ASEN 5044, Fall 2018

Statistical Estimation for Dynamical Systems

Lecture 25 [Special Topic #5]: Bayesian Recursive Updating; The General Bayes Filter

Prof. Nisar Ahmed (Nisar.Ahmed@Colorado.edu)

Friday 11/2/2018

University of Colorado Boulder

Aerospace Engineering

Overview

- Recursive Bayesian updating and estimation

- General probabilistic forward prediction/propagation of dynamical states
  - **The Chapman-Kolmogorov equation**

- General probabilistic measurement updating of dynamical states
  - **The general Bayes filter**

Recursive Bayesian Updating and Estimation

- Consider static x (no dynamics or process noise) with prior pdf p(x)
- Given T noisy independent measurements $y_{1:T}$ with known observation likelihoods $p(y_k|x)$, for steps k=1,…,T.
- **To get Bayesian estimate of x at step k:** need to minimize expected value of some desired cost fxn with respect to posterior pdf (see Lecture 20/Special topic #4)
- e.g. using Minimum Mean Squared Error cost function:

$$\hat{x}_{k,\text{MMSE}} = \arg\min_{x \in \mathbb{R}^n} E[(x - \hat{x}_k)^T (x - \hat{x}_k) \mid y_{1:k}]$$

$$\rightarrow \hat{x}_{k,\text{MMSE}} = E[x| \; y_{1:k}] \longrightarrow \text{what is the mean } p(x|y_1,\dots,y_k), \; k \le T \text{ Any}$$

$\longrightarrow$ Need to find $p(x|y_{1:k}) = p(x|y_1,\dots,y_k)$

$\propto p(x) \cdot p(y_{1:k}|x) \longleftrightarrow$ "batch"

$= p(x) \cdot \prod_{i=1}^{k} p(y_i|x) = p(x) \cdot p(y_1|x) \cdot p(y_2|x) \cdots p(y_k|x) \longleftrightarrow$ "recursive" basis

if meas. cond. ⫫ of each other given x $\Leftarrow$ if $y_i ⫫ y_j$ given x

Recursive Bayesian Updating and Estimation

- Alternative: what if data $y_1,\ldots,y_T$ absorbed sequentially, rather than in batch?
- As it turns out, we can use the posterior $p(x|y_1)$ as the prior for $p(x|y_{1:2})$…

…and then use the posterior $p(x|y_{1:2})$ as the prior for $p(x|y_{1:3})$, etc.

for $k=1$: $p(x|y_1) \propto p(x)\cdot p(y_1|x) \rightarrow p(x|y_1) = \dfrac{p(x)\cdot p(y_1|x)}{\int_{-\infty}^{\infty} p(x)\cdot p(y_1|x)\,dx}$ $\left(= \dfrac{p(x)\cdot p(y_1|x)}{p(y_1)}\right)$

for $k=2$: $p(x|y_1,y_2) \propto \underbrace{p(x)\,p(y_1|x)}\cdot p(y_2|x) \propto p(x|y_1)\cdot p(y_2|x)$ $= \dfrac{p(x)p(y_1|x)}{p(y_1)}$

$\rightarrow p(x|y_1,y_2) = \dfrac{p(x|y_1)\cdot p(y_2|x)}{\int_{-\infty}^{\infty} p(x|y_1)\cdot p(y_2|x)\,dx} = \dfrac{p(x|y_1)\cdot p(y_2|x)}{p(y_2|y_1)}$ $\left(= \dfrac{p(x)\cdot p(y_1|x)\cdot p(y_2|x)}{p(y_2,y_1)}\right)$

$\dfrac{p(y_2,y_1)}{p(y_1)}$

so Mathematically equivalent to batch Bayes' rule update!

