Problem 1

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ASEN 5050
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HW 4

Problem 1 → Given → at t_1 in Saturn-centered inertial frame, $r_{eq,sat} = 60,268$ km

$\vec{r}_1 = -720,000\hat{x} + 670,000\hat{y} + 310,000\hat{z}$ km

$\vec{v}_1 = 2.160\hat{x} - 3.360\hat{y} + 0.620\hat{z}$ km/s

Assume → $Gm_{saturn} = 3.794 \times 10^7$ km^3/s^2 → $\mu = G(m_{s/c} + M_{saturn})$

$M_{s/c} \ll M_{saturn}$ → $\mu = GM_{saturn}$, 2 Body Problem

a] t_2 is time at impact of s/c on Saturn surface

From HW 3, Problem 1 → $\theta_2^* = -13.1687°$, $h = 2.0768 \times 10^6$ km^2/s,

$e = 0.9102$, $r_2 = r_{eq,saturn} = 60,268$ km, $r_1 = 1.0312 \times 10^6$ km

$p = h^2/\mu = 1.1368 \times 10^5$ km

$\theta_1^* = \cos^{-1}\left[\frac{1}{e}\left(\frac{h^2/\mu}{r_1} - 1\right)\right] = -167.8322°$

$\Delta\theta^* = \theta_2^* - \theta_1^* = 154.6635°$

$f = 1 - \frac{r_2}{p}(1 - \cos(\Delta\theta^*)) = -0.0093$

$g = \frac{r_1 r_2}{\sqrt{\mu p}}[\sin(\Delta\theta^*)] = 1.2806 \times 10^4$ s

$\dot{f} = \sqrt{\frac{\mu}{p}} \tan\left(\frac{\Delta\theta^*}{2}\right)\left(\frac{1-\cos(\Delta\theta^*)}{p} - \frac{1}{r_2} - \frac{1}{r_1}\right) = -6.6275 \times 10^{-5}$ $\frac{1}{s}$

$\dot{g} = 1 - \frac{r_1}{p}(1 - \cos(\Delta\theta^*)) = -16.2695$

$$\boxed{\begin{aligned}\vec{r}_2 &= f\vec{r}_1 + g\vec{v}_1 = 3.4356\times10^4\hat{x} - 4.9258\times10^4 + 5.0576\times10^3 \text{ km}\\ \vec{v}_2 &= \dot{f}\vec{r}_1 + \dot{g}\vec{v}_1 = 12.5761\hat{x} + 10.2611\hat{y} - 30.6325\hat{z} \text{ km/s}\end{aligned}}$$

b] $P = 2\pi\sqrt{\frac{a^3}{\mu}} = 5.5041 \times 10^5$ s, $n = \frac{2\pi}{P} = 1.1415 \times 10^{-5}$ $\frac{1}{s}$

$E_1 = \tan^{-1}\left[\sqrt{\frac{1-e}{1+e}} \tan\left(\frac{\theta_1^*}{2}\right)\right] \cdot 2 = -2.2278$ rad

$E_2 = \tan^{-1}\left[\sqrt{\frac{1-e}{1+e}} \tan\left(\frac{\theta_2^*}{2}\right)\right] \cdot 2 = -0.05$ rad

<1 revolution $\rightarrow t_{12} = \frac{1}{n}\left[E_2 - e\sin(E_2) - E_1 + e\sin(E_2)\right] = \boxed{1.3162\times10^5\ s}$

$\boxed{t_1 - t_I = 1.3162\times10^5 \text{ seconds}}$

Problem 2 $\rightarrow$ Given: t_1 – August 29, 1996 in Jupiter-centered inertial frame

$\vec{r}_1 = 5.352950\times10^6\,\hat{x} + 2.053778\times10^5\,\hat{y} - 4.0579700\times10^5\,\hat{z}$ km

$\vec{v}_1 = -4.164248\,\hat{x} + 1.963690\,\hat{y} + 3.191257\times10^{-7}\,\hat{z}$ km/s

$r_{eq,\,jup} = 71{,}492$ km

Assume: $GM_{jupiter} = 1.268\times10^8$ km^3/s^2, $M_{s/c} \ll M_{jupiter}$ – $\mu = GM_{jupiter}$

2 Body problem

a) $r_1 = |\vec{r}_1| = 5.4145\times10^6$ km, $v_1 = |\vec{v}_1| = 4.6151$ km/s

$\vec{e} = \frac{1}{\mu}\left[\left(v_1^2 - \frac{\mu}{r}\right)\vec{r}_1 - (\vec{r}_1\cdot\vec{v}_1)\vec{v}_1\right] = -0.7803\,\hat{x} + 0.3140\,\hat{y} + 0.0597\,\hat{z}$

