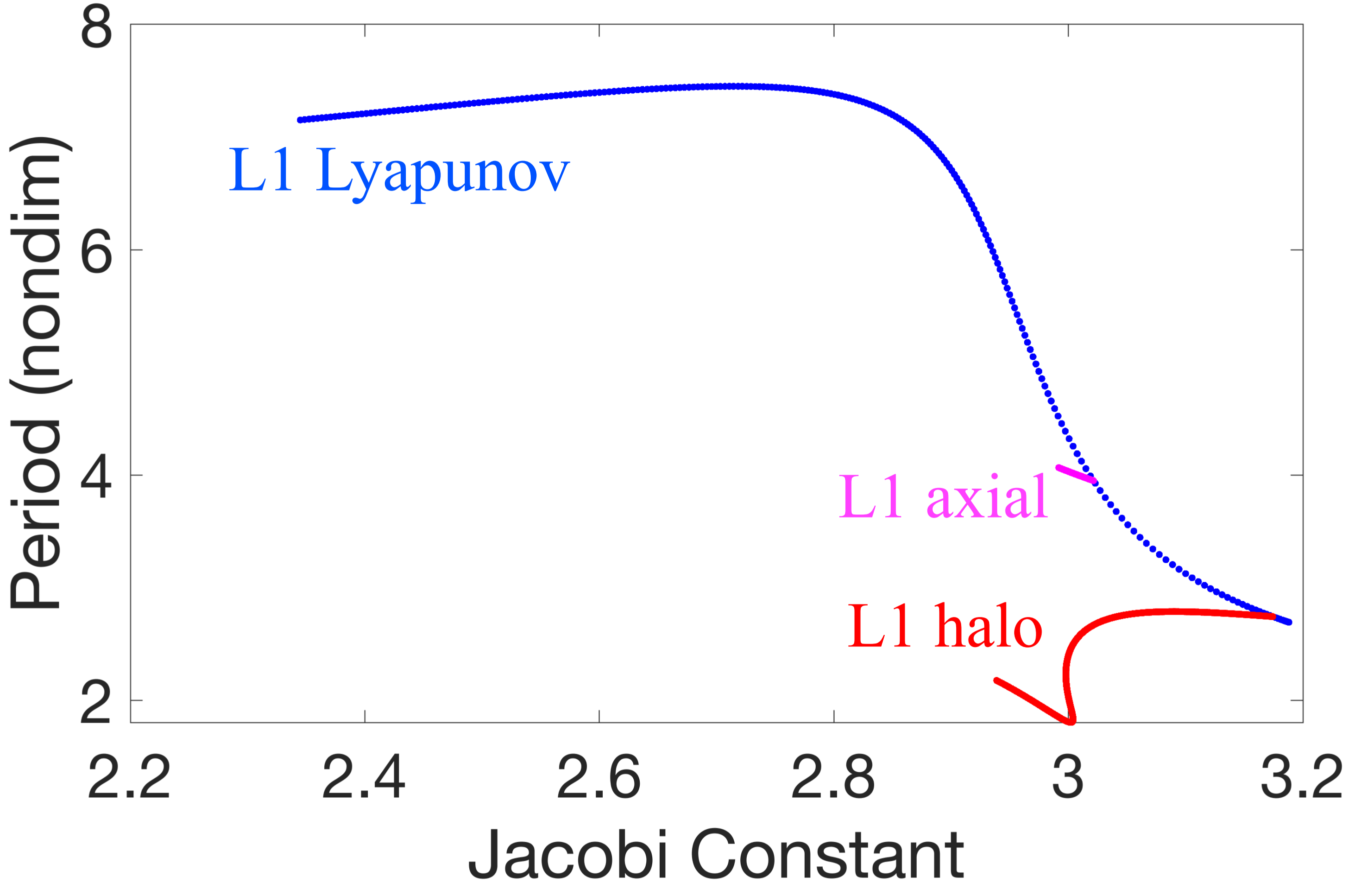

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