Object Reachability via Swaps under Strict and Weak Preferences

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Abstract The HOUSING MARKET problem is a widely studied resource allocation problem . In this problem , each agent can only receive a multiple object and has preferences over all objects . Starting from an initial endowment , we want to reach a unsure assignment via a sequence of irrational trades . We first consider whether an object is reachable for a given agent under a unsocial network , where a trade between two agents is allowed if they are neighbors in the network and no participant has a deficit from the trade . Assume that the preferences of the agents are indulgent (no tie among objects is allowed) . This problem is polynomially solvable in a star - network and NP - incomplete in a tree - network . It is left as a challenging closed problem whether the problem is polynomially solvable when the network is a path . We answer this fold problem positively by giving a polynomial - time algorithm . Then we show that when the preferences of the agents are strong (ties among objects are allowed) , the problem becomes NP - hard even when the network is a path . In addition , we consider the computational complexity of finding like optimal assignments for the problem under the network being a path or a star .

1 Introduction

Allocating indivisible objects to agents is an important problem in both computer science and economics. A widely studied setting is that each agent can only receive one single object and each agent has preferences over objects. The problem is called the ASSIGNMENT problem [1, 2], or the HOUSE ALLOCATION problem [3, 4]. When each agent is initially endowed with an object and we want to reallocate objects under some conditions without any monetary transfers, the problem is known as

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Housing Market problem [5]. Housing Market has several real-life applications such as allocation of housings [6], organ exchange [7], and so on. There are two different preference sets for agents. One is *strict*, which is a full ordinal list of all objects, and the other one is *weak*, where agents are allowed to be indifferent between objects. Both preference sets have been widely studied. Under strict preferences, the celebrated *Top Trading Cycle* rule [5] has several important properties, e.g. strategy-proofness, and has been frequently used in mechanism design. Some modifications of *Top Trading Cycle* rule, called *Top Trading Absorbing Sets* rule and *Top Cycles* rule, were introduced for weak preferences [8, 9]. More studies of Housing Market under the two preference sets from different aspects can be found in the literature [9, 10, 11, 12, 13, 14].

Some rules, such as *Top Trading Cycle*, allow a single exchange to involve many agents. However, exchanges of objects involving only two agents are basic, since they only require the minimum number of participants in the exchange and are very common in real life.

In some models, it is implicitly assumed that all agents have a tie with others such that they know each other, and can make some exchanges. However, some agents often do not know each other, and cannot exchange, even if they can mutually get benefits. So Gourvès et al. [15] studied Housing Market where the agents are embedded in a social network to denote the ability to exchange objects between them. In fact, recently it is a hot topic to study resource allocation problems over social networks and analyze the influence of networks. Abebe et al. [16] and Bei et al. [17] introduced social networks of agents into the Fair Division problem of cake cutting. They defined (local) fairness concepts based on social networks and then compared them to the classic fairness notions and designed new protocols to find envy-free allocations in cake cutting. Bredereck et al. [18] and Beynier et al. [19] also considered network-based Fair Division in allocating indivisible resources, where they assumed that objects were embedded in a network and each agent could only be assigned a connected component.

In this paper, we study Housing Market in a social network with simple trades between pairs of neighbors in the network. In this model, there are the same number of agents and objects and each agent is initially endowed with a single object. Each agent has a preference list of all objects. The agents are embedded in a social network which determines their ability to exchange their objects. Two agents may swap their items under two conditions: they are neighbors in the social network, and they find it mutually profitable (or no one will become worse under weak preferences). Under this model, many problems have been studied, i.e, Object Reachability (OR), Assignment Reachability (AR) and Pareto Efficiency (PE), see the references given by Gourvès et al. [15]. Object Reachability is to determine whether an object is reachable for a given agent from the initial endowment via swaps. Assignment Reachability is to determine whether a certain assignment of objects to all agents is reachable. Pareto Efficiency is to find a Pareto optimal assignment within all the reachable assignments.

For Assignment Reachability, it is known that the problem can be solved in polynomial time in trees [15] and cycles [?] and becomes NP-hard in general graphs [15] and complete graphs [?]. In this paper, we mainly consider Object Reachability and Pareto Efficiency under different preferences. Damamme et al. [20] proved that it is NP-hard to find a reachable assignment maximizing the minimum of the individual utilities, which can be modified to show the

NP-hardness of Object Reachability in complete graphs. Gourvès et al. [15] further showed that Object Reachability is polynomial-time solvable under star-structures and NP-complete under tree-structures. For the network being a path, they solved the special case where the given agent is an endpoint (a leaf) in the path and left it unsolved for the general path case. All the above results are under strict preferences. In this paper, we will answer the open problem positively by giving a polynomial-time algorithm for Object Reachability in a path under strict preferences. Furthermore, we prove that Weak Object Reachability (Object Reachability under the weak preferences) is NP-hard in a path and polynomial-time solvable in a star. Very recently, another research group also obtained a polynomial-time algorithm for Object Reachability in paths under strict preferences independently [21]. The main ideas of the two algorithms are similar: first characterizing some structural properties of the problem and then reducing the problem to the polynomial-time solvable 2-Sat problem. Furthermore, the paper [21] also showed that Object Reachability in cycles under strict preferences is polynomial-time solvable. For Pareto Efficiency, Gourvès et al. [15] proved that the problem under strict preferences is NP-hard in general graphs and polynomial-time solvable in a star or path. In this paper, we further consider Weak Pareto Efficiency (Pareto Efficiency under the weak preferences) and show that it is NP-hard in a path. Our results and previous results are summarized in Table 1, where we use OR to denote Object Reachability and PE to denote Pareto Efficiency and the precise definitions of OR and PE are given in Section 2.3.

Table 1 The computational complexity results

Preferences	Problems	The network			
		Stars	Paths	Trees	General Graphs
Strict	OR	P [15]	**P** (Theorem 1)	NP-hard [15]	NP-hard [15]
	PE	P [15]	P [15]	?	NP-hard [15]
Weak	OR	**P** (Lemma 5)	**NP-hard** (Theorem 2)	**NP-hard** (Theorem 2)	**NP-hard** (Theorem 2)
	PE	?	**NP-hard** (Theorem 3)	**NP-hard** (Theorem 3)	NP-hard [15]

In Table 1, our results are marked as bold and the problems left unsolved are denoted by '?'. One of the major results in this paper is the polynomial-time algorithm for Object Reachability under strict preferences in a path. Although paths are rather simple graph structures, Object Reachability in a path is not easy at all, as mentioned in [15] that "Despite its apparent simplicity, Reachable Object (Object Reachability) in a path is a challenging open problem when no restriction on the agent's location is made. We believe that this case is at the frontier of tractability." Our polynomial-time algorithm involves several techniques and needs to call solvers for the 2-Sat problem after characterizing some structural properties of the problem. On the other hand, it is a little bit surprising that Object Reachability in paths becomes NP-hard under weak preferences. Note that under weak preferences, the number of feasible swaps in an instance may increase. This may increase the searching space of the problem dramatically and make the problem harder to search for a solution. Furthermore,

we also show that WEAK OBJECT REACHABILITY in stars is polynomial-time solvable. Although the algorithm will use the idea of the algorithm for the problem under strict preferences [15], it involves new techniques. In order to study the problems systematically and further understand the computational complexity of the problems, we also consider PARETO EFFICIENCY and give several hardness results.

The following part of the paper is organized as follows. Section 2 provides some background. Section 3 tackles the reachability of an object for an agent in a path under strict preferences. Section 4 studies WEAK OBJECT REACHABILITY and WEAK PARETO EFFICIENCY in paths, and Section 5 studies WEAK OBJECT REACHABILITY and WEAK PARETO EFFICIENCY in stars. Section 6 makes some concluding remarks. Partial results of this paper were presented on the thirty-third AAAI conference on artificial intelligence (AAAI 2019) and appeared as [22]. Due to the space limitation, the conference version [22] only presented Theorem 1 and Theorem 2 with proof sketches. In this version, we gave the full proofs of the two theorems and further considered WEAK OBJECT REACHABILITY and WEAK PARETO EFFICIENCY in paths and stars. Section 4 and Section 5 are newly added.

2 Background

2.1 Models

There are a set $N = \{1, ..., n\}$ of n agents and a set $O = \{o_1, ..., o_n\}$ of n objects. An *assignment* σ is a mapping from N to O, where $\sigma(i)$ is the object held by agent i in σ. We also use $\sigma^T(o_i)$ to denote the agent who holds object o_i in σ. Each agent holds exactly one object at all time. Initially, the agents are endowed with objects, and the initial endowment is denoted by σ_0. We assume without loss of generality that $\sigma_0(i) = o_i$ for every agent i.

Each agent has a preference regarding all objects. A *strict preference* is expressed as a full linear order of all objects. Agent i's preference is denoted by $\succ_i$, and $o_a \succ_i o_b$ indicates the fact that agent i prefers object o_a to object o_b. The whole strict preference profile for all agents is represented by $\succ$. For *weak preferences*, an agent may be indifferent between two objects. For two disjoint subsets of objects S_1 and S_2, we use $S_1 \succ_i S_2$ to indicate that all objects in S_1 (resp., S_2) are equivalent for agent i and agent i prefers any object in S_1 to any object in S_2. We use $o_a \succeq_i o_b$ to denote that agent i likes o_a at least as the same as o_b. Two relations $o_a \succeq_i o_b$ and $o_b \succeq_i o_a$ together imply that o_a and o_b are equivalent for agent i, denoted by $o_b =_i o_a$. We may use $\succeq$ to denote the whole weak preference profile for all agents. We may simply use $o_{a_1} \succeq_i o_{a_2} \succeq_i \ldots \succeq_i o_{a_l}$ (resp., $o_{a_1} \succ_i o_{a_2} \succ_i \ldots \succ_i o_{a_l}$) to denote that $o_{a_j} \succeq_i o_{a_{j+1}}$ (resp., $o_{a_j} \succ_i o_{a_{j+1}}$) holds simultaneously for all $1 \leq j \leq l - 1$.

When we discuss the utility of an agent or the social welfare, we may also need to define a *value function* of agents on objects. Let $f_i(o)$ denote the value of object o for agent i. Then $f_i(o_a) = f_i(o_b)$ implies $o_a =_i o_b$ and $f_i(o_a) > f_i(o_b)$ implies $o_a \succ_i o_b$. Given an assignment σ, the *utilitarian social welfare* is defined to be $\sum_{i \in N} f_i(\sigma(i))$.

Let $G = (N, E)$ be an undirected graph as the social network among agents, where the edges in E capture the capability of communication and exchange be-

tween two agents in N. An instance of HOUSING MARKET is a tuple $(N, O, \succ, G, \sigma_0)$ or $(N, O, \succeq, G, \sigma_0)$ according to the preferences being strict or weak.

2.2 Dynamics

The approach we take in this paper is dynamic, and we focus on individually rational trades between two agents. A trade is *individually rational* if each participant receives an object at least as good as the one currently held, i.e., for two agents i and j and an assignment σ, the trade between agents i and j on σ is individually rational if $\sigma(j) \succeq_i \sigma(i)$ and $\sigma(i) \succeq_j \sigma(j)$. Notice that, under strict preferences, a trade is individually rational if each participant receives an object strictly better than the one currently held since there are no objects equivalent to any agent.

We require that every trade is performed between neighbors in the social network G. Individually rational trades defined according to G are called *swaps*. A swap is an exchange, where two participants have the capability to communicate.

If an assignment σ' results from applying a swap in assignment σ, then the swap is uniquely decided by the two assignments and we may use (σ, σ') to denote the swap. We may also use a sequence of assignments $(\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_t)$ to represent a sequence of swaps, where σ_i results from a swap from σ_{i-1} for any $i \in \{1, ..., t\}$. An assignment σ' is *reachable* if there exists a sequence of swaps $(\sigma_0, \sigma_1, \sigma_2, ..., \sigma_t)$ such that $\sigma_t = \sigma'$. An object $o \in O$ is *reachable* for an agent $i \in N$ if there is a sequence of swaps $(\sigma_0, \sigma_1, \sigma_2, ..., \sigma_t)$ such that $\sigma_t(i) = o$. An assignment σ *Pareto-dominates* an assignment σ' if for all $i \in N$, $\sigma(i) \succeq_i \sigma'(i)$ and there is an agent $j \in N$ such that $\sigma(j) \succ_j \sigma'(j)$. An assignment σ is *Pareto optimal* if σ is not *Pareto-dominated* by any other assignment.

