ASEN 5050
SPACEFLIGHT DYNAMICS
Initial Orbit Determination

Objectives:

- Brief introduction to initial orbit determination based on distinct types of observations

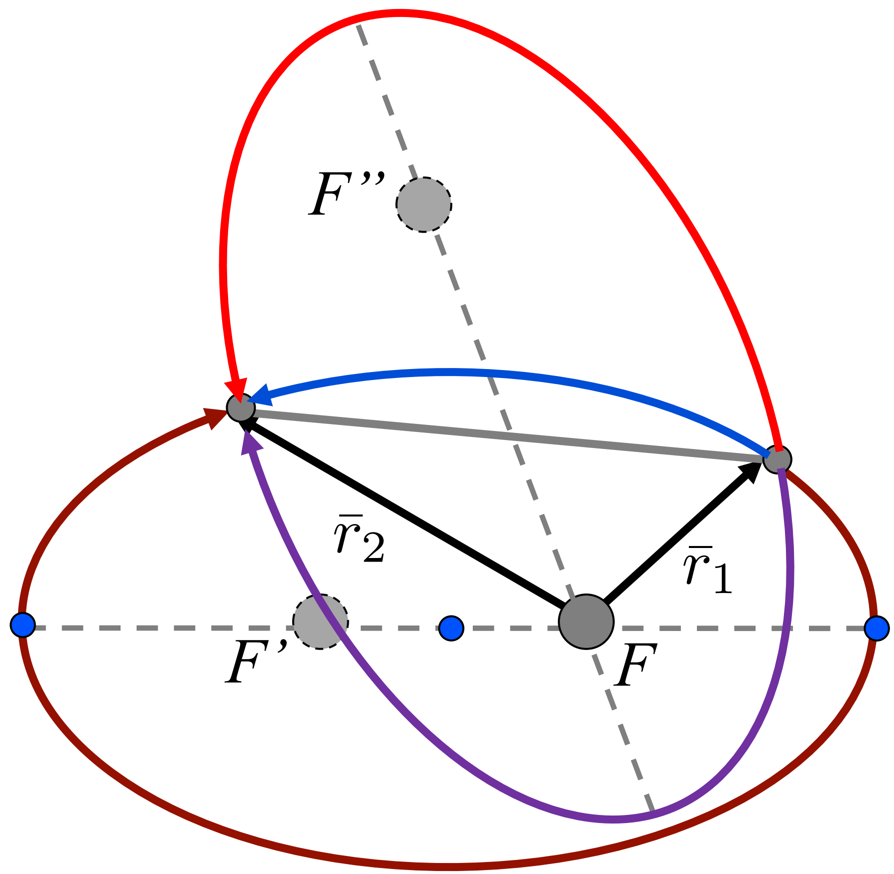
Ch. 7 in the Vallado textbook has an extensive overview of IOD methods; Ch. 10 discusses precise orbit determination

Initial Orbit Determination Methods

- Observations of range, azimuth, and elevation
- Angles-only observations
  - Laplace's Method
  - **Gauss's Technique**
  - Double r-iteration
- Mixed Observations
  - Range and Range-Rate Processing
  - Range-only Processing
- Three Position Vectors and Time
  - **Gibbs Method**
  - Herrick-Gibbs
- Two Position Vectors and Time
  - Lambert's problem

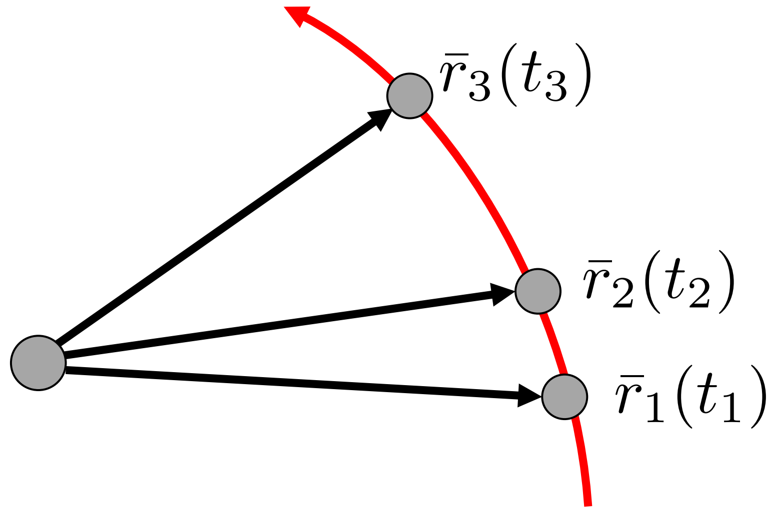
Two Position Vectors and Time

Can use Lambert's problem:



Three Position Vectors and Time

Given three position vectors (no velocity vectors) corresponding to sequential time instants on natural path, extract orbit information



Gibbs method: leverages conservation of angular momentum along an orbit to compute a velocity vector

Assumes: Coplanar, sequential position vectors (+2BP applies)

Limited accuracy when vectors are too close together (<1 deg)

Gibbs Method

1. Calculate the magnitude of the position vectors:

$$r_1 = |\bar{r}_1| \qquad r_2 = |\bar{r}_2| \qquad r_3 = |\bar{r}_3|$$

Note: Check angles between them as $\cos(\alpha_{i,i+1}) = \dfrac{\bar{r}_i \cdot \bar{r}_{i+1}}{|\bar{r}_i||\bar{r}_{i+1}|}$
but challenge is quadrant check

2. Calculate the vectors normal to each pair of position vectors

$$\bar{C}_{12} = \bar{r}_1 \times \bar{r}_2 \qquad \bar{C}_{23} = \bar{r}_2 \times \bar{r}_3 \qquad \bar{C}_{31} = \bar{r}_3 \times \bar{r}_1$$

Each vector must be perpendicular to all three position vectors, offering a straightforward check if coplanar assumption reasonable

Ideally: $\hat{C}_{ij} \cdot \hat{r}_k = 0$ *i,j,k* distinct

In practice: there may be some angle between these unit vectors

Gibbs Method

3. Calculate three vectors, $\bar{N}, \bar{D}, \bar{S}$, as a function of the position vectors

Write the second position vector as a linear combination of the first and third position vector

$$\bar{r}_2 = c_1\bar{r}_1 + c_3\bar{r}_3 \qquad \text{(A)}$$

Take the dot product of this expression with the eccentricity vector (to incorporate orbit dynamics)

$$\bar{r}_2 \cdot \bar{e} = c_1\bar{r}_1 \cdot \bar{e} + c_3\bar{r}_3 \cdot \bar{e} \qquad \text{(E)}$$

And recall that the conic equation can be written as:

$$r = \frac{h^2/\mu}{1 + e\cos(\theta^*)} \qquad \bar{r} \cdot \bar{e} = \frac{h^2}{\mu} - r$$

Gibbs Method

Substitute conic equation into equation (E)

$$\frac{h^2}{\mu} - r_2 = c_1 \left(\frac{h^2}{\mu} - r_1 \right) + c_3 \left(\frac{h^2}{\mu} - r_3 \right) \qquad \text{(B)}$$

Take the cross product of equation (A) with each of $\bar{r}_1$ and $\bar{r}_3$:

$$\bar{r}_2 \times \bar{r}_1 = c_3 \left(\bar{r}_3 \times \bar{r}_1 \right) \text{ (C)} \qquad \bar{r}_2 \times \bar{r}_3 = c_1 \left(\bar{r}_1 \times \bar{r}_3 \right) = -c_1 \left(\bar{r}_3 \times \bar{r}_1 \right) \text{ (D)}$$

Multiply equation (B) by the cross product $\left(\bar{r}_3 \times \bar{r}_1 \right)$

$$\frac{h^2}{\mu} \left(\bar{r}_3 \times \bar{r}_1 \right) - r_2 \left(\bar{r}_3 \times \bar{r}_1 \right) = c_1 \left(\bar{r}_3 \times \bar{r}_1 \right) \left(\frac{h^2}{\mu} - r_1 \right) + c_3 \left(\bar{r}_3 \times \bar{r}_1 \right) \left(\frac{h^2}{\mu} - r_3 \right)$$

Substitute expressions (C) and (D)

$$\frac{h^2}{\mu} \left(\bar{r}_3 \times \bar{r}_1 \right) - r_2 \left(\bar{r}_3 \times \bar{r}_1 \right) = - \left(\bar{r}_2 \times \bar{r}_3 \right) \left(\frac{h^2}{\mu} - r_1 \right) + \left(\bar{r}_2 \times \bar{r}_1 \right) \left(\frac{h^2}{\mu} - r_3 \right)$$

