ASEN 5044 - HW 7, Fall 2024, Jash Bhalavat

ASEN 5044
Fall 2024
Jash Bhalavat

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$\boxed{\text{HW 7}}$

Problem 1 → $\ddot{\xi} = -\Omega\dot{\eta}, \quad \ddot{\eta} = \Omega\dot{\xi}$

$$x = \begin{bmatrix} \xi \\ \dot{\xi} \\ \eta \\ \dot{\eta} \end{bmatrix} \rightarrow \dot{x} = \begin{bmatrix} \dot{\xi} \\ \ddot{\xi} \\ \dot{\eta} \\ \ddot{\eta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\Omega \\ 0 & 0 & 0 & 1 \\ 0 & \Omega & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \dot{\xi} \\ \eta \\ \dot{\eta} \end{bmatrix} = Ax$$

confirming → $e^{A\Delta t} = I + A\Delta t + A^2/2\ \Delta t^2 + \dots$

$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\Omega \\ 0 & 0 & 0 & 1 \\ 0 & \Omega & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\Omega \\ 0 & 0 & 0 & 1 \\ 0 & \Omega & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -\Omega \\ 0 & -\Omega^2 & 0 & 0 \\ 0 & \Omega & 0 & 0 \\ 0 & 0 & 0 & -\Omega^2 \end{bmatrix} \qquad A^4 = \begin{bmatrix} 0 & 0 & 0 & \Omega^3 \\ 0 & \Omega^4 & 0 & 0 \\ 0 & -\Omega^3 & 0 & 0 \\ 0 & 0 & 0 & \Omega^4 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 & -\Omega \\ 0 & -\Omega^2 & 0 & 0 \\ 0 & \Omega & 0 & 0 \\ 0 & 0 & 0 & -\Omega^2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\Omega \\ 0 & 0 & 0 & 1 \\ 0 & \Omega & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\Omega^2 & 0 & 0 \\ 0 & 0 & 0 & \Omega^3 \\ 0 & 0 & 0 & -\Omega^2 \\ 0 & -\Omega^3 & 0 & 0 \end{bmatrix}$$

$$e^{A\Delta t} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\Omega \\ 0 & 0 & 0 & 1 \\ 0 & \Omega & 0 & 0 \end{bmatrix} \Delta t + \begin{bmatrix} 0 & 0 & 0 & -\Omega \\ 0 & -\Omega^2 & 0 & 0 \\ 0 & \Omega & 0 & 0 \\ 0 & 0 & 0 & -\Omega^2 \end{bmatrix} \frac{\Delta t^2}{2} + \begin{bmatrix} 0 & -\Omega^3 & 0 & 0 \\ 0 & 0 & 0 & \Omega^3 \\ 0 & 0 & 0 & -\Omega^2 \\ 0 & -\Omega^3 & 0 & 0 \end{bmatrix} \frac{\Delta t^3}{3!}$$

$$e^{A\Delta t} = \begin{bmatrix} 1 & \Delta t - \frac{\Omega^2}{3!}\Delta t^3 + \dots & 0 & -\Omega\frac{\Delta t^2}{2} + \Omega^3\frac{\Delta t^4}{4!} + \dots \\ 0 & 1 - \frac{\Omega^2}{2}\Delta t^2 + \dots & 0 & -\Omega\Delta t + \frac{\Omega^3}{3!}\Delta t^3 + \dots \\ 0 & \frac{\Omega}{2}\Delta t^2 - \frac{\Omega^3}{4!}\Delta t^4 + \dots & 1 & \Delta t - \frac{\Omega^2}{3!}\Delta t^3 + \dots \\ 0 & \Omega\Delta t - \frac{\Omega^3}{3!}\Delta t^3 + \dots & 0 & 1 - \frac{\Omega^2}{2}\Delta t^2 + \frac{\Omega^4}{4!}\Delta t^4 + \dots \end{bmatrix}$$

$$\sin(\Omega\Delta t) = \Omega\Delta t - \frac{\Omega^3\Delta t^3}{3!} + \dots$$

$$\cos(\Omega\Delta t) = 1 - \frac{\Omega^2\Delta t^2}{2!} + \frac{\Omega^4\Delta t^4}{4!} - \dots$$

$$e^{A\Delta t} = \begin{bmatrix} 1 & \frac{\sin(\Omega\Delta t)}{\Omega} & 0 & -\frac{1-\cos(\Omega\Delta t)}{\Omega} \\ 0 & \cos(\Omega\Delta t) & 0 & -\sin(\Omega\Delta t) \\ 0 & \frac{1-\cos(\Omega\Delta t)}{\Omega} & 1 & \frac{\sin(\Omega\Delta t)}{\Omega} \\ 0 & \sin(\Omega\Delta t) & 0 & \cos(\Omega\Delta t) \end{bmatrix}$$

$$\boxed{\therefore A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\Omega \\ 0 & 0 & 0 & 1 \\ 0 & \Omega & 0 & 0 \end{bmatrix}}$$

$$\boxed{x_{k+1} = F x_k \quad \text{where } F = e^{A\Delta t}}$$

$$\begin{bmatrix} \xi_{k+1} \\ \dot{\xi}_{k+1} \\ \eta_{k+1} \\ \dot{\eta}_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & \sin(\Omega\Delta t)/\Omega & 0 & -(1-\cos(\Omega\Delta t))/\Omega \\ 0 & \cos(\Omega\Delta t) & 0 & -\sin(\Omega\Delta t) \\ 0 & (1-\cos(\Omega\Delta t))/\Omega & 1 & \sin(\Omega\Delta t)/\Omega \\ 0 & \sin(\Omega\Delta t) & 0 & \cos(\Omega\Delta t) \end{bmatrix} \begin{bmatrix} \xi_k \\ \dot{\xi}_k \\ \eta_k \\ \dot{\eta}_k \end{bmatrix}$$

Problem 2 → $\Delta t = .5s$, $\Omega = 0.045 \frac{rad}{s}$, $x(0) \sim \mathcal{N}(\mu(0), P(0))$

a) $x_1 = F x_0 \rightarrow x_2 = F x_1 = F(F x_0) = F^2 x_0 \rightarrow x_K = F^K x_0$

For $K = 0, 1, 2, 3, \ldots N$, the recursive formula to predict the mean of linear Gauss dynamical system into the future is:

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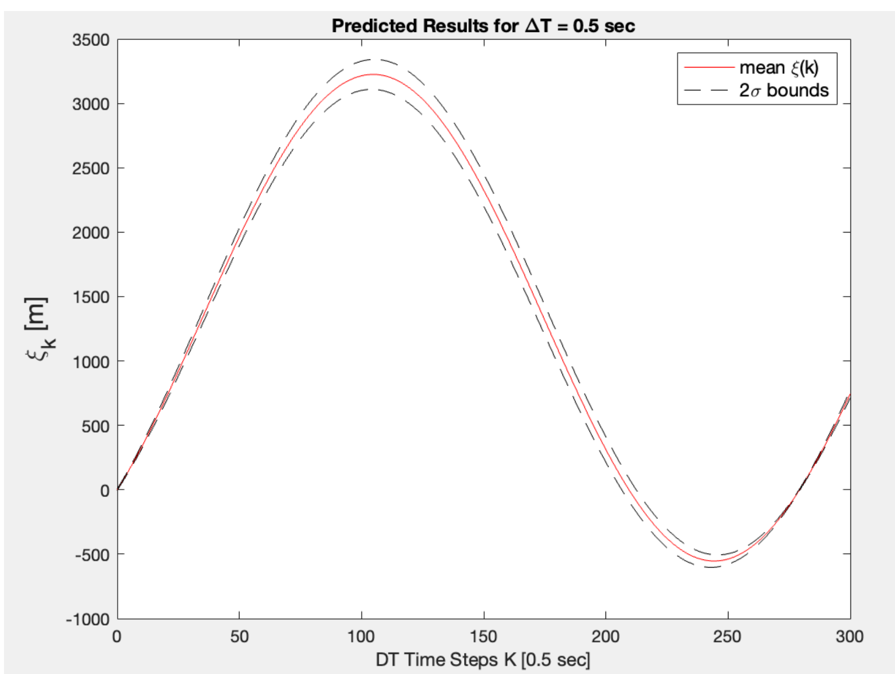
$\mu(k+1) = F\mu(k) + \cancel{G u(k)}^{\,0} = F\mu_k = F^K \mu(0)$ | $P_{K+1} = F P_K F^T + \cancel{Q}^{\,0} \rightarrow P_1 = F P_0 F^T$