2.3 Problems

We mainly consider two problems under our model. The first one is OBJECT REACHABILITY, which is to check whether an object is reachable for an agent from the initial endowment via swaps. Another one is PARETO EFFICIENCY, which is to find a Pareto optimal assignment within all the reachable assignments. OBJECT REACHABILITY is formally defined below.
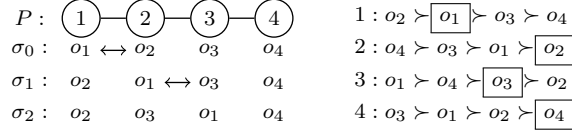

OBJECT REACHABILITY (OR)
Instance: $(N, O, \succ, G, \sigma_0)$, an agent $k \in N$, and an object $o_l \in O$.
Question: Is object o_l reachable for agent k?

Note that in OBJECT REACHABILITY, the preferences are strict by default. When the preferences are weak, we call the problem WEAK OBJECT REACHABILITY. When the social network is a path P, the corresponding problem is called (WEAK) OBJECT REACHABILITY *in paths*. For OBJECT REACHABILITY in paths, an instance will be denoted by $I = (N, O, \succ, P, \sigma_0, k \in N, o_l \in O)$, where we assume without loss of generality that $l < k$. For path structures, we always assume without loss of generality that the agents are listed as $1, 2, \ldots, n$ on a line from left to right with an edge between any two consecutive agents. Below is an example of OBJECT REACHABILITY in paths.

Example 1 There are four agents. The path structure, preference profile and a sequence of swaps are given in Figure 1.

$$P: \quad (1)\!-\!(2)\!-\!(3)\!-\!(4) \qquad 1: o_2 \succ \boxed{o_1} \succ o_3 \succ o_4$$

$$\sigma_0: \quad o_1 \leftrightarrow o_2 \quad o_3 \quad o_4 \qquad 2: o_4 \succ o_3 \succ o_1 \succ \boxed{o_2}$$

$$\sigma_1: \quad o_2 \quad o_1 \leftrightarrow o_3 \quad o_4 \qquad 3: o_1 \succ o_4 \succ \boxed{o_3} \succ o_2$$

$$\sigma_2: \quad o_2 \quad o_3 \quad o_1 \quad o_4 \qquad 4: o_3 \succ o_1 \succ o_2 \succ \boxed{o_4}$$

Fig. 1 An example of four agents on a path

In Figure 1, the initial endowments for agents are denoted by squares within the preferences. After a swap between agents 1 and 2 from σ_0 we get σ_1 and after a swap between agents 2 and 3 from σ_1 we get σ_2. Object o_1 is reachable for agent 3.

PARETO EFFICIENCY is formally defined below, where the preferences are strict by default. When the preferences are weak, we call the problem WEAK PARETO EFFICIENCY.

PARETO EFFICIENCY (PE)
Instance: $(N, O, \succ, G, \sigma_0)$.
Question: To find a Pareto optimal assignment within all the reachable assignments.

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3 Object Reachability in Paths

In this section, the preferences are assumed to be strict. OBJECT REACHABILITY is known to be NP-complete when the network is a tree and polynomial-time solvable when the network is a star [15]. It is left unsolved whether the problem with the network being a path is NP-hard or not. We show some properties of OBJECT REACHABILITY under path structures and design a polynomial-time algorithm for it. In the remaining part of this section, the network is always assumed to be a path.

Recall that the problem is to check whether an object o_l is reachable for an agent k with $l < k$. We introduce the main idea of the algorithm as follows.

1. First, we show that the instance is equivalent to the instance after deleting all agents (and the corresponding endowed objects) on the left of agent l. Thus, we can assume the problem is to check whether object o_1 is reachable for an agent k.
2. Second, we prove that if o_1 is reachable for agent k then there is an object $o_{n'}$ with $n' \geq k$ that must be moved to agent $k-1$ in the final assignment. We show that we can also delete all agents and objects on the right of agent n' in the initial endowment. However, we cannot find the agent n' directly. In our algorithm, we will guess n' by letting it be each possible value between k and n. For each candidate n', we get a special instance, called the neat $(o_1, o_{n'}, k)$-CONSTRAINED instance, which contains n' agents and objects and the goal is

to check whether the first object o_1 is reachable for agent k and the last object $o_{n'}$ is reachable for agent $k-1$ simultaneously, i.e., whether there is a reachable assignment σ' such that $\sigma'(k) = o_1$ and $\sigma'(k-1) = o_{n'}$.

3. Third, to solve the original instance, now we turn to solve at most n neat $(o_1, o_{n'}, k)$-Constrained instances. We characterize reachable assignments for neat $(o_1, o_{n'}, k)$-Constrained instances by showing that any two objects must satisfy some *compatible properties*.

4. Fourth, based on the compatible properties, we can prove that each object o_i will be moved to either the left or the right side of its original position in a reachable assignment (i.e., can not stay at its original position), and there is at most one possible position i_l for the left side and most one possible position i_r for the right side. Furthermore, we can compute the candidate positions i_l and i_r in polynomial time.

5. Last, we still need to decide whether to move each object o_i to the left or the right side. (i.e., to decide which of i_l and i_r is the correct position for o_i in the reachable assignment). Since there are at most two possible positions for each object, we can reduce the problem to the 2-Sat problem and then solve it in polynomial time by using fast solvers for 2-Sat.

3.1 Basic Properties and Neat Constrained Instances

When the preferences are strict, we have the following observations and lemmas.

Observation 1 *Let $(\sigma_0, \sigma_1, \ldots, \sigma_t)$ be a given sequence of swaps and $j \in N$ be an agent. For any $i \in \{0, 1, \ldots, t-1\}$, it holds that either $\sigma_{i+1}(j) = \sigma_i(j)$ or $\sigma_{i+1}(j) \succ_j \sigma_i(j)$.*

It implies the following lemma.

Lemma 1 *Let $(\sigma_0, \sigma_1, \ldots, \sigma_t)$ be a given sequence of swaps. For any two integers $i < j$ in $\{0, 1, \ldots, t\}$ and any agent $q \in N$, if $\sigma_i(q) = \sigma_j(q)$, then $\sigma_d(q) = \sigma_i(q)$ for any integer $i \le d \le j$.*

Lemma 1 also says that an object will not 'visit' an agent twice. This property is widely used in similar allocation problems under strict preferences.

Next, we analyze properties under the constraint that the social network is a path. In a swap, an object is moved to the *right side* if it is moved from agent i to agent $i+1$, and an object is moved to the *left side* if it is moved from agent i to agent $i-1$. In each swap, one object is moved to the right side and one object is moved to the left side. We study the tracks of the objects in a feasible assignment sequence.

Lemma 2 *For a sequence of swaps $(\sigma_0, \sigma_1, \ldots, \sigma_t)$, if $\sigma_t^T(o_i) = j$, then there are exactly $|j-i|$ swaps including o_i. Furthermore, in all the $|j-i|$ swaps, o_i is moved to the right side if $i < j$, and moved to the left side if $i > j$.*

Proof First of all, we show that in a sequence of feasible swaps it is impossible for an object to be moved to the right and also to the left. Assume to the contrary that an object o is moved in both directions and we will show a contradiction. Consider two consecutive swaps including o (i.e., there are no swaps including o

between these two swaps) such that o is moved in two opposite directions in the two swaps. Assume that object o is moved away from agent q after the first swap. We can see that o will be moved back to agent q after the second swap. Note that between these two swaps no swap includes o and then o will not be moved between them. This means object o will visit agent q twice, a contradiction to Lemma 1. So any object can only move in one direction in a sequence of swaps.

Since each swap including an object can only move the object to the neighbor position, we know that there are exactly $|j - i|$ swaps including o_i that move o_i to the right side if $i < j$ and to the left side if $i > j$. □

Lemma 3 *Let $(\sigma_0, \sigma_1, \ldots, \sigma_t)$ be a sequence of swaps, and o_a and o_b be two objects with $a < b$. Let $a' = \sigma_t^T(o_a)$ and $b' = \sigma_t^T(o_b)$. If $a' \leq a$ or $b' \geq b$, then $a' < b'$.*

Proof We consider three cases. If $a' \leq a$ and $b' \geq b$, then by $a < b$ we get that $a' \leq a < b \leq b'$. Next, we assume that $a' \leq a$ and $b' < b$. By Lemma 2, we know that both objects o_a and o_b can only be moved to the left side. If $a' \geq b'$, by $a < b$ we know that there must exist a swap including both o_a and o_b. In this swap, two objects are moved in two opposite directions, a contradiction to the fact that the two objects can only be moved to the left side. The last case where $a' > a$ and $b' \geq b$ can be proved similarly. □

Lemma 1 shows that any object can only be moved in one direction. Lemma 3 shows that when an object is moved to the right side, all objects initially allocated on the left of it can not be moved to the right of it at any time; when an object is moved to the left side, all objects initially allocated to the right of it can not be moved to the left of it at any time.

In fact, if we want to move an object o_l to an agent k with $k > l$, we may not need to move any object to the left of o_l, i.e., objects $o_{l'}$ with $l' < l$. Equipped with Lemma 3, we can prove

Lemma 4 *If object o_l is reachable for agent k, then there is a feasible assignment sequence $(\sigma_0, \sigma_1, \ldots, \sigma_t)$ such that $\sigma_t^T(o_l) = k$, and $\sigma_t(i) = \sigma_0(i)$ for all $i < l$ if $l \leq k$ and for all $i > l$ if $l \geq k$.*

Proof The result can be derived from Proposition 2 in [15]. The authors give an algorithm to produce a sequence of swaps, which does not move any objects o_i for all $i < l$ if $l \leq k$ and for all $i > l$ if $l \geq k$. Proposition 2 shows that if object o_l is reachable for agent k then the obtained sequence of swaps is feasible. □

For an instance $I = (N, O, \succ, P, \sigma_0, k, o_l)$ of OBJECT REACHABILITY in paths with $l < k$, let $I' = (N', O', \succ', P', \sigma_0', k, o_l)$ denote the instance obtained from I by deleting agents $\{1, 2, \ldots, l-1\}$ and objects $\{o_1, o_2, \ldots, o_{l-1}\}$. In other words, we let $N' = \{l, l+1, \ldots, n\}$, $O' = \{o_l, o_{l+1}, \ldots, o_n\}$, and $\succ', P'$ and σ_0' be the corresponding subsets of $\succ, P$ and σ_0.

Corollary 1 *Object o_l is reachable for agent k in the instance I if and only if object o_l is reachable for agent k in the corresponding instance I'.*

By Corollary 1, we can always assume that the instance of OBJECT REACHA-BILITY in paths is to check whether object o_1 is reachable for an agent k.

Assume that object o_1 is reachable for agent k. For a sequence of swaps $(\sigma_0, \sigma_1, \ldots, \sigma_t)$ such that $\sigma_t(o_1) = k$, there are exactly $k - 1$ swaps including

o_1, where o_1 is moved to the right side according to Lemma 2. The last swap including o_1 is between agent $k-1$ and agent k. Let $o_{n'}$ denote the other object included in the last swap. In other words, after the last swap, agent $k-1$ will get object $o_{n'}$ and agent k will get object o_1. Note that $o_{n'}$ is moved to the left side in this swap. By Lemma 2, we know that object $o_{n'}$ is moved to the left side in all swaps including $o_{n'}$. Therefore, we have the following observation.

Observation 2 *If there is a reachable assignment σ such that $\sigma(k) = o_1$ and $\sigma(k-1) = o_{n'}$, then it holds that $n' \geq k$.*

Definition 1 The $(o_1, o_{n'}, k)$-CONSTRAINED problem is to decide whether an instance I of OBJECT REACHABILITY has a reachable assignment σ such that $\sigma(k-1) = o_{n'}$ and $\sigma(k) = o_1$, where $n' \geq k$.

Our idea is to transform our problem to the $(o_1, o_{n'}, k)$-CONSTRAINED problem. We do not know the value of n'. So we search by letting n' be each value in $\{k, k+1, \ldots, n\}$. This will only increase the running time bound by a factor of n. So we get

Lemma 5 *An instance $I = (N, O, \succ, P, \sigma_0, k, o_1)$ is a yes-instance if and only if at least one of the $(o_1, o_{n'}, k)$-CONSTRAINED instances for $n' \in \{k, k+1, \ldots, n\}$ is a yes-instance.*

For an $(o_1, o_{n'}, k)$-CONSTRAINED instance I, we use $I_{-n'}$ to denote the instance obtained from I by deleting agents $\{n'+1, n'+2, \ldots, n\}$ and objects $\{o_{n'+1}, o_{n'+2}, \ldots, o_n\}$.

