ASEN 5044
Fall 2024
HW 3
Jash Bhalavat

Problem 1

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$\boxed{\text{HW 3}}$

Problem 1 →
$$\dot{x} = Ax(t) + Bu(t) = \begin{bmatrix} \dot{q}_1 \\ \ddot{q}_1 \\ \dot{q}_2 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_1-k_2}{m_1} & 0 & k_2/m_1 & 0 \\ 0 & 0 & 0 & 1 \\ k_2/m_2 & 0 & \frac{-k_2-k_3}{m_2} & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1/m_1 & 0 \\ 0 & 0 \\ 1/m_1 & 1/m_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = Cx(t) + Du(t) = \begin{bmatrix} q_1 \\ \dot{q}_1 - \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

a) $k_1 = k_2 = k_3 = 1 \frac{N}{m}$, $m_1 = m_2 = 1$ kg, $\Delta t = 0.05$ sec

From lecture 6, slide 4 →
$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$y = Cx + Du = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\hat{A} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \rightarrow e^{\hat{A}\Delta t} = e^{\hat{A}(0.05)} = \begin{bmatrix} F & G \\ 0 & I \end{bmatrix}$$ ← using matlab for matrix exponential, $H = C$, $M = D$

$$x_{k+1} = \begin{bmatrix} 0.9975 & 0.05 & 0.0012 & 0.0 \\ -0.0999 & 0.9975 & 0.0499 & 0.0012 \\ 0.0012 & 0.0 & 0.9975 & 0.05 \\ 0.0499 & 0.0012 & -0.0999 & 0.9975 \end{bmatrix} \begin{bmatrix} q_1(k) \\ \dot{q}_1(k) \\ q_2(k) \\ \dot{q}_2(k) \end{bmatrix} + \begin{bmatrix} -0.0012 & 0.0 \\ -0.0499 & 0.0 \\ 0.0012 & 0.0012 \\ 0.0499 & 0.05 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$
$$= Fx(k) + Gu(k)$$

$$y_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} u(k) = Hx(k) + Mx(k)$$

As stated in Lecture 6, slide 14, the natural modes of this system are 2.72 Hz and 1.57 Hz. The nyquist limit is twice both those modes i.e. 5.44 Hz and 3.14 Hz respectively. The sampling rate of 0.05 s converts to 20 Hz which is more than the nyquist limit. That’s why aliasing should not be a concern.

b) $$O = \begin{bmatrix} H \\ HF \\ HF^2 \\ HF^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0.9975 & 0.05 & 0.0012 & 0 \\ -0.1498 & 0.9963 & 0.1498 & -0.9963 \\ 0.99 & 0.0997 & 0.005 & 0.0002 \\ -0.2985 & 0.9850 & 0.2985 & -0.9850 \\ 0.9776 & 0.1489 & 0.012 & 0.0000 \\ -0.4450 & 0.9664 & 0.4450 & -0.9664 \end{bmatrix}$$

$\text{Rank}(O) = 4$, $n = 4$

$\because \text{Rank}(O) = n$, the DT system is observable!