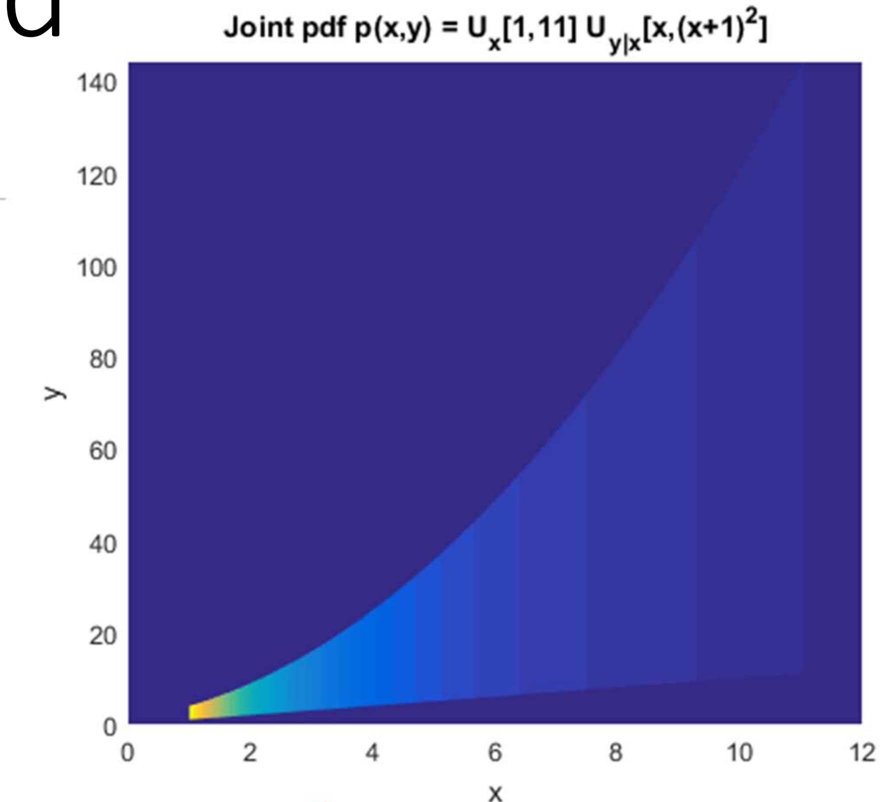
For step $k=j$:

$$\boxed{p(x|y_1,y_2,\ldots,y_j) = \frac{p(x|y_1,\ldots,y_{j-1})\cdot p(y_j|x)}{\int_{-\infty}^{\infty} p(x|y_{1:j-1})\cdot p(y_j|x)\,dx} \propto p(x|y_{1:j-1})\cdot p(y_j|x)}$$

(*) Recursive Bayesian update

has "memory" of previous update $\rightarrow$ posterior from $j-1$ encodes all past info

