Optimal Trajectories ASEN 6020

[Definition, Computation + Analysis of the optimal design, guidance + control of space trajectories and the properties of these trajectories.]

Optimal → Depends on the situation + goals

Design + Control → Where do you want to go? What technology are you using?

Guidance → Are you on track? What's the best way to correct errors?

Space Trajectory → Motion in a vacuum, interface w/ "top" of atmosphere . . .

Properties → Is a trajectory stable or unstable?
Is a trajectory truly optimal?

Analysis → Math. properties
Computation → Realization of the theory.

2 basic approaches

Parametric Optimization

- Set up a sequence of 2 body arcs & optimize total ΔV
- Specify a control law & choose $\vec{a}_i$

$$\vec{a} = \vec{a}_0 + \vec{a}_1 t + \vec{a}_2 t^2 + \cdots$$

Functional Optimization

- How to continuously choose your control law $\vec{a}(t)$ to satisfy optimality conditions ...

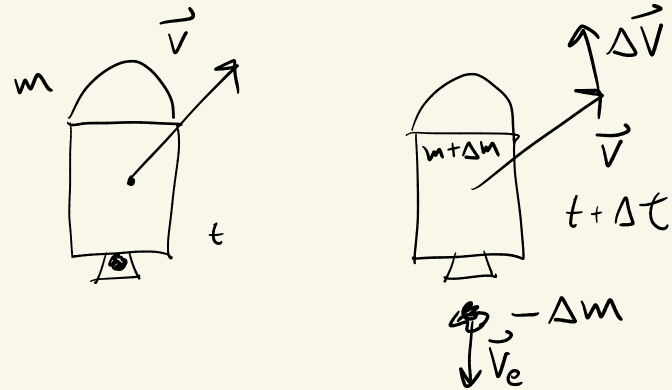
$\vec{a}(t)$ "infinite dimensional"

- Ex: Min. fuel traj. from LEO to Lunar orbit

Basics

Rocket Equation

$\vec{g} \equiv \vec{0}$



$m_f < m_0$

Cons. of Momentum

$$m\vec{V} = (m+\Delta m)(\vec{V}+\Delta\vec{V}) - \Delta m(\vec{V}_e+\vec{V})$$

$$\Downarrow$$

$$\boxed{\Delta m\,\vec{V}_e = m\,\Delta\vec{V}} \xRightarrow{\text{turn into}} \text{a rate} \quad \overbrace{\frac{dm}{dt}\vec{V}_e}^{\vec{T}} = m\frac{d\vec{V}}{dt} = \text{Thrust}$$

$$\frac{d\vec{V}}{dt} = \frac{1}{m}\frac{dm}{dt}\vec{V}_e \;; \quad \frac{dm}{dt} < 0, \; \vec{V}_e \text{ opposes acceleration}$$

Assume $\vec{V}_e \sim$ constant $\Rightarrow d\vec{V} = \frac{1}{m}dm\,\vec{V}_e \Rightarrow \boxed{\Delta\vec{V} = \vec{V}_e \ln\left(\frac{m_f}{m_0}\right) = -\vec{V}_e \ln\left(\frac{m_0}{m_f}\right)}$

$|\vec{V}_e| = c \sim I_{sp}$

$|\Delta\vec{V}| = \Delta V = c \ln\left(\frac{m_0}{m_f}\right)$ $\Rightarrow$ Direct link between ΔV + Payload

Series of ΔV's $\quad \sum_{i=0}^{N} \Delta V_i = c \sum_{i=0}^{N} \ln\left(\frac{m_i}{m_{i-1}}\right) = c \ln\left(\frac{m_0}{m_{N-1}}\right)$

General results

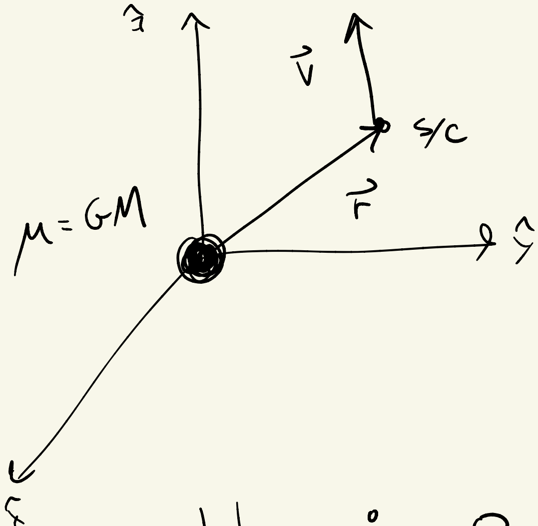
$$\boxed{\frac{d\vec{V}}{dt} = \frac{1}{m} \dot{m} \underbrace{\vec{V}_e}_{\vec{c}}}$$

