AQ4

ASEN 5044
Fall 2024
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$\boxed{\text{HW 4}}$

AQ4 → $P(x) = \sum_{i=1}^{M} w_i \cdot \mathcal{U}[a_i, b_i]$, $w_i \geq 0$, $\sum_{i=1}^{M} w_i = 1$, $a_i < b_i$

a) PDF of $\mathcal{U}[a_i, b_i] = \begin{cases} \frac{1}{b_i - a_i} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$

$$\mu_i = \int_{-\infty}^{\infty} x f(x)\, dx = \int_a^b x \left(\frac{1}{b-a}\right) dx = \frac{1}{b-a}\int_a^b x\, dx = \frac{1}{b-a}\left[\frac{1}{2}x^2\right]\Big|_a^b$$

$$= \frac{1}{b-a}\left[\frac{1}{2}b^2 - \frac{1}{2}a^2\right] = \frac{1}{b-a}\left(\frac{1}{2}\right)(b^2 - a^2) = \frac{1}{2}\left(\cancel{\frac{1}{b-a}}\right)\cancel{(b-a)}(a+b) = \frac{1}{2}(a_i + b_i) = \mu_i$$

$$\sigma_i^2 = \int_{-\infty}^{\infty} (x - \mu_i)^2 f(x)\, dx = \int_a^b \left(x - \frac{1}{2}(a+b)\right)^2 \frac{1}{b-a}\, dx = \int_a^b \left(x^2 - \frac{x}{2}(a+b) + \frac{1}{4}(a+b)^2\right) \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\left[\int_a^b x^2\, dx - \int_a^b \frac{x}{2}(a+b)\, dx + \int_a^b \frac{1}{4}(a+b)^2\, dx\right]$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\Big|_a^b - \frac{(a+b)}{2}\left(\frac{x^2}{2}\right)\Big|_a^b + \frac{1}{4}(a+b)^2(b-a)\right]$$

$$= \frac{1}{b-a}\left[\frac{b^3}{3} - \frac{a^3}{3} - \frac{a+b}{2}\left(\frac{b-a}{2}\right)^2 + \frac{1}{4}(a+b)^2(b-a)\right]$$

$$= \left(\frac{1}{b-a}\right)\left(\frac{b^3 - a^3}{3}\right) - \frac{a+b(b-a)}{2} + \frac{1}{4}(a+b)^2 = \frac{b^3 - a^3}{3b - 3a} - \frac{a+b(b-a)}{2} + \frac{1}{4}(a^2 + 2ab + b^2)$$

$$= \frac{b^3 - a^3}{3(b-a)} \cdot \frac{4}{4} - \frac{(a+b) \cdot b(b-a)^2}{2(6)(b-a)} + \frac{3(a^2 + 2ab + b^2)}{4 \cdot 3(b-a)} = \frac{(b^3 - a^3)4}{12(b-a)} - \frac{(a+b)(b^2 - 2ab + a^2)}{12(b-a)} + \frac{3(a^2 + 2ab + b^2)}{12(b-a)}$$