Gibbs Method

Rearrange and define $\bar{N}$ and $\bar{D}$ vectors:

$$\frac{h^2}{\mu}\left(\bar{r}_1 \times \bar{r}_2 + \bar{r}_2 \times \bar{r}_3 + \bar{r}_3 \times \bar{r}_1\right) = r_1(\bar{r}_2 \times \bar{r}_3) + r_2(\bar{r}_3 \times \bar{r}_1) + r_3(\bar{r}_1 \times \bar{r}_2)$$

$$\bar{N} = r_1(\bar{r}_2 \times \bar{r}_3) + r_2(\bar{r}_3 \times \bar{r}_1) + r_3(\bar{r}_1 \times \bar{r}_2)$$

$$\bar{D} = \bar{r}_1 \times \bar{r}_2 + \bar{r}_2 \times \bar{r}_3 + \bar{r}_3 \times \bar{r}_1$$

This expression is compactly written as:

$$\frac{h^2}{\mu}\bar{D} = \bar{N} \qquad\qquad h = \sqrt{\mu\frac{N}{D}}$$

Incorporate knowledge of the perifocal frame unit vectors:

$$\hat{P} = \frac{\bar{e}}{e} \qquad\qquad \hat{W} = \frac{\bar{h}}{h} = \frac{\bar{D}}{D} \qquad\qquad \hat{Q} = \hat{W} \times \hat{P} = \frac{\bar{D} \times \bar{e}}{De}$$

where $$\hat{Q} = \frac{1}{De}\left[(\bar{r}_1 \times \bar{r}_2) \times \bar{e} + (\bar{r}_2 \times \bar{r}_3) \times \bar{e} + (\bar{r}_3 \times \bar{r}_1) \times \bar{e}\right]$$

Gibbs Method

Applying a vector identity:

$$(\bar{A} \times \bar{B}) \times \bar{C} = \bar{B}(\bar{A} \cdot \bar{C}) - \bar{A}(\bar{B} \cdot \bar{C})$$

Use this vector identity to recover:

$$\hat{Q} = \frac{1}{De}\bar{S}$$

Where: $\bar{S} = \bar{r}_1(r_2 - r_3) + \bar{r}_2(r_3 - r_1) + \bar{r}_3(r_1 - r_2)$

Recap definitions of three vectors:

$$\bar{N} = r_1(\bar{r}_2 \times \bar{r}_3) + r_2(\bar{r}_3 \times \bar{r}_1) + r_3(\bar{r}_1 \times \bar{r}_2)$$

$$\bar{D} = \bar{r}_1 \times \bar{r}_2 + \bar{r}_2 \times \bar{r}_3 + \bar{r}_3 \times \bar{r}_1$$

$$\bar{S} = \bar{r}_1(r_2 - r_3) + \bar{r}_2(r_3 - r_1) + \bar{r}_3(r_1 - r_2)$$

Gibbs Method

4. Calculate the velocity vector

Recall the definition of the eccentricity vector, rearranged:

$$\bar{v} \times \bar{h} = \mu \left(\frac{\bar{r}}{r} + \bar{e} \right)$$

Take cross product with angular momentum vector:

$$\bar{h} \times \left(\bar{v} \times \bar{h} \right) = \mu \left(\frac{\bar{h} \times \bar{r}}{r} + \bar{h} \times \bar{e} \right)$$

Use vector identity and rearrange to recover:

$$\bar{v} = \frac{\mu}{h^2} \left(\frac{\bar{h} \times \bar{r}}{r} + \bar{h} \times \bar{e} \right) = \frac{\mu}{h} \left(\frac{\hat{W} \times \bar{r}}{r} + e\hat{Q} \right)$$

Substitute **N, D, S** vectors into terms with perifocal unit vectors:

$$\bar{v} = \sqrt{\frac{\mu}{ND}} \left(\frac{\bar{D} \times \bar{r}}{r} + \bar{S} \right)$$

Herrick-Gibbs Method

- Useful if vectors are close together
- In addition to the three position vectors, uses epochs at each instant of time
- Still assume coplanar position vectors

- Uses Taylor series expansion to write second velocity vector as:

$$\bar{v}_2 = -\Delta t_{32}\left(\frac{1}{\Delta t_{21}\Delta t_{31}} + \frac{\mu}{12 r_1^3}\right)\bar{r}_1 + (\Delta t_{32} - \Delta t_{21})\left(\frac{1}{\Delta t_{21}\Delta t_{32}} + \frac{\mu}{12 r_2^3}\right)\bar{r}_2$$

$$+ \Delta t_{21}\left(\frac{1}{\Delta t_{32}\Delta t_{31}} + \frac{\mu}{12 r_3^3}\right)\bar{r}_3$$