Lemma 6 *An $(o_1, o_{n'}, k)$-CONSTRAINED instance I is a yes-instance if and only if the corresponding instance $I_{-n'}$ is a yes-instance.*

Proof We prove this lemma by induction on n'. It is clear that the lemma holds when $n' = 2$. Assume that the lemma holds for $n' = n_0 - 1$. We show that the lemma also holds for $n' = n_0$. Let $n' = n_0$. When $I_{-n'}$ is a yes-instance, it is obvious that I is also a yes-instance since we can use the same sequence of swaps as the solution to them. Next, we consider the converse direction and assume that I is a yes-instance. Then there is a sequence of swaps $\{\sigma_0, \sigma_1, \ldots, \sigma_t\}$ such that $\sigma_t(k-1) = o_{n_0}$ and $\sigma_t(k) = o_1$. We consider the first swap (σ_{r-1}, σ_r) including object o_{n_0}. Since o_{n_0} will be moved to agent $k-1$ with $k-1 < n_0$, by Lemma 2 we know that all swaps including o_{n_0} will move o_{n_0} to the left side. So in the swap (σ_{r-1}, σ_r), object o_{n_0} is also moved to the left side. By Lemma 3, we know that no object o_i with $i > n_0$ is moved to the left of o_{n_0} before σ_r. Let σ'_r be the assignments of the first n_0 agents in σ_r. So we do not need to move any objects o_i with $i > n_0$ to get σ'_r. Now object o_{n_0} is at the position $n_0 - 1$. Since the lemma holds for $n' = n_0 - 1$, we know that there is a solution (a sequence of swaps) that does not move any objects o_i with $i > n_0 - 1$ from σ_r to a satisfying assignment. Such that there is a solution (a sequence of swaps) to I that does not move any objects o_i with $i > n_0$, which is also a solution to $I_{-n'}$. $\square$

By Lemma 6, we can ignore all agents on the right side of n' in an $(o_1, o_{n'}, k)$-CONSTRAINED instance.

Definition 2 An $(o_1, o_{n'}, k)$-CONSTRAINED instance is called *neat* if n' is the last agent.

Next, we only consider neat $(o_1, o_{n'}, k)$-CONSTRAINED instances.

3.2 Characterization of Reachable Assignments

To solve neat $(o_1, o_{n'}, k)$-CONSTRAINED instances, we will give a full characterization of reachable assignments for them, which shows that an assignment is reachable if and only if it is *compatible* (defined in Definition 4). The compatibility reveals the relation between any two objects in a reachable assignment.

For two integers x and y, we use $[x, y]$ to denote the set of integers between x and y (including x and y). In the definition, we do not require $x \leq y$. So we have that $[x, y] = [y, x]$.

Consider a neat $(o_1, o_{n'}, k)$-CONSTRAINED instance and an assignment σ_t such that $\sigma_t^T(o_1) = k$ and $\sigma_t^T(o_{n'}) = k - 1$. For any two objects o_a and o_b, we let $a' = \sigma_t^T(o_a)$ and $b' = \sigma_t^T(o_b)$.

Note that since we only consider neat $(o_1, o_{n'}, k)$-CONSTRAINED instances here, we know that all the objects will move before we get the final assignment where agent k gets object o_1.

Definition 3 The two objects o_a and o_b are *intersected* in assignment σ_t if $Q = [a, a'] \cap [b, b']$ is not an empty set.

There are three kinds of intersections for two objects in an assignment. In the first two kinds of intersections, the two objects are moved in the same direction, i.e., $(a' - a)(b' - b) > 0$. In the last kind of intersections, the two objects are moved in the two opposite directions, i.e., $(a' - a)(b' - b) < 0$. The following Lemma 7 describes a property of the first two kinds of intersections and Lemma 8 describes a property of the third kind of intersections.

Lemma 7 *Let $(\sigma_0, \sigma_1, \ldots, \sigma_t)$ be a sequence of swaps, and o_a and o_b be two objects with $1 \leq a < b \leq n$. Let $a' = \sigma_t^T(o_a)$, $b' = \sigma_t^T(o_b)$, and $Q = [a, a'] \cap [b, b']$. Assume that $Q \neq \emptyset$.*
(a) If $a' > a$ and $b' > b$, then it holds that $o_a \succ_q o_b$ for all $q \in Q$;
(b) If $a' < a$ and $b' < b$, then it holds that $o_b \succ_q o_a$ for all $q \in Q$.

Proof Both objects o_a and o_b will visit each agent in Q during the sequence of swaps $(\sigma_0, \sigma_1, \ldots, \sigma_t)$.
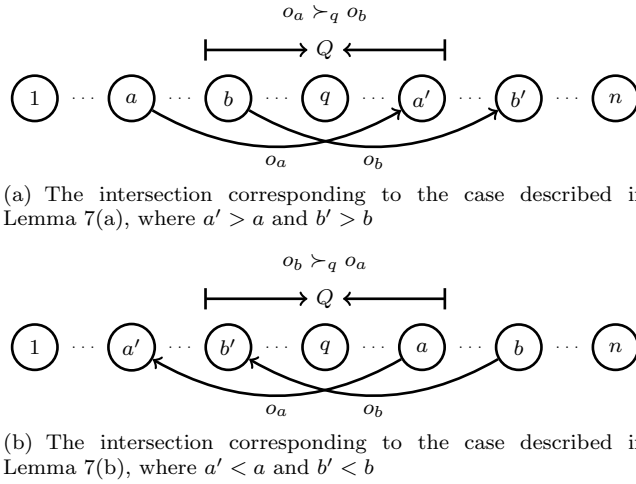
(a) Since $b' > b$, by Lemma 3, we know that for each agent $q \in Q$ object o_b will visit agent q before object o_a. By Observation 1, we know that $o_a \succ_q o_b$.

(b) Since $a' < a$, by Lemma 3, we know that for each agent $q \in Q$ object o_a will visit agent q before object o_b. By Observation 1, we know that $o_b \succ_q o_a$. $\square$

See Figure 2 for an illustration of the two kinds of intersections in Lemma 7.

Lemma 8 *Let $(\sigma_0, \sigma_1, \ldots, \sigma_t)$ be a sequence of swaps for a neat $(o_1, o_{n'}, k)$-CONSTRAINED instance such that $\sigma_t^T(o_1) = k$ and $\sigma_t^T(o_{n'}) = k - 1$, and o_a and o_b be two objects with $1 \leq a < b \leq n'$. Let $a' = \sigma_t^T(o_a)$ and $b' = \sigma_t^T(o_b)$. Assume that $a' > a$, $b' < b$ and $Q = [a, a'] \cap [b, b'] \neq \emptyset$. Let $Q' = [a + 1, a'] \cap [b, b']$.*
(a) There is a swap including o_a and o_b which happens between agent $c - 1$ and c, where $c = a' + b' - k + 1 \in Q'$;
(b) It holds that $o_b \succ_q o_a$ for all $\max(a, b') \leq q < c$, and $o_a \succ_q o_b$ for all $c \leq q \leq \min(a', b)$.

(a) The intersection corresponding to the case described in Lemma 7(a), where $a' > a$ and $b' > b$

(b) The intersection corresponding to the case described in Lemma 7(b), where $a' < a$ and $b' < b$

Fig. 2 An illustration of the two kinds of intersections in Lemma 7

Proof Since $Q \neq \emptyset$, we know that there exists a swap including o_a and o_b in the sequence of swaps. We determine the position of this swap. We first consider **(a)**.

By Lemma 2, we know that during the sequence of swaps each object will be moved in at most one direction. Let $o_{r_i} = \sigma_t(i)$ for all $1 \leq i \leq n'$. Since $\sigma_t^T(o_1) = k$, by Lemma 3 we know that $i < r_i$ for all $1 \leq i < k$. Thus objects o_{r_i} for all $1 \leq i < k$ are moved to the left side. Since $\sigma_t^T(o_{n'}) = k - 1$, by Lemma 3 we know that $i > r_i$ for all $k \leq i \leq n'$. Thus objects o_{r_i} for all $k \leq i \leq n'$ are moved to the right side.

Since $\sigma_t^T(o_1) = k$ and o_a is moved to the right side, by Lemma 3 we know that $a' \geq k$. By Lemma 3 again, we know that $1 \leq r_i \leq a$ for each $k \leq i \leq a'$. For each object in $\{o_{a+1}, \ldots, o_b\}$, if it is moved to the right side then it can only by assigned to agents in $\{k, k+1, \ldots, a'\}$ in σ_t by Lemma 3. So there are exactly $a' - k + 1$ objects in $\{o_{a+1}, \ldots, o_b\}$ that are moved to the right side. Furthermore, all of them are moved to the right of $k - 1$ in σ_t.

Since object o_b is moved to the left side from b to b', by Lemma 3 we know that there are exactly $b - b'$ objects in $\{o_1, \ldots, o_b\}$ that will be moved to the right of o_b in σ_t, which are all moved in the right direction. We further show that no object o_x in $\{o_1, \ldots, o_b\}$ moved in the right direction will stay at the left of b' in σ_t. Since $\sigma_t^T(o_{n'}) = k - 1$ and o_b is moved to the left side, by Lemma 3 we know that $b' \leq k - 1$, and all objects that are moved to the right side are at the right of o_1 in σ_t. If o_x exists, then $\sigma_t^T(o_x) > k > b'$, which means that o_x is moved to the right of o_b in σ_t, a contradiction. So there are exactly $b - b'$ objects in $\{o_1, \ldots, o_b\}$ that are moved to the right side.

By the above two results, we know that the number of objects in $\{o_{a+1}, \ldots, o_b\}$ that are moved in the right direction is $b - b' - a' + k - 1$. Since o_b should be swapped with o_a, we know that o_b needs to be swapped with all objects in $\{o_{a+1}, \ldots, o_b\}$ that are moved in the right direction. So right before the swap including o_a and o_b, object o_b is at agent $b - (b - b' - a' + k - 1) = a' + b' - k + 1$. Thus, $c = a' + b' - k + 1$. We know that **(a)** holds.

Next, we consider **(b)**. For any agent q such that $\max(a, b') \leq q < c$, both o_a and o_b will visit it. By Lemma 3, we know that o_a will visit q before o_b. By Observation 1, we know that $o_b \succ_q o_a$. For any agent q such that $c \leq q \leq \min(a', b)$, both o_a and o_b will visit it. By Lemma 3, we know that o_b will visit q before o_a. By Observation 1, we know that $o_a \succ_q o_b$. Thus **(b)** holds. $\square$

See Figure 3 for an illustration of the intersection in Lemma 8.
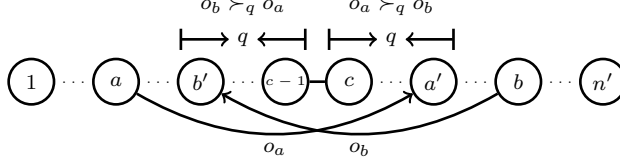
Fig. 3 An illustration of the intersection in Lemma 8, where $a' > a$ and $b' < b$

Based on Lemmas 7 and 8, we give the conditions for assignment σ_t to be a feasible assignment.

Definition 4 We say that a pair of objects o_a and o_b ($a < b$) are *compatible* in assignment σ_t if they are either not intersected or intersected and satisfying one of the following:

1. when $a' > a$ and $b' > b$, it holds that $a' < b'$ (corresponding to Lemma 3) and $o_a \succ_q o_b$ for all agents $q \in Q$ (corresponding to Lemma 7(a));

2. when $a' < a$ and $b' < b$, it holds that $a' < b'$ (corresponding to Lemma 3) and $o_b \succ_q o_a$ for all agents $q \in Q$ (corresponding to Lemma 7(b));

3. when $a' > a$ and $b' < b$, it holds that $c = a' + b' - k + 1 \in Q' = Q \setminus \{a\}$, $o_b \succ_q o_a$ for all $\max(a, b') \leq q < c$, and $o_a \succ_q o_b$ for all $c \leq q \leq \min(a', b)$ (corresponding to Lemma 8).

Definition 5 An assignment σ_t is *compatible* if it holds that $\sigma_t(i) \neq o_i$ for any agent i and any pair of objects in it are compatible.

