ASEN 5044, Fall 2024

Statistical Estimation for Dynamical Systems

Lecture 07:
Nyquist Rate, Linear System Stability;
DT LTI Observability, Deterministic State Estimation

Prof. Nisar Ahmed (Nisar.Ahmed@Colorado.edu)

Tuesday 09/17/2024

Ann and H.J. Smead
Aerospace Engineering Sciences
UNIVERSITY OF COLORADO BOULDER

Announcements

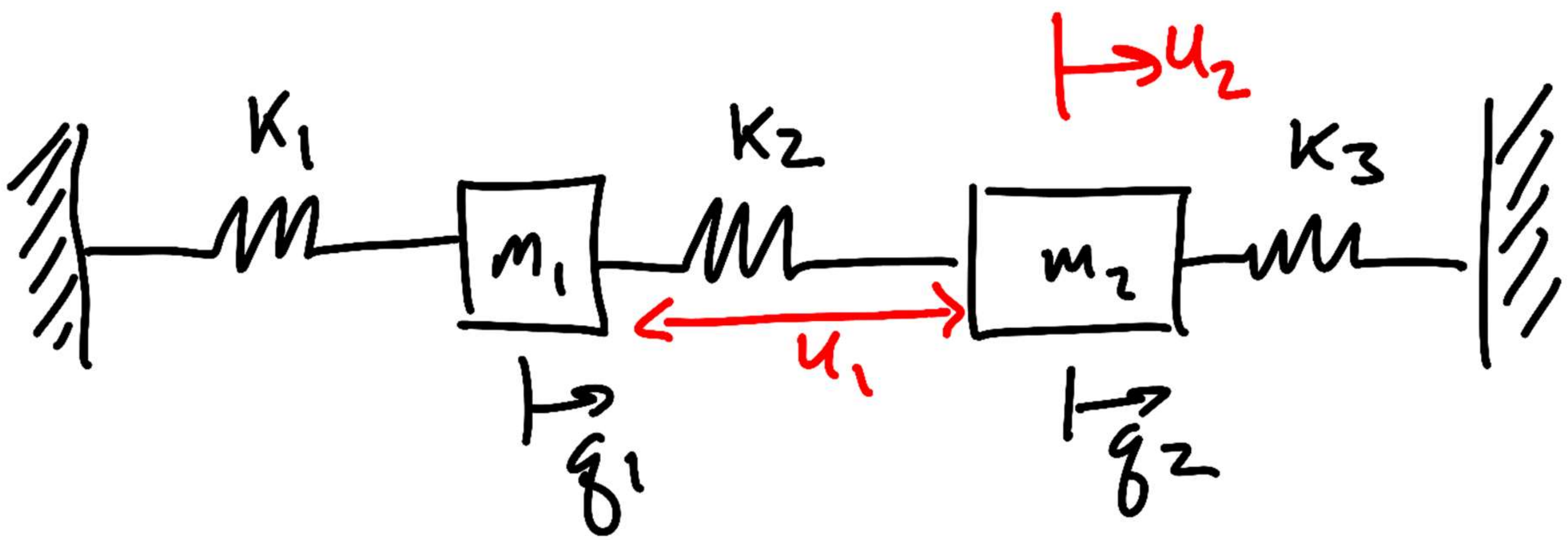
- **Prof. Ahmed out of country this week (SPIE Defense & Security Conference in UK)**
  - **No live classes Tues 09/17 or Thurs 09/19, BUT pre-recorded Lecture Videos to be posted – WATCH THEM!! (will need them for HW 2)**
  - **TF Aidan to cover Prof. Ahmed's hours next Wed 09/18 4:30-6 pm, AERO N353**
  - **Regular in-person lectures to resume Tues 09/24**
- **HW 2 due tomorrow Fri 9/20 at 11:59 pm**
- HW 3 to be posted this Thurs 09/19
- NO QUIZ THIS WEEK
- **MIDTERM 1 – TO BE RELEASED Thurs 10/03, DUE Thurs 10/10**
  - **Take home exam, to focus on material covered in HWs1-4 + quizzes up to that point**

Last Time…

- Conversion of Continuous Time (CT) LTI SS → DT LTI SS
- Computing the DT G matrix
  - Go through spring-mass example one more time…
  - How does choice of time discretization step size Δt affect results?
- **(Also covered CT to DT example + Nyquist rate last time in 2024 lecture/notes [should also be in pending 2024 Lec 06 video], but it will be covered again in this video…)**

(Last Time) Example: Convert CT SS model to DT SS model

- System of 2 masses and 3 springs: 2 actuator inputs u and 2 sensor outputs y



$$x = [q_1(t), \dot{q}_1(t), q_2(t), \dot{q}_2(t)]^T$$

$$u = [u_1(t), u_2(t)]^T$$

$$y = [q_1(t), q_2(t)]^T$$

For $k_1 = k_2 = k_3 = 1$ N/m and $m_1 = m_2 = 1$ kg, use simple physics to get CT linear SS model

$$\left.\begin{aligned} \dot{x} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}\right\}$$