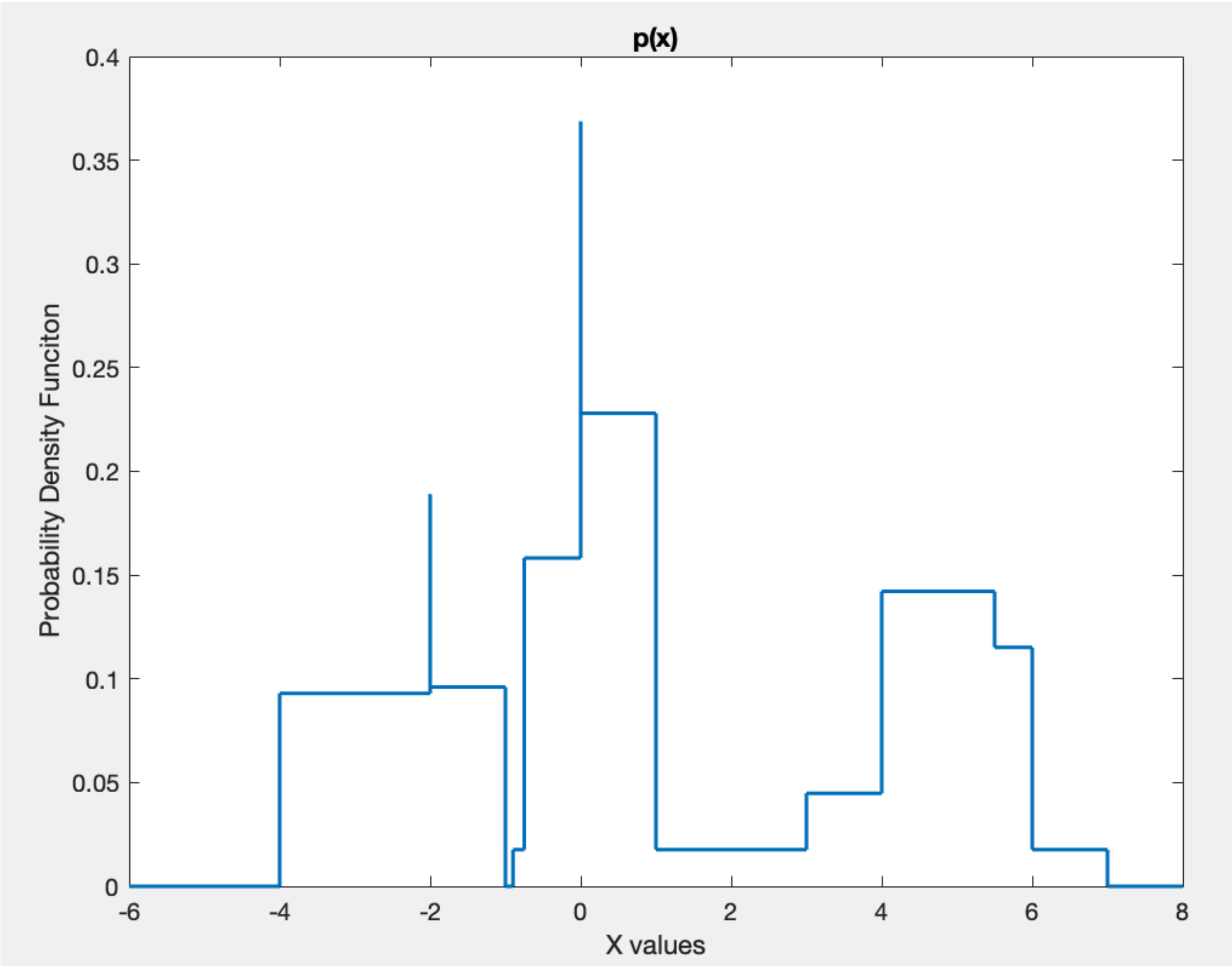
$$= \frac{4b^3 - 4a^3 - [6ab^2 - 12a^2b + 6a^3 + 6b^3 - 12ab^2 + 6a^2b] + 3a^2 + 6ab + 3b^2}{12(b-a)}$$

$$= \frac{b^3 - a^3 - 3b^2a + 3ba^2}{12(b-a)} = \frac{(b-a)^{\cancel{3}\,2}}{12\cancel{(b-a)}} = \frac{1}{12}(b-a)^2$$

$$E[x] = \sum_{i=1}^{M} x\, w_i\, \mathcal{U}[a_i, b_i] = \sum_{i=1}^{M} w_i\left[x\, \mathcal{U}[a_i, b_i]\right] = \boxed{\sum_{i=1}^{M} w_i \frac{1}{2}(a_i + b_i) = E[x]}$$

