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Illustration of the Primer Vector in Time-Fixed Orbit Transfer

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In optimal (fuel minimum) impulsive orbit transfer, few results are known for the "time-fixed" case (prescribed transfer time) in comparison with the known "time-open" solutions,[1] such as the Hohmann transfer. Time-fixed results have been reported in Refs. 3 and 4.

In this Note a technique for obtaining optimal time-fixed impulsive solutions, applied to orbital rendezvous in Ref. 4, is illustrated by applying it to a very simple system. The technique is based on satisfying a set of necessary conditions (NC) derived by Lawden[2] and described in terms of the primer vector, the vector of adjoint variables associated with the vehicle velocity vector. A simple illustrative example allows an uncomplicated presentation of the technique and yields optimal solutions that are verifiable by physical reasoning. In addition it is shown that for this example the NC are also sufficient.

The simple example treated is a one-dimensional harmonic oscillator. In terms of the displacement from equilibrium z, a nondimensional time $\tau = \omega t$, and a state vector $x^T = [z\ z']$, $[(\ )' \triangleq d(\ )/d\tau]$, the system differential equation is of the form $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{1}$$

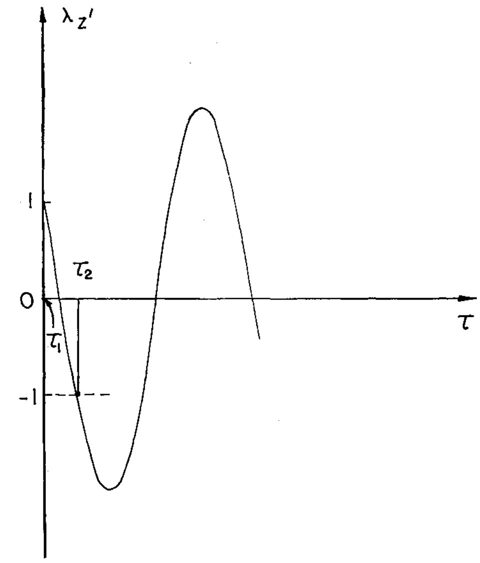
Defining $\tau_{ji} = \tau_j - \tau_i$, the state transition matrix is

$$\Phi_{ji} = \begin{bmatrix} \cos\tau_{ji} & \sin\tau_{ji} \\ -\sin\tau_{ji} & \cos\tau_{ji} \end{bmatrix} \tag{2}$$

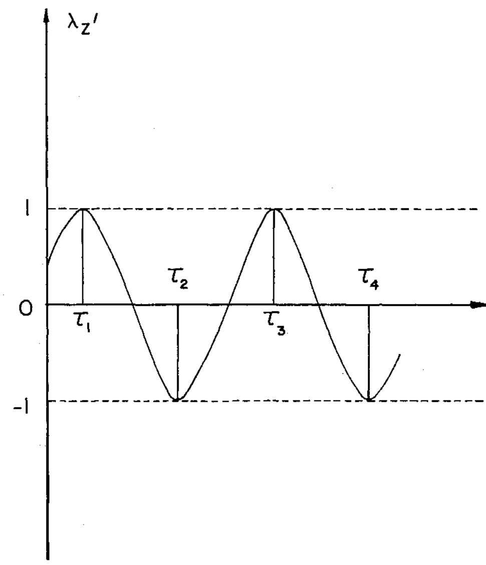
An orbital mechanics problem described by this system is the linearized out-of-plane motion near a circular reference orbit. In this context, the time-fixed optimal solutions can be interpreted as achieving prescribed out-of-plane position and velocity optimally in a prescribed time from arbitrary initial conditions.

Partitioning the vector adjoint to the state vector as $\boldsymbol{\lambda}^T = [\lambda_z \lambda_{z'}]$ one identifies $\lambda_{z'}$ as the primer vector, since it is the adjoint variable associated with the velocity z'. In this example the primer vector appears as a scalar since its direction is always parallel to the line of motion of the oscillator; its sense is determined by the sign of $\lambda_{z'}$. In nonlinear systems the primer is obtained from the adjoint to the state variational equations.

The NC of Lawden[2] are 1) $|\lambda_{z'}| \leq 1$ during the transfer; 2) impulses are applied at times for which $|\lambda_{z'}| = 1$; 3) the sign of each velocity change $\Delta z'$, due to a thrust impulse, is the sign of $\lambda_{z'}$ at these times. For this example $\lambda_{z'}$ varies sinusoidally. Figure 1 shows construction of the two possible types of primer solutions which satisfy the NC. These solutions differ in the boundary conditions on $\lambda_{z'}$. Figure 1a is a two-impulse transfer of duration $\tau_2 - \tau_1$ with impulses occurring at the terminal times; no terminal coasting period is allowed since the primer exceeds unit magnitude outside the interval $[\tau_1, \tau_2]$. Figure 1b is a transfer in which any number of impulses and unlimited terminal coasting periods are allowed since $|\lambda_{z'}| \leq 1$ for all time. Note that in both cases the impulses alternate direction.



a) $\tau_1 = \tau_0 = 0;\ 0 < \tau_{F0} < \pi.$



b) $\tau_n - \tau_1 = (n-1)\pi$
$\tau_0 = 0$

Fig. 1 Primer vector solution.