Range, Azimuth, Elevation

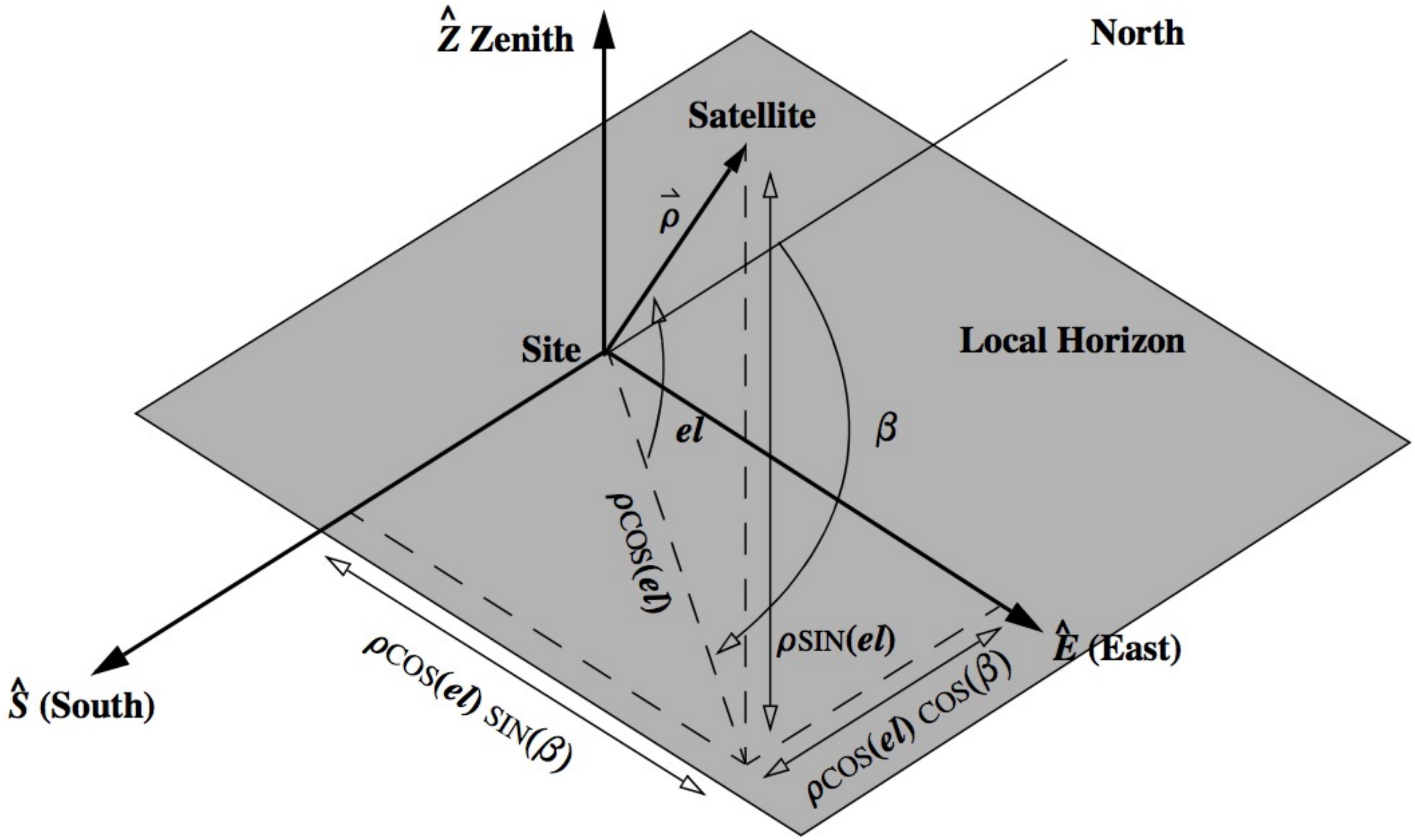


Figure 7-2. Geometry of the Sensor Site. This figure shows the site geometry from a different perspective. Elevation is measured positive up from the local horizon. Notice the positive definition because we measure azimuth from north.