$\Rightarrow$ mass can become an additional "state"

$(\vec{r}, \vec{v}) \Rightarrow (\vec{r}, \vec{v}, m)$

$$\boxed{\dot{m} = f(m, \vec{r}, t)}$$

2-Body Orbit Mechanics



$$\vec V = \frac{d\vec r}{dt} = \dot{\vec r}$$

$$m\dot{\vec V} = -\frac{\mu m}{r^3}\vec r + \vec T$$

$$\boxed{\dot{\vec V} = -\frac{\mu}{r^3}\vec r + \underbrace{\frac{1}{m}\dot m \vec c}_{\boxed{\vec a(t)}}}$$ = "acceleration control"

When $\dot m = 0 \Rightarrow \dot{\vec V} = -\frac{\mu}{r^3}\vec r \Rightarrow$ Integrate (completely solved)

Impulsive Maneuvers: when $\dot m \neq 0$ over a short time interval, & if we can assume $\vec r$ is relatively stationary, then......

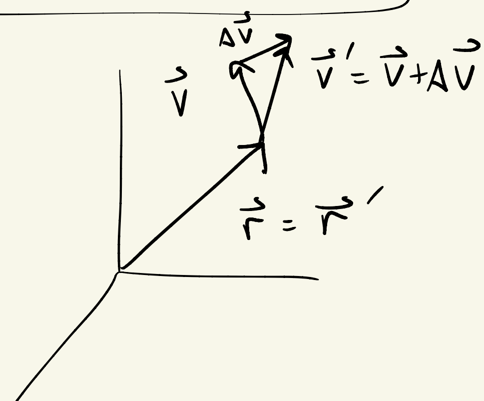
$$\boxed{\frac{d\vec{v}}{dt} = -\frac{\mu}{r^3}\vec{r} + \frac{1}{m}\vec{c}\,\frac{dm}{dt}} \Rightarrow \boxed{d\vec{v} = -\frac{\mu}{r^3}\vec{r}\,dt + \frac{1}{m}\vec{c}\,dm}$$

$$\int_0^{\Delta t} d\vec{v} = \int_0^{\Delta t} -\frac{\mu}{r^3}\vec{r}\,dt + \int_0^{\Delta t} \underbrace{\frac{\vec{c}}{m}\,dm}_{T} \Rightarrow \boxed{|\vec{g}| << |\vec{T}|}$$

$$\boxed{\Delta\vec{v} = -\vec{c}\,\ln\left(\frac{m_0}{m_f}\right)} \Rightarrow$$

"Impulsive Change" in velocity

Instantaneous Change in velocity.



Kepler's Problem

Integrals of Motion:

$$\begin{cases} E = \frac{1}{2}\vec{V}\cdot\vec{V} - \frac{\mu}{r} & \text{Energy} \\ \vec{H} = \vec{r}\times\vec{V} & \text{Ang. Mom} \\ |\vec{H}| = |\vec{r}\times\vec{V}| = rV\cos\gamma & \gamma = \text{flight path angle} \\ \vec{e} = \frac{1}{\mu}\vec{V}\times\vec{H} - \hat{r} \end{cases}$$

6 orbit elements

$a = \frac{-\mu}{2E}$; $e^2 = 1 + \frac{2EH^2}{\mu^2}$; $e = |\vec{e}|$

$\cos\hat{i} = \hat{z}\cdot\hat{H}$ $\tan\Omega = \frac{\hat{x}\cdot\hat{H}}{\hat{y}\cdot\hat{H}}$