Two characteristic oscillatory motion "modes" (eigenvectors) corresponding to eigenvalues of A with natural frequencies:

$$\mathrm{eig}(A) = \ \lambda_{1,2} = \pm j1.73, \ \ \lambda_{3,4} = \pm j1.00$$

$$\rightarrow \ \omega_{n,1,2} = 1.73 \text{ rad/s}, \ \ \omega_{n,3,4} = 1.00 \text{ rad/s}$$

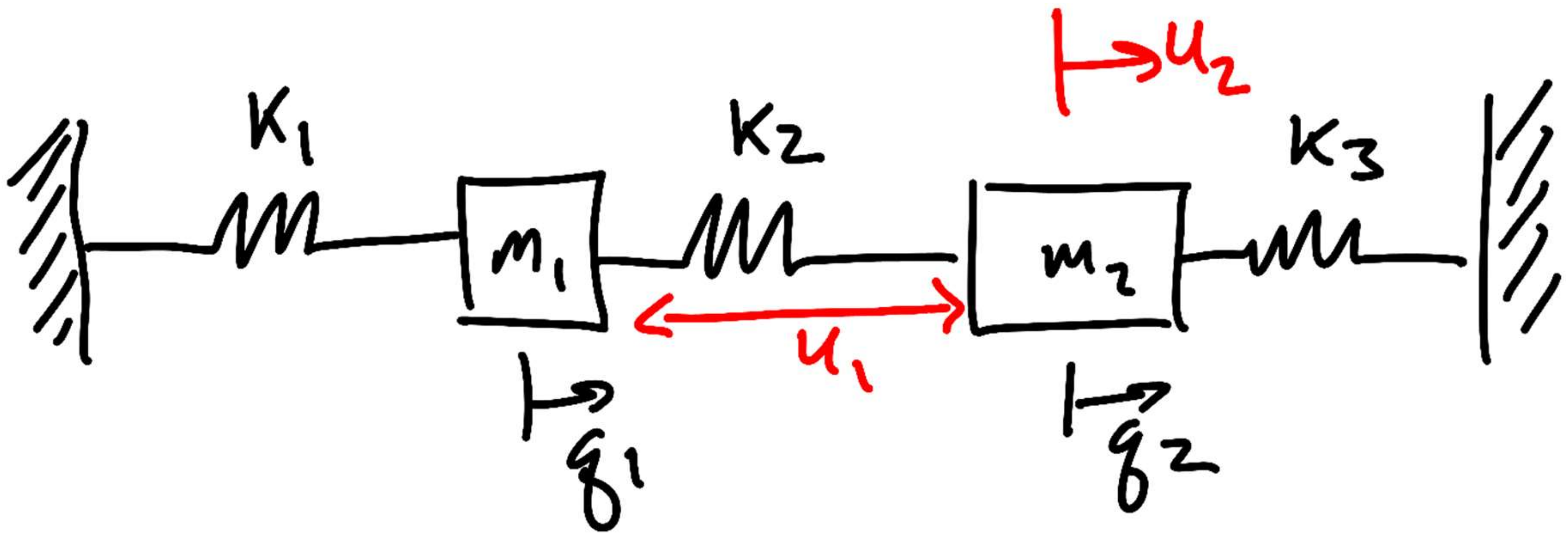
$$[2.72 \text{ Hz and } 1.57 \text{ Hz, resp.}]$$

$$A = \begin{bmatrix} 0 & 1.0 & 0 & 0 \\ -2.0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 \\ 1.0 & 0 & -2.0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\rightarrow \hat{A} = \left[\begin{array}{c|c} A & B \\ \hline 0 & 0 \end{array}\right]$$

(Last Time) Ex: Convert CT SS model to DT SS model (cont'd)

- System of 2 masses and 3 springs: 2 actuator inputs u and 2 sensor outputs y



$$x = [q_1(t), \dot{q}_1(t), q_2(t), \dot{q}_2(t)]^T$$

$$u = [u_1(t), u_2(t)]^T$$

$$y = [q_1(t), q_2(t)]^T$$

Converted to DT SS model **using the same state variables** with ZOH and sample rate Δt = 0.2 sec

$$x_{k+1} = Fx_k + Gu_k, \qquad x_k = [q_1(k), \dot{q}_1(k), q_2(k), \dot{q}_2(k)]^T$$

$$y_k = Hx_k + Mu_k, \qquad u_k = [u_1(k), u_2(k)]^T$$

$$y_k = [q_1(k), q_2(k)]^T$$

$$\hat{A} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \rightarrow e^{\hat{A}\Delta t} = \begin{bmatrix} F & G \\ 0 & I \end{bmatrix}$$