Credit: Vallado

Range, Azimuth, Elevation

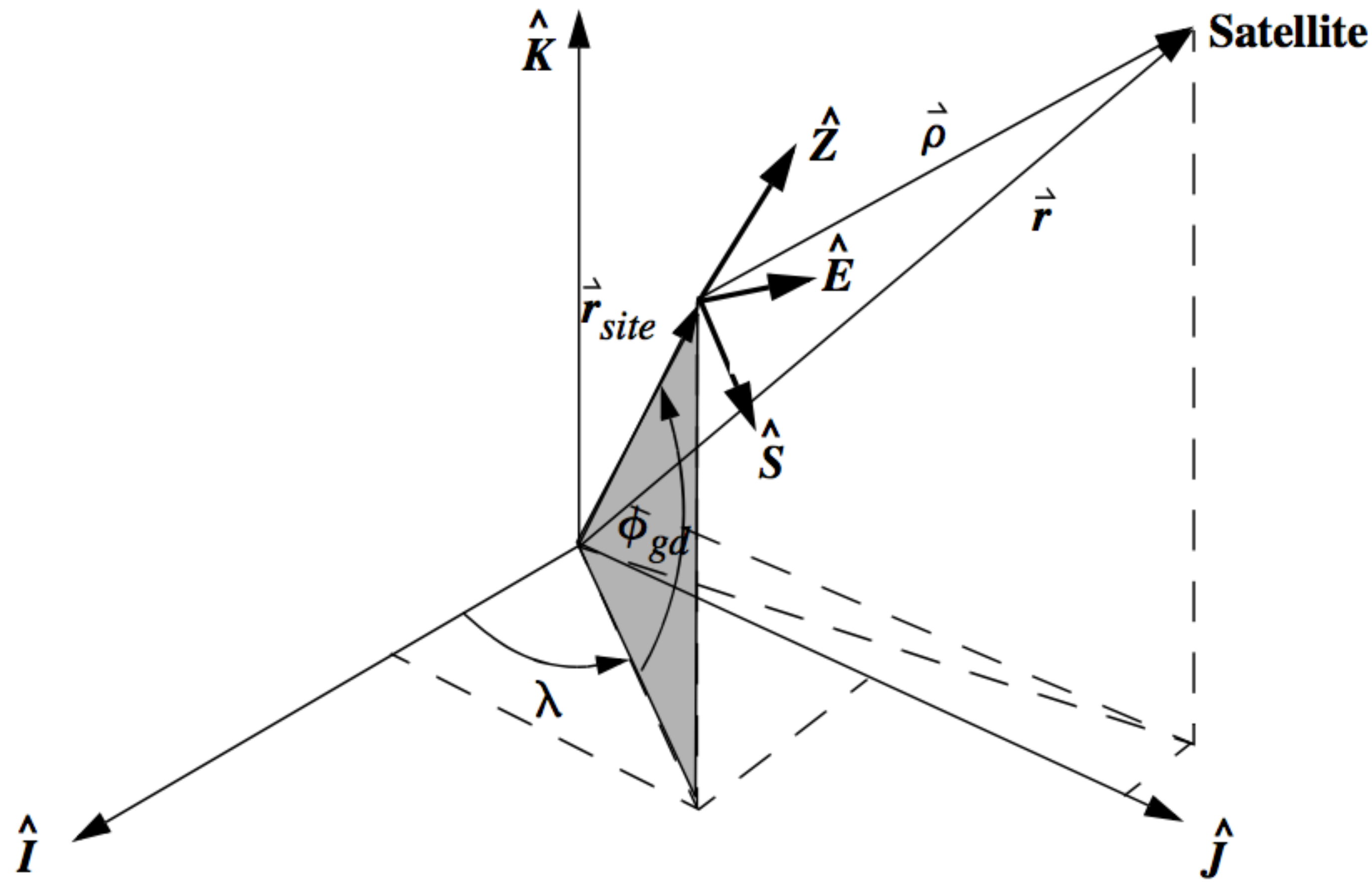


Figure 7-3. Site-to-Satellite Geometry. To complete ***SITE-TRACK***, we use the slant-range vector and the site vector and rotate through geodetic latitude, ϕ_{gd}, and longitude, λ.