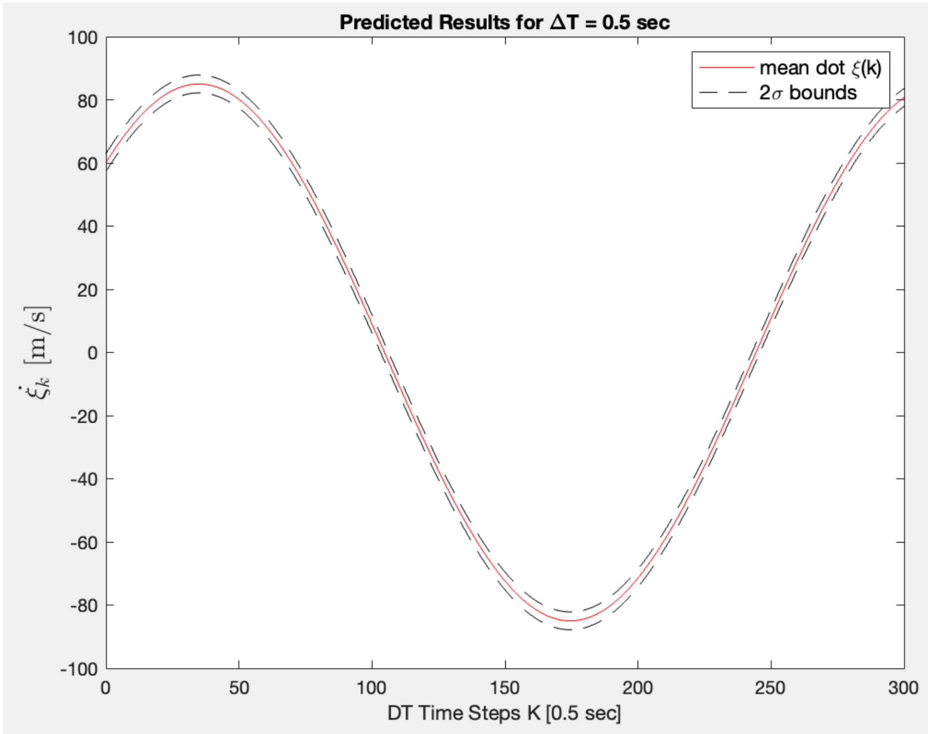
$\boxed{\mu(K+1) = F^{K+1}\mu(0)} \longrightarrow \boxed{\mu(N) = F^N \mu(0)}$ | $P_2 = F \cdot F P_0 F^T F^T = F^2 P_0 F^{T^2}$

$\boxed{P_{K+1} = F^{K+1} P_0 F^{T^{K+1}}} \longrightarrow \boxed{P_N = F^N P_0 (F^T)^N}$

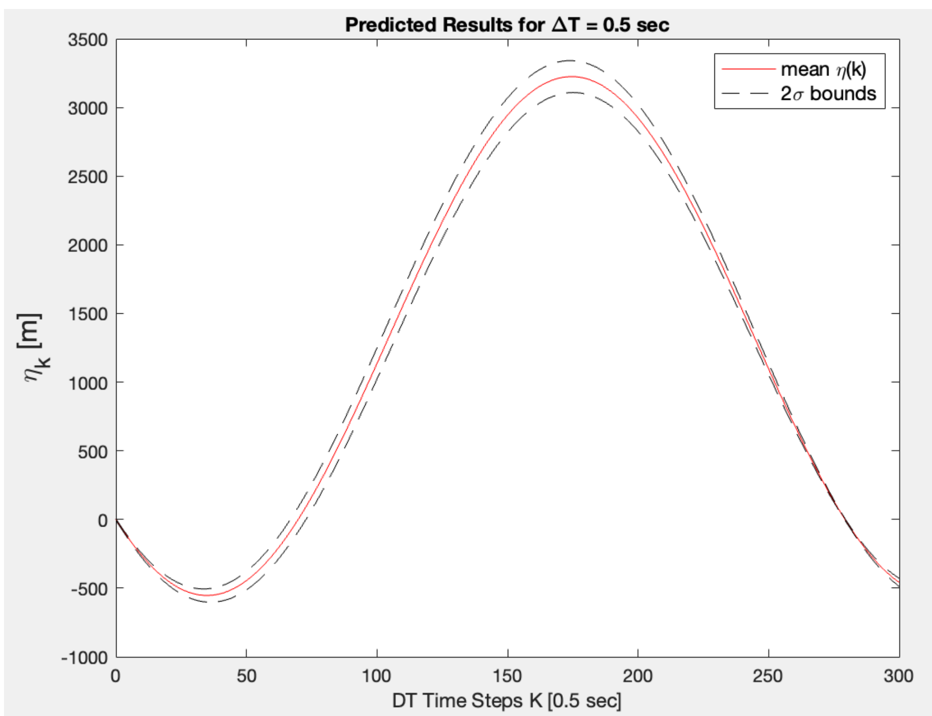
b) $\mu(0) = [0m, 85\cos(\frac{\pi}{4})\frac{m}{s}, 0m, -85\sin(\frac{\pi}{4})\frac{m}{s}]^T$