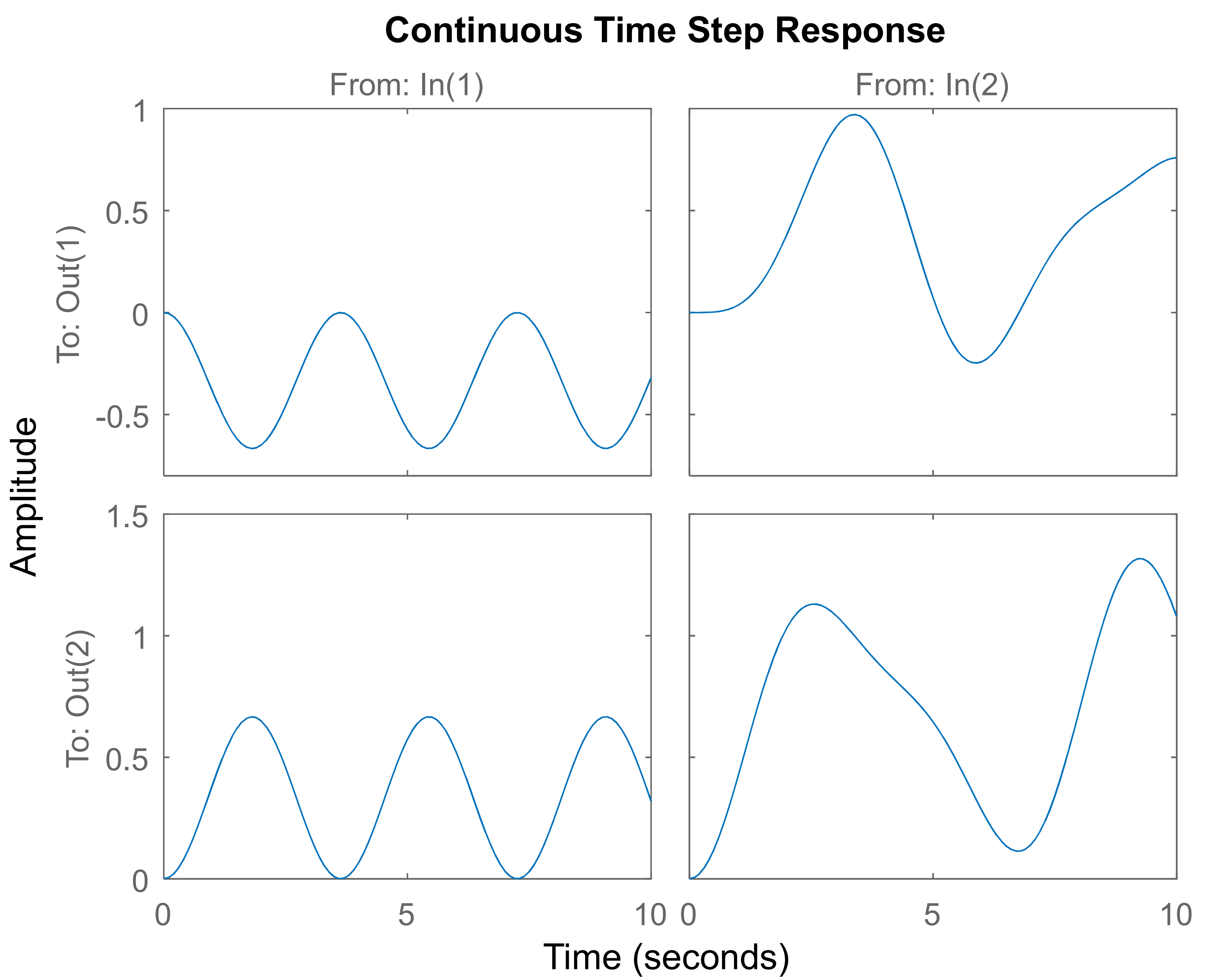
$$F = \begin{bmatrix} 9.6033e-01 & 1.9735e-01 & 1.9734e-02 & 1.3227e-03 \\ -3.9337e-01 & 9.6033e-01 & 1.9470e-01 & 1.9734e-02 \\ 1.9734e-02 & 1.3227e-03 & 9.6033e-01 & 1.9735e-01 \\ 1.9470e-01 & 1.9734e-02 & -3.9337e-01 & 9.6033e-01 \end{bmatrix} \quad G = \begin{bmatrix} -1.9801e-02 & 6.6312e-05 \\ -1.9602e-01 & 1.3227e-03 \\ 1.9801e-02 & 1.9867e-02 \\ 1.9602e-01 & 1.9735e-01 \end{bmatrix}$$