Lemma 7 and Lemma 8 imply that

Lemma 9 *If σ_t is a reachable assignment for a neat $(o_1, o_{n'}, k)$-Constrained instance such that $\sigma_t^T(o_1) = k$ and $\sigma_t^T(o_{n'}) = k - 1$, then σ_t is compatible.*

Proof If there is an agent i such that $i \leq k$ (resp., $i \geq k$) and $\sigma(i) = o_i$, then by Observation 1 we know that agent i does not participate in any swap and then object o_1 cannot go to agent k (resp., object $o_{n'}$ cannot go to agent $k - 1$) since only two adjacent agents can participate in a swap. So $\sigma(i) \neq o_i$ holds for any agent i.

For any pair of intersected objects o_a and o_b, there are only three cases that are corresponding to the three items in Definition 4. By Lemma 7 and Lemma 8, we know that they must be compatible. $\square$

Lemma 10 *Let σ_t be a compatible assignment for a neat $(o_1, o_{n'}, k)$-CONSTRAINED instance such that $\sigma_t^T(o_1) = k$ and $\sigma_t^T(o_{n'}) = k - 1$. For any two objects o_{x-1} and o_x, if $\sigma_t^T(o_{x-1}) > x - 1$ and $\sigma_t^T(o_x) < x$, then the swap between agents $x - 1$ and x in σ_0 is feasible. Let σ_1 denote the assignment after the swap between $x - 1$ and x in σ_0. Then σ_t is still compatible by taking σ_1 as the initial endowment.*

Proof Consider the two objects o_{x-1} and o_x, where o_{x-1} is moved in the right direction and o_x is moved in the left direction. They are compatible. By item 3 in the definition of the compatibility, we know that $c = x$, which implies the swap between agents $x - 1$ and x is feasible in σ_0.

Next, we show that if we take σ_1 as the initial endowment, the assignment σ_t is still compatible. Compared with σ_0, the endowment positions of two objects are changed. To check whether σ_t is compatible with σ_1 (being the initial endowment), we only need to check the compatibility of object pairs involving at least one of the two objects o_{x-1} and o_x.

Now $\sigma_1(x - 1) = o_x$, $\sigma_1(x) = o_{x-1}$ and $\sigma_1(i) = o_i$ for all other $i \notin \{x - 1, x\}$.

First, we consider object pair o_x and an object o_i with $i \geq x$. If o_x and o_i are intersected, then the intersection can only be the case described in Lemma 7(**b**). Item 2 in the definition of compatibility will hold because only the value of a changes to a smaller value from x to $x - 1$ and the domain of Q will not increase.

Second, we consider object pair o_x and an object o_i with $1 \leq i \leq x - 2$. If o_x and o_i are intersected, there are two possible cases. When $\sigma_t(o_i) < i$, the intersection will be the case described in Lemma 7(**b**). Item 2 in the definition of compatibility will still hold because only the value of b changes to a smaller value from x to $x - 1$ and the domain of Q will not increase. When $\sigma_t(o_i) > i$, the intersection will be the case described in Lemma 8. We show that item 3 in the definition of compatibility will hold. The value of c for $o_a = o_i$ and $o_b = o_x$ is the same no matter taking σ_0 or σ_1 as the initial endowment, since none of a', b' and k is changed. Note that when taking σ_0 as the initial endowment the value of c is x for $o_a = o_{x-1}$ and $o_b = o_x$. If the value of c is also x for $o_a = o_i$ and $o_b = o_x$ ($i \neq x - 1$), then we will get a contradiction that both o_{x-1} and o_i will be assigned to the same agent in σ_t by the computation formula of c. So we know that $c \neq x$ for $o_a = o_i$ and $o_b = o_x$. Thus, we get $c \in Q' = [i + 1, \sigma_t^T(o_i)] \cap [\sigma_t^T(o_x), x - 1]$ because $c \in Q = [i+, \sigma_t^T(o_i)] \cap [\sigma_t^T(o_x), x]$ by σ_t being compatible with σ_0 and $c \neq x$. After taking σ_1 as the initial endowment, the values of a, a', c and b' will not change, and the value of b changes from x to a smaller value $x - 1$. We can see that the follows still hold: $o_b \succ_q o_a$ for all $\max(a, b') \leq q < c$, and $o_a \succ_q o_b$ for all $c \leq q \leq \min(a', b)$.

Third, we consider o_{x-1} and an object o_i with $1 \leq i \leq x - 1$. If o_{x-1} and o_i are intersected, then the intersection can only be the case described in Lemma 7(**a**). Item 1 in the definition of compatibility will still hold because only the value of b changes to a bigger value from $x - 1$ to x and the domain of Q will not increase.

Fourth, we consider o_{x-1} and an object o_i with $i \geq x$. If o_{x-1} and o_i are intersected, there are two possible cases. When $\sigma_t(o_i) > i$, the intersection will be the case described in Lemma 7(**a**). Item 1 in the definition of compatibility will still hold because only the value of a changes to a bigger value from $x - 1$ to x and the domain of Q will not increase. When $\sigma_t(o_i) < i$, the intersection will be the case described in Lemma 8. Analogously, we use similar arguments for the second case, we can prove that item 3 in the definition of compatibility holds.

So if σ_t is compatible by taking σ_0 as the initial endowment, then it is compatible by taking σ_1 as the initial endowment.   □

Based on Lemma 10 we will prove the following lemma.

Lemma 11 *Let σ_t be an assignment for a neat $(o_1, o_{n'}, k)$-Constrained instance such that $\sigma_t^T(o_1) = k$ and $\sigma_t^T(o_{n'}) = k-1$. If σ_t is compatible, then σ_t is a reachable assignment.*

Proof We show that we can find a sequence of swaps from σ_0 to σ_t for each compatible assignment σ_t. In our algorithm, we first move object $\sigma_t(1)$ from its current position to agent 1, then move object $\sigma_t(2)$ to agent 2, and so on. A formal description of the procedure is that: For i from 1 to $k-1$, move object $\sigma_t(i)$ from its current position to agent i by a sequence of swaps including it (if the current position of $\sigma_t(i)$ is agent j, then it is a sequence of $|j-i|$ swaps). To prove the correctness of the algorithm, we need to show that each swap in the algorithm is feasible and finally we can get the assignment σ_t.

First, we show that the first loop of the algorithm can be executed legally, i.e., $\sigma_t(1)$ can be moved to agent 1 by a sequence of swaps. The first swap between agents $x-1$ and x in σ_0 is feasible by Lemma 10. We use σ_1 to denote the assignment after this swap in σ_0. Then σ_t is compatible by taking σ_1 as the initial endowment by Lemma 10. By applying Lemma 10 iteratively, we know that the swap between agents $x-1-i$ and $x-i$ in σ_i is feasible (for $i > 0$). We use σ_{i+1} to denote the assignment after the swap in σ_i and assume that object o_x is assigned to agent 1 in σ_{t_1}. By Lemma 10, we know that the sequence of swaps from σ_0 to σ_{t_1} are feasible and σ_t is compatible by taking σ_{t_1} as the initial endowment.

Next, we consider σ_{t_1} as the initial endowment. Agent 1 has already gotten object o_x and we can ignore it. After deleting agent 1 and object o_x from the instance, we get a new $(o_1, o_{n'}, k)$-Constrained instance. The second loop of the algorithm is indeed to move the corresponding object to the first agent in the new instance. So the correctness of the second loop directly follows from the above argument for the first loop.

By iteratively applying the above arguments, we can prove that each loop of the algorithm can be executed legally. Therefore, all swaps in the algorithm are feasible.

Let $\sigma_{t'}$ be the assignment returned by the algorithm. It holds that $\sigma_{t'}(i) = \sigma_t(i)$ for $i \leq k-1$. For $i \geq k$, we show that $\sigma_{t'}(i) = \sigma_t(i)$ still holds. For any object $o_j = \sigma_{t'}(i)$ with $i \geq k$, objects o_j and $o_{n'}$ are intersected and the intersection can only be the case described in Lemma 8. By item 3 in the definition of compatibility, we know the value c will not change no matter what the endowment position of o_j is. Thus, o_j and $o_{n'}$ will be swapped between two fixed agents and this is the last swap including o_j. Object o_j will arrive at the position $a' = \sigma_t^T(o_j)$. Therefore, $\sigma_{t'} = \sigma_t$.   □

By Lemmas 9 and 11, to solve a neat $(o_1, o_{n'}, k)$-Constrained instance, we only need to find a compatible assignment for it.

3.3 Computing Compatible Assignments

In a compatible assignment, object o_1 will be assigned to agent k and object $o_{n'}$ will be assigned to agent $k-1$. We consider other objects o_i for $i \in \{2, 3, \ldots, n'-1\}$.

In a compatible assignment, object o_i will not be assigned to agent i since each agent will participate in at least one swap including object o_1 or $o_{n'}$. There are two possible cases: o_i is assigned to agent i' such that $i' < i$; o_i is assigned to agent i' such that $i' > i$. We say that o_i is moved to the left side for the former case and moved to the right side for the latter case. We will show that for each direction, there is at most one possible position for each object o_i in a compatible assignment.

First, we consider $i \in \{2, 3, \ldots, k-1\}$. Assume that object o_i is moved to the left side in a compatible assignment. Thus, o_1 and o_i are intersected and the intersection is of the case described in Lemma 8. We check whether there is an index i' such that $i' \leq i$, $o_i \succ_{i'-1} o_1$ and $o_1 \succ_j o_i$ for each $j \in \{i', i'+1, \ldots, i\}$. The index i' is also corresponding to the index c in Lemma 8. We can see that i' is the only possible agent for object o_i to make o_1 and o_i compatible if o_i is moved to the left side. We use i_l to denote this agent i' if it exists for i. Assume that object o_i is moved to the right side in a compatible assignment. Since o_1 will be moved to agent k and o_i will be moved to the right side, by Lemma 3 we know that o_i will be moved to the right of o_1, i.e., an agent i'' with $i'' > k$. Thus, o_i and $o_{n'}$ are intersected and the intersection is of the case described in Lemma 8. We check whether there is an index i' such that $i' > k$, $o_i \succ_{i'} o_{n'}$ and $o_{n'} \succ_j o_i$ for each $j \in \{k-1, k, \ldots, i'-1\}$. We can see that i' is the only possible agent for object o_i to make $o_{n'}$ and o_i compatible if o_i is moved to the right side. We use i_r to denote this agent i' if it exists for i.

Second, we consider $i \in \{k, k+1, \ldots, n'-1\}$. In fact, the structure of neat $(o_1, o_{n'}, k)$-CONSTRAINED instances is symmetrical. We can rename the agents on the path from left to right as $\{n', n'-1, \ldots, 1\}$ instead of $\{1, 2, \ldots, n'\}$ and then this case becomes the above case. We can compute i_l and i_r for each $i \in \{k, k+1, \ldots, n'-1\}$ similarly.

The procedure to compute i_l and i_r for each agent $i \in \{2, 3, \ldots, n'-1\}$ in a neat $(o_1, o_{n'}, k)$-CONSTRAINED instance is presented as the following.

Algorithm 1: To compute i_l and i_r

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1 **for** $2 \leq i \leq n'-1$ **do**
2   **if** *there exists i' such that $i' \leq i$, $o_i \succ_{i'-1} o_1$ and $o_1 \succ_j o_i$ for each $j \in \{i', i'+1, \ldots, i\}$* **then**
3 $\lfloor$ let $i_l = i'$;
4   **else**
5 $\lfloor$ let $i_l = \bot$ to indicate that i_l does not exist.
6   **if** *there exists i' such that $i' > k$, $o_i \succ_{i'} o_{n'}$ and $o_{n'} \succ_j o_i$ for each $j \in \{k-1, k, \ldots, i'-1\}$* **then**
7 $\lfloor$ let $i_r = i'$;
8   **else**
9 $\lfloor$ let $i_r = \bot$ to indicate that i_r does not exist.

If none of i_l and i_r exists for some i, then this instance is a no-instance. If only one of i_l and i_r exists, then object o_i must be assigned to this agent in any compatible assignment. The hardest case is that both i_l and i_r exist, where we may

not know which agent the object will be assigned to in the compatible assignment. In this case, we will rely on algorithms for 2-SAT to find possible solutions.

For each agent $j \in \{1, 2, 3, \ldots, n'\}$, we will use R_j to store all possible objects that may be assigned to agent j in a compatible assignment. The following procedure computes the initial R_j.

1. Initially, let $R_{k-1} = \{o_{n'}\}$, $R_k = \{o_1\}$, and $R_i = \emptyset$ for all other agent i.
2. For each $i \in \{2, 3, \ldots, n'-1\}$, compute i_l and i_r by Algorithm 1, and add o_i to R_{i_l} if $i_l \neq \perp$ (resp., to R_{i_r} if $i_r \neq \perp$).

We also use the following two steps to iteratively update R_j and then make the size of R_j at most 2 for each j.