$P(0) = diag([10m^2, 2(\frac{m}{s})^2, 10m^2, 2(\frac{m}{s})^2])$ $N = 300$

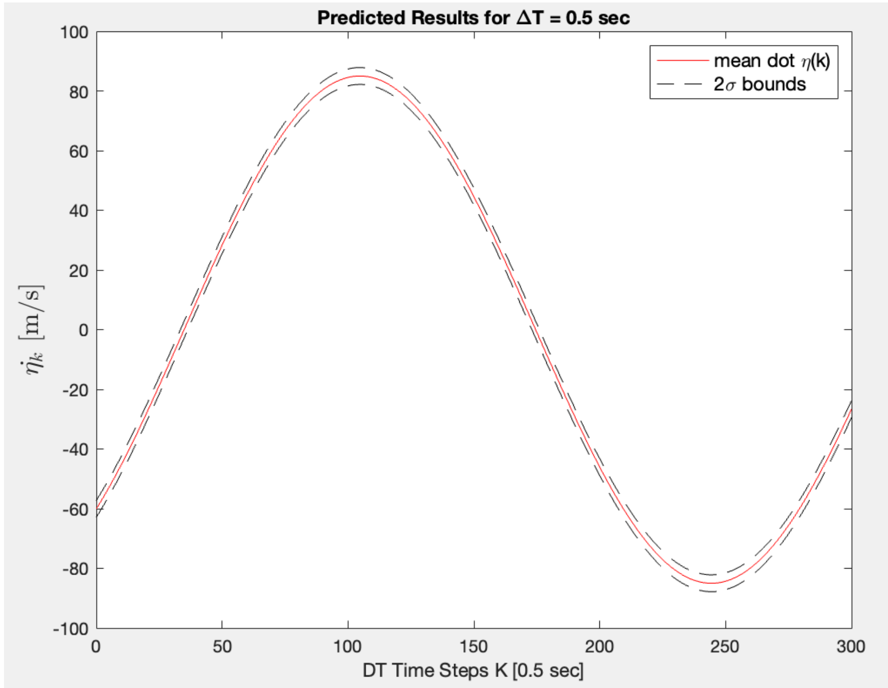


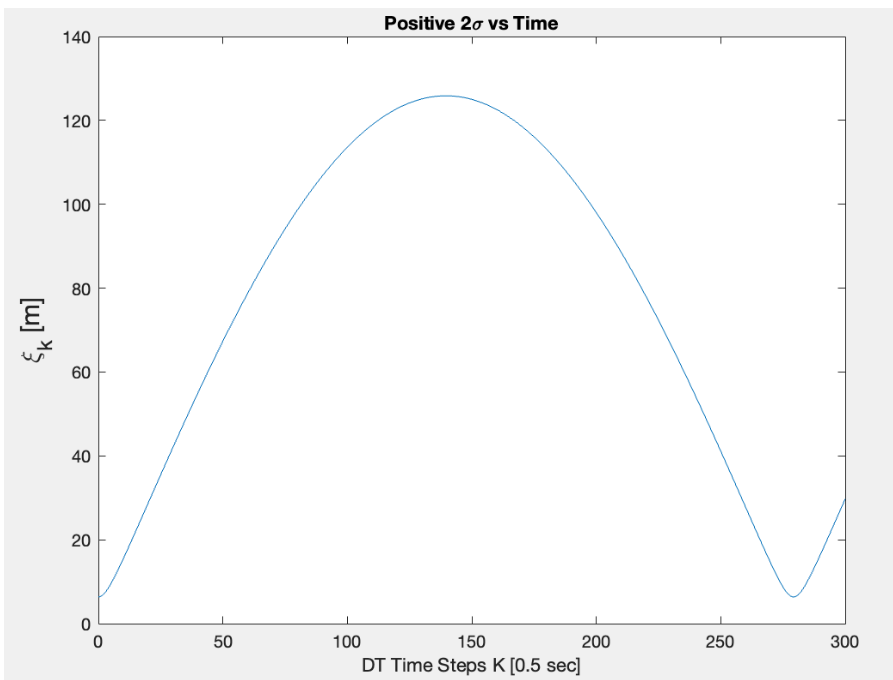


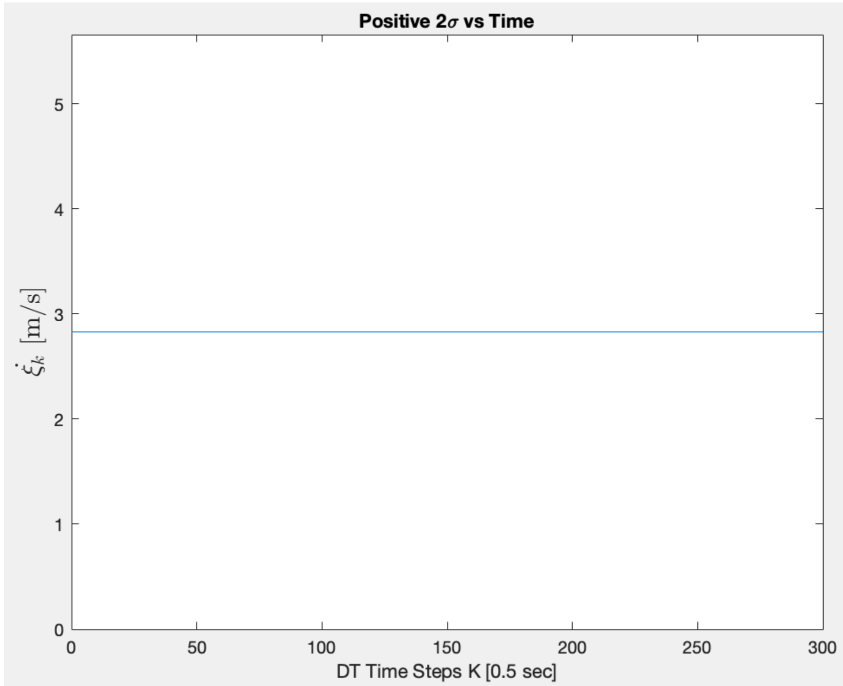
Predicted Results for ΔT = 0.5 sec
mean dot ξ(k)
2σ bounds
$\dot{\xi}_k$ [m/s]
100
80
60
40
20
0
-20
-40
-60
-80
-100
0
50
100
150
200
250
300
DT Time Steps K [0.5 sec]



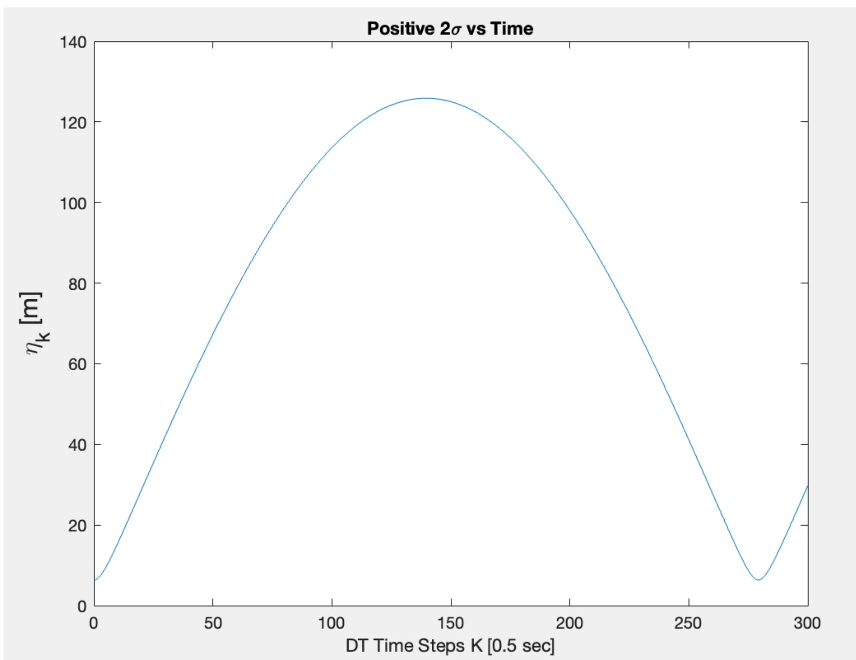
Predicted Results for ΔT = 0.5 sec
mean η(k)
2σ bounds
η_k [m]
3500
3000
2500
2000
1500
1000
500
0
-500
-1000
0
50
100
150
200
250
300
DT Time Steps K [0.5 sec]


Predicted Results for ΔT = 0.5 sec
mean dot η(k)
2σ bounds
η̇_k [m/s]
100
80
60
40
20
0
-20
-40
-60
-80
-100
0
50
100
150
200
250
300
DT Time Steps K [0.5 sec]