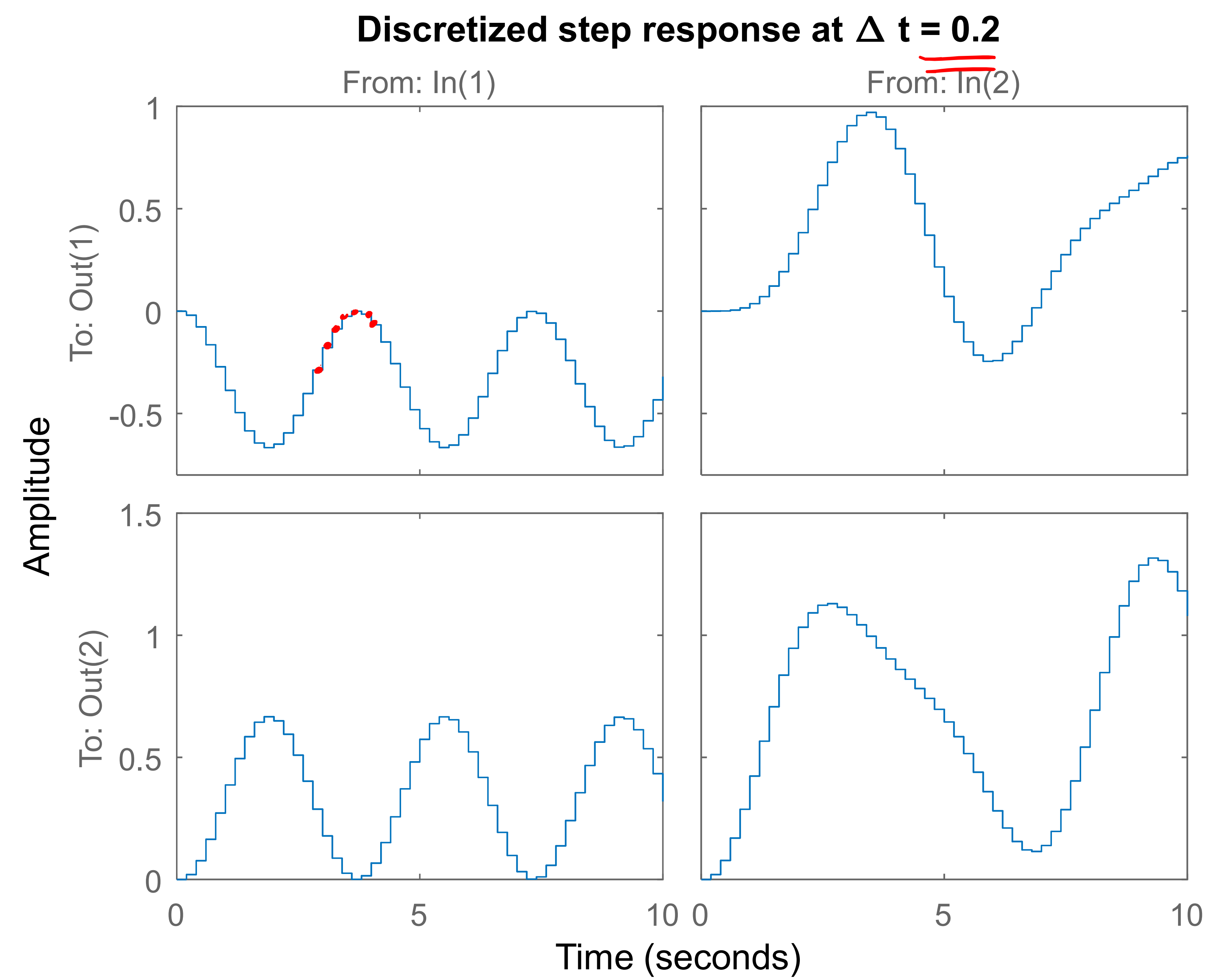
$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