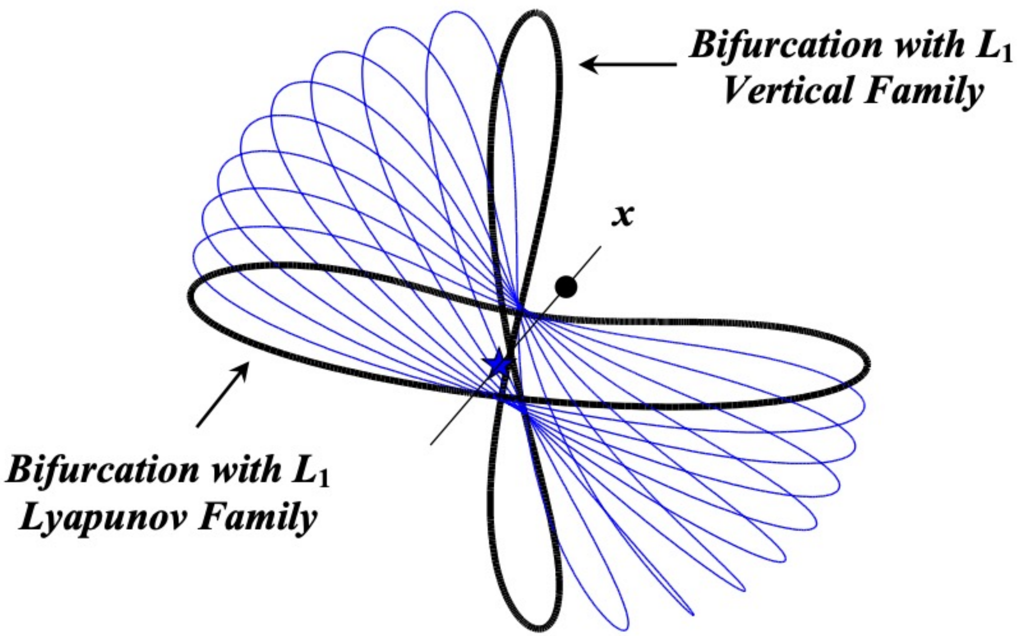
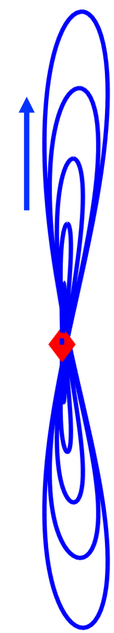
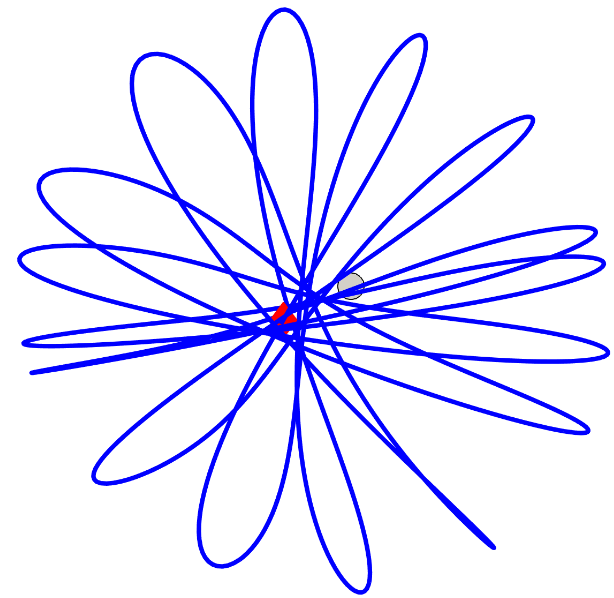