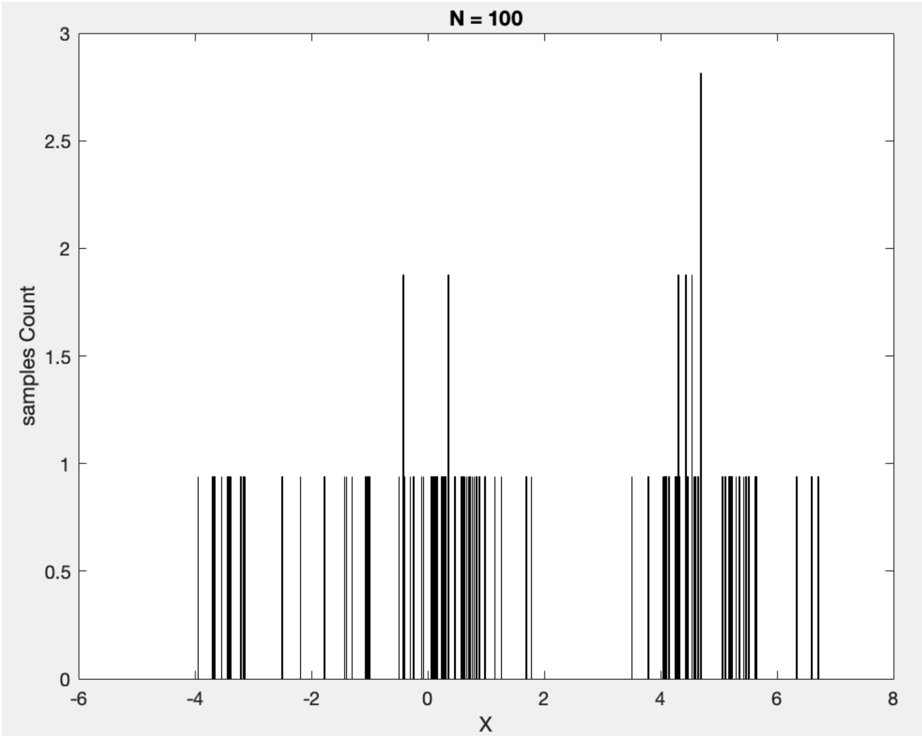
$$\text{var}(x) = \sum_{i=1}^{M} (x - \mu_x)^2 w_i\, \mathcal{U}[a_i, b_i] = \sum_{i=1}^{M} w_i (x - \mu_x)^2\, \mathcal{U}[a_i, b_i] = \boxed{\sum_{i=1}^{M} w_i \frac{(b_i - a_i)^2}{12} = \text{var}(x)}$$

