ASEN 5050
ADVANCED ASTRODYNAMICS
Multi-Body Dynamics

Objectives:

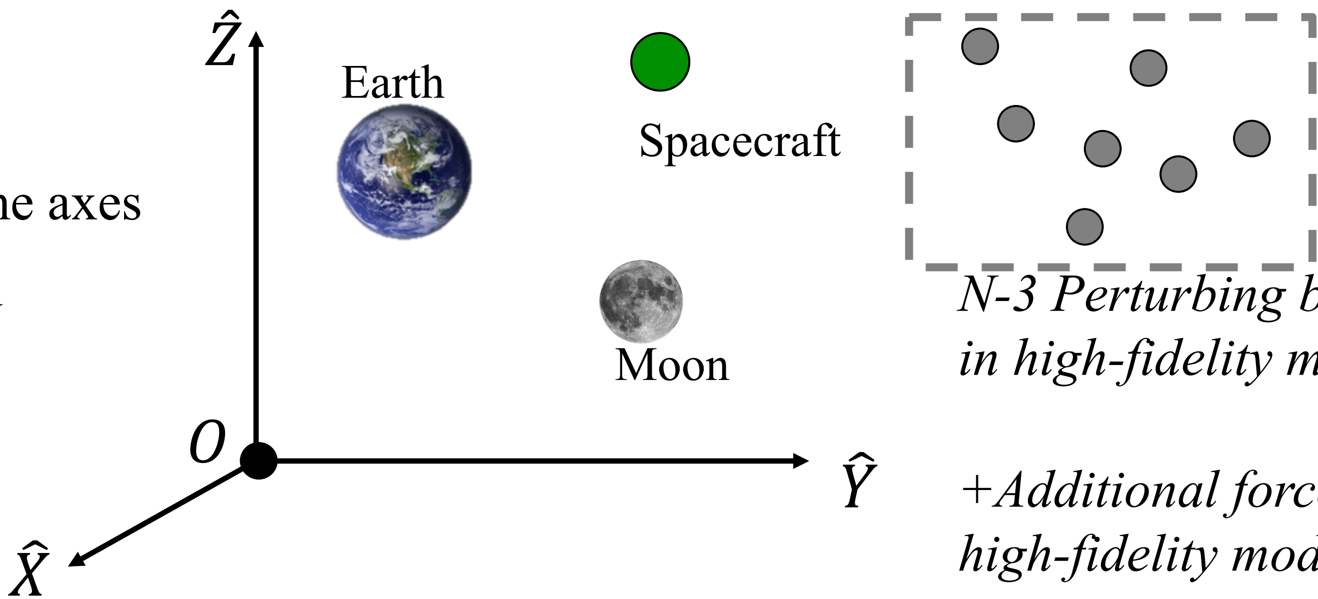
- Define the CR3BP
- Introduce the assumptions and nondimensionalization
- Derive the EOMs and constant of motion
- Introduce types of motion

Multi-Body Systems

Consider the following configuration of bodies, where each body can follow any general path