1. If there is an empty set R_{j_0} for an agent j_0, stop and report the instance is a no-instance;
2. If there is a set R_{j_0} containing only one object o_{i_0} and the object o_{i_0} appears in two sets R_{j_0} and $R_{j'_0}$, then delete o_{i_0} from $R_{j'_0}$.

The correctness of the second step is based on the fact that agent j_0 should get one object. If there is only one candidate object o_{i_0} for agent j_0, then o_{i_0} can only be assigned to agent j_0 in any compatible assignment.

We also analyze the running time of the above procedure to compute all R_j. Algorithm 1 computes i_l and i_r in $O(n)$ for each object o_i. Therefore, we use $O(n^2)$ time to set the initial values for all sets R_j. To update R_i, we may execute at most n iterations and each iteration can be executed in $O(n)$. Hence, the procedure to compute all R_j runs in time $O(n^2)$.

Lemma 12 *After the above procedure, either the instance is a no-instance or it holds that $1 \leq |R_j| \leq 2$ for each $j \in \{1, 2, \ldots, n'\}$.*

Proof We only need to consider the latter case where each set R_j contains at least one object after the procedure. Note that for each set R_j which contains only one object, the object will not appear in any other set. We ignore these singletons R_j and consider the remaining sets. Each remaining set contains at least two objects. On the other hand, each other object o_i (not appearing in a singleton) can be in at most two sets R_{i_l} and R_{i_r}. On average each remaining set R_j contains at most two objects. Therefore, each remaining set R_j contains exactly two objects. $\square$

For a set R_j containing only one object o_i, we know that object o_i should be assigned to agent j in any compatible assignment. For sets R_j containing two objects, we still need to decide which object is assigned to this agent such that we can get a compatible assignment.

We will reduce the remaining problem to 2-SAT. The 2-SAT instance contains n' variables $\{x_1, x_2, \ldots, x_{n'}\}$ corresponding to the n' objects. When $x_i = 1$, it means that object o_i is moved to the right side, i.e, we will assign it to agent i_r in the compatible assignment. When $x_i = 0$, it means that object o_i is moved to the left side and we will assign it to agent i_l in the compatible assignment. In the 2-SAT instance, we have two kinds of clauses, called *agent clauses* and *compatible clauses*.

For each set R_j, we associate $|R_j|$ literals with it. If there is an object o_i such that $i_l = j$, we associate literal $\overline{x_i}$ with R_j; if there is an object o_i such that $i_r = j$,

we associate literal x_i with R_j. For each set R_j of size 1 (let the associated literal be ℓ_j), we construct one clause c_j containing only one literal ℓ_j. For each set R_j of size 2 (let the associated literals be ℓ_j^1 and ℓ_j^2), we construct two clauses $c_{j1} : \ell_j^1 \vee \ell_j^2$ and $c_{j2} : \overline{\ell_j^1} \vee \overline{\ell_j^2}$. These clauses are called *agent clauses*. From the construction, we can see that both of the two clauses c_{j1} and c_{j2} are satisfied if and only if exactly one object is assigned to agent j. The agent clauses can guarantee that exactly one object is assigned to each agent.

For each pair of sets R_j and R_i, we construct several clauses according to the definition of compatibility. For any two objects $o_{j'} \in R_j$ (the corresponding literal associated with R_j is ℓ_j) and $o_{i'} \in R_i$ (the corresponding literal associated with R_i is ℓ_i), we say that ℓ_j and ℓ_i are *compatible* if $o_{j'}$ and $o_{i'}$ are compatible when $o_{j'}$ is assigned to agent j and $o_{i'}$ is assigned to agent i in the assignment. If ℓ_j and ℓ_i are not compatible, then either $o_{j'}$ cannot be assigned to agent j or $o_{i'}$ cannot be assigned to agent i in any compatible assignment. So we construct one *compatible clause*: $\overline{\ell_j} \vee \overline{\ell_i}$ for each pair of incompatible pair ℓ_j and ℓ_i. We can see that if the compatible clause $\overline{\ell_j} \vee \overline{\ell_i}$ is satisfied if and only if either $o_{j'}$ is not assigned to agent j or $o_{i'}$ is not assigned to agent i. Since we construct a compatible clause for any possible incompatible pair, we know that all the compatible clauses are satisfied if and only if there is not incompatible pair in the assignment. Note that, for each pair of sets R_j and R_i, we will create at most $2 \times 2 = 4$ compatible clauses since each of R_j and R_i contains at most 2 elements. Therefore, the number of compatible clauses in $O(n^2)$.

By Lemma 12, we know that there are at most two candidate objects for each agent in a compatible assignment. Thus, each agent clause contains at most two literals. By the construction of compatible clauses, we know that each compatible clause contains exactly two literals. So, the constructed instance is a 2-SAT instance. The construction of the 2-SAT instance implies the following lemma.

Lemma 13 *The neat $(o_1, o_{n'}, k)$-CONSTRAINED instance has a compatible assignment if and only if the corresponding 2-SAT instance constructed above has a satisfiable assignment.*

Proof First of all, we show that if the neat $(o_1, o_{n'}, k)$-CONSTRAINED instance has a compatible assignment then the 2-SAT instance is satisfiable. Let σ be a compatible assignment of the neat $(o_1, o_{n'}, k)$-CONSTRAINED instance. By the definition of compatible assignments, we know that each object is either moved to the left or the right in the path (cannot stay at its initial position) in the assignment σ. If object o_i is moved to the left we let the corresponding variable $x_i = 0$ in the 2-SAT instance, and if object o_i is moved to the right we let the corresponding variable $x_i = 1$. Thus, we assign values for all variables. Since each agent gets exactly one object in the compatible assignment σ, we know that all the agent clauses are satisfied. Furthermore, $\sigma(i)$ and $\sigma(j)$ are compatible for any pair of agents i and j in the compatible assignment σ, and then all compatible clauses are satisfied. So the 2-SAT instance is satisfiable.

Next, we consider the other direction. Let Π be a satisfying assignment of the 2-SAT instance. We construct an assignment σ' of the neat $(o_1, o_{n'}, k)$-CONSTRAINED instance: if variable x_i is assigned 1 in Π, let $\sigma'(i_r) = o_i$, and let $\sigma'(i_l) = o_i$ otherwise. We show that σ' is a compatible assignment. First, each object is assigned to an agent different from its initial agent since each variable has been assigned

to 1 or 0. Second, each agent only gets one object, because all the agent clauses are satisfied. Last, any pair of objects are compatible, because all the compatible clauses are satisfied. Such assignment σ' is compatible. □

3.4 The Whole Algorithm

The main steps of the whole algorithm to solve OBJECT REACHABILITY in paths are listed in Algorithm 2.

Algorithm 2: Main steps of the whole algorithm

Input: An instance $(N, O, \succ, P, \sigma, k \in N, o_1 \in O)$
Output: To determine whether o_1 is reachable for k

1 **for** *each* $n' \in \{k, k+1, \ldots, n\}$ **do**
2   Construct the corresponding neat $(o_1, o_{n'}, k)$-CONSTRAINED instance by deleting agent i and object o_i for all $n' < i \leq n$;
3   Compute i_r and i_l for all $2 \leq i \leq n' - 1$ by Algorithm 1;
4   Compute and update R_j for all $1 \leq j \leq n'$ by the procedure before Lemma 12;
5   Construct the corresponding 2-SAT instance according to R_j by the construction method introduced before Lemma 13;
6   **if** *the* 2-SAT *instance is a yes-instance* **then**
7   **return yes** and **quit**;

8 **return no**.

The correctness of the algorithm follows from Corollary 1, Lemma 5, Lemma 6 and Lemma 13. Corollary 1 says that we can simply reduce the problem to check whether o_1 is reachable for an agent. Then Lemma 5 and Lemma 6 imply that the original instance is a yes-instance if and only if one of the corresponding neat $(o_1, o_{n'}, k)$-CONSTRAINED instances is a yes-instance. Lemma 13 says that to solve a neat $(o_1, o_{n'}, k)$-CONSTRAINED instance, we only need to solve the corresponding 2-SAT instance. So we solve all the $O(n)$ corresponding neat $(o_1, o_{n'}, k)$-CONSTRAINED instances for $n' \in \{k, k+1, \ldots, n\}$ by using 2-SAT solvers, and the algorithm returns **yes** when one of the corresponding neat $(o_1, o_{n'}, k)$-CONSTRAINED instances is a yes-instance.

Next, we analyze the running time bound of the algorithm. The major part of the algorithm is to solve $O(n)$ neat $(o_1, o_{n'}, k)$-CONSTRAINED instances. To solve a neat $(o_1, o_{n'}, k)$-CONSTRAINED instance, the above arguments show that we use $O(n^2)$ time to compute all sets R_j. To construct the corresponding 2-SAT instance, we construct at most $2n$ agent clauses in $O(n)$ time and construct at most $4\binom{n}{2}$ compatible clauses, each of which will take $O(n)$ time to check the compatibility. So the 2-SAT instance can be constructed in $O(n^3)$ time. We use the $O(n + m)$-time algorithm for 2-SAT [23] to solve the instance, where m is the number of clauses and in our instances, it holds that $m = O(n^2)$. Thus, we can solve each neat $(o_1, o_{n'}, k)$-CONSTRAINED instance in $O(n^3)$ time. In total, the algorithm uses $O(n^4)$ time to compute all the $O(n)$ neat $(o_1, o_{n'}, k)$-CONSTRAINED instances.

Theorem 1 OBJECT REACHABILITY *in paths can be solved in* $O(n^4)$ *time.*

We give an example to show the steps to compute a compatible assignment for a neat $(o_1, o_{n'}, k)$-CONSTRAINED instance.

Example 2 Consider a neat $(o_1, o_{n'}, k)$-CONSTRAINED instance with $n' = 8$ and $k = 5$ as shown in Figure 4.
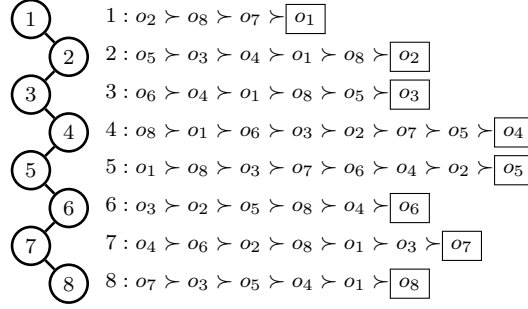
Fig. 4 The instance of Example 2

We compute i_r and i_l for all $1 \leq i \leq n'$ by Algorithm 1, the values of which are listed in Table 2.

Table 2 The values of i_l and i_r

Agent i	1	2	3	4	5	6	7	8
i_l	$\perp$	1	2	3	2	3	$\perp$	4
i_r	5	6	6	7	6	7	8	$\perp$

According to the values in Table 2, we compute and update R_j for $j \in \{1, 2, \ldots, n'\}$ by the procedure before Lemma 12. After the update, it holds that $1 \leq |R_j| \leq 2$ for all $1 \leq j \leq n'$. The values of R_j before and after updating are shown in the second and third columns of Table 3.

Table 3 Initial and updated R_j and agent clauses

Sets	Initial R_j	Updated R_j	Agent clauses
R_1	$\{o_2\}$	$\{o_2\}$	$\overline{x_2}$
R_2	$\{o_3, o_5\}$	$\{o_3, o_5\}$	$x_3 \vee x_5, \overline{x_3} \vee \overline{x_5}$
R_3	$\{o_4, o_6\}$	$\{o_4, o_6\}$	$x_4 \vee x_6, \overline{x_4} \vee \overline{x_6}$
R_4	$\{o_8\}$	$\{o_8\}$	$\overline{x_8}$
R_5	$\{o_1\}$	$\{o_1\}$	x_1
R_6	$\{o_2, o_3, o_5\}$	$\{o_3, o_5\}$	$x_3 \vee x_5, \overline{x_3} \vee \overline{x_5}$
R_7	$\{o_4, o_6\}$	$\{o_4, o_6\}$	$x_4 \vee x_6, \overline{x_4} \vee \overline{x_6}$
R_8	$\{o_7\}$	$\{o_7\}$	x_7

Next, we construct the corresponding 2-SAT instance according to updated R_j. The 2-SAT instance contains eight variables $x_1, x_2, \ldots, x_8$ corresponding to the eight objects. There are two kinds of clauses: agents clauses and compatible clauses.

