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Delta-V-Based Analysis of Spacecraft Pursuit–Evasion Games

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This work explores a ΔV-based approach to analyzing spacecraft pursuit–evasion games with large initial separation between the spacecraft. Reachable sets in semimajor axis, eccentricity, and inclination space are generated using time-free two-impulse transfers and set intersections are used as a basis for understanding ΔV optimal pursuit and evasion strategies. The evading spacecraft attempts to exit the reachable set of the pursuing spacecraft for a conservative guarantee that capture cannot occur. For cases where an evading spacecraft does not know the extent of the pursuing spacecraft's reachable set, optimal evasion strategies are found. Nonlinear analysis is also performed to find any evasion orbits that are advantageous for the evading spacecraft to escape to. Finally, endgame strategies are briefly explored as well to understand the applicability of this analysis as the distance between the spacecraft decreases.

Nomenclature

F	=	implicit function used to solve for η^*
i	=	orbit inclination
J_{total}	=	sum of J^* values for all spacecraft to transfer to a single final orbit
J^*	=	optimal two-impulse cost for one spacecraft to transfer from one orbit to another
J_0	=	transfer cost J^* from the initial orbit of the pursuing spacecraft to the initial orbit of the evading spacecraft
$[J_e]$	=	optimal costs J^* from the evading spacecraft's initial orbit to a grid of final orbits
$[J_p]$	=	optimal costs J^* from the pursuing spacecraft's initial orbit to a grid of final orbits
$[J_{p-e}]$	=	differenced grid
n_i	=	number of discrete values of i_f for which to calculate J_{total}
n_Q	=	number of discrete values of Q_f for which to calculate J_{total}
n_q	=	number of discrete values of q_f for which to calculate J_{total}
Q	=	orbit radius of apoapsis
q	=	orbit radius of periapsis
V_f	=	spacecraft speed after the second Hohmann transfer maneuver
V_{1t}	=	spacecraft speed after the first Hohmann transfer maneuver
V_{2t}	=	spacecraft speed before the second Hohmann transfer maneuver
V_0	=	spacecraft speed before first Hohmann transfer maneuver
$\Delta V'_{\text{total}}$	=	planar Hohmann transfer cost
ΔV_1	=	first maneuver ΔV for nonplanar two-impulse transfer
ΔV_2	=	second maneuver ΔV for nonplanar two-impulse transfer
η	=	fraction of total inclination change performed at first maneuver in two impulse 3D sequence

Subscripts

f	=	final orbit element
0	=	initial orbit element

I. Introduction

THE pursuit–evasion game is a classic example of a noncooperative multi-agent differential game. In the pursuit–evasion game, a pursuing agent attempts to "capture" an evading agent that attempts to avoid capture. When capture is defined as having the Cartesian position of both agents match (intercept), such games are not only of interest for scenarios where an optimal evasion strategy is desired to protect an agent from an actively adversarial opponent; they are also of interest in understanding the worst case evasion strategy for avoiding objects that may not be actively pursuing the agent of interest. For the orbital case, such objects include orbital debris, which have uncertain dynamics and can drastically impact an agent's ability to perform its primary functions if proper impact avoidance maneuvers are not made. However, if capture is defined as rendezvous (i.e., matching Cartesian position and velocity), a pursuit–evasion game is primarily of interest for analyzing fully antagonistic scenarios. Example scenarios addressed by a pursuit–evasion rendezvous game include an unwanted close inspection of a spacecraft by another spacecraft, or an unwanted modification of a spacecraft by another (e.g., disabling instruments, stealing resources).

The orbital pursuit–evasion game with spacecraft as the agents has been explored in a number of different studies. Recently, Shen and Casalino used an indirect optimal control approach to solve the fully nonlinear three-dimensional (spatial) pursuit–evasion differential game [1]. Similarly, Pontani and Conway used semidirect collocation with nonlinear programming (semi-DCNLP) to solve three-dimensional pursuit–evasion games [2]. Hafer et al. addressed a similar scenario as [2] but with the use of sensitivity methods to solve the problem [3]. Stupik et al. explored pursuit–evasion game solutions with linearized Hill–Clohessy–Wiltshire (HCW) dynamics [4] that apply to more localized problems. Generally, previous approaches have formulated the orbital pursuit–evasion game as an indirect or direct differential game in Cartesian (or equivalent) space and then have attempted to numerically solve the problem with various methods. Each of the preceding works and many studies not included here have also used time to intercept as the objective function; the evading spacecraft maximizes time to intercept, whereas the pursuing spacecraft minimizes it. A time-optimal game is equivalent to a mass- and ΔV-optimal game under the assumption that both spacecraft will continuously thrust at maximum levels throughout the entire game; this assumption was proven to be a feature of a time-optimal game in [1]. Notably, in such a scenario, coasting arcs cannot be used to

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enable a more efficient strategy because efficiency is not a primary goal of that formulation. Some authors have also used a feedback control law approach to explore pursuit–evasion games as well [5,6]. Another common approach is for the pursuer and evader to minimize and maximize the terminal miss distance, respectively [7,8]. Each of these studies is ultimately focused on final capture, where the pursuing agent reaches some threshold distance from the evading agent or it matches the evading spacecraft's spatial position or full state. Previous work on spacecraft evasion has also considered minimum fuel or minimum ΔV evasion maneuvers, but with a pursuer that cannot react to evasive maneuvers [9–11]. Instead, a given radial distance from the nominal interception point is used for evasion in those studies.

The term "capture" representing the goal of the pursuing spacecraft could mean reaching a relative distance within a missile's blast radius, reaching the same position as the evading spacecraft (intercept), or reaching the same position and velocity as the evading spacecraft (rendezvous). Although a majority of previous work has focused on intercept or close approach in position space, this work focuses on rendezvous as the final goal. The use of "capture" throughout will mean rendezvous, whereas the term "intercept" will be used to refer to the more commonly seen intercept goal for pursuit–evasion games.

Though expressing objective functions in terms of continuous variables is incredibly useful in enabling analysis of problems, we note that the fundamental objective for each agent in a pursuit–evasion game is often binary [12]. The evading spacecraft wishes to avoid capture, whereas the pursuing spacecraft wishes to achieve capture. In the present work, a longer-term, strategic approach to pursuit–evasion games is explored with specific interest in understanding optimal strategies for spacecraft with a large initial separation. An approach solely based on time to intercept, as in much of the previous work, may potentially cause either agent to needlessly use fuel when natural dynamics can still be used to their advantage in minimizing fuel use, especially when the initial separation between the agents is large. From a perspective of simply avoiding or achieving capture, maintaining more fuel reserves can make this binary objective more likely for either agent and may allow either agent to have more fuel available to complete other objectives beyond the pursuit–evasion game. In this work ΔV is used instead of time to capture when constructing and evaluating each agent's cost function, and impulsive maneuvers are also assumed. This formulation prioritizes the preservation of fuel over the minimization/maximization of time to intercept, and it is important in enabling solutions where long coasting periods using natural dynamics are allowed. Such a scenario may perhaps be the first phase of a pursuit–evasion game where a pursuing spacecraft allows natural dynamics to do much of the work in coming close to the evading spacecraft, after which in the terminal phase it switches to a continuously on thrust control law based on optimizing time to intercept to ensure that it can capture the evading spacecraft. This approach enables a high-level analysis of an evading spacecraft's risk posture and evasion strategies given different initial orbital placements of a pursuing and evading spacecraft.

Given this focus on using ΔV-based objectives for the pursuing and evading spacecraft, we use ΔV optimal orbit transfers to explore the pursuit–evasion problem where a pursuing spacecraft attempts to match the state of (rendezvous with) an evading spacecraft. Specifically, time-free, impulsive orbit transfers are used because they represent the lowest possible ΔV cost transfers that can be used. In a point mass gravity field perturbed by secular J_2 effects, ΔV must only be expended to match a target semimajor axis a, eccentricity e, and inclination i. Because the J_2 perturbation causes a secular drift rate in right ascension of the ascending node (RAAN, Ω), argument of periapsis (ω), and anomaly (M), any desired final values of those orbit parameters can theoretically be achieved by waiting until natural dynamics perturb them into certain values at the start of the $a-e-i$ transfer such that $\Omega-\omega-M$ drift into the final desired values at the end of the $a-e-i$ transfer [13,14]. Note that the time for this to occur may be infinite (allowable for time-free transfers), and this will only occur if no two angular rates are equal or commensurate with one another. Although in a full pursuit–evasion game some balance between time optimality and ΔV optimality must be found, in this work we specifically focus on the ΔV optimal aspect of the problem as part of an early fuel-conserving phase of the pursuit–evasion problem when the separation between the spacecraft is large in $a-e-i$ space. Thus, the orbit transfers used here only consider costs to achieve a certain $a-e-i$, and the evading spacecraft attempts to ensure that the pursuing spacecraft cannot match its $a-e-i$. We note that these optimal costs can always be achieved to reach a given set of six orbit elements if the spacecraft orbit is placed in a properly phased initial orbit. In a full rendezvous pursuit–evasion game, additional ΔV will potentially need to be expended to match or avoid the matching of $\Omega-\omega-M$, but the strategies described here address considerations for the orbit elements $a-e-i$.