$\hat{X}\hat{Y}\hat{Z}$: Inertial frame axes

O, Inertially fixed basepoint



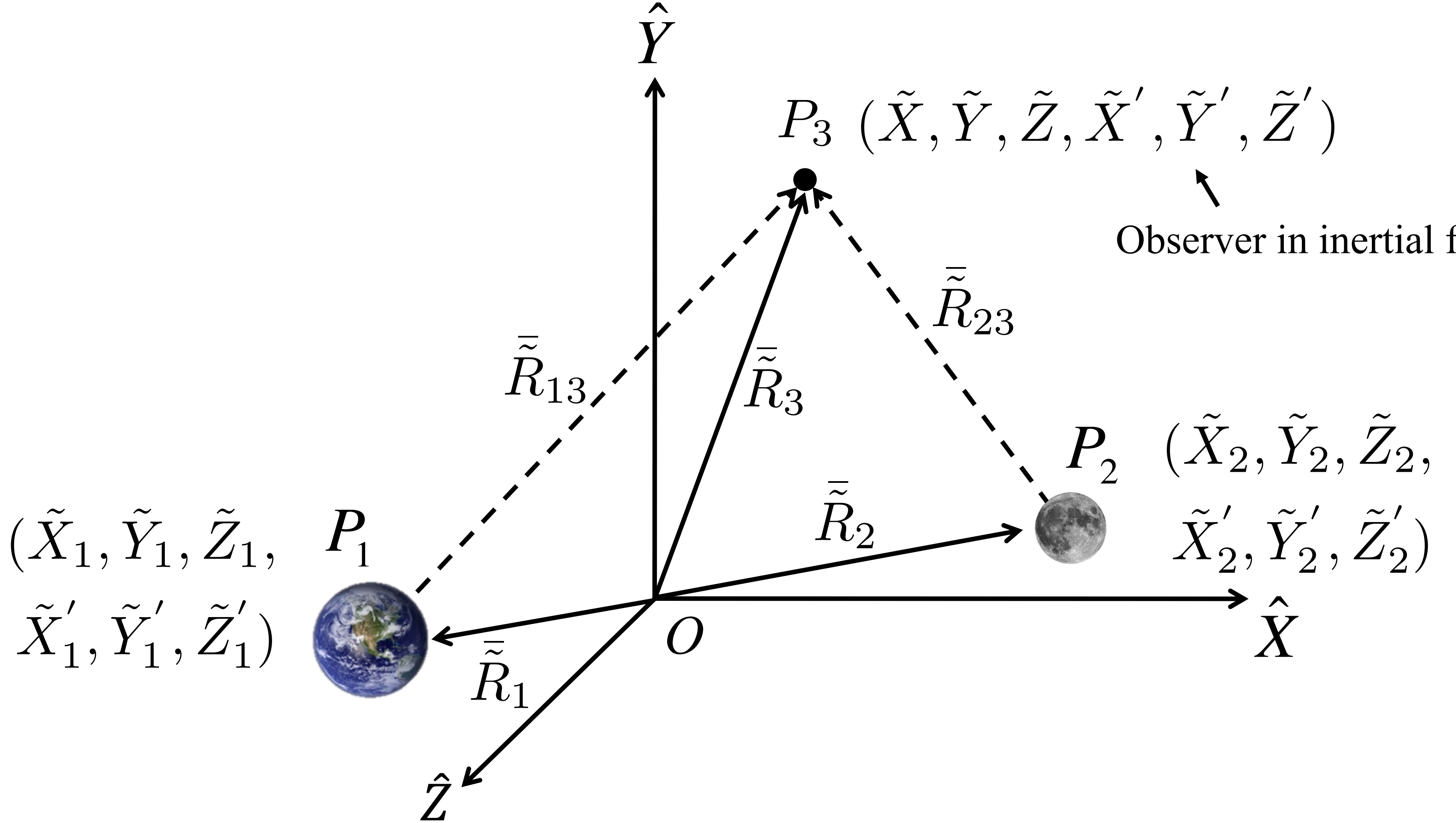
N-3 Perturbing bodies in high-fidelity model

+Additional forces in high-fidelity model

Interested in motion of spacecraft within Earth-Moon system

System Configuration

Reduce the complexity of this model by assuming only three bodies, modeled with same gravity as point masses, in the system:



Deriving EOMs for Three-Body Problem

Derive the equations of motion using the potential function for Body 3, per unit mass $\tilde{M}_3$, due to the gravity of two point masses:

$$\tilde{U}_3 = \frac{\tilde{G}\tilde{M}_1}{\tilde{R}_{13}} + \frac{\tilde{G}\tilde{M}_2}{\tilde{R}_{23}}$$

Where the tilde denotes dimensional quantities and

$$\bar{\tilde{R}}_3'' = \tilde{X}''\hat{X} + \tilde{Y}''\hat{Y} + \tilde{Z}''\hat{Z}$$

$$\tilde{R}_{13} = \sqrt{\left(\tilde{X} - \tilde{X}_1\right)^2 + \left(\tilde{Y} - \tilde{Y}_1\right)^2 + \left(\tilde{Z} - \tilde{Z}_1\right)^2}$$

$$\tilde{R}_{23} = \sqrt{\left(\tilde{X} - \tilde{X}_2\right)^2 + \left(\tilde{Y} - \tilde{Y}_2\right)^2 + \left(\tilde{Z} - \tilde{Z}_2\right)^2}$$

To derive the equations of motion (EOMs), note the force per unit mass (i.e., acceleration) acting on Body 3 is written as:

$$\bar{\nabla}\tilde{U}_3 = \bar{\tilde{R}}_3'' \qquad \bar{\nabla}\tilde{U}_3 = \frac{\partial\tilde{U}_3}{\partial\tilde{X}}\hat{X} + \frac{\partial\tilde{U}_3}{\partial\tilde{Y}}\hat{Y} + \frac{\partial\tilde{U}_3}{\partial\tilde{Z}}\hat{Z}$$

