ASEN 6060
ADVANCED ASTRODYNAMICS
Quasi-Periodic Trajectories

Objectives:
- Define properties of quasi-periodic trajectories
- Very high-level and brief overview of well-known approaches to numerically computing a quasi-periodic trajectory

Quasi-Periodic Trajectories

Quasi-periodic trajectory / quasi-periodic orbit (QPO):

- Nonperiodic, bounded motion, lies on surface of an invariant torus
- Exists near a periodic orbit with oscillatory modes
- Existence covered by Kolmogorov-Arnold-Moser (KAM) theory
- Terminology sometimes used more loosely for trajectories that are bounded over finite time intervals

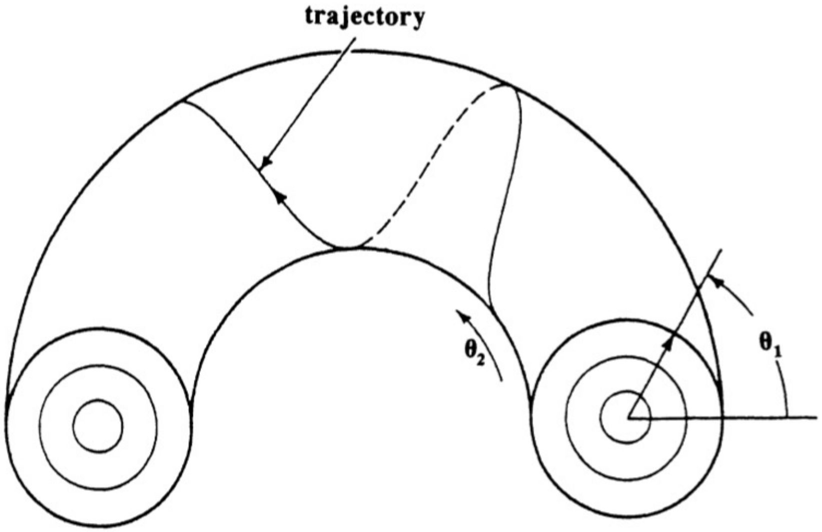


Image credit: Contopoulos, 2004, "Order and Chaos in Dynamical Astronomy"

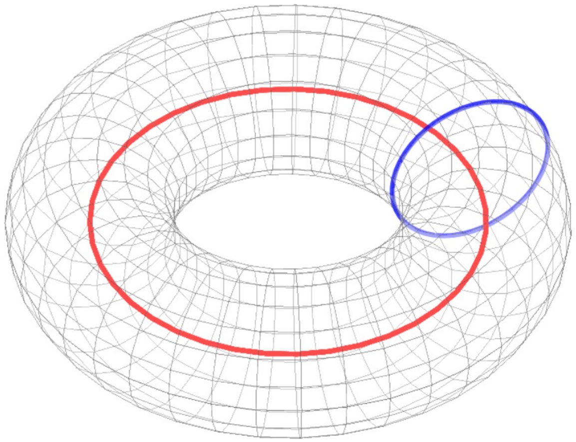


Image credit: Bosanac 2016

Quasi-Periodic Trajectories

Quasi-periodic trajectory on the surface of a two-dimensional torus defined by two incommensurate, fundamental frequencies: ω_1, ω_2

- ω_1 = Central frequency
- ω_2 = Transverse frequency

(Note: notation opposite to picture below!)

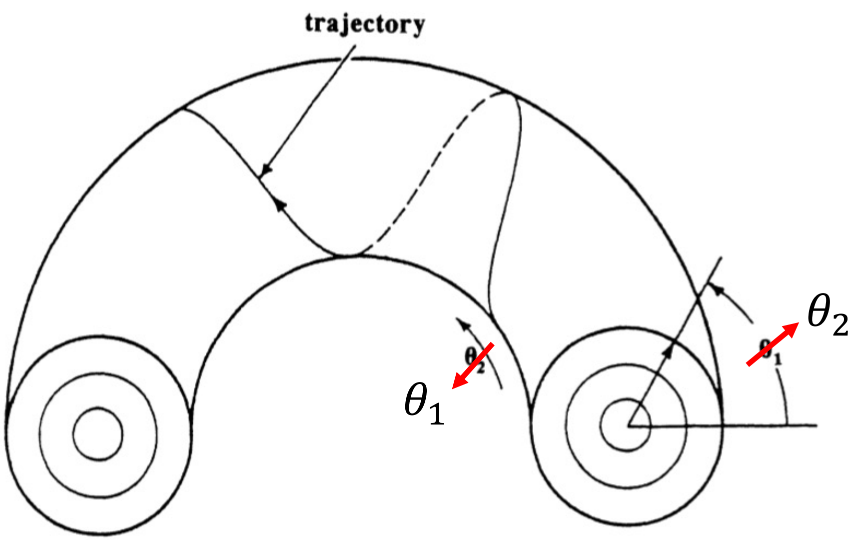


Image credit: Contopoulos, 2004, "Order and Chaos in Dynamical Astronomy"

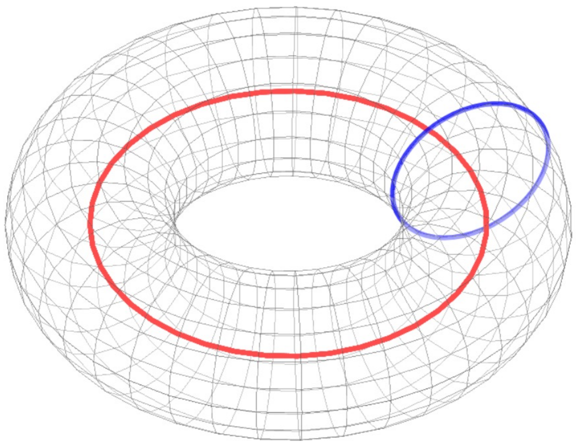


Image credit: Bosanac 2016

Frequency Decomposition

Frequency decomposition of a periodic trajectory:

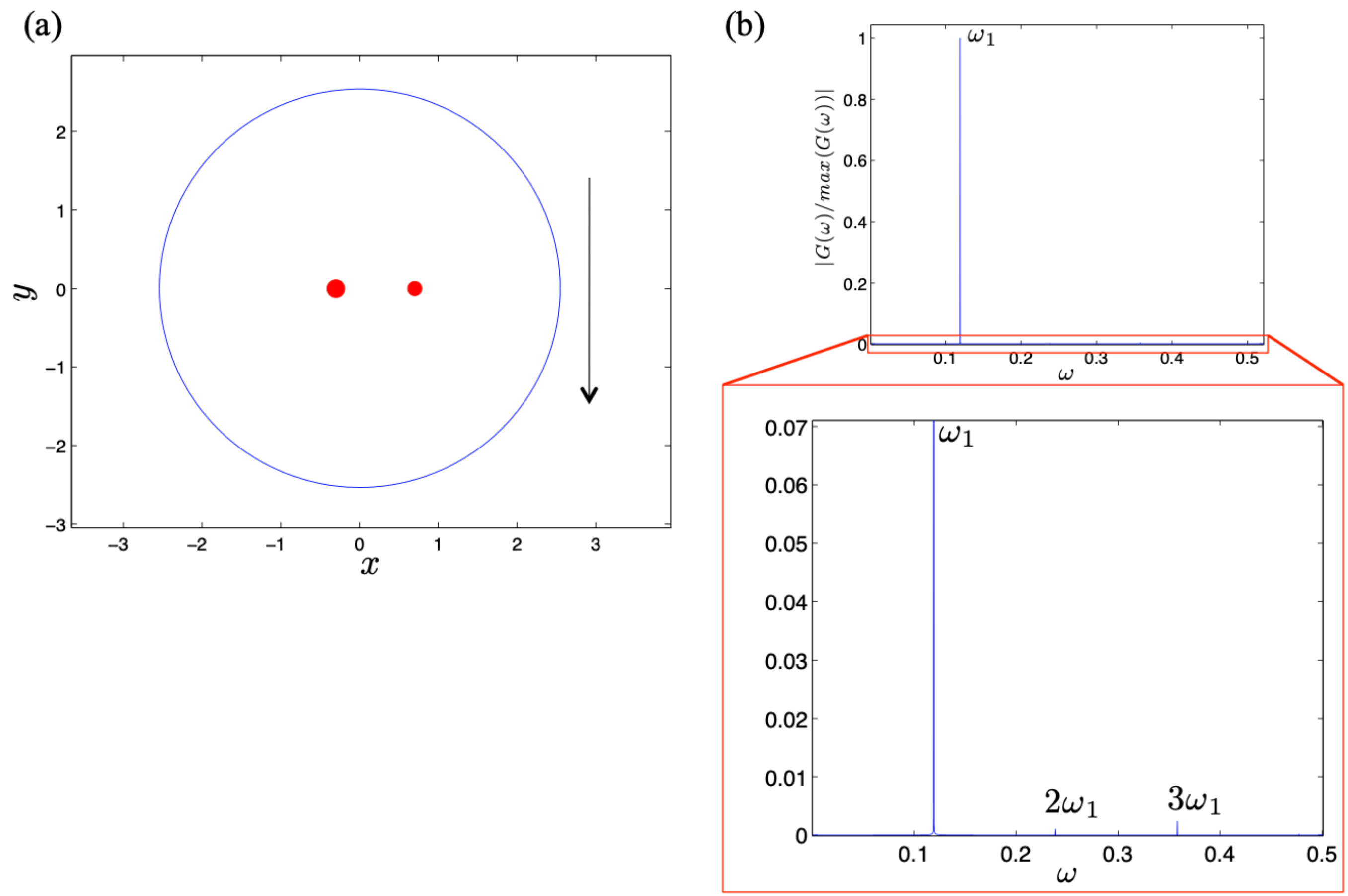


Image credit:
Bosanac 2012

Frequency Decomposition

Frequency decomposition of a quasi-periodic trajectory:

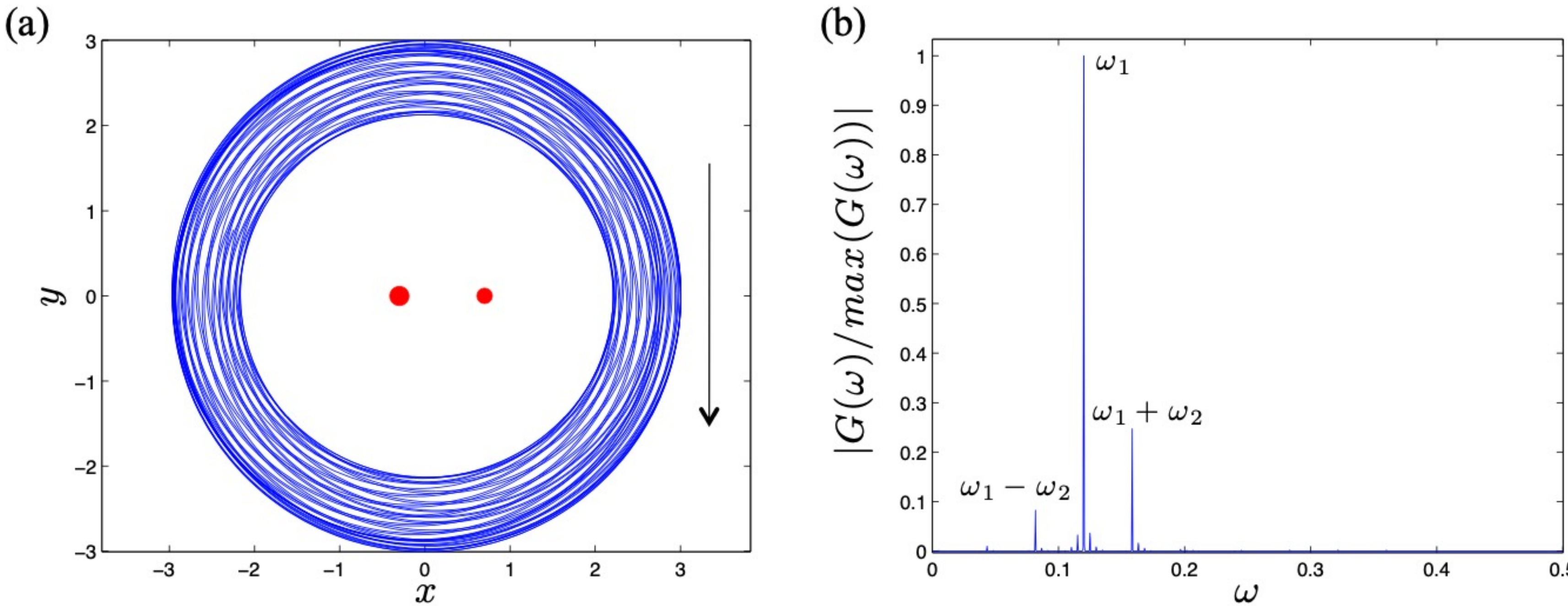
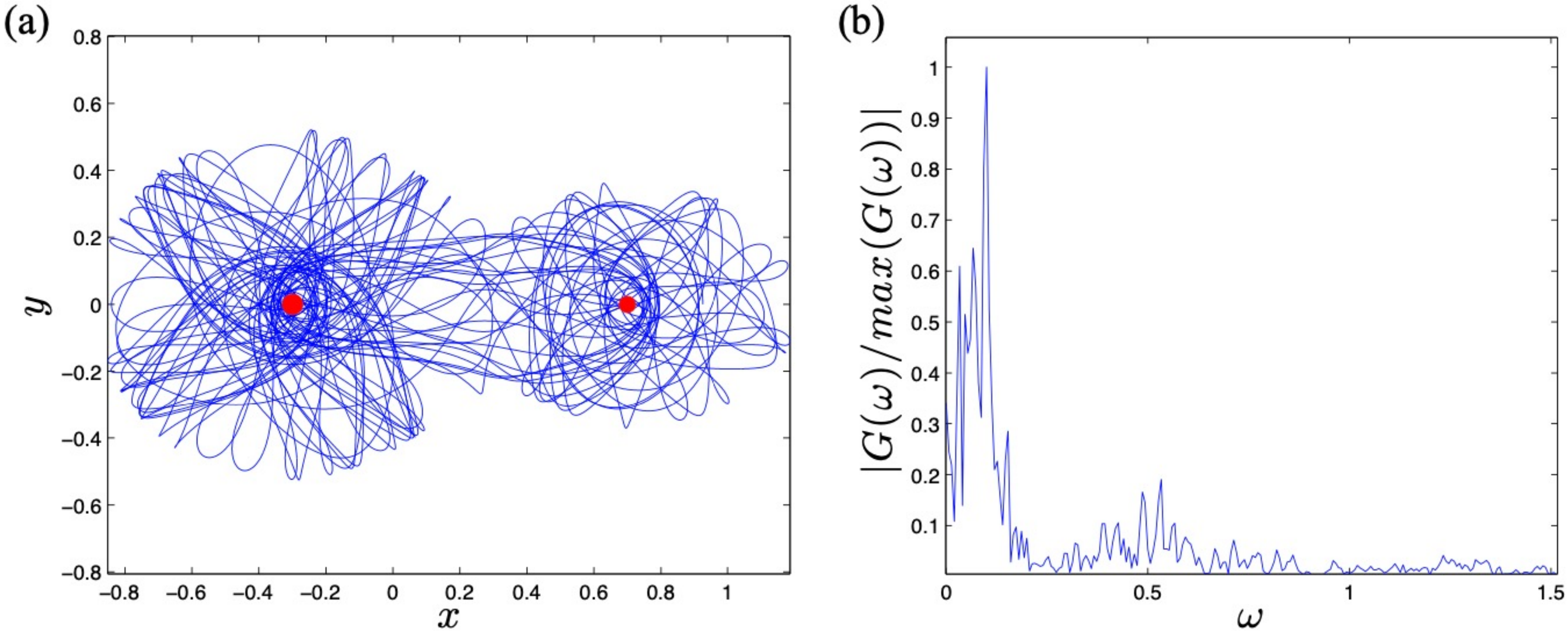