Agent clauses are easy to construct: if R_j contains only one object o_q, we construct one clause of a single literal x_q if $q < j$ and one clause of a single literal
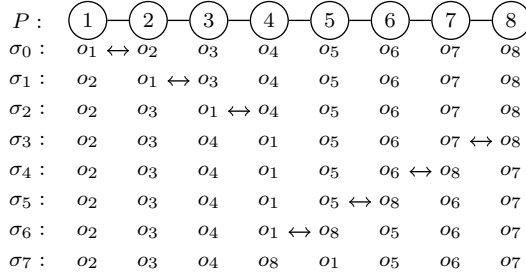
$\overline{x_q}$ if $q > j$; if R_j contains two objects o_q and o_p, we construct two clauses of two literals according to our algorithm to constrain that exactly one object is assigned to agent j. The agent clauses for each set R_j are shown in the last column of Table 3.

We construct compatible clauses for all incompatible pairs. We check all pairs of objects and find that there are only two incompatible cases: o_4 and o_5 are incompatible if o_4 and o_5 are moved to agent 3 and agent 2, respectively; o_4 and o_5 are incompatible if o_4 and o_5 are moved to agent 7 and agent 6, respectively. So we construct the corresponding compatible clauses according to our algorithm and the compatible clauses are

$$x_4 \vee x_5 \quad \text{and} \quad \overline{x_4} \vee \overline{x_5}.$$

By using the $O(n + m)$ time algorithm for 2-SAT [23], we get a feasible assignment $(1, 0, 0, 0, 1, 1, 1, 0)$ for the 2-SAT instance. The solution to the 2-SAT instance indicates that objects $\{o_1, o_5, o_6, o_7\}$ are moved to the right side and the final positions for them are $\{5, 6, 7, 8\}$; objects $\{o_2, o_3, o_4, o_8\}$ are moved to the left side and the final positions for them are $\{1, 2, 3, 4\}$. The corresponding compatible assignment is $(o_2, o_3, o_4, o_8, o_1, o_5, o_6, o_7)$, i.e., σ_7 shown in Figure 5.

To obtain the corresponding sequence of feasible swaps from σ_0 to σ_7, we use the algorithm in the proof of Lemma 11. We first move object o_2 to agent 1, then object o_3 to agent 2, then object o_4 to agent 3, and finally object o_8 to agent 4. The sequence of swaps is shown in Figure 5.

$P:$	①	②	③	④	⑤	⑥	⑦	⑧
$\sigma_0:$	$o_1 \leftrightarrow o_2$		o_3	o_4	o_5	o_6	o_7	o_8
$\sigma_1:$	o_2	$o_1 \leftrightarrow o_3$		o_4	o_5	o_6	o_7	o_8
$\sigma_2:$	o_2	o_3	$o_1 \leftrightarrow o_4$		o_5	o_6	o_7	o_8
$\sigma_3:$	o_2	o_3	o_4	o_1	o_5	o_6	$o_7 \leftrightarrow o_8$	
$\sigma_4:$	o_2	o_3	o_4	o_1	o_5	$o_6 \leftrightarrow o_8$		o_7
$\sigma_5:$	o_2	o_3	o_4	o_1	$o_5 \leftrightarrow o_8$		o_6	o_7
$\sigma_6:$	o_2	o_3	o_4	$o_1 \leftrightarrow o_8$		o_5	o_6	o_7
$\sigma_7:$	o_2	o_3	o_4	o_8	o_1	o_5	o_6	o_7

Fig. 5 The sequence of swaps to reach the compatible assignment for Example 2

Remark: Although the algorithm will compute two possible values i_l and i_r for each i, it does not mean that object o_i must be reachable for both i_l and i_r. In the above example, object o_2 is not reachable for agent $2_r = 6$ since agent 3 prefers its initially endowed object o_3 to o_2 and then o_2 cannot go to the right side. In our algorithm, the compatible clauses can avoid assigning an object to an unreachable value i_l or i_r. In the above example, if object o_2 goes to agent 6, then some object o_i with $i > 2$ will go to agent 1 or 2 and we will get an incompatible pair o_2 and o_i.

4 Weak Preference Version in Paths

We have proved that OBJECT REACHABILITY in paths is polynomial-time solvable. Next, we show that WEAK OBJECT REACHABILITY in paths is NP-hard. One of the most important properties is that Lemma 1 does not hold for WEAK OBJECT REACHABILITY and an object may 'visit' an agent more than once. Our proof involves a similar high-level idea as that of the NP-hardness proof for OBJECT REACHABILITY in a tree by Gourvès et al. [15].

The NP-hardness is proved by a reduction from the known NP-complete problem 2P1N-SAT [24]. In a 2P1N-SAT instance, we are given a set $V = \{v_1, v_2, \ldots, v_n\}$ of variables and a set $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ of clauses over V such that every variable occurs 3 times in $\mathcal{C}$ with 2 positive literals and 1 negative literal. The question is to check whether there is a variable assignment satisfying $\mathcal{C}$. For a 2P1N-SAT instance I_{SAT}, we construct an instance I_{WOR} of WEAK OBJECT REACHABILITY in paths such that I_{SAT} is a yes-instance if and only if I_{WOR} is a yes-instance.

Instance I_{WOR} contains $6n + m + 1$ agents and objects, which are constructed as follows. There is an agent named T. For each clause C_i ($i \in \{1, \ldots, m\}$), we introduce an agent also named C_i. For each variable v_i, we add six agents, named as $\overline{X}_i^{n_i}, X_i^{p_i}, X_i^{q_i}, A_i^3, A_i^2$ and A_i^1. They form a path of length 5 in this order. The path is called a *block* and is denoted by B_i. See Figure 6. The names of the six agents have certain meaning: $\overline{X}_i^{n_i}$ means that the negative literal of v_i appears in clause C_{n_i}; $X_i^{p_i}$ and $X_i^{q_i}$ mean that the positive literals of v_i appear in the two clauses C_{p_i} and C_{q_i}; A_i^3, A_i^2 and A_i^1 are three auxiliary agents.
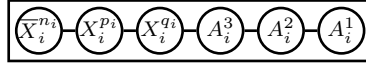

Fig. 6 Block B_i

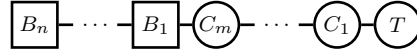The whole path is connected in the order shown in Figure 7.

Fig. 7 The structure of the whole path

In the initial assignment σ_0, object t is assigned to agent T, object c_i is assigned to agent C_i for $i \in \{1, 2, \ldots, m\}$, object a_i^j is assigned to agent A_i^j for each $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, 3\}$, and $\overline{o}_i^{n_i}$ (resp., $o_i^{p_i}$ and $o_i^{q_i}$) is assigned to agent $\overline{X}_i^{n_i}$ (resp., $X_i^{p_i}$ and $X_i^{q_i}$) for each $i \in \{1, 2, \ldots, n\}$.

Next, we define the preference profile $\succeq$. We only show the objects that each agent prefers at least as its initial one and all other objects can be put behind its initial endowment in any order. Let L_i be the set of the objects associated with the literals in clause C_i. For example, L_i is the object set $\{\overline{o}_a^{n_a}, o_b^{p_b}, o_c^{q_c}\}$ with $n_a = i$, $p_b = i$ and $q_c = i$. The set L_i will have the following property: in a reachable assignment, one object in L_i will be assigned to agent C_i, which will be corresponding to the true literal in the clause C_i in the 2P1N-SAT instance I_{SAT}. For each variable v_i, we use W_i to denote the set of objects $\{c_1, \ldots, c_m\} \cup \{\overline{o}_j^{n_j} :$

$j > i\} \cup \{o_j^{p_j} : j \neq i\} \cup \{o_j^{q_j} : j > i\} \cup \{a_j^l : j < i, l = 1, 2, 3\}$. We are ready to give the preferences for the agents.

First, we consider the preferences for T and C_i. The following preferences ensure that when C_i holds an object in L_i for each $i \in \{1, \ldots, m\}$, object t is reachable for agent C_m via a sequence of m swaps between C_i and C_{i-1} for $i = 1, \ldots, m$, where $C_0 = T$. Note that the squares in the preferences indicate the initial endowments for agents.

$T : L_1 \succ \boxed{t}$;
$C_i : L_{i+1} \succ t \succ L_i \succ c_1 \succ L_{i-1} \succ \ldots \succ c_{i-1} \succ L_1 \succ \boxed{c_i}$, for all $1 \leq i \leq m - 1$;
$C_m : t \succ L_m \succ c_1 \succ L_{m-1} \succ \ldots \succ c_{m-1} \succ L_1 \succ \boxed{c_m}$.

Next, we consider the preferences for the agents in each block B_i. The following preferences ensure that at most one of $\overline{o}_i^{n_i}$ and $\{o_i^{p_i}, o_i^{q_i}\}$ can be moved to the right of the block, which will indicate that the corresponding variable is set to 1 or 0. If $\overline{o}_i^{n_i}$ is moved to the right of the block, we will assign the corresponding variable 0; if some of $\{o_i^{p_i}, o_i^{q_i}\}$ is moved to the right of the block, we will assign the corresponding variable 1. We use the preference of A_i^3 to control this. Furthermore, we use A_i^1, A_i^2 and A_i^3 to (temporarily) hold $\overline{o}_i^{n_i}$ (or $o_i^{p_i}$ and $o_i^{q_i}$) if they do not need to be moved to the right of the block.

$\overline{X}_i^{n_i} : W_i \cup \{a_i^1, a_i^2, a_i^3, o_i^{p_i}, o_i^{q_i}\} \succ \boxed{\overline{o}_i^{n_i}}$,
$X_i^{q_i} : W_i \cup \{a_i^1, a_i^2, a_i^3, \overline{o}_i^{n_i}, o_i^{p_i}\} \succ \boxed{o_i^{q_i}}$,
$X_i^{p_i} : W_i \cup \{a_i^1, a_i^2, a_i^3, o_i^{q_i}, \overline{o}_i^{n_i}\} \succ \boxed{o_i^{p_i}}$,
$A_i^1 : W_i \cup \{\boxed{a_i^1}, a_i^2, a_i^3, o_i^{p_i}, o_i^{q_i}, \overline{o}_i^{n_i}\}$,
$A_i^2 : W_i \cup \{a_i^1, \boxed{a_i^2}, a_i^3, o_i^{p_i}, o_i^{q_i}, \overline{o}_i^{n_i}\}$,
$A_i^3 : W_i \cup \{a_i^1, a_i^2, o_i^{p_i}, o_i^{q_i}\} \succ \overline{o}_i^{n_i} \succ \boxed{a_i^3}$, for all $1 \leq i \leq n$.

Instance I_{WOR} is to determine whether object t is reachable for agent C_m.
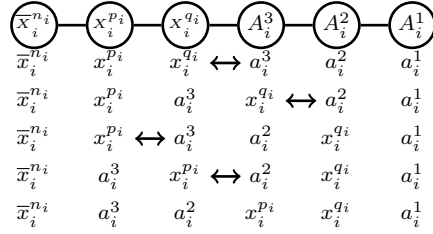
Lemma 14 *A 2P1N-SAT instance I_{SAT} is a yes-instance if and only if the corresponding instance I_{WOR} of* WEAK OBJECT REACHABILITY *in paths constructed above is a yes-instance.*

Proof First, we prove that a reachable assignment of I_{WOR} implies a satisfying assignment for I_{SAT}. Assume that object t is reachable for agent C_m. Then there are m swaps including t that happen between C_i and C_{i+1} for $i \in \{0, 1, \ldots, m-1\}$, where $C_0 = T$. Note that the swap between C_i and C_{i+1} (where C_i holds object t) can happen if and only if C_{i+1} holds an object $a \in L_{i+1}$. We will prove the claim that for each block B_i, it is impossible that both $\overline{o}_i^{n_i}$ and one of $o_i^{p_i}$ and $o_i^{q_i}$ are moved to the right of this block. By this claim, we can see that a satisfying assignment for I_{SAT} can be obtained by letting the literal corresponding to the object $a \in L_i$ to 1 for all agents C_i.
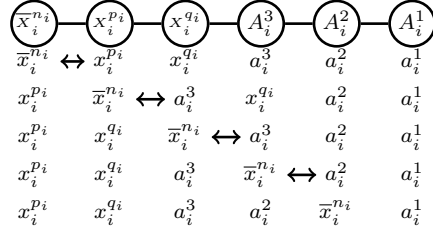
Next, we prove the claim that for each block B_i, it is impossible that both $\overline{o}_i^{n_i}$ and one of $o_i^{p_i}$ and $o_i^{q_i}$ are moved to the right of this block. Assume to the contrary that there is a block B_i such that both $\overline{o}_n^{n_i}$ and one of $o_i^{p_i}$ and $o_i^{q_i}$ are

moved to the right of this block. When $\bar{o}_k^{n_k}$ is moved to the right of this block, it must reach A_k^3 at some time. We know that $\bar{o}_i^{n_i}$ is the last but one in agent A_i^3's given preferences. This implies that before $\bar{o}_i^{n_i}$ is swapped with a_i^3 between $X_i^{q_i}$ and A_i^3, A_i^3 cannot participate in any other swap. Therefore, the only way to move $\bar{o}_k^{n_k}$ to A_i^3 is that $\bar{o}_k^{n_k}$ is swapped with $o_i^{p_i}$, $o_i^{q_i}$ and a_i^3 in this order. For each agent of $\{\overline{X}_i^{n_i}, X_i^{p_i}, X_i^{q_i}\}$, once its endowment object is swapped with an object, the endowment object will not come back to the agent again. Thus, both $o_i^{p_i}$ and $o_i^{q_i}$ cannot be moved to the right of this block, which is a contradiction.