$e = |\vec{e}| = 0.8432$

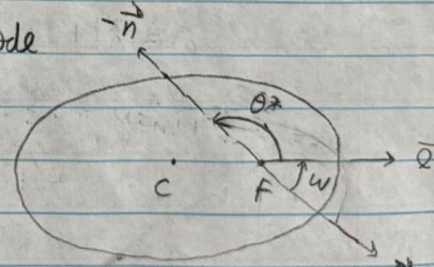
$\theta_1^* = \arccos\left(\frac{\vec{e}\cdot\vec{r}_1}{e\cdot r_1}\right) = \pm150.6541^\circ$

$\vec{r}_1\cdot\vec{v}_1 = -2.1035\times10^7 \rightarrow \because \vec{r}_1\cdot\vec{v}_1 < 0 \rightarrow \boxed{\theta_1^* = -150.6541^\circ}$

$E_1 = \tan^{-1}\left[\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta_1^*}{2}\right)\right] = \boxed{-1.6748 \text{ rad} = E_1}$

b) $t_2 \rightarrow$ s/c crosses descending node

General orbit to help with problem. Not representative of Problem 2. $\rightarrow$



Orbit tells us that $\omega + \theta^* = 180^\circ$ if s/c is at descending node.

$\therefore \theta_2^* = 180^\circ - \omega$

$\vec{h} = \vec{r}_1\times\vec{v}_1 = 1.0223\times10^6\,\hat{x} - 1.7704\times10^4\,\hat{y} + 1.3449\times10^7\,\hat{z}$ km^2/s

$|\vec{h}| = h = 1.3488\times10^7$ km^2/s

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$\vec{n} = \vec{z} \times \vec{h}$ where $\vec{z} = 0\hat{x} + 0\hat{y} + 1\hat{z}$

$= 1.7704 \times 10^4 \hat{x} + 1.0223 \times 10^6 \hat{y} + 0\hat{z}$ km²/s

$\hat{n} = 0.0173\hat{x} + 0.9999\hat{y} + 0\hat{z}$

$\omega = \arccos\left(\frac{\vec{n} \cdot \vec{e}}{|\vec{n}||\vec{e}|}\right) = \pm 69.1290°$, $\because e_3 = 0.0597 > 0 \rightarrow \omega = 69.1290°$

$\theta_2^* = 180° - \omega = \boxed{110.8710° = \theta_2^*}$

$E_2 = \tan^{-1}\left[\sqrt{\frac{1-e}{1+e}} \tan\left(\frac{\theta_2^*}{2}\right)\right] = \boxed{0.8009 \text{ rad} = E_2}$

c) $\mathcal{E} = \frac{v_1^2}{2} - \frac{\mu}{r_1} = -12.7693$ km²/s²

$a = -\frac{\mu}{\mathcal{E}} = 4.9650 \times 10^6$ km

$P = 2\pi\sqrt{a^3/\mu} = 6.1731 \times 10^6$ s $\rightarrow n = \frac{2\pi}{P} = 1.0178 \times 10^{-6}$

$\because E_1 < 0$, $(t_1 - t_p) < 0$ and denote time until periapsis

$(t_1 - t_p) = \frac{1}{n}[E_1 - e\sin(E_1)] = 8.2527 \times 10^6$ s

$\because E_2 > 0$, $(t_2 - t_p) > 0$ and denote time from periapsis

$\therefore (t_2 - t_p) = \frac{1}{n}[E_2 - e\sin(E_2)] = 1.9204 \times 10^5$ s

$t_{12} = t_2 + t_1 = 1.0173 \times 10^6$ s $= \boxed{11.7744 \text{ days} = t_1 \text{ to } t_2}$

d) $p = h^2/\mu = 1.4347 \times 10^6$ km

$\Delta\theta^* = \theta_2^* - \theta_1^* = 261.5251°$

$r_2 = \frac{p}{1 + e\cos\theta_2^*} = 2.0508 \times 10^6$ km

$f = 1 - \frac{r_2}{p}(1 - \cos(\Delta\theta^*)) = -0.6401$

$g = \frac{r_2 r_1}{\sqrt{\mu p}} \sin(\Delta\theta^*) = -8.1426 \times 10^{-4}$ s

