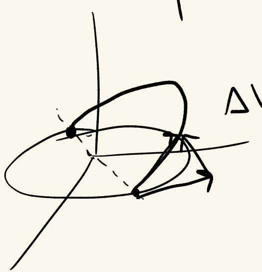
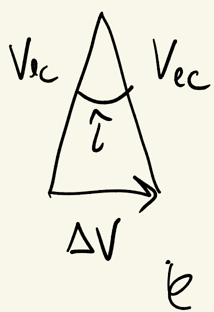
Optimal Plane Changes

In addition to Transfers between circular orbits, other specialized situations where we can save propellant:

Planetary Escape, Planetary Capture, plane changes, combined plane & orbit changes, ...

Simplest is a 1-impulse plane change at the line of nodes.





$$\boxed{\Delta V_i = 2 V_{ec} \sin\left(\tfrac{1}{2}\hat{i}\right)}$$

Recall... $\dot{H}_{max}$ occurs @ large r.... if $r \to \infty$, plane changes cost ~ 0.

Define Bi-Parabolic Plane Change.

$$\Delta V_{ip} = V_{ec}(\sqrt{2}-1) + \cancel{\Delta V} + V_{ec}(\sqrt{2}-1)$$

$$\boxed{\Delta V_{ip} = 2V_{ec}(\sqrt{2}-1)}$$

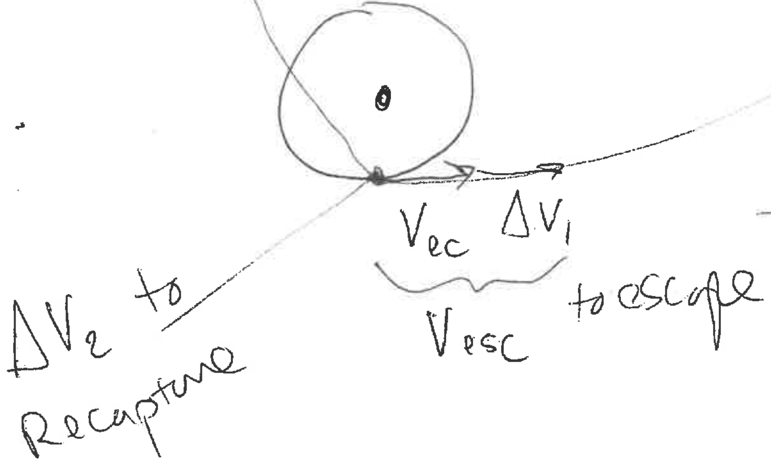
Compare w/ 1-impulse....

when is $\Delta V_{ip} \leq \Delta V_i$

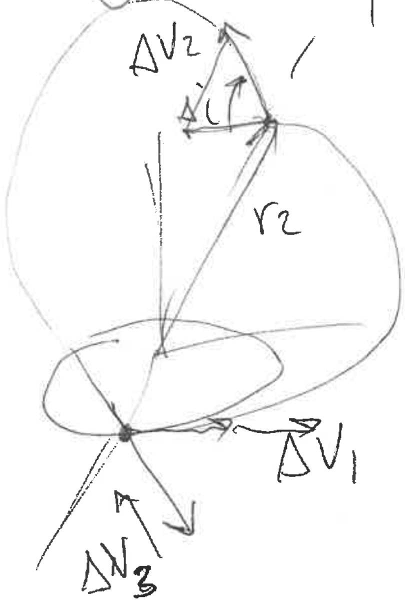
$$\cancel{2V_{ec}}(\sqrt{2}-1) \leq \cancel{2V_{ec}} \sin(i/2)$$

$$(\sqrt{2}-1) \leq \sin(i/2)$$

$$2\,\mathrm{asin}[\sqrt{2}-1] \leq i \Rightarrow \boxed{48.9^\circ \geq \hat{i}}$$



More general approach ⇒ "restricted" Bi-elliptic transfer



r_2 is a parameter

$$\Delta V_1 = \sqrt{\frac{2\mu}{r_1+r_2}\frac{r_2}{r_1}} - \sqrt{\frac{\mu}{r_1}} = \Delta V_3$$

$$\Delta V_2 = 2\sqrt{\frac{2\mu}{r_1+r_2}\frac{r_1}{r_2}}\sin(i/2)$$

Normalize by $\sqrt{\frac{\mu}{r_1}}$ & Look at ratio $\mathcal{R} = \frac{r_2}{r_1}$

$$\boxed{\Delta \tilde{V}_{iB} = 2\left[\sqrt{\frac{2\mathcal{R}}{1+\mathcal{R}}} - 1 + \sqrt{\frac{2}{\mathcal{R}(1+\mathcal{R})}}\sin(i/2)\right]}$$

For what values of $\mathcal{R}$ & $\hat{i}$ is

$$\boxed{\Delta \tilde{V}_{iB} \le \Delta \tilde{V}_i}$$

$$r\left[\sqrt{\frac{2r}{1+r}} + \sqrt{\frac{2}{r(1+r)}}\,\sin(i/2) - 1\right] \le 2\sin(i/2)$$

$$\sqrt{\frac{2r}{1+r}} - 1 \le \left[1 - \sqrt{\frac{2}{r(1+r)}}\right] \sin(i/2)$$

$$\boxed{\sin(i/2) \geq \frac{\sqrt{\frac{2r}{1+r}} - 1}{1 - \sqrt{\frac{2}{r(1+r)}}}} = \frac{f(r)}{g(r)}$$

$$\lim_{r\to\infty} \frac{f(r)}{g(r)} = \frac{\sqrt{2}-1}{1} \Rightarrow i \geq 48.9^\circ$$