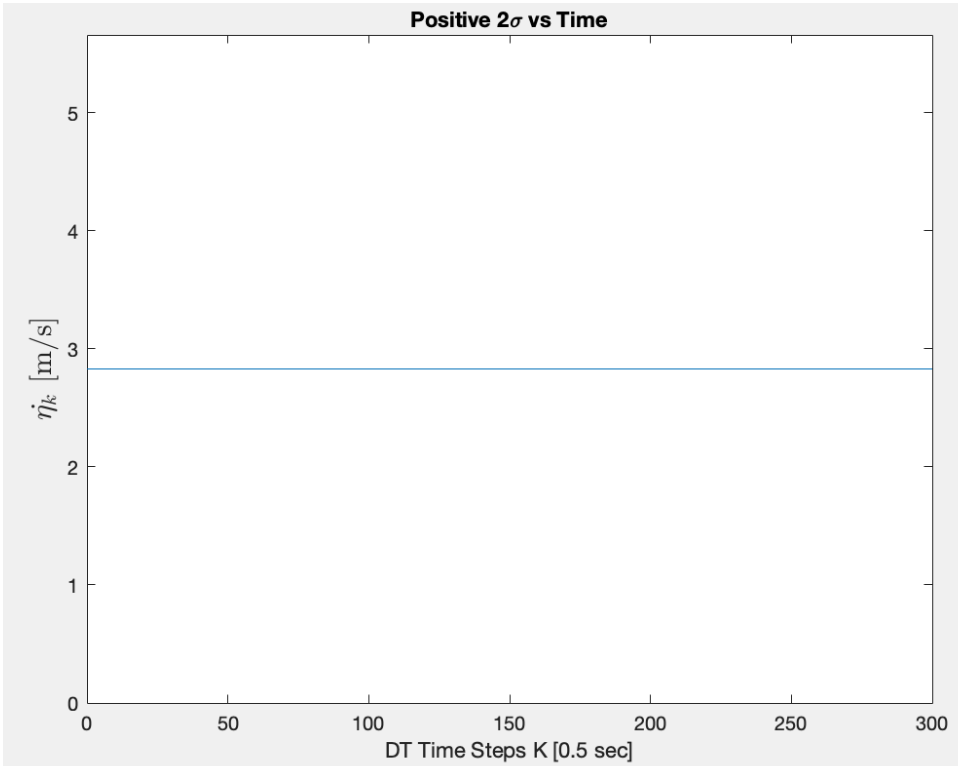

Positive 2σ vs Time
ξ_k [m]
140
120
100
80
60
40
20
0
0
50
100
150
200
250
300
DT Time Steps K [0.5 sec]



Positive 2σ vs Time
$\dot{\xi}_k$ [m/s]
0
1
2
3
4
5
0
50
100
150
200
250
300
DT Time Steps K [0.5 sec]



Positive 2σ vs Time
η_k [m]
0
20
40
60
80
100
120
140
0
50
100
150
200
250
300
DT Time Steps K [0.5 sec]



Positive 2σ vs Time
$\dot{\eta}_k$ [m/s]
DT Time Steps K [0.5 sec]
0
1
2
3
4
5
0
50
100
150
200
250
300

Problem 3 → a) $r_c(k) = r_a(k) - r_b(k) \rightarrow r_a(k) = [\xi_a(k), \eta_a(k)]^T$, $r_b(k) = [\xi_b(k), \eta_b(k)]^T$

$E[r_c(k)] = E[r_a(k) - r_b(k)] = E[r_a(k)] - E[r_b(k)]$

$x_a(0) \sim \mathcal{N}(\mu_a(0), P_a(0)) \rightarrow$ From prob 2 $\rightarrow x_a(k) \sim \mathcal{N}(F_a^k \mu_a(0), F_a^k P_a(0) F_a^{Tk})$

similarly $\rightarrow x_b(k) \sim \mathcal{N}(F_b^k \mu_b(0), F_b^k P_b(0) F_b^{Tk})$

$\mu_{ra}(k) = [F_a^k \mu_a(0)]_{1,3} \leftarrow$ 1st, 3rd component of vector

$P_{ra}(k) = [F_a^k P_a(0) F_a^{Tk}]_{(1,1),(3,3)} \leftarrow$ 1st, 3rd diagonal components of matrix

$\mu_{rb}(k) = [F_b^k \mu_b(0)]_{1,3}$

$P_{rb}(k) = [F_b^k P_b(0) F_b^{Tk}]_{(1,1),(3,3)}$

$\therefore \mu_{rc}(k) = \mu_{ra}(k) - \mu_{rb}(k)$

$\sigma^2_{rc}(k) = P_{ra}(k) + P_{rb}(k) = P_{rc}(k)$ $\Big] \rightarrow r_c(k) \sim \mathcal{N}(\mu_{rc}(k), P_{rc}(k))$

b) $P(-\infty < r_c(k) \leq X) = \int_{-\infty}^{X} p(r_c(k))\, dr_c(k) = C(X)$

Aircrafts will collide when $\Delta\xi_k \in [-\xi_R, \xi_R]$ and $\Delta\eta_k \in [-\eta_R, \eta_R]$

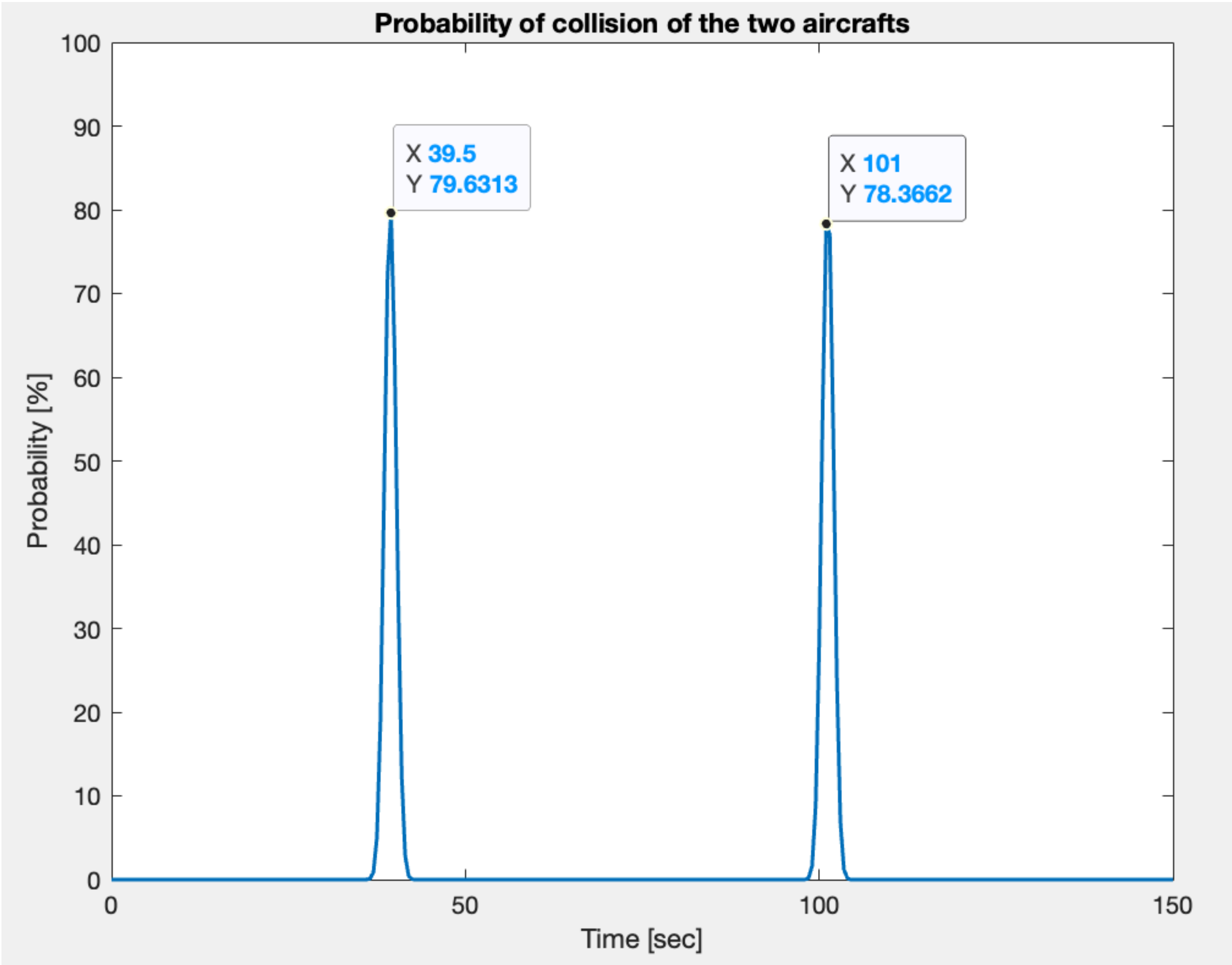
$r_c(k) = [\Delta\xi(k), \Delta\eta(k)]^T$, let $X_R = [\xi_R, \eta_R]^T$

so, aircrafts will collide if $-X_R \leq r_c(k) \leq X_R$

$P(-X_R \leq r_c(k) \leq X_R) = \int_{-X_R}^{X_R} p(r_c(k))\, dr_c(k)$ where $r_c(k) \sim \mathcal{N}(\mu_{Rc}(k), P_{Rc}(k))$

$$\boxed{P(-X_R \leq r_c(k) \leq X_R) = C(X_R) - C(-X_R)}$$

Part c



- The aircrafts are in the greatest danger of colliding at time 39.5 seconds and 101 seconds.
- Outside of these two times (and some margin around them), the probability of collision is almost always zero. That makes sense because these two aircrafts are moving almost circularly in opposite directions which causes certain times to have a very high probability of collision and other times to have zero probability of collision.