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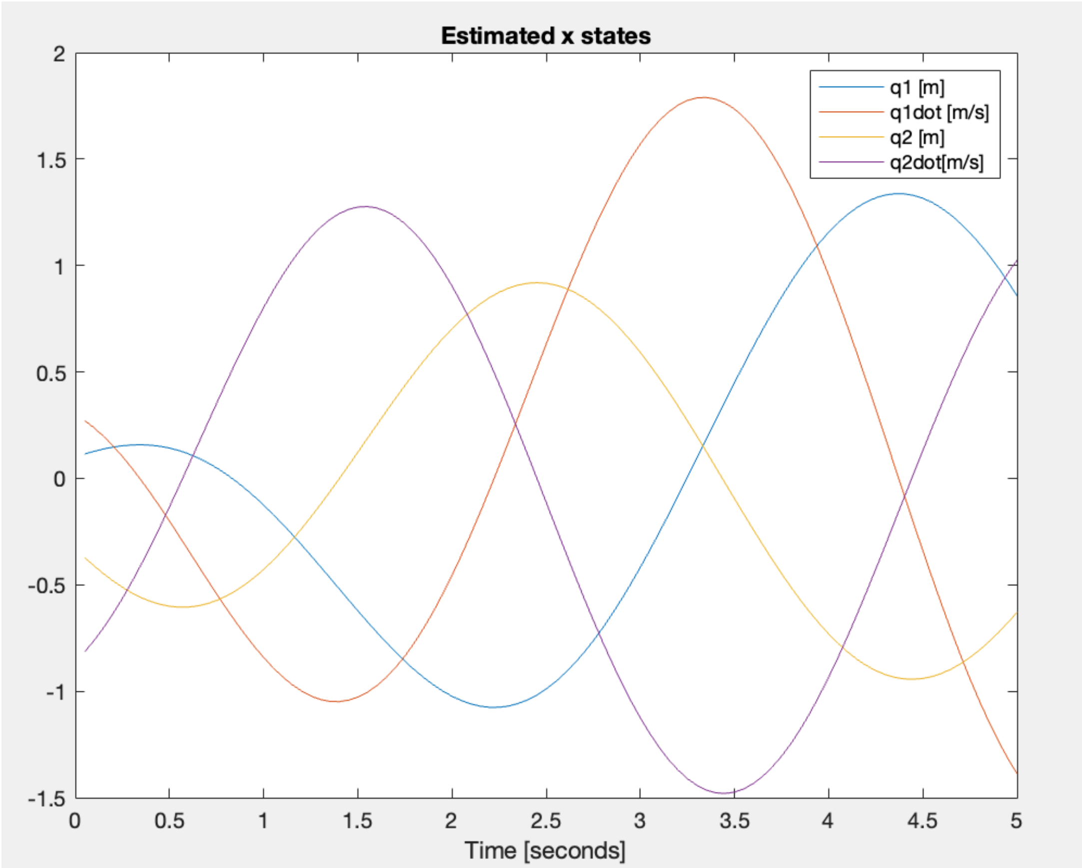
Problem 1 → c) $x_1 = Fx_0 + Gu_0 \rightarrow y_1 = Hx_1 = HFx_0 + HGu_0$

$x_2 = Fx_1 + Gu_1 \rightarrow y_2 = Hx_2 = HFx_1 + HGu_1 = HF^2 x_0 + HFGu_0 + HGu_1$

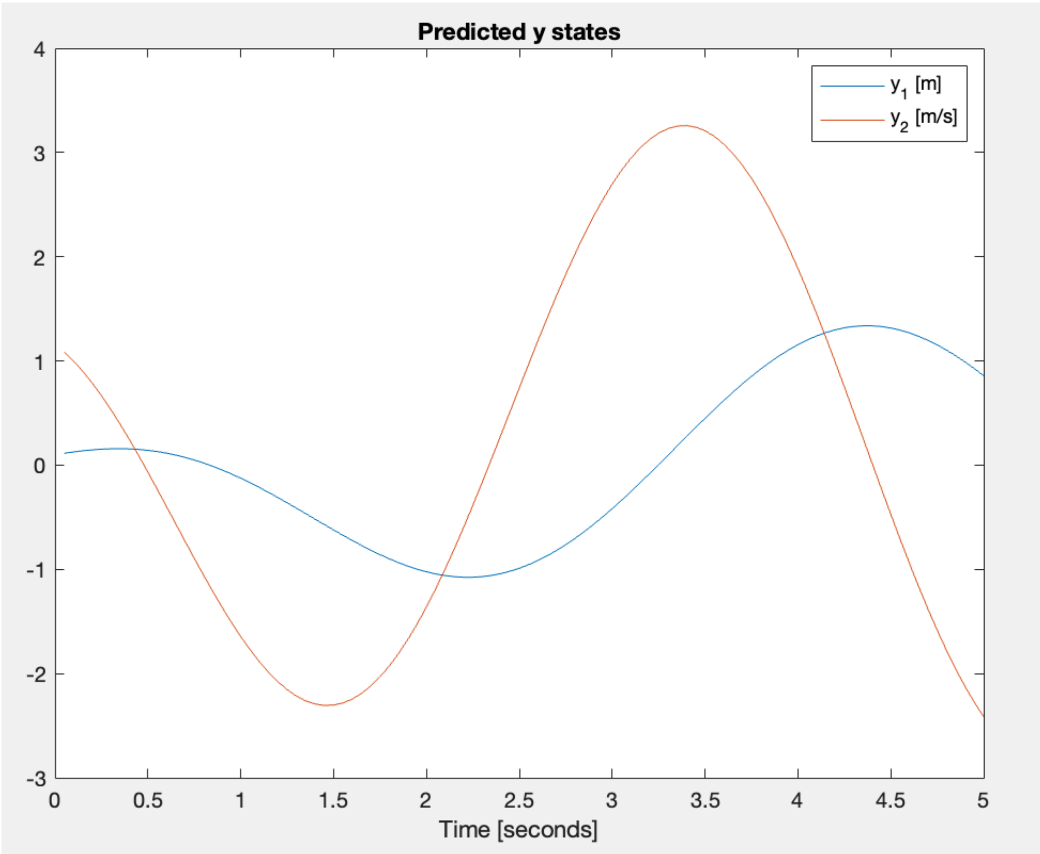
$$\underline{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} HF \\ HF^2 \\ \vdots \\ HF^n \end{bmatrix} x_0 + \begin{bmatrix} HG & \cdots & & 0 \\ HFG & HG & & \\ HF^2G & HG & & \\ HF^{(n-1)}G & HF^{(n-2)}G & \cdots & HG \end{bmatrix} \begin{bmatrix} u_0 \\ \vdots \\ u_{n-1} \end{bmatrix} = \underline{O} x_0 + \underline{G} u_k$$

