①

Josh Bhalawat
ASEN 5044
Fall 2024

HW1

Problem 1 → a) Let $a = \begin{bmatrix} a_1 \\ a_2 \\ a_n \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \\ b_n \end{bmatrix}$, a, b are non-zero vectors, $c, d \in \mathbb{R}$ (non zero)

$$ab^T = \begin{bmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & & \\ \vdots & & \ddots & \\ a_nb_1 & & & a_nb_n \end{bmatrix} \quad \begin{matrix} ca_1b_1 + da_1b_n = 0 \rightarrow cb_1 = -db_n \rightarrow \frac{c}{d} = -b_n/b_1 \\ ca_nb_1 + da_nb_n = 0 \rightarrow cb_1 = -db_n \rightarrow \frac{c}{d} = -b_n/b_1 \end{matrix}$$

If $b_i = 0$, i^{th} column is 0. So, there are constants c, d that cause the rows to add to 0. Hence, the rows are linearly dependent ∴ The rank of matrix ab^T cannot be >1 ∴ $\boxed{\text{The rank is 1}}$

b) Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & & a_{mn} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & & b_{1m} \\ \vdots & & \\ b_{n1} & & b_{nm} \end{bmatrix}$, $AB = \begin{bmatrix} (a_{11}b_{11} + \cdots + a_{1n}b_{n1}) & & \\ & \ddots & \\ & & (a_{m1}b_{1m} + \cdots + a_{mn}b_{nm}) \end{bmatrix}$

$tr(AB) = a_{11}b_{11} + \cdots + a_{1n}b_{n1} + \cdots + a_{m1}b_{1m} + \cdots + a_{mn}b_{nm}$

$BA = \begin{bmatrix} (a_{11}b_{11} + \cdots + a_{m1}b_{1m}) & & \\ & \ddots & \\ & & (a_{1n}b_{n1} + a_{mn}b_{nm}) \end{bmatrix}$, $tr(BA) = a_{11}b_{11} + a_{m1}b_{1m} + \cdots + a_{1n}b_{n1} + a_{mn}b_{nm}$

∴ $\boxed{tr(AB) = tr(BA)}$

Problem 2 →

$m_1\ddot{q}_1 = -k_1q_1 - k_2(q_1 - q_2) - u_1 \rightarrow \ddot{q}_1 = -\frac{k_1}{m_1}q_1 - \frac{k_2}{m_1}q_1 + \frac{k_2}{m_1}q_2 - \frac{u_1}{m_1}$

$m_2\ddot{q}_2 = -k_2(q_2 - q_1) - k_3q_2 + u_1 + u_2 \rightarrow \ddot{q}_2 = -\frac{k_2}{m_2}q_2 + \frac{k_2}{m_2}q_1 - \frac{k_3}{m_2}q_2 + \frac{u_1}{m_1} + \frac{u_2}{m_2}$

a) $x = \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix}$, $\dot{x} = \begin{bmatrix} \dot{q}_1 \\ \ddot{q}_1 \\ \dot{q}_2 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ q_1\left(\frac{-k_1 - k_2}{m_1}\right) + \frac{k_2}{m_1}q_2 - \frac{u_1}{m_1} \\ \dot{q}_2 \\ k_2/m_2\, q_1 + \left(\frac{-k_2 - k_3}{m_2}\right)q_2 + \frac{u_1}{m_2} + \frac{u_2}{m_2} \end{bmatrix}$

$$\dot{x} = \begin{bmatrix} \dot{q}_1 \\ \ddot{q}_1 \\ \dot{q}_2 \\ \ddot{q}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_1-k_2}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & \frac{-k_2-k_3}{m_2} & 0 \end{bmatrix}}_{A} \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ -\frac{1}{m_1} & 0 \\ 0 & 0 \\ \frac{1}{m_1} & \frac{1}{m_2} \end{bmatrix}}_{B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{D} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