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```
clear; clc; close all;

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syms delta_t
syms omega

A = [0 1 0 0; 0 0 0 -omega; 0 0 0 1; 0 omega 0 0];

e_at = expm(A * delta_t);

delta_t = 0.5;
omega = 0.045;

% Omega times delta_t
odt = omega * delta_t;

e_at_given = [1 sin(odt)/omega 0 -(1 - cos(odt))/omega; 0 cos(odt) 0
-sin(odt); 0 (1-cos(odt))/omega 1 sin(odt)/omega; 0 sin(odt) 0 cos(odt)];

u_0 = [0; 85*cos(pi/4); 0; -85*sin(pi/4)];
P_a_0 = diag([10, 2, 10, 2]);

T = 300;

F = e_at_given;

mu = [u_0];
P = [P_a_0];
P_1_2sig = 2 * sqrt(P(1, 1));
P_2_2sig = 2 * sqrt(P(2, 2));
P_3_2sig = 2 * sqrt(P(3, 3));
P_4_2sig = 2 * sqrt(P(4, 4));

for i = 1:T
    mu(:, i+1) = F^i * u_0;
    F_t = F';
    P_temp = F^i * P_a_0 * F_t^i;
    P(:, i+1) = get_diag(P_temp);
    P_1_2sig(i+1) = 2 * sqrt(P(1, i+1));
    P_2_2sig(i+1) = 2 * sqrt(P(2, i+1));
    P_3_2sig(i+1) = 2 * sqrt(P(3, i+1));
    P_4_2sig(i+1) = 2 * sqrt(P(4, i+1));
end
```
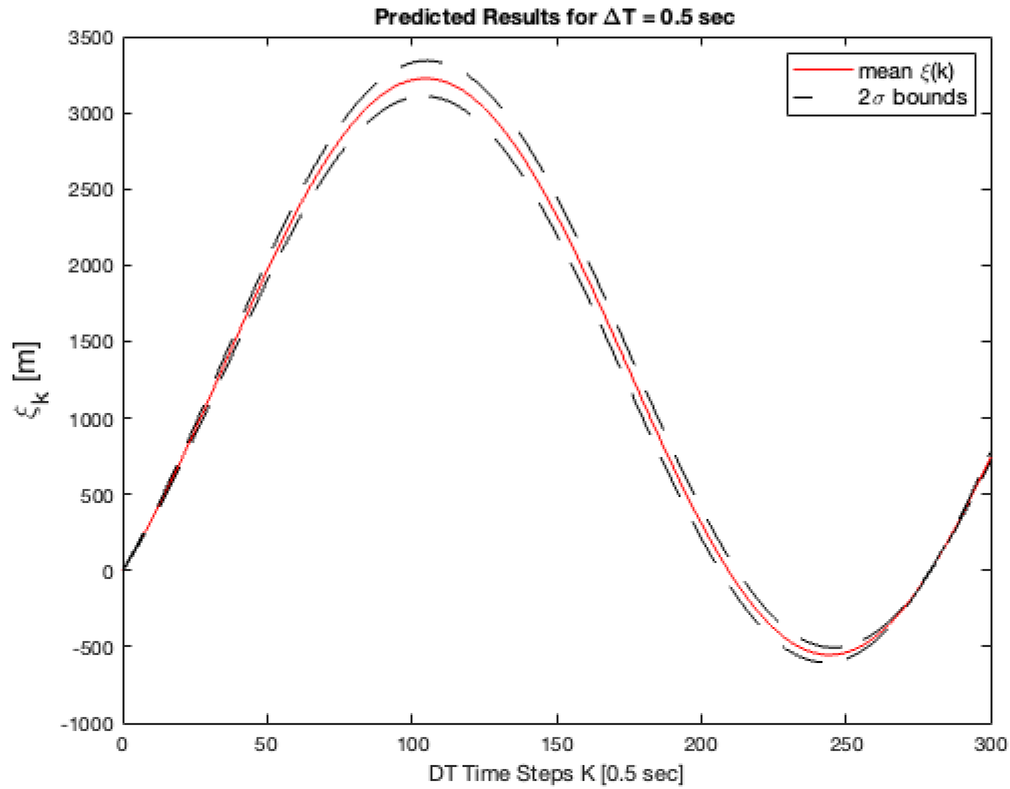
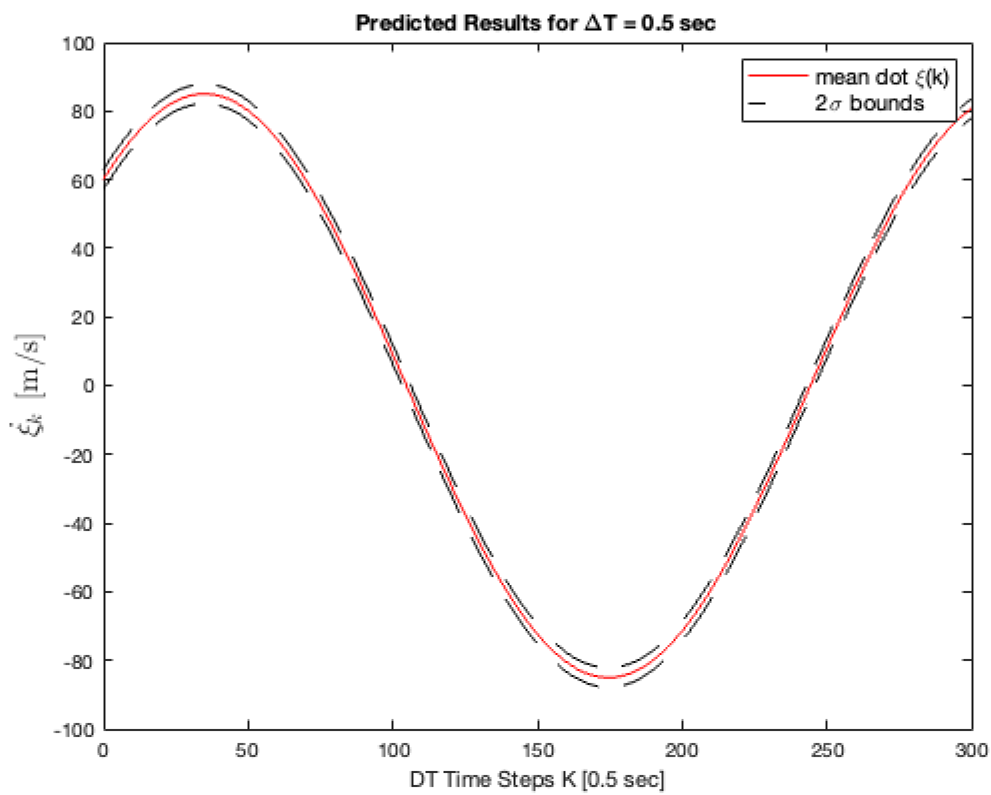

