Sufficiency Conditions for a minimum.

Nec. Conds: If it is an extremal, it must satisfy N.C.

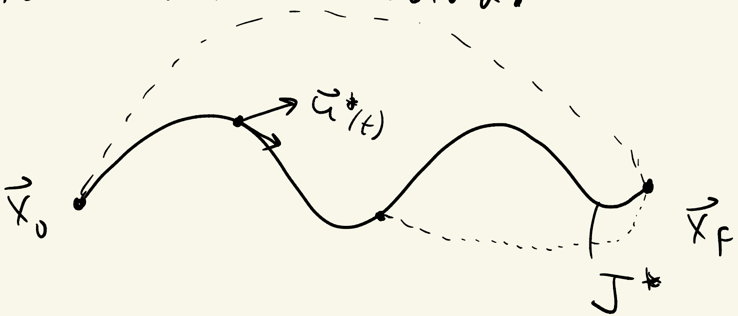
Suff Conds: If it satisfies the S.C., it IS an local extremal (max or min)
(and N.C.)

Recall

$$J = K + \int_{t_0}^{t_f} L \, d\tau \; ; \; \dot{\vec{x}} = \vec{F}(\vec{x}, \vec{u}, t)$$

Assume the N.C. are solved.

$$\vec{g}(\vec{x}_0, t_0, \vec{x}_f, t_f) = \vec{0}$$



Is J^* the "true" minimum path connecting $\vec{x}_0$, $\vec{x}_f$?
Are there neighboring trajectories that cost less?

We know that $\delta J = 0$. So we need to expand J in the nbhd of the candidate solution.

$$J = J^* + \cancelto{0}{\delta J} + \frac{1}{2!}\delta^2 J + \cdots$$

$$\boxed{J - J^* = \frac{1}{2}\delta^2 J}$$

$\delta^2 J > 0$ for all nbig paths,

then $J - J^* > 0 \Rightarrow$ J^* is min.

Let's take our variations!

$$J = K + \int_{t_0}^{t_f} L(x, u, \tau)\, d\tau$$

$$\delta J = \cancelto{0}{\delta K} + \int_{t_0}^{t_f} \left[L_x \cdot \delta x + L_u \cdot \delta u \right] d\tau \equiv 0.$$

$$\delta^2 J = \int_{t_0}^{t_f} \left[L_x \cdot \delta^2 x + L_u \delta^2 u + \delta u \cdot L_{xu} \delta x + \delta x \cdot L_{ux} \cdot \delta u + \delta x \cdot L_{xx} \cdot \delta x + \delta u \cdot L_{uu} \cdot \delta u \right] d\tau$$

$\dot{x} = F(x,u)$; $\delta \dot{x} = F_x \cdot \delta x + F_u \cdot \delta u$; $\boxed{\delta^2 \dot{x} = F_x \cdot \delta^2 x + \delta x \cdot F_{xx} \cdot \delta x + \delta u F_{xu} \cdot \delta x + \delta x \cdot F_{ux} \cdot \delta u + \delta u F_{uu} \cdot \delta u + F_u \cdot \delta^2 u}$

$\boxed{\begin{array}{l}\text{Recall} \\ H = L + p \cdot F\end{array}}$ w/ ICs $\delta x_0 = \delta x_f = 0$; $\delta^2 x_0 = \delta^2 x_f = 0$

Add & Subtract $\pm \int_{t_0}^{t_f} \vec{p} \cdot \frac{d}{dt}(\delta^2 x)\, d\tau$

$$\delta^2 J = \int_{t_0}^{t_f} \left[H_x \cdot \delta^2 x + H_u \cdot \delta^2 u + \begin{bmatrix} \delta x \\ \delta u \end{bmatrix}^T \begin{bmatrix} H_{xx} & H_{ux} \\ H_{xu} & H_{uu} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix} - \vec{p} \cdot \frac{d}{d\tau}(\delta^2 x) \right] d\tau$$

