ASEN 5044, Fall 2024

Statistical Estimation for Dynamical Systems

Lecture 06:
Discrete Time Linear State Space Systems

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Thursday 9/12/2024

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Announcements

- **Prof. Ahmed out of country next week (SPIE Defense & Security Conference in UK)**
  - **No live classes next Tues 09/17 or Thurs 09/19, BUT pre-recorded Lecture Videos to be posted – WATCH THEM!! (will need them for HW 2)**
  - **TF Aidan to cover Prof. Ahmed's hours next Wed 09/18 4:30-6 pm, AERO N353**
  - **Regular in-person lectures to resume Tues 09/24**
- **HW 1 due tomorrow Fri 9/13 at 11:59 pm**
- **Submit to Gradescope (linked via Canvas) –**
  - **All submissions must be legible!!! – zero credit otherwise**
- Advanced Questions: these are optional/extra credit (follow instructions)
- HW 2 will be posted today, due Fri 09/20; HW 3 to be posted next Thurs 09/19
- Quiz 3: this Friday-~~Sun~~ Tues day via Canvas; NO QUIZ NEXT WEEK
- **MIDTERM 1 – TO BE RELEASED Thurs 10/03, DUE Thurs 10/10**
  - **Take home exam, to focus on material covered in HWs1-4 + quizzes up to that point**

Overview

Last time: Linearization of nonlinear to linear SS models

- Please read slides 13-16 of Lec 05 on your own!! (and work out the example on your own!)

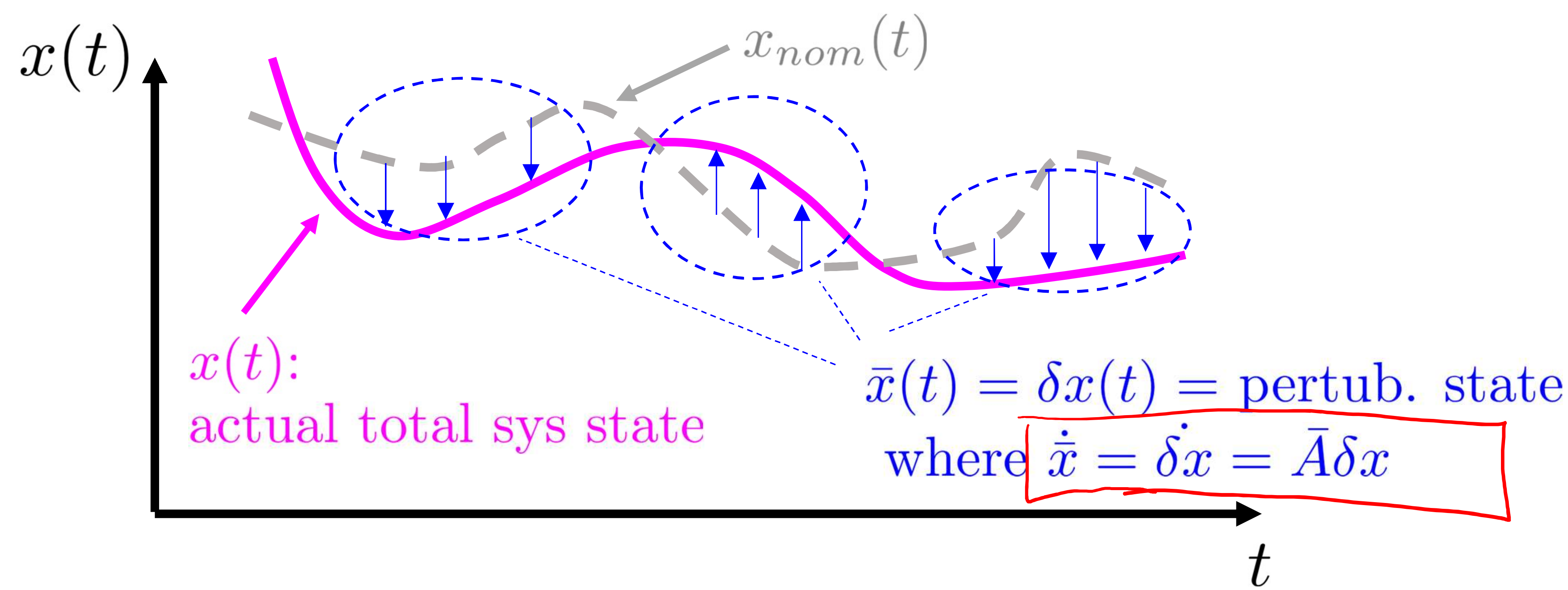
$$* \quad \bar{x} = \delta x = \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = \text{perturbation state vector} \quad (\neq \text{total state vector } x(t)!!)$$

$\delta x(t) \triangleq x(t) - x_{nom}(t)$

$\rightarrow$ What is actual total state of NL system at any time (w.r.t. op. pt.)?

$$x(t) = x_{nom}(t) + \delta x(t)$$

$$= \begin{bmatrix} x_{nom,1}(t) \\ x_{nom,2}(t) \end{bmatrix} + \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$$



Overview

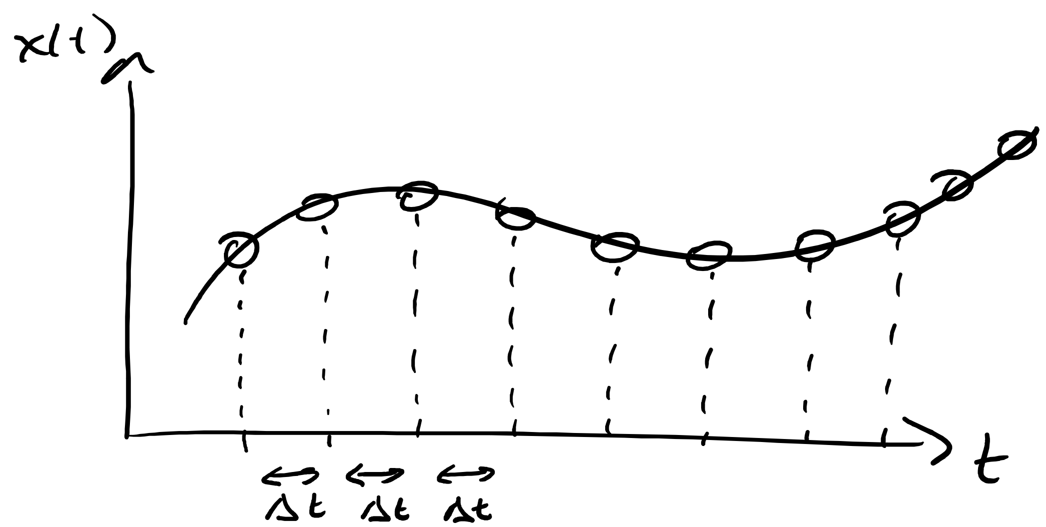
Today:

- **Discrete time (DT) linear systems,**
- **Converting continuous time (CT) systems to DT systems**