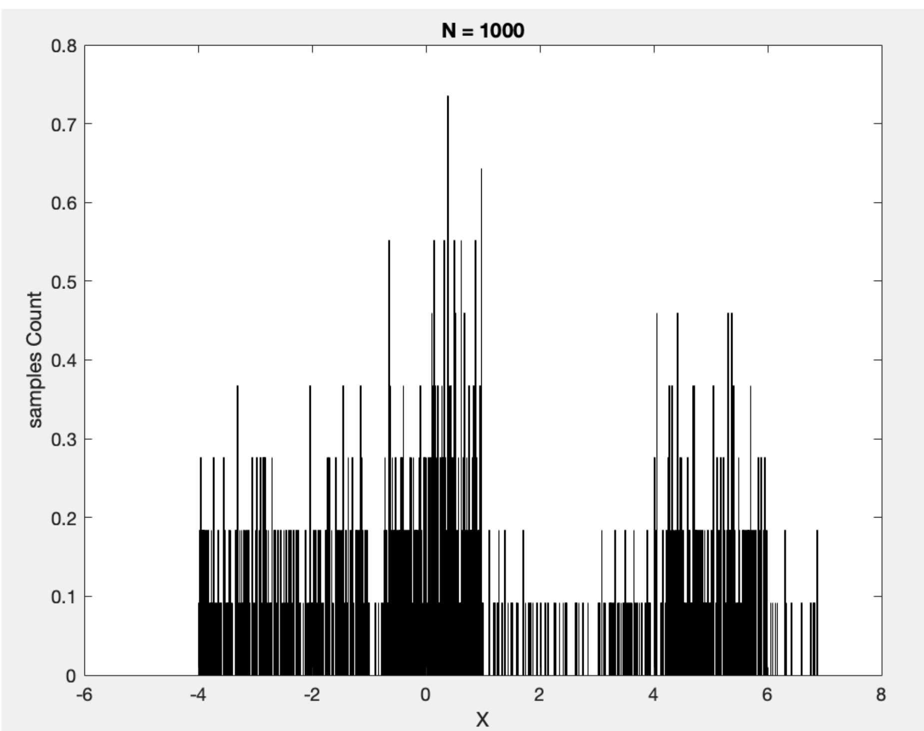
- Part b

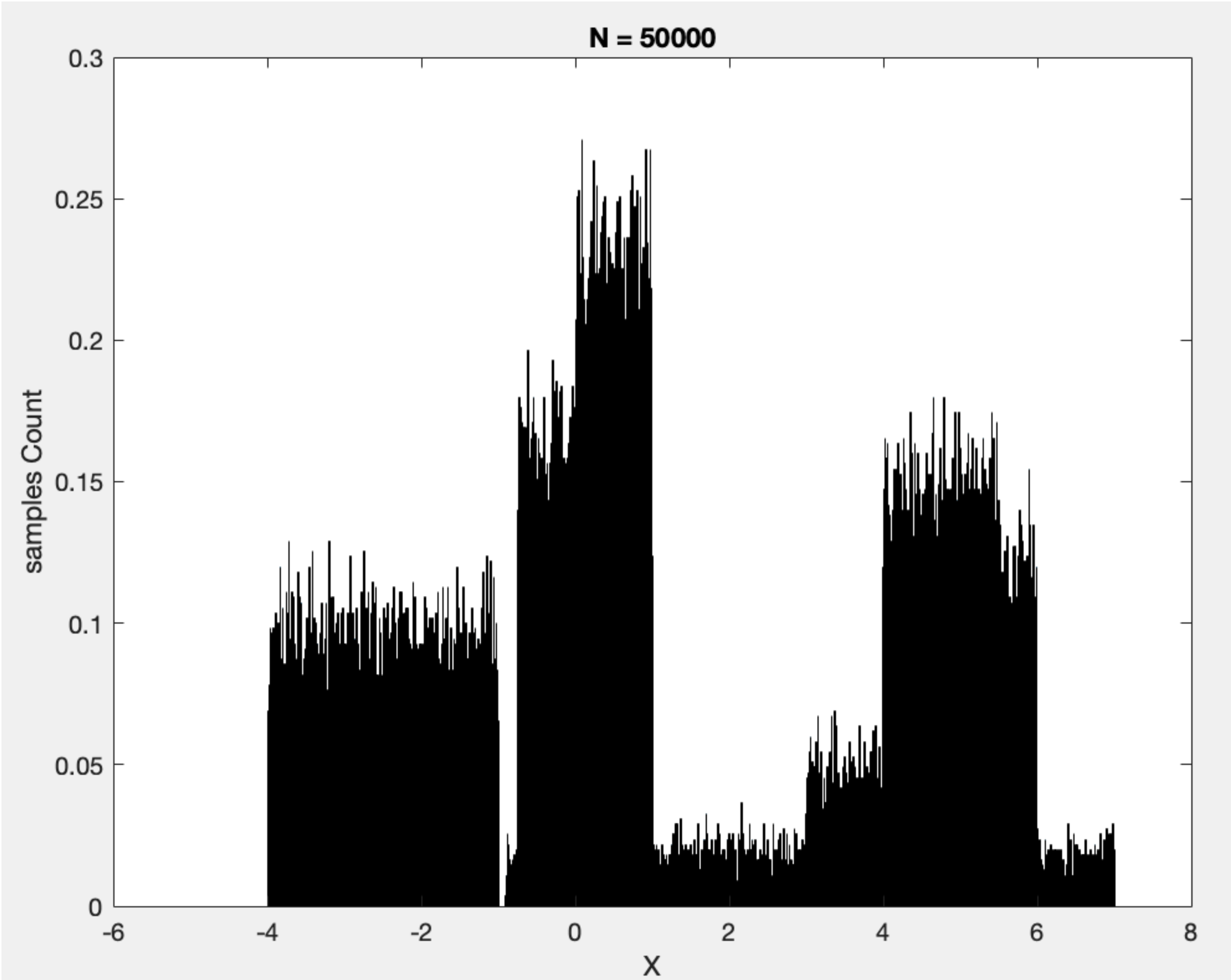


- Part c
    - E[x]
        - E[x] computed analytically is 1.0518
        - E[x] computed numerically is 1.0516
        - There isn't a major difference between the two values.
    - var(x)
        - var(x) computed analytically is 0.9172
        - var(x) computed numerically is 9.1682
        - Looking forward at part e, seems like 9.1682 is the more accurate answer and the analytical calculation is faulty.