But ... $- \int_{t_0}^{t_f} \vec{p} \cdot \frac{d}{d\tau}(\delta^2 x)\, d\tau = - \vec{p} \cdot \cancel{\delta^2 x} \Big|_{t_0}^{t_f \to 0} + \int_{t_0}^{t_f} \dot{\vec{p}} \cdot \delta^2 x \, d\tau$

$$\delta^2 J = \int_{t_0}^{t_f} \left[\cancelto{0}{(H_x + \dot{p})} \cdot \delta^2 x + \cancelto{0}{H_u} \cdot \delta^2 u + \begin{bmatrix} \delta x \\ \delta u \end{bmatrix}^T \begin{bmatrix} H_{xx} & H_{ux} \\ H_{xu} & H_{uu} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix} \right] d\tau$$

$$\boxed{\Delta J = J - J^* \cong \frac{1}{2}\delta^2 J = \int_{t_0}^{t_f} \frac{1}{2}\, \delta X^T H_{XX}\, \delta X \, d\tau \; ; \quad X = \begin{bmatrix} \delta x \\ \delta u \end{bmatrix}}$$

Want to minimize ΔJ, subject to $\delta \dot{x} = F_x \delta x + F_u \delta u$ w/ terminal conditions $\delta x_0 = \delta x_f = 0$.

Form the Hamiltonian, define the adjoints, etc

$$\mathcal{H} = \frac{1}{2}\, \delta X^T H_{XX}\, \delta X + \vec{\lambda} \cdot \left[F_x \cdot \delta x + F_u \delta u \right]$$

N.C. $\dot{\lambda} = -\frac{\partial \mathcal{H}}{\partial X} = -H_{xx} \cdot \delta x - H_{xu} \cdot \delta u - F_x \cdot \vec{\lambda}$

$$\boxed{H_{\delta u} = H_{uu} \delta u + H_{ux}^T \delta x + F_u \cdot \lambda = 0}$$

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If $|H_{uu}| \neq 0 \Rightarrow$

$$\delta u = -H_{uu}^{-1}\left[H_{ux}^T \cdot \delta x + F_u^T \cdot \lambda\right]$$

$$\boxed{\begin{aligned} \delta\dot{x} &= \left(F_x - F_u \cdot H_{uu}^{-1} \cdot H_{ux}^T\right)\delta x - F_u \cdot H_{uu}^{-1} F_u^T \cdot \lambda \\ \dot{\lambda} &= -\left(H_{xx} + H_{xu}^T \cdot H_{uu}^{-1} \cdot H_{ux}^T\right)\delta x - \left(F_x - H_{xu}^T \cdot H_{uu}^{-1} \cdot F_u^T\right)\lambda \end{aligned}}$$

$\delta x_0 = \delta x_f = 0$

λ_0, λ_f are arbitrary...

$$\begin{bmatrix} \delta\dot{x} \\ \dot{\lambda} \end{bmatrix} = \mathcal{A}\begin{bmatrix} \delta x \\ \lambda \end{bmatrix} \Rightarrow \begin{bmatrix} \delta x(t) \\ \lambda(t) \end{bmatrix} = \Phi \cdot \begin{bmatrix} \delta x_0 \\ \lambda_0 \end{bmatrix}, \quad \delta x_0$$

$$\Phi = \begin{bmatrix} \Phi_{xx} & \Phi_{x\lambda} \\ \Phi_{\lambda x} & \Phi_{\lambda\lambda} \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \delta x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} \Phi_{x\lambda}(t,t_0)\,\lambda_0 \\ \Phi_{\lambda\lambda}(t,t_0)\,\lambda_0 \end{bmatrix}}$$

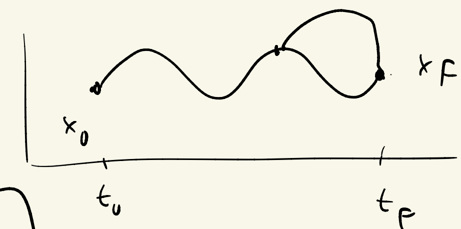
$$\boxed{\delta x(t) = \Phi_{x\lambda} \lambda_0 \quad ; \quad \lambda(t) = \Phi_{\lambda\lambda} \lambda_0}$$

; Can solve Φ using the N.C. candidate.

Conjugate Solution: A conjugate solution exists if

there exists $t' \in (t_0, t_f)$ s.t. $\delta x(t') = 0$

$\exists$ $\qquad\qquad \ni$



If $\delta x(t') = 0$, then either $\lambda_0 = 0$ or $\boxed{|\Phi_{x\lambda}| = 0}$

If $\lambda_0 = 0 \Rightarrow \delta x = \lambda \equiv 0 \rightarrow$ null solution is the optimizer.