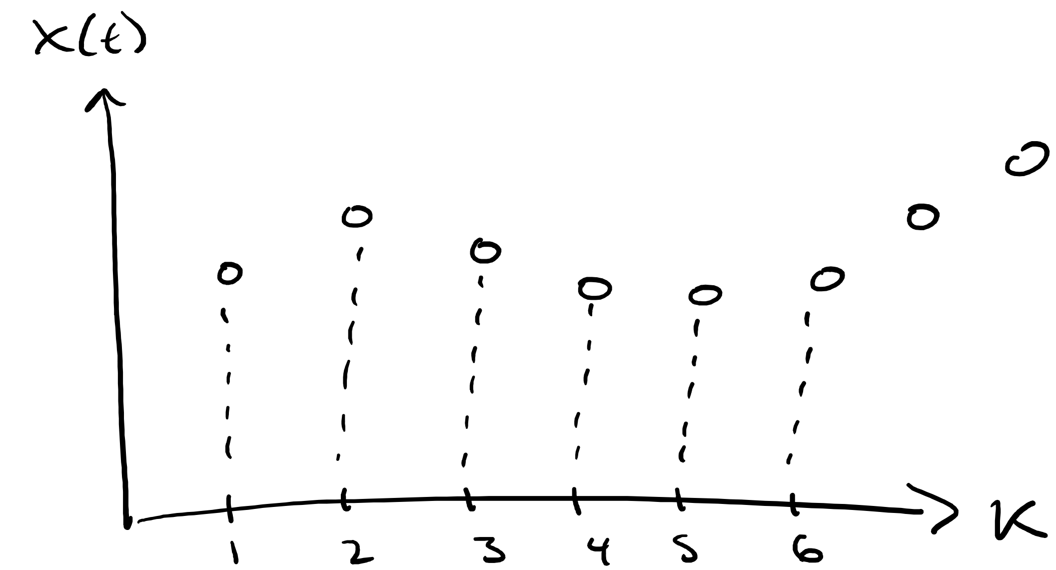
READ: Chapter 2.1 in Simon book (intro to probability)

Discrete Time Dynamic System Models

- State vector ="internal memory" of what system is doing at any given time
- In applications: only care to know what system is doing at fixed time instants, esp. sampled intervals for digital systems



$\Rightarrow$



- Some systems are naturally "episodic", i.e. agnostic to physical time
    - Baseball innings; rounds of poker, pool, squash, boxing, negotiation…
    - Finite state automata for computing, event-based systems
    - Often naturally described by finite difference equations (FDEs)

Discrete Time (DT) Dynamic System Models

- Convenient to specify dynamics as updates to internal system memory (i.e. state vector) from one discrete time step to another
- Linear DT models: matrices **summarize changes between integer time steps k**

$k = 0, 1, 2, 3, \ldots$

$$x(k+1) = x_{k+1} = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = F(k)\,x(k) + G(k)\,u(k), \quad u(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \\ \vdots \\ u_m(k) \end{bmatrix}$$

Linear Time Varying DT SS Model

$$y(k+1) = y_{k+1} = \begin{bmatrix} y_1(k+1) \\ \vdots \\ y_p(k+1) \end{bmatrix} = H(k+1)\,x(k+1) + M(k+1)\,u(k+1)$$

$F(k)$: State Transition Matrix (STM) $\in \mathbb{R}^{n \times n}$

$G(k)$: Control effect matrix $\in \mathbb{R}^{n \times m}$

$H(k+1)$: Sensing matrix $\in \mathbb{R}^{p \times n}$

$M(k+1)$: direct transmission matrix $\in \mathbb{R}^{p \times m}$

→ If no dependence on k (time step), then we get LTI DT SS : F, G, H, M, i.e. $x_{k+1} = F x_k + G u_k$ etc.

Example: Linear Car Dealer Model (episodic LTI DT system)

- You are haggling with car dealer Gary Slick for a used Ferrari
- At negotiation round k, **Slick's offer = $x_1(k)$** and **your offer = $x_2(k)$**
- Offering algorithm (finite difference equation, FDE):
  - At each round k, you both lay down offers simultaneously
  - For round k+1, you update by adding fraction μ of difference to $x_2(k)$
  - For round k+1, Slick updates by subtracting fraction λ of difference from $x_1(k)$

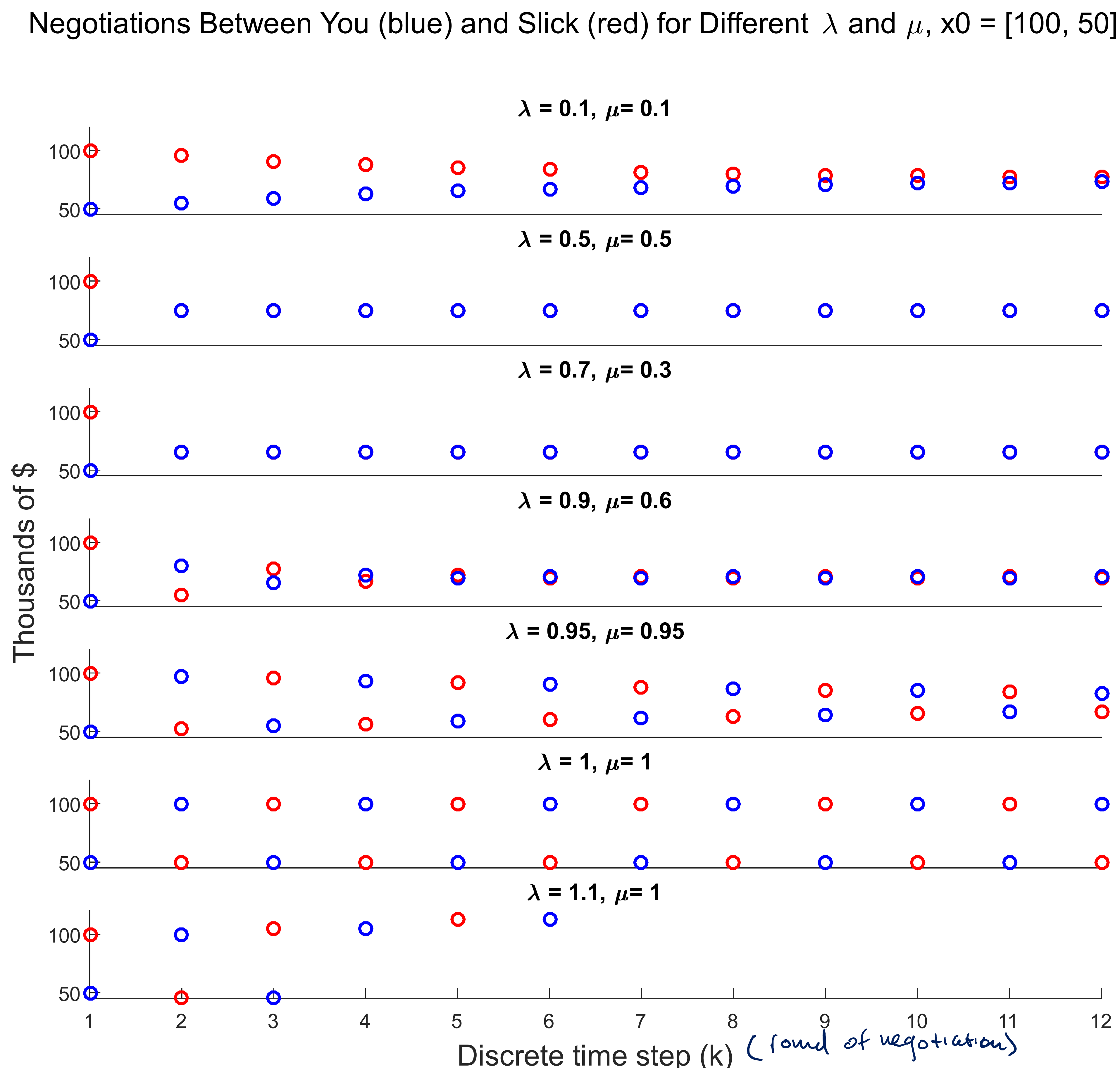
FDE: let $\Delta_k = x_1(k) - x_2(k)$ (offer difference)

