Attitude Control Examples

ASEN 5010

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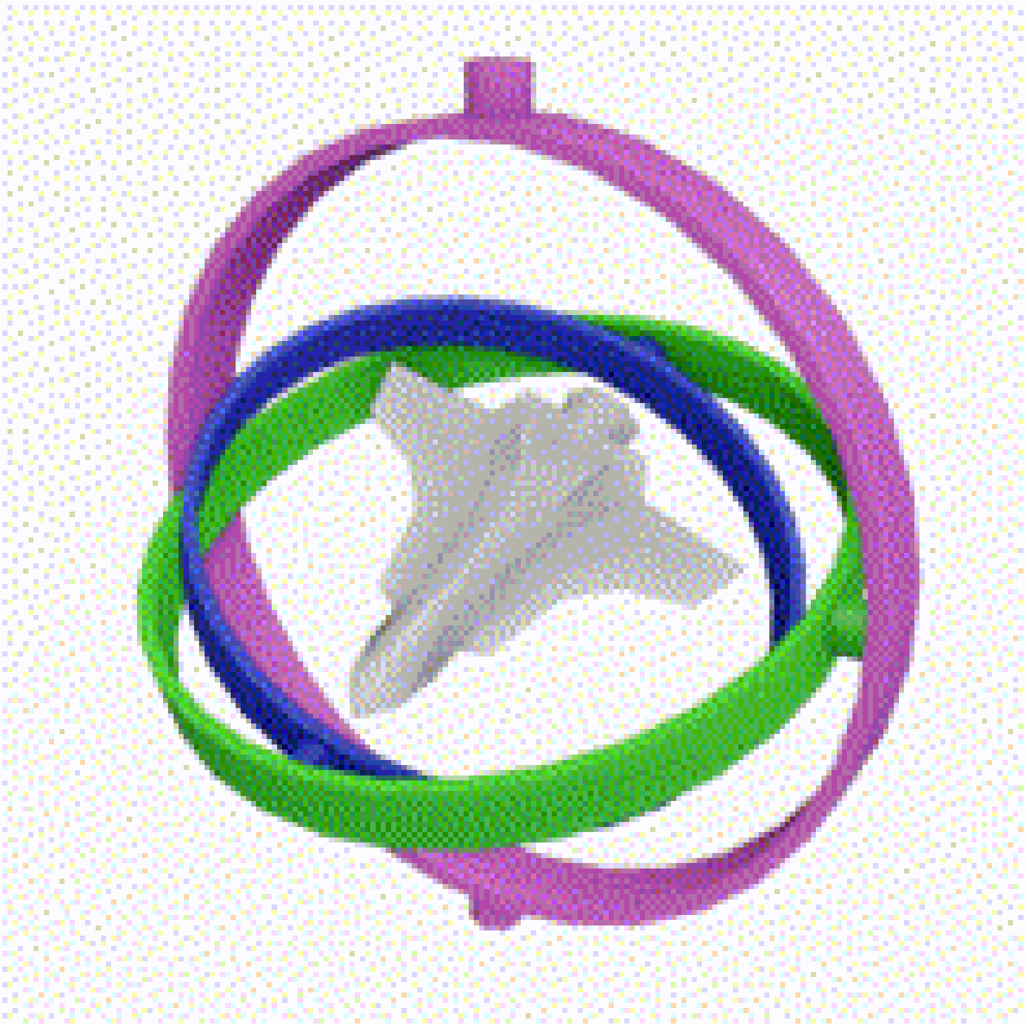
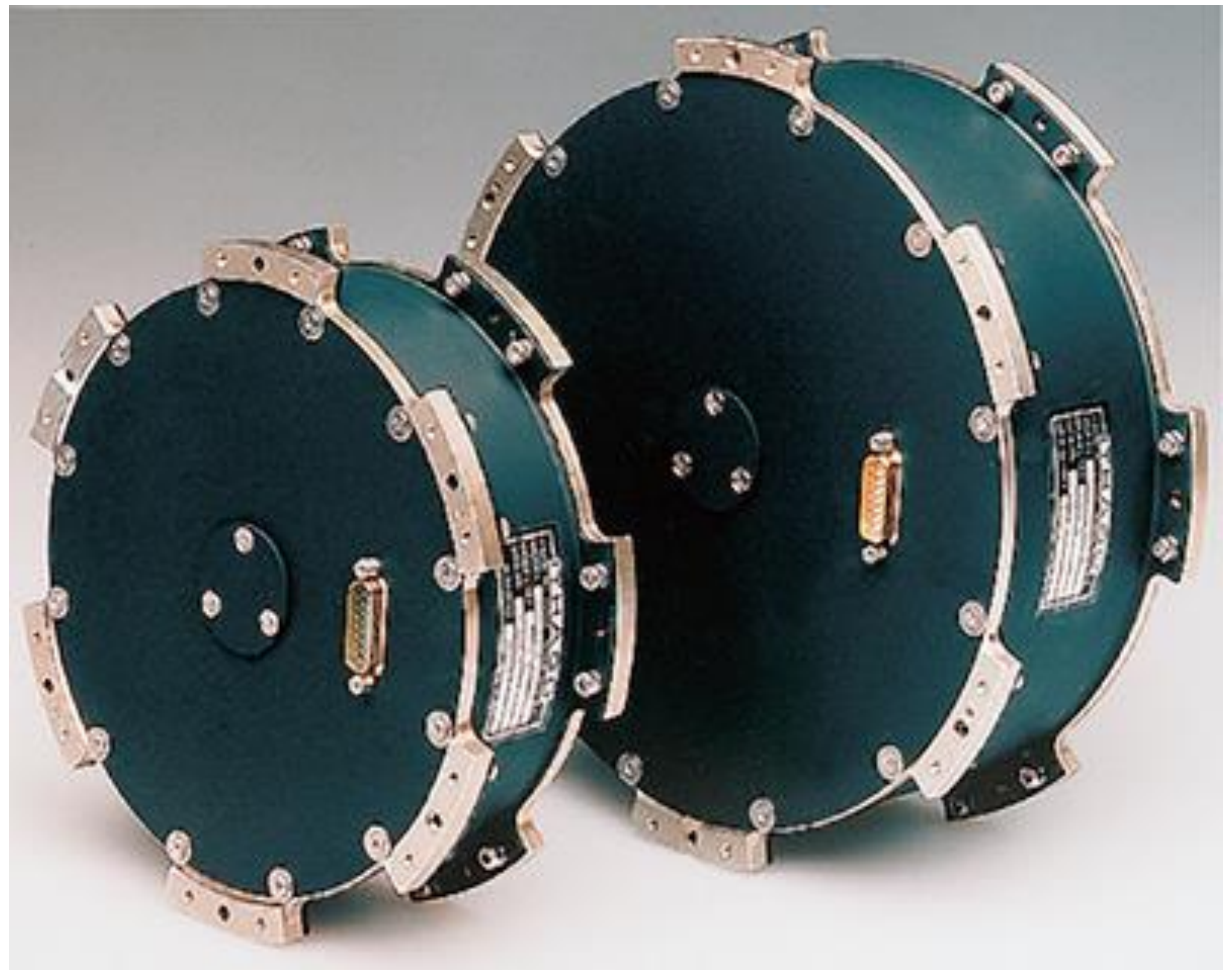
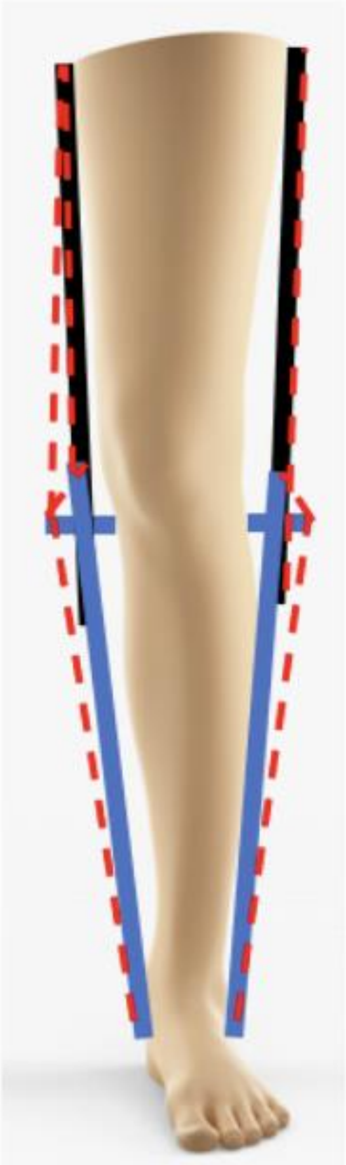
Some Ordinary Examples

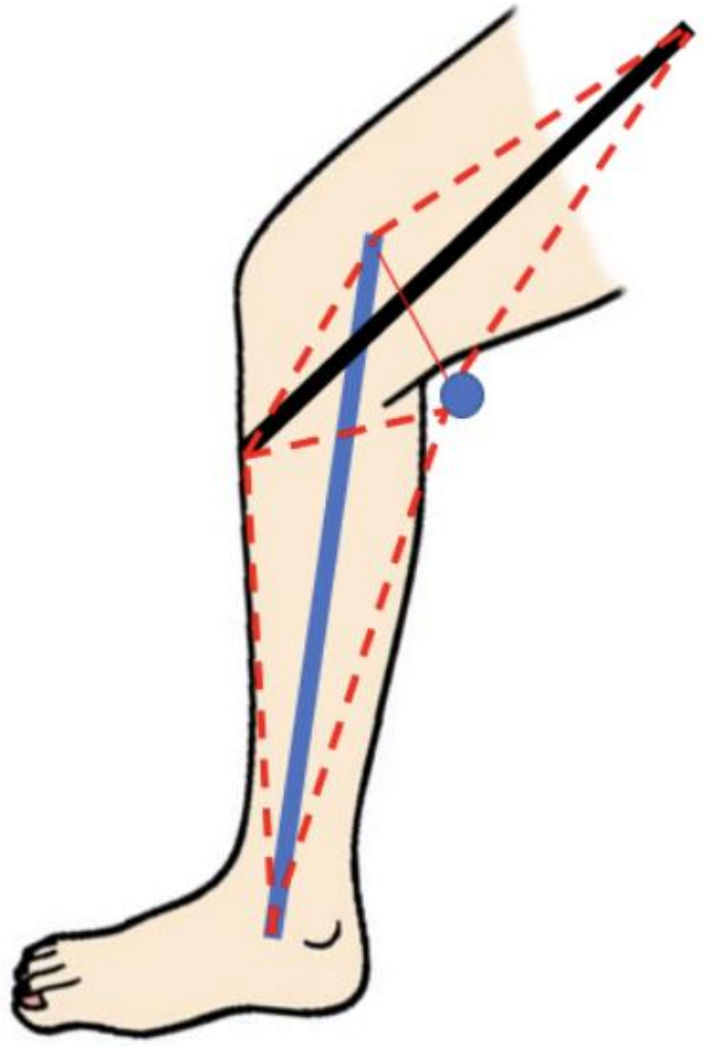
- Tensegrity Robot
- Humans in Space
- Momentum Control Devices



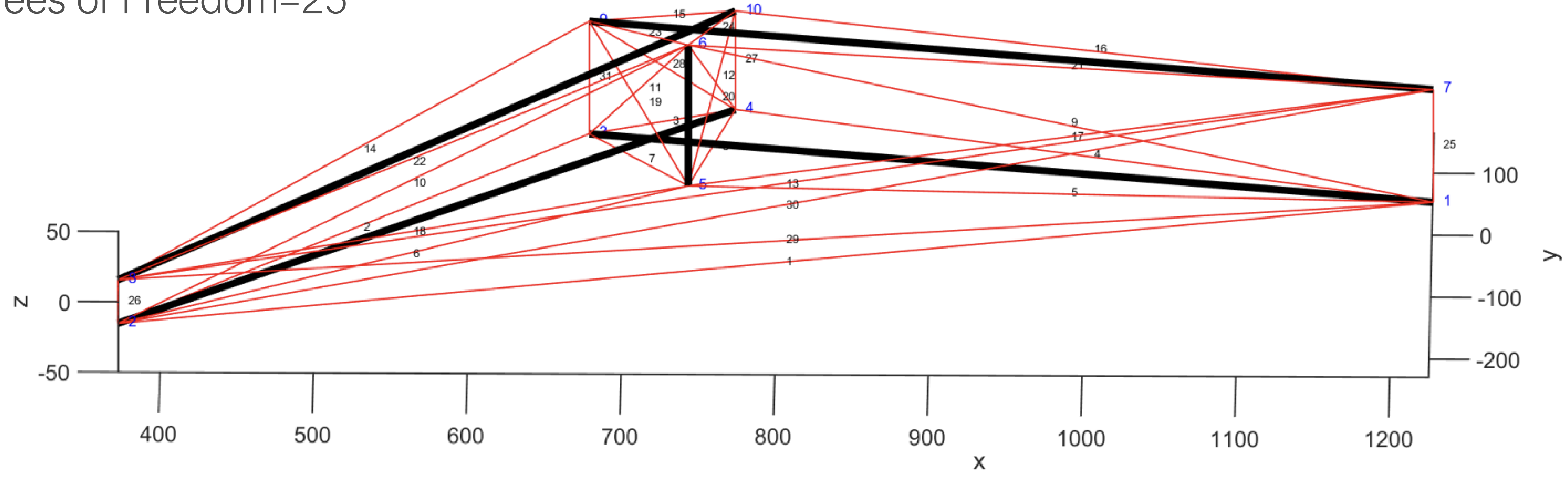


Tensegrity Robots





Degrees of Freedom=25



$$\ddot{E} + \Psi \dot{E} + \Theta E = 0$$

$$L\ddot{N}R + \Psi L\dot{N}R + \Theta(LNR - \bar{Y}) = 0$$

$$L(W + \Omega P^T - NK)M^{-1}R + \Psi L\dot{N}R + \Theta(LNR - \bar{Y}) = 0$$

Tensegrity Dynamics

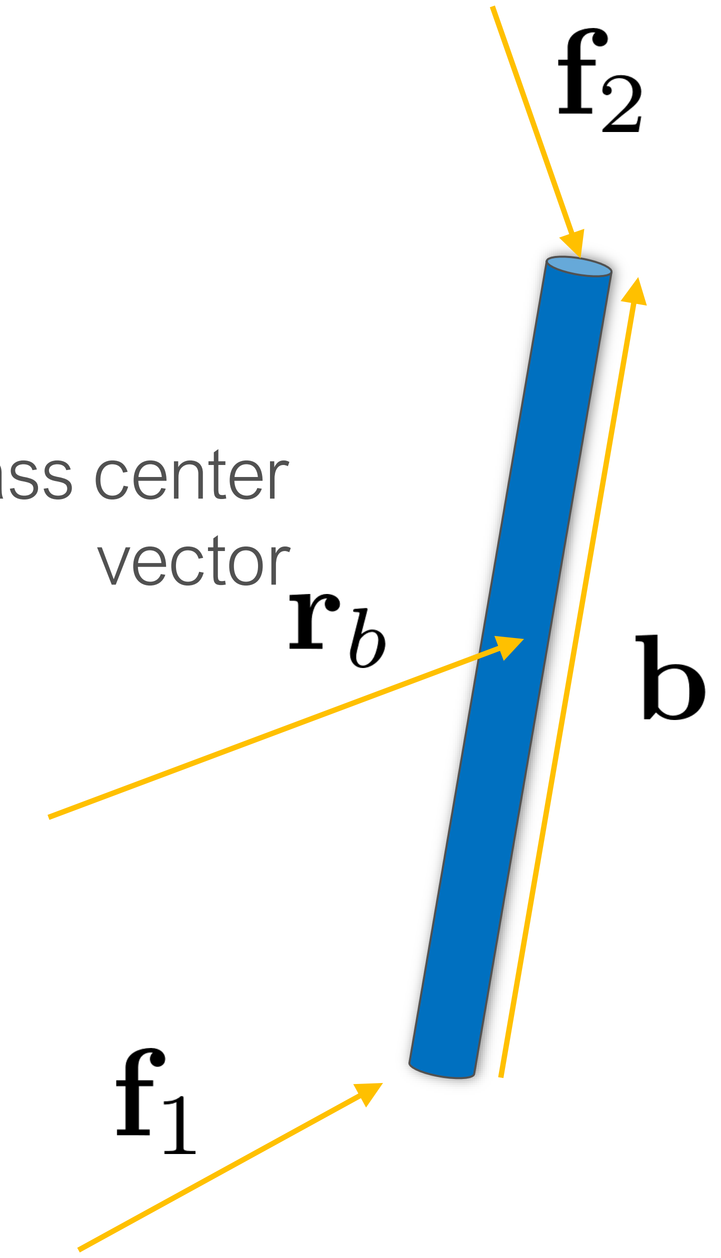
- For a rigid rod, using Euler equation

$$\dot{h} = \sum_i \tau_i$$
$$\frac{m_b}{12}\tilde{b}\ddot{b} = \frac{1}{2}\tilde{b}(f_2 - f_1)$$