Once a primer solution satisfying the NC has been constructed, the optimal times and directions of the thrust impulses are known. What initial state the solution corresponds to and what size thrust impulses are required to achieve the desired final state are determined by solving a boundary-value problem. The final state $\mathbf{x}_F$ can be expressed in terms of the initial state $\mathbf{x}_0$, and the n velocity changes $\Delta z'_j$ as

$$\mathbf{x}_F = \Phi_{F0}\mathbf{x}_0 + \sum_{j=1}^{n} \Phi_{Fj} \begin{bmatrix} 0 \\ \Delta z'_j \end{bmatrix} \tag{3}$$

Since both z and z' are to be described at the final time, a theorem by Potter[5] places the maximum number of impulses necessary to realize the optimal solution at two. Let the

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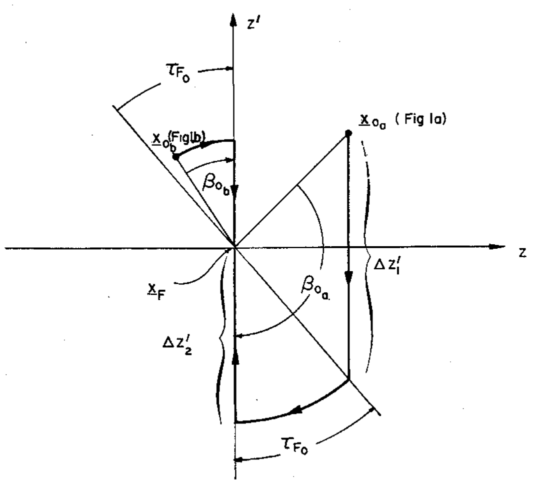

Fig. 2 Phase plane plot showing optimal transfers from two initial states, x_{0a} and x_{0b}, in fixed time τ_{F0}.

desired final state be rest equilibrium ($z = z' = 0$). For two impulses, Eq. (3) can be rewritten in terms of $\tau_{Fk} \triangleq \tau_F - \tau_k$ ($k = 0, 1, 2$) as

$$\begin{bmatrix} \sin\tau_{F1} & \sin\tau_{F2} \\ \cos\tau_{F1} & \cos\tau_{F2} \end{bmatrix} \begin{bmatrix} \Delta z'_1 \\ \Delta z'_2 \end{bmatrix} = - \begin{bmatrix} \cos\tau_{F0} & \sin\tau_{F0} \\ -\sin\tau_{F0} & \cos\tau_{F0} \end{bmatrix} \begin{bmatrix} z_0 \\ z'_0 \end{bmatrix} \quad (4)$$

For solutions of the form of Fig. 1a, $\tau_{F0} = \tau_2 - \tau_1$ (no terminal coasts). The solution for the velocity changes can be written in terms of β_0, the angle between the initial state vector in the phase plane and the next crossing of $z = 0$ (Fig. 2) ($0 \leq \beta_0 < \pi$) ($z'_0 = -z_0 \cot\beta_0$):

$$\begin{bmatrix} \Delta z'_1 \\ \Delta z'_2 \end{bmatrix} = z_0 \begin{bmatrix} -\cot\tau_{F0} & 1 \\ \csc\tau_{F0} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \cot\beta_0 \end{bmatrix} \quad (5)$$

Since $0 < \tau_{F0} < \pi$ (Fig. 1a), $\Delta z'_2/z_0 > 0$, requiring $\Delta z'_1/z_0 < 0$ to satisfy the NC. This requires $\cot\beta_0 < \cot\tau_{F0}$ implying that $\tau_{F0} < \beta_0$. A nondimensional fuel cost is obtained in terms of $\rho_0 = (z_0{}^2 + z'_0{}^2)^{1/2}$ as

$$\frac{\Delta V}{\rho_0} \triangleq \frac{1}{\rho_0}\left(|\Delta z'_1| + |\Delta z'_2|\right) = \sin\beta_0 \left[\cot\left(\frac{\tau_{F0}}{2}\right) - \cot\beta_0\right] \quad (6)$$

For a primer solution of the form (1b), Eq. (3) yields a unique one-impulse solution for $\beta_0 \leq \tau_{F0} < \beta_0 + \pi$. In this case $\tau_{F0} = \tau_{F1} + \beta_0$ (one coasts to $z = 0$ before applying the single impulse). The fuel cost is $\Delta V/\rho_0 = 1$, the absolute minimum fuel cost.