Slick's offer at time $k+1$: $x_1(k+1) = x_1(k) - \lambda \cdot \Delta_k \ = \ x_1(k) - \lambda[x_1(k) - x_2(k)]$

Your offer at time $k+1$: $x_2(k+1) = x_2(k) + \mu \cdot \Delta_k \ = \ x_2(k) + \mu[x_1(k) - x_2(k)]$

$\rightarrow$ Simplify into LTI DT SS Model: Define $x(k+1) = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} (1-\lambda)x_1(k) + \lambda x_2(k) \\ \mu x_1(k) + (1-\mu)x_2(k) \end{bmatrix}$

$\rightarrow$ Rewrite in F matrix times $x(k)$ form: $x(k+1) = \begin{bmatrix} (1-\lambda) & \lambda \\ \mu & (1-\mu) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = Fx(k), \ x(k) \in \mathbb{R}^{2\times 1}$


Negotiations Between You (blue) and Slick (red) for Different λ and μ, x0 = [100, 50]
λ = 0.1, μ= 0.1
λ = 0.5, μ= 0.5
λ = 0.7, μ= 0.3
λ = 0.9, μ= 0.6
λ = 0.95, μ= 0.95
λ = 1, μ= 1
λ = 1.1, μ= 1
Thousands of $
100
50
1
2
3
4
5
6
7
8
9
10
11
12
Discrete time step (k)
(round of negotiation)

Converting CT Linear Models to Sampled DT Linear Models

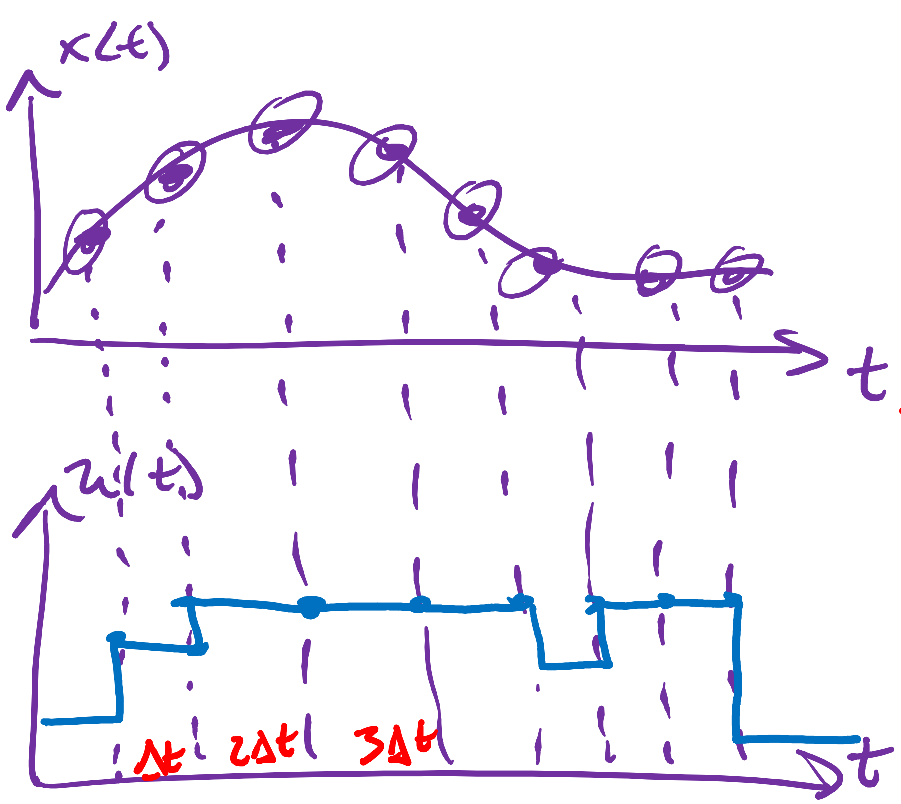
- How to translate from CT model (linear system of ODEs) for a given system?

$$\dot{x} = Ax(t) + Bu(t) \qquad\qquad x_{k+1} = Fx_k + Gu_k$$
$$y(t) = Cx(t) + Du(t) \quad \xrightarrow{??} \quad y_k = Hx_k + Mu_k$$

- Suppose u(k) follows a **zero-order hold (ZOH) discretization of u(t):**

$$u(t) = \text{some const.}, t \in [t_k, t_{k+1})$$

- Recall: general state solution x(t) is (for given $x(t_0)$):



Zero order Hold (ZOH)

$$x(t) = \underbrace{e^{A(t-t_0)}}_{n \times n} x(t_0) \; + \int_{t_0}^{t} \underbrace{e^{A(t-\tau)}}_{n \times n} \underbrace{B}_{n \times m} \underbrace{u(\tau)}_{m \times 1} d\tau$$

$\rightarrow$ If we use ZOH input $u(t)$ with fixed Δt sample time, then:

$$x(t) = x(t_0 + \Delta t) = e^{A\Delta t} x(t_0) \; + \boxed{\left[\int_{t_0}^{t} e^{A(t-\tau)} d\tau\right] B} \; u(t_0)$$

$$x(k+1) = Fx(k) + Gu(k)$$

Converting CT Linear Models to Sampled DT Linear Models

- **FACT:** if CT LTI SS model has ZOH input u(t) applied for fixed sample periods $\Delta t = t - t_0$, then can explicitly find DT LTI matrices (F,G,H,M) such that:

$$\dot{x} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$$\boxed{\begin{aligned} x_{k+1} &= Fx_k + Gu_k \\ y_k &= Hx_k + Mu_k \end{aligned}}$$