Deriving EOMs for Three-Body Problem

Taking these derivatives:

$$\frac{\partial \tilde{U}_3}{\partial \tilde{X}} = \frac{\partial}{\partial \tilde{X}} \left(\frac{\tilde{G}\tilde{M}_1}{\tilde{R}_{13}} + \frac{\tilde{G}\tilde{M}_2}{\tilde{R}_{23}} \right)$$

$$\tilde{R}_{13} = \left((\tilde{X} - \tilde{X}_1)^2 + (\tilde{Y} - \tilde{Y}_1)^2 + (\tilde{Z} - \tilde{Z}_1)^2 \right)^{0.5}$$

$$\frac{\partial}{\partial \tilde{X}} \left(\frac{1}{\tilde{R}_{13}} \right) = \frac{\partial}{\partial \tilde{X}} \left[\left((\tilde{X} - \tilde{X}_1)^2 + (\tilde{Y} - \tilde{Y}_1)^2 + (\tilde{Z} - \tilde{Z}_1)^2 \right)^{-0.5} \right]$$

$$\frac{\partial}{\partial \tilde{X}} \left(\frac{1}{\tilde{R}_{13}} \right) = -\frac{1}{2} 2(\tilde{X} - \tilde{X}_1) \left((\tilde{X} - \tilde{X}_1)^2 + (\tilde{Y} - \tilde{Y}_1)^2 + (\tilde{Z} - \tilde{Z}_1)^2 \right)^{-3/2}$$

$$\frac{\partial \tilde{U}_3}{\partial \tilde{X}} = \frac{\partial}{\partial \tilde{X}} \left(\frac{\tilde{G}\tilde{M}_1}{\tilde{R}_{13}} + \frac{\tilde{G}\tilde{M}_2}{\tilde{R}_{23}} \right) = -\frac{\tilde{G}\tilde{M}_1(\tilde{X} - \tilde{X}_1)}{\tilde{R}_{13}^3} - \frac{\tilde{G}\tilde{M}_2(\tilde{X} - \tilde{X}_2)}{\tilde{R}_{23}^3}$$

Deriving EOMs for Three-Body Problem

Using Newton's 2nd law, EOMs for the three-body problem:

$$\tilde{X}'' = -\frac{\tilde{G}\tilde{M}_1(\tilde{X}-\tilde{X}_1)}{\tilde{R}_{13}^3} - \frac{\tilde{G}\tilde{M}_2(\tilde{X}-\tilde{X}_2)}{\tilde{R}_{23}^3}$$

$$\tilde{Y}'' = -\frac{\tilde{G}\tilde{M}_1(\tilde{Y}-\tilde{Y}_1)}{\tilde{R}_{13}^3} - \frac{\tilde{G}\tilde{M}_2(\tilde{Y}-\tilde{Y}_2)}{\tilde{R}_{23}^3}$$

$$\tilde{Z}'' = -\frac{\tilde{G}\tilde{M}_1(\tilde{Z}-\tilde{Z}_1)}{\tilde{R}_{13}^3} - \frac{\tilde{G}\tilde{M}_2(\tilde{Z}-\tilde{Z}_2)}{\tilde{R}_{23}^3}$$

