ASEN 6020 - HW 2
Spring 2025
Jash Bhalavat

Problem 1

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HW #2

Problem 1 → $f(x) = \sqrt{\vec{x} \cdot \vec{x}}$ $\quad \vec{x} \in \mathbb{R}^n$

Let $n=1$ → $x \in \mathbb{R}^1$ (x is a scalar) → $f(x) = \sqrt{x^2}$

$df_+(x,u) = \lim_{\lambda \to 0^+} \frac{1}{\lambda}\left[f(x+\lambda u) - f(x)\right] = \lim_{\lambda \to 0^+} \frac{1}{\lambda}\left[\sqrt{(x+\lambda u)^2} - \sqrt{x^2}\right]$

Only when $x=0$ → $\lim_{\lambda \to 0^+} \frac{1}{\lambda}\left[\sqrt{(\lambda u)^2}\right] = \lim_{\lambda \to 0^+} \frac{1}{\cancel{\lambda}} \cancel{\lambda}|u|$ (can cancel λ because approaching from +ve side)

$\boxed{df_+(x,u) = |u| \geq 0}$

Let $n>1$ → $\vec{x} \in \mathbb{R}^n$ → $f(x) = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{|\vec{x}|^2} = |\vec{x}|$ – where $|\vec{x}|$ represents the magnitude of $\vec{x}$ and is ≥ 0 always

$df_+(\vec{x},\vec{u}) = \lim_{\lambda \to 0^+} \frac{1}{\lambda}\left[|\vec{x}+\lambda\vec{u}| - |\vec{x}|\right]$

If $|\vec{x}|$ is non-zero → $df_+(\vec{x},\vec{u}) = \lim_{\lambda \to 0^+} \frac{1}{\lambda}|\vec{x}+\lambda\vec{u}| - \frac{|\vec{x}|}{\lambda}$

$= \lim_{\lambda \to 0^+} \frac{|\vec{x}+\lambda\vec{u}|}{\lambda} - \lim_{\lambda \to 0^+} \frac{|\vec{x}|}{\lambda}$

$\lim_{\lambda \to 0^+} \frac{|\vec{x}|}{\lambda}$ – diverges.

If $|\vec{x}|$ is 0 → $df_+(\vec{x},\vec{u}) = \lim_{\lambda \to 0^+} \frac{1}{\cancel{\lambda}}|\cancel{\lambda}\vec{u}| = \boxed{|\vec{u}| \geq 0 = df_+(\vec{x},\vec{u})}$

(can cancel λ because λ approaches 0 from +ve side)

$\therefore$ $\boxed{df_+(\vec{x},\vec{u}) \geq 0 \text{ for all } \vec{u} \text{ when } \vec{x} = \vec{0} \text{ (for } \vec{x} \in \mathbb{R}^n)}$, $\vec{x}=\vec{0}$ is minimizer

Problem 2

Problem 2 →

$$f(x) = \sum_{i,j=1}^{n} q_{ij} x_i x_j = \vec{x}^T Q \vec{x}$$

a) $$f(x) = \sum_{i,j=1}^{n} q_{ij} x_i x_j = \frac{1}{2} \cdot 2 \sum_{i,j=1}^{n} q_{ij} x_i x_j = \frac{1}{2}\left(\sum_{i,j=1}^{n} q_{ij} x_i x_j + \sum_{i,j=1}^{n} q_{ij} x_i x_j\right)$$

$$\sum_{i,j=1}^{n} q_{ij} x_i x_j \rightarrow \text{Replace } i,j \rightarrow \sum_{i,j=1}^{n} q_{ji} x_j x_i = \sum_{i,j=1}^{n} q_{ij} x_i x_j$$ (Both representations are the same).

$$\therefore f(x) = \frac{1}{2}\left(\sum_{i,j=1}^{n} q_{ij} x_i x_j + \sum_{i,j=1}^{n} q_{ji} x_j x_i\right) = \frac{1}{2}\left(\vec{x}^T Q \vec{x} + \vec{x}^T Q^T \vec{x}\right) = \boxed{\frac{1}{2}\vec{x}^T (Q + Q^T)\vec{x}}$$

Demonstrating further with a 2D example: (this can be expanded to n)

Let $\vec{x} = [a, b]^T$, $Q = \begin{bmatrix} c & d \\ e & f \end{bmatrix}$, $a, b, c, d, e, f \in \mathbb{R}$