Section II summarizes the two-impulse optimal orbit transfers used in the present work to calculate the cost for either agent to transfer to any other orbit. In Sec. III we discuss the pursuit–evasion game when the total amount of ΔV available to each spacecraft is known, and the reachable sets of both spacecraft can be used to determine if capture is possible. Though only ΔV costs for changing a, e, and i are considered here, reachable sets based on those transfers provide conservative keep-out zones for the evading spacecraft to avoid. If the evading spacecraft remains outside of the pursuing spacecraft's reachable set, it guarantees that capture cannot occur. Section IV explores evasion strategies in the case where the pursuing spacecraft's available ΔV is unknown. In that case, it is beneficial to explore what maneuvers are the most efficient for the evader that would allow it to potentially escape the pursuing spacecraft's reachable set. Finally, in Sec. V, potential endgame strategies in the context of the maneuvers used here are briefly considered. These strategies focus on the terminal phase where the pursuing spacecraft is close to achieving capture.

II. Time- and Orientation-Free Optimal Transfers and Reachable Sets

This work makes use of the two-impulse optimal time-free transfers with Keplerian dynamics that are described in more detail in [14]. Notably, these transfers are ΔV optimal two-impulse transfers in $a-e-i$ space when transfer time is unconstrained and secular J_2 perturbations to Keplerian dynamics are present. We equivalently use periapsis radius q and apoapsis radius Q in place of a and e because optimal transfers occur along lines of constant q and Q. Here, we briefly summarize the relevant aspects of the transfers as they pertain to this pursuit–evasion analysis. In short, these transfers are a combination of elliptical Hohmann transfers for changes in semimajor axis a and eccentricity e as well as dogleg maneuvers to distribute plane changes in inclination i across both impulses for a minimum ΔV transfer from one orbit in $a-e-i$ space to another. Component maneuvers used here are also shown to be allowable ΔV optimal maneuvers for creating optimal reachable sets as proven in [13]. Maneuvers occur tangentially at periapsis and apoapsis, which must also be at a node ($\omega = 0$). Even if assumptions were relaxed to allow impulses at any point and in any direction outside of those restrictions, such maneuvers would not be strictly ΔV optimal for an orbit transfer [14] or ΔV optimal for maximizing the reachable set of orbits [13]. If transferring to an orbit with a larger apoapsis radius, the elliptical Hohmann transfer sequence is to first change apoapsis to the desired level with a tangential maneuver at periapsis, then to change periapsis to the desired level with a tangential maneuver at apoapsis. If transferring to an orbit with a smaller apoapsis, the sequence is reversed; first periapsis is changed and then apoapsis is changed. For a purely planar transfer with no inclination change, each of the two impulses can be split into an infinite number of smaller impulsive maneuvers as long as the entirety of one apsis change is performed before maneuvering to change the other apsis. For example, if a transfer to a larger apoapsis orbit first requires an apoapsis change maneuver of 50 m/s, this can be split into five maneuvers of 10 m/s at different passes of periapsis all performed before making any changes to the periapsis radius. In the three-dimensional case with a dogleg maneuver, however, all of the maneuvering must be performed within two impulses. If strictly ΔV optimal maneuvers of three or more impulses were allowed, maneuvers through infinity (e.g., biparabolic transfer) would be permitted and would result in unbounded reachable sets. Even for orbit transfers where bounded three-impulse transfers are optimal, the required ΔV is large enough to allow escape to infinity as shown in the survey work of Gobetz and Doll [15]. Allowing such transfers would result in unbounded reachable sets, whereas the present analysis is limited to the bounded reachable set case. The assumptions on maneuver number and placement are of course highly restrictive, especially in a pursuit evasion game, but these maneuvers are considered to be part of a high-efficiency phase where spacecraft are attempting to avoid or chase one another in a maximally efficient way, before perhaps transitioning to a more aggressive posture with fewer limits on maneuver types if necessary.

A. Optimal Transfer Cost Calculation

The planar transfer cost is denoted as

$$\Delta V'_{\text{total}} = \Delta V'_1 + \Delta V'_2 \tag{1}$$

where

$$\Delta V'_1 = |V_{1t} - V_0| \tag{2}$$

$$\Delta V'_2 = |V_f - V_{2t}| \tag{3}$$

and subscript t is for the transfer orbit. V_0 is the velocity on the initial orbit immediately before the first impulse, and V_f is the velocity on the final orbit immediately after the second impulse. The optimal two-impulse transfer sequence and ΔV cost relies on the values of the initial and final radius of apoapsis Q_0 and Q_f, where q_0 and q_f are the initial and final periapsis radii. For $Q_f > Q_0$,

$$V_0 = \sqrt{\frac{2\mu}{q_0 + Q_0}\frac{Q_0}{q_0}} \tag{4}$$

$$V_{1t} = \sqrt{\frac{2\mu}{q_0 + Q_f}\frac{Q_f}{q_0}} \tag{5}$$

$$V_{2t} = \sqrt{\frac{2\mu}{q_0 + Q_f}\frac{q_0}{Q_f}} \tag{6}$$

$$V_f = \sqrt{\frac{2\mu}{q_f + Q_f}\frac{q_f}{Q_f}} \tag{7}$$

For $Q_f \le Q_0$,

$$V_0 = \sqrt{\frac{2\mu}{q_0 + Q_0}\frac{q_0}{Q_0}} \tag{8}$$

$$V_{1t} = \sqrt{\frac{2\mu}{q_f + Q_0}\frac{q_f}{Q_0}} \tag{9}$$

$$V_{2t} = \sqrt{\frac{2\mu}{q_f + Q_0}\frac{Q_0}{q_f}} \tag{10}$$

$$V_f = \sqrt{\frac{2\mu}{q_f + Q_f}\frac{Q_f}{q_f}} \tag{11}$$

Equations (2) and (3) can be rewritten without absolute value operators with knowledge of the initial and final states. This is easier to see if Eqs. (4–11) are rewritten in a different form. For example, Eq. (4) can be rewritten as

$$V_0 = \sqrt{2\mu\left(\frac{1}{q_0} - \frac{1}{q_0 + Q_0}\right)} \tag{12}$$

The general expression for the optimal ΔV cost to transfer from one elliptical orbit $\boldsymbol{A}_0 = [q_0, Q_0, i_0]$ to another elliptical orbit $\boldsymbol{A}_f = [q_f, Q_f, i_f]$ in a different plane using a two-impulse sequence is then

$$\begin{aligned}\Delta V_{\text{total}}(\boldsymbol{A}_0, \boldsymbol{A}_f) = \Delta V_{0,f} &= \Delta V_1 + \Delta V_2 \\ &= \sqrt{V_{1t}^2 + V_0^2 - 2V_{1t}V_0\cos(\eta\Delta i)} \\ &\quad + \sqrt{V_f^2 + V_{2t}^2 - 2V_{2t}V_f\cos((1-\eta)\Delta i)}\end{aligned} \tag{13}$$

This equation combines the planar and out-of-plane maneuvers for each impulse using the law of cosines. Fractional inclination changes at each of the two impulses are dictated by the parameter η, which is defined in the range $0 \le \eta \le 1$. The transfer paths in $q - i$ space are shown in Fig. 1, where the initial spacecraft orbit is shown as a black square and paths to various final orbits (solid black circles) are shown with arrows. The open circles in gray indicate the intermediate orbits used in each transfer; they are the orbits reached after the first impulse. For a view of the transfers in $q - Q$ space, see the inset in Fig. 4.

Also note that

$$\Delta i = |i_f - i_0| \tag{14}$$

The η^* that minimizes ΔV_{total} can be found by taking the partial derivative of Eq. (13) with respect to η and equating it to 0. The result,

$$F = \frac{\partial \Delta V_{\text{total}}}{\partial \eta} = \frac{\Delta i V_0 V_{1t}\sin(\eta^*\Delta i)}{\Delta V_1} - \frac{\Delta i V_f V_{2t}\sin((1-\eta^*)\Delta i)}{\Delta V_2} = 0 \tag{15}$$

cannot be explicitly solved for η. The implicit function can, however, be solved using a variety of techniques; in this work a bisection method is used to find the correct value for each individual transfer.

B. Reachable Sets

A reachable set here is described as the set of all possible orbits a given spacecraft can reach given its initial orbit $\boldsymbol{A}_0$ and maximum available $\Delta V_{\max}$. Because transfers here are considered in $q - Q - i$ space, a reachable set R can be written as

$$R(\boldsymbol{A}_0, \Delta V_{\max}) = \{\boldsymbol{A}_f | \Delta V_{\text{total}}(\boldsymbol{A}_0, \boldsymbol{A}_f) \le \Delta V_{\max}\} \tag{16}$$

The reachable set is considered here to include all orbits that cost less than $\Delta V_{\max}$ for the spacecraft to transfer to. That is, the set includes both the extremal surface as well as the interior of the surface.