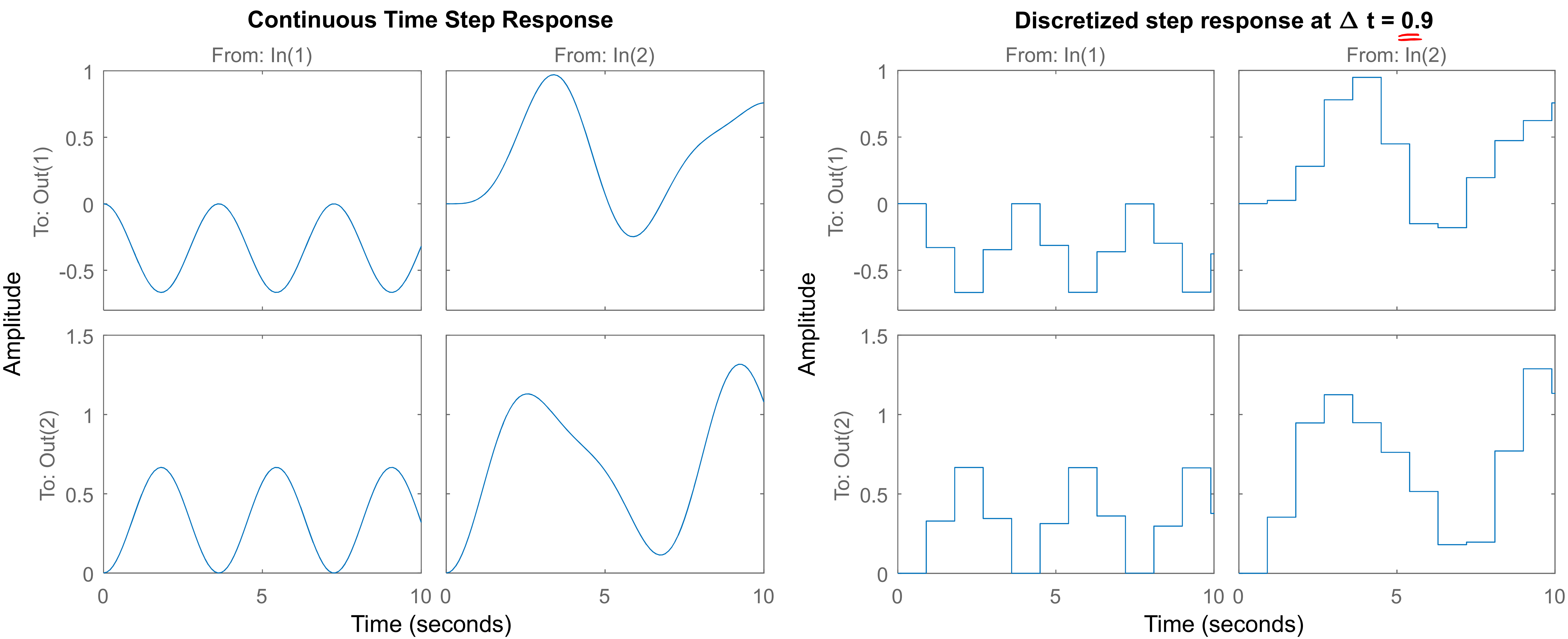
(Last Time) Sample Input Step Response Output from DT vs CT

lsim [matlab]