Simplifying Assumptions

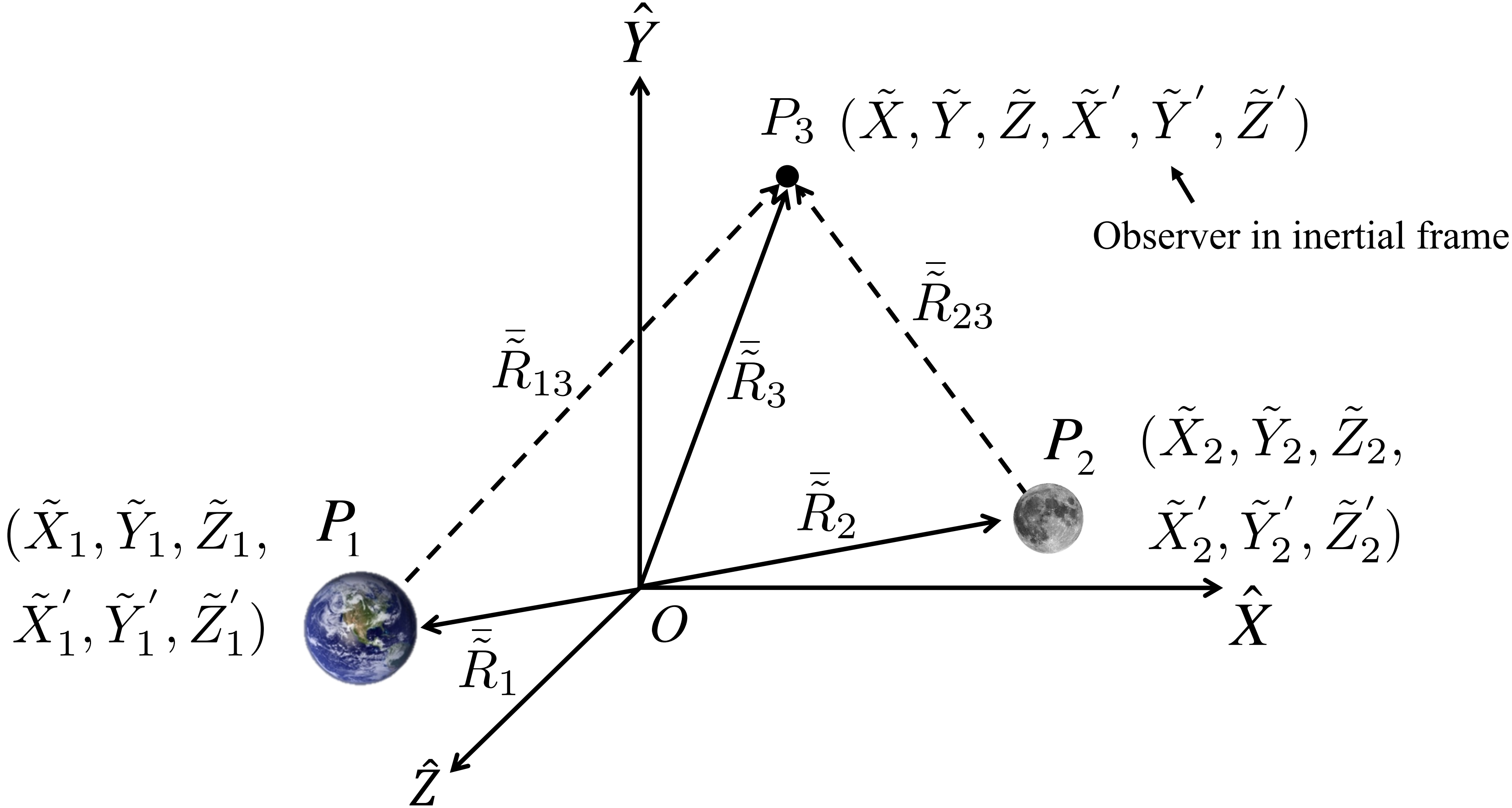
To reduce the complexity of modeling and analyzing the dynamics, we will use all the following assumptions:

1. Mass of P_3 << Masses of each of P_1, P_2
2. P_3 does not influence the paths of P_1 and P_2, so both primaries travel on conics about their mutual barycenter
3. P_1 and P_2 follow circular orbits
4. Model each body with gravity field equivalent to a point mass with constant mass

When these assumptions are used, we call the dynamical model the circular restricted three-body problem (CR3BP)

System Configuration

Reduce the complexity of this model by assuming only three bodies, modeled with same gravity as point masses, in the system:



Nondimensionalization

In practice, time, mass and length quantities can have very different orders of magnitude. Also useful to compare quantities with similar ratios of the mass of the primaries

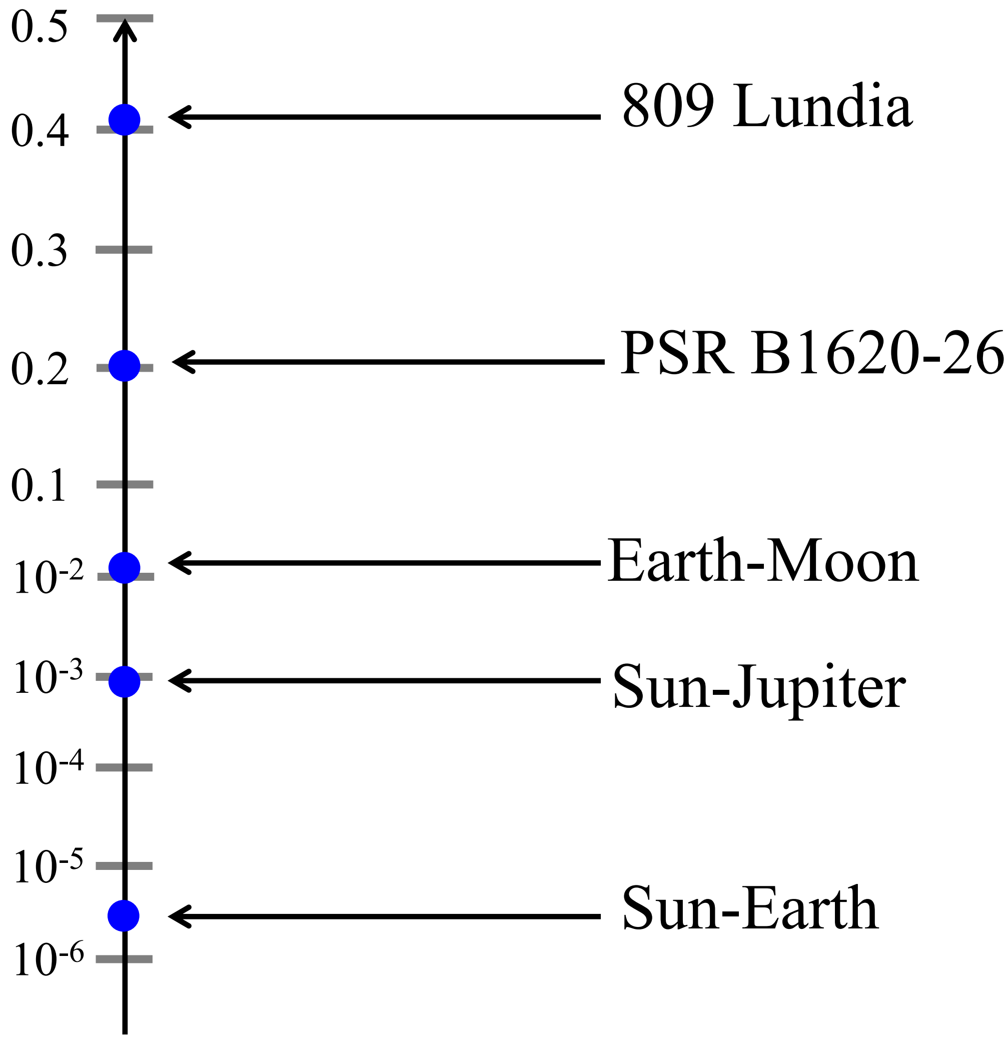
Introduce characteristic quantities m^*, l^*, t^* and nondimensional quantities have no tilde.