Image credit:
Bosanac 2012

When frequencies are resonant, produce higher-order periodic orbit

Frequency Decomposition

Frequency decomposition of a chaotic trajectory:



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Bosanac 2012

Quasi-Periodic Trajectories in the CR3BP

QPOs exist in families near periodic orbits throughout a multi-body systems

Tori near planar 3:1 interior resonance in EM system

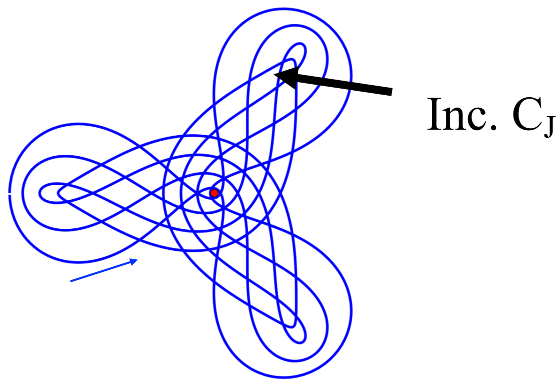
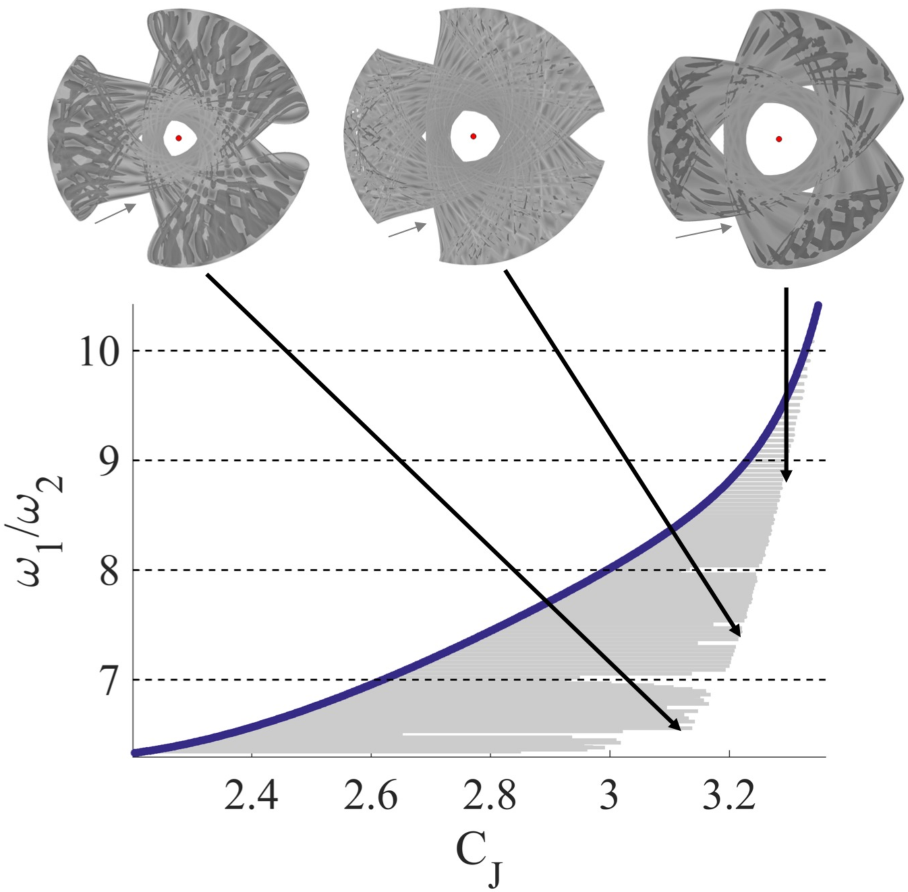


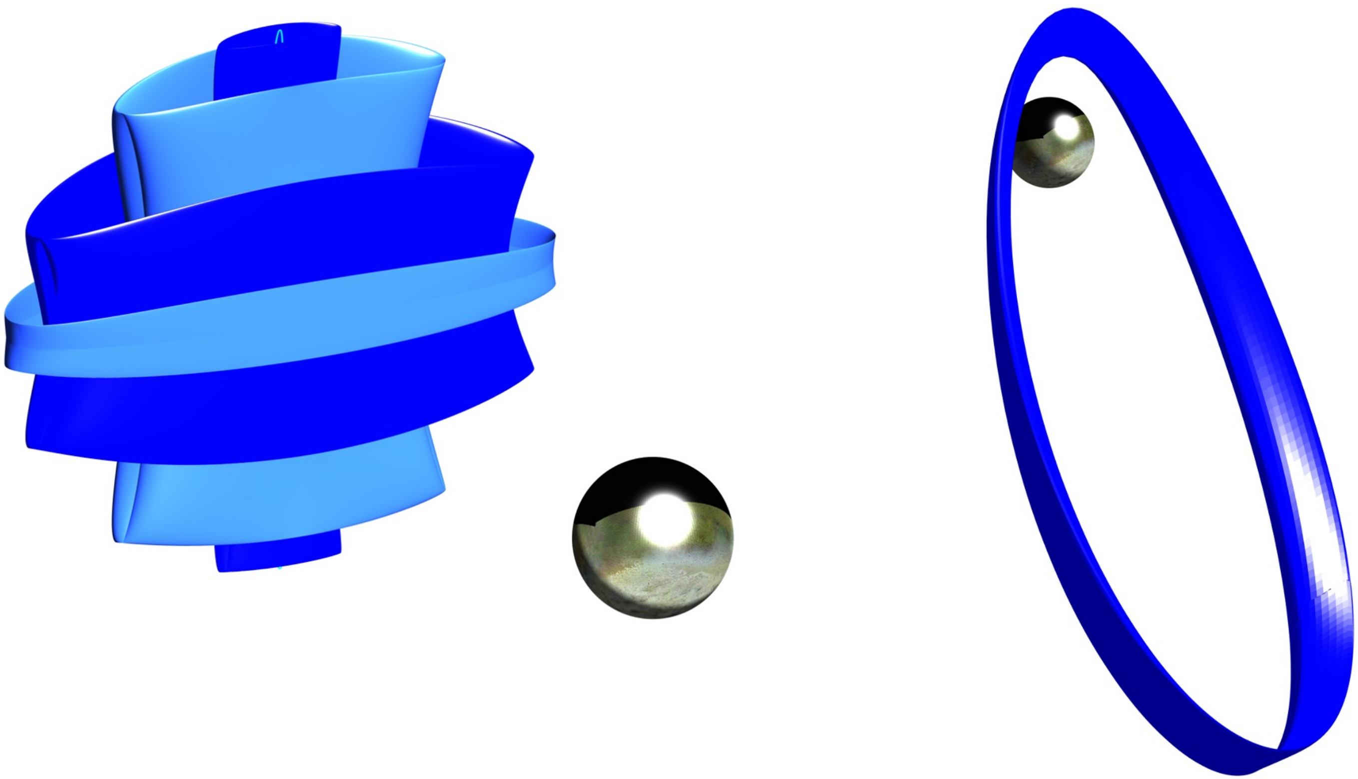
Image credit: Bosanac, N, 2018, “Bounded Motions Near Resonant Orbits in the Earth-Moon and Sun-Earth Systems” AAS/AIAA Spaceflight Mechanics Meeting