$L_1$ vertical orbits



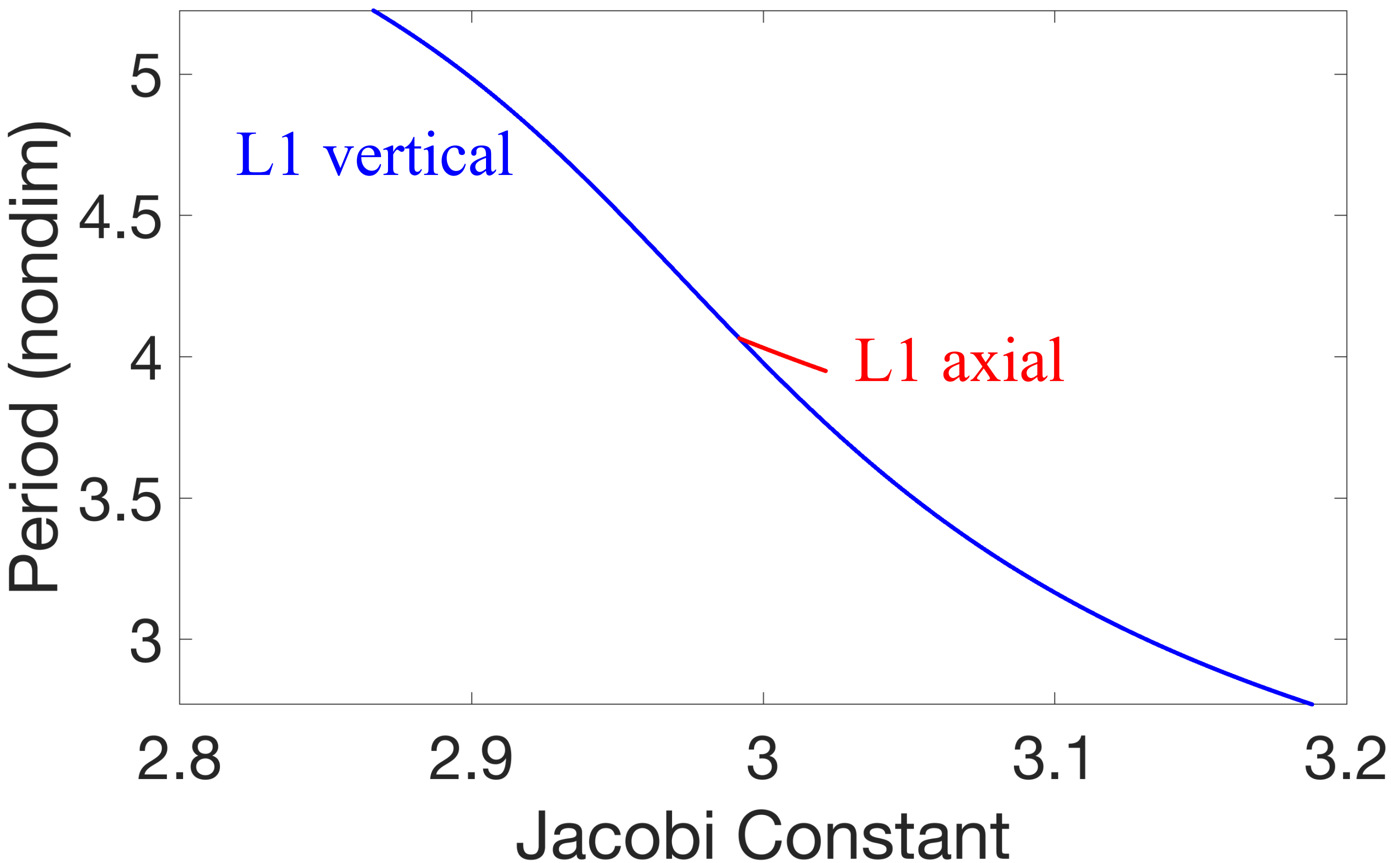
$L_1$ axial orbits



Left image credit: Grebow, D. 2006, "GENERATING PERIODIC ORBITS IN THE CIRCULAR RESTRICTED THREE- BODY PROBLEM WITH APPLICATIONS TO LUNAR SOUTH POLE COVERAGE", MS Thesis

# *Periodic Orbit Families near $L_1$*

Bifurcation along $L_1$ vertical orbit family correspond to intersection with $L_1$ axial orbit family