let $k_1 = k_2 = k_3 = 1\,N/m$, $m_1 = m_2 = 1\,kg$

$$\boxed{\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = Ax + Bu}$$

$$\boxed{y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = Cx + Du}$$

b) $$\tilde{x} = \begin{bmatrix} q_1 - q_2 \\ \dot{q}_1 - \dot{q}_2 \\ q_1 + q_2 \\ \dot{q}_1 + \dot{q}_2 \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}}^{T} \overbrace{\begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix}}^{x} \rightarrow \tilde{x} = Tx, \quad \dot{\tilde{x}} = \frac{d}{dt}(Tx) = T\dot{x} + \cancel{\frac{d}{dt}(T)x} \rightarrow 0$$

$$\dot{x} = Ax + Bu, \quad \dot{\tilde{x}} = T(Ax + Bu)$$

$$\tilde{x} = Tx \rightarrow x = T^{-1}\tilde{x} \longrightarrow \dot{\tilde{x}} = TAT^{-1}\tilde{x} + TBu$$

$$\dot{\tilde{x}} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_1-k_2}{m_1} & 0 & k_2/m_1 & 0 \\ 0 & 0 & 0 & 1 \\ k_2/m_2 & 0 & \frac{-k_2-k_3}{m_2} & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ -1/2 & 0 & 1/2 & 0 \\ 0 & -1/2 & 0 & 1/2 \end{bmatrix} \tilde{x} + \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1/m_1 & 0 \\ 0 & 0 \\ 1/m_1 & 1/m_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ \frac{-k_1-2k_2}{2m_1} & 0 & -k_1/2m_1 & 0 \\ 0 & -1/2 & 0 & 1/2 \\ \frac{2k_2+k_3}{2m_2} & 0 & -k_3/2m_2 & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 & 0 \\ -\frac{2}{m_1} & -\frac{1}{m_2} \\ 0 & 0 \\ 0 & 1/m_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\dot{\tilde{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_1-2k_2}{2m_1} - \frac{(2k_2+k_3)}{2m_2} & 0 & -\frac{k_1}{2m_1} + \frac{k_3}{2m_2} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k_1-2k_2}{2m_1} + \frac{(2k_2+k_3)}{2m_2} & 0 & -\frac{k_1}{2m_1} - \frac{k_3}{2m_2} & 0 \end{bmatrix}}_{\tilde{A}} \tilde{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ -\frac{2}{m_1} & -\frac{1}{m_2} \\ 0 & 0 \\ 0 & 1/m_2 \end{bmatrix}}_{\tilde{B}} u$$

$$y = \underbrace{\begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ -1/2 & 0 & 1/2 & 0 \end{bmatrix}}_{\tilde{C}} \begin{bmatrix} q_1 - q_2 \\ \dot{q}_1 - \dot{q}_2 \\ q_1 + q_2 \\ \dot{q}_1 + \dot{q}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\tilde{D}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where $\boxed{\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u \text{ and } y = \tilde{C}\tilde{x} + \tilde{D}u}$

$$y = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

(2)

Jash Bhalavat
ASEN 5044
Fall 2024

$\boxed{\text{HW 1}}$

Problem 3 → $\Delta\dot{p} = \frac{M_x}{I_x}$, $\Delta\dot{q} = \frac{p_0(I_x - I_z)\Delta r + M_y}{I_y}$, $\Delta\dot{r} = \frac{p_0(I_y - I_x)\Delta q + M_z}{I_z}$, $x = [\Delta p, \Delta q, \Delta r]^T$

a)

$$\dot{x} = \begin{bmatrix} \Delta\dot{p} \\ \Delta\dot{q} \\ \Delta\dot{r} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{p_0(\cancel{I_y}_x - I_z)}{I_y} \\ 0 & \frac{p_0(I_y - I_x)}{I_z} & 0 \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta q \\ \Delta r \end{bmatrix} + \begin{bmatrix} \frac{1}{I_x} & 0 & 0 \\ 0 & \frac{1}{I_y} & 0 \\ 0 & 0 & \frac{1}{I_z} \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$$