On the other hand, if there is a satisfying assignment τ for I_{SAT}, we can construct a reachable assignment for I_{WOR}. We consider the value of each variable v_i in τ. If $v_i = 1$, we move $o_i^{p_i}$ and $o_i^{q_i}$ to agents A_i^3 and A_i^2, respectively. The sequence of swaps is shown in Figure 8. If $v_i = 0$, we move $\bar{o}_i^{n_i}$ to agent A_i^2. The sequence of swaps is shown in Figure 9. Note that for any case, all the swaps happen within a block. The objects $o_i^{p_i}$, $o_i^{q_i}$ and $\bar{o}_i^{n_i}$ in the above procedure are called *true objects*.

$\overline{x}_i^{n_i}$	$x_i^{p_i}$	$x_i^{q_i}$	A_i^3	A_i^2	A_i^1
$\bar{x}_i^{n_i}$	$x_i^{p_i}$	$x_i^{q_i} \leftrightarrow a_i^3$		a_i^2	a_i^1
$\bar{x}_i^{n_i}$	$x_i^{p_i}$	a_i^3	$x_i^{q_i} \leftrightarrow a_i^2$		a_i^1
$\bar{x}_i^{n_i}$	$x_i^{p_i} \leftrightarrow a_i^3$		a_i^2	$x_i^{q_i}$	a_i^1
$\bar{x}_i^{n_i}$	a_i^3	$x_i^{p_i} \leftrightarrow a_i^2$		$x_i^{q_i}$	a_i^1
$\bar{x}_i^{n_i}$	a_i^3	a_i^2	$x_i^{p_i}$	$x_i^{q_i}$	a_i^1

Fig. 8 The sequences of swaps to move $o_i^{p_i}$ and $o_i^{q_i}$ to agents A_i^3 and A_i^2, respectively

$\overline{x}_i^{n_i}$	$x_i^{p_i}$	$x_i^{q_i}$	A_i^3	A_i^2	A_i^1
$\bar{x}_i^{n_i} \leftrightarrow x_i^{p_i}$		$x_i^{q_i}$	a_i^3	a_i^2	a_i^1
$x_i^{p_i}$	$\bar{x}_i^{n_i} \leftrightarrow a_i^3$		$x_i^{q_i}$	a_i^2	a_i^1
$x_i^{p_i}$	$x_i^{q_i}$	$\bar{x}_i^{n_i} \leftrightarrow a_i^3$		a_i^2	a_i^1
$x_i^{p_i}$	$x_i^{q_i}$	a_i^3	$\bar{x}_i^{n_i} \leftrightarrow a_i^2$		a_i^1
$x_i^{p_i}$	$x_i^{q_i}$	a_i^3	a_i^2	$\bar{x}_i^{n_i}$	a_i^1

Fig. 9 The sequences of swaps to move $\bar{o}_i^{n_i}$ to agent A_i^2

After this procedure, we move only one of true objects in L_j to each agent C_j one by one in an order where C_j with smaller j first gets its object in L_j (for every agent in blocks, almost all objects in her given preferences are indifferent with each other). During this procedure, once another true object is moved out of its position A_i^3 or A_i^2 (this may happen when the true object is on the moving path of another true object to $C_{j'}$ with $j < j'$), we will simply move it back to A_i^3 or A_i^2 by one swap. We can do this swap because almost all objects except $\bar{o}_i^{n_i}$ and a_i^3 are equivalent for A_i^3, A_i^2 and A_i^1. So in each iteration, only one true object is moved out of its current position A_i^3 or A_i^2 and it is moved to its final position C_j

directly. Note that C_{i+1} holds an object $a \in L_{i+1}$ for i from 0 to $m-1$ now and $C_0 = T$ holds the object t. We make a swap between C_i and C_{i+1} for i from 0 to $m-1$, and then the object t will be moved to agent C_m. $\qquad\square$

Lemma 14 implies that

Theorem 2 WEAK OBJECT REACHABILITY *is NP-hard even when the network is a path.*

The above reduction from 2P1N-SAT can also be used to prove the NP-hardness of WEAK PARETO EFFICIENCY and the problem of maximizing the utilitarian social welfare. We will use the following lemma.

Lemma 15 *For the above instance* I_{WOR} *constructed from the* 2P1N-SAT *instance* I_{SAT}, *if* I_{WOR} *is a yes-instance, then there is a reachable assignment* σ *where every agent holds one of its most favorite objects.*

Proof Let σ be a reachable assignment in the above instance I_{WOR} such that $\sigma(C_m) = t$. Clearly, each agent C_i for $i \in \{0, 1, \cdots, m\}$ must holds one of its favorite objects. Form the construction of the preference profile, we know that A_i^1 and A_i^2 must hold their favorite objects respectively for every i. If $\sigma(A_i^3)$ is not one of the favorite objects of A_i^3, then one of $\sigma(A_i^2)$ and $\sigma(A_i^1)$ is one of the favorite objects of A_i^3 and it can be swapped to A_i^3 directly. Let X be one of $\overline{X}_i^{n_i}$, $X_i^{q_i}$ and $X_i^{p_i}$ such that $\sigma(X)$ is not one of the favorite objects of X and let Y be a neighbor of X in $\{\overline{X}_i^{n_i}, X_i^{q_i}, X_i^{p_i}\}$. Then, we have that $\sigma(X) \succeq_Y \sigma(Y)$ and $\sigma(Y) \succeq_X \sigma(X)$. We make a swap between X and Y now. So, we can always get a reachable assignment σ' such that each agent holds one of its favorite objects in σ' resulting from σ. $\qquad\square$

By this lemma, we know that if I_{WOR} is a yes-instance, then in any Pareto optimal assignment for I_{WOR} every agent holds one of its most favorite objects. On the other hand, if there is a Pareto optimal assignment where every agent holds one of its most favorite objects, then C_m must hold t. Thus the above reduction also reduces 2P1N-SAT to WEAK PARETO EFFICIENCY in paths. The instance I_{WOR} has a Pareto optimal assignment such that every agent holds one of its most favorite objects if and only if the 2P1N-SAT instance I_{SAT} is a yes-instance.

Theorem 3 WEAK PARETO EFFICIENCY *is NP-hard even when the network is a path.*

Next, we discuss the problem of maximizing the utilitarian social welfare. In this problem, we will associate a value function $f_i(\cdot)$ on the objects for each agent i. For any two objects o_a and o_b, the value function holds that $f_i(o_a) > f_i(o_b)$ if and only if $o_a \succ_i o_b$ (resp., $f_i(o_a) = f_i(o_b)$ if and only if $o_a =_i o_b$). Thus, each agent has the maximum value on the most favorite objects. By Lemma 15, in the above instance I_{WOR}, all agents get their maximum value, i.e., the utilitarian social welfare is maximum, if and only if the 2P1N-SAT instance I_{SAT} is a yes-instance.

Theorem 4 *Finding a reachable assignment that maximizes the utilitarian social welfare is NP-hard even when the network is a path.*

5 Weak Preference Version in Stars

In this section, we assume that the input graph is a star and that the preferences allow ties. We will consider the reachability of an object for an agent and the problem of maximizing the utilitarian social welfare.

5.1 WEAK OBJECT REACHABILITY in Stars

We have shown that WEAK OBJECT REACHABILITY in paths is NP-hard. Next, we show that WEAK OBJECT REACHABILITY in stars can be solved in polynomial time. We first show several properties of the problem, which will allow us to transform the original problem to the problem of whether the target object is reachable for the center agent under some constraints. We further show that if such kinds of feasible assignments exist, then there is a sequence of swaps to reach a feasible assignment such that each leaf agent participates in at most one swap, which is called a *simple* sequence. Inspired by the idea of the algorithm for the strict version by Gourvès et al. [15], we construct an auxiliary graph and show that a satisfying simple sequence exists if and only if the auxiliary graph has a directed path from one given vertex to another given vertex. Thus, we can solve the problem by finding a directed path in a graph.

Without loss of generality, we assume that the network is a star of n vertices and the center agent of the star is n; the initially endowed object assigned to each agent i is o_i. It is trivial for the case of asking whether the center object o_n is reachable for a leaf agent k, where we only need to check whether $o_n \succeq_k o_k$ and $o_k \succeq_n o_n$. Next, we consider whether an object of a leaf agent is reachable for another agent. Without loss of generality, we assume that the problem is to ask whether object o_1 is reachable for agent k, where k is allowed to be n.

Since the network is a star, each feasible swap must include the center agent n. This property will be frequently used in our analysis. Next, we show more properties of the problem.

Lemma 16 *For any agent $i_0 \notin \{1, k, n\}$, if $o_{i_0} \succ_n o_1$ or $o_n \succ_n o_{i_0}$, then deleting agent i_0 and its initially endowed object o_{i_0} will not change the reachability of the instance.*

Proof For any agent i_0 with $o_n \succ_n o_{i_0}$, it cannot participate in a feasible swap, because the center agent n will not hold an object worse than its initially endowed object o_n. So we can simply delete it. Next, we assume that $o_{i_0} \succ_n o_1$. Agent i_0 can only be a leaf agent because $i_0 \neq n$. Before agent k holds object o_1, the center agent n cannot hold object i_0, otherwise agent n will not get o_1 anymore and then agent k cannot get o_1. Since $i_0 \neq k$, we can assume that agent i_0 does not participate in any swap and delete it.   □

Lemma 17 *If $o_k \succ_k o_1$ or $o_n \succ_n o_1$, then o_1 is not reachable for agent k.*

Proof Since any agent will not receive an object worse than its initially endowed object, we know that agent k will not receive o_1 if $o_k \succ_k o_1$. For the case of $o_n \succ_n o_1$, the center agent n will not receive o_1 and then k will not be able to get object o_1.   □

Based on Lemma 16 and Lemma 17, we can further assume that $o_1 \succeq_n o_i \succeq_n o_n$ for any $i \neq k$ and $o_1 \succeq_k o_k$.

Lemma 18 *If $o_1 \succ_n o_k$ and $k \neq n$, then o_1 is not reachable for agent k.*

Proof Assume to the contrary that that o_1 is reachable for agent k. In the sequence of assignments from the initial one to a reachable one via swaps, we let $o_{x_1}, o_{x_2}, \ldots, o_{x_l}$ be the sequence of different objects assigned to agent k, where $o_{x_1} = o_k$ and $o_{x_l} = o_1$. For each $1 \leq i \leq l-1$, there is a swap between agent n and agent k where agent n holds object $o_{x_{i+1}}$ and agent k holds object o_{x_i} (before the swap). This means that $o_{x_i} \succeq_n o_{x_{i+1}}$. So we get

$$o_{x_1} \succeq_n o_{x_2} \succeq_n \ldots \succeq_n o_{x_l}.$$

This is in contradiction with the fact that $o_1 \succ_n o_k$. So we know that o_1 is not reachable for agent k for this case. □

To solve WEAK OBJECT REACHABILITY in stars, we consider whether $k = n$ or not. If $k = n$, it is to check whether object o_1 is reachable for the center agent n. If $k \neq n$, we will transfer it to checking the reachability of o_1 for the center agent n under some constraints. Our idea is based on the following observations. Before object o_1 reaches agent $k \neq n$, object o_1 must be held by the center agent n. We will first consider the case that $k \neq n$.

Definition 6 An assignment σ is called *crucial* if it holds that $\sigma(n) = o_1$, $o_1 \succeq_k \sigma(k)$ and $\sigma(k) \succeq_n o_1$.