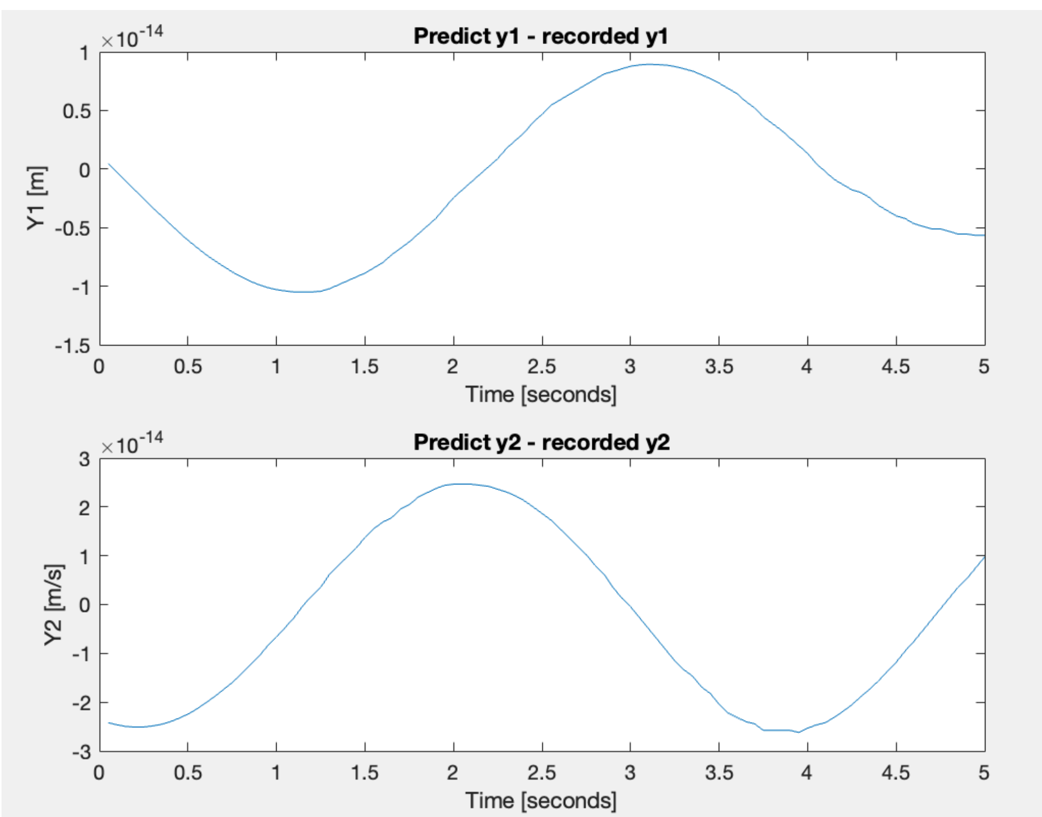
$$\underline{O} x_0 = \underline{Y} - \underline{G} u_k \rightarrow x_0 = (\underline{O}^T \underline{O})^{-1} \underline{O}^T (\underline{Y} - \underline{G} u_k)$$

- Part d
    - x0 = [0.1, 0.3, -0.33, -0.86]^T where “^T” signifies transpose



Estimated x states
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Time [seconds]
q1 [m]
q1dot [m/s]
q2 [m]
q2dot[m/s]