# *EM Libration Point Orbit Families*

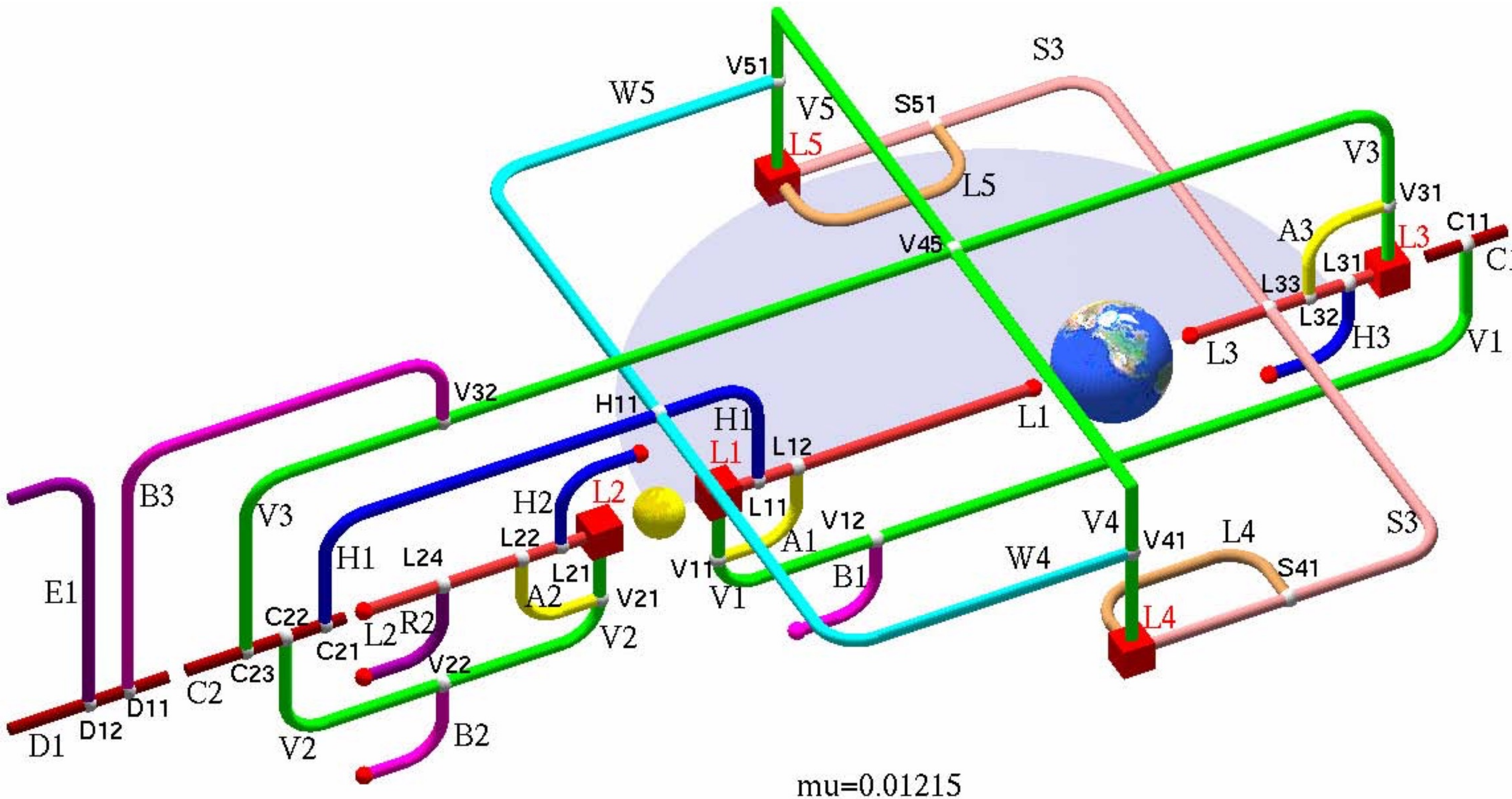



Image credit: Doedel et. al., 2007, “Elemental Periodic Orbits Associated with the Libration Points in the Circular Restricted Three-Body Problem” (See Canvas for link to paper)