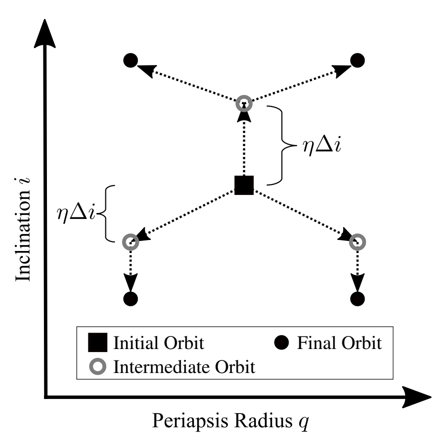


Fig. 1 Three-dimensional orbit transfer paths.

Although reachable sets can be quite difficult to calculate, the analytic expression for the optimal transfer cost in Eq. (13) can be used to relatively quickly calculate reachable sets for spacecraft in $a - e - i$ space. To generate these reachable sets, costs for each spacecraft to transfer to each orbit in a discretized grid of final orbits is calculated. Using that grid, the geometric surface that represents the reachable set of orbits in $q - Q - i$ space can be interpolated given the amount of ΔV available to each spacecraft. The surface defined by any given $\Delta V_{\max}$ value must be found by interpolating between grid points where the actual ΔV has been calculated, so higher-resolution grids with more points will produce more accurate reachable set approximations. This reachable set computation method gives similar results to the method described in [13], with a major difference being that these sets assume a two-impulse structure for orbit transfers (see [14] for more detail), whereas [13] does not. The benefit of this method is a faster, more flexible method of computation that makes the analysis of multiple reachable sets easier and enables some analytic insight that will be seen in later sections.

III. Known Reachable Set Approach to Pursuit–Evasion Games

First, a scenario where the evading spacecraft has knowledge of the pursuing spacecraft's available ΔV is considered. If a reachable set of orbits can be calculated for a spacecraft given its initial orbit and its available ΔV, the reachable sets dictate whether an evading spacecraft can be guaranteed to successfully avoid capture in a pursuit–evasion game. Given the evading spacecraft's reachable set of orbits E and the pursuing spacecraft's reachable set of orbits P, and assuming that both spacecraft maneuver optimally, evasion cannot be guaranteed if

$$E \subseteq P \tag{17}$$

(E is a subset of P). If

$$P \subset E \tag{18}$$

(P is a proper subset of E), then the evading spacecraft is guaranteed to have the ability to avoid capture. If

$$P \cap E = \varnothing \tag{19}$$

(the intersection of P and E is empty, P and E are disjoint), then the evading spacecraft is guaranteed to never be captured because there is no orbit reachable to both agents. In short, the condition $P \cap E = \varnothing$ is a necessary but not sufficient condition for capture to occur; alternatively, the evading spacecraft has a guaranteed ability to escape capture if any part of its reachable set lies outside of the reachable set of the pursuing spacecraft ($E \subseteq P$). A minimum ΔV evasion strategy for guaranteed avoidance of capture would be for the evading spacecraft to shrink its reachable set (by reducing ΔV) until exactly one orbit remains outside of the reachable set of the pursuing spacecraft. That orbit would be a minimum ΔV evasion orbit that guarantees that the pursuing spacecraft cannot reach it. The intersection of both reachable sets $P \cap E$ can be used as a safe keep-out zone for the evading spacecraft to ensure that it cannot be captured.

Notably, in the case where E is a subset of P, we can definitively state that capture is possible when the pursuit–evasion game is a sequential, two-stage game. In the sequential, two-stage game, the evading spacecraft is first allowed to make a two-impulse move to transfer orbits, after which the second spacecraft can make a two-impulse move to attempt to match the orbit of the evading spacecraft. Given this scenario, the evading spacecraft can only maneuver to some other orbit that lies within the reachable set of the pursuing spacecraft, so the pursuing spacecraft can always reach the evading spacecraft. A differential game formulation is better able to account for more complex interactions between the pursuing and evading spacecraft when both spacecraft are allowed to maneuver any number of times and to maneuver in response to actions taken by the other. When any number of maneuvers are allowed and when additional factors such as uncertainty in the other agent's intent and state are introduced, the evader may be able to avoid capture in certain scenarios even when the initial reachable set $E_0 \subset P_0$. Pioneering work by Isaacs [12] presents a useful discourse on the nature of differential games as compared with discrete games with small numbers of actions/maneuvers/moves allowed for each player. A differential game approach is out of scope for the present work.

Figure 2 shows various examples of what the reachable sets look like for different amounts of ΔV available to each spacecraft. Figure 2a shows a disjoint example where $P \cap E = \varnothing$ (capture is impossible), Fig. 2b shows an example where $E \subset P$ (evasion cannot be guaranteed), Fig. 2c shows an example where $P \cap E \neq \varnothing$ (evasion can be guaranteed), and Fig. 2d shows an example where $P \subset E$ (evasion can be guaranteed).

IV. Unknown Pursuer Reachable Set Approach to Pursuit–Evasion Games

Alternatively, if the evading spacecraft has no knowledge of the pursuing spacecraft's total available ΔV, it is important to understand how an evading spacecraft can most efficiently expend ΔV to avoid the pursuing spacecraft. In this scenario, given a pursuing spacecraft with initial orbital state $[\,q_{p,0} \quad Q_{p,0} \quad i_{p,0}\,]$ and an evading spacecraft with initial orbital state $[\,q_{e,0} \quad Q_{e,0} \quad i_{e,0}\,]$, we define the objective function

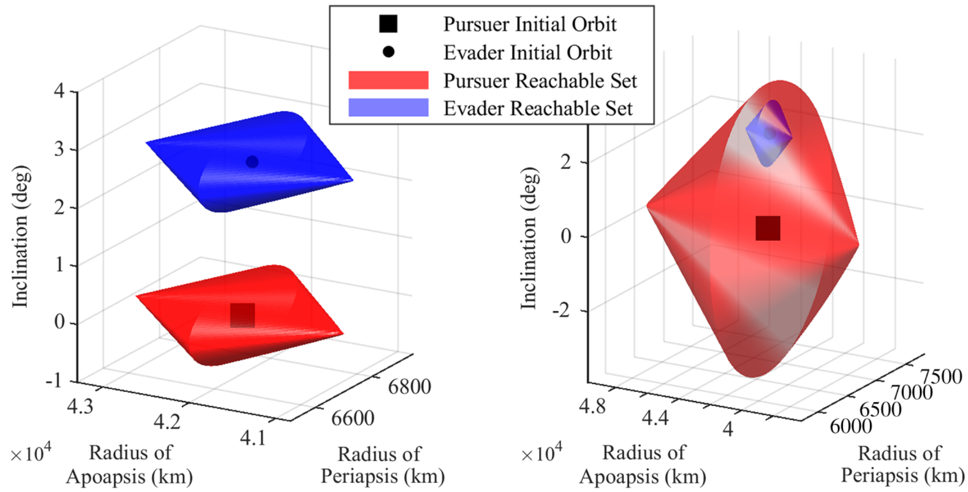
$$J = \Delta V_{p_0,e_f} - \Delta V_{e_0,e_f} \tag{20}$$

which the evading spacecraft seeks to maximize. $\Delta V_{p_0,e_f}$ is the cost for the pursuer to transfer from its initial orbit to the final orbit e_f of the evading spacecraft, and $\Delta V_{e_0,e_f}$ is the cost for the evader to transfer from its initial orbit to its final orbit. The effort to maximize this cost function is in effect an effort to make capture as expensive as possible for the pursuing spacecraft. However, the inclusion of the second term $-\Delta V_{e_0,e_f}$ balances the cost for the evader to transfer to orbit e_f against the increase in $\Delta V_{p_0,e_f}$ that e_f provides. In other words, given that both spacecraft will be expending ΔV as e_f is varied, the evading spacecraft seeks to expend less ΔV to outlast the pursuing spacecraft. The evading spacecraft successfully outlasts the pursuer if the pursuing spacecraft runs out of fuel before rendezvous occurring. For this reason, the relative amount of total fuel used by each spacecraft in kilograms is not of primary importance; the focus on ΔV is in the context of attempting to make an agent use more or less of its total control authority ΔV_{total}.

The pursuer, in contrast, is potentially only attempting to minimize its cost $\Delta V_{p_0,e_f}$ without regard for the amount of fuel expended by the evading spacecraft. Alternatively, it may seek to minimize Eq. (20), which could potentially cause the evading spacecraft to run out of fuel and enable capture. The pursuer is, however, at the mercy of the evader in that it must go to whichever orbit the evading spacecraft transfers to. In this section, we explore both linearized and nonlinear approaches to evaluating strategies for the evading spacecraft to maximize J. In the linearized analyses, we take partial derivatives of J as e_f is varied away from e_0 to understand local evasion strategies. In the nonlinear analysis, we perform grid searches and evaluate J across a wide array of e_f values to understand optimal evasion strategies.