1. Mass: $$m^* = \tilde{M}_1 + \tilde{M}_2$$

$$\mu = M_2 = \frac{\tilde{M}_2}{m^*} \qquad\qquad 1 - \mu = M_1 = \frac{\tilde{M}_1}{m^*}$$

Common Mass Ratios

$$\mu = \frac{\tilde{M}_2}{\tilde{M}_1 + \tilde{M}_2}$$



Nondimensionalization

2. Length

$$l^* = \tilde{R}_1 + \tilde{R}_2$$

3. Time $\quad t^* = \left(\dfrac{(l^*)^3}{\tilde{G} m^*} \right)^{1/2}$

$$t = \frac{\tau}{t^*}$$

Dimensional time

Nondimensional time

Recall from 2BP: $\quad n = \sqrt{\dfrac{\mu_{2BP}}{a^3}}$

Nondimensional gravitational constant: $\quad G = \dfrac{\tilde{G} m^* (t^*)^2}{(l^*)^3} = 1$

Writing Nondimensional EOMs

Relating dimensional and nondimensional components

$$X = \frac{\tilde{X}}{l*} \qquad Y = \frac{\tilde{Y}}{l*} \qquad Z = \frac{\tilde{Z}}{l*}$$

$$R_{13} = \frac{\tilde{R}_{13}}{l*} \qquad R_{23} = \frac{\tilde{R}_{23}}{l*}$$

$$\tilde{X}'' = \frac{d}{d\tau}\left(\frac{d\tilde{X}}{d\tau}\right) = \frac{d}{d(tt^*)}\left(\frac{d(Xl^*)}{d(tt^*)}\right) = \frac{l^*}{(t^*)^2}\frac{d^2X}{dt^2}$$

$$\tilde{Y}'' = \frac{l^*}{(t^*)^2}\frac{d^2Y}{dt^2} \qquad \tilde{Z}'' = \frac{l^*}{(t^*)^2}\frac{d^2Z}{dt^2}$$

Writing Nondimensional EOMs

$$\tilde{X}'' = -\frac{\tilde{G}\tilde{M}_1(\tilde{X} - \tilde{X}_1)}{\tilde{R}_{13}^3} - \frac{\tilde{G}\tilde{M}_2(\tilde{X} - \tilde{X}_2)}{\tilde{R}_{23}^3}$$

$$\frac{l^*}{(t^*)^2}X'' = -\frac{\tilde{G}(1-\mu)m^* l^*(X - X_1)}{(l^*)^3 R_{13}^3} - \frac{\tilde{G}(\mu)m^* l^*(X - X_2)}{(l^*)^3 R_{23}^3}$$

$$X'' = \frac{\tilde{G}(t^*)^2 m^*}{(l^*)^3}\left[-\frac{(1-\mu)(X - X_1)}{R_{13}^3} - \frac{\mu(X - X_2)}{R_{23}^3}\right]$$