Quasi-Periodic Trajectories in the CR3BP

QPOs exist in families near periodic orbits throughout a multi-body system

Tori near L_1

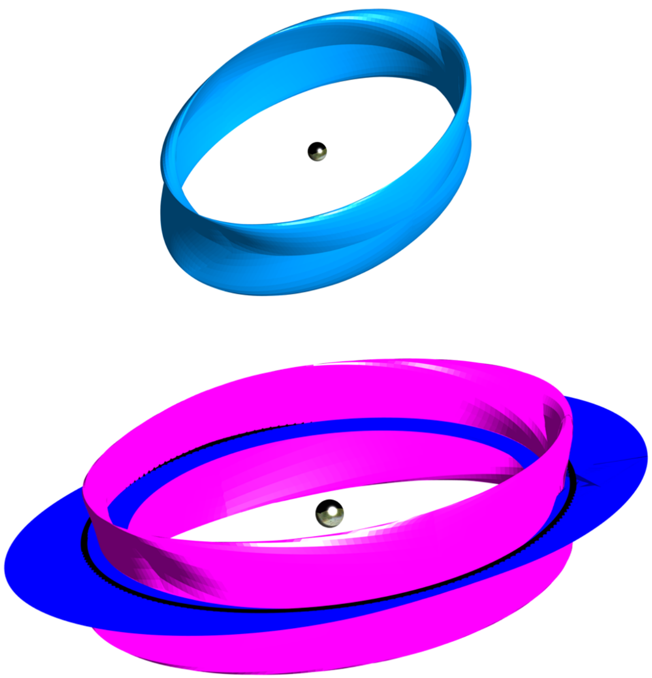


* In an outer planet system!

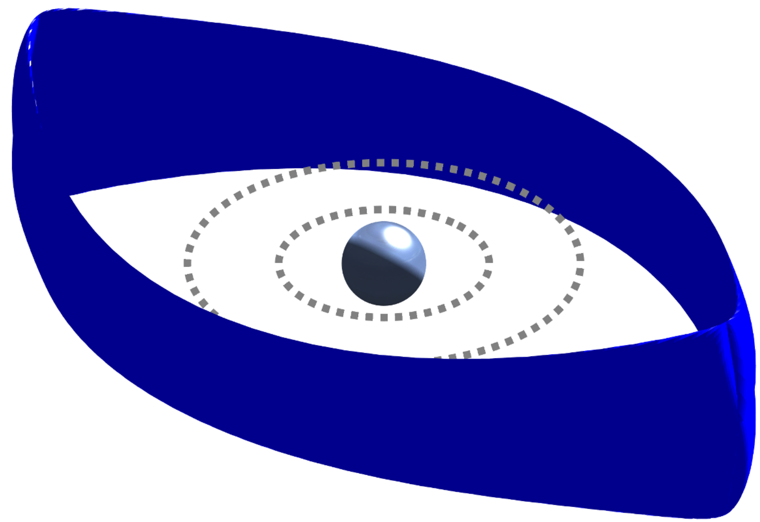
Quasi-Periodic Trajectories in the CR3BP

QPOs exist in families near periodic orbits throughout a multi-body system

Torus near DRO

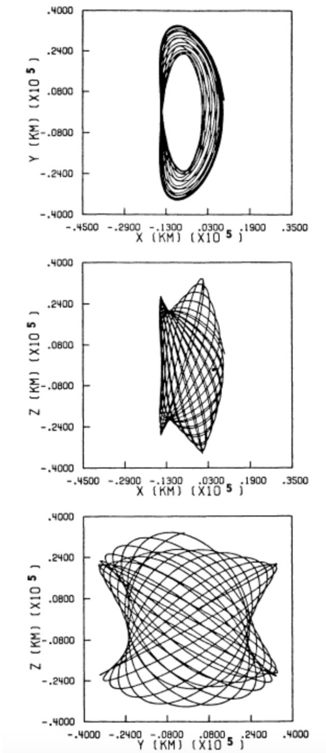


Torus near 2:1 resonance



* In an outer planet system!

Computing Quasi-Periodic Trajectories



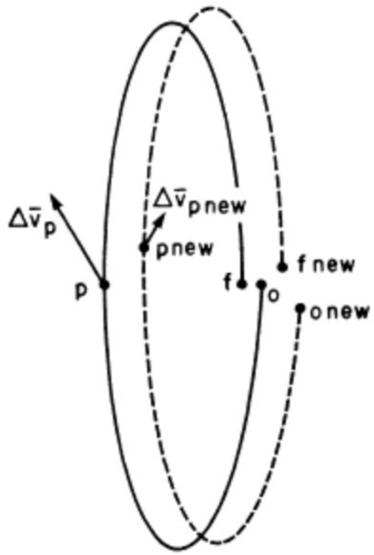


Image credits: Howell, Pernicka, 1988, “Numerical Determination of Lissajous Trajectories in the Restricted Three-Body Problem,” *Celestial Mechanics*

Corrections to produce <u>bounded</u> trajectory over selected time interval (no guarantee to lie on a torus):

- Initial guess, options:
  - Small step into center eigenspace of state along periodic orbit
  - Analytical approximation
- Discretize trajectory into multiple arcs for multiple revolutions
- Correct using multiple shooting to produce a bounded trajectory (should resemble initial guess if good)

Action-Angle Variables

Canonical transformation

$$(\bar{q}, \bar{p}) \rightarrow (\bar{\theta}, \bar{I})$$

where

- θ_i is angle for ith frequency (generalized coordinate)
- I_i is the ith action (generalized momentum)

Allows Hamiltonian to only be expressed in terms of the actions

As a result

$$\dot{\theta}_i = \frac{\partial H}{\partial I_i} = \omega_i \quad \dot{I}_i = -\frac{\partial H}{\partial \theta_i} = 0$$

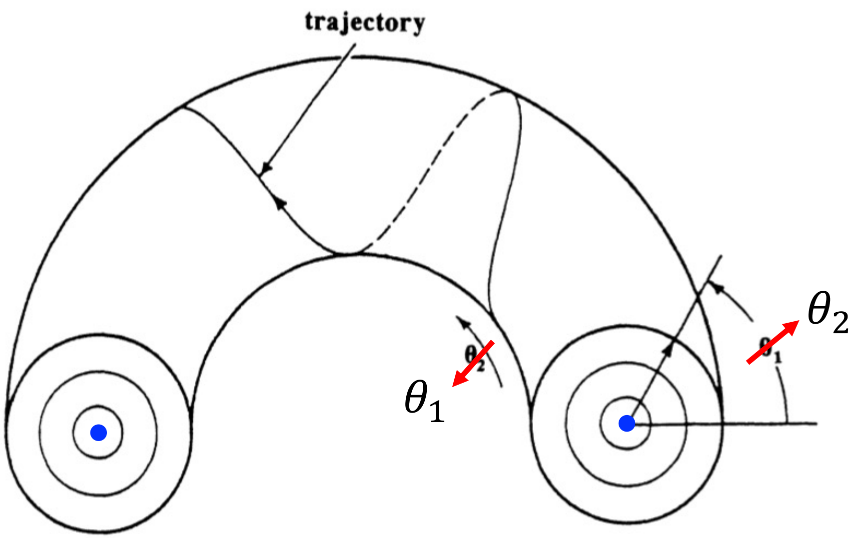


Image credit: Contopoulos, 2004, “Order and Chaos in Dynamical Astronomy”