Credit: Vallado

Angles-Only Observations

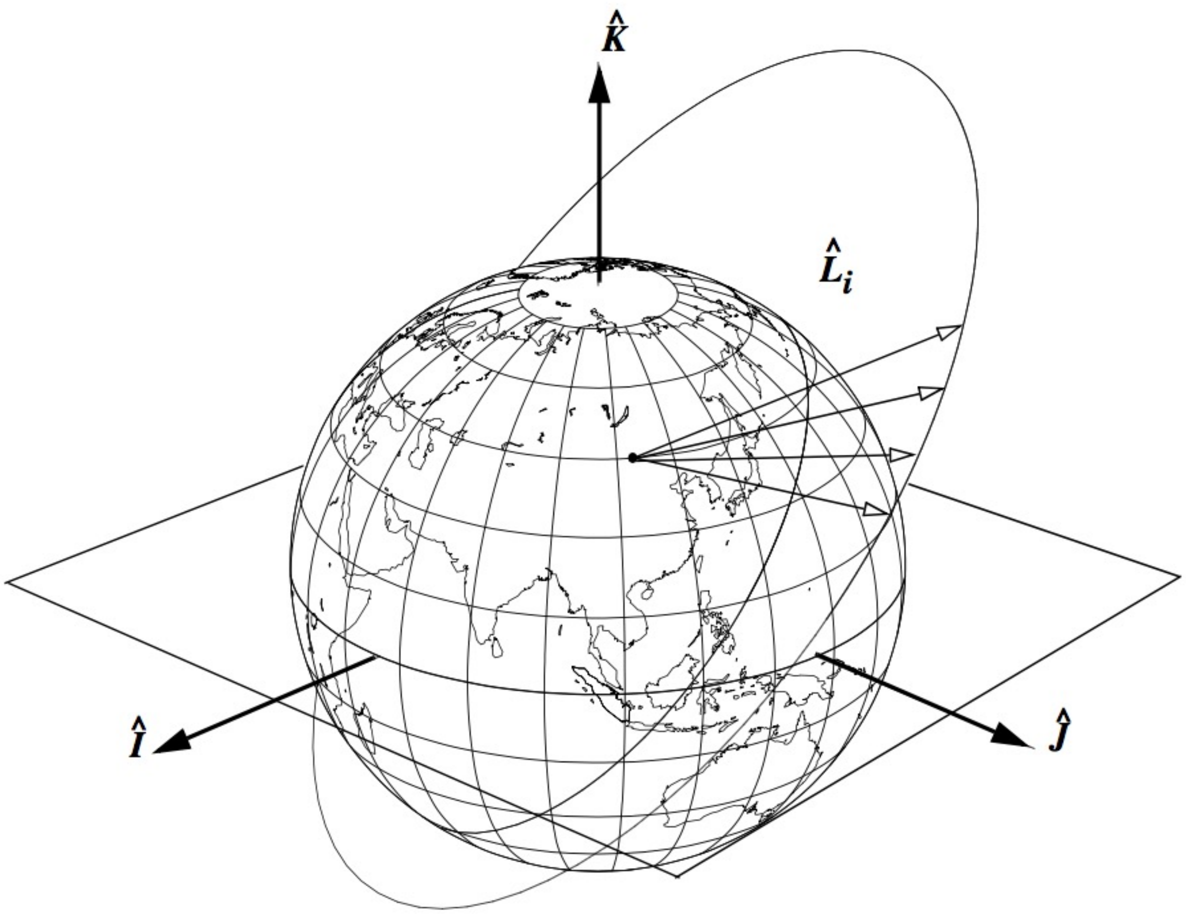


Figure 7-4. Geometry of Angles-only Observations. The underlying principle for the angles-only technique is the use of topocentric angles to form line-of-sight vectors. For objects that are very distant, the distinction for topocentric values diminishes.

Credit: Vallado

Gauss' Method

Position vector from center of Earth to satellite is equal to:

$$\bar{r} = \rho\hat{L} + \bar{r}_{site}$$

where line of sight unit vector is written in terms of right ascension and declination as:

$$\hat{L} = \cos(\delta)\cos(\alpha)\hat{X} + \cos(\delta)\sin(\alpha)\hat{Y} + \sin(\delta)\hat{Z}$$

Then, recall that all three position vectors lie in same plane in 2BP:

$$c_1\bar{r}_1 + c_2\bar{r}_2 + c_3\bar{r}_3 = \bar{0}$$

$$\bar{r}_2 \times \bar{r}_1 = c_3\left(\bar{r}_3 \times \bar{r}_1\right) \qquad \bar{r}_2 \times \bar{r}_3 = c_1\left(\bar{r}_1 \times \bar{r}_3\right) = -c_1\left(\bar{r}_3 \times \bar{r}_1\right)$$