- Part d


N = 100
samples Count
X
3
2.5
2
1.5
1
0.5
0
-6
-4
-2
0
2
4
6
8


N = 1000
samples Count
X
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-6
-4
-2
0
2
4
6
8



N = 50000
samples Count
X
0.3
0.25
0.2
0.15
0.1
0.05
0
-6
-4
-2
0
2
4
6
8

- Part e
    - E[x]

N = 100	N = 1000	N = 50000	Part c
0.8408	0.9940	1.0406	1.0516

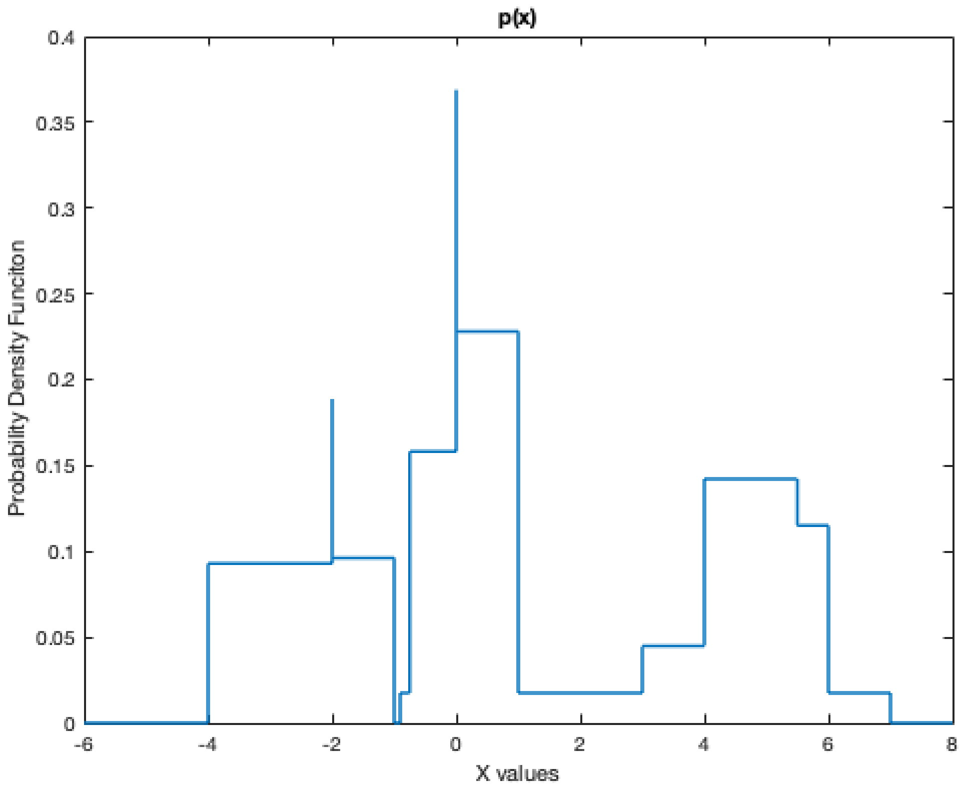
    - var(x)