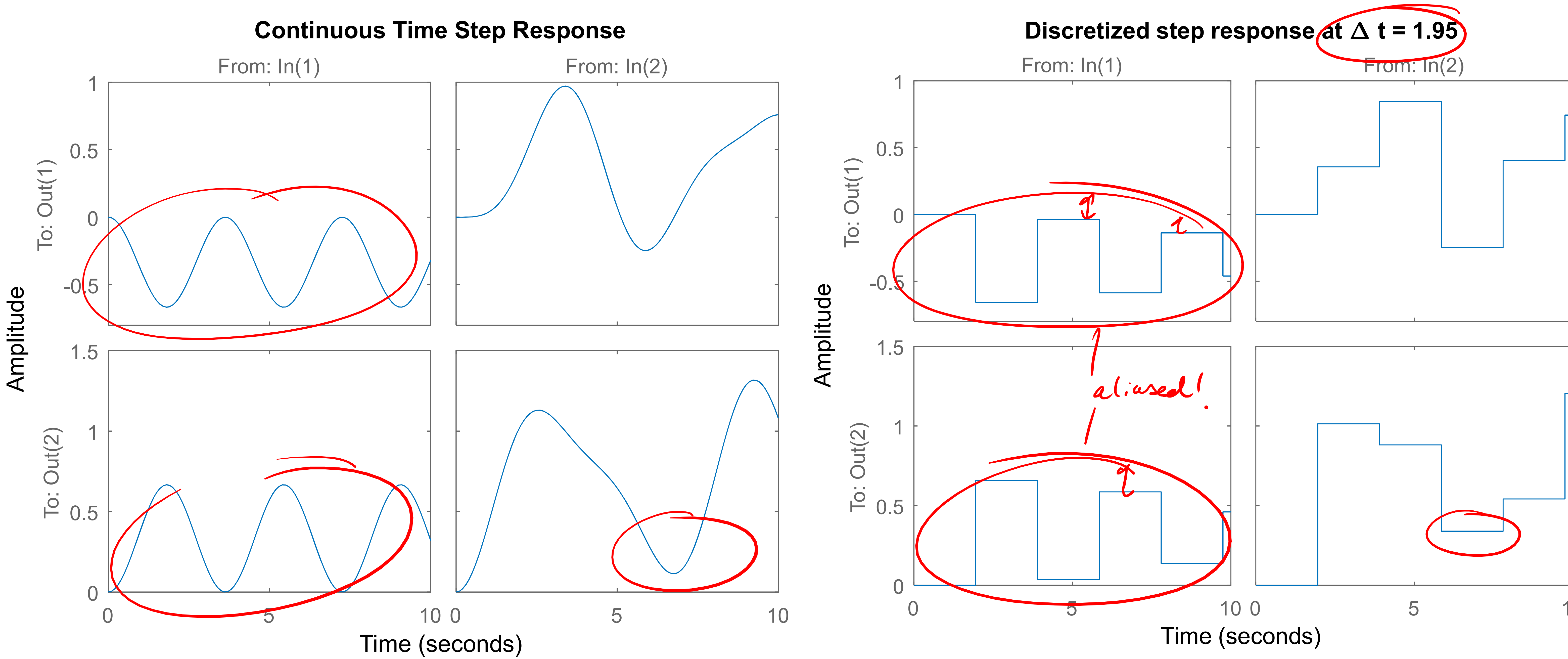




(Last Time) Sample Input Step Response Output from DT vs CT



(Last Time) Sample Input Step Response Output from DT vs CT



Nyquist Rate and CT System Natural Frequencies

- **WARNING FOR CT → DT conversions: cannot just pick any old Δt !!!**
- For LTI systems: fundamental upper bound on how large Δt should be
- <u>Nyquist Sampling Criterion:</u> if sample rate (in rad/s) is $\omega_{sample} = \frac{2\pi}{\Delta t}$,

$$\text{need } \omega_{sample} > 2\omega_{sys,max} \;\Rightarrow\; \Delta t < \frac{\pi}{|\lambda_{A,max}|}$$

where $|\lambda_{A,max}|$ is largest $\underbrace{\text{complex magnitude}}_{\text{(natural freq./ time constant)}}$ among all eigenvalues of A

(e.g. max Δt for 2 mass/3 spring example system is ~1.82 sec → Δt larger than this leads to aliasing…)

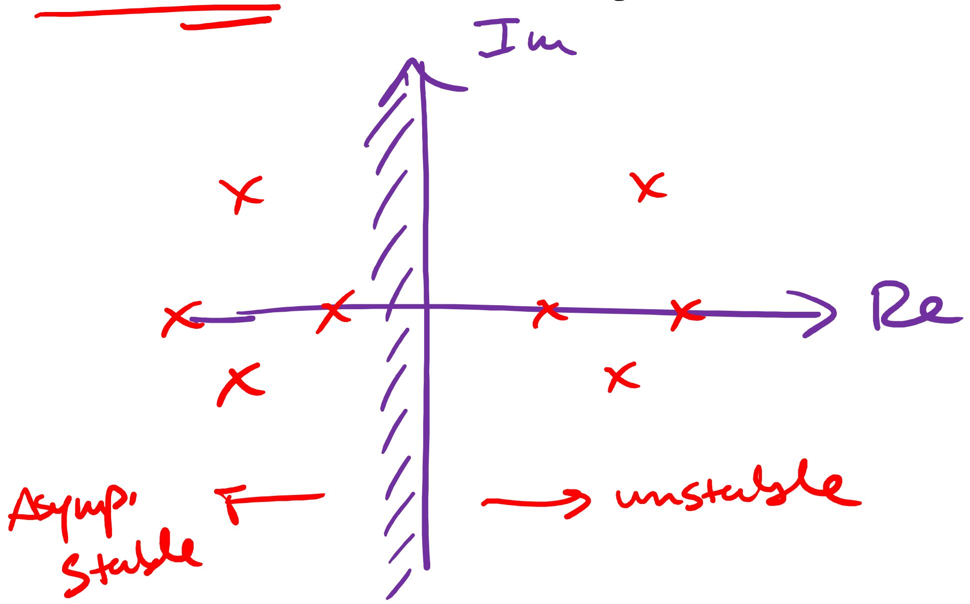
Today…

- Nyquist rate (done) ✓
- Rest of lecture:
  - Stability of CT/DT linear systems
  - Observability of DT linear systems and deterministic state estimation: how to find x(k=0) from *some* finite sequence of measurements y(0),y(1),…,y(K)?

READ: Chapter 2.1-2.2 in Simon book (probability)

Asymptotic Stability for CT LTI Systems

- Necessary & sufficient condition for **CT LTI asymptotic stability: e'vals of A matrix must all lie strictly in the left half plane** (e'vals must have strictly negative real part)



$$\text{eig}(A) = \{\lambda_i\}_{i=1}^{n} \quad [n \text{ \# states}]$$

$$\lambda_i = \underbrace{\sigma_i}_{\text{Re}(\lambda_i)} \pm j\underbrace{\omega_i}_{\text{Im}(\lambda_i)}, \quad i = 1, \dots, n$$

$\rightarrow$ If $\sigma_i < 0 \;\forall \lambda_i$, then A a.s. (asymp. stable)