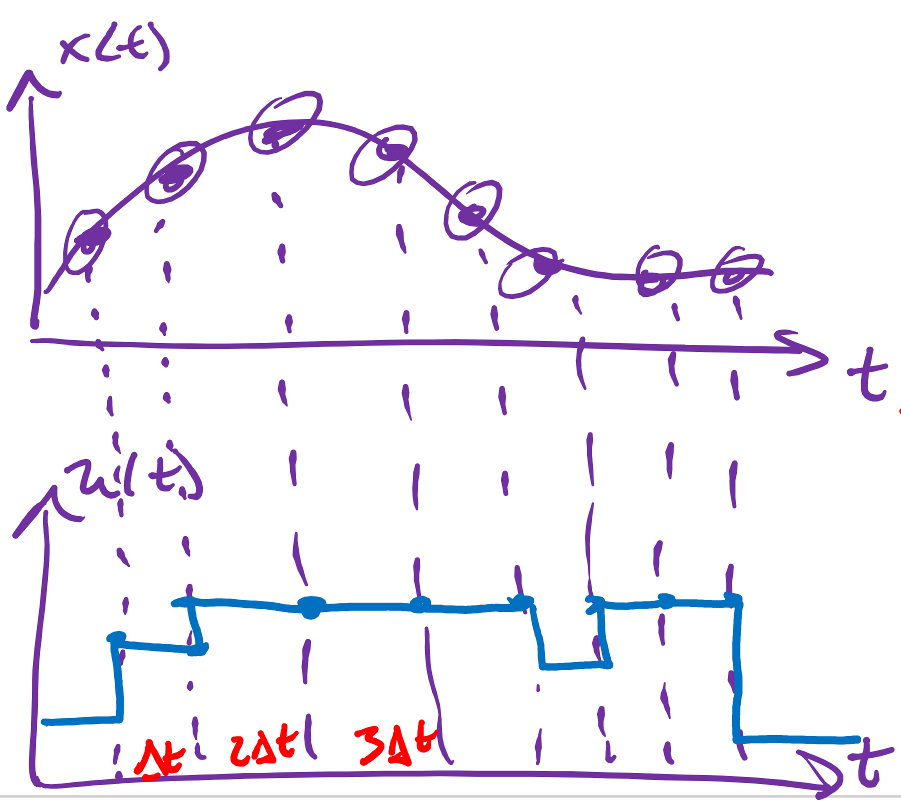
$$\boxed{\begin{aligned} F &= e^{A\Delta t} \\ \boxed{G} &= \left[\int_{t_0}^{t} e^{A(t-\tau)} d\tau\right] B \\ H &= C \\ M &= D \end{aligned}}$$

How to get this??

Chapter 1 of Simon book gives formula for G if A is invertible…

…but what if A is singular generally?

e.g. $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ → Inverse doesn't exist! (note: this is A matrix for the simple block mass problem in Lec 3!)



Zero order Hold (ZOH)

Computing the G matrix

- How to actually compute the G matrix integral? $G = \left[\int_{t_0}^{t} e^{A(t-\tau)} d\tau \right] \cdot B$
- First look at expansion of the integral:

$$\int_{t_0}^{t} e^{A(t-\tau)} d\tau = \int_0^{\Delta t} e^{A(\Delta t - \tau)} d\tau$$

$\longrightarrow$ Now, plug in ∞ series def. of matrix exp.:

$$= \int_0^{\Delta t} \sum_{i=0}^{\infty} A^i \frac{(\Delta t - \tau)^i}{i!} d\tau$$

$$= \sum_{i=0}^{\infty} \int_0^{\Delta t} A^i \frac{(\Delta t - \tau)^i}{i!} d\tau = \sum_{i=0}^{\infty} A^i \int_0^{\Delta t} \frac{(\Delta t - \tau)^i}{i!} d\tau$$

easy to show:

$$= \sum_{i=1}^{\infty} A^{i-1} \frac{\Delta t^i}{i!}$$

$\longrightarrow$ so $G = \left[\sum_{i=1}^{\infty} A^{i-1} \frac{\Delta t^i}{i!} \right] \cdot B$ $\longrightarrow$ what does this converge to?

How to compute??

Computing the G matrix

- Turns out there is a sneaky trick to computing this series for <u>ZOH</u>
- Note that ZOH assumption implies that for any $t \in [t_0, t_0 + \Delta t)$ that $\dot{u}(t) = 0$

Therefore we have $\dot{x}(t) = A x(t) + B u(t)$, $x(t_0) = x_0$
$\dot{u}(t) = 0$, $u(t_0) = u_0$ [const. over Δt interval]

Define: Augmented State vector $x_a \triangleq \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$

$$\text{s.t.} \quad \dot{x}_a = \underbrace{\left[\begin{array}{c|c} A & B \\ \hline 0 & 0 \end{array}\right]}_{=\hat{A}} \begin{bmatrix} x(t) \\ \hline u(t) \end{bmatrix}, \quad x_a(t_0) = \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}$$

$\rightarrow \dot{x}_a = \hat{A} x_a$, where $\hat{A} \in \mathbb{R}^{[n+m]\times[n+m]}$ $\quad n = \#\ x$ states, $m = \#\ u$ inputs

$\rightarrow$ Sol'n for $t_0 \rightarrow t_0 + \Delta t$ ($t =$): $\boxed{x_a(t) = e^{\hat{A}\Delta t} x_a(t_0)}$ (!)

Computing the G matrix

- But if we expand the matrix exponential in this case:

$$e^{\hat{A}\Delta t} = I + \hat{A}\Delta t + \hat{A}^2 \frac{\Delta t^2}{2!} + \cdots$$

where
$$\hat{A}^2 = \left[\begin{array}{c|c} A & B \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} A^2 & AB \\ \hline 0 & 0 \end{array}\right]$$

$$\hat{A}^3 = \left[\begin{array}{c|c} A & B \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} A^2 & AB \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} A^3 & A^2B \\ \hline 0 & 0 \end{array}\right]$$

$$\vdots$$

$$\hat{A}^i = \left[\begin{array}{c|c} A^i & A^{i-1}B \\ \hline 0 & 0 \end{array}\right]$$

$=$ Series expansion of $\int_{t_0}^{t_0+\Delta t} e^{A(t-\tau)}\, d\tau$ from before!!