Rotational Dynamics

$$\text{where, } \tilde{b} = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}$$

Also, moment of inertia along the axis of the bar is assumed zero



- Bar Length Constraints $b^T b = l^2, \quad b^T \dot{b} = 0, \quad b^T \ddot{b} = -\dot{b}^T \dot{b}.$

$$\begin{bmatrix} \tilde{b} \\ b^T \end{bmatrix} \ddot{b} = \begin{bmatrix} \frac{6}{m_b}\tilde{b}(f_2 - f_1) \\ -\dot{b}^T \dot{b} \end{bmatrix}$$

Where, l is length of the bar

$$\ddot{b} = \frac{6}{m_b l^2}(l^2 I - bb^T)(f_2 - f_1) - b\frac{\dot{b}^T \dot{b}}{l^2}$$

Cheong, J., & Skelton, R. E. (2015). Nonminimal Dynamics of General Class k Tensegrity Systems, *15*(2), 1–22. https://doi.org/10.1142/S0219455414500424

Tensegrity Dynamics

$$m_b\ddot{r}_b = f_1 + f_2$$

Translational Dynamics

Matrix Formulation

- Node matrix $N = [n_1 \quad n_2 \quad n_3 \quad \cdots \quad n_{2\beta}]$ β: Number of rods σ: Number of strings
- Bar matrix $B = N_b C_b^T$ where, $B = [b_1 \quad b_2 \quad b_3 \quad \cdots \quad b_\beta]$
- Bar Center of Mass matrix $R_b = N_b C_r^T$ where, $R_b = [r_{b1} \quad r_{b2} \quad r_{b3} \quad \cdots \quad r_{b\beta}]$
- String matrix $S = N C_s^T$ where, $S = [s_1 \quad s_2 \quad s_3 \quad \cdots \quad s_\sigma]$
- Force Density

$$\|t_i\| = \boxed{k_i(\|s_i\| - s_{0i})} + \boxed{c_i \frac{s_i^T \dot{s}_i}{\|s_i\|}}$$

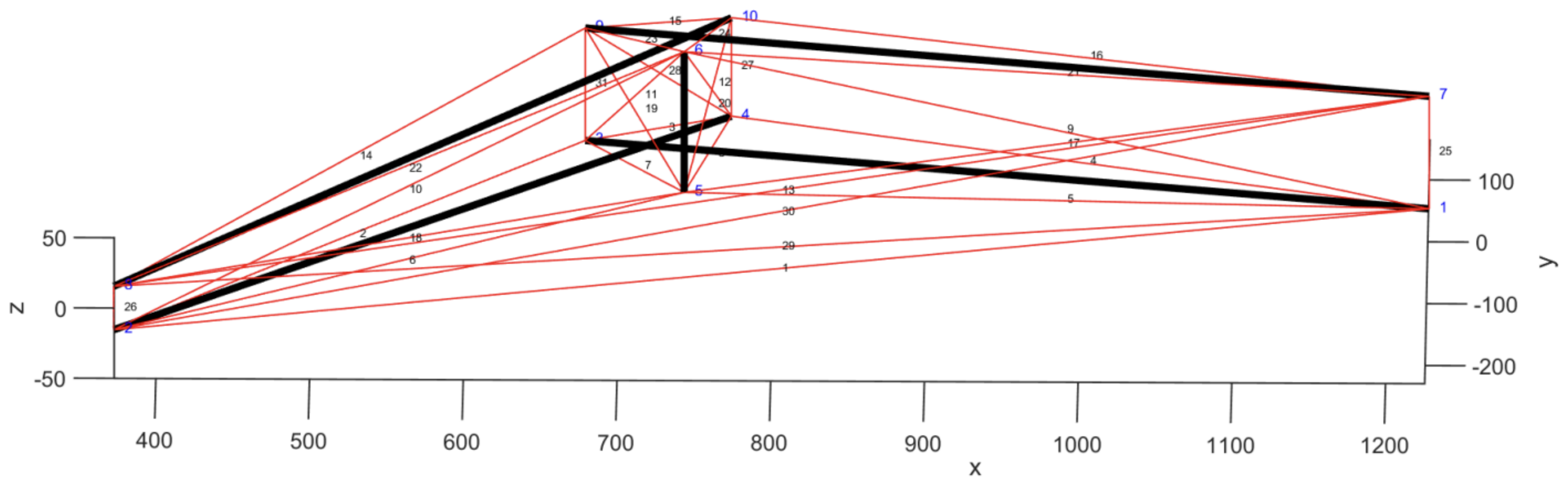
Hooke's law

String Damping

$$\gamma_i = \frac{\|t_i\|}{\|s_i\|} = k_i\left(1 - \frac{s_{0i}}{\|s_i\|}\right) + c_i \frac{s_i^T \dot{s}_i}{\|s_i\|^2}$$

- External Force matrix $W = [w_1 \quad w_2 \quad w_3 \quad \cdots \quad w_{2\beta}]$

$$F = W - TC_s = W - NC_s^T \hat{\gamma} C_s$$



Tensegrity Dynamics

$$[\ddot{B} \quad \ddot{R}_b] \begin{bmatrix} \frac{1}{12}\hat{m}_b & 0 & 0 \\ 0 & \hat{m}_b & 0 \\ 0 & 0 & \hat{m}_s \end{bmatrix} + [B \quad R_b] \begin{bmatrix} -\hat{\lambda} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [\frac{1}{2}FC_b^T \quad 2FC_r^T]$$

Where $\hat{\lambda}$ is the Force density in a bar

$$\hat{\lambda} = -\frac{1}{12}\hat{m}_b\hat{l}^{-2}\lfloor \dot{B}^T\dot{B} \rfloor - \frac{1}{2}\hat{l}^{-2}\lfloor B^T(W - S\hat{\gamma}C_s)C_{nb}^T C_b^T \rfloor$$

$$\ddot{N}M + NK = W$$

$$M = [C_b^T(\tfrac{1}{12}C_b^T\hat{m}C_b + C_r^T\hat{m}C_r)]$$

$$K = [C_s^T\hat{\gamma}C_s - C_b^T\hat{\lambda}C_b]$$

For static, $\ddot{N} = 0$ $\quad NK = W \longrightarrow A\begin{bmatrix} \gamma \\ \lambda \end{bmatrix} = w \quad , \gamma \geq 0$ $\quad$ Linear in γ

where, A is $[(C_s \otimes I_3)\hat{S}, \ \ (-C_b \otimes I_3)\hat{B}]$

Reduced Order Dynamics Equation

$$P = U\Sigma V^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix} \begin{bmatrix} V^T \end{bmatrix}$$

$$\eta = \begin{bmatrix} \eta_1 & \eta_2 \end{bmatrix} \triangleq NU = \begin{bmatrix} NU_1 & NU_2 \end{bmatrix}$$

$$NP = NU\Sigma V^T = \begin{bmatrix} \eta_1 & \eta_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix} \begin{bmatrix} V^T \end{bmatrix} = D$$

$$\eta_1 = DV\Sigma_1^{-1}, \quad \dot{\eta}_1 = 0, \quad \ddot{\eta}_1 = 0$$

$$\ddot{N}M + NK = W + \Omega P^T$$

$$NK(\Omega)M^{-1}U_1 - \boxed{\Omega} P^T M^{-1} U_1 = WM^{-1}U_1$$

Final reduced second order differential equation:

$$\boxed{\ddot{\eta}_2 U_2^T M U_2 + \eta_2 U_2^T K U_2 = W U_2 - \eta_1 U_1^T K U_2}$$

Control for Tensegrity System

- Writing a second order differential equation from Nodal position errors

$$Y = LNR$$

L: axis (x,y,z)
R: Nodes to control

$$E = Y - \bar{Y}$$

$$E = LNR - \bar{Y},$$

$$\ddot{E} + \Psi \dot{E} + \Theta E = 0$$

Control Parameters
ψ, Θ

$$L\ddot{N}R + \Psi L\dot{N}R + \Theta(LNR - \bar{Y}) = 0$$

$$L(W + \Omega P^T - NK)M^{-1}R + \Psi L\dot{N}R + \Theta(LNR - \bar{Y}) = 0$$

$$K = (C_s^{\mathsf{T}} \hat{\gamma} C_s - C_b^{\mathsf{T}} \hat{\lambda} C_b)$$

- We know, $\hat{\lambda} = -\frac{1}{12}\hat{m}_b \hat{l}^{-2} \lfloor \dot{B}^T \dot{B} \rfloor - \frac{1}{2}\hat{l}^{-2} \lfloor B^T (W - S\hat{\gamma}C_s) C_b^T \rfloor$

$$\underbrace{\lambda_i = -m_{bi} l_i^{-2} ||\dot{b}_i||^2 - \frac{1}{2} l_i^{-2} b_i^{\mathsf{T}} (W + \Omega P^{\mathsf{T}}) C_b^{\mathsf{T}} e_i + \frac{1}{2} l_i^{-2} b_i^{\mathsf{T}} S(C_s \widehat{C_b^{\mathsf{T}} e_i}) \gamma}_{\boxed{\lambda = \Lambda \gamma + \tau}}$$

Reduced-Order Controller

- Final equation for calculating control variable (Force density in strings)

$$\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_{n_r} \end{bmatrix} \gamma = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{n_r} \end{bmatrix},$$

where

$$\Gamma_i = LNC_s^{\mathsf{T}}(C_s\widehat{MRe_i}) - LNC_b^{\mathsf{T}}(C_b\widehat{MRe_i})\Lambda$$

$$\mu_i = \mathcal{C}e_i + LNC_b^{\mathsf{T}}C_b\widehat{MRe_i})\tau$$

$$\mathcal{C} = LWU_2M_2^{-1}U_2^{\mathsf{T}}R + \alpha L\dot{\eta}_2U_2^{\mathsf{T}}R + \Theta[L(\eta_1U_1^{\mathsf{T}} + \eta_2U_2^{\mathsf{T}})R - \bar{Y}]$$