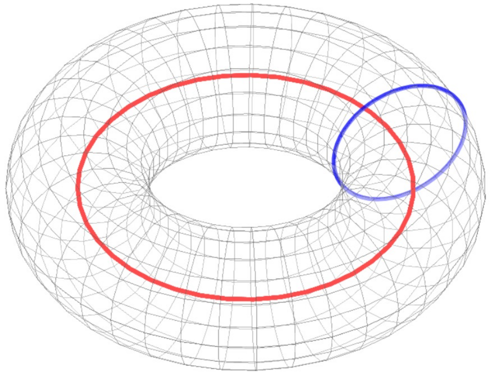


Image credit: Bosanac 2016

Computing Quasi-Periodic Trajectories

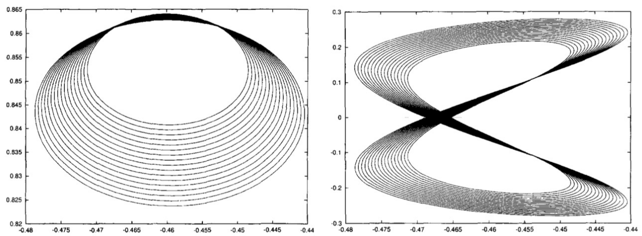


Fig. 4. Periodic orbits of the vertical family obtained from the truncated normal form. They can be easily computed by using a standard continuation method, but in the normal form are trivial to obtain putting $I_1 = I_2 = 0$, and using I_0 as a parameter in the family. Here, we plot the projections (x, y) (left) and (x, z) (right) of the orbits corresponding to I_0 from -8×10^{-3} to 8×10^{-3} with step 10^{-3}. We recall that the orbit with $I_0 = 0$ is the initial one.

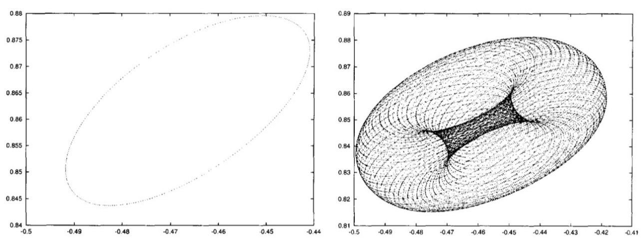


Fig. 5. These figures correspond to a 2D torus near the periodic orbit, given by $I_0 = I_1 = 0$ and $I_2 = 1 \times 10^{-6}$, and plotted up to time 5000. Left: (x, y) projection of the Poincaré section $z = 0$ of the torus. Right: (x, y) projection of the same orbit, plotting a point every 0.1 units of time. The intrinsic frequencies of this torus (with the determination given by the truncated normal form) are $\omega_0 = 0.9995541833$ and $\omega_2 = 0.3315682317$. The normal one is $\omega_1 = -0.2528624981$. The error in the determination of this torus (see Section 3.6.3) computed by comparing points in the RTBP is (up to time 10000) of order 1×10^{-6}. The errors on the actions I_0, I_1 and I_2, up to the same time, are of order 1.5×10^{-12}, 1.2×10^{-20} and 3.4×10^{-12}, respectively.

Image credit: Jorba, Villanueva, 1997, “Numerical Computation of Normal Forms Around Some Periodic Orbits of the Restricted Three-Body Problem,” *Physica D*

Nonlinear approximation of local neighborhood of an equilibrium point or periodic orbit:

- Normal forms, reduction to center manifold
- Write expansion about reference motion to selected degree in Hamiltonian form and include action-angle variables
- Numerically compute change of variables using generating functions

Computing Quasi-Periodic Trajectories

Compute an approximation of torus via boundary value problem

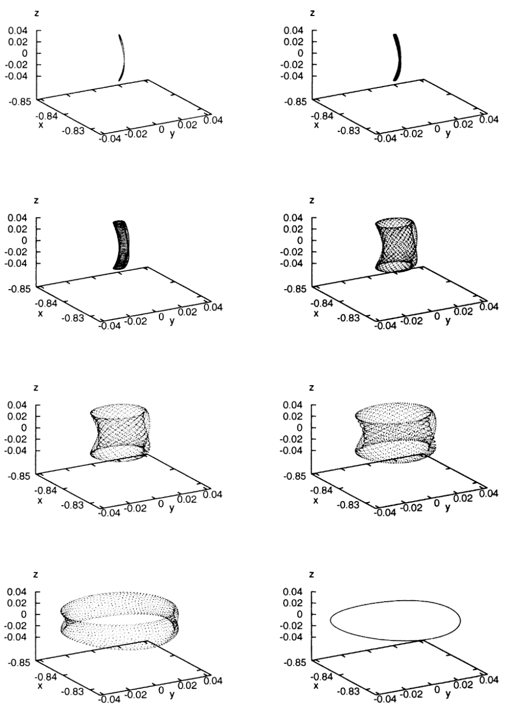


Image credit: Gomez, Mondelo, 2001, "The dynamics around the collinear equilibrium points of the RTBP", *Physica D*

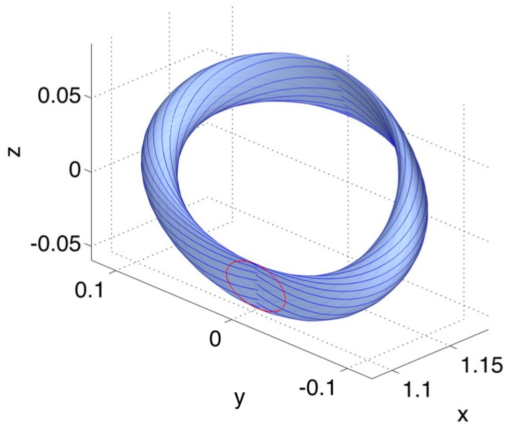


Image credit: Olikara, 2016, "Computation of Quasi-Periodic Tori and Heteroclinic Connections in Astrodynamics Using Collocation Techniques"

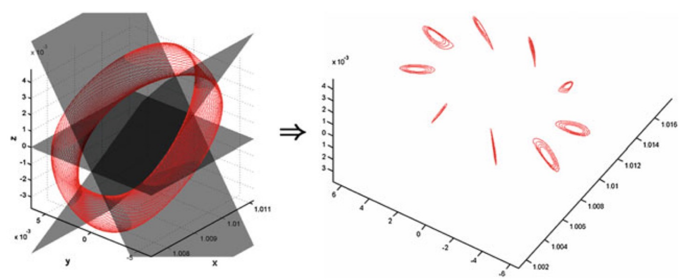


Image credit: Kolemen, Kasdin, Gurfil, 2011, "Multiple Poincaré Sections Method for Finding the Quasiperiodic Orbits of the Restricted Three-Body Problem," *Celestial Mechanics & Dynamical Astronomy*

Computation via Invariance Condition

- Method presented by Olikara and Scheeres (2012)
  - In-depth discussion in: Olikara, Z.P, 2016, "Computation of Quasi-Periodic Tori and Heteroclinic Connections in Astrodynamics Using Collocation Techniques"
- Builds on work by Schilder et al, Gomez and Mondelo, and Jorba
- Leverages an invariance condition to numerically recover a torus
- Can implement with single shooting, multiple shooting, collocation
- Can also assess stability of torus to produce their stable/unstable manifolds