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$$f(x) = \vec{x}^T Q \vec{x} = [a \;\; b] \begin{bmatrix} c & d \\ e & f \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = [a \;\; b] \begin{bmatrix} ac + bd \\ ae + bf \end{bmatrix} = \boxed{a^2c + abd + abe + b^2 f}$$

$$\frac{1}{2}\vec{x}^T (Q + Q^T)\vec{x} = \frac{1}{2}[a \;\; b] \begin{bmatrix} 2c & d+e \\ d+e & 2f \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{2}[a \;\; b] \begin{bmatrix} 2ac + bd + be \\ ad + ae + 2bf \end{bmatrix} = a^2 c + \frac{abd + abe}{2} + \frac{abd + abe}{2} + b^2 f$$

$$= \boxed{a^2 c + abd + abe + b^2 f = f(x)}$$

$$\therefore \boxed{f(x) = \frac{1}{2}\vec{x}^T (Q + Q^T)\vec{x}}$$

b) $\frac{\partial f}{\partial x} = \frac{d}{dx}\left(\sum_{i,j=1}^{n} q_{ij} x_i x_j\right) = \sum_{i,j=1}^{n} q_{ij}(x_j + x_i)$

$\sum_{i,j=1}^{n} x_i + x_j$ is essentially adding the 2 vectors $= 2\sum_{i,j=1}^{n} x_i$

$= 2 \sum_{i,j=1}^{n} q_{ij} x_i$

(from part a) $\rightarrow q_{ij} = \frac{1}{2}(q_{ij} + q_{ji})$

$= \sum_{i,j=1}^{n} (q_{ij} + q_{ji}) x_i = \boxed{(Q+Q^T)\vec{x} = \frac{\partial f}{\partial x}}$

Similar to part a, let's further verify this with a 2D example:

Let $\rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $x_1, x_2, a, b, c, d \in \mathbb{R}$

$f(x) = [x_1 \; x_2] \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1 \; x_2] \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = ax_1^2 + bx_1x_2 + cx_1x_2 + dx_2^2$

$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2ax_1 + bx_2 + cx_2 \\ bx_1 + cx_1 + 2dx_2 \end{bmatrix}$

$(Q+Q^T)x = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2ax_1 + bx_2 + cx_2 \\ bx_1 + cx_1 + 2dx_2 \end{bmatrix}$

$\boxed{\frac{\partial f}{\partial x} = (Q+Q^T)\vec{x}}$

Problem 3

Problem 3 → $f(x) = -\sin(x)\cos(y)$

$\because$ $\sin(x)$ & $\cos(y)$ are continuous, $f(x)$ is continuous

$\because$ $\sin(x)$ & $\cos(y)$ are smooth, $f(x)$ is smooth

To get local minimizer → $\frac{\partial f}{\partial \vec{x}}\Big|_{\vec{x}^*} = \vec{0}$ & $\frac{\partial^2 f}{\partial \vec{x}^2}\Big|_{\vec{x}^*} = F^*$ has to be pos-definite

$$\frac{\partial f}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} -\cos(x)\cos(y) \\ \sin(x)\sin(y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the following candidates emerge → $\vec{x}^* = \begin{bmatrix} 0 \\ \frac{\pi}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{3\pi}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{\pi}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{3\pi}{2} \end{bmatrix}, \begin{bmatrix} \pi/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3\pi/2 \\ 0 \end{bmatrix}, \ldots$

$$\frac{\partial^2 f}{\partial \vec{x}^2} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial y}\frac{\partial f}{\partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} \sin(x)\cos(y) & \cos(x)\sin(y) \\ \cos(x)\sin(y) & \sin(x)\cos(y) \end{bmatrix} = F^*$$

for the definite matrix → $|F^*| > 0$

$|F^*| = \sin^2(x)\cos^2(y) - \cos^2(x)\sin^2(y) > 0$

Testing the candidates →

$\vec{x}_1^* = [0, \pi/2] \rightarrow |F^*| = -1$, $\quad \vec{x}_5^* = [\pi/2, 0] \Rightarrow |F^*| = 1$

$\vec{x}_2^* = [0, 3\pi/2] \rightarrow |F^*| = -1$, $\quad \vec{x}_6^* = [3\pi/2, 0] \rightarrow |F^*| = 1$

$\vec{x}_3^* = [0, -\pi/2] \rightarrow |F^*| = -1$, $\quad \vec{x}_7^* = [-\pi/2, 0] \Rightarrow |F^*| = 1$

$\vec{x}_4^* = [0, -3\pi/2] \rightarrow |F^*| = -1$, $\quad \vec{x}_8^* = [-3\pi/2, 0] \rightarrow |F^*| = 1$

$\vec{x}_5^*, \vec{x}_6^*, \vec{x}_7^*, \vec{x}_8^*$ meet 2^{nd} order necessary condition.