$$M = U_2M_s^{-1}U_2^{\mathsf{T}}$$

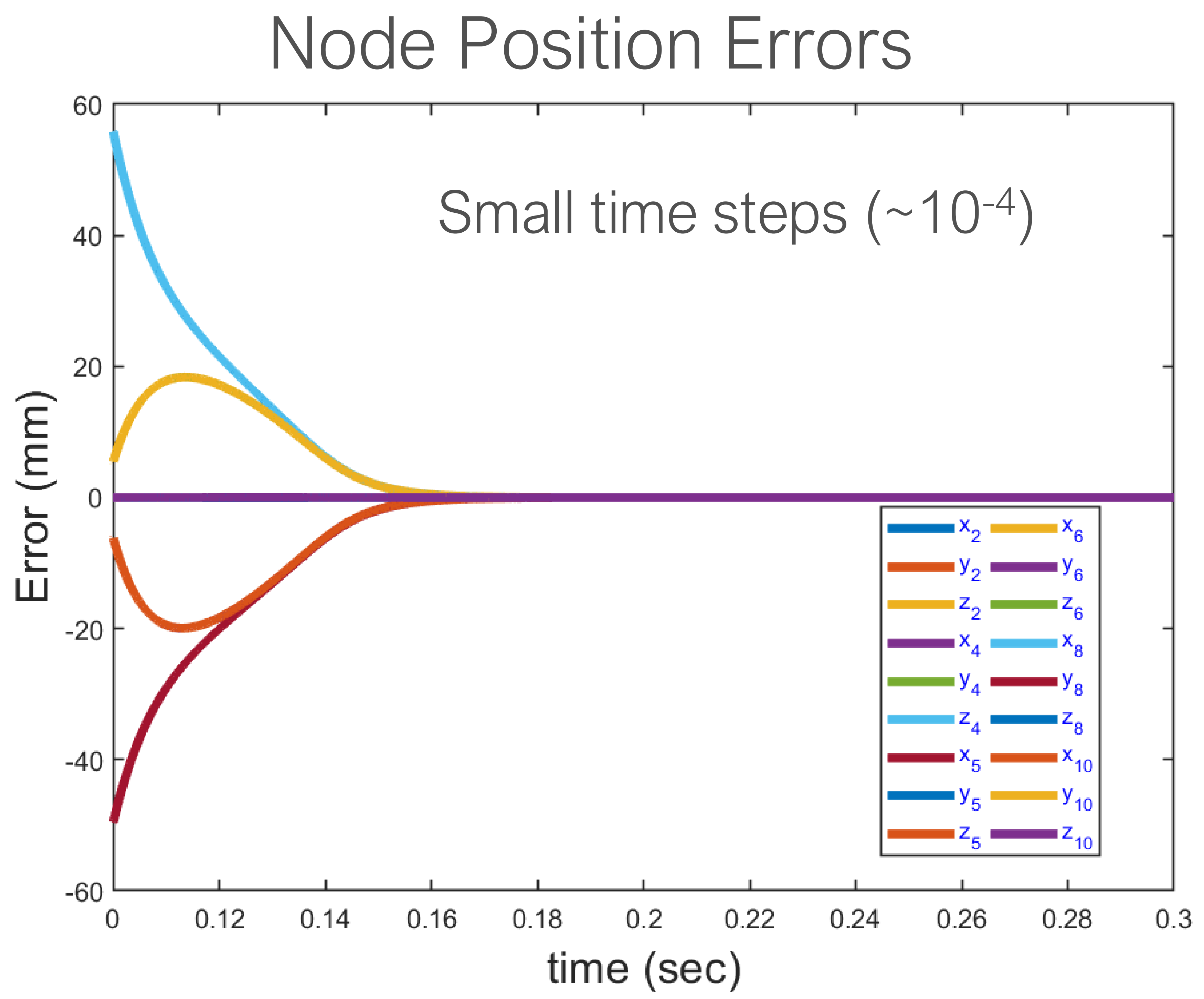
$$i = 1, 2 \ldots n_r$$

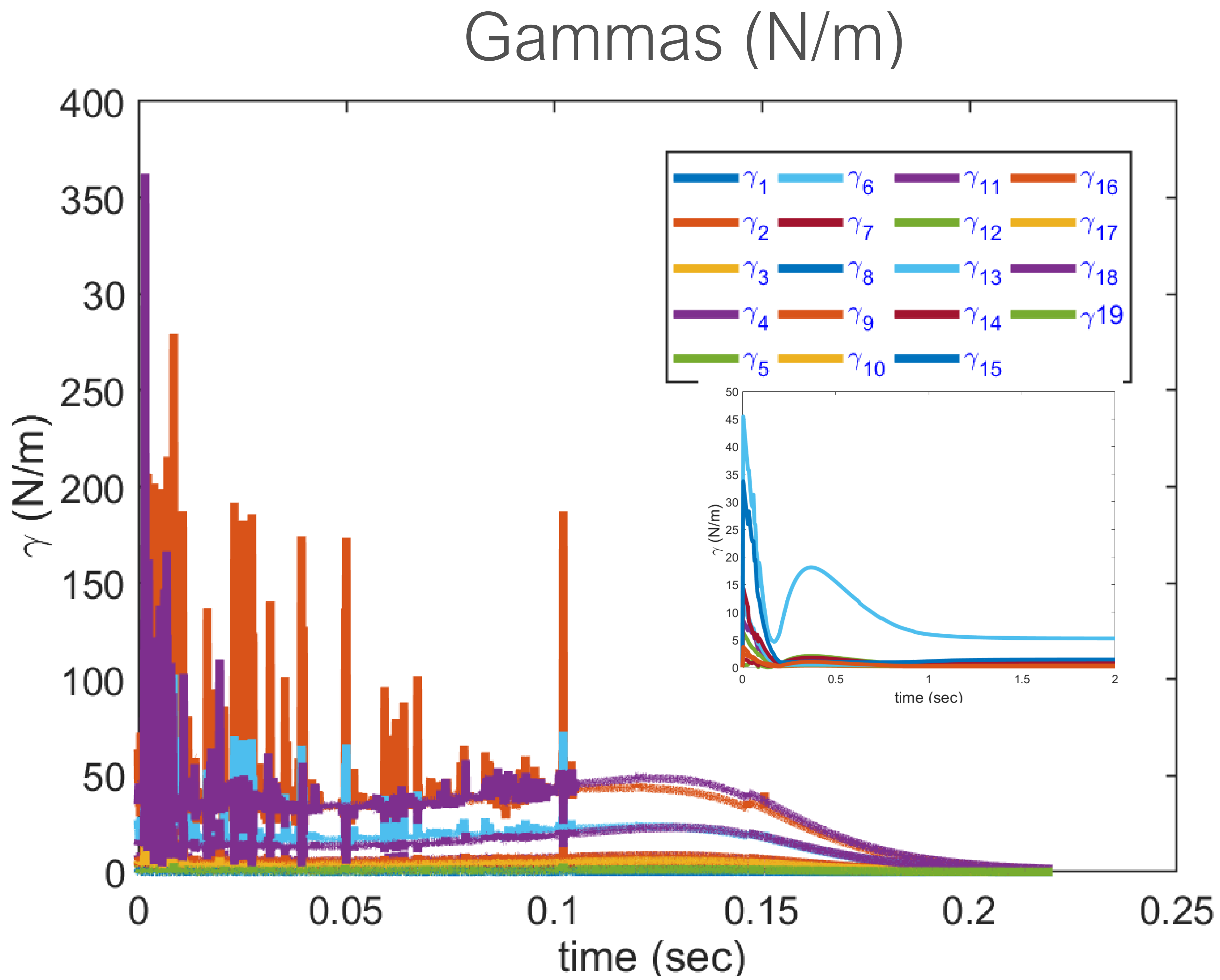
$$\underset{\gamma}{Min} \quad ||\Gamma\gamma - \mu||_2$$

$$s.t. \ \gamma \geq 0$$

MATLAB lsqlin is used to find γ for every time step for finding least square solution to the problem $\Gamma\gamma = \mu$ for $\gamma \geq 0$.