Computation via Invariance Condition

- Recall: ω_1 = central frequency; ω_2 = transverse frequency
- States along QPO and beginning at an invariant circle return to this invariant circle when propagated for stroboscopic mapping time, T_1, but rotated by angle ρ

$$T_1 = 2\pi/\omega_1 \qquad\qquad \rho = 2\pi\omega_2/\omega_1$$

- Numerically calculate states along the invariant circle, T_1, ρ

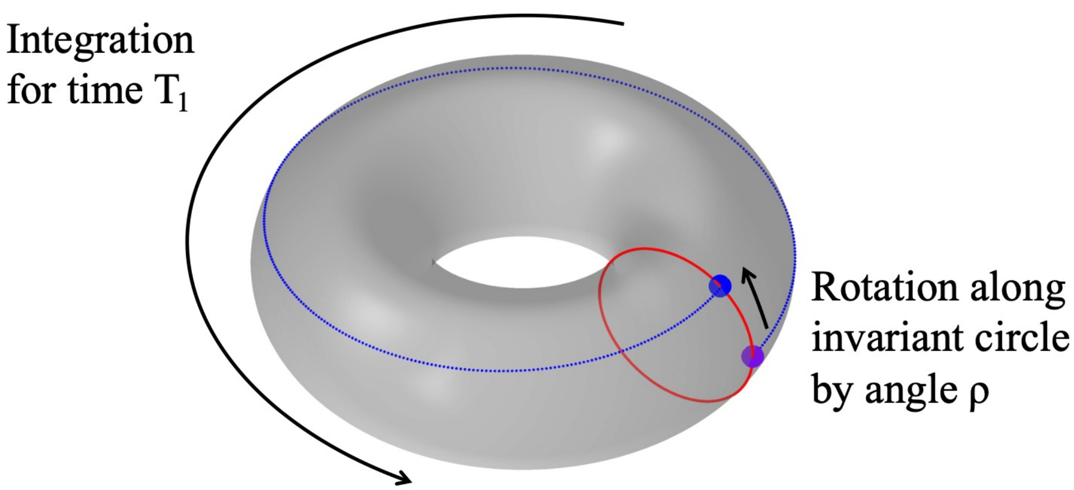


Image credit: Bosanac 2016

Computation via Invariance Condition

1. Generate initial guess for invariant circle by stepping into center eigenspace with N values of $\theta_2 = [0, 2\pi)$:

$$\bar{u}_{IG} = \bar{x}_{PO} + \epsilon\left[Re(\bar{v}_C)\cos(\theta_2) + Im(\bar{v}_C)\sin(\theta_2)\right]$$

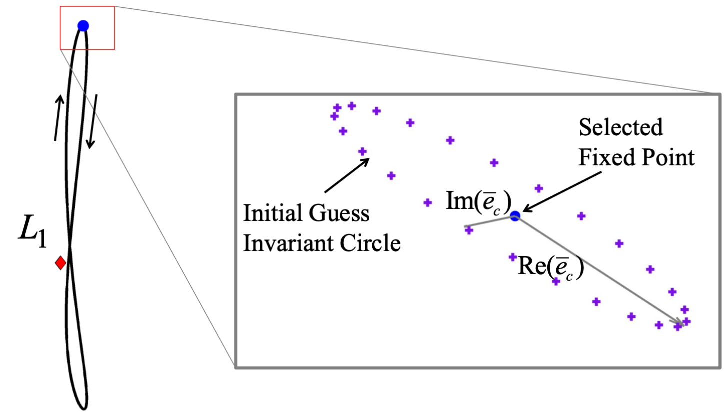
2. Propagate each state forward for stroboscopic mapping time, T_1, to produce $\bar{f}(\bar{u}(\theta_2), T_1)$

3. Approximate the invariant circle using a Discrete Fourier Transform (DFT)

4. Rotate states computed after T_1 along the invariant circle using ρ

$$[R(-\rho)]\bar{f}(\bar{u}(\theta_2), T_1)$$

where $[R(-\rho)]$ is calculated from DFT



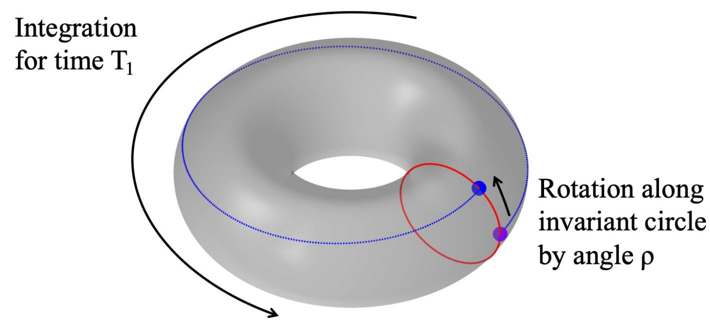


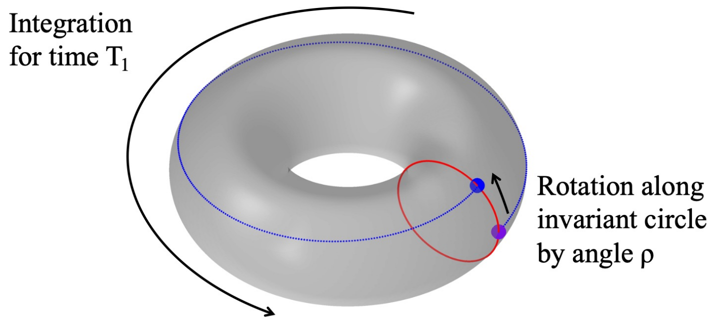
Image credit: Bosanac 2016

Computation via Invariance Condition

5. Evaluate the invariance condition at all N states associated with selected values of

$\theta_2 = [0, 2\pi)$:

$$[R(-\rho)]\bar{f}(\bar{u}(\theta_2), T_1) - \bar{u}(\theta_2) = \bar{0}$$



6. Add constraints such as Jacobi constant, frequency ratio, location of invariant circle

Solve using Newton's method by updating states along invariant circle, T_1, ρ

Image credit: Bosanac 2016

Approximating QPOs near L1/2/3

Rewriting the equations of motion relative to a collinear eq. pt. and linearizing produces the following equations (where c is a constant):

$$\ddot{x} - 2\dot{y} - (1+2c)x = 0 \qquad \ddot{y} + 2\dot{x} + (c-1)y = 0 \qquad \ddot{z} + cz = 0$$

Denoting the in-plane and out-of-plane imaginary eigenvalues as λ and ν, respectively, an analytical approximation for bounded motion near a collinear eq. pt. is:

$$\left.\begin{aligned} x &= -kA_y \cos(\lambda t + \phi) \\ y &= A_y \sin(\lambda t + \phi) \\ z &= A_z \sin(\nu t + \psi) \end{aligned}\right\}$$

When λ and ν are:

- Not commensurate: quasi-periodic Lissajous trajectories
- Equal: halo orbits

Can construct higher-order approximations, incorporating lunar eccentricity and other gravitational effects (e.g., Sun)

Citation: Howell, K.C. and Pernicka, H.J., “Numerical Determination Of Lissajous Trajectories In The Restricted Three-body Problem,” Celestial Mechanics, Vol. 41, 1988, pp. 107-124