$\therefore$ $\vec{x}^* = \begin{bmatrix} \pi/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3\pi/2 \\ 0 \end{bmatrix}, \begin{bmatrix} -\pi/2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3\pi/2 \\ 0 \end{bmatrix}$ are local minimizers

The domain of f(x,y) can be reduced to be x → [0, π] and y → [-1, 1] and then [π/2, 0] will be the unique global minimizer.

Problem 4

Problem 4 →

$f(x) = a + b^T x + x^T Q x$

$f(x)$ is continuous and smooth. ∴ Necessary condition for $f(x)$ using 1st & 2nd order Taylor series expansion

$\left.\frac{\partial f}{\partial \vec{x}}\right|_{\vec{x}^*} = \vec{0}$, $\frac{\partial f}{\partial \vec{x}} = 0 + \vec{x} + (Q + Q^T)\vec{x} = \vec{b} + 2Q\vec{x}$ $\quad (\because Q^T = Q)$

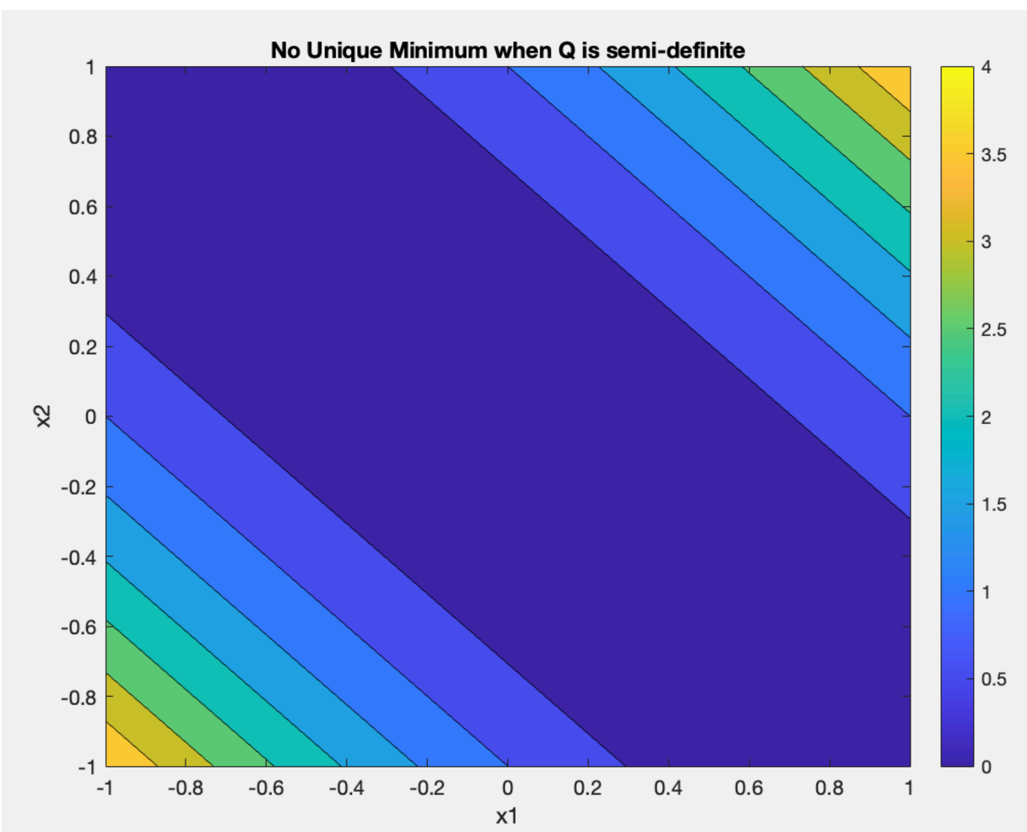
$\therefore \boxed{\vec{b} + 2Q\vec{x} = 0}$ – A

$\left.\frac{\partial^2 f}{\partial \vec{x}^2}\right|_{\vec{x}^*} \geq 0 \rightarrow \frac{\partial^2 f}{\partial \vec{x}^2} = 2Q \geq 0 \rightarrow \boxed{Q \geq 0}$ – $\boxed{B1}$ → If Q is positive semi-definite, $\vec{x}^*$ is a local minimizer

$Q > 0$ – $\boxed{B2}$ – If Q is positive definite, $\vec{x}^*$ is unique local minimizer

Sufficient condition → $\vec{x}^*$ satisfies $\boxed{A}$ & $\boxed{B2}$

If Q is [[1, 1], [1, 1]], then it is semi-definite and the cost function looks like this:



As seen above the plot has a lot of minimizers and no unique minimizers.