Intuition: if A were diagonalizable (for instance) & $\dot{x} = Ax$, $x(0) = x_0$

$$\rightarrow x(t) = e^{At}x(0) = \sum_{i=1}^{n} \alpha_i \cdot v_i \cdot e^{\lambda_i t}$$

where α_i = some scalar constants

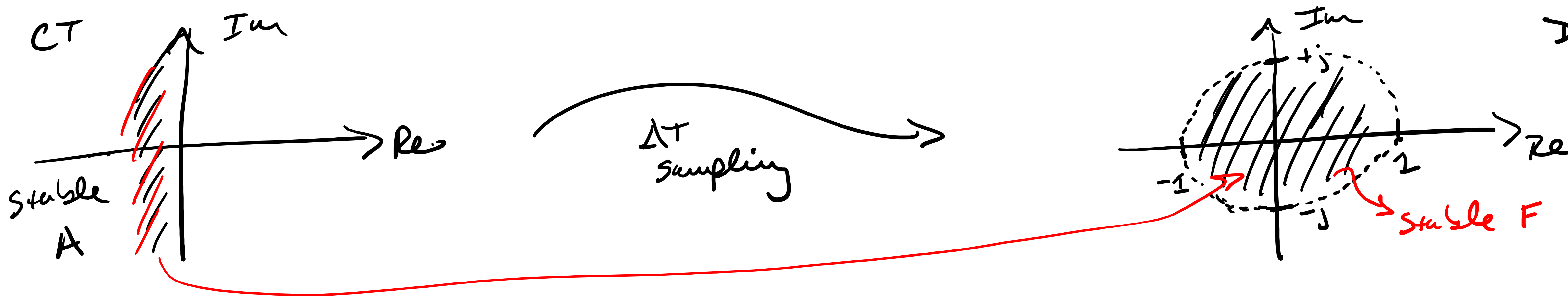
v_i = e'vec of A for λ_i e'val

superposition of modal responses

gives some insight about Nyquist rate!
have to sample "fast enough" in time to reconstruct the fastest modes!

Asymptotic Stability for DT LTI Systems

- FACT: DT linear system **asymptotically stable** if & only if **eigenvalues of the F matrix lie in the unit circle**



- Idea: x(k+1) = F*x(k) only "settles down" if F does not force magnitude of x(k) to grow as k→ infinity
- Simple example: consider if F is diagonal – each iteration through k scales elements of the state vectors:

$$F_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \text{ if } x(k) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow$$

$$x(k+1) = F\,x(k) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$x(k+2) = F\,x(k+1) = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$$

$$x(k+3) = F\,x(k+2) = \begin{bmatrix} 8 \\ 27 \end{bmatrix}$$

$$\vdots \longrightarrow x(k+N) \text{ blows up as } N \to \infty!$$

$$F_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix} \longrightarrow$$

$$x(k+1) = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}$$

$$x(k+2) = \begin{bmatrix} 0.04 \\ 0.09 \end{bmatrix}$$

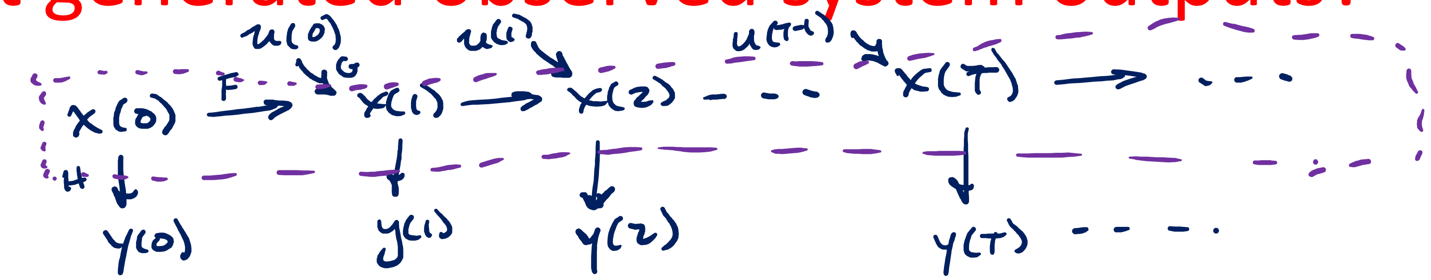
$$\vdots \longrightarrow \text{eventually goes to } 0 \text{ as } k \to \infty$$

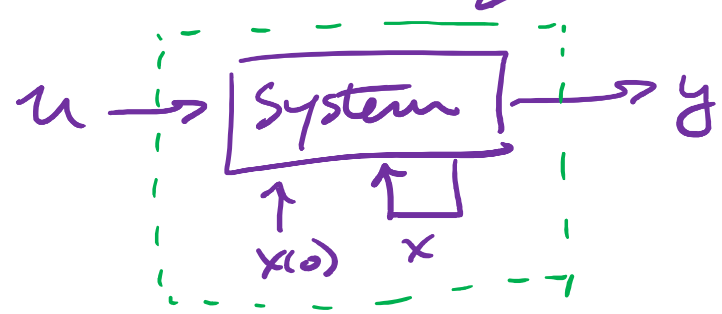
Deterministic DT LTI State Estimation

How to recover sequence of states that generated observed system outputs?

If $x(k+1) = F x(k) + G u(k)$

$y(k) = H x(k)$





we'd like to know which sequence of $x(k)$'s explains the observations $y(k)$

[& hopefully / ideally: sequence of $x(k)$'s is unique!]