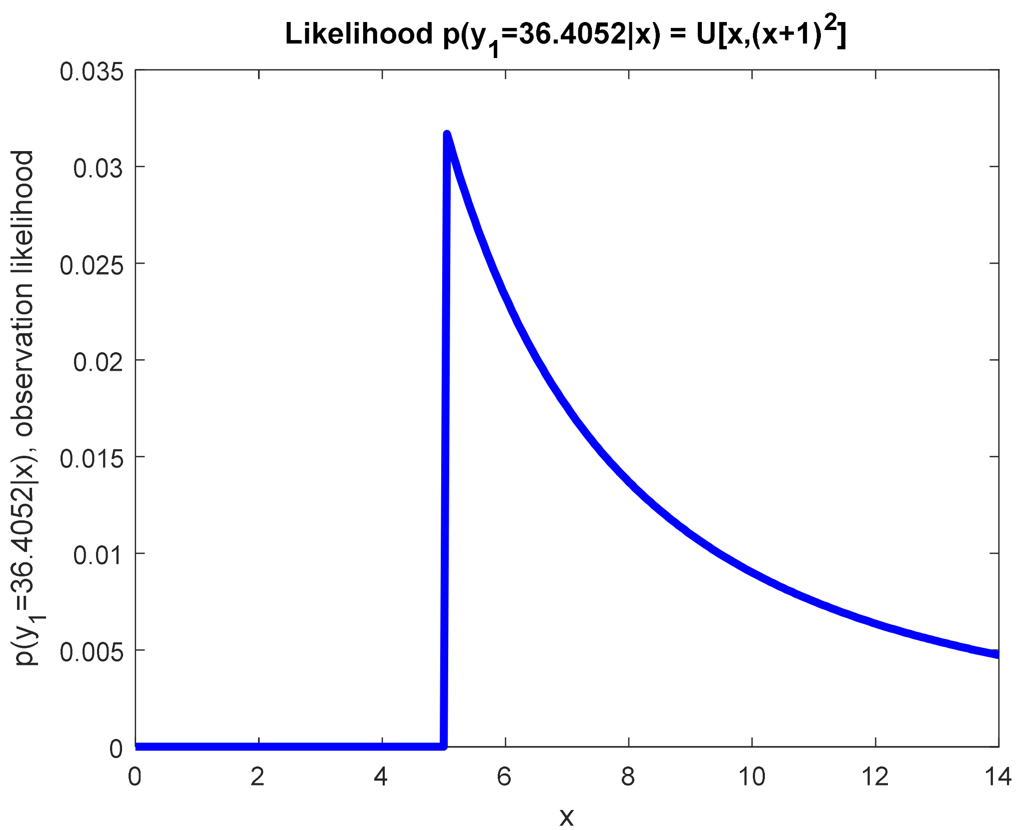
Example: Conditional Uniform PDF Revisited (Lec 14, Special Topic #2)