```
function out = get_diag(mat)
    len = length(mat);
    for i = 1:len
        out(i, 1) = mat(i, i);
    end
end


k = 0:T;

figure()
plot(k, mu(1, :), 'r')
hold on
plot(k, mu(1, :) + P_1_2sig, '--k')
plot(k, mu(1, :) - P_1_2sig, '--k')
hold off
title("Predicted Results for \DeltaT = 0.5 sec")
legend("mean \xi(k)", "2\sigma bounds", 'FontSize', 11)
xlabel("DT Time Steps K [0.5 sec]")
ylabel("\xi_k [m]", 'FontSize', 15)


figure()
plot(k, mu(2, :), 'r')
hold on
plot(k, mu(2, :) + P_2_2sig, '--k')
plot(k, mu(2, :) - P_2_2sig, '--k')
hold off
title("Predicted Results for \DeltaT = 0.5 sec")
legend("mean dot \xi(k)", "2\sigma bounds", 'FontSize', 11)
xlabel("DT Time Steps K [0.5 sec]")
ylabel('$\dot{\xi_k}$ [m/s]', 'Interpreter','latex', 'FontSize', 15)


figure()
plot(k, mu(3, :), 'r')
hold on
plot(k, mu(3, :) + P_3_2sig, '--k')
plot(k, mu(3, :) - P_3_2sig, '--k')
hold off
title("Predicted Results for \DeltaT = 0.5 sec")
legend("mean \eta(k)", "2\sigma bounds", 'FontSize', 11)
xlabel("DT Time Steps K [0.5 sec]")
ylabel("\eta_k [m]", 'FontSize', 15)


figure()
plot(k, mu(4, :), 'r')
hold on
plot(k, mu(4, :) + P_4_2sig, '--k')
plot(k, mu(4, :) - P_4_2sig, '--k')
hold off
title("Predicted Results for \DeltaT = 0.5 sec")
```

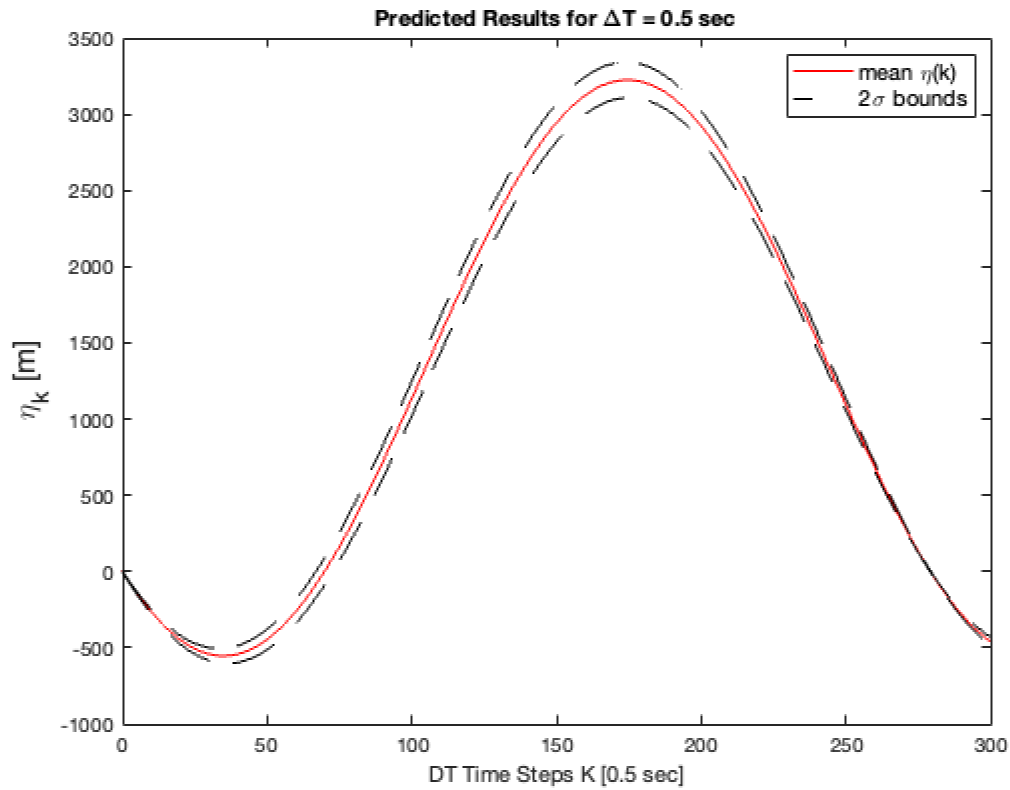
```
legend("mean dot \eta(k)", "2\sigma bounds", 'FontSize', 11)
xlabel("DT Time Steps K [0.5 sec]")
ylabel('$\dot{\eta_k}$ [m/s]', 'Interpreter','latex', 'FontSize', 15)


figure()
plot(k, P_1_2sig)
title("Positive 2\sigma vs Time")
xlabel("DT Time Steps K [0.5 sec]")
ylabel("\xi_k [m]", 'FontSize', 15)

figure()
plot(k, P_2_2sig)
title("Positive 2\sigma vs Time")
xlabel("DT Time Steps K [0.5 sec]")
ylabel('$\dot{\xi_k}$ [m/s]', 'Interpreter','latex', 'FontSize', 15)
ylim([0 2*P_2_2sig(1)])

figure()
plot(k, P_3_2sig)
title("Positive 2\sigma vs Time")
xlabel("DT Time Steps K [0.5 sec]")
ylabel("\eta_k [m]", 'FontSize', 15)