→ plug into ∞ series:

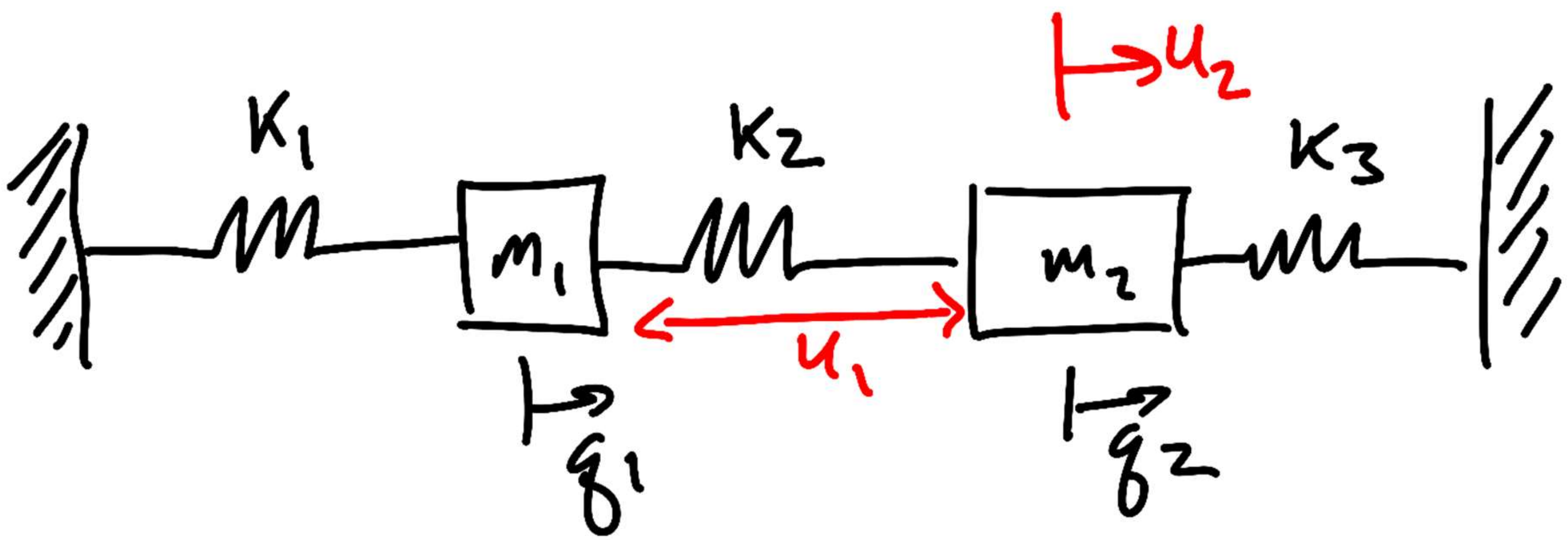
$$e^{\hat{A}\Delta t} = \left[\begin{array}{c|c} e^{A\Delta t} & \left[\sum_{i=1}^{\infty} A^{i-1}\frac{\Delta t^i}{i!}\right] B \\ \hline 0 & I_{m\times m} \end{array}\right] = \left[\begin{array}{c|c} F & G \\ \hline 0 & I_{m\times m} \end{array}\right]$$

$$= \text{expm}(\hat{A}\cdot\Delta t) \in \mathbb{R}^{[n+m]\times[n+m]}$$

[in Matlab]

Example: Convert CT SS model to DT SS model

- System of 2 masses and 3 springs: 2 actuator inputs u and 2 sensor outputs y



$$x = [q_1(t), \dot{q}_1(t), q_2(t), \dot{q}_2(t)]^T$$

$$u = [u_1(t), u_2(t)]^T$$

$$y = [q_1(t), q_2(t)]^T$$

For $k_1 = k_2 = k_3 = 1$ N/m and $m_1 = m_2 = 1$ kg, use simple physics to get CT linear SS model

$$\dot{x} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Two characteristic oscillatory motion "modes" (eigenvectors) corresponding to eigenvalues of A with natural frequencies:

$$\mathrm{eig}(A) = \ \lambda_{1,2} = \pm j1.73, \ \ \lambda_{3,4} = \pm j1.00$$
$$\rightarrow \ \omega_{n,1,2} = 1.73 \text{ rad/s}, \ \ \omega_{n,3,4} = 1.00 \text{ rad/s}$$

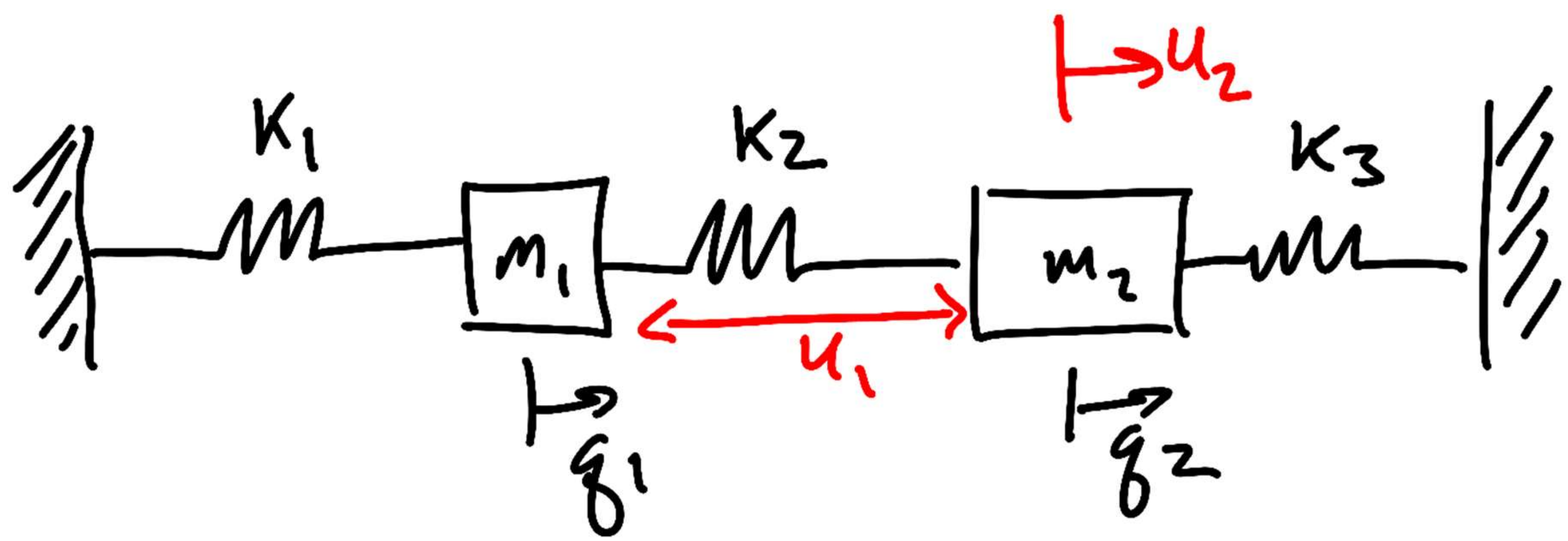
[2.72 Hz and 1.57 Hz, resp.]

$$A = \begin{bmatrix} 0 & 1.0 & 0 & 0 \\ -2.0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 \\ 1.0 & 0 & -2.0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\rightarrow \hat{A} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$$

Example: Convert CT SS model to DT SS model (cont'd)