Plug the f and g functions into these expressions:

$$\bar{r}_i = f_i\bar{r}_2 + g_i\bar{v}_2$$

Gauss' Method

This step produces

$$c_1 = \frac{g_3}{f_1 g_3 - f_3 g_1} \qquad\qquad c_3 = \frac{-g_1}{f_1 g_3 - f_3 g_1}$$

But with limited information, cannot completely evaluate these expressions. Instead, use a series approximation of f & g functions:

$$f = 1 - \frac{u}{2}\tau^2 - \frac{\dot{u}}{6}\tau^3 - \frac{\ddot{u} - u^2}{24}\tau^4 - \frac{\dddot{u} - 4u\dot{u}}{120}\tau^5 - \frac{\overset{\text{iv}}{u} - 7u\ddot{u} + u^3 - 4\dot{u}^2}{720}\tau^6$$

$$- \frac{\overset{\text{v}}{u} - 15\dot{u}\ddot{u} - 11u\dddot{u} + 9u^2\dot{u}}{5040}\tau^7$$

$$- \frac{\overset{\text{vi}}{u} - 15\ddot{u}^2 - 26\dot{u}\dddot{u} - 16u\overset{\text{iv}}{u} + 28u\dot{u}^2 + 22u^2\ddot{u} - u^4}{40{,}320}\tau^8$$

$$u = \frac{\mu}{r^3}$$

$$g = \tau - \frac{u}{6}\tau^3 - \frac{\dot{u}}{12}\tau^4 - \frac{3\ddot{u} - u^2}{120}\tau^5 - \frac{2\dddot{u} - 3u\dot{u}}{360}\tau^6$$

$$- \frac{5\overset{\text{iv}}{u} - 13u\ddot{u} - 10\dot{u}^2 + u^3}{5040}\tau^7 - \frac{6\overset{\text{v}}{u} - 48\dot{u}\ddot{u} - 24u\dddot{u} + 12u^2\dot{u}}{40{,}320}\tau^8$$

Image credit: Vallado "Fundamentals of Astrodynamics"

Gauss' Method

Produces coefficients

$$c_1 \approx \frac{\tau_3}{\tau_3 - \tau_1} + \frac{u\tau_3((\tau_3 - \tau_1)^2 - \tau_3^2)}{6(\tau_3 - \tau_1)} = a_1 + a_{1,u}u$$

$$c_3 \approx -\frac{\tau_1}{\tau_3 - \tau_1} - \frac{u\tau_1((\tau_3 - \tau_1)^2 - \tau_1^2)}{6(\tau_3 - \tau_1)} = a_3 + a_{3,u}u$$

Recalling the linear combination and position vector definition:

$$c_1(\bar{\rho}_1 + \bar{r}_{site,1}) + c_2(\bar{\rho}_2 + \bar{r}_{site,2}) + c_3(\bar{\rho}_3 + \bar{r}_{site,3}) = \bar{0}$$

$$c_1\bar{\rho}_1 + c_2\bar{\rho}_2 + c_3\bar{\rho}_3 = -c_1\bar{r}_{site,1} - c_2\bar{r}_{site,2} - c_3\bar{r}_{site,3}$$

In matrix form:

$$[\hat{L}_1, \hat{L}_2, \hat{L}_3] \begin{bmatrix} c_1\rho_1 \\ c_2\rho_2 \\ c_3\rho_3 \end{bmatrix} = [\bar{r}_{site,1}, \bar{r}_{site,2}, \bar{r}_{site,3}] \begin{bmatrix} -c_1 \\ -c_2 \\ -c_3 \end{bmatrix}$$

Gauss' Method

Invert the matrix system of equations

$$\begin{bmatrix} c_1\rho_1 \\ c_2\rho_2 \\ c_3\rho_3 \end{bmatrix} = \underbrace{[\hat{L}_1, \hat{L}_2, \hat{L}_3]^{-1}[\bar{r}_{site,1}, \bar{r}_{site,2}, \bar{r}_{site,3}]}_{[M]} \begin{bmatrix} -c_1 \\ -c_2 \\ -c_3 \end{bmatrix}$$

Rewrite the middle value of slant range as

$$\rho_2 = M_{21}a_1 - M_{22} + M_{23}a_3 + (M_{21}a_{1,u} + M_{23}a_{3,u})u = d_1 + d_2 u$$

