Numerical Solution Techniques to the Optimal Control Problem

Generally mean solving the <u>Nec. Conds.</u>

$$\dot{\vec{x}} = \frac{\partial H^*}{\partial \vec{p}} \; ; \; \dot{\vec{p}} = -\frac{\partial H^*}{\partial \vec{x}} \; ; \; \vec{u}^* = \underset{\vec{u} \in \Omega}{\operatorname{argmin}} \, H(\vec{x}, \vec{p}, \vec{u}, t)$$

"Hard Constraint" problem

$$\left.\begin{aligned} \vec{x}(t_0) &= \vec{x}_0 \\ \vec{x}(t_f) &= \vec{x}_f \end{aligned}\right\}$$

Many other conditions possible, but always results in a 2 point boundary value problem (2PBVP).

Transversality conditions are $\vec{P}(t_0)$, $\vec{P}(t_f)$, H_0^*, H_f^* are all unknown.

<u>Indirect Methods</u>: Knowledge of the correct value of $\vec{P}_0$ or $\vec{P}_f$ "solves" the O.C. problem.

Consider the joint state

$$\mathbf{X} = \begin{bmatrix} \vec{x} \\ \vec{p} \end{bmatrix} \in \mathbb{R}^{2n}.$$

Solution of the N. C. gives us

$$\boxed{\mathbf{X}(t) = \varphi(t; \vec{x}_0, \vec{p}_0, t_0)}$$

Mathematical Representation of a solution to the IVP.

Target Solution:

$$\mathbf{X}(t_f) = \begin{bmatrix} \vec{x}_f \\ \vec{p}_f \end{bmatrix} = \begin{bmatrix} \varphi_x(t_f; \vec{x}_0, \vec{p}_0, t_0) \\ \varphi_p(t_f; \vec{x}_0, \vec{p}_0, t_0) \end{bmatrix}$$

Don't care about, unless we choose to go backwards in time.

Need to satisfy $\boxed{\vec{x}_f = \varphi_x(t_f; \vec{x}_0, \vec{p}_0, t_0)}$

If we relax the "hard constraint" problem by letting x_{F_i} be "Free",
it then requires us to target specific values of $p_{F_i} = 0$ ..

In general we do not know $\vec{p}_0$ & must solve for it.

This defines the 2PBVP

$$\vec{x}_F = \varphi_x(t_F; \vec{x}_0, \vec{p}_0, t_0),$$ solving for $\vec{p}_0$.

If a good initial guess is available, we just need to solve

$$\vec{x}_F = \varphi_x(t_F; \vec{x}_0, \vec{p}_0 + \delta\vec{p}_0, t_0) = \boxed{\varphi_x\Big|_0 + \frac{\partial \varphi_x}{\partial \vec{p}}\Big|_0 \cdot \delta\vec{p}_0 + \cdots \mathcal{O}(\delta p_0^2)}$$

$$\vec{p}_F + \delta p_F = \varphi_p(t_F; \vec{x}_0, \vec{p}_0 + \delta\vec{p}_0, t_0)$$

Implying that $|\delta\vec{p}_0| \ll 1$.

Given the nominal & the STM quadrate, $\dfrac{\partial \psi_x}{\partial \vec{p}}$,

Solve for $\delta \vec{p}_0$ as

$$\delta \vec{p}_0 = \left(\frac{\partial \psi_x}{\partial \vec{p}} \right)\Bigg|_{0,f}^{-1} \cdot \left[\vec{x}_f - \psi_x(t_f; \vec{x}_0, \vec{p}_0, t_0) \right]$$

$\vec{p}_0{}' = \vec{p}_0 + \delta \vec{p}_0 \Rightarrow$ repeat the process iteratively.

"Seldomly" works well, especially for higher dimensional systems, unless we have a good initial guess.

IF $\vec{p}_0 \in \mathbb{R}^1 \Rightarrow$ sweep through p_0

$\vec{p}_0 \in \mathbb{R}^2 \Rightarrow$ start to have issues, but simpler

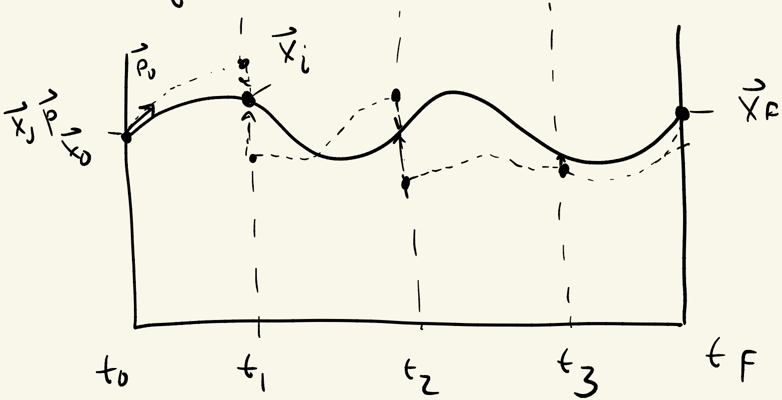
$\vdots$

$\vec{p}_0 \in \mathbb{R}^6 \Rightarrow$ real world....

Variations on "Single Shooting" Method outlines above.