Problem 5

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Problem 5 →

Plane $P \rightarrow \vec{b}\cdot\vec{x} + a = 0, \quad \vec{b} \in \mathbb{R}^3, \vec{x} \in \mathbb{R}^3, a \in \mathbb{R}$, Let $\vec{b} = [c, d, e]^T$, $\vec{x} = [x, y, z]^T$

$$J = \frac{1}{2}\vec{x}\cdot\vec{x}, \quad g(\vec{x}) = cx + dy + ez + a = 0$$

$$J = \frac{1}{2}(x^2 + y^2 + z^2), \quad g(\vec{x}) = cx + dy + ez + a = 0$$

$$L = \frac{1}{2}(x^2 + y^2 + z^2) + \lambda(cx + dy + ez + a)$$

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= x + c\lambda = 0 \\ \frac{\partial L}{\partial y} &= y + d\lambda = 0 \\ \frac{\partial L}{\partial z} &= z + e\lambda = 0 \end{aligned}\right\} \rightarrow \boxed{\begin{aligned} x &= -c\lambda \\ y &= -d\lambda \\ z &= -e\lambda \end{aligned}}$$

$$\frac{\partial L}{\partial \lambda} = cx + dy + ez + a = 0 \rightarrow -c^2\lambda - d^2\lambda - e^2\lambda + a = 0 \rightarrow \boxed{\lambda = \frac{a}{c^2 + d^2 + e^2}}$$

$$\boxed{x^* = \frac{-ac}{c^2+d^2+e^2}, \quad y^* = \frac{-da}{c^2+d^2+e^2}, \quad z^* = \frac{-ea}{c^2+d^2+e^2}}$$

Now, use $J = \sqrt{\vec{x}\cdot\vec{x}} = \boxed{\sqrt{x^2+y^2+z^2} = m}$, $g(\vec{x})$ is the same

$$L = \sqrt{x^2+y^2+z^2} + \lambda(cx + dy + ez + a)$$

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= \frac{x}{m} + \lambda c = 0 \\ \frac{\partial L}{\partial y} &= \frac{y}{m} + \lambda d = 0 \\ \frac{\partial L}{\partial z} &= \frac{z}{m} + \lambda e = 0 \end{aligned}\right\} \rightarrow \boxed{\begin{aligned} x &= -\lambda c m \\ y &= -\lambda d m \\ z &= -\lambda e m \end{aligned}}$$

$$x^2 + y^2 + z^2 = \cancel{m^2} = \lambda^2 \cancel{m^2}(c^2 + d^2 + e^2)$$

$$\boxed{\lambda = \frac{1}{\sqrt{c^2+d^2+e^2}}}$$

$$\frac{\partial L}{\partial \lambda} = cx + dy + ez + a = 0 \rightarrow -c^2 m\lambda - d^2 m\lambda - e^2 m\lambda + a = 0$$

$$m\lambda(c^2 + d^2 + e^2) = a \rightarrow \lambda m = \frac{a}{c^2+d^2+e^2} \rightarrow m = \frac{1}{\lambda}\cdot\frac{a}{c^2+d^2+e^2} \Rightarrow \boxed{\frac{a}{\sqrt{c^2+d^2+e^2}} = m^*}$$

$$\boxed{x^* = \frac{-ac}{c^2+d^2+e^2}, \quad y^* = \frac{-ad}{c^2+d^2+e^2}, \quad z^* = \frac{-ae}{c^2+d^2+e^2}}$$

Problem 6

Problem 6 →

$\vec{x} = [x, y, z]^T$, $\vec{b} = [c, d, e]^T$, $h(\vec{x}) = cx + dy + ez + a \geq 0$

$J = \frac{1}{2}\vec{x}\cdot\vec{x} = \frac{1}{2}(x^2 + y^2 + z^2)$

Use $\vec{x}^*$ from Problem 5 and evaluate $h(\vec{x}^*)$ to determine activeness.

$$h(\vec{x}^*) = -\frac{ac^2}{c^2+d^2+e^2} - \frac{ad^2}{c^2+d^2+e^2} - \frac{ae^2}{c^2+d^2+e^2} + a = -a\left[\cancel{\frac{c^2+d^2+e^2}{c^2+d^2+e^2}}\right] + a = \cancel{-a} + \cancel{a} = 0$$