Predicted y states
y1 [m]
y2 [m/s]
Time [seconds]


Predict y1 - recorded y1
×10^-14
Y1 [m]
Time [seconds]
Predict y2 - recorded y2
×10^-14
Y2 [m/s]
Time [seconds]

- Part e
  - At least 2 vector measurements of Y (k=1, k=2) are needed in order to get an estimate for x(0). So, not all available measurements are needed to estimate x(0).
  - Yes, this is consistent with the observability matrix because if only one measurement is used, O = [H*F], and H*F is a 2x4 matrix which means that the rank of O cannot be more than 2, but n = 4 which means that O is not full rank. So, a unique solution to x(0) does not exist. Additionally, if O doesn't have full rank, then its gramian (O^T*O) also doesn't have full rank. And, the system doesn't have a unique solution.
  - If H is updated to be as follows:

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

  - then, rank of H is 1. Multiplying H by F leads to rank of 1 because F has all linearly independent vectors and H has all dependent vectors. This means multiplying it by F 4 times will lead to a full rank matrix. Hence, O has to be [H*F, H*F*F, H*F*F*F, H*F*F*F*F]. Hence, 4 Y measurements (k = 1, 2, 3, 4) are needed in order to estimate x(0).
  - *All the k discrete values start from 1 because Y(k=0) is not provided. Assuming that the same assumption spans throughout this problem.*

Problem 2

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Problem 2 → $z_1(k+1) = z_1(k) + \lambda[z_1(k) - z_2(k)]$
$z_2(k+1) = z_2(k) + \mu[z_1(k) - z_2(k)]$ → $z(k) = [z_1(k), z_2(k)]^T$

a) $x(k) = [\lambda, \mu]^T$, $y(k+1) = [g(k+1), p(k+1)]^T = [z_1(k+1) - z_1(k), z_2(k+1) - z_2(k)]^T$

∵ λ, μ are constants

$$x(k+1) = x(k) = \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \lambda \\ \mu \end{bmatrix} = Fx, \quad G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$y(k+1) = \begin{bmatrix} z_1(k+1) - z_1(k) \\ z_2(k+1) - z_2(k) \end{bmatrix} = \begin{bmatrix} \lambda(z_1(k) - z_2(k)) \\ \mu(z_1(k) - z_2(k)) \end{bmatrix} = \begin{bmatrix} z_1(k) - z_2(k) & 0 \\ 0 & z_1(k) - z_2(k) \end{bmatrix}\begin{bmatrix} \lambda \\ \mu \end{bmatrix}$$

$$= H(k)x, \quad M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

b) If $z_1(k) = z_2(k)$ then $H(k) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $z(k+1) = z(k)$. In this case λ and μ don't affect $z(k)$ so they cannot be determined from given $z(k)$ values. In the given values, this represents $z(5)$. ∴ $z(5)$ cannot be used to calculate λ and μ.

Also, $y(k+1) = H(k)x$ → where $y(k+1)$ is 2×1 and $H(k)$ is 2×2. So, this is a "Nice" linear system of equation (as in lecture 2, slide 8). So, λ and μ can be determined if $|H(k)| \neq 0$.

c)

Not using $z(5)$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_5 \end{bmatrix} = \begin{bmatrix} H_0 \\ H_1 \\ \vdots \\ H_4 \end{bmatrix} x \rightarrow \begin{bmatrix} z_1(1) - z_1(0) \\ z_2(1) - z_2(0) \\ z_1(2) - z_1(1) \\ z_2(2) - z_2(1) \\ \vdots \end{bmatrix} = \begin{bmatrix} z_1(0) - z_2(0) & 0 \\ 0 & z_1(0) - z_2(0) \\ z_1(1) - z_2(1) & 0 \\ 0 & z_1(1) - z_2(1) \\ \vdots & \vdots \end{bmatrix} x \rightarrow \underline{Y} = \underline{H}x$$

$$\begin{bmatrix} -56.334 \\ 19.2815 \\ -3.0873 \\ 1.0567 \\ -0.1692 \\ 0.0579 \\ -0.0093 \\ 0.0032 \\ -5\times10^{-4} \\ 2\times10^{-4} \end{bmatrix} = \begin{bmatrix} 80 & 0 \\ 0 & 80 \\ 4.3843 & 0 \\ 0 & 4.3843 \\ 0.2403 & 0 \\ 0 & 0.2403 \\ 0.0132 & 0 \\ 0 & 0.0132 \\ 7\times10^{-4} & 0 \\ 0 & 7\times10^{-4} \end{bmatrix}\begin{bmatrix} \lambda \\ \mu \end{bmatrix} \Rightarrow$$