$$G = \frac{\tilde{G}m^*(t^*)^2}{(l^*)^3} = 1$$

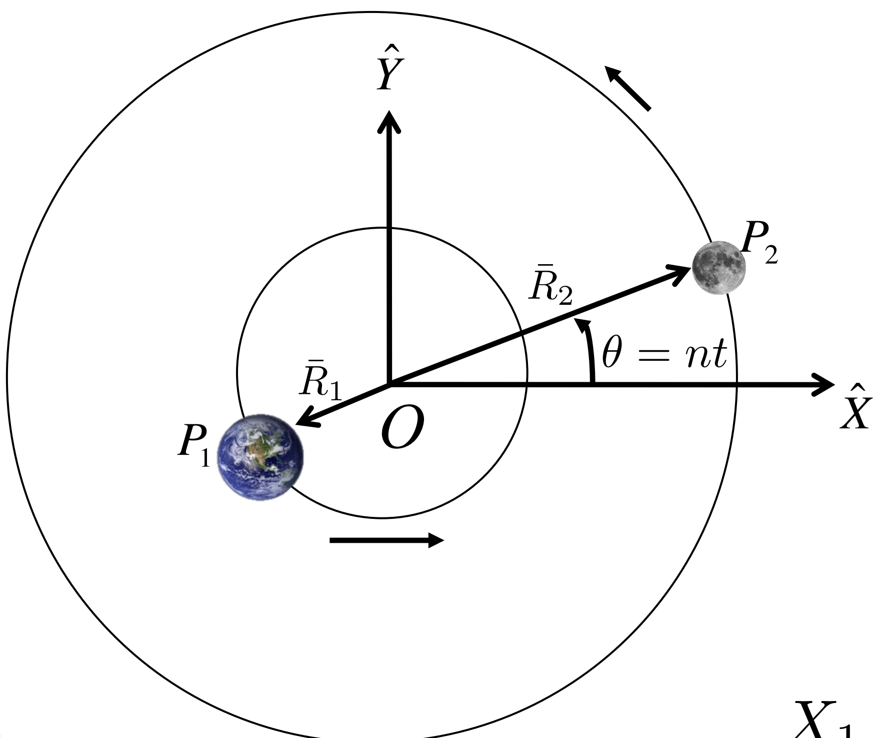
Writing Nondimensional EOMs

$$X^{''} = -\frac{(1-\mu)(X-X_1)}{R_{13}^3} - \frac{\mu(X-X_2)}{R_{23}^3}$$

Repeat for *Y*, *Z* equations

$$\bar{R}^{''} = -\frac{(1-\mu)}{R_{13}^3}(\bar{R}-\bar{R}_1) - \frac{\mu}{R_{23}^3}(\bar{R}-\bar{R}_2)$$

Path of Primaries



Center of mass lies along line between primaries:

$$\bar{0} = \frac{M_1\bar{R}_1 + M_2\bar{R}_2}{M_1 + M_2}$$

$$(1-\mu)\bar{R}_1 = -\mu\bar{R}_2$$

Write coordinates of P_1, P_2 in inertial frame, relative to barycenter

$$X_1 = -\mu\cos(t) \qquad X_2 = (1-\mu)\cos(t)$$

$$Y_1 = -\mu\sin(t) \qquad Y_2 = (1-\mu)\sin(t)$$

$$Z_1 = 0 \qquad Z_2 = 0$$

Update EOMs

Plug in known coordinates of primaries:

$$X'' = -\frac{(1-\mu)(X+\mu\cos(t))}{R_{13}^3} - \frac{\mu(X-(1-\mu)\cos(t))}{R_{23}^3}$$