In practice, we must deal with some complications:

- model errors, nonlinearities
- sensor noise and state disturbances } starting next lec (8)
- ability of states to "reveal" themselves via measurements and dynamics

• For now, assume LTI models and ignore random noise/state/model errors…

• When is it actually possible to <u>uniquely</u> compute internal state sequence?

Observability: DT Definition

- A system is **observable** if for any initial state x(0) and *some* final time T, the initial state x(0) can be **uniquely determined** from knowledge of u(k) and y(k) alone for k=0,1,…,T.

$$[y(0), \dots, y(T)] \rightarrow \boxed{\text{"Some function" [TBD...]}} \rightarrow x(0) \text{ [unique]} \rightarrow \boxed{\text{Dyn}} \rightarrow x(1) \rightarrow \boxed{\text{Dyn}} \rightarrow x(2) \rightarrow \dots$$

$$[u(0), \dots, u(T-1)] \rightarrow$$

Can I first estimate/infer x(0) [using "some fcn"] & then reconstruct x(1), x(2), etc. from Dyn. model?

- Key idea: is it **always** possible to **perfectly reconstruct internal states** x from only inputs u and sensed outputs y over some finite time interval? (i.e. can there ever be enough info to "invert" the state space model?)

- Since state x(k) at any time k is initial condition to x(k+1), suffices to examine whether possible to recover any arbitrary x(0) for deterministic state estimation

DT Observability: Another View

Suppose $u(k) = 0$ for all $k > 0$ (ignore inputs for now). [Simplification, but not strictly necessary]

A state $x(0) = x$ is *unobservable* for the system (F, H) if

$$y(k) = HF^k x(0) = HF^k x = 0 \text{ for every } k \geq 0.$$

Let $R_{\bar{o}}$ = set of all unobservable states x = *unobservable subspace* of (F, H).

System (F, H) is *observable* if $x = 0$ is only unobservable state, i.e. if $R_{\bar{o}} = \{0\}$.

→ How do we know if a given (F, H) is observable?

→ How does this help us recover $x(0)$?

→ If (F, H) unobservable, how to know what $R_{\bar{o}}$ is?

Assessing DT Observability

- Consider zero input case for finding x(0) from y(0),…,y($\overset{\text{\# of states}}{n}$-1) [n sequential px1 measurements]

Start w/ $y(0) = H\,x(0)$ & fact that $x(k+1) = F\,x(k)$

→ since $x(k=1) = F\,x(0)$, we have that

$$y(1) = H\,x(1) = H\,F\,x(0)$$

→ Likewise: $y(2) = H\,x(2) = H\,[F\,x(1)]$

but we know that $x(1) = F\,x(0)$, so it follows that

$$y(2) = H\,F \cdot F\,x(0) = H\,\underbrace{F^2}_{n \times n \text{ sq. matrix!}}\,x(0)$$

→ using similar reasoning, easy to show that:

$$y(3) = H\,F^3\,x(0)$$

$$\vdots$$

$$y(n-1) = H\,\underbrace{F^{n-1}}_{n \times n \text{ sq. matrix!}}\,x(0)$$

Solution to Deterministic State Estimation Problem

- Now, stack up all the (linear) equations for observed $y(0), \dots, y(n-1)$:

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix} = \begin{bmatrix} H\,x(0) \\ HF\,x(0) \\ \vdots \\ HF^{n-1}x(0) \end{bmatrix} \longrightarrow \underbrace{\underline{Y} = \begin{bmatrix} y(0) \\ \vdots \\ y(n-1) \end{bmatrix}}_{(n\cdot p\times 1)} = \underbrace{\begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix}}_{(n\cdot p)\times n} \underbrace{x(0)}_{n\times 1}$$

actual observed data

Model/prediction of observations based on unknown $x(0)$

→ looks like an overdetermined sys. of linear equations! [generally more eqs. than unknowns → see lec #2]

→ Rewrite as: $\underline{Y} = \mathcal{O}\cdot x(0)$, where $\mathcal{O} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} \in \mathbb{R}^{np\times n}$

→ **Recall**: that unique solution to $x(0)$ is possible if $\text{rank}\,\mathcal{O} = n$ [full col. rank]

→ If this is the case, then $\underbrace{x(0)}_{n\times 1} = (\mathcal{O}^T\mathcal{O})^{-1}\mathcal{O}^T\underline{Y}$ $[n\times np][np\times n]\,[n\times p]\;np\times 1$ (if Gram matrix $\mathcal{O}^T\mathcal{O}$ non singular)

→ If $\text{rank}(\mathcal{O}) < n$, then Null space $(\mathcal{O})$ is non-trivial, i.e. $\exists\, x \neq 0 \in \mathbb{R}^n$ s.t. $\mathcal{O}x = 0$

null space $(\mathcal{O})$ $= R_{\bar{O}}$ (unobs. subspace of F, H)