figure()
plot(k, P_4_2sig)
title("Positive 2\sigma vs Time")
xlabel("DT Time Steps K [0.5 sec]")
ylabel('$\dot{\eta_k}$ [m/s]', 'Interpreter','latex', 'FontSize', 15)
ylim([0 2*P_4_2sig(1)])
```

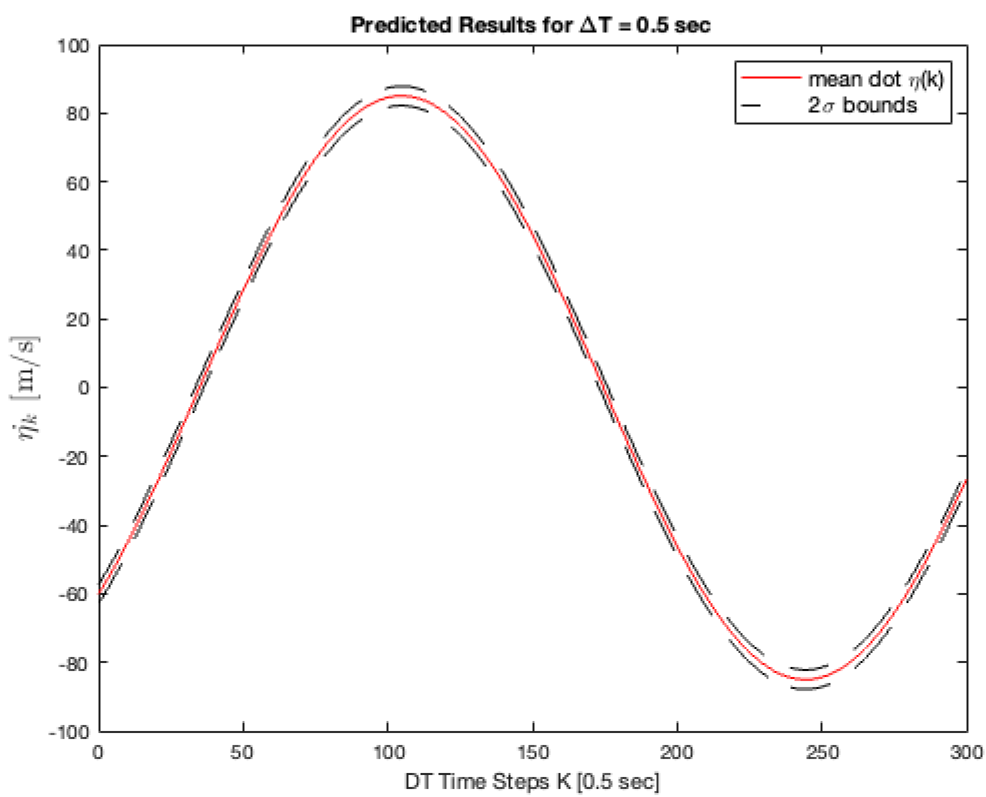


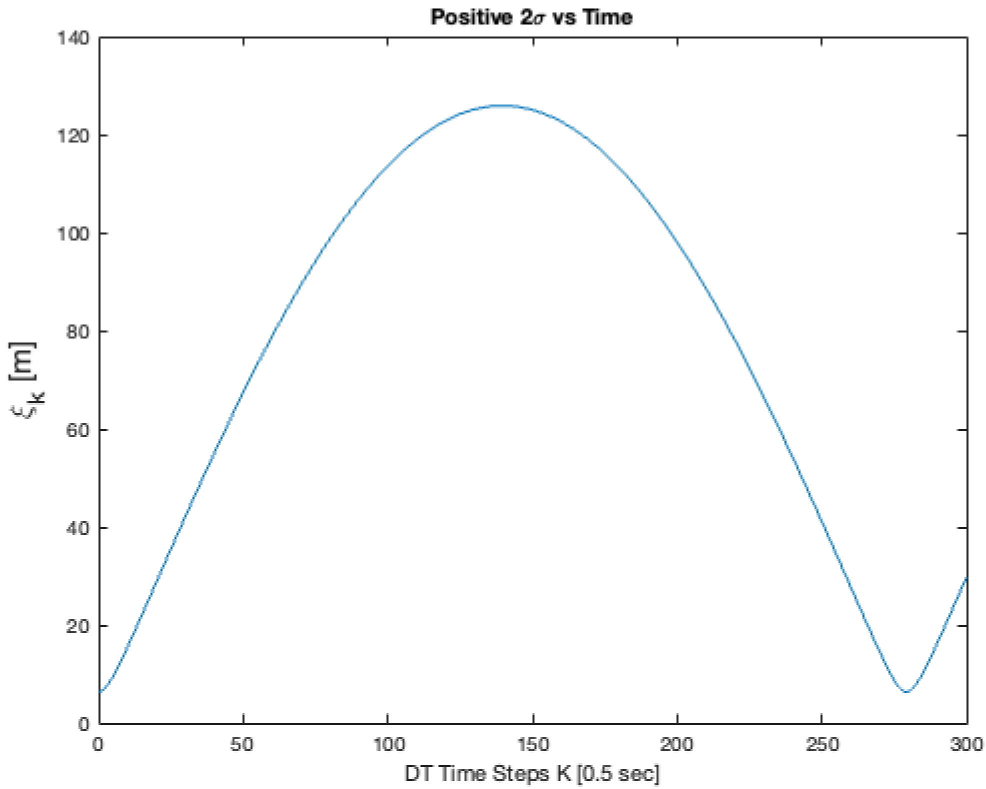
Predicted Results for ΔT = 0.5 sec
mean ξ(k)
2σ bounds
ξ_k [m]
DT Time Steps K [0.5 sec]

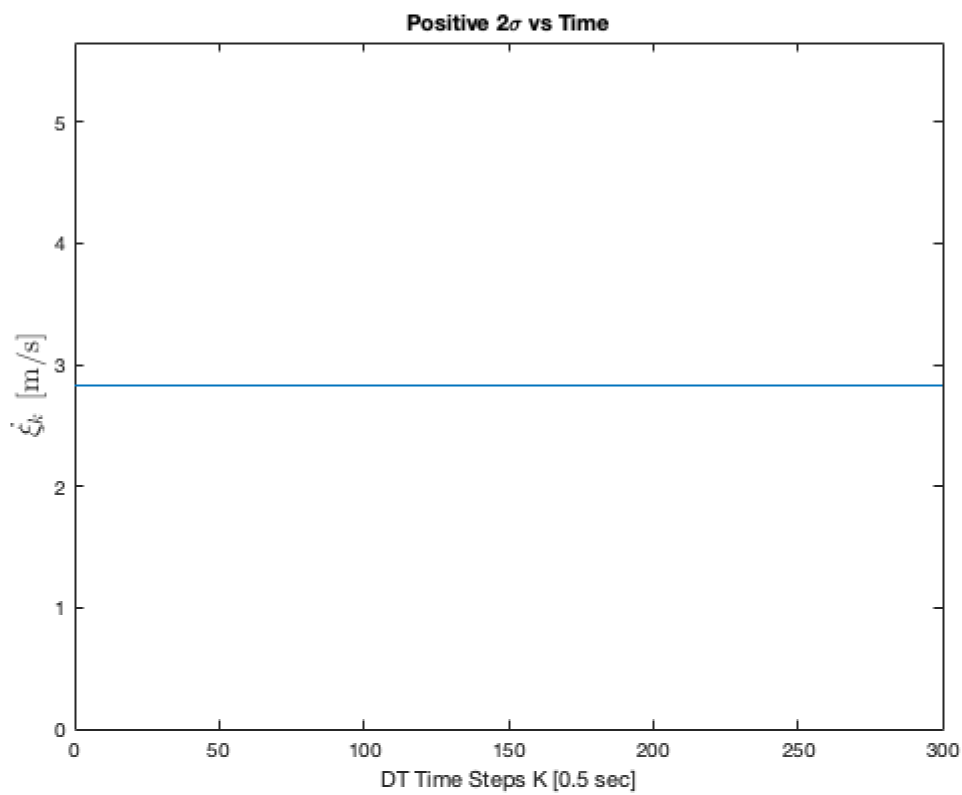


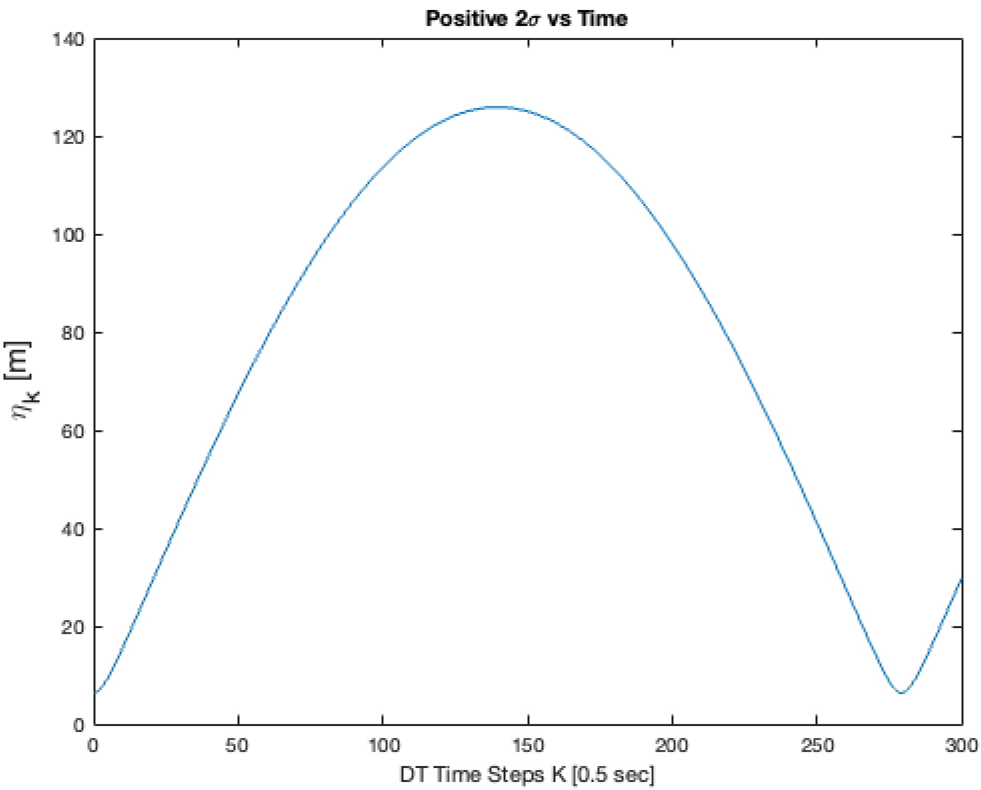
Predicted Results for ΔT = 0.5 sec
mean dot ξ(k)
2σ bounds
ξ̇_k [m/s]
DT Time Steps K [0.5 sec]


Predicted Results for ΔT = 0.5 sec
mean η(k)
2σ bounds
η_k [m]
DT Time Steps K [0.5 sec]
3500
3000
2500
2000
1500
1000
500
0
-500
-1000
0
50
100
150
200
250
300

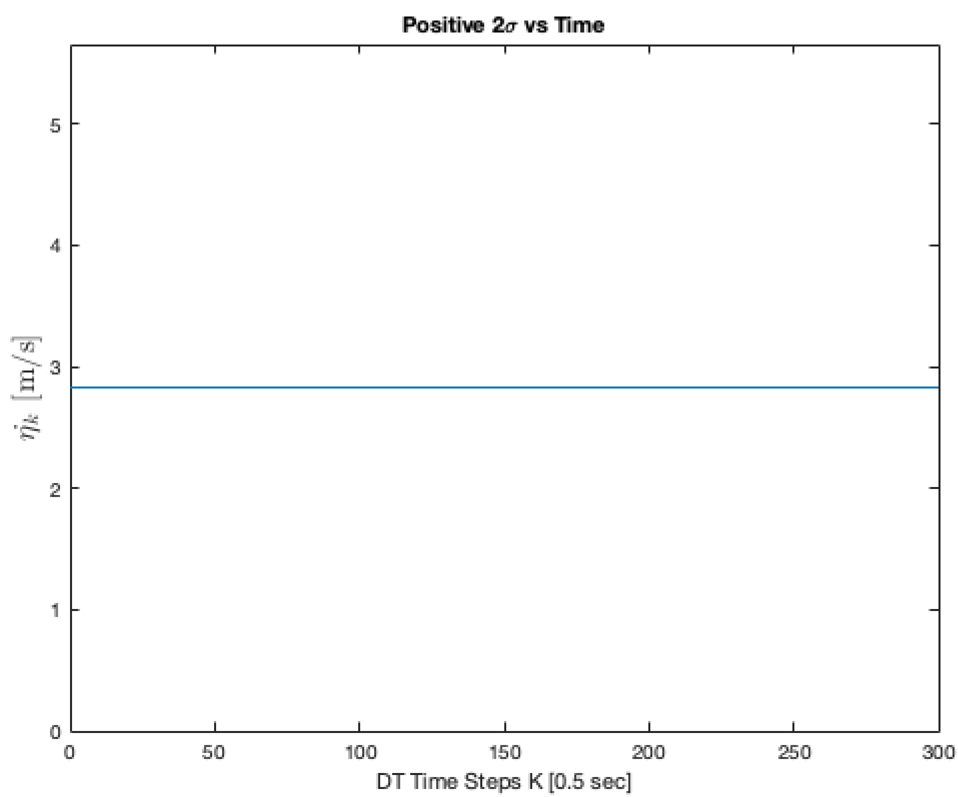

Predicted Results for ΔT = 0.5 sec
mean dot η(k)
2σ bounds
η̇_k [m/s]
DT Time Steps K [0.5 sec]
100
80
60
40
20
0
-20
-40
-60
-80
-100
0
50
100
150
200
250
300


Positive 2σ vs Time
ξ_k [m]
DT Time Steps K [0.5 sec]


Positive 2σ vs Time
$\dot{\xi}_k$ [m/s]
DT Time Steps K [0.5 sec]



Positive 2σ vs Time
140
120
100
80
60
40
20
0
η_k [m]
0
50
100
150
200
250
300
DT Time Steps K [0.5 sec]



Positive 2σ vs Time
5
4
3
2
1
0
$\dot{\eta}_k$ [m/s]
0
50
100
150
200
250
300
DT Time Steps K [0.5 sec]

Problem 3