$$\lim_{r\to 1} \frac{f(r)}{g(r)} = \frac{0}{0}\ !\ ;\ \lim_{r\to 1} \frac{f'(r)}{g'(r)} = \lim_{r\to 1} \frac{\sqrt{2}\left[\frac{1}{2}\frac{1}{\sqrt{r(1+r)}} - \frac{1}{2}\sqrt{\frac{r}{(1+r)^3}}\right]}{\frac{\sqrt{2}}{2}\left[\frac{1}{\sqrt{r^3(1+r)}} + \frac{1}{\sqrt{r(1+r)^3}}\right]}$$

$$\lim_{r \to 1} \frac{f'(r)}{g'(r)} = \frac{1/4}{3/4} = \frac{1}{3} \Rightarrow \quad \hat{i} = 2\,\mathrm{asin}(1/3)$$

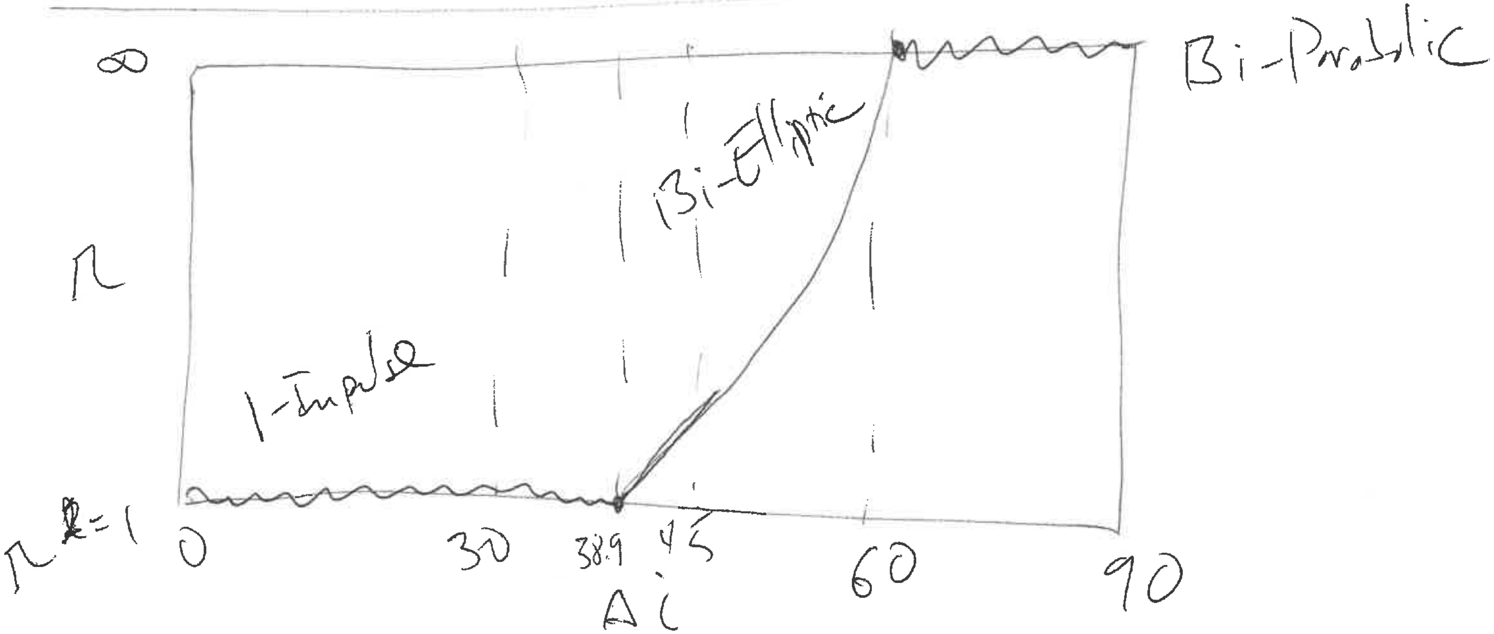
$$\boxed{\hat{i} = 38.94^\circ}$$

For $i \geq 38.94^\circ \Rightarrow$ Bi-elliptic is optimal

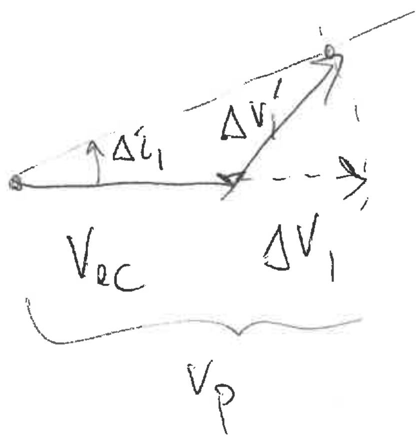
1-Impulse optimal $\quad i \leq 38.94^\circ$

Bi-elliptic " $\quad 38.94^\circ \leq i \leq 48.9^\circ \Rightarrow 60^\circ$

Bi-Parabolic $\quad i \geq 60^\circ$



"Restricted" Bi-Elliptic means that the entire plane change occurs at apoapsis r_2. Optimal is to perform "Dog-Leg" Maneuvers at ΔV_1 + ΔV_3.



$$\Delta V_1^2 = V_{ec}^2 + V_p^2 - 2 V_{ec} V_p \cos(\Delta i_1)$$

$$\Delta \hat{i} = \underline{\Delta i_1} + \underbrace{\Delta i_2}_{\underline{\text{still larger}}} + \underline{\Delta i_3}$$