# *Impact of Mass Ratio*

Existence of some families depends on mass ratio

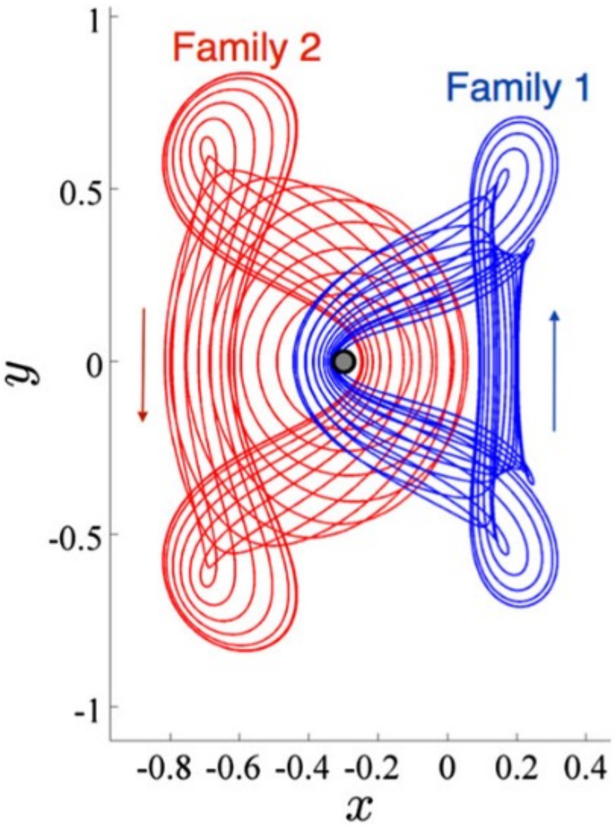
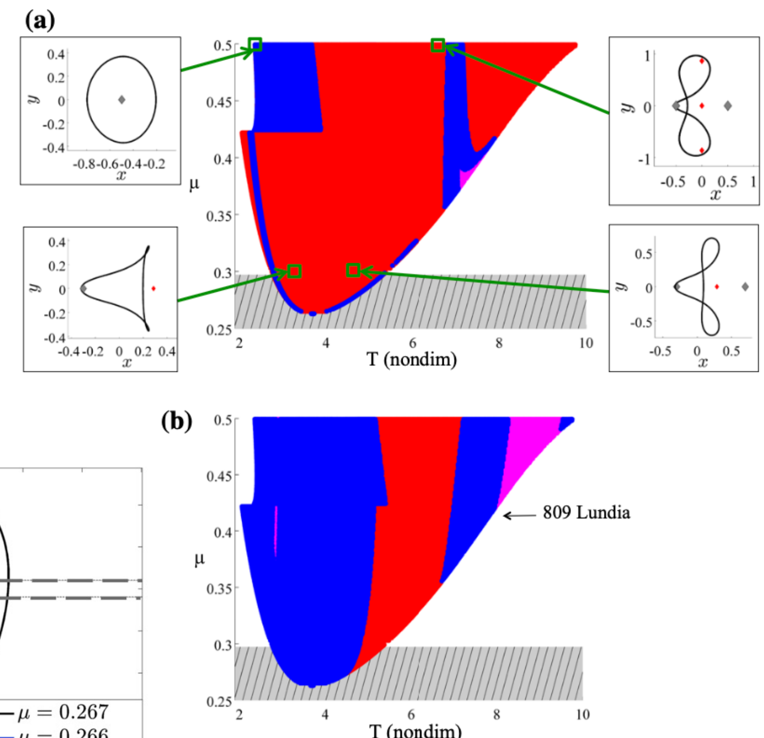



**Fig. 11** Stability representation for 'family 1', comprised of prograde orbits about $P_1$. Orbital stability is indicated via color: stable (*blue*), positive unstable (*red*), and negative unstable (*purple*). **a** In-plane stability. **b** Out-of-plane stability



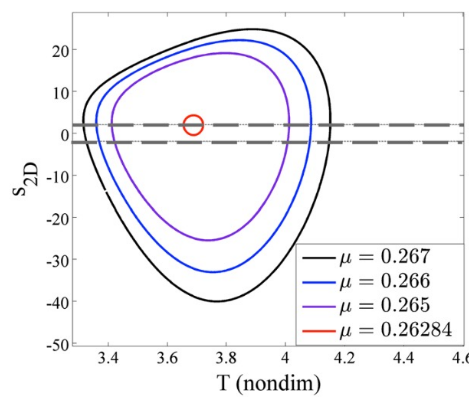




Image credits: Bosanac, N; Howell, K.C; Fischbach, E, 'Stability of Orbits Near Large Mass Ratio Binary Systems,' May 2015, Volume 122, pp. 27-52, Celestial Mechanics and Dynamical Astronomy.