- System of 2 masses and 3 springs: 2 actuator inputs u and 2 sensor outputs y



$$x = [q_1(t), \dot{q}_1(t), q_2(t), \dot{q}_2(t)]^T$$

$$u = [u_1(t), u_2(t)]^T$$

$$y = [q_1(t), q_2(t)]^T$$

Converted to DT SS model **using the same state variables** with ZOH and sample rate Δt = 0.2 sec

$$x_{k+1} = Fx_k + Gu_k,$$
$$y_k = Hx_k + Mu_k,$$

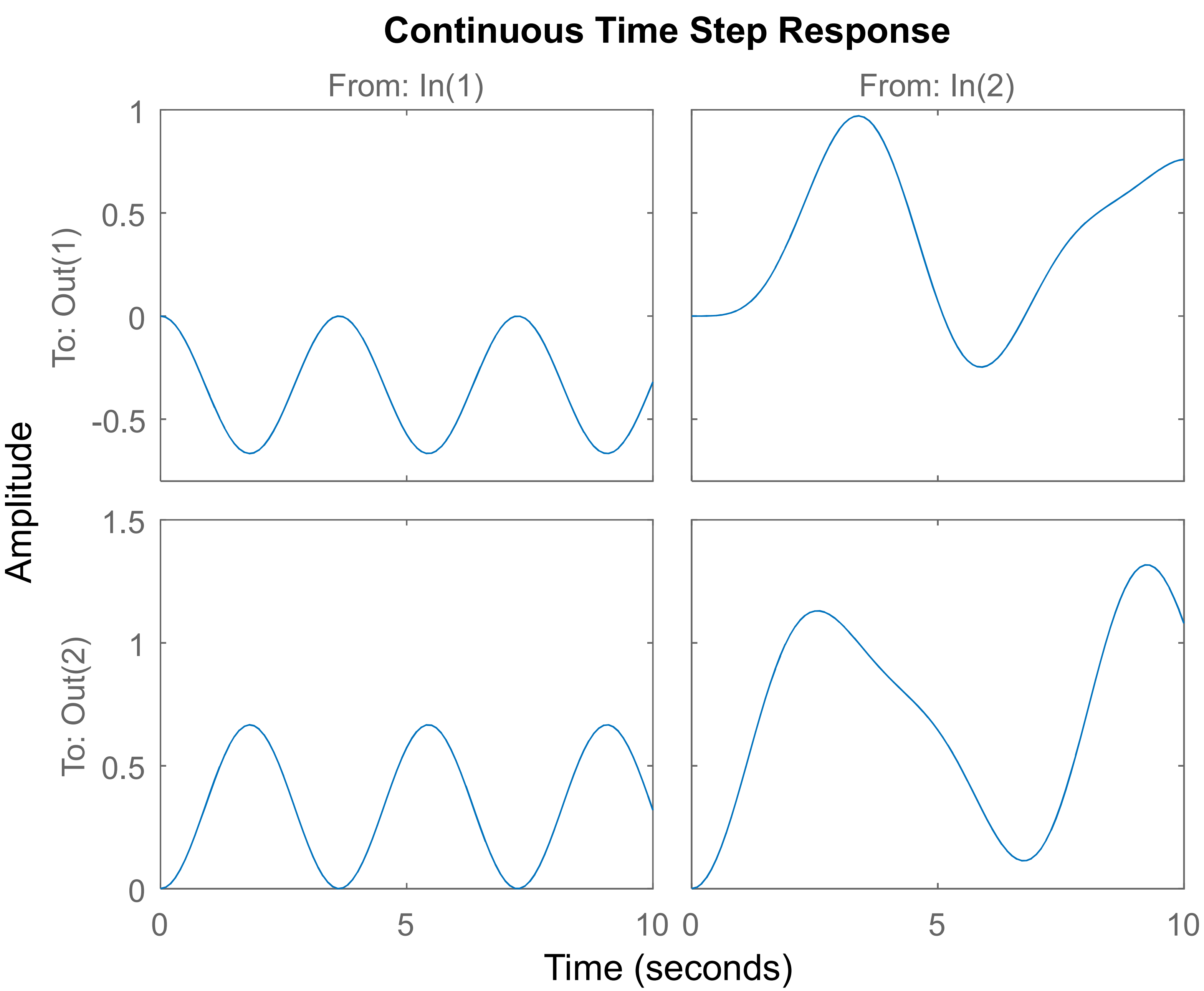
$$x_k = [q_1(k), \dot{q}_1(k), q_2(k), \dot{q}_2(k)]^T$$
$$u_k = [u_1(k), u_2(k)]^T$$
$$y_k = [q_1(k), q_2(k)]^T$$

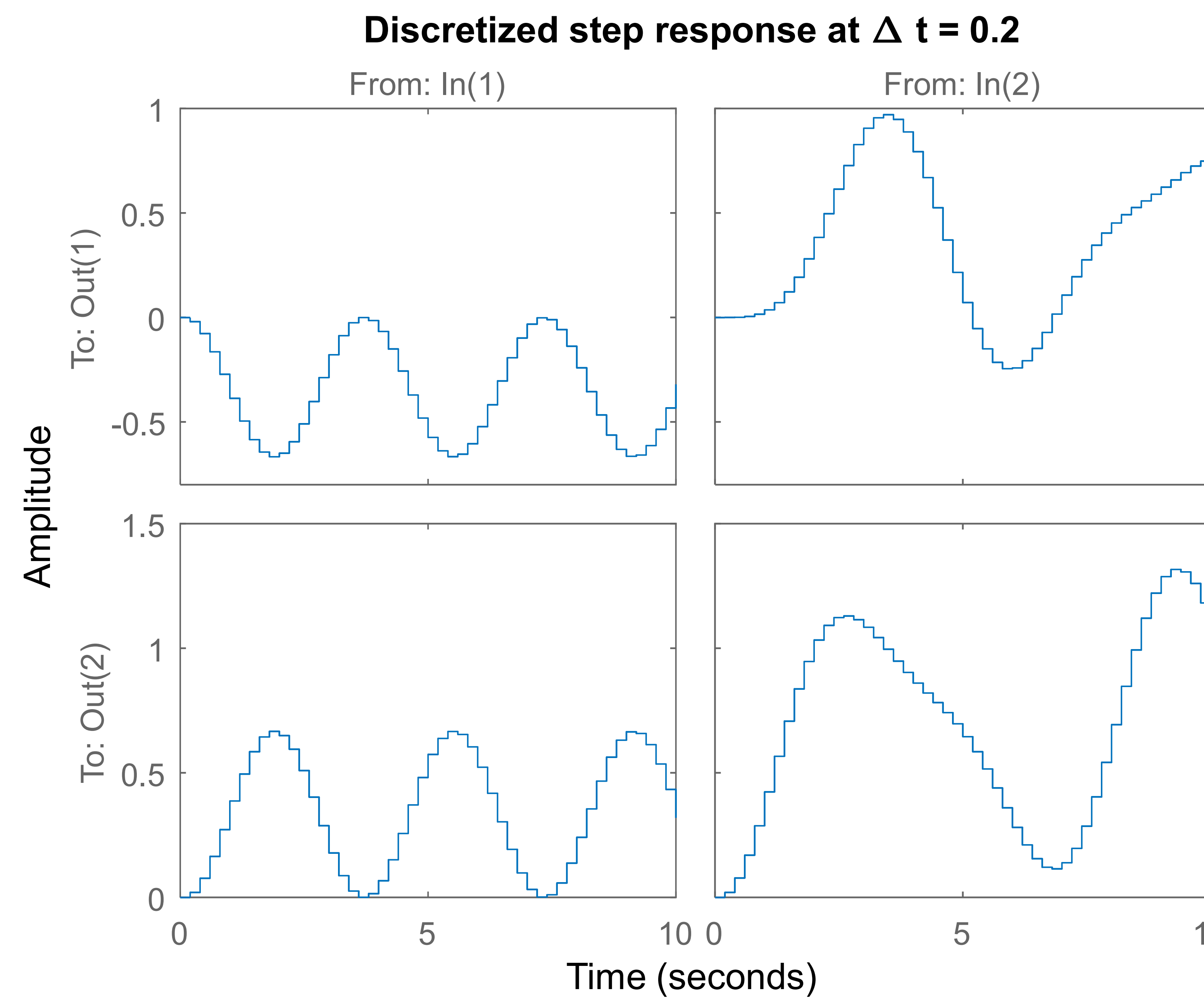
$$\hat{A} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \rightarrow e^{\hat{A}\Delta t} = \begin{bmatrix} F & G \\ 0 & I \end{bmatrix}$$