$\therefore$ $\boxed{\text{Inequality is active}}$

$$L = \frac{1}{2}(x^2+y^2+z^2) + \sigma(cx + dy + ez + a)$$

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= x + c\sigma = 0 \\ \frac{\partial L}{\partial y} &= y + d\sigma = 0 \\ \frac{\partial L}{\partial z} &= z + e\sigma = 0 \end{aligned}\right\} \rightarrow \boxed{\begin{aligned} x &= -c\sigma \\ y &= -d\sigma \\ z &= -e\sigma \end{aligned}} \rightarrow \vec{x} = -\vec{b}\sigma$$

$$h(\vec{x}) \geq 0 \rightarrow cx + dy + ez + a \geq 0 \rightarrow -c^2\sigma - d^2\sigma - e^2\sigma + a \geq 0$$

$$a \geq \sigma(c^2+d^2+e^2) \rightarrow \boxed{\frac{a}{c^2+d^2+e^2} \geq \sigma} \rightarrow \sigma \leq \frac{a}{\vec{b}\cdot\vec{b}}$$

$$\boxed{x^* \leq -\vec{b}\cdot\frac{a}{\vec{b}\cdot\vec{b}}}$$

Problem 7

Problem 7 → $f(\vec{r}) = \frac{1}{2}\vec{r}^T \cdot Q \cdot \vec{r}$, $g_1(\vec{r}) = \hat{a}_1 \cdot \vec{r} + b_1 = 0$, $g_2(\vec{r}) = \hat{a}_2 \cdot \vec{r} + b_2 = 0$

$\vec{r} = [r_1, r_2, r_3]^T$, $\hat{a}_1 = [a, b, c]^T$, $\hat{a}_2 = [d, e, f]^T$, $\vec{\lambda} = [\lambda_1, \lambda_2]^T$

$$f(\vec{r}) = \frac{1}{2}[r_1 \; r_2 \; r_3]\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \frac{1}{2}[r_1 \; r_2 \; r_3]\begin{bmatrix} r_1 - r_2 + r_3 \\ r_1 + r_2 - r_3 \\ -r_1 + r_2 + r_3 \end{bmatrix} = \frac{1}{2}\left[r_1^2 - \cancel{r_1 r_2} + \cancel{r_1 r_3} + \cancel{r_1 r_2} + r_2^2 - \cancel{r_2 r_3} - \cancel{r_1 r_3} + \cancel{r_2 r_3} + r_3^2\right]$$

$$= \frac{1}{2}\left[r_1^2 + r_2^2 + r_3^2\right]$$

$$L = \frac{1}{2}[r_1^2 + r_2^2 + r_3^2] + \lambda_1(ar_1 + br_2 + cr_3 + b_1) + \lambda_2(dr_1 + er_2 + fr_3 + b_2)$$

a) We want $\frac{\partial g_1}{\partial \vec{r}}$ and $\frac{\partial g_2}{\partial \vec{r}}$ to be linearly independent:

$\frac{\partial g_1}{\partial \vec{r}} = \hat{a}_1 + b_1$

$\frac{\partial g_2}{\partial \vec{r}} = \hat{a}_2 + b_2$

$\rightarrow m(\hat{a}_1 + b_1) + n(\hat{a}_2 + b_2) \neq 0$ where $m, n \in \mathbb{R}$

b)

$$\left.\begin{aligned} \frac{\partial L}{\partial r_1} &= r_1 + a\lambda_1 + d\lambda_2 = 0 \\ \frac{\partial L}{\partial r_2} &= r_2 + b\lambda_1 + e\lambda_2 = 0 \\ \frac{\partial L}{\partial r_3} &= r_3 + c\lambda_1 + f\lambda_2 = 0 \end{aligned}\right\} \rightarrow \boxed{\begin{aligned} r_1 &= -a\lambda_1 - d\lambda_2 \\ r_2 &= -b\lambda_1 - e\lambda_2 \\ r_3 &= -c\lambda_1 - f\lambda_2 \end{aligned}} = -\hat{a}_1\lambda_1 - \hat{a}_2\lambda_2$$

$$\frac{\partial L}{\partial \lambda_1} = ar_1 + br_2 + cr_3 + b_1 = 0 = -a^2\lambda_1 - ad\lambda_2 - b^2\lambda_1 - be\lambda_2 - c^2\lambda_1 - cf\lambda_2 + b_1 = 0$$

$$b_1 = \lambda_1(a^2 + b^2 + c^2) + \lambda_2(ad + be + cf) \rightarrow \boxed{\lambda_1 = \frac{b_1 - \lambda_2(ad + be + cf)}{a^2 + b^2 + c^2}}$$

