ASEN 6020: Optimal Trajectories
Necessary Conditions for Optimal Control
D.J. Scheeres, scheeres@colorado.edu

- State equations: $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t)$ where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{u} \in U \subset \mathbb{R}^m$ and $\boldsymbol{f} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$.
- Performance index:

$$J = K(\boldsymbol{x}_o, t_o, \boldsymbol{x}_f, t_f) + \int_{t_o}^{t_f} L(\boldsymbol{x}, \boldsymbol{u}, \tau) d\tau$$

  where K is a scalar function of the terminal states and times and L is a scalar function of the state, controls and time in the interval $[t_o, t_f]$.
- Terminal constraints: $\boldsymbol{g}(\boldsymbol{x}_o, t_o, \boldsymbol{x}_f, t_f) = \boldsymbol{0}$ where $\boldsymbol{g} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^l$, $l \leq 2n+2$.
- Hamiltonian:

$$H(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{u}, t) = L(\boldsymbol{x}, \boldsymbol{u}, t) + \boldsymbol{p} \cdot \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t)$$

  where H is a scalar function and $\boldsymbol{p} \in \mathbb{R}^n$ are the adjoints.
- Optimal control policy:

$$\left.\frac{\partial H}{\partial \boldsymbol{u}}\right|_{\boldsymbol{u}^*} = \boldsymbol{0} \text{ if } \boldsymbol{u} \text{ is in the interior of } U$$

  or

$$\boldsymbol{u}^*(\boldsymbol{x}, \boldsymbol{p}, t) = \arg\min_{\boldsymbol{u}} H(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{u}, t) \text{ if } \boldsymbol{u} \text{ is in the boundary or interior of } U$$

  Leading to the Hamiltonian:

$$H^*(\boldsymbol{x}, \boldsymbol{p}, t) = H(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{u}^*(\boldsymbol{x}, \boldsymbol{p}, t), t)$$

- Dynamics of an optimal control trajectory are defined by the differential equations:

$$\dot{\boldsymbol{x}} = \frac{\partial H^*}{\partial \boldsymbol{p}}$$
$$\dot{\boldsymbol{p}} = -\frac{\partial H^*}{\partial \boldsymbol{x}}$$

- Transversality Conditions:

$$\boldsymbol{p}_o = -\frac{\partial K}{\partial \boldsymbol{x}_o} - \boldsymbol{\lambda} \cdot \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}_o}$$
$$H_o = \frac{\partial K}{\partial t_o} + \boldsymbol{\lambda} \cdot \frac{\partial \boldsymbol{g}}{\partial t_o}$$
$$\boldsymbol{p}_f = \frac{\partial K}{\partial \boldsymbol{x}_f} + \boldsymbol{\lambda} \cdot \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}_f}$$
$$H_f = -\frac{\partial K}{\partial t_f} - \boldsymbol{\lambda} \cdot \frac{\partial \boldsymbol{g}}{\partial t_f}$$

  where $\boldsymbol{\lambda} \in \mathbb{R}^l$ are the constant Lagrange multipliers associated with the constraints $\boldsymbol{g} = \boldsymbol{0}$.