Substitute into

$$r_2 = \sqrt{\rho^2 + 2\rho\hat{L}_2 \cdot \bar{r}_{site,2} + r^2_{site,2}} \qquad C = \hat{L}_2 \cdot \bar{r}_{site,2}$$

$$r_2^8 - (d_1^2 + 2Cd_1 + r^2_{site,2})r_2^6 - 2\mu(Cd_2 + d_1 d_2)r_2^3 - \mu^2 d_2^2 = 0$$

Gauss' Method

$$r_2^8 - (d_1^2 + 2Cd_1 + r_{site,2}^2)r_2^6 - 2\mu(Cd_2 + d_1 d_2)r_2^3 - \mu^2 d_2^2 = 0$$

Calculate a real root for r_2, update $u = \dfrac{\mu}{r^3}$

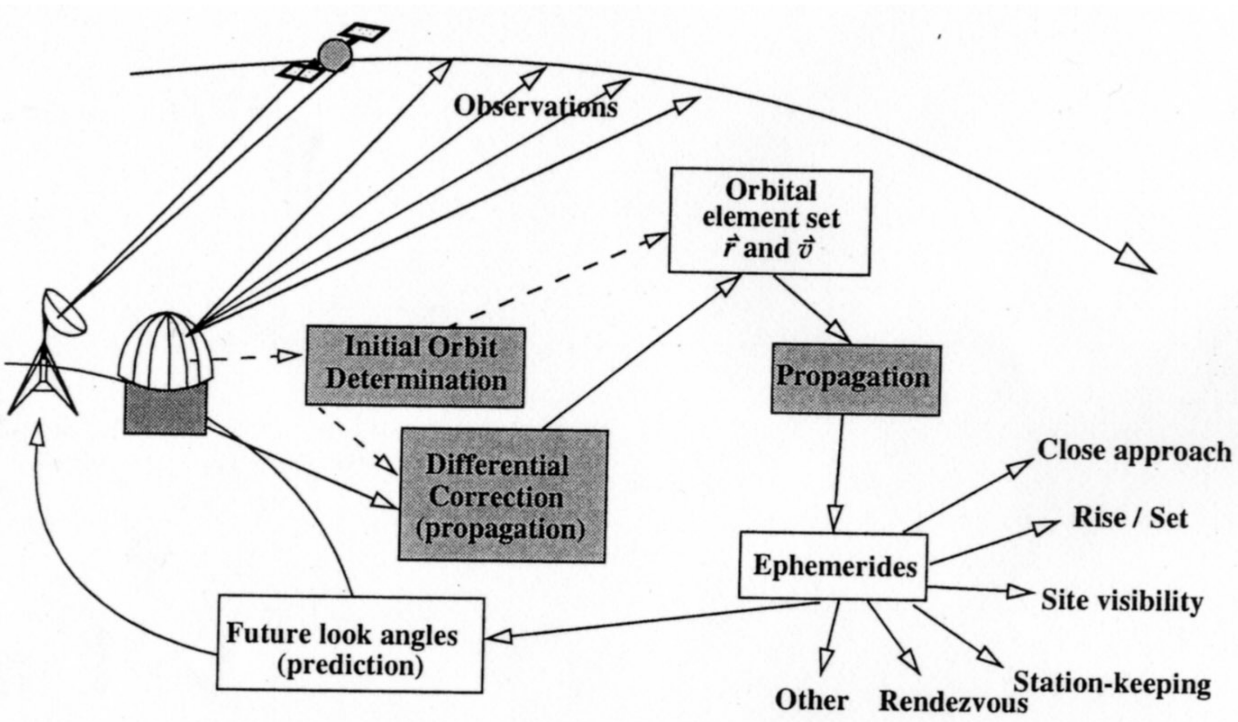
Then, calculate c_i

$$c_1 = a_1 + a_{1,u}u \qquad c_3 = a_3 + a_{3,u}u$$

Solve matrix expression for slant ranges

To improve accuracy: can now calculate position and then velocity via Gibbs/Herrick-Gibbs methods and iterate with more accurate values of f & g until slant ranges converge to values.

Orbit Determination



Credit: Vallado

Predictions from orbit determination strategies will not match actual state information because:

– Method limitations in accuracy

– Observational errors

– Model errors or limitations in accuracy

– Limited knowledge of spacecraft parameters