A. Linearized Pursuit–Evasion Analysis

First-order insight into the pursuit–evasion game can be gained by taking partial derivatives of the analytic expressions for the ΔV cost required for each spacecraft to change orbits. From the evading spacecraft's perspective, the strategies of interest are incremental maneuvers it can perform at its initial orbit to most efficiently escape the pursuing spacecraft. Because it is already known that ΔV optimal in-plane maneuvers will be to change periapsis or apoapsis, these are the candidate evasion strategies with which to compare the pursuing spacecraft's ΔV costs to. An example scenario is shown in Fig. 3,



a) Capture impossible

b) Evasion not guaranteed

c) Evasion can be guaranteed

d) Evasion can be guaranteed

Fig. 2 Various scenarios for pursuit–evasion reachable sets.

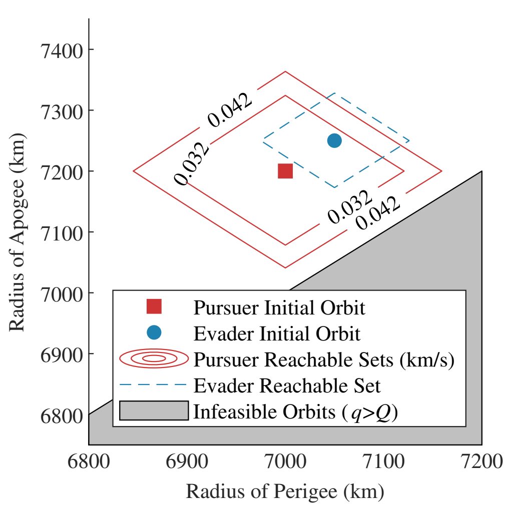


Fig. 3 Potential evasion scenario showing possible extents of pursuer reachable set.

where the incremental maneuvers found in this section will give the optimal evasion direction for the evading spacecraft to move in order to possibly exit the unknown reachable set of the pursuing spacecraft. In the case where the evading spacecraft maneuvers to change apoapsis, the relation of interest is

$$J_Q \equiv \frac{\partial J}{\partial Q_f}\bigg|_{Q_f=Q_{0,e}} = \frac{\partial \Delta V'_p}{\partial Q_f}\bigg|_{Q_f=Q_{0,e}} - \frac{\partial \Delta V'_e}{\partial Q_f}\bigg|_{Q_f=Q_{0,e}} \tag{21}$$

where a p subscript indicates the ΔV magnitude of the pursuing spacecraft, and an e subscript indicates the ΔV magnitude of the evading spacecraft. Q_f for the pursuing spacecraft is the evading spacecraft's initial apoapsis $Q_{0,e}$, because nominally the pursuing spacecraft must reach the initial orbit of the evading spacecraft. Thus, to explore incremental maneuvers from its nominal state, Q_f for the evading spacecraft is its own initial apoapsis $Q_{0,e}$. The quantity J_Q represents how much more ΔV a pursuing spacecraft must expend than an evading spacecraft to exact the same change in radius of apoapsis from the nominal $Q_f = Q_{0,e}$. A positive value indicates an advantage for the evading spacecraft, a negative value indicates an advantage for the pursuing spacecraft, and a zero value indicates that both spacecraft maneuver with the same efficiency in that direction. Similarly, the relation of interest when the evading spacecraft maneuvers to change periapsis is

$$J_q \equiv \frac{\partial J}{\partial q_f}\bigg|_{q_f=q_{0,e}} = \frac{\partial \Delta V'_p}{\partial q_f}\bigg|_{q_f=q_{0,e}} - \frac{\partial \Delta V'_e}{\partial q_f}\bigg|_{q_f=q_{0,e}} \tag{22}$$

The sign of the quantity J_q has the same interpretation as the quantity J_Q, with $q_f = q_{0,e}$, but is directed in the periapsis direction. The analytic expressions for the optimal ΔV cost of in-plane maneuvers allow the direct evaluation of these partial derivatives. However,

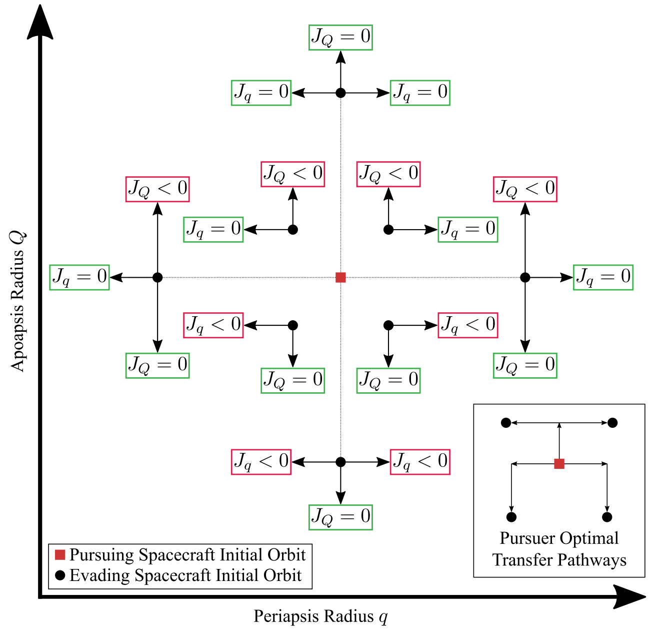


Fig. 4 J_Q and J_q values for different evasion strategies and different initial pursuer and evader orbits.

because the costs in Eqs. (2) and (3) have absolute value operators, the partials will be evaluated separately for each case of initial evading spacecraft orbit location relative to the initial orbit of the pursuing spacecraft. The final values for J_Q and J_q are shown in Fig. 4 for different relative positions of pursuer and evader, as well as for different evasion directions. Note that the zero value directions all occur in the same directions as optimal transfers noted in the bottom right corner of Fig. 4. For example, if a spacecraft is pursuing an evading spacecraft with a smaller q and larger Q, then the zero value direction is along the direction of the second transfer arrow for an optimal transfer to an orbit with a smaller q and larger Q. Importantly, no positive values of J_Q and J_q are found, so the evading spacecraft can at best maneuver as efficiently as the pursuing spacecraft. Figure 5 summarizes Fig. 4 by showing the evasion directions where J_q and J_Q are zero for different relative initial orbits between the pursuer and evader.

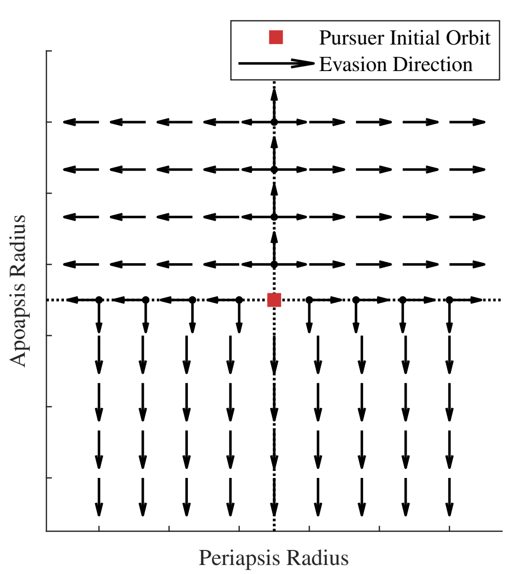


Fig. 5 Strategy for the evading spacecraft given different initial conditions.

1. Cases Where the Evading Spacecraft Has a Larger Apoapsis

When $Q_e > Q_p, J_q = 0$ for all relative values of q_e and q_p. That is, both spacecraft maneuver equally as efficiently in the periapsis direction. To exact one unit of change in periapsis radius, both spacecraft must expend the same amount of ΔV. The parameter J_Q has also been numerically found to be negative in general for $Q_e > Q_p$, though in the special case where $q_e = q_p, J_Q = 0$. Thus, in general the pursuing spacecraft has an advantage in the apoapsis direction. To exact one unit of change in apoapsis radius, the pursuing spacecraft must expend less ΔV than the evading spacecraft.