Swing Phase Simulation

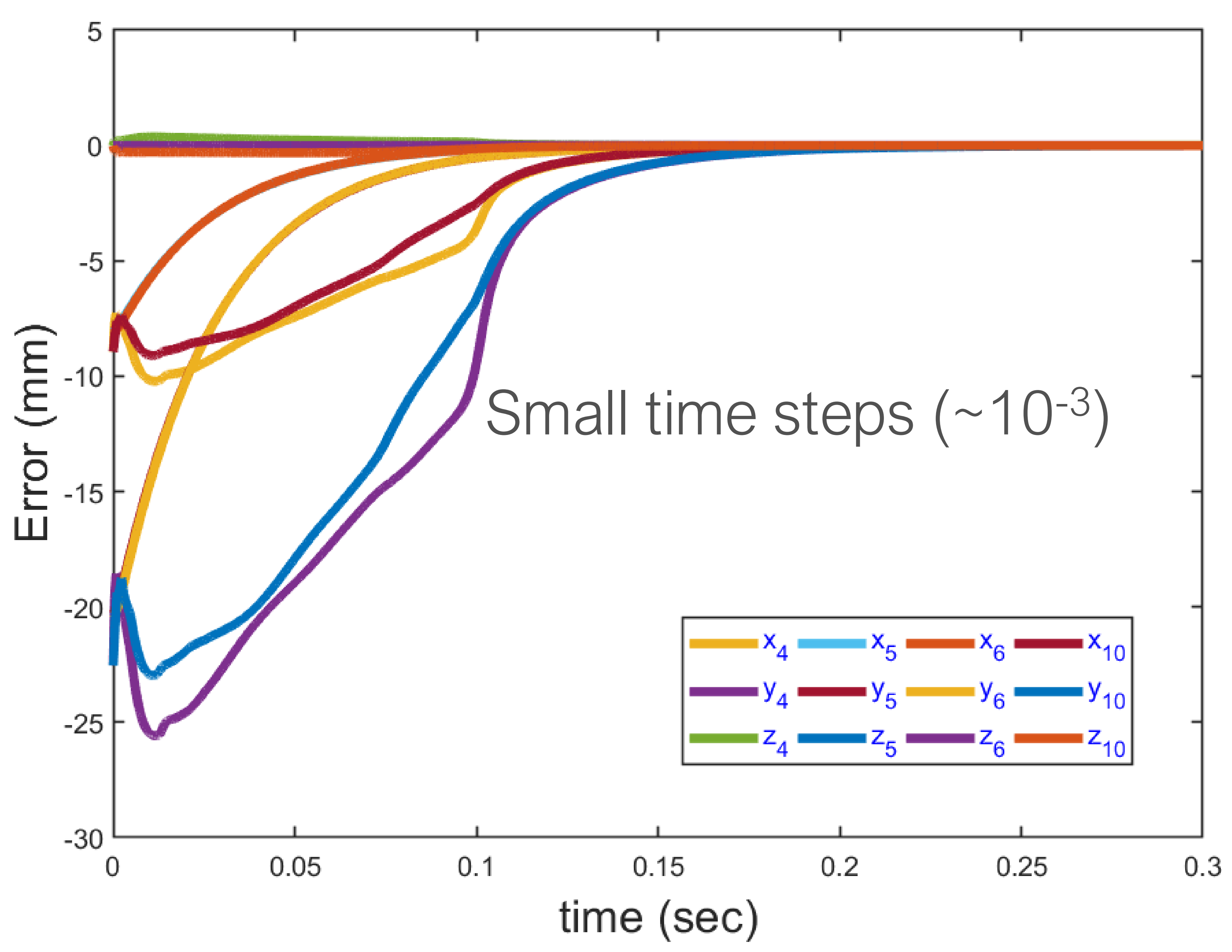


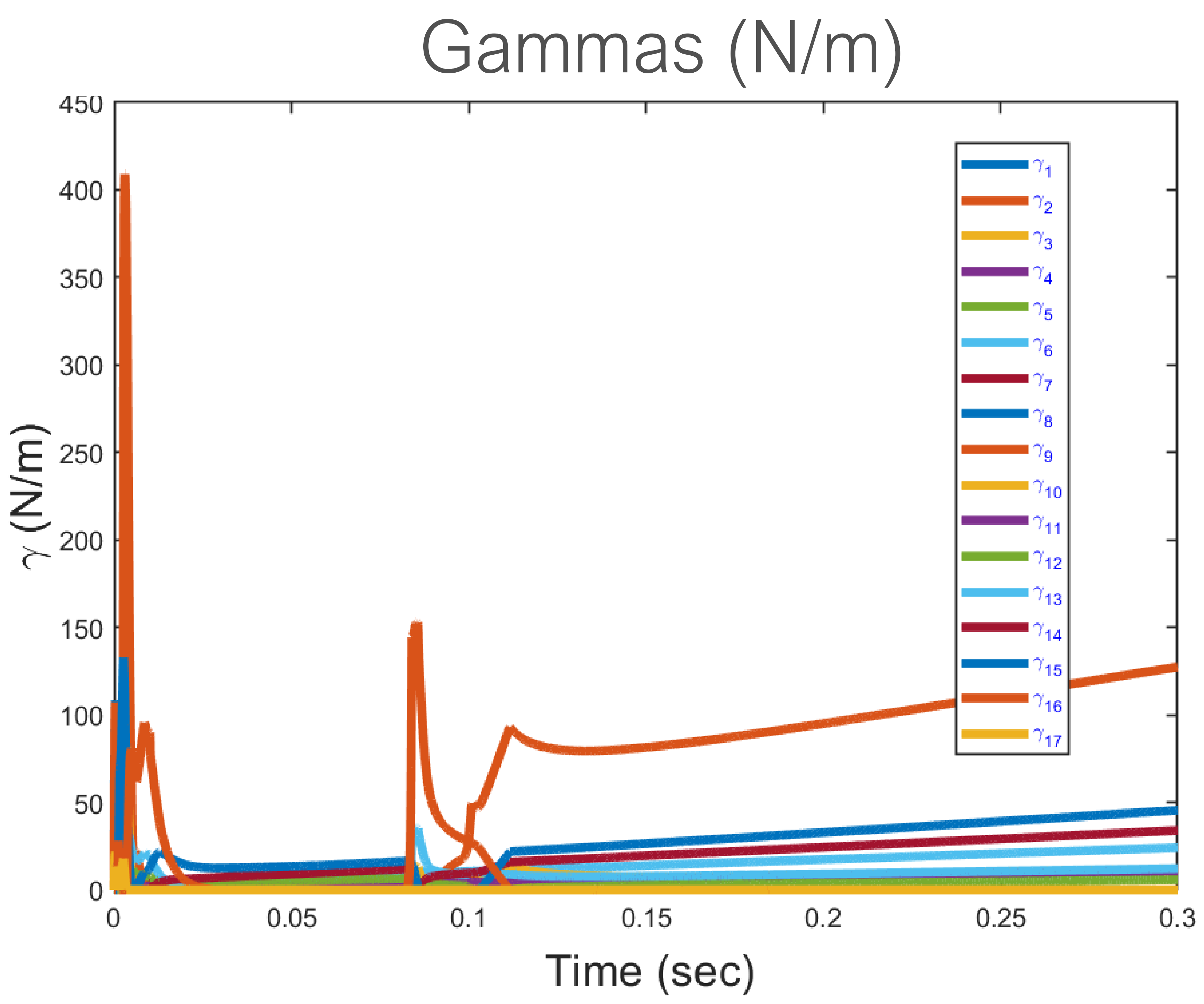


Strings with maximum force density are $S_{13},\ S_{22},\ S_{23},\ S_{25}$

Stance Phase Simulation

Node Position Errors





Strings with maximum force density are S_9, S_{15}, S_{14}, S_6

ProTense

Human Attitude Control in Space

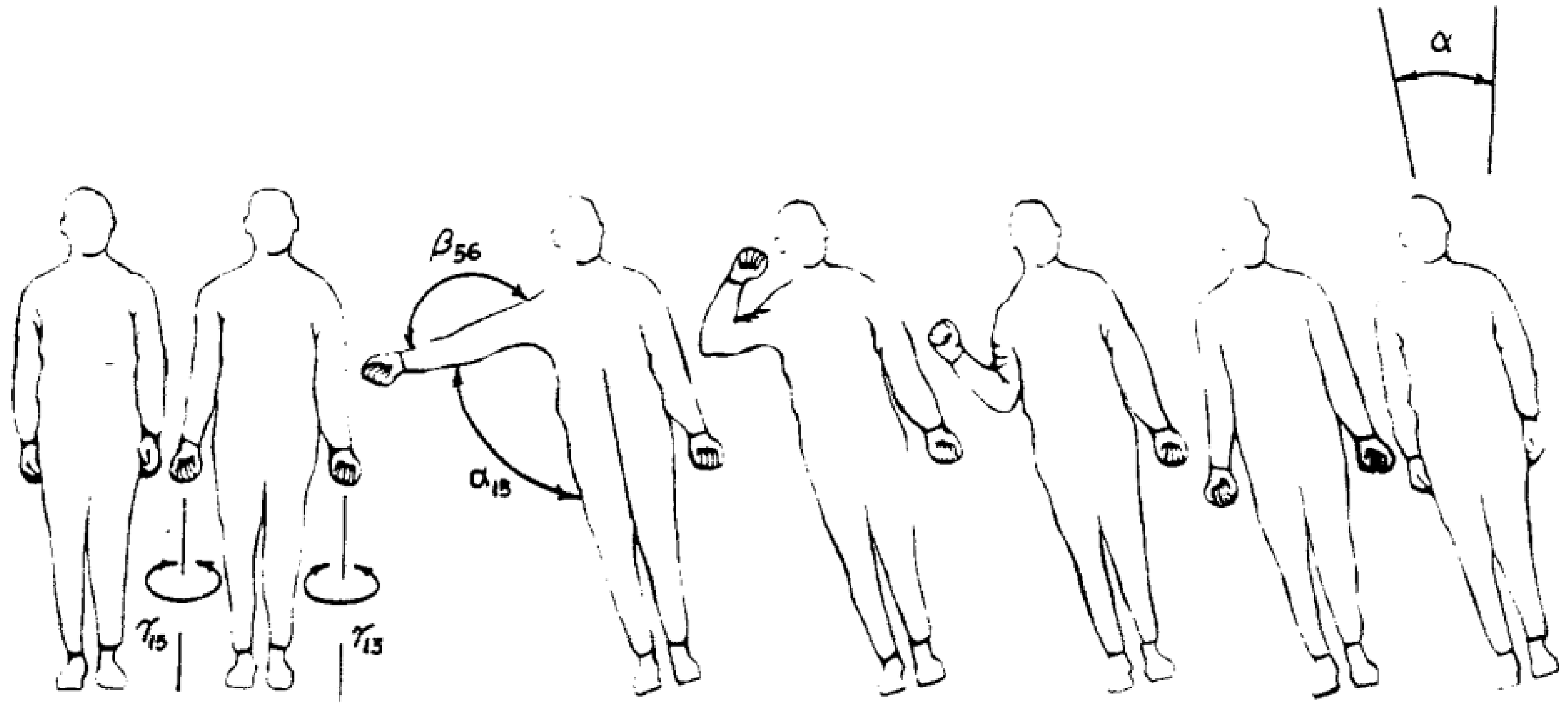
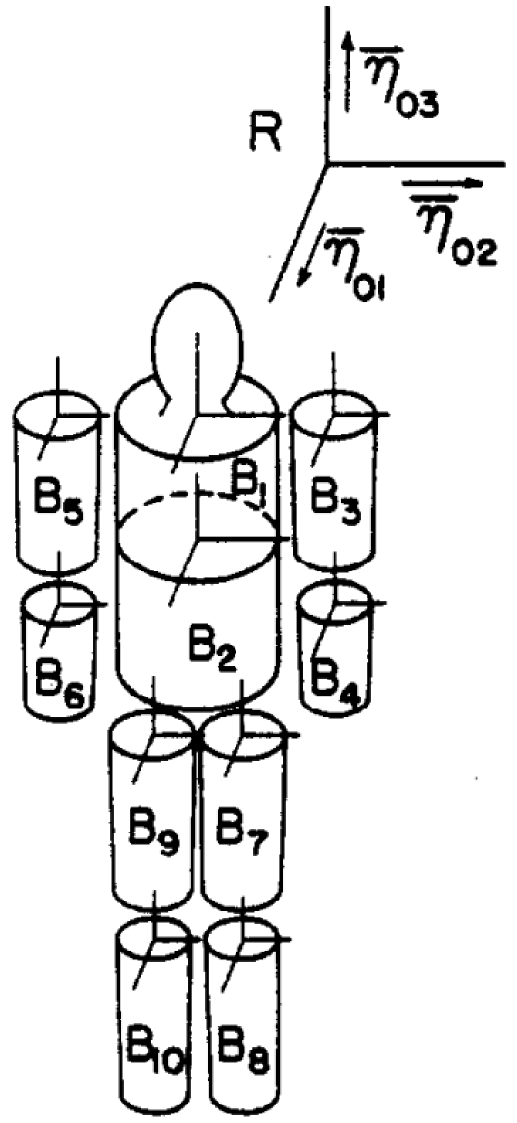


Fig. 5. Yaw maneuver.



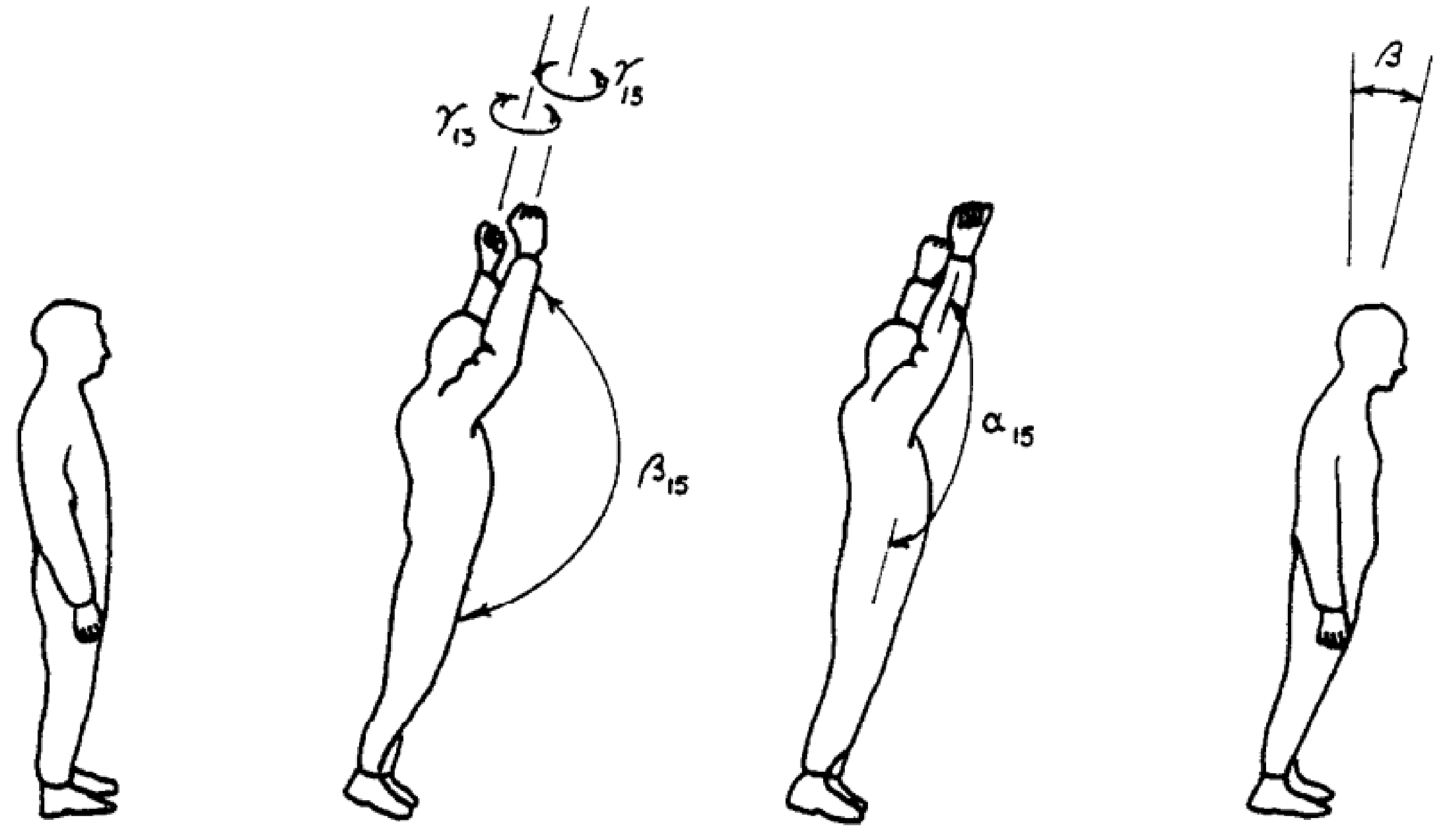


Fig. 6. Pitch maneuver.

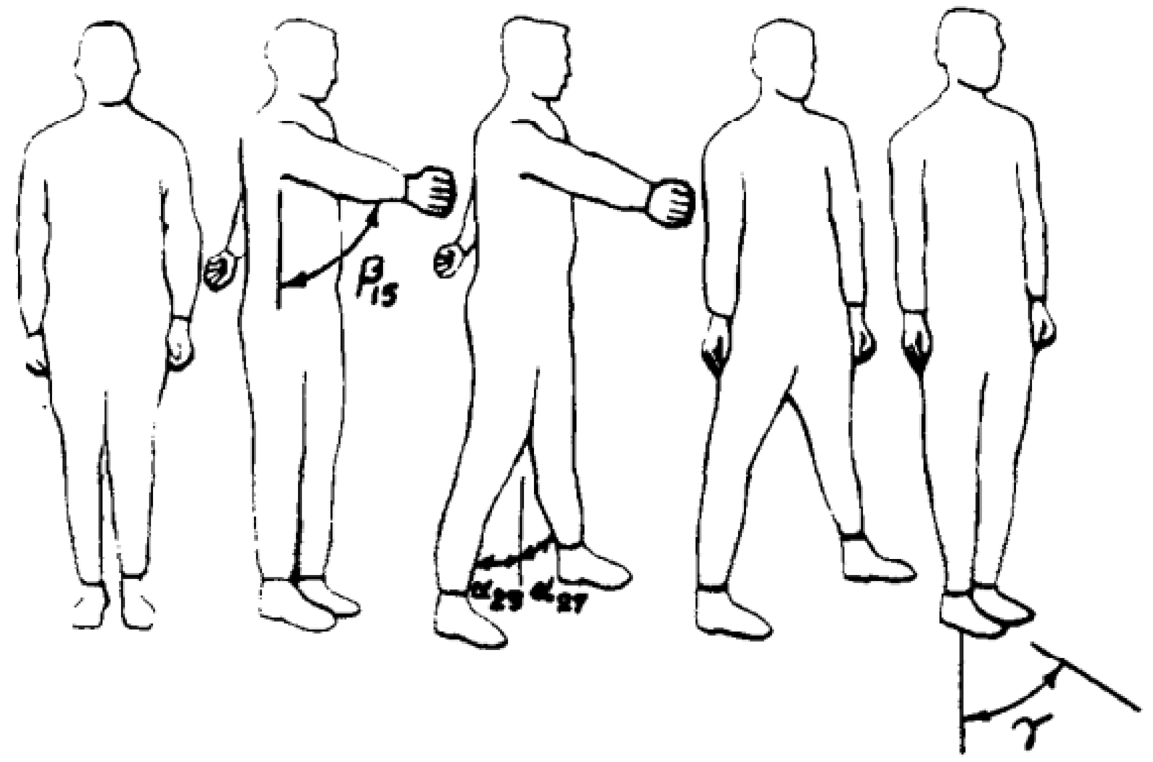


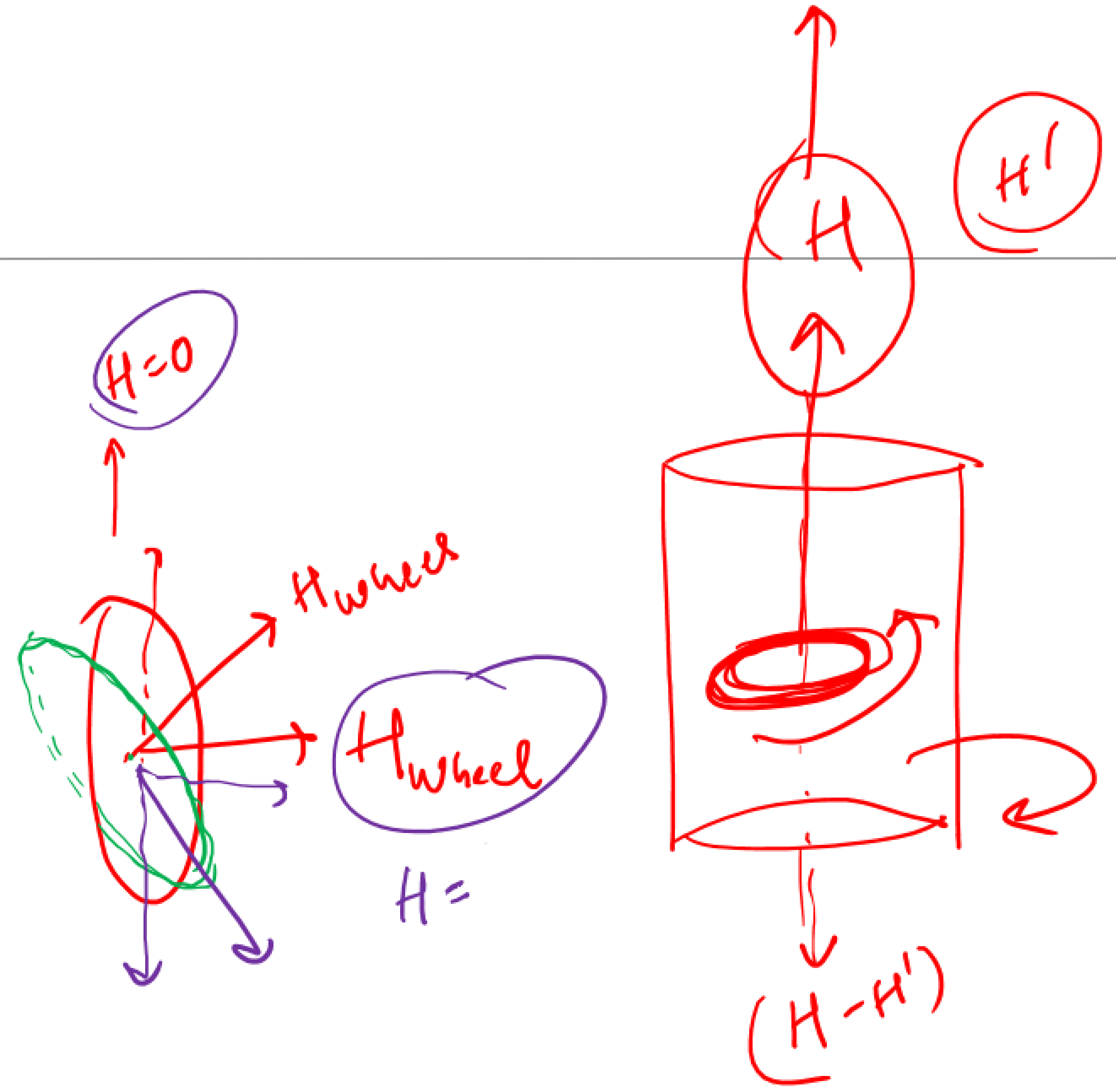
Fig. 7. Roll maneuver.