N = 100	N = 1000	N = 50000	Part c
8.7553	8.9524	9.1047	9.1682

    - H(p(x)) = E[-log(p(x))]

N = 100	N = 1000	N = 50000	Part c
-0.4255 - 1.3509i	-0.3546 - 1.2881i	-0.3942 - 1.2691i	N/A

```
clear; clc; close all;

w = [0.1859, 0.0961, 0.1055, 0.2104, 0.0678, 0.1950, 0.1393];
a = [-4, -2, -0.75, 0, 3, 4, -0.9];
b = [-2, -1, 0, 1, 5.5, 6, 7];

x = -6:0.001:8;

for i = 1:length(x)
    px(i) = 0;
    for j = 1:length(w)
        px(i) = px(i) + w(j) * uni_dist_pdr(a(j), b(j), x(i));
    end
end

figure()
plot(x, px, 'LineWidth', 1.5)
xlabel("X values")
ylabel("Probability Density Funciton")
title("p(x)")

analytical_mean = 0;
analytical_var = 0;
for k = 1:length(w)
    analytical_mean = analytical_mean + w(k)*1/2*(a(k) + b(k));
    analytical_var = analytical_var + w(k) * (b(k) - a(k))^2/12;
end

numerical_mean = mean(px);
numerical_var = std(px)^2;

function uniform_pdf = uni_dist_pdr(a, b, x)
    if x >= a && x <= b
        uniform_pdf = 1/(b-a);
    else
        uniform_pdf = 0;
    end
end
```