To find the values J_q and J_Q, partial derivatives of the cost J are derived and then evaluated. Partial derivatives of orbital speed at periapsis and apoapsis with respect to Q and q are frequently found in the equations of interest, so we first define some convenience functions. Partial derivatives of orbital speed at apoapsis V_Q are defined as

$$\Gamma_1(q,Q) \equiv \frac{\partial V_Q}{\partial q} = \frac{\mu(q+Q)^{-2}}{\sqrt{2\mu[Q^{-1}-(Q+q)^{-1}]}} = \frac{\mu(q+Q)^{-2}}{V_Q} \tag{23}$$

$$\Gamma_2(q,Q) \equiv \frac{\partial V_Q}{\partial Q} = \frac{\mu[(q+Q)^{-2}-Q^{-2}]}{\sqrt{2\mu[Q^{-1}-(Q+q)^{-1}]}} = \frac{\mu[(q+Q)^{-2}-Q^{-2}]}{V_Q} \tag{24}$$

and partial derivatives of orbital speed at periapsis V_q are defined as

$$\Gamma_3(q,Q) \equiv \frac{\partial V_q}{\partial Q} = \frac{\mu(q+Q)^{-2}}{\sqrt{2\mu[q^{-1}-(Q+q)^{-1}]}} = \frac{\mu(q+Q)^{-2}}{V_q} \tag{25}$$

$$\Gamma_4(q,Q) \equiv \frac{\partial V_q}{\partial q} = \frac{\mu[(q+Q)^{-2}-q^{-2}]}{\sqrt{2\mu[q^{-1}-(Q+q)^{-1}]}} = \frac{\mu[(q+Q)^{-2}-q^{-2}]}{V_q} \tag{26}$$

For cases where $q_f \geq q_0$ and $Q_f > Q_0$, Eqs. (2–7) give the optimal cost. However, the absolute value operators in Eqs. (2) and (3) can be removed, because the quantities within the operators will be positive for these values of q and Q. Evaluating the partial derivatives gives the following results:

$$\frac{\partial \Delta V'_{\text{total}}}{\partial q_f} = \Gamma_1(q_f, Q_f) \tag{27}$$

$$\frac{\partial \Delta V'_{\text{total}}}{\partial Q_f} = \Gamma_3(q_0, Q_f) + \Gamma_2(q_f, Q_f) - \Gamma_2(q_0, Q_f) \tag{28}$$

For cases where $q_f < q_0$ and $Q_f \geq Q_0$, the only change is to Eq. (3). The absolute value operators in Eqs. (2) and (3) can be removed, but Eq. (3) must be multiplied by -1 to ensure that $\Delta V'_2$ remains positive. Re-evaluating the partial derivatives gives the following results:

$$\frac{\partial \Delta V'_{\text{total}}}{\partial q_f} = -\Gamma_1(q_f, Q_f) \tag{29}$$

$$\frac{\partial \Delta V'_{\text{total}}}{\partial Q_f} = \Gamma_3(q_0, Q_f) - \Gamma_2(q_f, Q_f) + \Gamma_2(q_0, Q_f) \tag{30}$$

Notice that Eqs. (27) and (29) have no dependence on initial condition, which is the only factor that differentiates ΔV_{total} between the pursuing and evading spacecraft when both are changing q_f in the same direction. Thus, for $Q_e \geq Q_p, J_q = 0$, and both spacecraft must expend ΔV at the same rate while the evading spacecraft maneuvers to change its periapsis radius away from the pursuing spacecraft. The value of J_Q is less clear in this case; J_Q is evaluated as follows:

$$J_Q = \frac{\partial \Delta V'_p}{\partial Q_f}\bigg|_{\substack{q_0=q_p,Q_0=Q_p,\\ q_f=q_e,Q_f=Q_e}} - \frac{\partial \Delta V'_e}{\partial Q_f}\bigg|_{\substack{q_0=q_e,Q_0=Q_e,\\ q_f=q_e,Q_f=Q_e}} \tag{31}$$

Because the initial and final orbits are the same for

$$\frac{d\Delta V'_e}{dQ_f}\bigg|_{\substack{q_0=q_e,Q_0=Q_e,\\ q_f=q_e,Q_f=Q_e}}$$

the value of the partial derivative is different based on whether Q_f is being increased or decreased. Here it is presupposed that $Q_e > Q_p$, so Q_e must be increased to move in a direction that exits the pursuing spacecraft's reachable set. For this reason, Eq. (28) or (30) can be used to evaluate Eq. (31). For the partial derivative of the evading spacecraft's total ΔV, Eqs. (28) and (30) are equivalent because $q_0 = q_f = q_e$, but the partial derivative of the pursuing spacecraft's total ΔV changes depending on whether $q_e > q_p$ [use Eq. (28)] or $q_e < q_p$ [use Eq. (30)].

Evaluating J_Q for $q_e > q_p$ and $Q_e > Q_p$ gives

$$J_Q = \Gamma_3(q_p, Q_e) + \Gamma_2(q_e, Q_e) - \Gamma_2(q_p, Q_e) - \Gamma_3(q_e, Q_e) \tag{32}$$

noting that the Γ_2 terms for the evading spacecraft cancel each other out.

For $q_e < q_p$ and $Q_e > Q_p$, the expression for J_Q is

$$J_Q = \Gamma_3(q_p, Q_e) - \Gamma_2(q_e, Q_e) + \Gamma_2(q_p, Q_e) - \Gamma_3(q_e, Q_e) \tag{33}$$

Although inspecting Eqs. (32) and (33) does not immediately reveal any clear understanding of the sign of J_Q in the general case, it is clear that $J_Q = 0$ for $q_e = q_p$. This is because terms 2 and 3 as well as terms 1 and 4 cancel out in both equations. Thus, when $q_e = q_p$, the evading spacecraft can maneuver either to increase/decrease periapsis or increase apoapsis without disadvantaging itself.

For low-Earth-orbit test cases where $q_e < q_p$ or $q_e > q_p$, J_Q appears to always be negative in numeric test cases. Despite considerable effort, we were unable to find any cases where $J_Q > 0$. This indicates that, in general, for an evading spacecraft with a larger apoapsis than a pursuing spacecraft, it should maneuver to change its periapsis away from the periapsis of the pursuing spacecraft. That is, it should increase periapsis if $q_e > q_p$, or decrease periapsis if $q_e < q_p$. In the special case where $q_e = q_p$ and $Q_e > Q_p$, the evading spacecraft can either maneuver to increase apoapsis or it can maneuver to change periapsis in either direction. Other evasion strategies are certainly possible, but such strategies will either move the evading spacecraft closer to the pursuing spacecraft in $Q - q$ space, or it will require the evading spacecraft to expend more ΔV than a pursuing spacecraft would need to cause the same changes in orbit.

2. *Cases Where the Evading Spacecraft Has a Smaller Apoapsis*

For cases $Q_e \le Q_p$ the results essentially mirror the previous case. That is, $J_Q = 0$ for all relative values of q_e and q_p, and J_q has been numerically found to be negative in general for $Q_e < Q_p$. However, for the special case where $Q_e = Q_p$, $J_q = 0$. Thus, in general the pursuing spacecraft has an advantage in the periapsis direction.

For cases where $q_f \ge q_0$ and $Q_f \le Q_0$, Eqs. (2), (3), and (8–11) give the optimal cost. However, under these constraints, Eqs. (2) and (3) can be written without the absolute value operators because the results will still be positive. Then, evaluating the partial derivatives gives the following results:

$$\frac{\partial \Delta V'_{\text{total}}}{\partial q_f} = \Gamma_1(q_f, Q_0) + \Gamma_4(q_f, Q_0) - \Gamma_4(q_f, Q_f) \tag{34}$$

$$\frac{\partial \Delta V'_{\text{total}}}{\partial Q_f} = -\Gamma_3(q_f, Q_f) \tag{35}$$

For cases where $q_f < q_0$ and $Q_f < Q_0$, the absolute value operators in Eqs. (2) and (3) are removed, and Eq. (2) is multiplied by -1 before the partial derivatives are evaluated. This gives the following results:

$$\frac{\partial \Delta V'_{\text{total}}}{\partial q_f} = -\Gamma_1(q_f, Q_0) + \Gamma_4(q_f, Q_0) - \Gamma_4(q_f, Q_f) \tag{36}$$

$$\frac{\partial \Delta V'_{\text{total}}}{\partial Q_f} = -\Gamma_3(q_f, Q_f) \tag{37}$$

Similar to the value of J_q for $Q_e \ge Q_p$, Eqs. (35) and (37) show that for $Q_e < Q_p$, $J_Q = 0$.

Finding J_q is more complex for $Q_e < Q_p$, and is evaluated as follows:

$$J_q = \frac{\partial \Delta V'_p}{\partial q_f}\bigg|_{\substack{q_0=q_p,Q_0=Q_p,\\ q_f=q_e,Q_f=Q_e}} - \frac{\partial \Delta V'_e}{\partial q_f}\bigg|_{\substack{q_0=q_e,Q_0=Q_e,\\ q_f=q_e,Q_f=Q_e}} \tag{38}$$

In this case, the sign of $\partial \Delta V_e / \partial q_f$ will change depending on whether q_e is greater than or less than q_p, but it will have the same magnitude because $Q_0 = Q_f = Q_e$. Evaluating J_q for $q_e > q_p$ and $Q_e < Q_p$ gives

$$J_q = \Gamma_1(q_e, Q_p) + \Gamma_4(q_e, Q_p) - \Gamma_4(q_e, Q_e) - \Gamma_1(q_e, Q_e) \tag{39}$$

When $q_e < q_p$ and $Q_e < Q_p$, evaluating J_q is slightly different because the evading spacecraft will be decreasing periapsis to move away from the pursuing spacecraft. Equation (36) gives the partial derivative for positive changes in q, so when using Eq. (36) to evaluate J_q it must be multiplied by -1 to account for the fact that both spacecraft will be making negative changes in q. Thus, evaluating J_q for $q_e < q_p$ and $Q_e < Q_p$ where the evading spacecraft is decreasing periapsis gives

$$J_q = -\Gamma_1(q_e, Q_p) + \Gamma_4(q_e, Q_p) - \Gamma_4(q_e, Q_e) + \Gamma_1(q_e, Q_e) \tag{40}$$

Similar to what was found for J_Q when $Q_e > Q_p$, J_q always appears to be negative. This indicates that the most prudent direction for the evader to move is to decrease apoapsis.