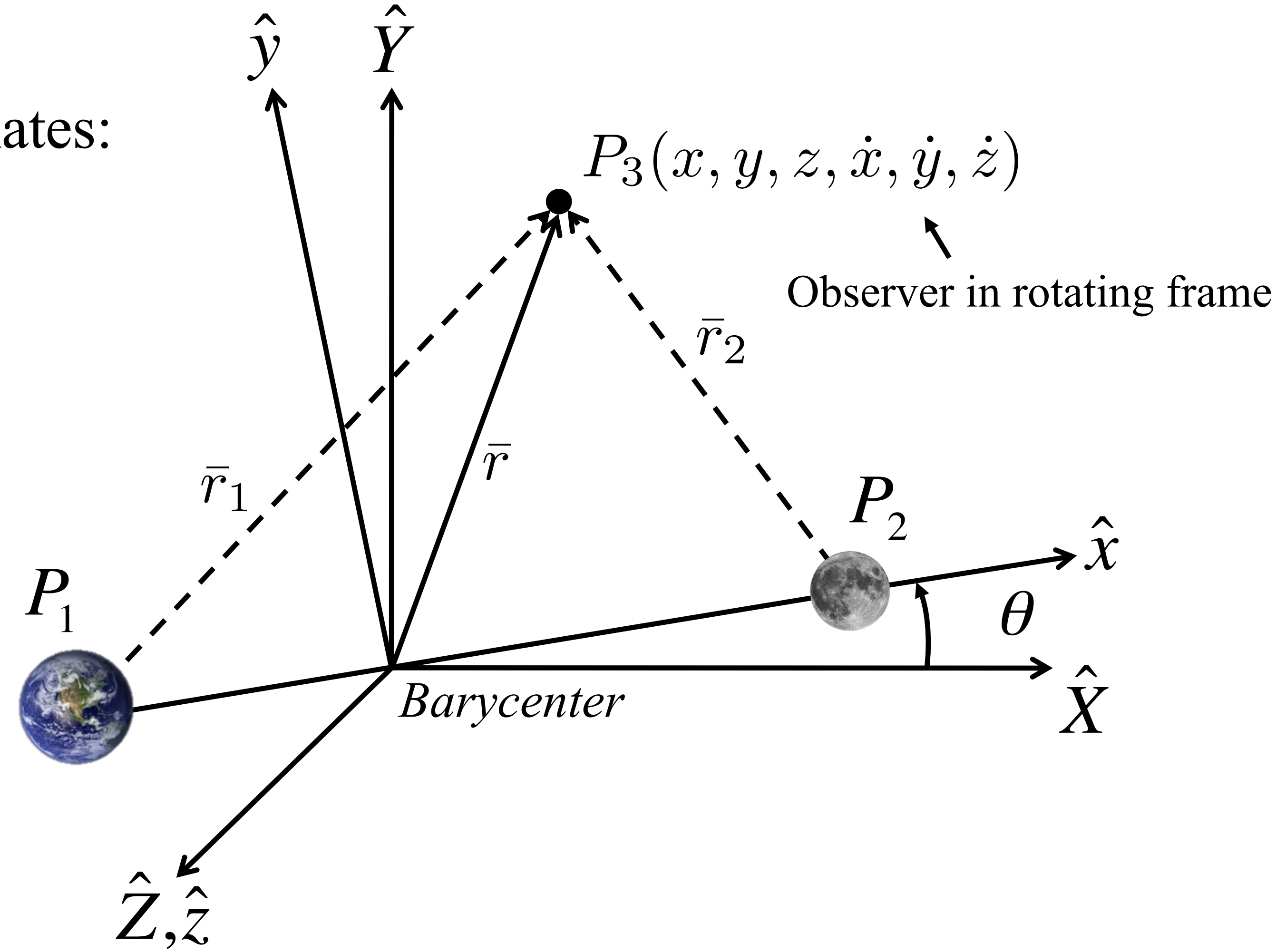
$$Y'' = -\frac{(1-\mu)(Y+\mu\sin(t))}{R_{13}^3} - \frac{\mu(Y-(1-\mu)\sin(t))}{R_{23}^3}$$

$$Z'' = -\frac{(1-\mu)Z}{R_{13}^3} - \frac{\mu Z}{R_{23}^3}$$

Rotating Frame

Introduce a frame that rotates with the two primaries

Lowercase coordinates: correspond to rotating frame



Rotating Frame

Transformation written as simple rotation:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos(t) & \sin(t) & 0 \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

In the rotating frame, the primaries are located at:

P_1: $(-\mu, 0, 0)$ P_2: $(1-\mu, 0, 0)$

Then: $\bar{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$$\bar{r}_1 = (x+\mu)\hat{x} + y\hat{y} + z\hat{z}$$

$$\bar{r}_2 = (x-1+\mu)\hat{x} + y\hat{y} + z\hat{z}$$

Rewriting EOMs in Rotating Frame

First step is to convert the acceleration to use rotating frame components and an observer in the rotating frame

$$\bar{r} = x\hat{x} + y\hat{y} + z\hat{z} \qquad {}^{R}\bar{v} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z} \qquad {}^{I}\bar{\omega}^{R} = n\hat{z} = +1\hat{z}$$

$$\frac{{}^{I}d\bar{r}}{dt} = \frac{{}^{R}d\bar{r}}{dt} + {}^{I}\bar{\omega}^{R} \times \bar{r}$$

$$\frac{{}^{I}d\bar{r}}{dt} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z} + \hat{z} \times [x\hat{x} + y\hat{y} + z\hat{z}]$$

$$\frac{{}^{I}d\bar{r}}{dt} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z} + x\hat{y} - y\hat{x}$$

$$\frac{{}^{I}d\bar{r}}{dt} = (\dot{x} - y)\hat{x} + (\dot{y} + x)\hat{y} + \dot{z}\hat{z}$$