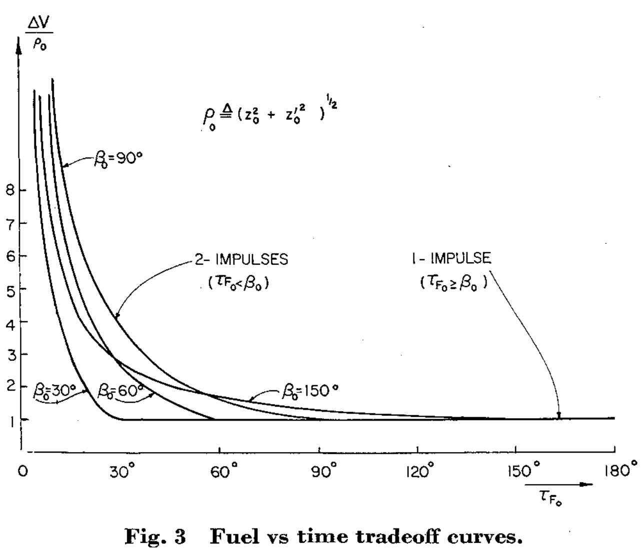

Fig. 3 Fuel vs time tradeoff curves.

Nonunique n impulse solutions also exist for $\tau_{F0} \geq \beta_0 + (n - 1)\pi$. These transfers require the absolute minimum fuel and can be accomplished by the $n = 1$ solution described previously. This is consistent with the result of Potter,[5] which says that fuel cannot be saved by using more than two impulses. The n velocity changes, applied at $z = 0$ and directed toward the origin, have arbitrary magnitudes as long as the total fuel cost is the absolute minimum. This nonuniqueness is demonstrated for $n = 2$ by the fact that the velocity-change coefficient matrix in Eq. (4) is singular. The initial coast required for these solutions is of duration β_0 with a final coast (waiting at the origin for the specified time to elapse) of duration $\tau_{F0} - \beta_0 - (n - 1)\pi$. Thus, this solution is similar in several ways to a Hohmann transfer for rendezvous between close circular orbits[4]; the initial coast is analogous to waiting for the correct phase angle between the target and vehicle.

Since the form (1a) corresponds to $\tau_{F0} < \beta_0$ and (1b) to $\tau_{F0} \geq \beta_0$, one can plot a one-parameter (β_0) family of time-fuel tradeoff curves as shown in Fig. 3. For each prescribed transfer time τ_{F0} and initial conditions β_0 one can determine the fuel cost and number of impulses required.

To show that the NC are also sufficient, one uses Eq. (3) and the property of the adjoint system, $\boldsymbol{\lambda}^T = \boldsymbol{\lambda}_F{}^T\Phi_{Fj}$ to show that

$$a \triangleq \boldsymbol{\lambda}_F{}^T(\mathbf{x}_F - \Phi_{F0}\mathbf{x}_0) = \sum_{j=1}^{n} \lambda_{z'_j}\Delta z'_j$$

Furthermore, if the NC are satisfied, $|\lambda_{z'_j}| \leq 1$, and

$$\sum_{j=1}^{n} \lambda_{z'_j}\Delta z_j \leq \sum_{j=1}^{n} |\lambda_{z'_j}||\Delta z'_j| \leq \sum_{j=1}^{n} |\Delta z'_j| \triangleq \Delta V$$

with equalities only if each $\lambda_{z'_j}$ and its corresponding $\Delta z'_j$ have the same sign and $|\lambda_{z'_j}| = 1$. These are precisely the remaining NC. Therefore, if the NC are satisfied, the fuel cost ΔV equals a. Otherwise $\Delta V > a$.

The primer vector has the physical interpretation of being a measure of the fuel efficiency of the times and directions of the thrust impulses. The NC prescribed the times and directions for the impulses that use fuel most efficiently. For the oscillator, the most efficient times to impulsively change the total energy of the system are those times at which the velocity is a maximum ($z = 0$); to diminish the energy (as in the example) the impulses are directed counter to the motion of the oscillator. This is a physical interpretation of Fig. 1b (case b in Fig. 2), where $\tau_1 = \beta_0$. In this case $|\lambda_{z'}|$ can be interpreted as $|z'/z'_{max}|$, with the impulses occurring when $|\lambda_{z'}| = 1$.

In the case in which the fixed transfer time is not large enough to allow the oscillator to coast to a state of maximum velocity, an impulse must be applied when $z \neq 0$. In this situation one applies the smallest impulse which will transfer the oscillator to $z = 0$ within the allotted time; a second impulse then occurs at $z = 0$ to achieve the final boundary conditions. This is a physical interpretation of Fig. 1a (case a of Fig. 2).

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