All results for J_q and J_Q are summarized in Table 1.

3. *Limiting Cases*

When the evading spacecraft has the same apoapsis as the pursuing spacecraft, it has two evasion strategies where either J_Q or J_q is zero. Because it is on the edge between the two cases $Q_e < Q_p$ and $Q_e > Q_p$ explored above, it has the same properties as the $Q_e < Q_p$ case when decreasing apoapsis, and the same properties as the $Q_e > Q_p$

Table 1 Summary of partial derivative expressions

	$q_e < q_p$	$q_e > q_p$
$Q_e > Q_p$	$J_q = 0, J_Q = \Gamma_3(q_p, Q_e) - \Gamma_2(q_e, Q_e) + \Gamma_2(q_p, Q_e) - \Gamma_3(q_e, Q_e)$	$J_q = 0, J_Q = \Gamma_3(q_p, Q_e) + \Gamma_2(q_e, Q_e) - \Gamma_2(q_p, Q_e) - \Gamma_3(q_e, Q_e)$
$Q_e < Q_p$	$J_Q = 0, J_q = -\Gamma_1(q_e, Q_p) + \Gamma_4(q_e, Q_p) - \Gamma_4(q_e, Q_e) + \Gamma_1(q_e, Q_e)$	$J_Q = 0, J_q = \Gamma_1(q_e, Q_p) + \Gamma_4(q_e, Q_p) - \Gamma_4(q_e, Q_e) - \Gamma_1(q_e, Q_e)$

when changing periapsis away from the pursuing spacecraft. When decreasing apoapsis, $J_Q = 0$ for the same reasons discussed for $Q_e < Q_p$. When moving periapsis away from the pursuing spacecraft, $J_q = 0$ for the same reasons discussed for $Q_e > Q_p$. However, when testing values of J_Q as the evader increases apoapsis, we find that $J_Q < 0$.

As Q_e approaches ∞, we find that

$$\lim_{Q_e\to\infty} J_Q = 0 \tag{41}$$

regardless of the relative values of q_e and q_p.

4. Discussion

This analysis has demonstrated that the evading spacecraft has very specific directions along which it can maneuver as efficiently as the pursuing spacecraft; these directions coincide with the optimal orbit transfer directions for the evading spacecraft to move to an orbit farther away from the pursuing spacecraft in $Q - q$ space. If the evading spacecraft maneuvers in a manner inconsistent with the optimal transfer pathways, then it will be maneuvering in a non-ΔV optimal manner and shrinking its reachable set of orbits. Thus, we expect these alternative maneuver directions to not be ideal candidates for ΔV optimal evasion.

General strategies for pursuing and evading spacecraft can be designed based on these results. A limitation not previously discussed is the line of circular orbits where $Q = q$. Note that as seen in Fig. 4 the best evasion strategy for a spacecraft with a lower periapsis than a pursuing spacecraft ($Q_e < Q_p$) is to decrease apoapsis. However, once apoapsis is decreased to the point where $Q_e = q_e$, decreasing the radius of any point of the orbit would be equivalent to decreasing periapsis, which is not an efficient maneuver for the evading spacecraft ($J_q < 0$). This can also occur when the evader is in the northeast quadrant ($Q_e > Q_p$, and $q_e > q_p$), because the optimal strategy to increase periapsis can only occur until $q_e = Q_e$. Thus, an evader should avoid being in a circular orbit where $Q_e = q_e$, because from that orbit there is no prudent direction for it to transfer. The other limiting line to consider is where $q_e = r_c$, where r_c is the radius of the central body or the radius of the central body plus some altitude limit. Of course, if the optimal evasion strategy for the evading spacecraft is to decrease periapsis, it can only do so until periapsis is just above this limit. Thus, a pursuer ideally would initially position itself such that it leverages the limit of a circular orbit or the limit of q_e impacting the central body, whichever the evading spacecraft would be closer to. These limits are shown in Fig. 6 with red octagons. The only evasion direction without a limit is the scenario where the evading spacecraft has $Q_e > Q_p$ and $q_e = q_p$, but this is a highly constrained position that may not be possible while satisfying other mission objectives. The next best initial orbit for the evading spacecraft would give it the most runway possible before hitting a limit, giving the most potential to move outside of the reachable set of the pursuing spacecraft. That is, the initial orbit with the preferred risk posture would satisfy other mission constraints and maximize the ΔV between the orbit and the limit on the evasion strategy.

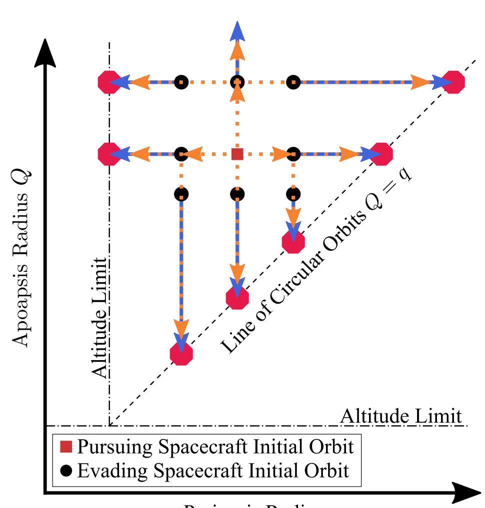


Fig. 6 Limits on evasion strategies.

The best-case strategies for an evading spacecraft found here only allow the evading spacecraft to maneuver equally as efficiently as a pursuing spacecraft. Thus, such a strategy is likely only viable in the case where the evading spacecraft believes that it has enough ΔV capability to outlast the pursuing spacecraft. This motivates future investigation into the use of orbit phasing maneuvers as an evasion tool in a pursuit/evasion game. Indeed, the strategies found here do not take into account the need for the pursuer to match the evader's orbit phasing for true capture, because time-free transfers are being used. The need to match phasing can allow an evader to destroy any exact timing used by a pursuing spacecraft to attempt rendezvous.

The preceding analysis has entirely focused on the planar case, but when transfers in inclination space are also considered then we have the additional quantity of interest

$$J_i \equiv \frac{\partial J}{\partial i_f} = \frac{\partial \Delta V_p}{\partial i_f} - \frac{\partial \Delta V_e}{\partial i_f} \tag{42}$$

Equations (21) and (22) for J_Q and J_q can similarly be evaluated in the three-dimensional case using the full ΔV_{total} equation in Eq. (13) to describe the ΔV cost instead of the planar $\Delta V'_{\text{total}}$ in Eq. (1). In the three-dimensional case, however, Eqs. (21) and (22) cannot be easily evaluated or reduced analytically. Through numerical calculation in sample scenarios, J_Q, J_q, and J_i appear to always be negative. That is, the pursuing spacecraft appears to always be more efficient, and there do not appear to be any special directions where both spacecraft are equally efficient. This, in part, motivates a nonlinear analysis of the objective function in order to gain a better understanding of evasion strategies.

B. Nonlinear Pursuit–Evasion Analysis

The previous method gives an understanding of local, incremental evasion strategies, but does not give a full understanding of relative advantages for each spacecraft to transfer to any given final orbit. The baseline scalar cost for the pursuing spacecraft to reach the evading spacecraft's initial orbit is denoted as $\Delta V_0 = \Delta V(q_{0,p}, Q_{0,p}, i_{0,p}, q_{0,e}, Q_{0,e}, i_{0,e})$. The p subscript denotes the pursuer, and the e subscript denotes the evader. A grid of pursuer optimal transfer costs is denoted as

$$[\Delta V_p] = \Delta V\left(q_{0,p}, Q_{0,p}, i_{0,p}, q_{f,1:n_q}, Q_{f,1:n_Q}, i_{f,1:n_i}\right) \tag{43}$$

and the grid of evader optimal transfer costs as

$$[\Delta V_e] = \Delta V\left(q_{0,e}, Q_{0,e}, i_{0,e}, q_{f,1:n_q}, Q_{f,1:n_Q}, i_{f,1:n_i}\right) \tag{44}$$

Both grids are of dimension $n_q \times n_Q \times n_i$. Using these grids, the differenced grid can be calculated as

$$[J_{p-e}] = [\Delta V_p] - [\Delta V_e] - \Delta V_0 \tag{45}$$

The differenced grid $[J_{p-e}]$ is a grid where each value is simply the objective function Eq. (20) with the baseline capture cost ΔV_0 subtracted out. In this grid, zero values indicate that both spacecraft must expend the same amount of additional ΔV to reach a given orbit. Negative values mean that an evading spacecraft must expend more ΔV to reach the same orbit as the pursuing spacecraft, and positive values mean that an evading spacecraft must expend less ΔV to reach the same orbit as the pursuing spacecraft. In other words, the differenced grid represents how much additional ΔV a pursuing spacecraft must expend as compared with the evading spacecraft in order to capture the evader at a given orbit.