$$y = \begin{bmatrix} \Delta p \\ \Delta q \\ \Delta r \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta q \\ \Delta r \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$$

where $\boxed{\dot{x} = Ax + Bu, \quad y = Cx + Du}$

b) $I_x = 500\ kg\,m^2$, $I_y = 750\ kg\,m^3$, $I_z = 1000\ kg\,m^3$, $p_0 = 20\ rad/s$, $\Delta t = 0.1s$

$$\Phi(t, t_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\frac{10\sqrt{6}}{3}\Delta t\right) & -2\frac{\sqrt{6}}{3}\sin\left(\frac{10\sqrt{6}}{3}\Delta t\right) \\ 0 & \frac{\sqrt{6}}{4}\sin\left(\frac{10\sqrt{6}}{3}\Delta t\right) & \cos\left(\frac{10\sqrt{6}}{3}\Delta t\right) \end{bmatrix}$$

where $\Delta t = t - t_0$

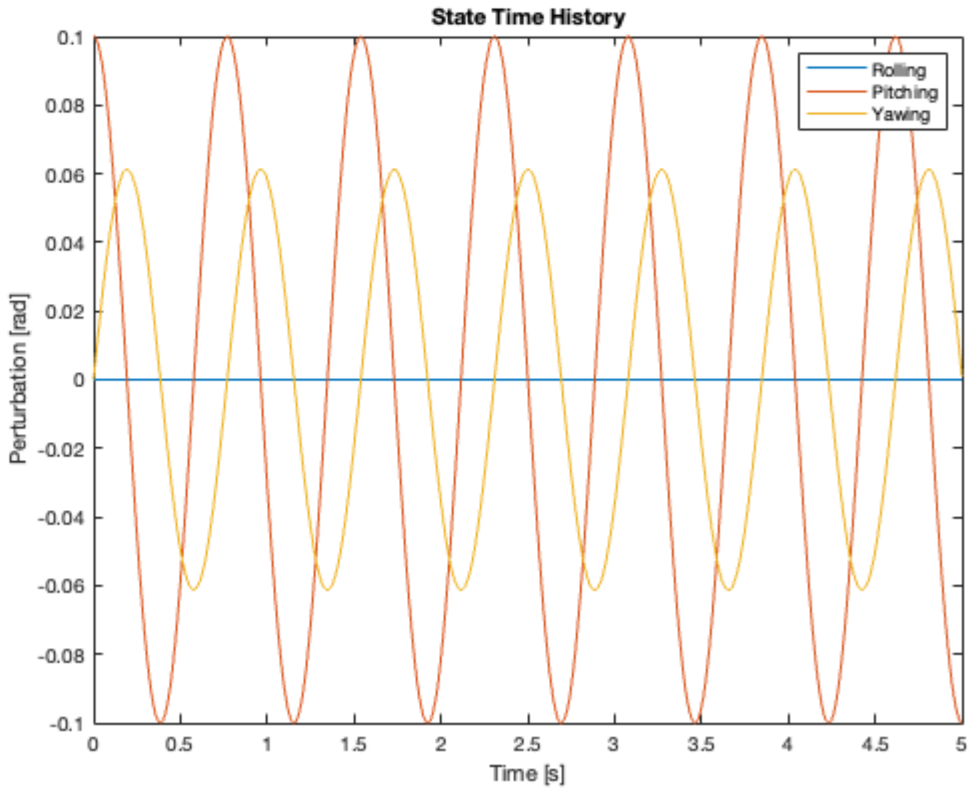
$$\boxed{\Phi(0.1, t_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\frac{\sqrt{6}}{3}\right) & -\frac{2\sqrt{6}}{3}\sin\left(\frac{\sqrt{6}}{3}\right) \\ 0 & \frac{\sqrt{6}}{4}\sin\left(\frac{\sqrt{6}}{3}\right) & \cos\left(\frac{\sqrt{6}}{3}\right) \end{bmatrix}}$$

c) Plot shown below.