$\tan w = \ldots\ldots$

a = semi-major axis

e = eccentricity

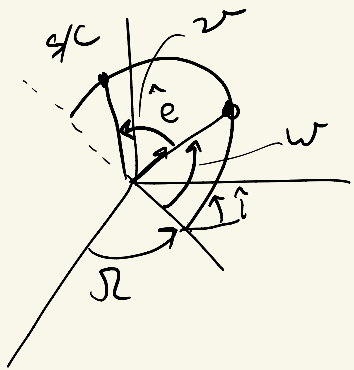
i = inclination

$$\tan\nu = \frac{\vec{r}\cdot(\hat{z}\times\vec{e})}{\vec{r}\cdot\vec{e}}$$

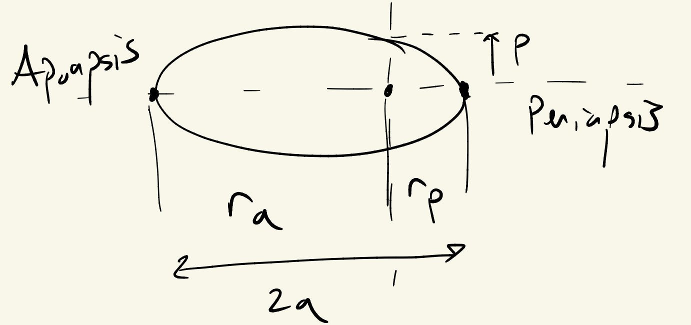
$\nu \equiv$ true anomaly

Ω = long. of asc. node

w = arg. of periapsis



Elliptic



$E < 0$

$a > 0$

$1 \geq e \geq 0$

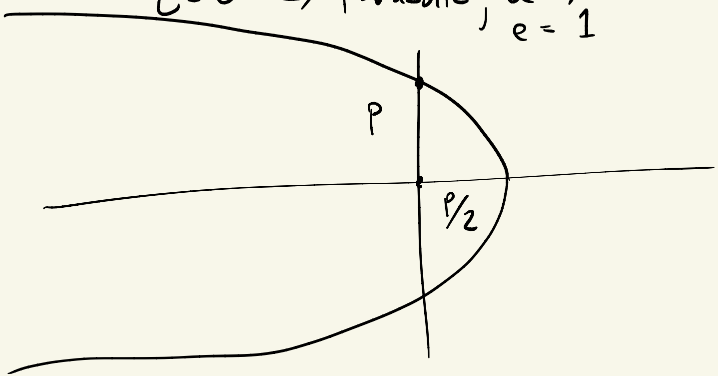
$$r = \frac{p}{1 + e\cos\nu}$$

$$\tan\gamma = \frac{e\sin\nu}{1 + e\cos\nu}$$

$$V = \sqrt{\frac{\mu}{p}}\sqrt{1 + 2e\cos\nu + e^2}$$

$$p = a(1-e^2) = H^2/\mu$$

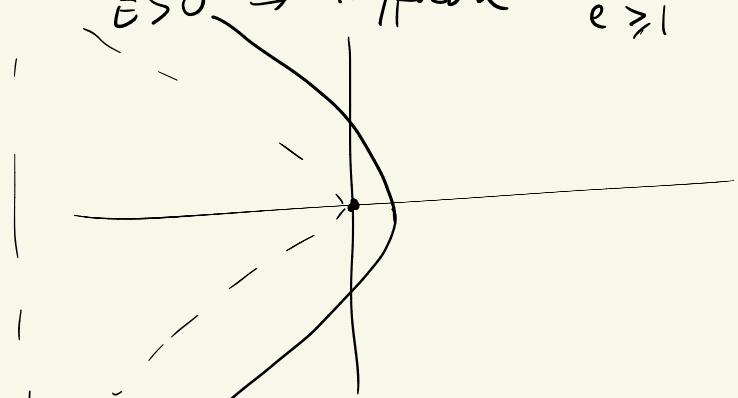
$E = 0 \Rightarrow$ Parabolic, $a \to \pm\infty$, $e = 1$



$E > 0 \Rightarrow$ Hyperbola

$a < 0$

$e \geq 1$



Important "points" on these orbits

Periapsis $\Rightarrow$ r_{min}, $\nu = 0$; $r_p = a(1-e) = \dfrac{p}{1+e}$

Always exists (r_p, q)

$$V_p = \sqrt{\frac{\mu}{p}}(1+e) = \sqrt{\frac{\mu}{a}\frac{(1+e)}{(1-e)}}$$

Apoapsis $\Rightarrow$ r_{max}, $\nu = \pi$ $\left(\left(= \operatorname{acos}\left(-\frac{1}{e}\right)\right) \quad r_{max} \to \infty\right)$

Does not always exist (r_A, Q)

$$r_a = \frac{p}{1-e} = a(1+e)$$

$$V_a = \sqrt{\frac{\mu}{p}}(1-e) = \sqrt{\frac{\mu}{a}\frac{(1-e)}{(1+e)}}$$

Note:

$$a = \frac{1}{2}(r_a + r_p)$$

$$e = \frac{r_a - r_p}{r_a + r_p}$$

$$V_p = \sqrt{\frac{2\mu}{(r_a+r_p)}\frac{r_a}{r_p}}$$

$$V_a = \sqrt{\frac{2\mu}{(r_a+r_p)}\frac{r_p}{r_a}}$$

Some special orbits:

For a given AM, what is the minimum energy orbit? Circular

$E = -\frac{\mu}{2a}$; $H = \sqrt{\mu a(1-e^2)} \Rightarrow \boxed{H^2 = \mu a(1-e^2)}$

$$\boxed{E = \boxed{-\frac{\mu}{2}\frac{\mu(1-e^2)}{H^2}} = \frac{-\mu^2}{2H^2} + \frac{\mu^2}{2H^2}e^2} \Rightarrow$$ drive $e \to 0 \Rightarrow$ circular orbit

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What is the minimum increment in velocity $(\Delta\vec{V})$ to make a s/c escape from a circular orbit of radius "r"?