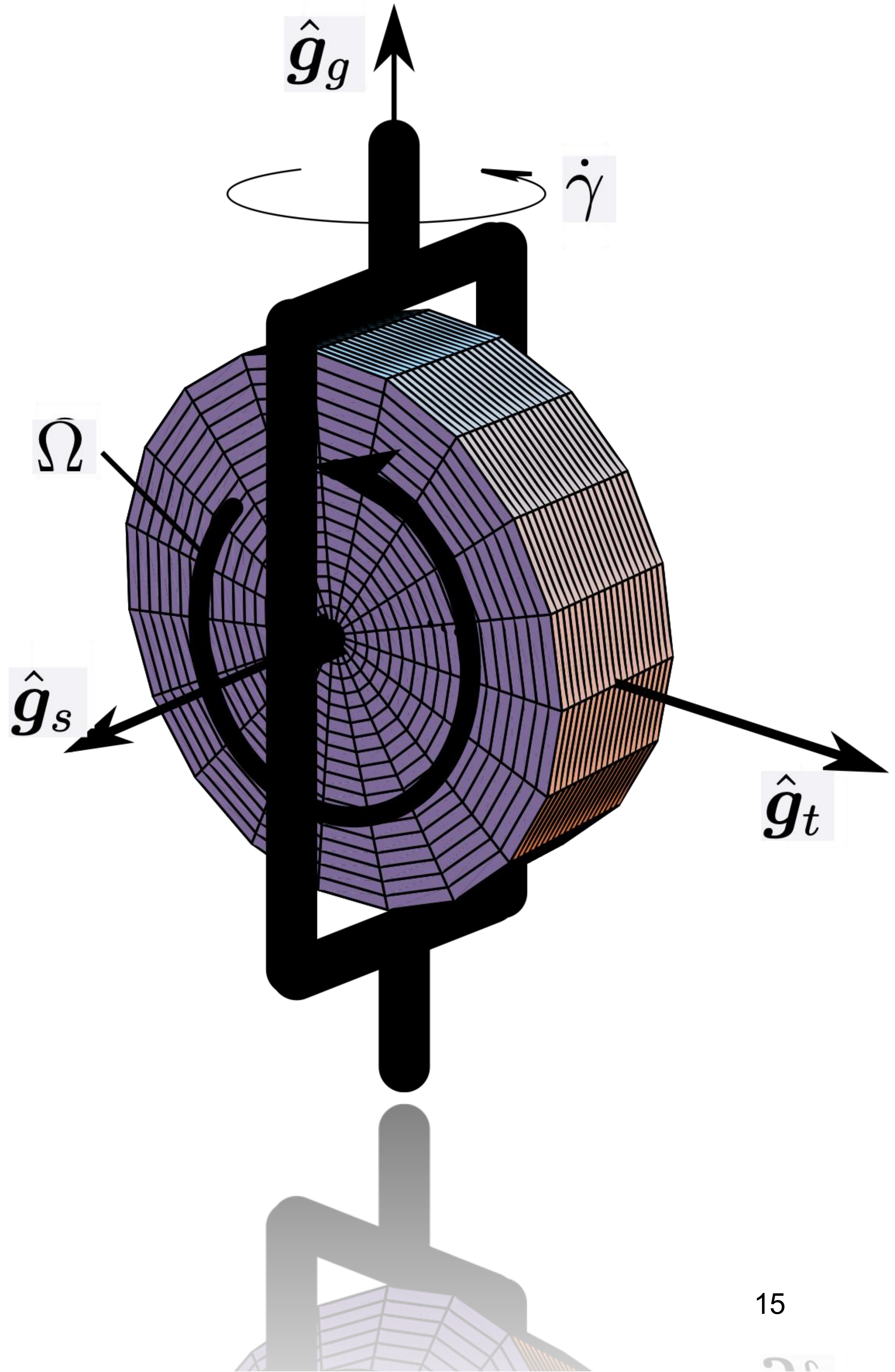
Momentum Control Devices

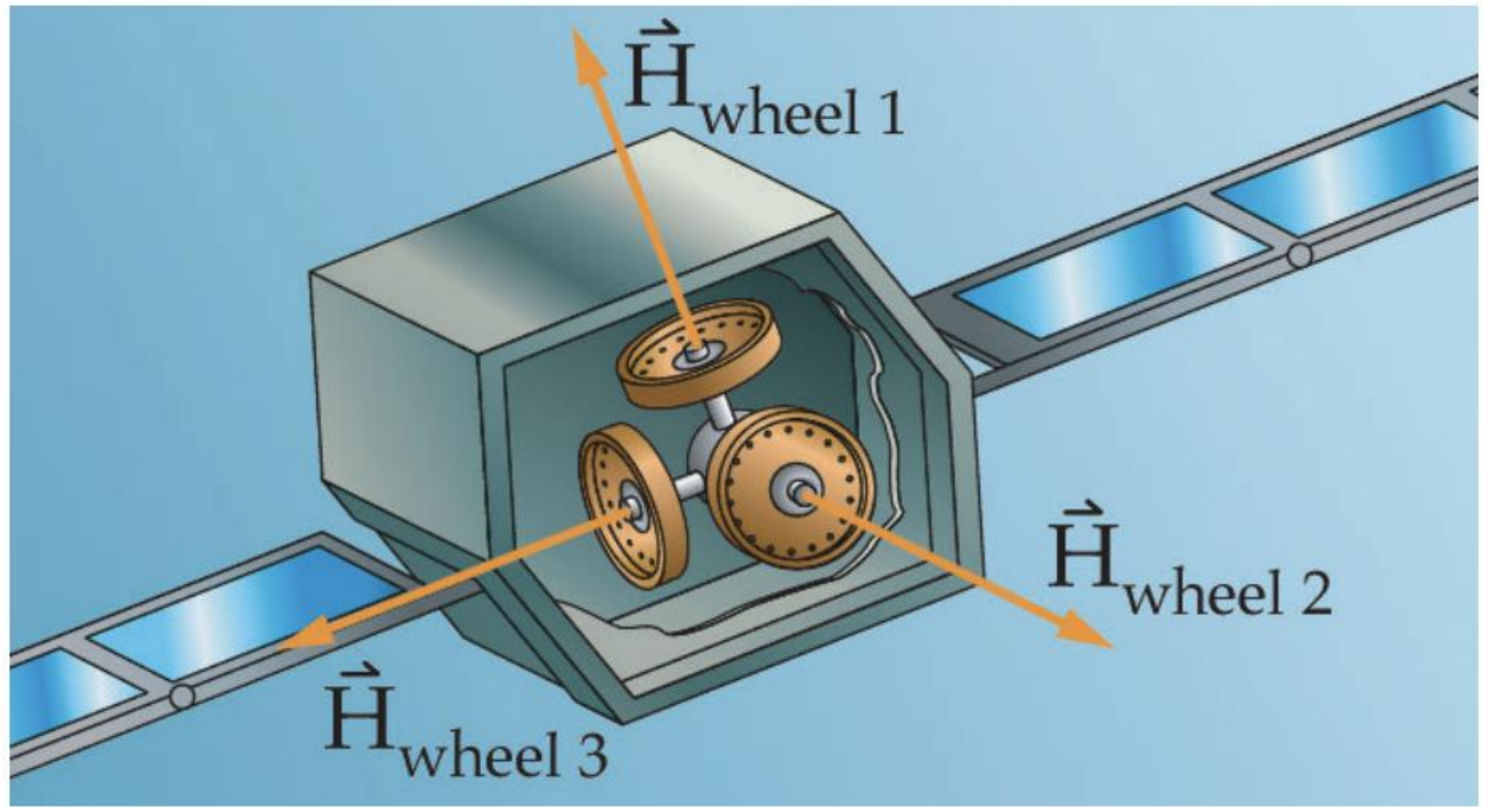
Spinning hardware “thingies” to rotate the spacecraft...

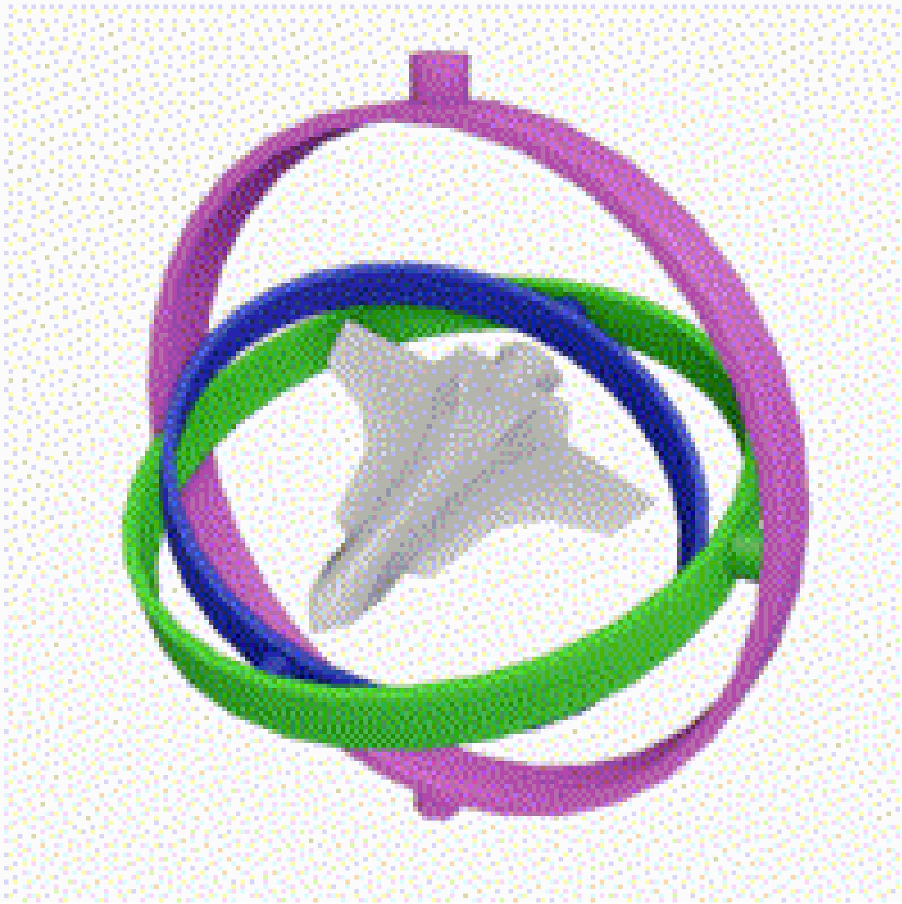
Outline

- Momentum Control Devices:
  - Reaction Wheels
  - CMGs
  - Variable Speed CMGs
- Single VSCMG EOM
- Momentum Device Control
  - Overview of RW control solution









Comparison Momentum Control Devices

Feature	Reaction Wheel (RW)	Control Moment Gyro (CMG)	VSCMG
Torque Level	Low	High	Medium-High
Control Input	Wheel Speed	Gimbal Angle	Both
Power Use	Moderate	Lower for same torque	Moderate
Complexity	Low	Medium	High
Singularities	None	Yes	Fewer / manageable
Use Case	Precision pointing	Agile maneuvers (e.g., ISS)	Agile + compact missions

Equations of Motion

Let’s learn to be one with the truth of gyroscopics...

Spacecraft with 1 VSCMG

- A Variable-Speed CMG is a classical CMG device where the disk speed is left to be variable.
- Think of a VSCMG device as a hybrid CMG/RW.
- Convenient when developing the equations of motion, since we get both the CMG and RW equations of motion by doing the work only once!!
- Researchers have started to look into actually building and flying a VSCMG devices.
  - Avoids classical CMG singularities
  - Highly redundant system (more robust to component failure)
  - Can be used as a combined power storage/attitude control device.

Battle Plan...

- To derive the equations of motion of a spacecraft with a single VSCMG, we recall Euler's equation

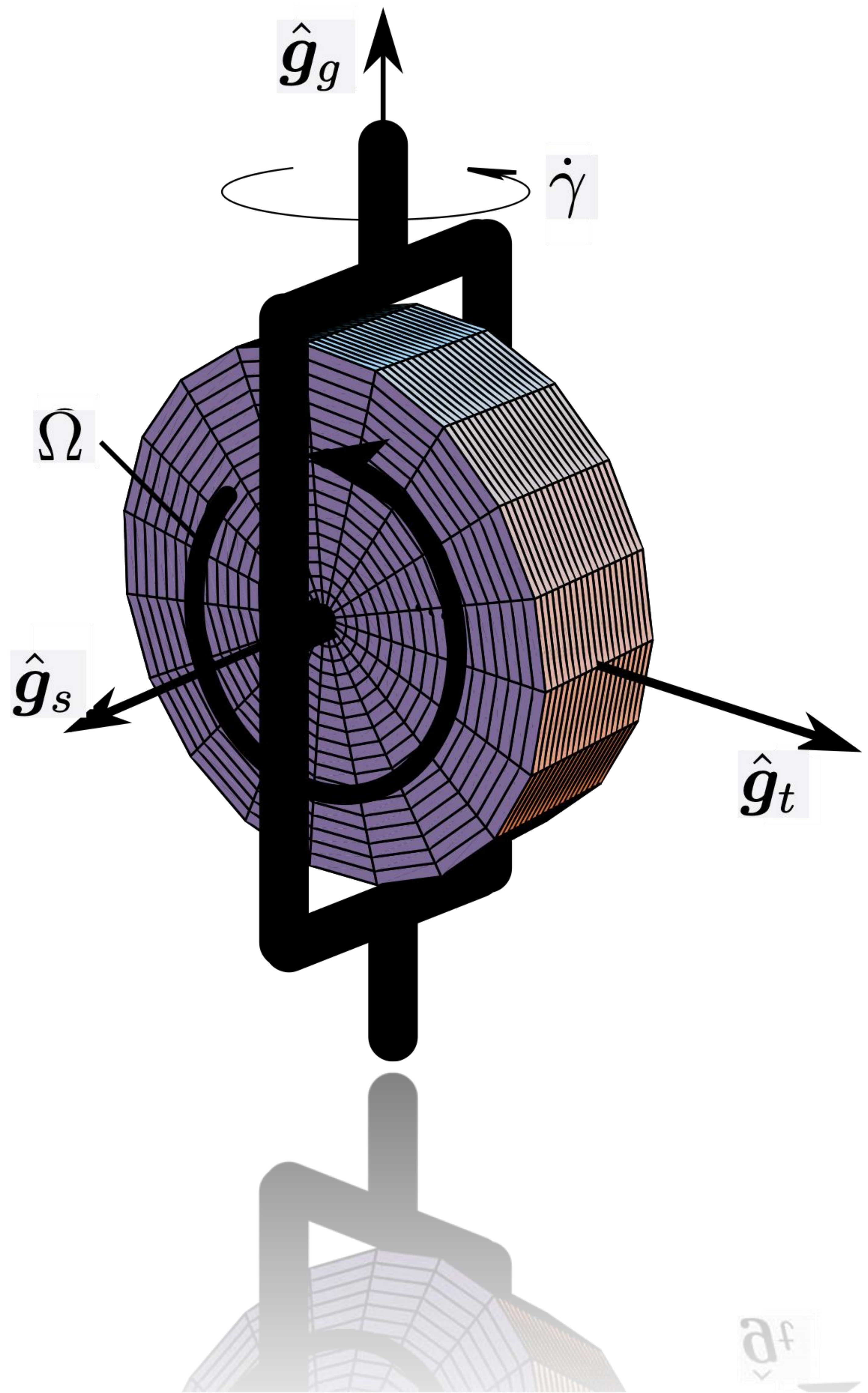
$$\dot{\boldsymbol{H}} = \boldsymbol{L}$$

- We will need to find the total angular momentum vector *H* for the combined spacecraft/VSCMG system. Once we have this expression, we can then differentiate it to get the desired equations of motion.
- To manage all this algebra, we will break up the whole system into the spacecraft part, the CMG momentum and the RW momentum.