Rewriting EOMs in Rotating Frame

$$\frac{{}^I d\bar{v}}{dt} = \frac{{}^R d\bar{v}}{dt} + {}^I \bar{\omega}^R \times \bar{v}$$

$$\frac{{}^I d\bar{v}}{dt} = (\ddot{x} - \dot{y})\hat{x} + (\ddot{y} + \dot{x})\hat{y} + \ddot{z}\hat{z} + \hat{z} \times [(\dot{x} - y)\hat{x} + (\dot{y} + x)\hat{y} + \dot{z}\hat{z}]$$

$$\frac{{}^I d\bar{v}}{dt} = (\ddot{x} - 2\dot{y} - x)\hat{x} + (\ddot{y} + 2\dot{x} - y)\hat{y} + \ddot{z}\hat{z}$$

Recall: $\bar{R}'' = -\frac{(1-\mu)}{R_{13}^3}(\bar{R} - \bar{R}_1) - \frac{\mu}{R_{23}^3}(\bar{R} - \bar{R}_2)$

Circular Restricted Three-Body Problem

Nondimensional equations of motion for spacecraft in the rotating frame, using coordinates relative to the barycenter:

$$\ddot{x} = 2\dot{y} + x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3}$$

$$\ddot{y} = -2\dot{x} + y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}$$

$$\ddot{z} = -\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3}$$

Where:

$$r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}$$

$$r_2 = \sqrt{(x-1+\mu)^2 + y^2 + z^2}$$

Pseudo-Potential Function

Define a "pseudo-potential", which is similar to a potential function and depends only on position variables but incorporates an additional term due to the rotation of the rotating frame

$$\ddot{x} = 2\dot{y} + \boxed{x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3}} \longleftarrow \frac{\partial U^*}{\partial x}$$

$$\ddot{y} = -2\dot{x} + \boxed{y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}} \longleftarrow \frac{\partial U^*}{\partial y}$$

$$\ddot{z} = \boxed{-\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3}} \longleftarrow \frac{\partial U^*}{\partial z}$$

Pseudo-Potential Function

Produces compact equations of motion:

$$\ddot{x} - 2\dot{y} = \frac{\partial U^*}{\partial x} \qquad \ddot{y} + 2\dot{x} = \frac{\partial U^*}{\partial y} \qquad \ddot{z} = \frac{\partial U^*}{\partial z}$$

$$U^* = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

Properties of the resulting dynamical system:

- Time-independent
- Chaotic solution space
- Possesses an integral of motion in rotating frame
- Symmetries exist

Deriving the Jacobi Constant

Derive the **Jacobi constant** using equations of motion:

$$\ddot{x} - 2\dot{y} = \frac{\partial U^*}{\partial x} \qquad \ddot{y} + 2\dot{x} = \frac{\partial U^*}{\partial y} \qquad \ddot{z} = \frac{\partial U^*}{\partial z}$$

$$U^* = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

Recall the acceleration and velocity vectors:

$$\bar{a} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}$$

$$\bar{v} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$$

Deriving the Jacobi Constant

Take the dot product of acceleration and velocity vectors:

$$\ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z} = \left(2\dot{y} + \frac{\partial U^*}{\partial x}\right)\dot{x} + \left(-2\dot{x} + \frac{\partial U^*}{\partial y}\right)\dot{y} + \left(\frac{\partial U^*}{\partial z}\right)\dot{z}$$

$$\ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z} = \frac{\partial U^*}{\partial x}\dot{x} + \frac{\partial U^*}{\partial y}\dot{y} + \frac{\partial U^*}{\partial z}\dot{z}$$

Because $\frac{\partial U^*}{\partial t} = 0$ the total derivative of U^* is equal to

$$\frac{dU^*}{dt} = \frac{\partial U^*}{\partial x}\frac{dx}{dt} + \frac{\partial U^*}{\partial y}\frac{dy}{dt} + \frac{\partial U^*}{\partial z}\frac{dz}{dt}$$

Then:

$$\ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z} = \frac{dU^*}{dt}$$

Deriving the Jacobi Constant

To rewrite the LHS, note $\frac{d}{dt}\left(\dot{x}^2\right) = 2\dot{x}\ddot{x}$

Then $\frac{1}{2}\frac{d}{dt}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{dU^*}{dt}$

Setting $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$

And integrating $v^2 = 2U^* - C$

Rearranging, the Jacobi constant is defined as

$$C = 2U^* - v^2 = \left(x^2 + y^2\right) + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - \dot{x}^2 - \dot{y}^2 - \dot{z}^2$$

Energy-like quantity