$E_{circ} = \frac{1}{2}\frac{\mu}{r} - \frac{\mu}{r} = -\frac{\mu}{2r}$

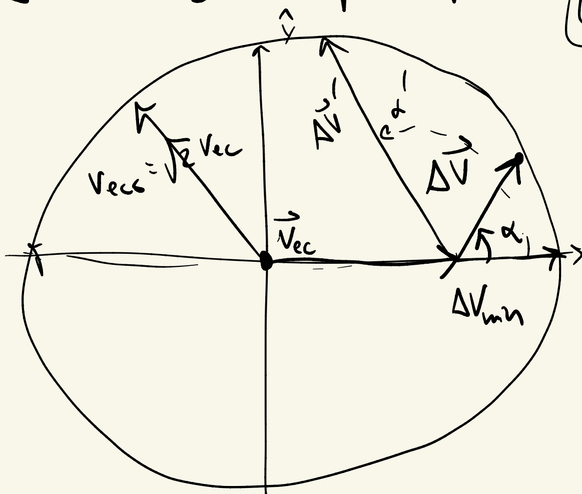
$$E' = \boxed{\frac{1}{2}(\vec{V}_{ec} + \Delta\vec{V})\cdot(\vec{V}_{ec} + \Delta\vec{V}) - \frac{\mu}{r} = 0} \Rightarrow$$

$\vec{V}_{ec} + \Delta\vec{V} = \vec{V}_{esc}$

$$\boxed{(\vec{V}_{ec}+\Delta\vec{V})\cdot(\vec{V}_{ec}+\Delta\vec{V}) = \frac{2\mu}{r}} = V_{esc}^2 \checkmark$$

$$\Delta V^2 + 2\vec{V}_{ec}\cdot\Delta\vec{V} + \cancel{\frac{\mu}{r}} - \frac{2\mu}{r} = \boxed{\Delta V^2 + 2V_{ec}\Delta V\cos\alpha - \frac{\mu}{r} = 0}$$

$$\Downarrow$$



$$\Delta V = V_{ec}\left[\sqrt{1+\cos^2\alpha} - \cos\alpha\right]$$

$$\frac{d\Delta V}{d\alpha} = V_{ec}\sin\alpha\left[1 - \frac{\cos\alpha}{\sqrt{1+\cos^2\alpha}}\right] = 0 \Rightarrow \boxed{\alpha = 0, \pi}$$

$$\frac{d^2\Delta V}{d\alpha^2} \simeq V_{ec}\cos\alpha\left[1 - \frac{\cos\alpha}{\sqrt{1+\cos^2\alpha}}\right] + V_{ec}\sin\alpha\left[\cdots\right]$$

$\alpha = \pi$ $\quad -V_{ec}\left[1 + \frac{1}{\sqrt{2}}\right] < 0$

$\alpha = 0$ $\quad V_{ec}\left[1 - \frac{1}{\sqrt{2}}\right] > 0 \Rightarrow$ minimum

$$\boxed{\Delta V_{min} = (\sqrt{2}-1)V_{ec}}$$

Other questions

Where is the best location to change Energy?

" " " " " " " A.M. ?

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r} + \vec{a}$$

$E \Rightarrow \dot{E} = \vec{V}\cdot\vec{a} \Rightarrow$ where $\underbrace{|\vec{V}| \text{ is max}}_{\text{periapsis}}$ + $\vec{a} \parallel \vec{V}$

$\vec{H} \Rightarrow \dot{\vec{H}} = \vec{r}\times\vec{a} \Rightarrow$ where $\underbrace{|\vec{r}| \text{ is max}}_{\text{apoapsis}}$ + $\vec{a} \perp \vec{V}$

$$\vec{e} \Rightarrow \boxed{\dot{\vec{e}} = \frac{1}{\mu}\left[\vec{a}\times(\vec{r}\times\vec{V}) + \vec{V}\times(\vec{r}\times\vec{a})\right]}$$

$$e^2 = 1 + \frac{2EH^2}{\mu^2} \Rightarrow \boxed{e\,\Delta e = \frac{4EH}{\mu^2}\Delta H + \frac{2H^2}{\mu^2}\Delta E}$$