$$F = \begin{bmatrix} 9.6033e-01 & 1.9735e-01 & 1.9734e-02 & 1.3227e-03 \\ -3.9337e-01 & 9.6033e-01 & 1.9470e-01 & 1.9734e-02 \\ 1.9734e-02 & 1.3227e-03 & 9.6033e-01 & 1.9735e-01 \\ 1.9470e-01 & 1.9734e-02 & -3.9337e-01 & 9.6033e-01 \end{bmatrix} \quad G = \begin{bmatrix} -1.9801e-02 & 6.6312e-05 \\ -1.9602e-01 & 1.3227e-03 \\ 1.9801e-02 & 1.9867e-02 \\ 1.9602e-01 & 1.9735e-01 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Sample Input Step Response Output from DT vs CT





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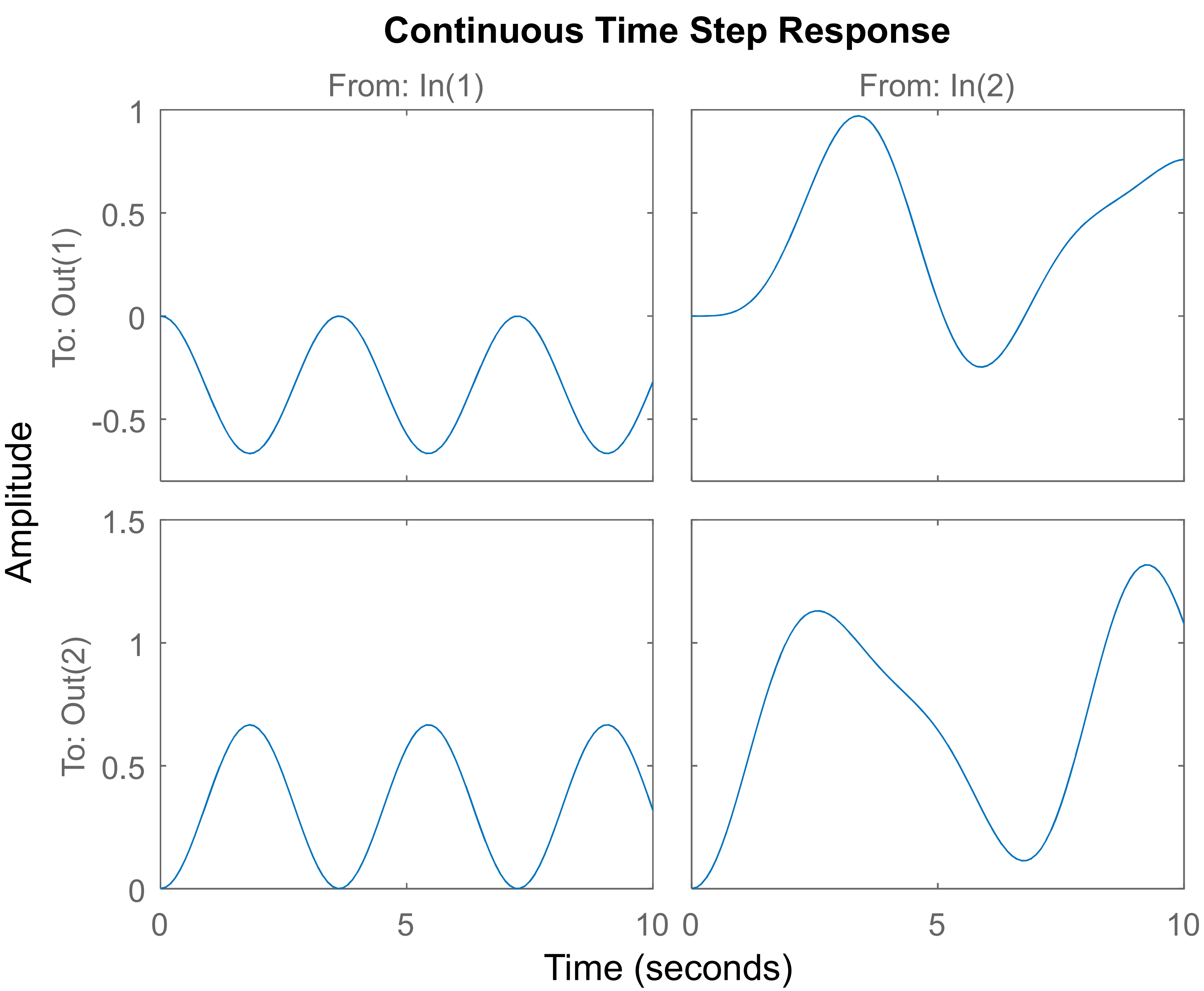
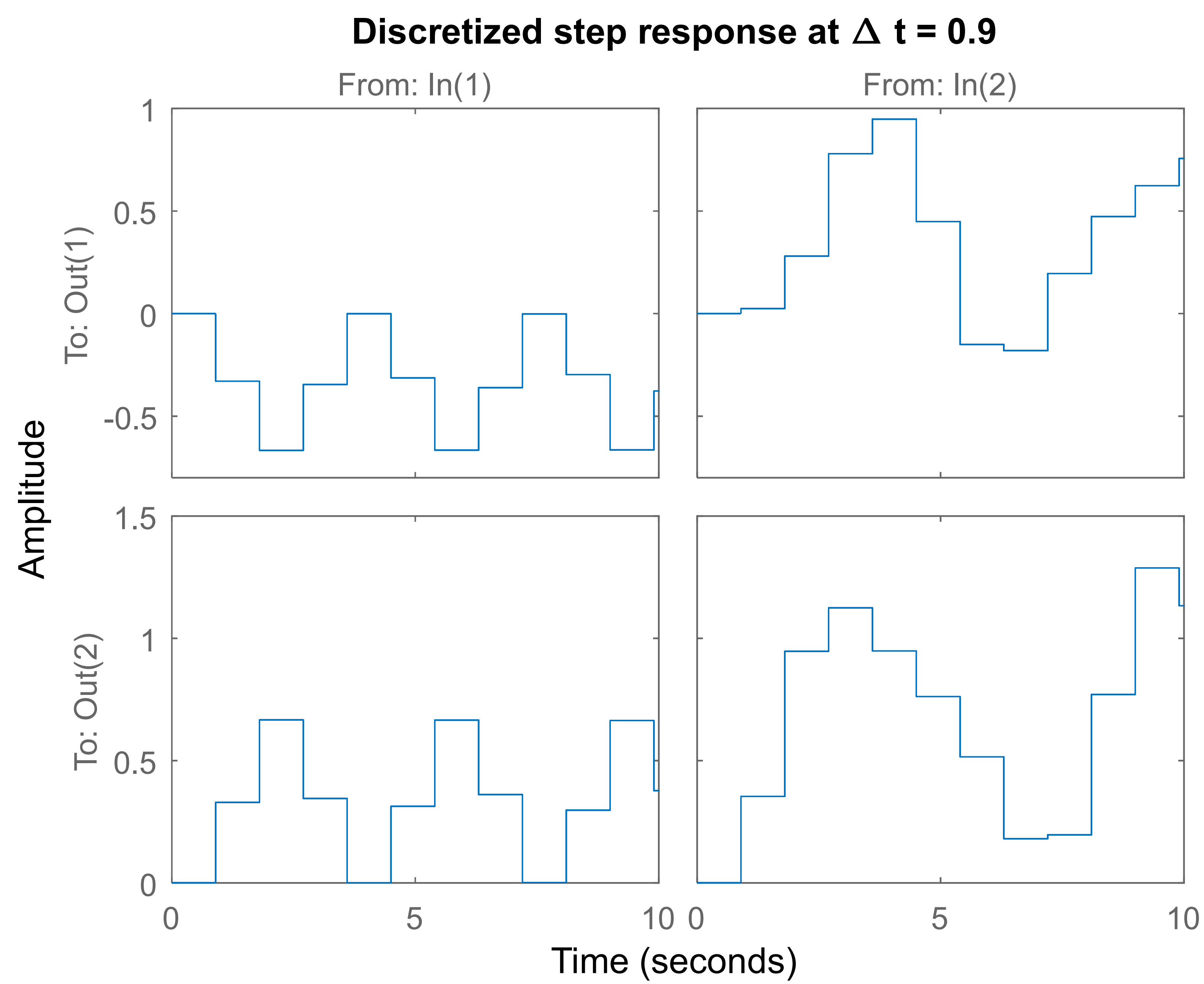
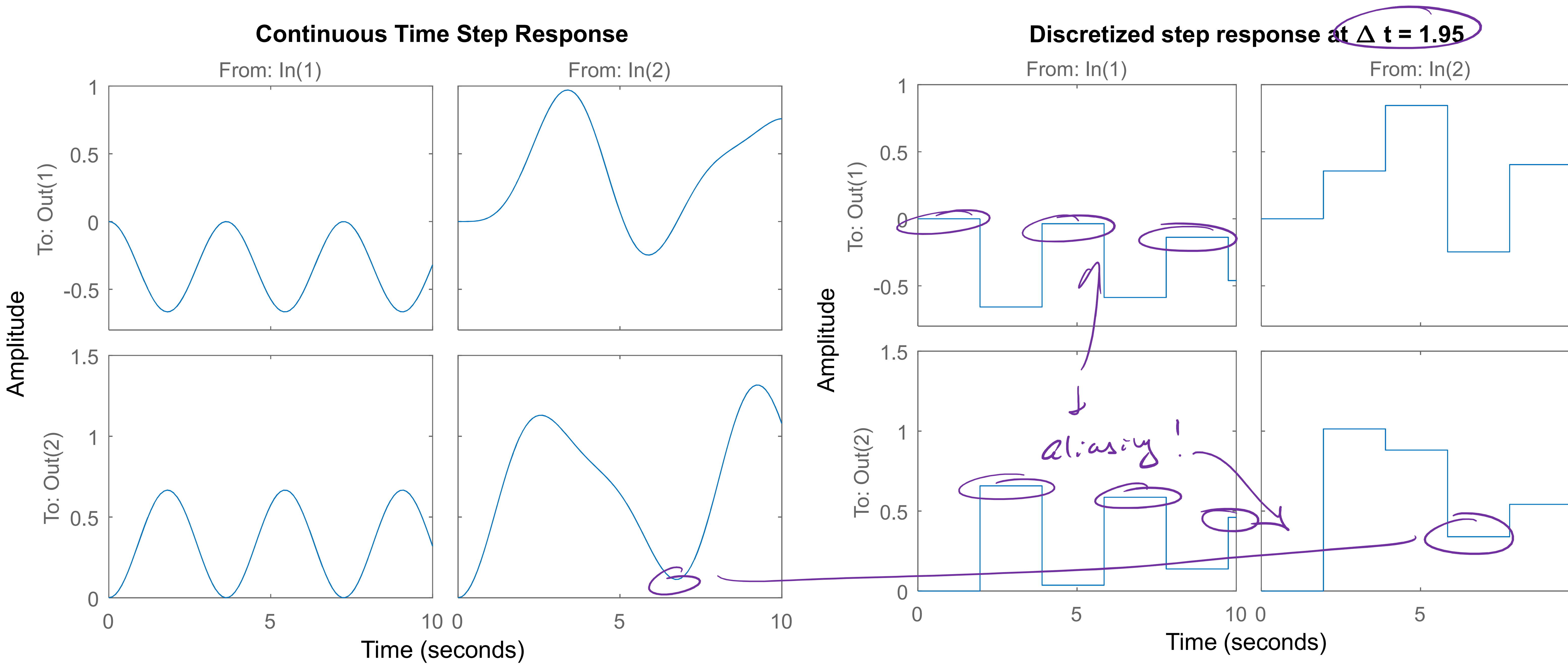




Sample Input Step Response Output from DT vs CT



Nyquist Rate and CT System Natural Frequencies

- **WARNING FOR CT → DT conversions: cannot just pick any old Δt !!!**
- For LTI systems: fundamental upper bound on how large Δt should be
- <u>Nyquist Sampling Criterion:</u> if sample rate (in rad/s) is $\omega_{sample} = \frac{2\pi}{\Delta t}$,

$$\text{need } \omega_{sample} > 2\omega_{sys,max} \;\; \Rightarrow \;\; \Delta t < \frac{\pi}{|\lambda_{A,max}|}$$

$$\text{where } |\lambda_{A,max}| \text{ is largest } \underbrace{\text{complex magnitude}}_{\text{(natural freq./ time constant)}} \text{ among all eigenvalues of } A$$

(e.g. max Δt for 2 mass/3 spring example system is ~1.82 sec → Δt larger than this leads to aliasing…)