$$\frac{\partial L}{\partial \lambda_2} = dr_1 + er_2 + fr_3 + b_2 = 0 = -ad\lambda_1 - d^2\lambda_2 - be\lambda_1 - e^2\lambda_2 - fc\lambda_1 - f^2\lambda_2 + b_2 = 0$$

$$b_2 = \lambda_1(ad + be + fc) + \lambda_2(d^2 + e^2 + f^2)$$

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$$b_2 = \frac{b_1 - \lambda_2 (ad + be + cf)^2}{a^2 + b^2 + c^2} + \lambda_2 (d^2 + e^2 + f^2) = \frac{b_1 - \lambda_2 (\hat{a}_1 \cdot \hat{a}_2)^2}{\hat{a}_1 \cdot \hat{a}_1} + \lambda_2 (\hat{a}_2 \cdot \hat{a}_2)$$

$$= \frac{b_1 - \lambda_2 (\hat{a}_1 \cdot \hat{a}_2)^2 + \lambda_2 (\hat{a}_2 \cdot \hat{a}_2)(\hat{a}_1 \cdot \hat{a}_1)}{\hat{a}_1 \cdot \hat{a}_1} \rightarrow b_2 \cdot \hat{a}_1 \cdot \hat{a}_1 - b_1 = \lambda_2 \left[(\hat{a}_2 \cdot \hat{a}_2)(\hat{a}_1 \cdot \hat{a}_1) - (\hat{a}_1 \cdot \hat{a}_2)^2 \right]$$

$$\boxed{\lambda_2 = \frac{b_2 \cdot \hat{a}_1 \cdot \hat{a}_1 - b_1}{(\hat{a}_2 \cdot \hat{a}_2)(\hat{a}_1 \cdot \hat{a}_1) - (\hat{a}_1 \cdot \hat{a}_2)^2}} \rightarrow \lambda_1 = \frac{b_1 - \lambda_2 (\hat{a}_1 \cdot \hat{a}_2)}{\hat{a}_1 \cdot \hat{a}_1} = \boxed{\frac{b_1 - \frac{\left[(b_2)(\hat{a}_1 \cdot \hat{a}_1) - b_1 \right] \hat{a}_1 \cdot \hat{a}_2}{(\hat{a}_2 \cdot \hat{a}_2)(\hat{a}_1 \cdot \hat{a}_1) - (\hat{a}_1 \cdot \hat{a}_2)^2}}{\hat{a} \cdot \hat{a}_1}}$$

$$\vec{r}^* = -\hat{a}_1 \lambda_1 - \hat{a}_2 \lambda_2$$

$$\boxed{\vec{r}^* = -\hat{a}_1 \left[\frac{b_2 (\hat{a}_1 \cdot \hat{a}_1) - b_1}{(\hat{a}_2 \cdot \hat{a}_2)(\hat{a}_1 \cdot \hat{a}_1) - (\hat{a}_1 \cdot \hat{a}_2)^2} \right] - \hat{a}_2 \left[\frac{b_1 - \frac{\left[b_2 (\hat{a}_1 \cdot \hat{a}_1) - b_1 \right] \hat{a}_1 \cdot \hat{a}_2}{(\hat{a}_2 \cdot \hat{a}_2)(\hat{a}_1 \cdot \hat{a}_1) - (\hat{a}_1 \cdot \hat{a}_2)^2}}{\hat{a}_1 \cdot \hat{a}_1} \right]}$$

Problem 8

Problem 8 →

$$GM(\vec{x}) = (x_1 x_2 \cdots x_n)^{1/n} = g \qquad AM(\vec{x}) = \frac{1}{n}\left[x_1 + x_2 + \ldots + x_n\right] = J$$

$$g(x) = (x_1 x_2 \cdots x_n)^{1/n} - g = 0, \quad J = \frac{1}{n}\left[x_1 + x_2 + \ldots + x_n\right]$$

$$L = \frac{1}{n}\left[x_1 + x_2 + \ldots + x_n\right] + \lambda\left[(x_1 x_2 \cdots x_n)^{1/n} - g\right]$$

$$\frac{\partial L}{\partial x_1} = \frac{1}{n} + \frac{\lambda}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_2 x_3 \cdots x_n) = 0 = \frac{1}{n} + \frac{\lambda}{n}\frac{(x_1 x_2 \cdots x_n)^{1/n}}{(x_1 \cancel{x_2 \cdots x_n})} \cdot \cancel{(x_2 x_3 \cdots x_n)}$$

$$\rightarrow \frac{1}{\cancel{n}} = -\frac{\lambda}{\cancel{n}}\frac{g}{x_1} \rightarrow x_1 = -\lambda g \rightarrow \lambda = -\frac{x_1}{g}$$