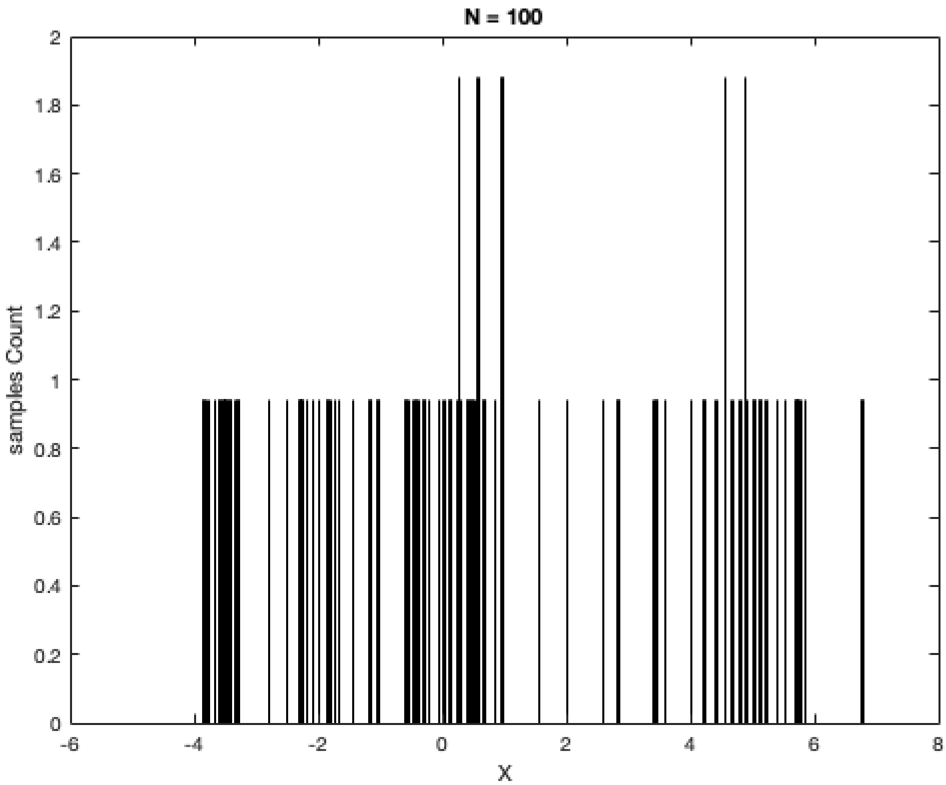
Part c

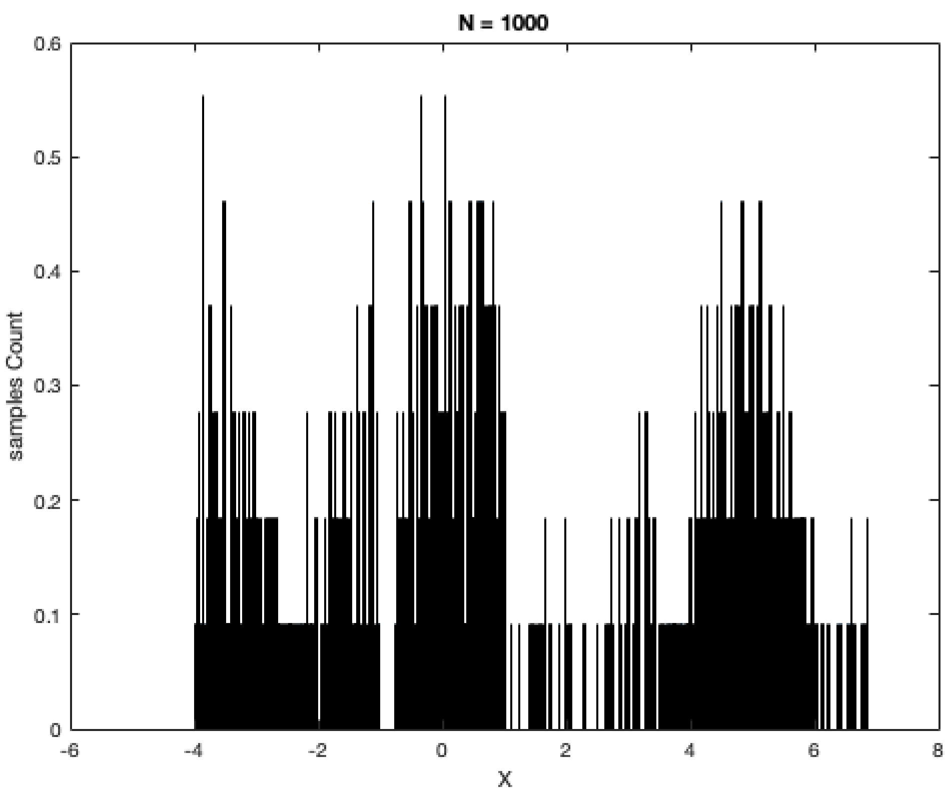
```
e_x = 0;
var_x = 0;
for i = 1:length(px)-1
    e_x = x(i) * (x(i+1) - x(i)) * ((px(i+1) + px(i))/2) + e_x;
end

for i = 1:length(px)-1
    var_x = (x(i) - e_x)^2 * (x(i+1) - x(i)) * ((px(i+1) + px(i))/2) + var_x;
end

diff_ent = 0;
for i = 1:length(px)-1
    if px(i) == 0
        out = 0;
    else
        out = log(px(i));
    end
    diff_ent = -out * (x(i+1) - x(i)) * ((px(i+1) + px(i))/2) + diff_ent;
end
```