VSCMG Frames

- The VSCMG spin axis is $\hat{\boldsymbol{g}}_s$
- The gimbal axis is $\hat{\boldsymbol{g}}_g$
- The disk spin rate is $\Omega(t)$
- The gimbal rate is $\dot{\gamma}(t)$
- The gimbal coordinate frame *G* is

$$\mathcal{G} : \{\hat{\boldsymbol{g}}_s, \hat{\boldsymbol{g}}_t, \hat{\boldsymbol{g}}_g\}$$



VSCMG Frames

- Note that the gimbal axis is fixed with respect to the spacecraft body frame *B*.
- The gimbal frame *G* angular velocity is

$\omega_{G/B} + \omega_{B/N} = \omega_{G/N}$

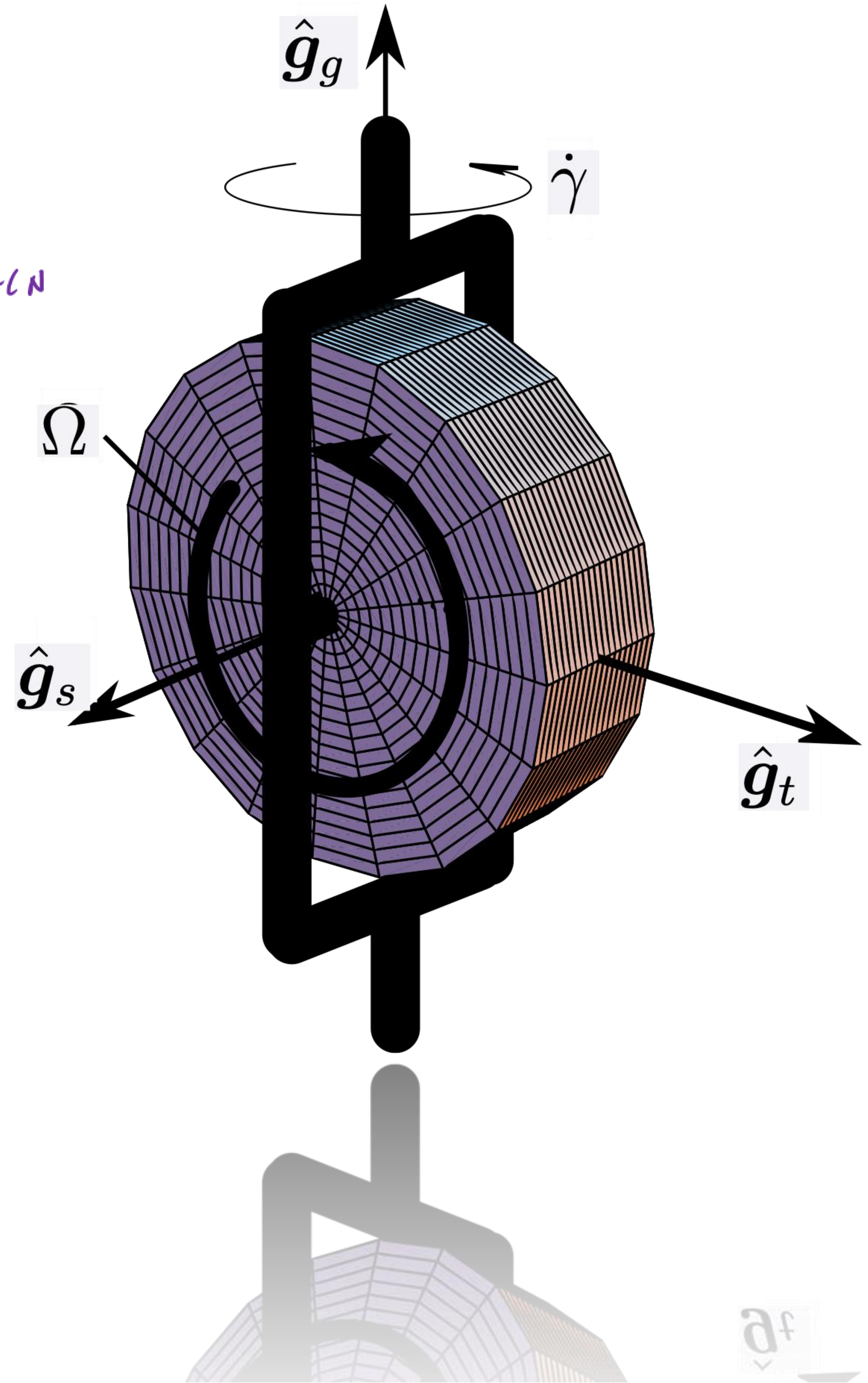
$$\boldsymbol{\omega}_{\mathcal{G}/\mathcal{B}} = \dot{\gamma}\hat{\boldsymbol{g}}_g$$

- Let *W* be a frame that tracks the motion of the reaction wheel.

$$\mathcal{W} : \{\hat{\boldsymbol{g}}_s, \hat{\boldsymbol{w}}_t, \hat{\boldsymbol{w}}_g\}$$

- It's angular velocity is

$$\boldsymbol{\omega}_{\mathcal{W}/\mathcal{G}} = \Omega\hat{\boldsymbol{g}}_s$$



VSCMG Inertias

- Let the gimbal frame inertia be

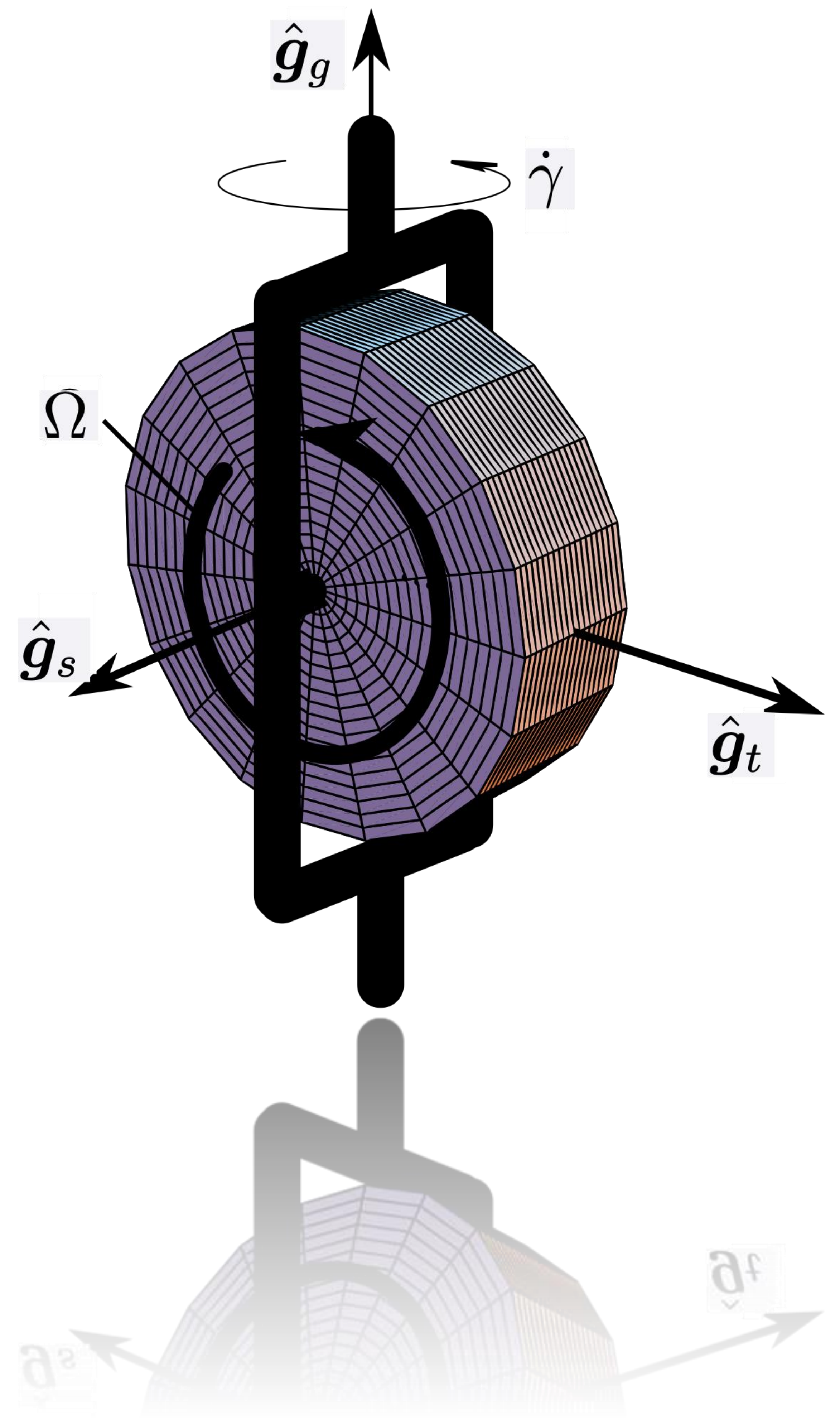
$$[I_G] = {}^{\mathcal{G}}[I_G] = {}^{\mathcal{G}}\begin{bmatrix} I_{G_s} & 0 & 0 \\ 0 & I_{G_t} & 0 \\ 0 & 0 & I_{G_g} \end{bmatrix}$$

- The wheel (disk) inertia is

$$[I_W] = {}^{\mathcal{W}}[I_W] = {}^{\mathcal{W}}\begin{bmatrix} I_{W_s} & 0 & 0 \\ 0 & I_{W_t} & 0 \\ 0 & 0 & I_{W_t} \end{bmatrix}$$

- Due to symmetry of the disk, we find that

$${}^{\mathcal{W}}[I_W] = {}^{\mathcal{G}}[I_W]$$



- Assuming the gimbal frame unit vectors are expressed in body frame vector components, then the rotation matrix [BG] can be expressed through

$$[BG] = \left[\hat{\boldsymbol{g}}_s \ \hat{\boldsymbol{g}}_t \ \hat{\boldsymbol{g}}_g\right]$$

- The gimbal frame and disk inertias (which were given in gimbal frame components), can be written in body frame components using

$$^{\mathcal{B}}[I_G] = [BG]^{\mathcal{G}}[I_G][BG]^T = I_{G_s}\hat{\boldsymbol{g}}_s\hat{\boldsymbol{g}}_s^T + I_{G_t}\hat{\boldsymbol{g}}_t\hat{\boldsymbol{g}}_t^T + I_{G_g}\hat{\boldsymbol{g}}_g\hat{\boldsymbol{g}}_g^T$$

$$^{\mathcal{B}}[I_W] = [BG]^{\mathcal{G}}[I_W][BG]^T = I_{W_s}\hat{\boldsymbol{g}}_s\hat{\boldsymbol{g}}_s^T + I_{W_t}\hat{\boldsymbol{g}}_t\hat{\boldsymbol{g}}_t^T + I_{W_g}\hat{\boldsymbol{g}}_g\hat{\boldsymbol{g}}_g^T$$

Angular Momentum...

- We are now ready to express the total angular momentum of the system using

$$\boldsymbol{H} = \boldsymbol{H}_B + \boldsymbol{H}_G + \boldsymbol{H}_W$$

- $\boldsymbol{H}_B$ is the angular momentum of the spacecraft itself, $\boldsymbol{H}_G$ is the angular momentum of the gimbal frame, while $\boldsymbol{H}_W$ is the angular momentum of the spinning disk.

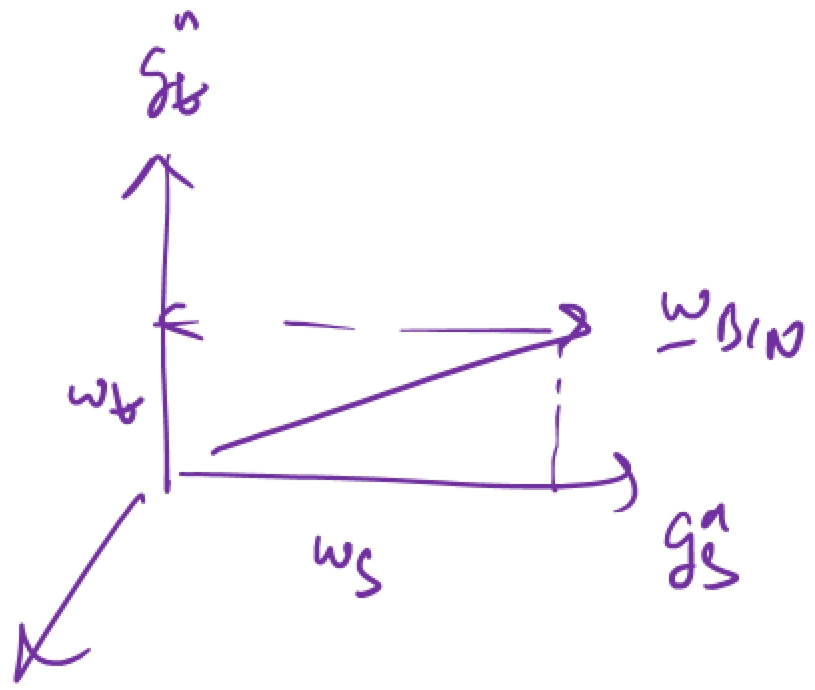
- The spacecraft angular momentum is simply that of a rigid body:

$$\boldsymbol{H}_B = [I_s]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$$

- The inertial angular momentum of the rigid gimbal frame is

$$\boldsymbol{H}_G = [I_G]\boldsymbol{\omega}_{\mathcal{G}/\mathcal{N}}$$

- where $\boldsymbol{\omega}_{\mathcal{G}/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{G}/\mathcal{B}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$. This can now be rewritten as



$$\boldsymbol{H}_G = \left(I_{G_s}\hat{\boldsymbol{g}}_s\hat{\boldsymbol{g}}_s^T + I_{G_t}\hat{\boldsymbol{g}}_t\hat{\boldsymbol{g}}_t^T + I_{G_g}\hat{\boldsymbol{g}}_g\hat{\boldsymbol{g}}_g^T\right)\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + I_{G_g}\dot{\gamma}\hat{\boldsymbol{g}}_g$$

- Let use introduce the angular velocity components taken along the gimbal frame axis directions:

$$\omega_s = \hat{\boldsymbol{g}}_s^T\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \qquad \omega_t = \hat{\boldsymbol{g}}_t^T\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \qquad \omega_g = \hat{\boldsymbol{g}}_g^T\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$$

$${}^{\mathcal{G}}\boldsymbol{\omega}_{B/N} = \omega_s\hat{\boldsymbol{g}}_s + \omega_t\hat{\boldsymbol{g}}_t + \omega_g\hat{\boldsymbol{g}}_g$$

- This allows us to write the gimbal frame angular momentum expression as

$$\boldsymbol{H}_G = I_{G_s}\omega_s\hat{\boldsymbol{g}}_s + I_{G_t}\omega_t\hat{\boldsymbol{g}}_t + I_{G_g}\left(\omega_g + \dot{\gamma}\right)\hat{\boldsymbol{g}}_g$$

- The inertial angular momentum of the disk is

$$\boldsymbol{H}_W = [I_W]\boldsymbol{\omega}_{\mathcal{W}/\mathcal{N}}$$

- where $\boldsymbol{\omega}_{\mathcal{W}/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{W}/\mathcal{G}} + \boldsymbol{\omega}_{\mathcal{G}/\mathcal{B}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$

- The momentum expression can be expanded using

$$\boldsymbol{H}_W = [I_W]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_W]\boldsymbol{\omega}_{\mathcal{G}/\mathcal{B}} + [I_W]\boldsymbol{\omega}_{\mathcal{W}/\mathcal{G}}$$

It is implied that all vectors are added with components in the same frame.