$\frac{\partial L}{\partial x_2} = 0 \rightarrow$ should lead to $\rightarrow \lambda = -\frac{x_2}{g}$, $\frac{\partial L}{\partial x_n} \rightarrow \lambda = -\frac{x_n}{g}$

λs are all equal $\rightarrow \therefore \lambda = \cancel{-}\frac{x_1}{\cancel{g}} = \cancel{-}\frac{x_2}{\cancel{g}} = \ldots = \cancel{-}\frac{x_n}{\cancel{g}} \rightarrow \boxed{x_1 = x_2 = \ldots = x_n}$

$\therefore$ $\boxed{\text{All } x_i \text{ should be equal}}$ for J to be minimum

Problem 9

Problem 9 $\rightarrow$ $f(x) = (x_1-2)^2 + x_2^2 + x_3^2, \quad g_1(x) = x_1^2 + x_2^2 + x_3^2 - 2 = 0, \quad g_2(x) = x_1^2 + x_2^2 - 1 = 0$

$L = (x_1-2)^2 + x_2^2 + x_3^2 + \lambda_1(x_1^2+x_2^2+x_3^2-2) + \lambda_2(x_1^2+x_2^2-1)$

$\frac{\partial L}{\partial x_1} = 2x_1 + 2\lambda_1 x_1 + 2\lambda_2 x_1 = 0 \quad \cancel{2x_1} = \cancel{2x_1} + 2\lambda_2 x_1 = 0$

$\frac{\partial L}{\partial x_2} = 2x_2 + 2\lambda_1 x_2 + 2\lambda_2 x_2 = 0 \quad \cancel{2x_2} = \cancel{2x_2} + 2\lambda_2 x_2 = 0$

$\rightarrow$ add $\rightarrow 2\lambda_2(x_1+x_2) = 0 \quad \therefore \lambda_2 \neq 0 \rightarrow \boxed{x_1 = -x_2}$

$\frac{\partial L}{\partial x_3} = 2x_3 + 2\lambda_1 x_3 = 0 \rightarrow \cancel{2x_3} = -2\lambda_1 x_3 \rightarrow \boxed{\lambda_1 = -1}$

$g_1(x) = x_1^2 + x_2^2 + x_3^2 - 2 = 0 \rightarrow 2x_1^2 + x_3^2 = 2 \rightarrow x_3^2 = 1 \rightarrow \boxed{x_3^* = \pm 1}$

$g_2(x) = x_1^2 + x_2^2 - 1 = 0 \rightarrow 2x_1^2 = 1 \rightarrow \boxed{x_1^* = \pm\frac{1}{\sqrt{2}} \rightarrow x_2^* = \mp\frac{1}{\sqrt{2}}}$

KKT conditions apply. $f(x), g_1(x), g_2(x)$ are C^1.

$$\frac{\partial^2 f}{\partial x^2} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} & \frac{\partial^2 f}{\partial x_1 x_3} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 x_3} \\ \frac{\partial^2 f}{\partial x_3 x_1} & \frac{\partial^2 f}{\partial x_3 x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \left|\frac{\partial^2 f}{\partial x^2}\right| = 8 > 0$$

$\frac{\partial^2 f}{\partial x^2}$ is Positive definite

$\therefore$ $f(x)$ is convex.

$$\vec{g}(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_1 & x_2 & 0 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \end{bmatrix} = 0 = \bar{A}\vec{x} + \vec{b} \rightarrow \therefore \boxed{x^* \text{ is the minimum}}$$

Problem 10

Problem 10 → $\vec{x} = [x, y, z]^T$, $R^2 = x^2 + y^2 + z^2$, $f(\vec{x}) = y^2 + z^2$, $g(\vec{x}) = x^2 + y^2 + z^2 - R^2 = 0$

$h(\vec{x}) = x - R \geq 0$, Assuming $R^2 = x^2 + y^2 + z^2$ & $R \geq 0$.

$g(\vec{x})$ & $h(\vec{x})$ are not linearly dependent i.e. they are linearly independent

because → $m \cdot g(\vec{x}) + n \cdot h(\vec{x}) \neq 0$ where $m, n \in \mathbb{R}$ $[m, n \neq 0]$

Similarly → $\frac{\partial g}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$, $\frac{\partial h}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\because$ There does not exist c_1, c_2 (non-zero) $\in \mathbb{R}$ s.t.

$c_1 \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \therefore \frac{\partial g}{\partial \vec{x}}$ & $\frac{\partial h}{\partial \vec{x}}$ are linearly independent.