Lissajous Orbit near L1

Lissajous orbit near Sun-Earth/Moon L1 after corrections

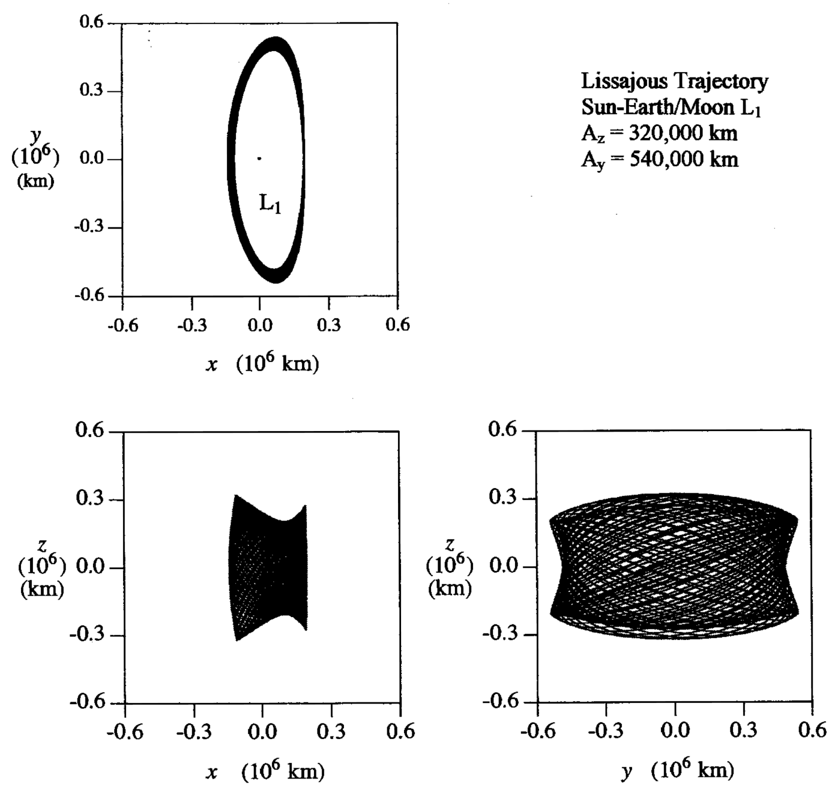


FIG. 4. Lissajous Trajectory.

Image credit: Howell, K.C., "Families of Orbits in the Vicinity of the Collinear Libration Points," The Journal of the Astronautical Sciences, Vol. 49, No. 1, Jan-March 2001, pp. 107-125

Lissajous orbits exist in families, e.g., at C = 3.18126 in the Earth-Moon system near L_1

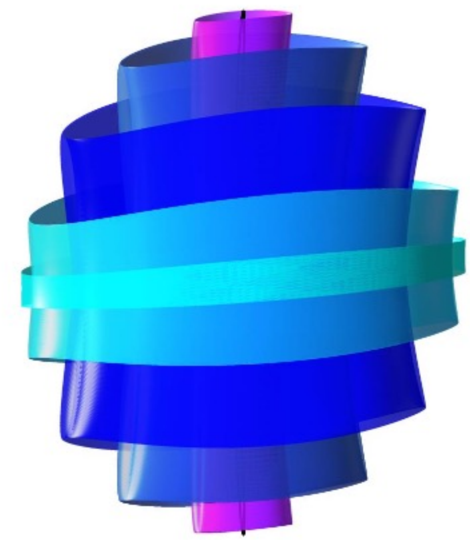


Image credit: Bosanac 2016

Bounded Motions Near L1

Poincaré map representation of intersections of bounded motion near L_1 with a surface of section defined as $\Sigma : z = 0$

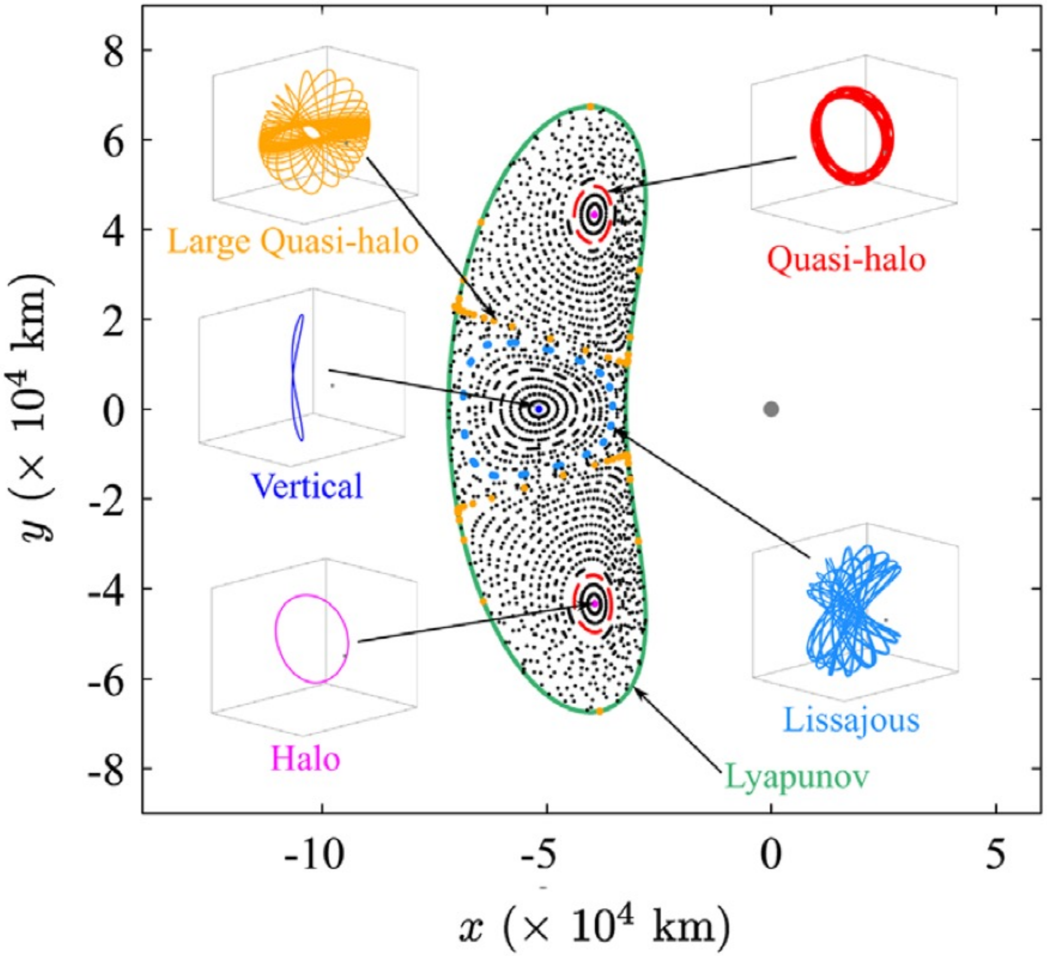


Image credit: Folta, D.C., et al. "Earth–Moon Libration Point Orbit Stationkeeping: Theory, Modeling, And Operations," Acta Astronautica, Vol. 94, No. 1, 2014, pp. 421-433

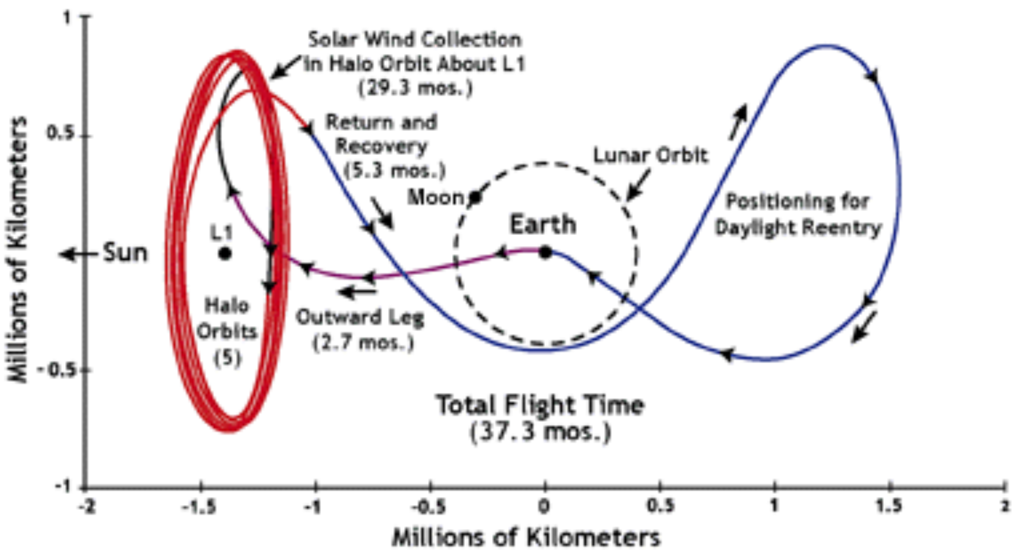
Mission Applications

Although a perfect quasi-periodic orbit does not exist in a high-fidelity dynamical model of a multi-body system, their characteristics are approximately retained.