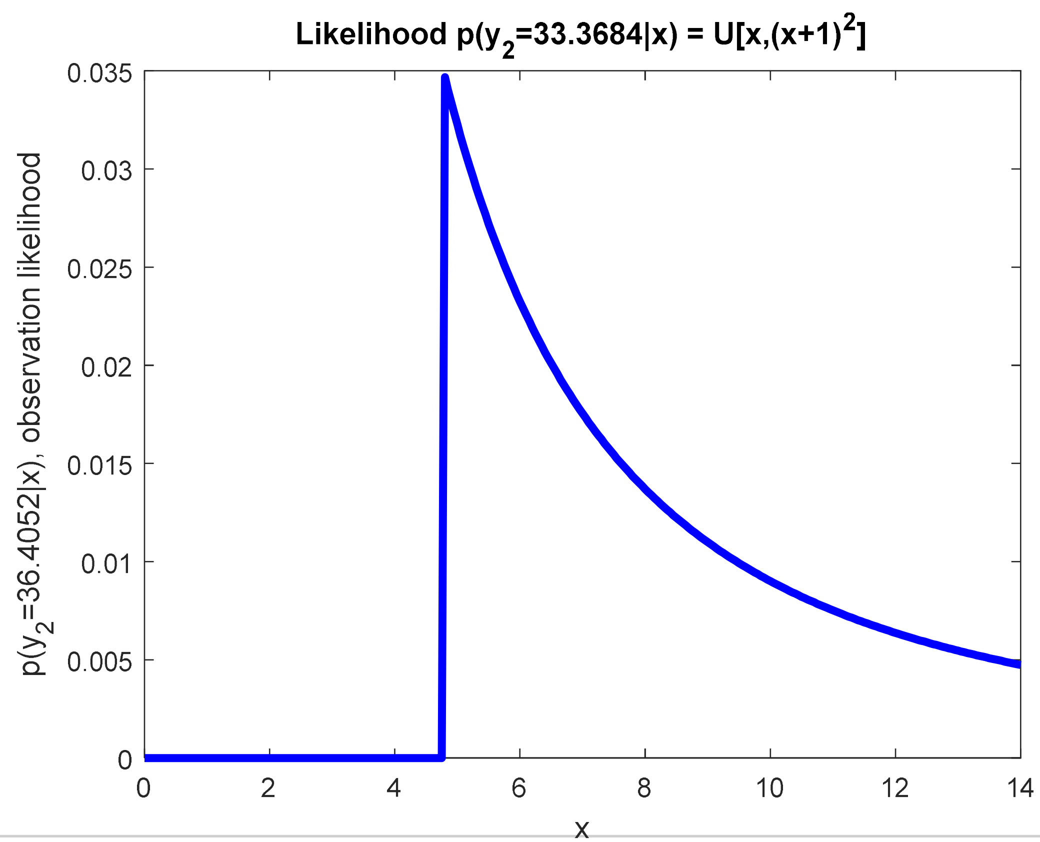


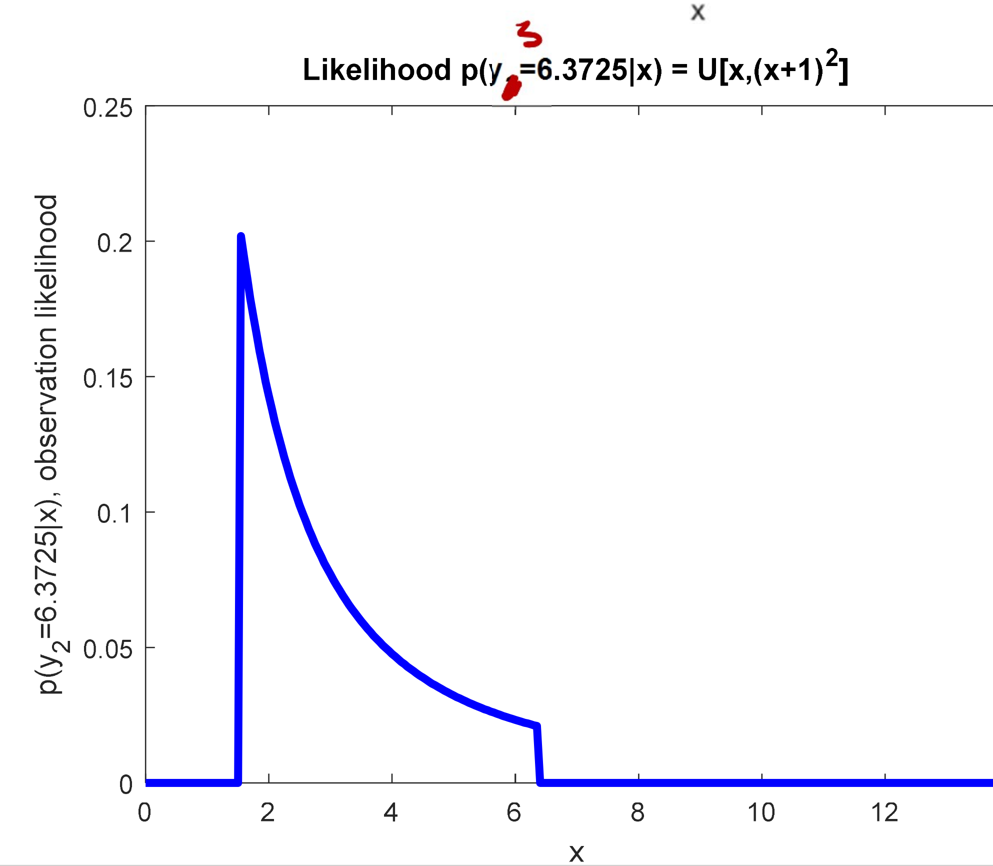
Given joint pdf $p(x,y) = p(x)\ p(y|x) = U_x[1,11]\ \ U_{y|x}[x,(x+1)^2]$

- Suppose true (unknown) $x \sim p(x) = U_x[1,11] \rightarrow x = 5.0723$
- 3 indep. observations in sequence, $y_k \sim p(y_k|x) = U_{y|x}[x,(x+1)^2]$:

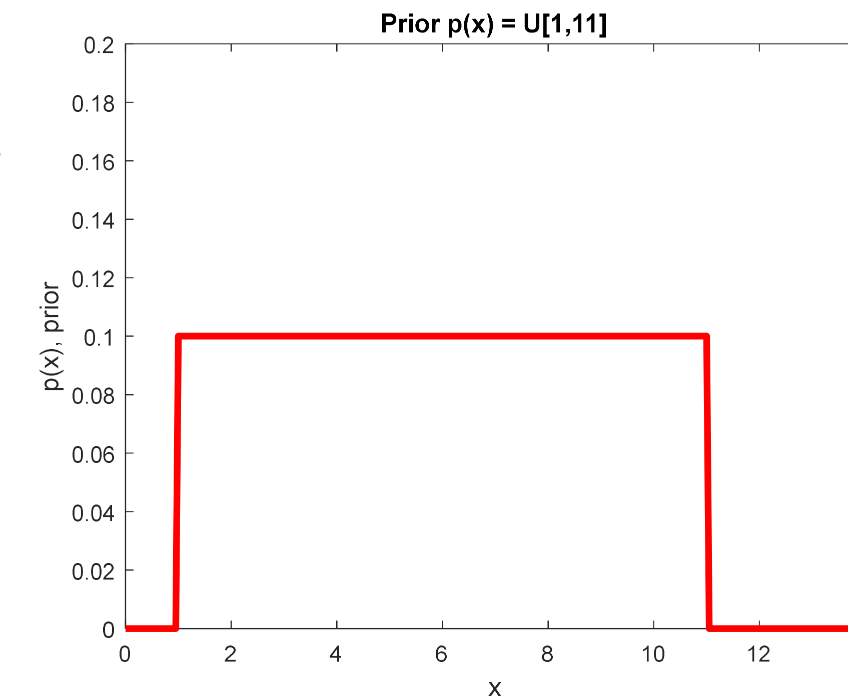
$y_1 = 36.4052,\ y_2 = 33.3684,\ y_3 = 6.3725$







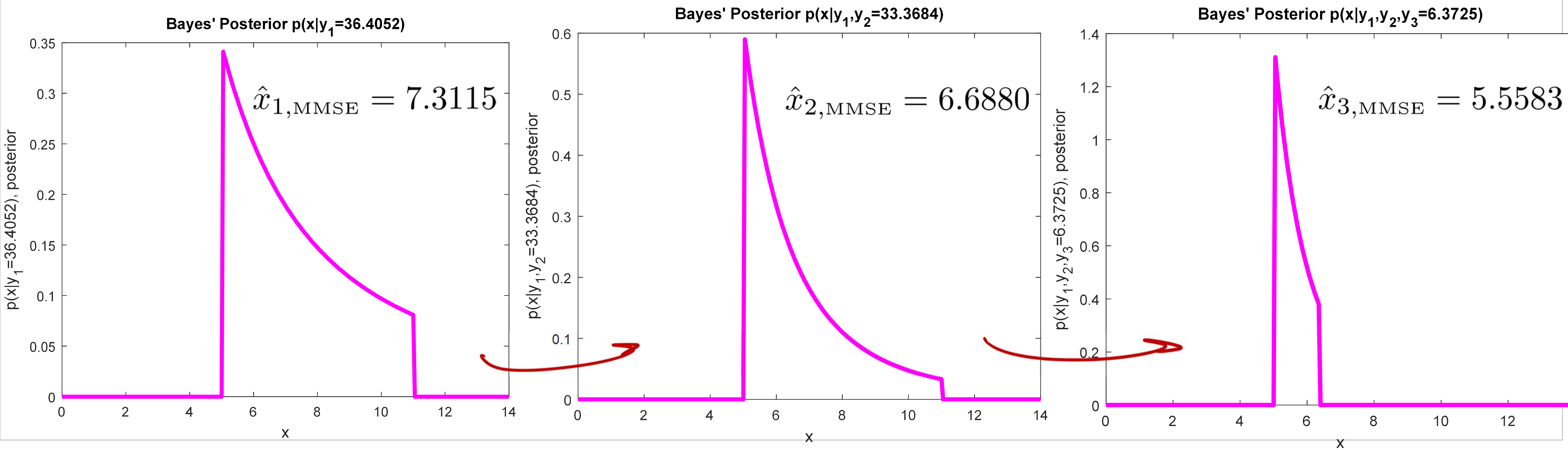
Example: Conditional Uniform PDF



Given joint pdf $p(x,y) = p(x)\ p(y|x) = U_x[1,11]\ \ U_{y|x}[x,(x+1)^2]$

- Suppose true (unknown) $x \sim p(x) = U_x[1,11]$ → $x = 5.0723$
- 3 indep. observations in sequence, $y_k \sim p(y_k|x) = U_{y|x}[x,(x+1)^2]$:

$y_1 = 36.4052,\ y_2 = 33.3684,\ y_3 = 6.3725$



General probabilistic forward state prediction

- Suppose now x is NOT a static random vector, but has dynamics and process noise
- How to deal with general (non-linear/non-Gaussian) forward dynamical state propagation in a probabilistic (Bayesian) way?

(stochastic state transition fxn)

Assumed 1st order Markov process for $x_k \to x_{k+1}$

- Suppose DT dynamics are $x_{k+1} = f(x_k, u_k, w_k, k)$, where $w_k \sim p(w_k)$
- Can generally define the conditional state transition pdf: $x_{k+1} | x_k \sim p(x_{k+1} | x_k)$
- Suppose x_k has known $p(x_k)$ $\longrightarrow$ what is $p(x_{k+1})$?

$\longrightarrow$ using law of total probability, we obtain the Chapman-Kolmogorov Eqn.

$$\circledast \quad \boxed{p(x_{k+1}) = \int_{-\infty}^{\infty} p(x_{k+1} | x_k) \cdot p(x_k)\, dx_k}$$

The basis of probabilistic recursive prediction: $p(x_{k+2}) = \int_{-\infty}^{\infty} p(x_{k+2} | x_{k+1}) \cdot p(x_{k+1})\, dx_{k+1} = \int_{-\infty}^{\infty} p(x_{k+2} | x_{k+1}) \cdot \left[\int_{-\infty}^{\infty} p(x_{k+1} | x_k) \cdot p(x_k)\, dx_k\right] dx_{k+1}$

The General Probabilistic Filtering Problem

- Given Set of measurements (online) $y_1, \ldots, y_k$

  & initial state pdf $p(x_0)$ & state transition pdfs $p(x_{k+1} | x_k)$

  → <u>Find:</u> $p(x_k | y_1, \ldots, y_k)$ for each time step $k = 1, 2, 3, \ldots$

  → & estimate x_k as $\underline{\underline{\hat{x}_k}} = \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \; E\left[c(x_k, \hat{x}_k) \right]_{p(x_k | y_{1:k})}$

The General Bayes Filter (Recursive!)

Initialized with some initial guess of state vector pdf at k=0:

given: $p(x_0)$ (& also know: $p(x_{k+1}|x_k)$ for all $k=0,1,2,\ldots$ & $p(y_k|x)$, & $y_i \perp y_j$ given x)

Consists of two key steps for subsequent time steps k=1,2,3,… :

1. Dynamic Prediction Step (Chapman-Kolmogorov)

@ $k=1$: $p(x_1) = \int_{-\infty}^{\infty} p(x_1|x_0)\cdot p(x_0)\cdot dx_0$

@ $k\geq 2$: $p(x_k|y_{1:k-1}) = \int_{-\infty}^{\infty} p(x_k|x_{k-1})\cdot p(x_{k-1}|y_{1:k-1})\, dx_{k-1}$

2. Bayesian Measurement update step

$$p(x_1|y_1) = \frac{p(x_1)\cdot p(y_1|x_1)}{\int_{-\infty}^{\infty} p(x_1)\cdot p(y_1|x_1)\,dx_1}$$

$$p(x_k|y_1,\ldots,y_k) = \frac{p(x_k|y_{1:k-1})\cdot p(y_k|x_k)}{\int_{-\infty}^{\infty} p(x_k|y_{1:k-1})\cdot p(y_k|x_k)\,dx_k}$$