$$G = \underline{H}^T\underline{H}$$

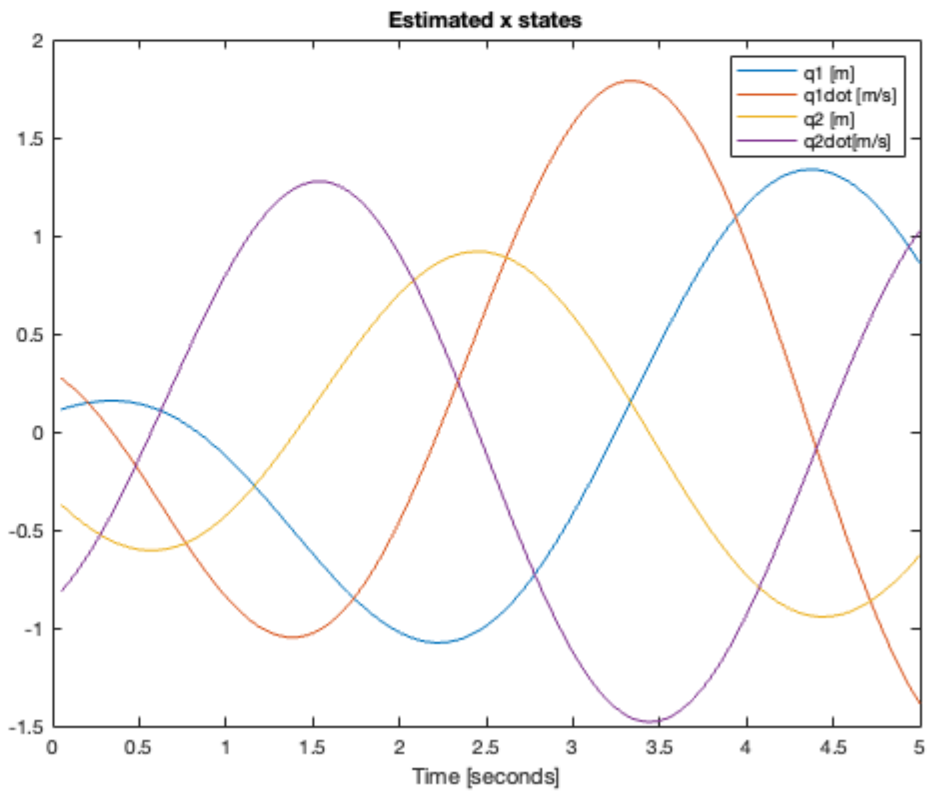
$$\therefore x = G^{-1}\underline{H}^T\underline{Y}$$

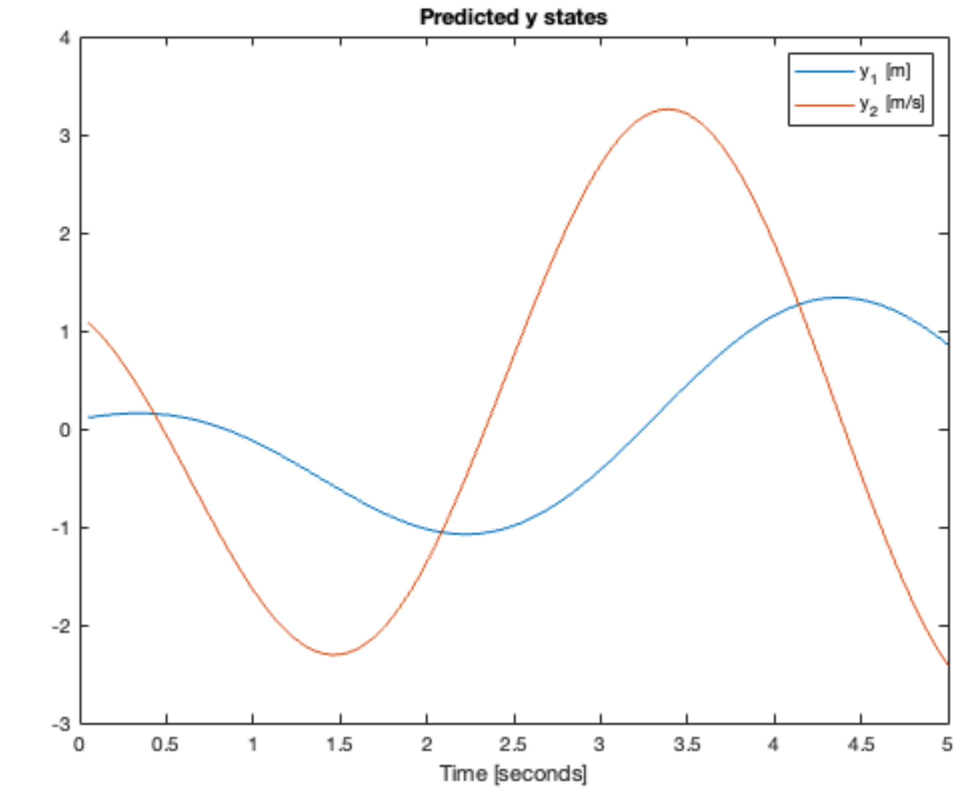
$$\boxed{x = \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} -0.7042 \\ 0.2410 \end{bmatrix}}$$