- The first term can be written as

$$\begin{aligned} [I_W]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} &= \left(I_{W_s}\hat{\boldsymbol{g}}_s\hat{\boldsymbol{g}}_s^T + I_{W_t}\hat{\boldsymbol{g}}_t\hat{\boldsymbol{g}}_t^T + I_{W_t}\hat{\boldsymbol{g}}_g\hat{\boldsymbol{g}}_g^T\right)\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ &= I_{W_s}\omega_s\hat{\boldsymbol{g}}_s + I_{W_t}\omega_t\hat{\boldsymbol{g}}_t + I_{W_g}\omega_g\hat{\boldsymbol{g}}_g \end{aligned}$$

- The second two terms can be written as

$$[I_W]\boldsymbol{\omega}_{\mathcal{G}/\mathcal{B}} = {}^{\mathcal{G}}\begin{bmatrix} I_{W_s} & 0 & 0 \\ 0 & I_{W_t} & 0 \\ 0 & 0 & I_{W_t} \end{bmatrix} {}^{\mathcal{G}}\begin{pmatrix} 0 \\ 0 \\ \dot{\gamma} \end{pmatrix} = I_{W_t}\dot{\gamma}\hat{\boldsymbol{g}}_g$$

$$[I_W]\boldsymbol{\omega}_{\mathcal{W}/\mathcal{G}} = {}^{\mathcal{W}}\begin{bmatrix} I_{W_s} & 0 & 0 \\ 0 & I_{W_t} & 0 \\ 0 & 0 & I_{W_t} \end{bmatrix} {}^{\mathcal{W}}\begin{pmatrix} \Omega \\ 0 \\ 0 \end{pmatrix} = I_{W_s}\Omega\hat{\boldsymbol{g}}_s$$

- Combining all these results, the spinning wheel inertial angular momentum is written as

$$\boldsymbol{H}_W = I_{W_s}\left(\omega_s + \Omega\right)\hat{\boldsymbol{g}}_s + I_{W_t}\omega_t\hat{\boldsymbol{g}}_t + I_{W_t}\left(\omega_g + \dot{\gamma}\right)\hat{\boldsymbol{g}}_g$$

Some final preparation...

- Before we begin to differentiate the system angular momentum vectors, we need to establish some useful relationships.

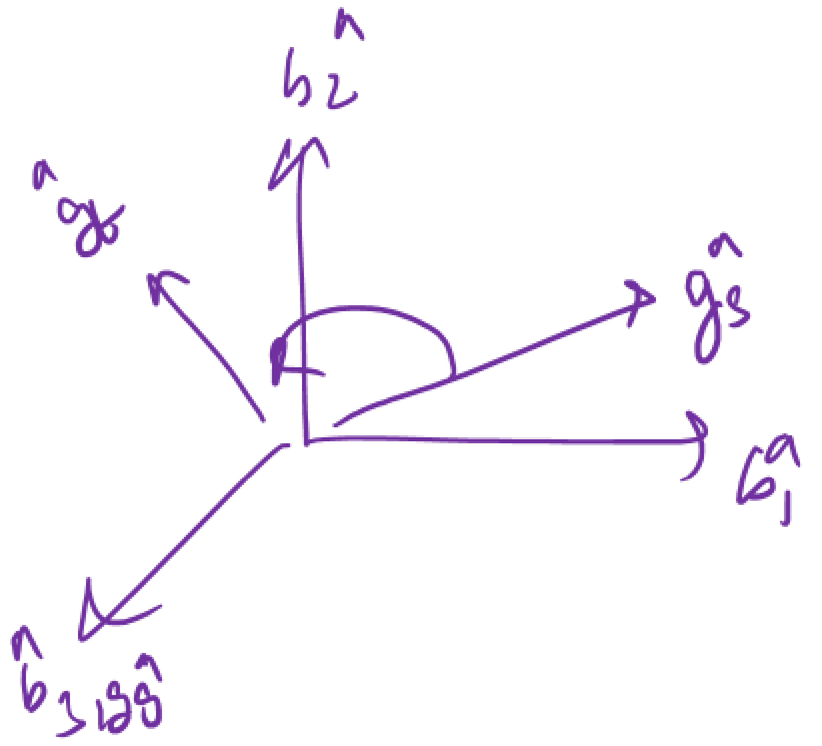
- The gimbal frame direction vectors can be written in terms of their initial orientations as

$$\hat{\boldsymbol{g}}_s(t) = \cos(\gamma(t) - \gamma_0)\,\hat{\boldsymbol{g}}_s(t_0) + \sin(\gamma(t) - \gamma_0)\,\hat{\boldsymbol{g}}_t(t_0)$$

$$\hat{\boldsymbol{g}}_t(t) = -\sin(\gamma(t) - \gamma_0)\,\hat{\boldsymbol{g}}_s(t_0) + \cos(\gamma(t) - \gamma_0)\,\hat{\boldsymbol{g}}_t(t_0)$$

$$\hat{\boldsymbol{g}}_g(t) = \hat{\boldsymbol{g}}_g(t_0)$$

$$M_3 = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



- Note that the B frame derivatives of the gimbal frame unit vectors are

$$\frac{^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\hat{\boldsymbol{g}}_s) = \dot{\gamma}\hat{\boldsymbol{g}}_t \qquad \frac{^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\hat{\boldsymbol{g}}_t) = -\dot{\gamma}\hat{\boldsymbol{g}}_s \qquad \frac{^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\hat{\boldsymbol{g}}_g) = 0$$

- The inertial derivatives of these vectors are

$$\dot{\hat{\boldsymbol{g}}}_s = \frac{^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\hat{\boldsymbol{g}}_s) + \boldsymbol{\omega} \times \hat{\boldsymbol{g}}_s = (\dot{\gamma} + \omega_g)\,\hat{\boldsymbol{g}}_t - \omega_t\hat{\boldsymbol{g}}_g$$

$$\dot{\hat{\boldsymbol{g}}}_t = \frac{^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\hat{\boldsymbol{g}}_t) + \boldsymbol{\omega} \times \hat{\boldsymbol{g}}_t = -\,(\dot{\gamma} + \omega_g)\,\hat{\boldsymbol{g}}_s + \omega_s\hat{\boldsymbol{g}}_g$$

$$\dot{\hat{\boldsymbol{g}}}_g = \frac{^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\hat{\boldsymbol{g}}_g) + \boldsymbol{\omega} \times \hat{\boldsymbol{g}}_g = \omega_t\hat{\boldsymbol{g}}_s - \omega_s\hat{\boldsymbol{g}}_t$$

use ${}^{\mathcal{G}}\boldsymbol{\omega} = \omega_s\hat{\boldsymbol{g}}_s + \omega_t\hat{\boldsymbol{g}}_t + \omega_g\hat{\boldsymbol{g}}_g$ to derive this result.

- Finally, the following expressions are derived:

$$\begin{aligned}
\dot{\omega}_s &= \dot{\hat{\boldsymbol{g}}}_s^T \boldsymbol{\omega} + \hat{\boldsymbol{g}}_s^T \dot{\boldsymbol{\omega}} = \dot{\gamma}\omega_t + \hat{\boldsymbol{g}}_s^T \dot{\boldsymbol{\omega}} \\
\dot{\omega}_t &= \dot{\hat{\boldsymbol{g}}}_t^T \boldsymbol{\omega} + \hat{\boldsymbol{g}}_t^T \dot{\boldsymbol{\omega}} = -\dot{\gamma}\omega_s + \hat{\boldsymbol{g}}_t^T \dot{\boldsymbol{\omega}} \\
\dot{\omega}_g &= \dot{\hat{\boldsymbol{g}}}_g^T \boldsymbol{\omega} + \hat{\boldsymbol{g}}_g^T \dot{\boldsymbol{\omega}} = \hat{\boldsymbol{g}}_g^T \dot{\boldsymbol{\omega}}
\end{aligned}$$

- The following combined gimbal and spinning disk inertia matrix will be useful to simplify some results:

$$[J] = [I_G] + [I_W] = {}^{\mathcal{G}}\begin{bmatrix} J_s & 0 & 0 \\ 0 & J_t & 0 \\ 0 & 0 & J_g \end{bmatrix}$$

And now, the fun...

- At this point we are ready to compute the terms in Euler's equation $\dot{\boldsymbol{H}} = \boldsymbol{L}$ We have all the required expressions and need to simply carry out the required algebra.

- Taking the inertial derivative of the spinning wheel angular momentum expression $\boldsymbol{H}_W$, we find

$$\begin{aligned}\dot{\boldsymbol{H}}_W = \hat{\boldsymbol{g}}_s \left[I_{W_s}\left(\dot{\Omega} + \hat{\boldsymbol{g}}_s^T\dot{\boldsymbol{\omega}} + \dot{\gamma}\omega_t\right)\right] + \hat{\boldsymbol{g}}_t\left[I_{W_s}\left(\dot{\gamma}(\omega_s + \Omega) + \Omega\omega_g\right)\right.\\ \left. + I_{W_t}\hat{\boldsymbol{g}}_t^T\dot{\boldsymbol{\omega}} + \left(I_{W_s} - I_{W_t}\right)\omega_s\omega_g - 2I_{W_t}\omega_s\dot{\gamma}\right]\\ + \hat{\boldsymbol{g}}_g\left[I_{W_t}\left(\hat{\boldsymbol{g}}_g^T\dot{\boldsymbol{\omega}} + \ddot{\gamma}\right) + \left(I_{W_t} - I_{W_s}\right)\omega_s\omega_t - I_{W_s}\Omega\omega_t\right]\end{aligned}$$

- Taking the derivative of the gimbal frame angular momentum expression $\boldsymbol{H}_G$, we find

$$
\begin{aligned}
\dot{\boldsymbol{H}}_G = \hat{\boldsymbol{g}}_s &\left(\left(I_{G_s} - I_{G_t} + I_{G_g}\right)\dot{\gamma}\omega_t + I_{G_s}\hat{\boldsymbol{g}}_s^T\dot{\boldsymbol{\omega}} + \left(I_{G_g} - I_{G_t}\right)\omega_t\omega_g\right) \\
+ \hat{\boldsymbol{g}}_t &\left(\left(I_{G_s} - I_{G_t} - I_{G_g}\right)\dot{\gamma}\omega_s + I_{G_t}\hat{\boldsymbol{g}}_t^T\dot{\boldsymbol{\omega}} + \left(I_{G_s} - I_{G_g}\right)\omega_s\omega_g\right) \\
+ \hat{\boldsymbol{g}}_g &\left(I_{G_g}\left(\hat{\boldsymbol{g}}_g^T\dot{\boldsymbol{\omega}} + \ddot{\gamma}\right) + \left(I_{G_t} - I_{G_s}\right)\omega_s\omega_t\right)
\end{aligned}
$$

- Finally, the spacecraft angular momentum inertial derivative is

$$
\dot{\boldsymbol{H}}_B = [I_s]\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times [I_s]\boldsymbol{\omega}
$$

- Let us define the time-varying total spacecraft inertia matrix [*I*]:

$$[I] = [I_s] + [J]$$

- Adding up all the terms, and substituting them into Euler's equation $\dot{\boldsymbol{H}} = \boldsymbol{L}$ we finally arrive at the desired equations of motion of a spacecraft with a single VSCMG.

0

$$\begin{aligned}[I]\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I]\boldsymbol{\omega} &- \hat{\boldsymbol{g}}_s \left(J_s \dot{\gamma} \omega_t + I_{W_s} \dot{\Omega} - (J_t - J_g)\, \omega_t \dot{\gamma} \right) \\ &- \hat{\boldsymbol{g}}_t \left(\boxed{(J_s \omega_s + I_{W_s} \Omega)\, \dot{\gamma}} - (J_t + J_g)\, \omega_s \dot{\gamma} + I_{W_s} \Omega \omega_g \right) \\ &- \hat{\boldsymbol{g}}_g \left(J_g \ddot{\gamma} - I_{W_s} \Omega \omega_t \right) + \boldsymbol{L}\end{aligned}$$

These equations of motion are valid for both a RW or CMG device!

$$\dot{\boldsymbol{H}}_W = \boldsymbol{L}_W = u_s \hat{\boldsymbol{g}}_s + \tau_{w_t} \hat{\boldsymbol{g}}_t + \tau_{w_g} \hat{\boldsymbol{g}}_g$$

$$u_s = I_{W_s} \left(\dot{\Omega} + \hat{\boldsymbol{g}}_s^T \dot{\boldsymbol{\omega}} + \dot{\gamma} \omega_t \right)$$