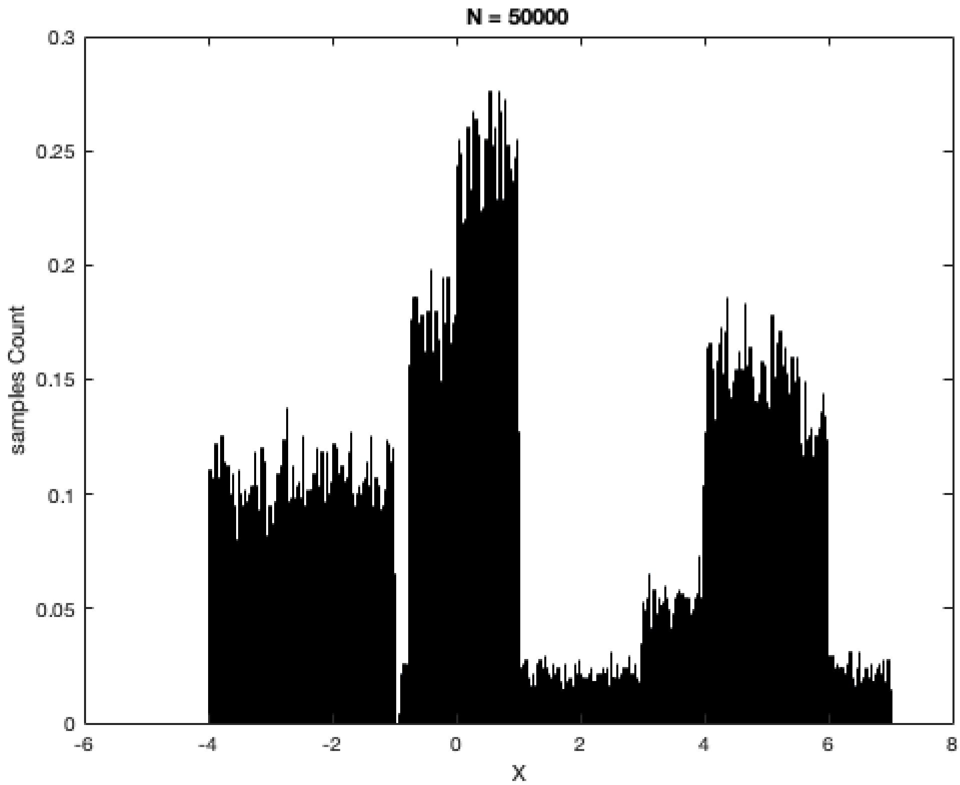
Part d

```
N1 = 100;
N2 = 1000;
```

```
N3 = 50000;
nbins = 1000;

var = 1:length(w);


C = randsample(var, 1, true, w);

N1_samples = part_d(N1, w, a, b, nbins);
N2_samples = part_d(N2, w, a, b, nbins);
N3_samples = part_d(N3, w, a, b, nbins);

function out = part_d(N, w, a, b, nbins)
    var = 1:length(w);
    for i = 1:N
        C = randsample(var, 1, true, w);
        samples(i) = a(C) + (b(C) - a(C))*rand;
    end

    figure()
    histogram(samples, nbins, 'Normalization','pdf')
    xlim([-6, 8])
    title("N = " + N)
    xlabel("X")
    ylabel("samples Count")

    out = samples;
end
```


N = 100
samples Count
X


N = 1000
samples Count
X



Part e

```
e_x_n1 = 1/N1 * sum(N1_samples);
e_x_n2 = 1/N2 * sum(N2_samples);
e_x_n3 = 1/N3 * sum(N3_samples);

var_x_n1 = 1/N1 * sum((N1_samples - e_x_n1).^2);
var_x_n2 = 1/N2 * sum((N2_samples - e_x_n2).^2);
var_x_n3 = 1/N3 * sum((N3_samples - e_x_n3).^2);

h_x_n1 = 1/N1 * sum(-log(N1_samples));
h_x_n2 = 1/N2 * sum(-log(N2_samples));
h_x_n3 = 1/N3 * sum(-log(N3_samples));
```

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