In the planar case we have not found any orbits with positive values in the differenced grid, which indicates that even in a nonlinear sense the evader cannot gain an advantage while in-plane with the pursuing spacecraft. The linear planar evasion strategies that have been found in the previous section can clearly be seen in the differenced grid, as shown in the example in Fig. 7 with the 0 contour line.

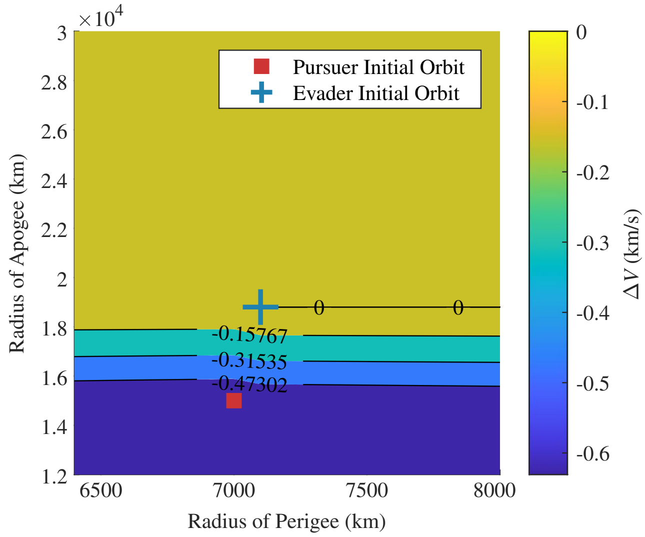
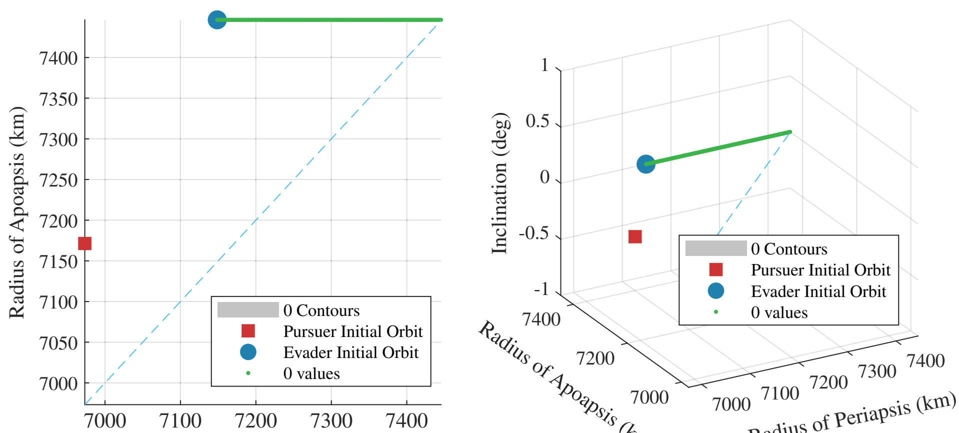


Fig. 7 Differenced grid contour lines.

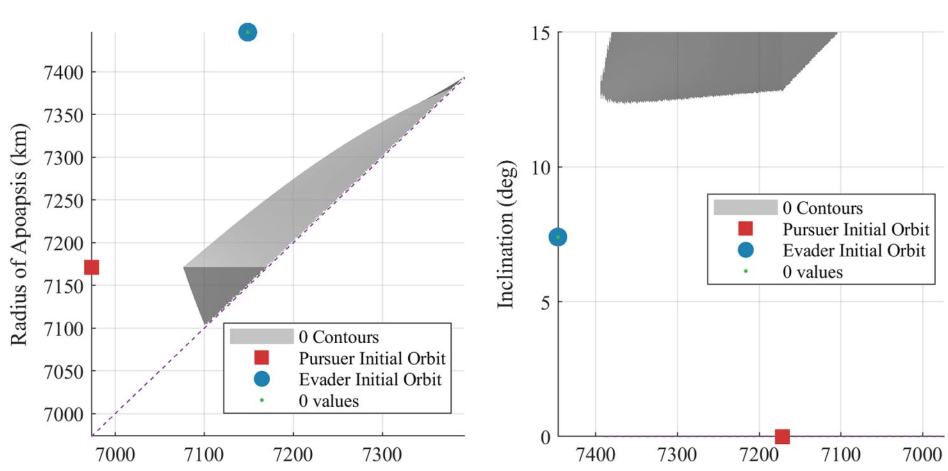
For the three-dimensional case, a majority of orbits in the differenced grid are found to have negative values while some orbits have 0 values, similar to the planar case. Figure 8 shows the resulting differenced grid for a sample scenario where the pursuer and evader begin in the same orbital plane ($i_p = i_e = 0$ deg). Zero values, indicated by green dots, follow exactly the continuous planar evasion strategy as found in Sec. IV.A by examining partial derivatives. The continuous line of 0 values indicates that the evading spacecraft can impulsively raise its periapsis radius multiple times as part of an evasion strategy that allows it to maneuver as efficiently as a pursuing spacecraft.

Interestingly, in certain cases there are regions where the evading spacecraft can transfer at a lower cost than the pursuing spacecraft. These regions are shown by 0 level contours in gray; the fact that the contours have nonzero volume indicates that there are positive values of the differenced grid within them. The contour is easily seen in the scenario shown in Fig. 9. In this particular example, there are two reasons this special region is found. First, the evading spacecraft has a larger apoapsis radius than the pursuing spacecraft and thus can perform more efficient inclination changes at slower speeds. The second, more general, reason is that the dogleg maneuver leads to nonintuitive effects on relative costs to transfer orbits. If the dogleg maneuver is not used, and instead the total inclination change for an orbit transfer is just performed at the largest (slowest) apoapsis encountered in the Hohmann transfer (giving a three-impulse transfer), then as expected the same planar evasion strategies as seen in Fig. 8 are found for nonplanar problems as in Fig. 9. In this case, both effects are necessary to create the evader-advantaged region; if the initial conditions of the pursuing spacecraft and evading spacecraft are switched but dogleg maneuvers are used, no positive-valued regions in the differenced grid are found.



a) b)

Fig. 8 Differenced grid for a planar pursuit–evasion game.



a) b)

Fig. 9 Differenced grid for a nonplanar pursuit–evasion game.

These positive regions, however, require quite large ΔV expenditures; in this case they can approach up to $0.13V_c$ (approx. 1 km/s), where V_c is the local circular velocity at periapsis of the evading spacecraft. Further, upon transferring to an orbit in this positive region, there still is no nearby 0 or positive point in the new differenced grid. Further, the advantage the evading spacecraft has in these positive region is quite low. In results generated for Fig. 9, the evader's ΔV advantages are generally on the order of $0.0001V_c$ (a few meters per second) or less. For very large maneuvers this small of an advantage over an evading spacecraft may not truly be useful.

It also appears that when the two spacecraft do not begin in the same plane there is no longer an incremental evasion strategy like that seen in the planar case. That is, there is no continuous line of 0 values seen extending from the evading spacecraft's initial orbit. This indicates that there is likely no such strategy directly in the q, Q, or i directions. However, it is certainly possible that there is a more complex, combined maneuver that allows the evader to incrementally maneuver as efficiently as the evading spacecraft. Because this combined maneuver strategy would not have a clean line of grid points evaluated along it, it would not be trivial to see the resulting 0 points in the differenced grids produced here. Differenced grids with very fine resolution centered closely around the evading spacecraft have been generated, but a continuous line of 0 points has not been found.

V. Endgame Strategy

In this section we explore the terminal phase of the rendezvous pursuit–evasion game given the types of maneuvers considered here. First, suppose that in the course of the pursuit–evasion game the evading spacecraft has expended all available fuel and can no longer maneuver. Further, suppose that the evading spacecraft remains within the reachable set of the pursuing spacecraft. In such a scenario, the terminal phase of the game will appear as in Fig. 10. Before this terminal phase, the pursuing spacecraft has already matched the right ascension of the ascending node Ω and argument of periapsis ω of the evading spacecraft and has timed its final coast arc so that it intersects the evading spacecraft at the correct time t_{q_0}. At time t_{q_0} the pursuing spacecraft matches the position of the evading spacecraft (with a slight offset), and then performs an impulsive ΔV maneuver to correct the final inclination difference of $(1-\eta)\Delta i_p$ and to match the apoapsis radius of the evading spacecraft. If rendezvous occurs at apoapsis, then a similar maneuver is performed but to match periapsis radius instead of apoapsis radius.