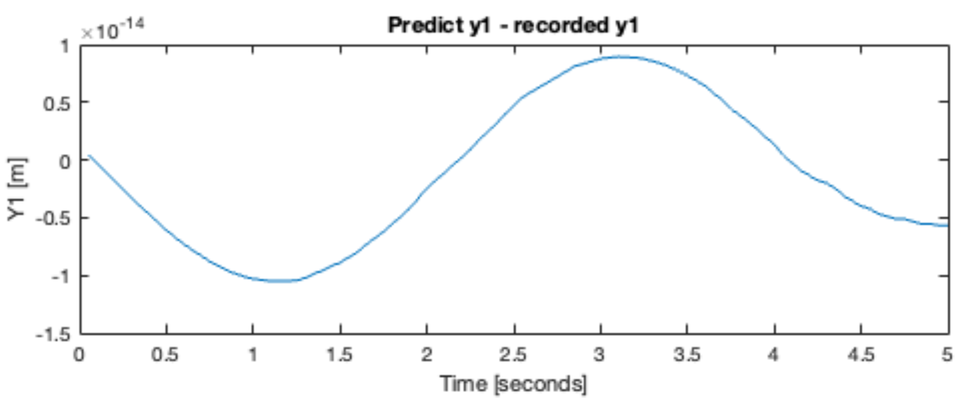
```
clear; clc; close all;

A = [0 1 0 0; -2 0 1 0; 0 0 0 1; 1 0 -2 0];
B = [0 0; -1 0; 0 0; 1 1];

C = [1 0 0 0; 0 1 0 -1];
D = [0 0; 0 0];

H = C;
M = D;

delta_t = 0.05;

% Part a
% A is always square matrix
n = length(A);
m = size(B,2);

A_hat = [A B; zeros([n - m, n+m])];

syms t

stm = expm(A_hat*t);
stmf(t) = stm;
f(t) = stm(1:4, 1:4);
g(t) = stm(1:4, 5:6);


% Part b
% is u = 0?

F = double(f(delta_t));
G = double(g(delta_t));

O = [H; H*F; H*F*F; H*F*F*F];

rank_O = rank(O);

% Part c
data = load("hw3problem1data.mat");
U = data.Udata;
Y = data.Ydata;

t0 = 0;
tf = 5;

time_steps = t0:delta_t:tf;

% Estimating x(0)
Y_temp = Y';
Y_sole = reshape(Y_temp, [2*length(Y_temp), 1]);
```