```
delta_t = 0.5;

omega_a = 0.045;
omega_b = -0.045;

odt_a = omega_a * delta_t;

F_a = [1 sin(odt_a)/omega_a 0 -(1-cos(odt_a))/omega_a;
        0 cos(odt_a) 0 -sin(odt_a);
        0 (1-cos(odt_a))/omega_a 1 sin(odt_a)/omega_a;
        0 sin(odt_a) 0 cos(odt_a)];

odt_b = omega_b * delta_t;

F_b = [1 sin(odt_b)/omega_b 0 -(1-cos(odt_b))/omega_b;
        0 cos(odt_b) 0 -sin(odt_b);
        0 (1-cos(odt_b))/omega_b 1 sin(odt_b)/omega_b;
        0 sin(odt_b) 0 cos(odt_b)];

u_a_0 = [0; 85*cos(pi/4); 0; -85*sin(pi/4)];
u_b_0 = [3200; 85*cos(pi/4); 3200; -85*sin(pi/4)];

u_a = [u_a_0];
u_b = [u_b_0];

P_a_0 = diag([10, 4, 10, 4]);
P_b_0 = diag([11, 3.5, 11, 3.5]);

P_a = [P_a_0];
P_b = [P_b_0];

T = 150;
t = 0:delta_t:T;

for i = 1:T/delta_t
    u_a(:, i+1) = F_a^i * u_a_0;
    P_a((4*i + 1):(4*(i+1)), :) = F_a^i * P_a_0 * (F_a')^i;

    u_b(:, i+1) = F_b^i * u_b_0;
    P_b((4*i + 1):(4*(i+1)), :) = F_b^i * P_b_0 * (F_b')^i;
end

u_c = u_a - u_b;
P_c = P_a + P_b;

u_r_c = [u_c(1, :); u_c(3, :)];
P_r_c = [P_c(1, 1), P_c(1, 3); P_c(3, 1), P_c(3, 3)];

xi_R = 100;
eta_R = 100;
x_R = [xi_R; eta_R];
```

```
cdf_x_R = mvncdf(x_R, u_r_c(:,1), P_r_c);
cdf_neg_x_R = mvncdf(-x_R, u_r_c(:,1), P_r_c);

probability_of_collision = cdf_x_R - cdf_neg_x_R;

for i = 1:T/delta_t
    P_c_temp = [P_c((4*i + 1):(4*(i+1)),:)];
    P_r_c_current = [P_c_temp(1,1), P_c_temp(1,3); P_c_temp(3,1),
P_c_temp(3,3)];
    P_r_c((2*i + 1):(2*(i+1)), :) = P_r_c_current;

    cdf_x_R = mvncdf(x_R, u_r_c(:,i), P_r_c_current);
    cdf_neg_x_R = mvncdf(-x_R, u_r_c(:,i), P_r_c_current);

    probability_of_collision(i+1) = cdf_x_R - cdf_neg_x_R;
end

figure()
plot(t, probability_of_collision*100, 'LineWidth',1.5)
xlabel("Time [sec]")
ylabel("Probability [%]")
ylim([0, 100])
title("Probability of collision of the two aircrafts")
```