Note that if the evader lies outside of the reachable set of the pursuing spacecraft, the scenario depicted in Fig. 10 is still possible with the difference being that the pursuing spacecraft does not have enough ΔV to perform a full rendezvous maneuver at the capture point. Because the rendezvous maneuver is not possible in this case, it is considered a successful evasion and a win for the evading spacecraft. This scenario, however, could potentially result in a successful intercept of the evading spacecraft where the two spacecraft collide. Given that the focus of this work is on the rendezvous pursuit–evasion game, we suppose that the pursuer is not interested in this even as a secondary outcome and would not destroy the evading spacecraft. Despite the non-cooperative and actively antagonistic nature of the problem, it is realistic to suppose that a pursuing agent may not want to destroy itself, may not want to generate excessive space debris with a collision, may not want to destroy a target without a chance to inspect or modify it, etc.

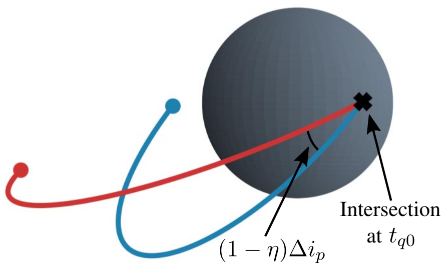


Fig. 10 Endgame scenario.

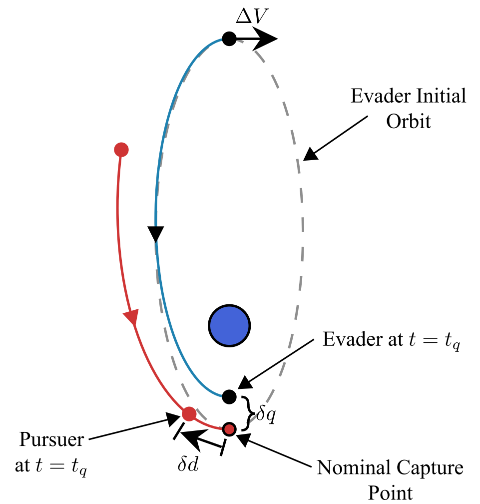


Fig. 11 Endgame scenario with evasive maneuver.

Now in a similar terminal scenario as in Fig. 10, suppose that the evading spacecraft has a small amount of fuel left, has a matching inclination with the pursuing spacecraft as well as Ω and ω, and has initial orbit parameters q_0 and Q_0. Figure 11 shows this scenario for the case where the pursuing spacecraft is attempting to capture the evading spacecraft at its periapsis. In this scenario, the evading spacecraft is initially at its apoapsis and can perform a tangential, impulsive maneuver to change its periapsis radius by an amount δq to ensure that the two orbits no longer intersect. The new time of periapsis passage for the evader after it performs a maneuver is t_q. A first-order estimate of the distance from the pursuer at time t_q to the position of the pursuer at time t_{q0} is δd. Here, we investigate the relative magnitudes of δq and δd to first order. The first-order quantity δd is

$$\delta d = (t_{q_0} - t_q)V_{q_0} = \delta T V_{q_0} \tag{46}$$

where V_{q_0} is the orbital speed at periapsis of the evader's initial orbit. To find δT, the partial derivative of the equation for orbital period is used to find

$$\frac{\partial T}{\partial q} = \frac{3\pi}{2\sqrt{\mu}} a_0^{1/2} \frac{\partial a}{\partial q} \tag{47}$$

$$\frac{\partial a}{\partial q} = \frac{1}{2} \tag{48}$$

$$\delta T = \frac{3\pi}{4\sqrt{\mu}} a_0^{1/2} \delta q \tag{49}$$

Combining this with Eq. (46) and the expression for V_{q_0} results in the ratio of interest,

$$\frac{\delta d}{\delta q} = \frac{3\pi}{4}\sqrt{\frac{Q_0}{q_0}} \tag{50}$$

Because $Q \geq q$ and $3\pi > 4$, the ratio $\delta d/\delta q$ is always greater than one. That implies that, to first order, one unit change in q will result in a larger change to d.

If the scenario is changed to have the evading spacecraft at its own periapsis at the initial time, and it makes a change δQ to avoid capture, the same ratio becomes

$$\frac{\delta d}{\delta q} = \frac{3\pi}{4}\sqrt{\frac{q_0}{Q_0}} \tag{51}$$

In this case, $(\delta d/\delta q) > 1$ if $\sqrt{(q_0/Q_0)} > [4/(3\pi)]$. The alternative, $\sqrt{(q_0/Q_0)} \leq [4/(3\pi)]$, indicates that the orbit q_0, Q_0 is highly elliptic with an eccentricity of 0.69 or greater. Thus, for most operational orbits the miss distance d will dominate.

This demonstrates one option the evading spacecraft may have to avoid capture while using a ΔV optimal maneuver. However, note that the ΔV cost of such a maneuver has not been discussed. Although it is straightforward to relate the changes δq and δQ to corresponding ΔV amounts for the evading spacecraft, the pursuing spacecraft cannot immediately respond with a maneuver because it likely will not be at periapsis or apoapsis of its own orbit. In this analysis, maneuvers can only occur at periapsis and apoapsis to maintain ΔV optimality. The pursuing spacecraft must now replan an interception strategy that it can only begin, at the earliest, once it reaches the nominal capture point at periapsis or apoapsis where the evading spacecraft no longer reaches. This replanned strategy is then again subject to being thwarted by changes in q and Q once the pursuer is again on its final coast arc. Thus, the inherent multistage nature of the pursuit–evasion game becomes apparent; the true game consists of both agents reacting to the actions of the other over time. A future analysis may consider using the maneuvers here in a carefully constructed multistage game, but such a scenario is difficult to pose in a meaningful way. Future work may also consider the use of orbit phase (anomaly), RAAN, and argument of periapsis changes as tools for optimal evasion strategies; orbit phase changes may be of particular interest due to their relatively low ΔV cost and their ability to eliminate the precise conditions a pursuer might use for a ΔV optimal transfer.

The limitations imposed in this analysis illustrate the usefulness of the differential game approach, especially for the endgame phase, because it inherently entails solving for the optimal control at all times with consideration for what the optimal maneuvers of the other agent will be as well. The differential game approach inherently takes into account that the pursuit–evasion game is not a discrete, multistage problem, but it is instead essentially an infinite-stage problem across time. A differential game approach would also not restrict maneuvers to only occur at periapsis and apoapsis, allowing the pursuing spacecraft to immediately respond to the evasion maneuver made by the evading spacecraft if such a strategy is optimal. An immediate response, especially for small δq or δQ, would likely be better than waiting to maneuver until the next apsis passage. Given an immediate response from the pursuing spacecraft, then it would likely be most prudent for the evading spacecraft to also maneuver in a manner that is not strictly ΔV optimal. The differential game or even closed-loop control approaches discussed in Sec. I are potential candidate methods that could be used to solve for controls in this endgame scenario if they are adaptable to the terminal condition of rendezvous (note that the vast majority of them address the intercept problem). Future work might consider analyzing a two-phase approach where the strategy outlined in the present work is used early on when the separation between the two spacecraft is large, before switching to a differential game or closed-loop control law approach to generate controls for both spacecraft. The exact threshold distance defining a large separation is not immediately clear, so the logic used in deciding when to switch strategies is an interesting topic to explore. The impact of the initial risk posture on the ability to evade capture is also an interesting avenue to explore in a two-phase game. Thus, while the maneuver strategy used in the present work could potentially be applied to the terminal phase of the pursuit–evasion game, the primary usefulness of the maneuver strategy is either when using it to maneuver out of the reachable set of the pursuing spacecraft for guaranteed evasion or when the separation between the two spacecraft is large and ΔV (fuel) optimality can be prioritized without adversely impacting the ability to perform or avoid final rendezvous.

VI. Conclusions

This work has illuminated some aspects of the time-free, ΔV optimal spacecraft rendezvous pursuit–evasion problem. If an evading spacecraft has knowledge of the total ΔV available to the pursuing spacecraft, it can use the knowledge of the pursuing spacecraft's reachable set of orbits to transfer to an orbit that is guaranteed to avoid capture. Without knowledge of the pursuing spacecraft's total available ΔV, alternative evasion strategies have also been identified. Notably, for planar pursuit–evasion games there are special directions for the evading spacecraft to move where it will match the efficiency of a pursuing spacecraft. In the three-dimensional $(q - Q - i)$ case, these special directions were not found, but the evading spacecraft can potentially find certain orbits that it can transfer to at a lower cost than the pursuing spacecraft. Although such regions may not be directly useful for practical applications due to the high ΔV of the transfers, they are of theoretical interest and perhaps may inform strategies that do not limit transfers to the time-free ΔV optimal transfers used here.

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