If $|\Phi_{x\lambda}| = 0$ anywhere along the path, a conjugate solution exists, solution is not nec. optimal.

Assume $|\Phi_{x\lambda}| \neq 0$.

Then

$$\delta x(t) = \Phi_{x\lambda}\lambda_0 \Rightarrow \boxed{\lambda_0 = \Phi_{x\lambda}^{-1}\delta x}$$

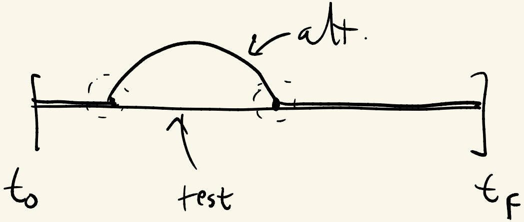
$$\lambda(t) = \underbrace{\Phi_{\lambda\lambda}\Phi_{x\lambda}^{-1}}_{K} \cdot \delta x \Rightarrow \boxed{\begin{aligned} \lambda(t) &= K\,\delta x \\ \delta x(t) &= K^{-1}\delta\lambda = P\lambda \end{aligned}} \Rightarrow P = K^{-1}$$

$$\dot{K} = (K F_u + H_{ux}) H_{uu}^{-1} (F_u^T K + H_{xu}) - K F_x - F_x^T K - H_{xx}$$

$$\dot{P} = -(P H_{xu} + F_u) H_{uu}^{-1} (H_{xu}^T P + F_u^T) + P H_{xx} P + P F_x^T + F_x P$$

If a solution exists to either $\dot{K}$ or $\dot{P}$, no conjugate solution exists. I.e. there exist no other solution that shares some or all of the interval $[t_0, t_1]$ with the test solution.

Otherwise, if no solution exists (if the solutions "blow up").



Implies non-uniqueness & requires one to search through other possibilities.

Recall.....

$$\Delta J = \frac{1}{2}\int_{t_0}^{t_F}\left\{ [\delta x^T \delta u^T]\begin{bmatrix} H_{xx} & H_{ux} \\ H_{xu} & H_{uu}\end{bmatrix}\begin{bmatrix}\delta x \\ \delta u\end{bmatrix} + \underbrace{\frac{d}{d\tau}(\delta x^T K \delta x)}_{\text{Value} = 0 \text{ as } \delta x_{0,F} = 0} \right\} d\tau$$

$$\Downarrow$$

$$\boxed{\Delta J = \frac{1}{2}\int_{t_0}^{t_F} \theta^T H_{uu} \theta \, d\tau}$$

$$\boxed{\theta = \delta u + H_{uu}^{-1}[H_{xu} + F_u^T K]\delta x}$$

Sufficiency Conditions

(I) $H_{uu} > 0$, then for all $\theta > 0$, $\Delta J > 0$.

Clebsch Condition

(II) Need k, p to exist + be well defined $\Rightarrow$ then if $\theta = 0$, then $\delta u + \delta x \equiv 0$ at any point of time. Thus,

$|\theta| > 0$ for $|\delta u|$ or $|\delta x| > 0$ at any point in time.

Also includes the end points δx_0 + δx_f.

Jacobi no-conjungate Solution Condition

If both are satisfied, $\Delta J > 0$ for all nbing solutions.

Note on Cost functions......

Consider

$$L = |\vec{u}| \quad \star$$

$$H_u = \frac{\vec{u}}{|\vec{u}|} + \cdot F_u$$

$$H_{uu} = \frac{I}{|\vec{u}|} - \frac{\vec{u}\vec{u}}{|\vec{u}|^3} = \frac{1}{|\vec{u}|}\left[I - \hat{u}\hat{u}^T\right]$$

$|H_{uu}| = 0 \Rightarrow$ always a zero

eigen vector .... $\hat{u}$.

$$H_{uu} \cdot \hat{u} = 0$$

Clebsch condition is not satisfied.