# *Impact of Mass Ratio*

Families can significantly change as mass ratio changes

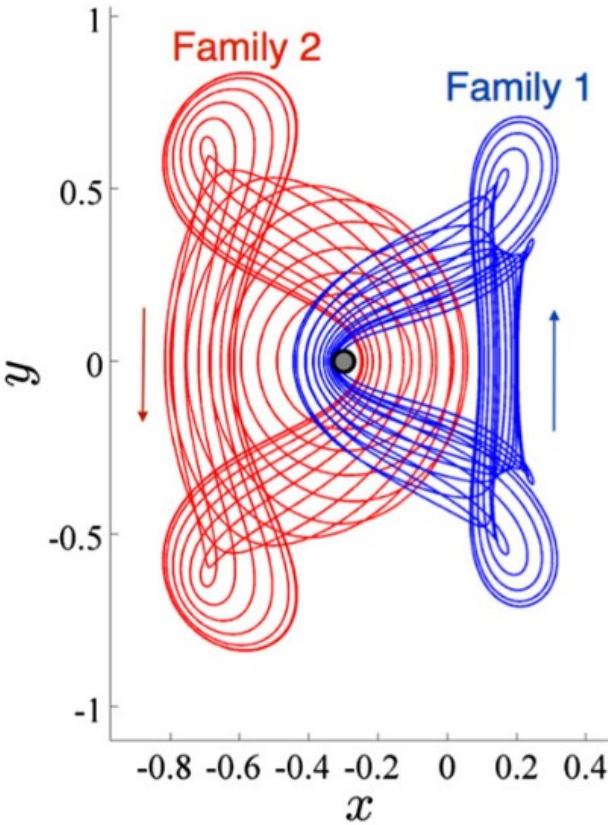
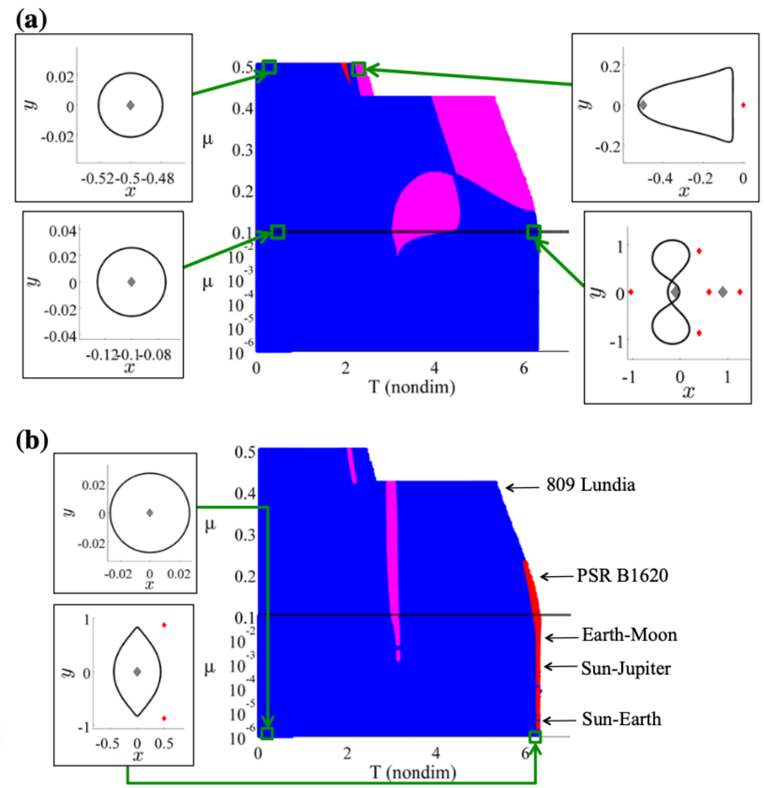
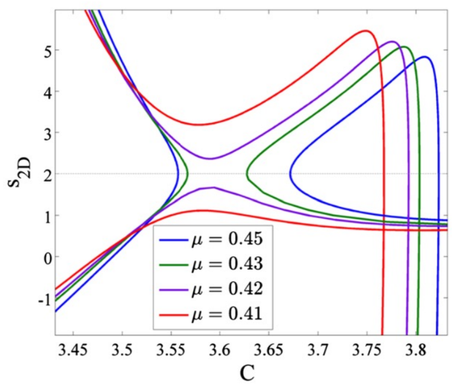



**Fig. 15** Stability representation for 'family 2', consisting of prograde orbits about $P_1$. Orbital stability is indicated via color: stable (*blue*), positive unstable (*red*), and negative unstable (*purple*). **a** In-plane stability. **b** Out-of-plane stability

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