There are many examples of mission orbits that resemble quasi-periodic orbits – here are just a few!

Genesis

Image credit: NASA, genesismission.jpl.nasa.gov



Mission Applications

TESS: Transiting Exoplanet Survey Satellite

Image credit: Dichmann, Parker, Williams, Mendelsohn, 2014, “Trajectory Design for the Transiting Exoplanet Survey Satellite”

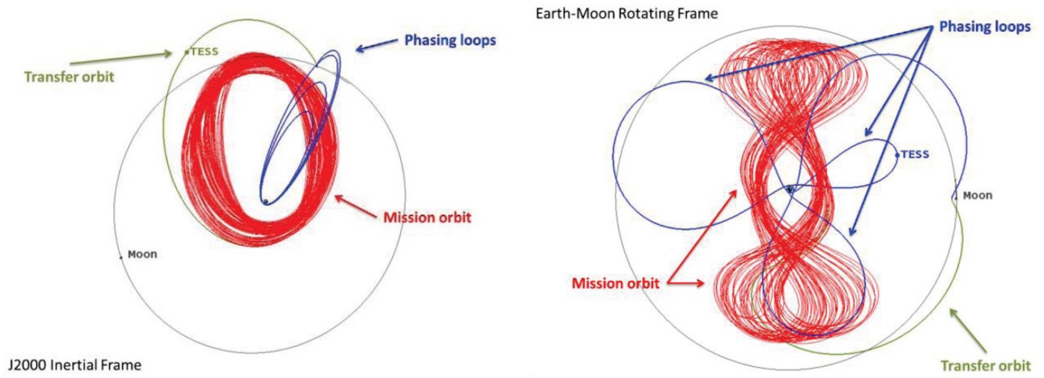


Figure 9: (a) Propagation in J2000 inertial frame; (b) Propagation in Earth-Moon rotating frame

Mission Applications

ARTEMIS: Acceleration, Reconnection, Turbulence and Electrodynamics of the Moon's Interaction with the Sun

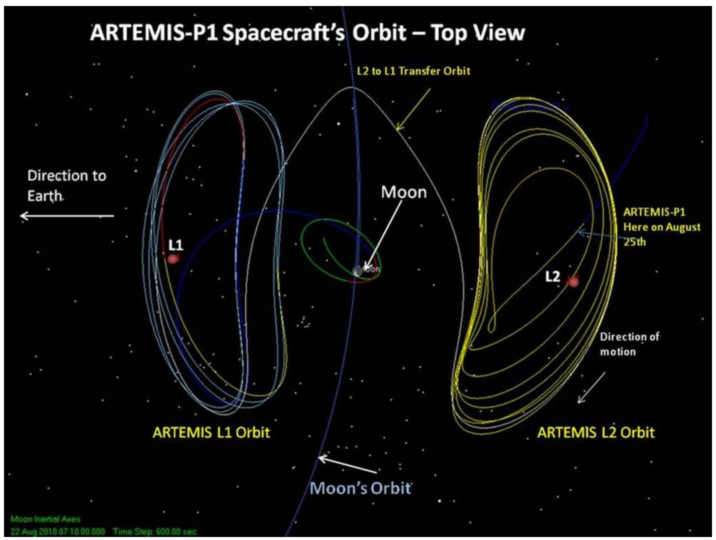
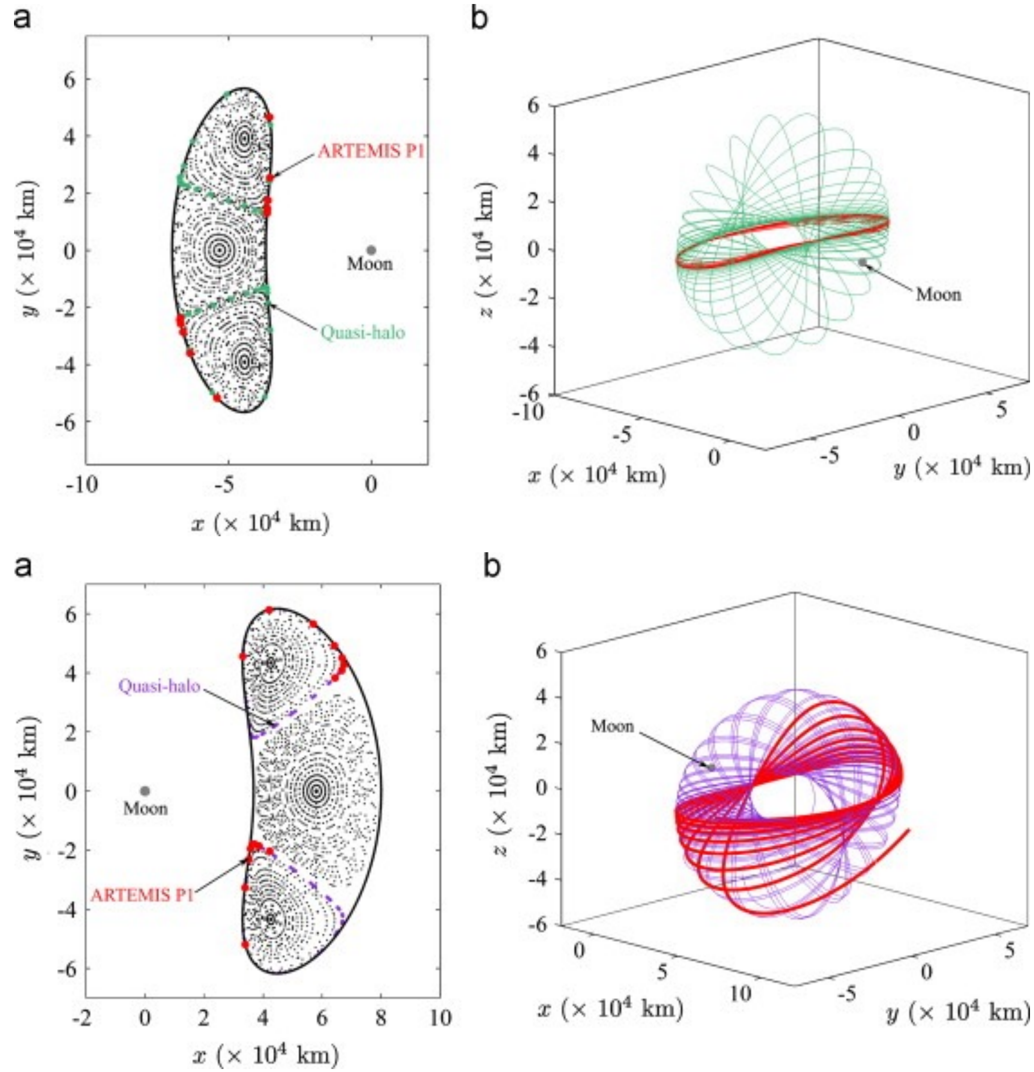


Image credit: NASA/Goddard Space Flight Center



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