The rolling perturbation is stable and stays at its initial conditions. But, pitching and yawing perturbations are oscillatory and unstable.

```
clear; clc; close all;

syms x

p0 = 20;
Ix = 500;
Iy = 750;
Iz = 1000;

A = [0 0 0; 0 0 x*p0*(Ix-Iz)/Iy; 0 x*p0*(Iy-Ix)/Iz 0];

stm = simplify(expm(A));
stmf(x) = stm;
stmf(0.1)

% Part c
time = linspace(0, 5, 1000);

x0 = [0 0.1 0]';

for i = 1:length(time)
    system(:, i) = stmf(time(i)) * x0;
end

figure()
plot(time, system(1, :))
hold on
plot(time, system(2, :))
plot(time, system(3, :))
hold off
legend("Rolling", "Pitching", "Yawing")
xlabel("Time [s]")
ylabel("Perturbation [rad]")
title("State Time History")
```

```
ans =

[1,                          0,                              0]
[0,              cos(6^(1/2)/3), -(2*6^(1/2)*sin(6^(1/2)/3))/3]
[0, (6^(1/2)*sin(6^(1/2)/3))/4,                cos(6^(1/2)/3)]
```



Published with MATLAB® R2024a

Problem 4 → a) $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, $P(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda - a & b \\ b & \lambda - c \end{vmatrix} = (\lambda - a)(\lambda - c) - b^2 = \lambda^2 - \lambda c - \lambda a + ac - b^2$

$P(\lambda) = \lambda^2 - (a+c)\lambda + ac - b^2$

$P(A) = A^2 - (a+c)A + ac - b^2 = \begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix} - (a+c)\begin{bmatrix} a & b \\ b & c \end{bmatrix} + (ac - b^2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} a^2+b^2 & ab+bc \\ ab+bc & b^2+c^2 \end{bmatrix} - \begin{bmatrix} a^2+ac & ab+bc \\ ab+bc & ac+c^2 \end{bmatrix} + \begin{bmatrix} ac-b^2 & 0 \\ 0 & ac-b^2 \end{bmatrix}$

$= \begin{bmatrix} \cancel{a^2}+\cancel{b^2}-\cancel{a^2}-\cancel{ac}+\cancel{ac}-\cancel{b^2} & \cancel{ab}+\cancel{bc}-\cancel{ab}-\cancel{bc} \\ \cancel{ab}+\cancel{bc}-\cancel{ab}-\cancel{bc} & \cancel{b^2}+\cancel{c^2}-\cancel{ac}-\cancel{c^2}+\cancel{ac}-\cancel{b^2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \therefore \boxed{P(A) = 0}$

b) i) $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, $P(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda - a & b \\ b & \lambda - c \end{vmatrix} = (\lambda - a)(\lambda - c) - b^2$

$\lambda^2 - (a+c)\lambda + (ac - b^2)$

$\lambda = \dfrac{(a+c) \pm \sqrt{(a+c)^2 - 4(ac - b^2)}}{4}$

λ is not real if $[(a+c)^2 - 4(ac-b^2)] \geq 0$, $(a+c)^2 \geq 4(ac - b^2)$

$a^2 + 2ac + c^2 \geq 4ac - 4b^2 \rightarrow a^2 - 2ac + c^2 \geq -4b^2 \rightarrow (a-c)^2 \geq -4b^2$

$(a-c)^2 \geq 0$, $b^2 \geq 0$, $-4b^2 \leq 0 \rightarrow \therefore (a-c)^2$ is always $\geq -4b^2$ when a, b, c are real and a, c are nonnegative. $\therefore$ $\boxed{\text{Eigenvalues of } A \text{ are real}}$

ii) For A to be positive semi-definite, $|A| \geq 0 \rightarrow ac - b^2 \geq 0$

$\boxed{ac \geq b^2}$