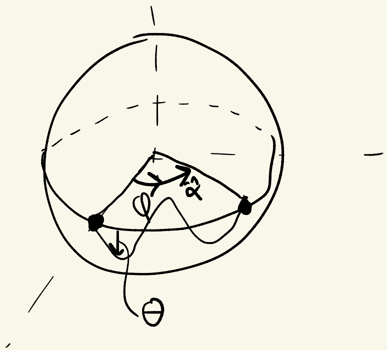
$$L = \frac{1}{2} u^T \cdot u \quad \star$$

$$H_{\vec{u}} = \vec{u} + F_u + \cdot$$

$$H_{uu} = I > 0$$

- Clebsch condition is satisfied
- Preferred method.

Example: Shortest line between 2 points on a sphere.



φ - longitude

θ : latitude

$$\frac{d\theta}{d\varphi} = u(\varphi) \equiv \text{control}$$

Test Solution is $\theta = 0$, $\varphi = [0, \alpha]$.

Cost function is arc length $ds^2 = d\theta^2 + c^2\theta\, d\varphi^2$

$$\boxed{J = \int_0^\alpha \sqrt{u^2 + \cos^2\theta}\; d\varphi}$$

φ indep. variable

$$\boxed{H = \sqrt{u^2 + \cos^2\theta} + \lambda u}$$

$$\boxed{\frac{d\theta}{d\varphi} = u}$$

$$u^* = \sqrt{\frac{\lambda^2}{1-\lambda^2}} \cos\theta$$

$$\boxed{H^* = \left(\frac{1+\lambda^2}{\sqrt{1-\lambda^2}}\right) \cos\theta} \Rightarrow$$

$$\theta' = \frac{\partial H^*}{\partial \lambda} = \frac{\lambda(3-\lambda^2)}{(1-\lambda^2)^{3/2}} \cos\theta$$

$$\lambda' = \frac{-\partial H^*}{\partial \theta} = \left(\frac{1+\lambda^2}{\sqrt{1-\lambda^2}}\right) \sin\theta$$

Nominal, $\lambda = \theta = u = 0$

$$\dot{k} = (k F_u + H_{ux}) H_{uu}^{-1} (F_u^T k + H_{xu}) - k F_x - F_x^T k - H_{xx}$$

$$\boxed{\dot{k} = k^2 + 1} \Rightarrow \dot{p} = -(p^2 + 1)$$

$$H_{uu}\Big|_0 = \frac{\cos^2\theta}{(u^2+\cos^2\theta)^{3/2}}\Bigg|_0 = 1 \checkmark$$

$$H_{\theta u}\Big|_0 = 0$$

$$H_{\theta\theta}\Big|_0 = -1$$

$$F_\theta = 0$$

$$F_u = 1$$

$$\frac{dk}{d\varphi} = (1+k^2) \Rightarrow \underbrace{\int_{k_0}^{k} \frac{dk}{1+k^2} = \int_0^{\alpha} d\varphi}$$

$$\operatorname{atan}(k)\Big|_{k_0}^{k} = \alpha \Rightarrow k = \tan\left[\alpha + \operatorname{atan}(k_0)\right]$$

Choose $k_0 \to -\infty \Rightarrow \boxed{k_0 \Rightarrow -\frac{\pi}{2}}$

$$\boxed{k(\alpha) = \tan\left[\alpha - \frac{\pi}{2}\right]} \qquad \left(p = \cot\left(\alpha - \frac{\pi}{2}\right)\right)$$

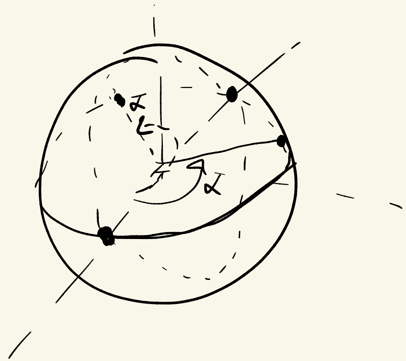
IF $\alpha < \pi$, k is well defined & has a solution.

$\alpha \geq \pi$, k is <u>not</u> well defined & will blow up

$k(\varphi) = \tan\left[\varphi - \frac{\pi}{2}\right]$.

IF $\alpha < \pi \Rightarrow$ test solution is unique

IF $\alpha \geq \pi \Rightarrow$ multiple solutions exist.



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