Indirect Multiple Shooting

Replace one O.C. 2PBVP with several by dividing the time domain.



Enforce Internal constraints

$$\varphi_{x_{\hat{i}}}(t; \vec{x}_{i-1}, \vec{p}_{i-1}, t_{i-1})$$

$$\varphi_{x_{\hat{i}}}(t_i) = \vec{x}_i$$

Find State

$$\underline{X}_{MS} = \begin{bmatrix} \vec{x}_0 \\ \vec{x}_1 \\ \vdots \\ \vec{x}_M \\ \hline \vec{p}_0 \\ \vec{p}_1 \\ \vdots \\ \vec{p}_M \end{bmatrix}$$

$\Rightarrow$ Although the state space is large, the associated gradients are "sparse" as each segment tends to be independent

Indirect Collocation/Transcription

Analogous to Mult. Shooting, but we choose time intervals to approx. Integrate the DEQs.

"Euler Step"

$$\vec{x}_{i+1} = \vec{x}_i + \Delta t \left.\frac{\partial H}{\partial \vec{p}}\right|_i$$

$$\vec{p}_{i+1} = \vec{p}_i - \Delta t \left.\frac{\partial H}{\partial \vec{x}}\right|_i$$

$\left.\frac{\partial H}{\partial \vec{p}}\right|_i$ & $\left.\frac{\partial H}{\partial \vec{x}}\right|_i$ are fixed with $\vec{u}_i$ chosen to minimize $H(\vec{x}_i, \vec{p}_i, \vec{u}, t_i)$.

"Step" can be generalized to more sophisticated approaches,

Ⓐ Devise higher order integration steps for mapping from state to state.

Ⓑ Utilize interpolating functions that can fit the states & adjoints across each time interval, Chebyshev Polynomials

(c) "collocation" methods called Pseudo-Spectral and expand states & adjoints in frequency space & solve the Multi Shooting Problem.

Indirect Methods provide the most precise & accurate description of the optimal traj. & control

$\boxed{\text{Direct Methods}}$ DM are more "practical." Instead of using the Nec. Conds to generate the control law, we presume that the control has a certain form:

$$\vec{u}^* = \underset{u \in U}{\operatorname{argmin}}\, H(\vec{x}, \vec{p}, \vec{u}, t)$$

vs

$$\vec{u} = \sum_i \underline{\vec{\alpha}_i} \Lambda_i(t)$$; $\Lambda_i(t)$ are some set of interpolating functions in time, $\vec{\alpha}_i$ are unknown coefficients.

Examples: $\Lambda_n = t^n \rightarrow$ polynomial

$= e^{int} \rightarrow$ Fourier Series

$=$ Chebyshev Polynomial, etc.

Two philosophies for choosing $\Lambda_i(t)$

- Choose $\Lambda_i(t)$ to match the approx. form of the O. C. Law.

  Launch trajectory $\theta(t) = \alpha_0 t + \alpha_1 t^2 + \cdots$

- Choose $\Lambda_i(t)$ to be able to "converge" to an arbitrary function of time.

  Use Chebyshev Polys, Fourier Series, etc...

$$J = K(\vec{x}_0, t_0, \vec{x}_f, t_f) + \int_{t_0}^{t_f} L(\vec{x}, \vec{u}, \tau)\, d\tau$$

$$\dot{\vec{x}} = \vec{F}(\vec{x}, \vec{u}, t)$$

$$\vec{g}(\vec{x}_0, t_0, \vec{x}_f, t_f) = 0 \checkmark$$

$$\boxed{\vec{u}(t) = \sum_{i=1}^{N} \vec{\alpha}_i \Lambda_i(t)}$$

; $\Lambda_i(t)$ basis functions we know.

$$\to J(\vec{\alpha}) \quad ; \quad \vec{\alpha} = \begin{bmatrix} \vec{\alpha}_1 \\ \vec{\alpha}_2 \\ \vec{\alpha}_3 \\ \vdots \\ \vec{\alpha}_N \end{bmatrix}$$

Minimize $J(\vec{\alpha})$ subject to constraints $\boxed{\vec{g}(\vec{\alpha}) = 0}$, integration of the EoM is implied.

Can incorporate inequality constraints such as $\vec{h}(\vec{\alpha}) \geq 0$

⋮

Nec. Conditions for a parametric optimization problem.

Generally we assume constraint qualification, so we can apply the KKT conditions.

Same range of methods exist to solve this problem.

Direct Shooting: $\vec{u}$ defined over the entire interval.

Multiple Shooting: $\vec{u}$ defined over mult. time sequences, solve deqs across each segment.

Transcription/Collocation: Use discrete dynamics approximations to map in time, simplest approach use constant $\vec{u}_i$ over each step.

See J.T. Betts, "Survey of Numerical Methods for Traj. Opt.", JGCD Vol 12(2) March-April 1998.

Has a SIAM Book as well.

Standard S/W packages for solving Direct Methods

(MPP)

- POST, SNOPT, TOMLAB.

Now programs for Indirect Solutions: Copernicus NASA

- Pseudo Spectral Methods:

DIDO , I.M. Ross (NPS)

* GPOPS , A. Rao (U.F.)