$\therefore \boxed{\lambda_0 = 1}$

$L = y^2 + z^2 + \lambda[x^2 + y^2 + z^2 - R^2] + \sigma(x - R)$

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= 2x\lambda + \sigma = 0 \\ \frac{\partial L}{\partial y} &= 2y + 2y\lambda = 0 \\ \frac{\partial L}{\partial z} &= 2z + 2z\lambda = 0 \end{aligned}\right\} \begin{aligned} & x = -\frac{\sigma}{2\lambda} \rightarrow x = +\frac{\sigma}{2} \rightarrow \sigma = 2x \\ & \left.\begin{aligned} 2y(1+\lambda) &= 0 \\ 2z(1+\lambda) &= 0 \end{aligned}\right] \rightarrow \lambda = -1 \rightarrow \boxed{\begin{aligned} y^* &= 0 \\ z^* &= 0 \end{aligned}} \end{aligned}$$

$x^2 + y^2 + z^2 - R^2 = 0 \rightarrow x^2 = R^2 \rightarrow \boxed{x^* = R}$ (where $R \geq 0$), $R \in \mathbb{R}$

Problem 11

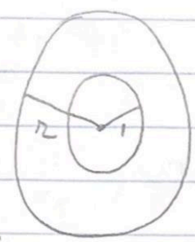
ASEN 6020
Spring 2021
Jash Bhalavat

(5)

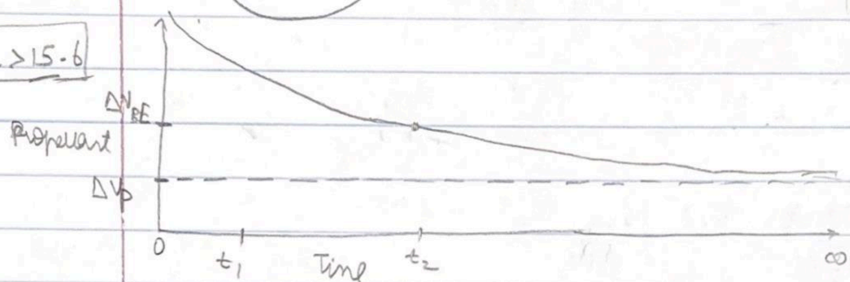
$\boxed{\text{HW \#2}}$

Problem 11 →

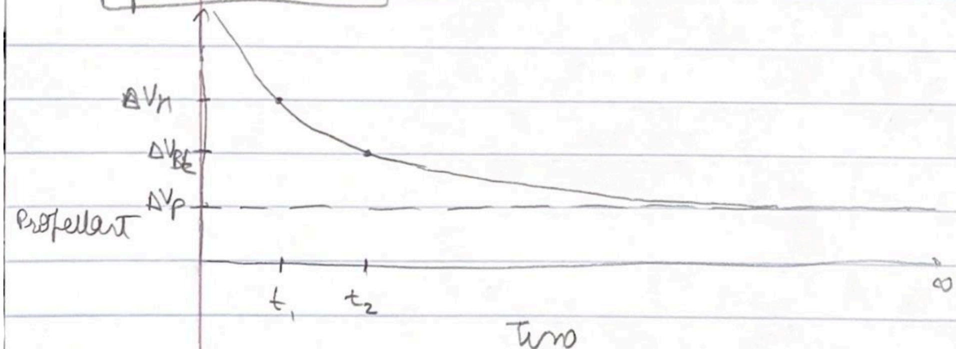
$r \geq 11.94$



$\boxed{\text{If } r > 15.6}$



$\boxed{\text{If } r \leq 15.6 \ \& \ r \geq 11.94}$



There are two scenarios when r > 11.94 for the Pareto front optimal ΔV-transfer time sketch. First is when r is > 11.94 AND r < 15.6. The second is when r > 15.6. The top plot shows the latter case (r > 15.6). When r > 15.6, there are only two optimal options - parabolic or bi-elliptic transfers. When time is at t1 (from the top figure), it's really close to 0 and a hyperbolic transfer is needed. But, at some point time t2 bi-elliptic transfers are the most optimal and as time increases, parabolic transfers are most optimal. When time is ∞, parabolic transfer is optimal.

Then, the second case is when r > 11.94 AND r < 15.6. Here, when time is really small (close to 0), hyperbolic transfers will still be the most optimal. But, as time increases, and reaches some time t1 (from the bottom figure), hohmann transfer is most optimal (there exists an intermediate ratio l where J_H < J_BE). Then, as time reaches time t2, bi-elliptic transfers are most optimal and same as the previous case, as time reaches ∞, parabolic transfer is optimal.