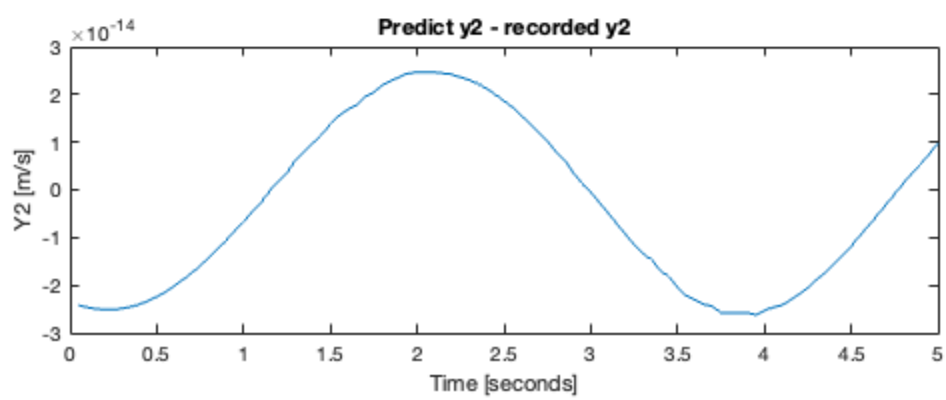
```
U_temp = U(1:end-1,:)';
U_sole = reshape(U_temp, [2*length(U_temp), 1]);

n = 3;

O_sole = [];
for i = 1:n
    O_sole = [O_sole; H*F^i];
end

p = 2;
HFG_sole = [];
for i = 1:n
    temp_vec = zeros([p, 2*n]);
    count = 1;
    for j = 1:i
        temp_vec(:,count:count+1) = H*F^(i-j)*G;
        count = count + 2;
    end
    HFG_sole = [HFG_sole; temp_vec];
end

LHS = Y_sole(1:2*n) - HFG_sole*U_sole(1:2*n);
gram = O_sole' * O_sole;
x0 = inv(gram) * O_sole' * LHS;

x = x0;

for i = 1:length(time_steps)
    x(:,i+1) = F * x(:,i) + G * U(i,:)';
    y_predicted(:,i) = H * x(:,i) + M * U(i,:)';
end

figure()
plot(time_steps(2:end), x(1,2:length(time_steps)))
hold on
plot(time_steps(2:end), x(2,2:length(time_steps)))
plot(time_steps(2:end), x(3,2:length(time_steps)))
plot(time_steps(2:end), x(4,2:length(time_steps)))
hold off
legend("q1 [m]", "q1dot [m/s]", "q2 [m]", "q2dot[m/s]")
xlabel("Time [seconds]")
title("Estimated x states")

figure()
plot(time_steps(2:end), y_predicted(1,2:end))
hold on
plot(time_steps(2:end), y_predicted(2,2:end))
hold off
legend("y_1 [m]", "y_2 [m/s]")
xlabel("Time [seconds]")
title("Predicted y states")

figure()
```

```
subplot(2,1,1)
plot(time_steps(2:end), Y(:,1)-y_predicted(1,2:end)')
xlabel("Time [seconds]")
ylabel("Y1 [m]")
title("Predict y1 - recorded y1")

subplot(2,1,2)
plot(time_steps(2:end), Y(:,2)-y_predicted(2,2:end)')
xlabel("Time [seconds]")
ylabel("Y2 [m/s]")
title("Predict y2 - recorded y2")


% Part e
H_single = [1 0 0 0; 1 0 0 0; 1 0 0 0];
O_single = [H_single*F; H_single*F*F; H_single*F*F*F; H_single*F*F*F*F];

rank_O_single = rank(O_single);
```




Predicted y states
y_1 [m]
y_2 [m/s]
Time [seconds]


Predict y1 - recorded y1
$\times 10^{-14}$
Y1 [m]
Time [seconds]


Predict y2 - recorded y2
$\times 10^{-14}$
Y2 [m/s]
Time [seconds]

```matlab
clear; clc; close all;

% Slick's offer - z1 (thousands of $)
% Prad Bitt's offer - z2 (thousands of $)

z0 = [100; 20];
z1 = [43.6658; 39.2815];
z2 = [40.5785; 40.3382];
z3 = [40.4093; 40.3961];
z4 = [40.4; 40.3993];
z5 = [40.3995; 40.3995];

z = [z0, z1, z2, z3, z4, z5];

for i = 2:6
    lambda(i) = (z(1,i) - z(1,i-1))/(z(1,i-1) - z(2,i-1));
    mu(i) = (z(2,i) - z(2,i-1))/(z(1,i-1) - z(2,i-1));
end

for i = 1:length(z)-1
    y(:,i) = [z(1,i+1) - z(1,i); z(2,i+1) - z(2,i)];
end
y = reshape(y, [10, 1]);

H = [];
for i = 1:length(z)-1
    H_temp = [z(1,i) - z(2,i), 0; 0, z(1,i)-z(2,i)];
    H = [H; H_temp];
end

gram = H' * H;
x = inv(gram) * H' * y;
```

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