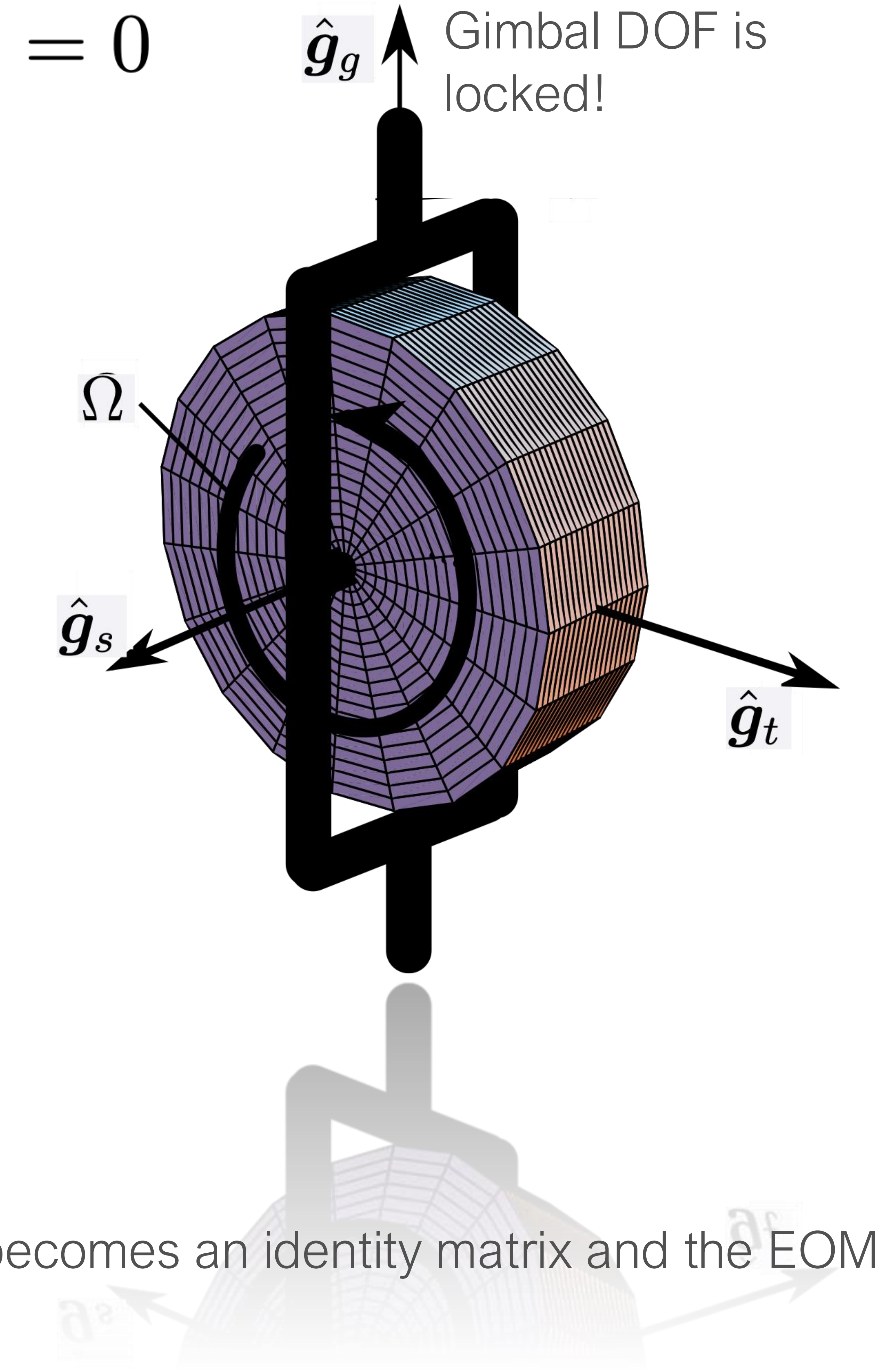
$$\dot{\boldsymbol{H}}_G + \dot{\boldsymbol{H}}_W = \boldsymbol{L}_G = \tau_{G_s} \hat{\boldsymbol{g}}_s + \tau_{G_t} \hat{\boldsymbol{g}}_t + u_g \hat{\boldsymbol{g}}_g$$

$$\begin{aligned}u_g = &J_g \left(\hat{\boldsymbol{g}}_g^T \dot{\boldsymbol{\omega}} + \ddot{\gamma} \right) \\ &- (J_s - J_t)\, \omega_s \omega_t - I_{W_s} \Omega \omega_t\end{aligned}$$

Example: multiple RW devices

$$\dot{\gamma} = 0$$

$$\ddot{\gamma} = 0$$



Inertia matrix definition:

$$[I_{RW}] = [I_s] + \sum_{i=1}^{N} \left(J_{t_i} \hat{\boldsymbol{g}}_{t_i} \hat{\boldsymbol{g}}_{t_i}^T + J_{g_i} \hat{\boldsymbol{g}}_{g_i} \hat{\boldsymbol{g}}_{g_i}^T \right)$$

Let us define the momentum vector $\boldsymbol{h}_s$ as:

$$\boldsymbol{h}_s = \begin{pmatrix} \vdots \\ \boxed{J_{s_i} \left(\omega_{s_i} + \Omega_i \right)} \\ \vdots \end{pmatrix}$$

The equations of motion then become:

I $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$[I_{RW}]\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I_{RW}]\boldsymbol{\omega} - \boldsymbol{\omega} \times \boxed{[G_s]\boldsymbol{h}_s} - [G_s]\boldsymbol{u}_s + \boldsymbol{L}$$

For the special case with 3 RWs aligned with the principal axis, $[G_s]$ becomes an identity matrix and the EOM reduce to

$$[I_{RW}]\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I_{RW}]\boldsymbol{\omega} - \boldsymbol{\omega} \times \boldsymbol{h}_s - \boldsymbol{u}_s + \boldsymbol{L}$$

Momentum-Device Control Laws

This is where the pudding starts to come together...

‘N’ RW Control Devices

- Let us now develop a feedback control law for a spacecraft with *N* reaction wheels with general orientation.

EOM:

$$[I_{RW}]\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I_{RW}]\boldsymbol{\omega} - \boldsymbol{\omega} \times [G_s]\boldsymbol{h}_s - [G_s]\boldsymbol{u}_s + \boldsymbol{L}$$

with

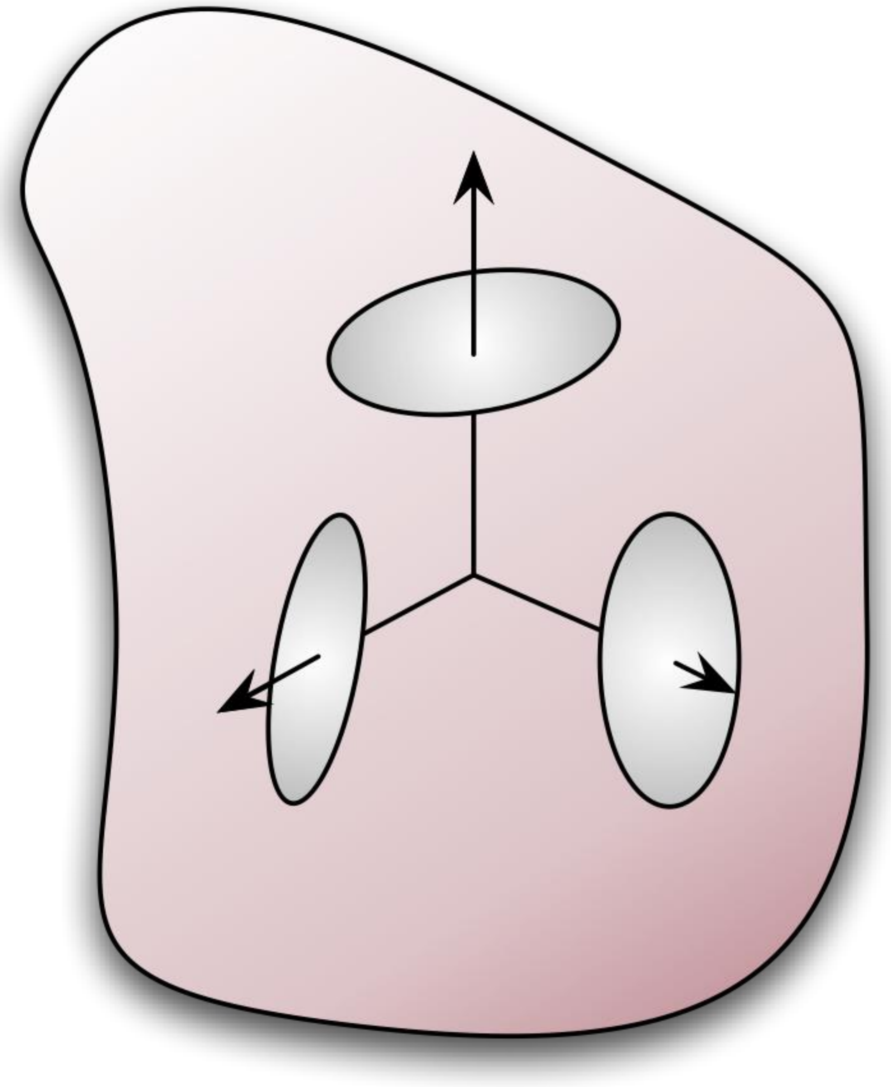
$$\boldsymbol{h}_s = \begin{pmatrix} \vdots \\ J_{s_i}\left(\omega_{s_i} + \Omega_i\right) \\ \vdots \end{pmatrix}$$

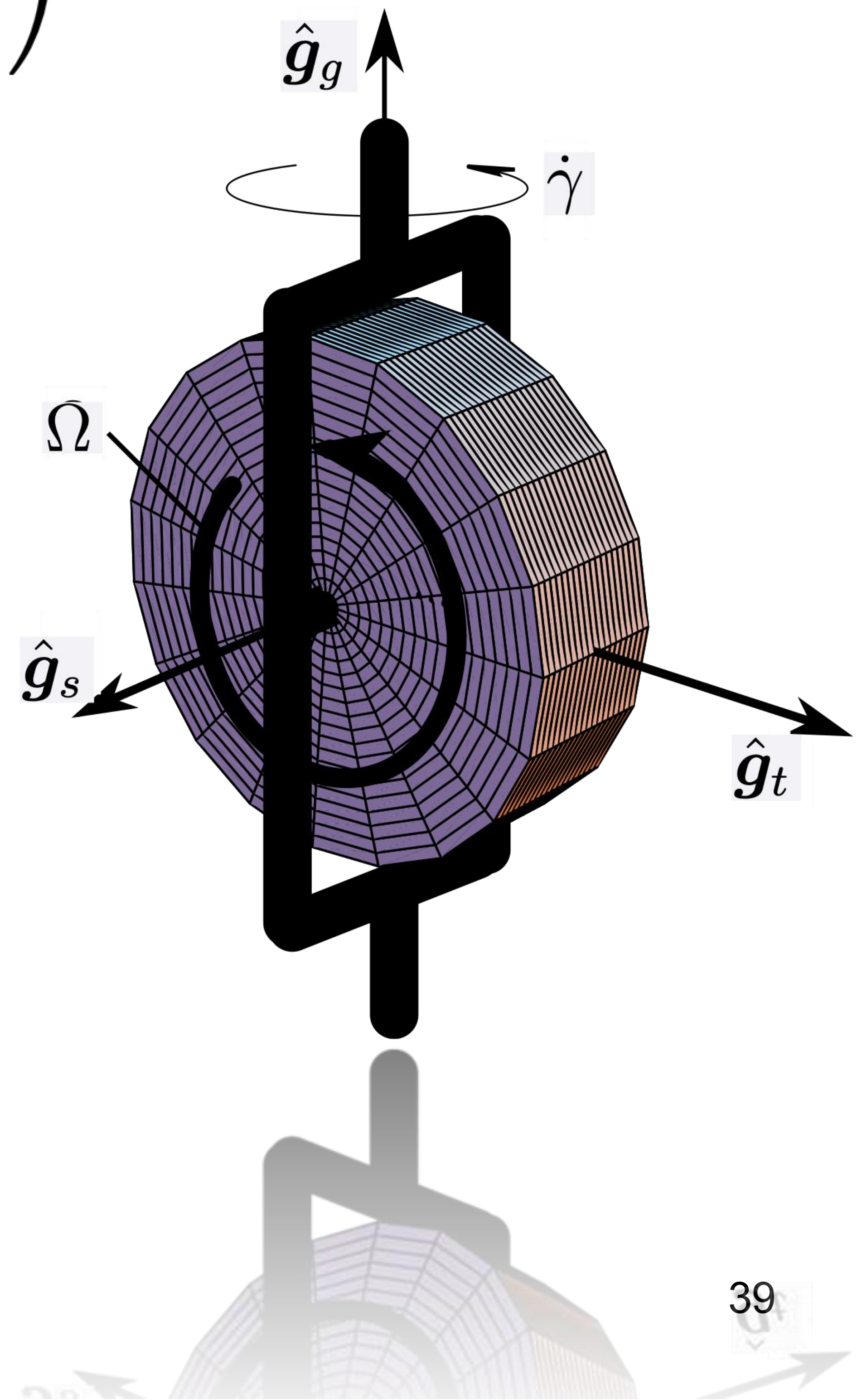
Inertia Matrix:

$$[I_{RW}] = [I_s] + \sum_{i=1}^{N} \left(J_{t_i}\hat{\boldsymbol{g}}_{t_i}\hat{\boldsymbol{g}}_{t_i}^T + J_{g_i}\hat{\boldsymbol{g}}_{g_i}\hat{\boldsymbol{g}}_{g_i}^T\right)$$

The RW motor control torque vector is:

$$\boldsymbol{u}_s = \begin{pmatrix} \vdots \\ J_{s_i}\left(\dot{\Omega}_i + \hat{\boldsymbol{g}}_{s_i}^T\dot{\boldsymbol{\omega}}\right) \\ \vdots \end{pmatrix}$$





Spacecraft Tracking Errors:

$\boldsymbol{\sigma}_{BR}$ - MRP vector of body frame relative to reference frame

$\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_r$ - body angular velocity tracking error vector

ω_{BR}

Lyapunov Function:

$$V(\boldsymbol{\sigma}, \delta\boldsymbol{\omega}) = \frac{1}{2}\delta\boldsymbol{\omega}^T[I_{RW}]\delta\boldsymbol{\omega} + 2K\ln\left(1 + \boldsymbol{\sigma}_{BR}^T\boldsymbol{\sigma}_{BR}\right)$$

components taken in the *B* frame

Let's set the Lyapunov Rate to:

$$\boxed{\dot{V} = -\delta\boldsymbol{\omega}[P]\delta\boldsymbol{\omega} \leq 0}$$

$$[I_{RW}]\frac{^{\mathcal{B}}\mathrm{d}}{\mathrm{dt}}(\delta\boldsymbol{\omega}) + K\boldsymbol{\sigma} + [P]\delta\boldsymbol{\omega} = 0$$

closed-loop dynamics

$$[I_{RW}]\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega}\times[I_{RW}]\boldsymbol{\omega} - \boldsymbol{\omega}\times[G_s]\boldsymbol{h}_s - [G_s]\boldsymbol{u}_s + \boldsymbol{L}$$

$$[G_s]\boldsymbol{u}_s \boxed{= K\boldsymbol{\sigma} + [P]\delta\boldsymbol{\omega} - [\tilde{\boldsymbol{\omega}}]([I_{\mathrm{RW}}]\boldsymbol{\omega} + [G_s]\boldsymbol{h}_s) - [I_{\mathrm{RW}}](\dot{\boldsymbol{\omega}}_r - \boldsymbol{\omega}\times\boldsymbol{\omega}_r) + \boldsymbol{L}}$$

$$-\boldsymbol{L}_r$$

Control condition:

$$[G_s]\boldsymbol{u}_s = -\boldsymbol{L}_r$$

Case 1: 3 RWs aligned with principal axes of spacecraft.

$$\boldsymbol{u}_s = -\boldsymbol{L}_r$$

Case 2: N RWs aligned generally.

$$\boldsymbol{u}_s = [G_s]^T\left([G_s][G_s]^T\right)^{-1}(-\boldsymbol{L}_r)$$

minimum-norm inverse

Energy rate:

$$\dot{T} = \boldsymbol{\omega}^T\boldsymbol{L} + \sum_{i=1}^{N}\Omega_i u_{s_i}$$

work/energy principle

Comments...

- By changing the wheel speed or by gimbaling the CMG devices, a torque is applied to the spacecraft and the corresponding attitude is changed.
- RW devices are simpler, but have limits on how large the spin speed Ω can grow.
- Adding the gimbaling mode clearly makes the mathematics much more fun and interesting :-)
- To generally control a spacecraft attitude, three of more of these devices would have to be attached to the spacecraft.

The End…

https://colorado.campuslabs.com/eval-home/