$\dot{f} = \sqrt{\frac{\mu}{p}} \tan\left(\frac{\Delta\theta^*}{2}\right) = -1.3897 \times 10^{-6}$ $\frac{1}{s}$

$\dot{g} = 1 - \frac{r_1}{p}(1 - \cos(\Delta\theta^*)) = -3.3302$

$\rightarrow$ $\vec{r}_2 = f\vec{r}_1 + g\vec{v}_1$, $\vec{v}_2 = \dot{f}\vec{r}_1 + \dot{g}\vec{v}_1$

$$\boxed{\begin{aligned}\vec{r}_2 &= -3.551 \times 10^4 \hat{x} - 2.0505 \times 10^6 \hat{y} - 5.8208 \times 10^{-11} \hat{z} \text{ km} \\ \vec{v}_2 &= 6.4287\hat{x} - 7.5196\hat{y} - 0.4986\hat{z} \text{ km/s}\end{aligned}}$$

The z-component of $\vec{r}_2$ is very very close to 0 indicating that the s/c is on the xy plane. $i = \text{acos}(h_3/h) = 4.3426^\circ \rightarrow \therefore i \neq 0$, s/c is not in an equatorial orbit and on the line of nodes. $\because \vec{v}_2$ z-component is -0.4986 km/s < 0, s/c is on the descending node.
Lastly, $\text{acos}(\text{dot}(\vec{r}_2, \hat{n})/|\vec{r}_2|) = 180^\circ$ indicating s/c is opposite ascending node $\rightarrow$ on the descending node.

Problem 2

- Part e
  - Section i
    - Part 1, 2

e] i] 1] $M = E - e\sin(E) \rightarrow g(E) = M - E + e\sin(E)$

$g'(E) = -1 + e\cos(E)$

$$E_{N+1} = E_N - \frac{g(E_N)}{g'(E_N)} = E_N - \frac{(M - E_N + e\sin(E_N))}{-1 + e\cos(E_N)} =$$

$$\boxed{E_{N+1} = E_N + \frac{(M - E_N + e\sin(E_N))}{1 - e\cos(E_N)}}$$

2] $E_0 = M + e\sin(M) + \frac{e^2}{2}\sin(2M)$

- Part 3
  - The stopping condition checks the difference between E_N_plus_1 and E_N to see if that difference is less than the tolerance selected.
- Part 4
  - A tolerance of 1e-8 is used because most computers now can handle doing a few more iterations to get more precise answers and since 4 digits seems like the appropriate number of decimal places for most solutions, this just doubles it.

- Section ii

```
function ecc_anomaly = kepler_solver_eclipse(time_past_periapsis, a, e, mu)
    % Numerically solve kepler's equation for eccentric anomaly using time
    % past periapsis, semi-major axis, eccentricity, system gravitational
    % parameter.
    % Following algorithm 2 in Vallado section 2.2.5

    tol = 1e-8;

    % Calculate period
    P = 2 * pi * sqrt(a^3/mu);

    % Mean anomaly - M = n(t-tp) = 2pi/p * (t-tp)
    M = (2*pi)/P * time_past_periapsis;

    % Using  series solution of E_N and truncating higher order
    % terms
    E_N = M + e*sin(M) + e^2/2 * sin(2*M);

    % E_N+1 can be obtained through the Newton-Raphson method
    E_N_plus_1 = E_N + (M - E_N + e*sin(E_N))/(1 - e*cos(E_N));

    % Keep applying the NR method until tolerance is met
    while abs(E_N_plus_1 - E_N) > tol
        E_N = E_N_plus_1;
        E_N_plus_1 = E_N + (M - E_N + e*sin(E_N))/(1 - e*cos(E_N));
    end

    % Output
    ecc_anomaly = E_N_plus_1;
end
```

- Part f

f) t_3 is 20 days after t_1. $t_1 - t_p$ indicate time to periapsis.

$(t_1 - t_p) = 9.5518$ days.

$t_3 - t_p$ is time since periapsis at t_3

$\therefore t_3 - t_p = 20 - (t_1 - t_p) = 10.4482 \text{ days} = 9.0273 \times 10^5 \text{ s}$

We have time since periapsis $(t_3 - t_p)$, a, e, μ. We can pass this in the Kepler eqn solver in part e for E_3

$\boxed{\therefore E_3 = 1.748741 \text{ rad}}$

$$\theta_3^* = \tan^{-1}\left[\sqrt{\frac{1+e}{1-e}} \tan\left(\frac{E_3}{2}\right)\right] \cdot 2 = \boxed{152.59034^\circ = \theta_3^*}$$