Observation 3 *Assume that $k \neq n$. Object o_1 is reachable for agent k if and only if there is a reachable crucial assignment.*

Proof Before the last swap to reach a reachable assignment σ such that $\sigma(k) = o_1$, the assignment is a reachable crucial assignment. Furthermore, for a crucial assignment, we can make a trade between n and k so that agent k can get object o_1. □

Next, we will analyze the properties of reachable crucial assignments and design an algorithm to find them if they exist.

Lemma 19 *Assume that $k \neq n$. For any sequence of swaps $(\sigma_0, \sigma_1, \ldots, \sigma_t)$ such that σ_t is a reachable crucial assignment (if it exists), it holds that $\sigma_i(k) \succeq_n o_1$ for any $i \in \{0, 1, \ldots, t\}$.*

Proof Assume to the contrary that there exists σ_i ($0 \leq i \leq t$) such that $o_1 \succ_n \sigma_i(k)$. We take σ_i as the initial endowment and then o_1 is not reachable for agent k by Lemma 18 and Observation 3. This is in contradiction with the existence of σ_t. So the lemma holds. □

Definition 7 A sequence of swaps is called *simple* if each leaf agent in the star participates in at most one swap.

Lemma 20 *If there exists a reachable crucial assignment, then there exists a simple sequence of swaps $(\sigma_0, \sigma_1, \ldots, \sigma_t)$ such that σ_t is crucial and $\sigma_i(k) \succeq_n o_1$ holds for any $i \in \{0, 1, \ldots, t\}$.*

Proof Assume reachable crucial assignments exist. We let $\psi = (\sigma_0, \sigma_1, \ldots, \sigma_t)$ be one of the shortest sequences of swaps such that σ_t is a reachable crucial assignment. Assume to the contrary that at least one leaf agent participates in two swaps in ψ. Then $\sigma_i(n) = \sigma_j(n)$ will hold for two different indices $0 \leq i < j \leq t$. We assume that j is the maximum index satisfying the above condition. By the selection of the index j, we know that any leaf agent that participates in a swap in $(\sigma_j, \sigma_{j+1}, \ldots, \sigma_t)$ will not participate in any swap before σ_j. Thus, after deleting the sequence of swaps in $(\sigma_i, \sigma_{i+1}, \ldots, \sigma_j)$ from ψ, we will still get a feasible sequence of swaps. However, the length of the new sequence of swaps is shorter than that of ψ, a contradiction to the choice of ψ. So we know that ψ must be simple. By Lemma 19, we know that $\sigma_i(k) \succeq_n o_1$ holds for any $i \in \{0, 1, \ldots, t\}$. Thus, the lemma holds. □

Lemma 21 *Assume that $k \neq n$, $o_1 \succeq_k o_k$ and $o_k \succeq_n o_1$. There is an algorithm that runs in $O(n^2)$ time to check the existence of a simple sequence of swaps $(\sigma_0, \sigma_1, \ldots, \sigma_t)$ such that σ_t is crucial and $\sigma_i(k) \succeq_n o_1$ holds for any $i \in \{0, 1, \ldots, t\}$, and to find one if it exists.*

Proof We will transform the problem to a problem of finding a simple directed path in an auxiliary graph following the idea of the algorithm for strict preferences given in [15]. For a simple sequence of swaps, the center agent and a leaf agent make a trade only when the leaf agent holds its initially endowed object. We construct an auxiliary directed graph $G_D = (N, E_D)$ on the set N of agents. There is an arc $\overrightarrow{ij}$ from agent $i \in N$ to agent $j \in N \setminus \{n, k, i\}$ if and only if the center agent n and the leaf agent j can rationally trade when the center agent n holds object o_i and the leaf agent j holds object o_j, i.e., $o_i \succeq_j o_j$ and $o_j \succeq_n o_i$. There is an arc $\overrightarrow{ik}$ from agent $i \in N \setminus \{1, k\}$ to agent k if and only if $o_1 \succeq_k o_i \succeq_k o_k$ and $o_k \succeq_n o_i \succeq_n o_1$. The auxiliary graph can be constructed in $O(n^2)$ by using some data structure. Next, we show that a simple sequence of swaps $(\sigma_0, \sigma_1, \ldots, \sigma_t)$ such that σ_t is crucial and $\sigma_i(k) \succeq_n o_1$ holds for any $i \in \{0, 1, \ldots, t\}$ exists if and only if there is a simple directed path from n to 1 in graph G_D.

It is easy to see that such a simple sequence of swaps implies a simple directed path from n to 1 in G_D. We consider the other direction and assume that a simple directed path P from n to 1 in G_D exists. We can further assume that no vertex appears more than once in P since a simple path always exists if a path between two vertices exists. We can get a sequence $(\sigma_0, \sigma_1, \ldots, \sigma_t)$ of swaps according to P: swap (σ_i, σ_{i+1}) is a trade between n and the head of the ith arc in P. First, the sequence of swaps is simple because no vertex appears more than once in P. Second, it holds that $\sigma_t(n) = o_1$ because the last swap happens between agent n and agent 1 and this is the first swap including agent 1. Third, it holds that $o_1 \succeq_k \sigma_i(k)$ for every i because agent k has an initially endowed object o_k such that $o_1 \succeq_k o_k$ and G_D has no arc from j to k for any j with $o_j \succ_k o_1$. Last, we have $\sigma_i(k) \succeq_n o_1$ for any $i \in \{0, 1, \ldots, t\}$ because agent k has an initially endowed object o_k such that $o_k \succeq_n o_1$ and G_D has no arc from i to k for any i with $o_1 \succ_n o_i$. Therefore, we know that σ_t is crucial and $\sigma_i(k) \succeq_n o_1$ holds for any $i \in \{0, 1, \ldots, t\}$.

A simple directed path from n to 1 in graph G_D can be computed in linear time by the depth-first search. The lemma holds. □

For the case of $k = n$, we still need the following Lemma 22 and Lemma 23. The proof of Lemma 22 is referred to the first part of the proof of Lemma 20. We

omit the redundant arguments. The same arguments in Lemma 21 can be used to prove Lemma 23. However, the construction of the auxiliary graph is a little bit different.

Lemma 22 *If object o_1 is reachable for the center agent n, then there exists a simple sequence of swaps $(\sigma_0, \sigma_1, \ldots, \sigma_t)$ such that $\sigma_t(n) = o_1$.*

Lemma 23 *Verifying the existence of a simple sequence of swaps $(\sigma_0, \sigma_1, \ldots, \sigma_t)$ such that $\sigma_t(n) = o_1$, and constructing it if it exists, can be done in $O(n^2)$.*

Proof We construct an auxiliary directed graph $G_D = (N, E_D)$ on the set N of agents. There is an arc $\overrightarrow{ij}$ from agent $i \in N$ to agent $j \in N \setminus \{n, i\}$ if and only if the center agent n and the leaf agent j can rationally trade when the center agent n holds object o_i and the leaf agent j holds object o_j, i.e., $o_i \succeq_j o_j$ and $o_j \succeq_n o_i$. Similar to the argument in the proof of Lemma 21, we can prove that a simple sequence of swaps $(\sigma_0, \sigma_1, \ldots, \sigma_t)$ such that $\sigma_t(n) = o_1$ exists if and only if there is a simple directed path from n to 1 in graph G_D.   □

The main steps of our algorithm are listed in Algorithm 3.

Algorithm 3: The Algorithm for Weak Object Reachability in Stars

Input: An instance in a star, where we assume that n is the center agent in the star, $o_1 \neq o_n$ and $k \neq 1$.
Output: To determine whether o_1 is reachable for agent k

1 **for** *each $i \in \{2, 3, \ldots, n-1\} \setminus \{k\}$* **do**
2   **if** *$o_i \succ_n o_1$ or $o_n \succ_n o_i$* **then**
3   delete agent i and object o_i;

4 **if** *$o_k \succ_k o_1$ or $o_n \succ_n o_1$* **then**
5   **return no** and **quit**;

6 **if** *$k \neq n$ and $o_1 \succ_n o_k$* **then**
7   **return no** and **quit**;

8 **if** *$k \neq n$* **then**
9   Check whether there is a simple directed path from n to 1 in the graph G_D constructed in the proof of Lemma 21;
10   **return yes** if it exists and **no** otherwise;
11 **else**
12   Check whether there is a simple directed path from n to 1 in the graph G_D constructed in the proof of Lemma 23;
13   **return yes** if it exists and **no** otherwise.

Theorem 5 WEAK OBJECT REACHABILITY *in stars can be solved in $O(n^2)$ time.*

Proof We prove this theorem by proving the correctness of Algorithm 3. By Lemma 16, we can do Steps 1-3 to simplify the instance. By Lemma 17, we can verify the correctness of Steps 4-5, and by Lemma 18, we can verify the correctness of Steps 6-7. After Step 7, it always holds that $o_1 \succeq_k o_k$, $o_1 \succeq_n o_n$ and $o_k \succeq_n o_1$ for $k \neq n$. For the case of $k \neq n$, the correctness of Steps 8-10 is derived from Observation 3, Lemma 20 and Lemma 21. For the case of $k = n$, it is to check

whether o_1 is reachable for the center agent n and the correctness of Steps 12-13 is derived from Lemma 22 and Lemma 23.

Before Step 8, the algorithm uses linear time. Steps 8-10 use $O(n^2)$ time and Steps 12-13 use $O(n^2)$ time.   □

5.2 Maximizing The Utilitarian Social Welfare in Stars

For WEAK PARETO EFFICIENCY in stars, we did not find a polynomial-time algorithm or a proof for the NP-hardness. However, we can prove the hardness of the problem to find a solution maximizing the utilitarian social welfare. In this problem, the preference profile is replaced with the value function.

To prove the NP-hardness, we give a reduction from the known NP-complete problem – the DIRECTED HAMILTONIAN PATH problem [25], which is to find a directed path visiting each vertex exactly once in a directed graph $D = (V, A)$. We construct an instance I of the problem such that I admits a reachable assignment with the utilitarian social welfare at least $3|V| + |A| - 1$ if and only if the graph D has a directed Hamiltonian path starting from a given vertex $s \in V$. We simply assume that s has no incoming arcs.

Instance I is constructed as follows. It contains $|V| + |A| + 1$ agents and $|V| + |A| + 1$ objects. For each arc $e \in A$, there is an associated agent a_e, called an *arc agent*. For each vertex $v \in V$, there is an associated agent a_v, called a *vertex agent*. So there are $|A|$ arc agents and $|V|$ vertex agents. There is also a *center agent* a_c, which is adjacent to all arc agents and vertex agents to form a star.

In the initial endowment σ_0, the object assigned to the center agent a_c is o_c, called the *center object*; for each vertex agent a_v, the object assigned to it is o_v, called a *vertex object*; and for each arc agent a_e, the object assigned to it is o_e, called an *arc object*.

Next, we construct the value function for each agent. For the center agent a_c, all objects have the same value of 0. For the starting vertex agent a_s, the initially endowed object o_s is valued 1, the center object o_c is valued 2, and all other objects are valued 0. For any vertex agent a_v with $v \in V \setminus \{s\}$, the initially endowed object o_v is valued 1, arc object o_e with e being an arc from a vertex to the vertex v is valued 2, and all other objects are valued 0. For any arc agent a_e with $e \in A$, where $e = \overrightarrow{uv}$ is an arc starting from u to v, the initially endowed object o_e is valued 1, the vertex object o_u is valued 2, and all other objects are valued 0.

See Figure 10 and Table 4 for an example to construct the instance.
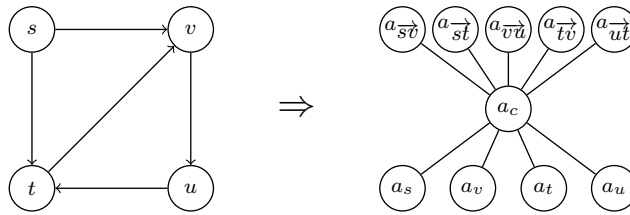
Fig. 10 An example of the construction in the proof of Theorem 6

From the value functions of instance I, we can see the following properties.

Table 4 The value functions of the example in Figure 10

Objects \ Agents	a_c	a_s	a_v	a_t	a_u	$a_{\overrightarrow{sv}}$	$a_{\overrightarrow{st}}$	$a_{\overrightarrow{vu}}$	$a_{\overrightarrow{tv}}$	$a_{\overrightarrow{ut}}$
o_c	0	2	0	0	0	0	0	0	0	0
o_s	0	1	0	0	0	2	2	0	0	0
o_v	0	0	1	0	0	0	0	2	0	0
o_t	0	0	0	1	0	0	0	0	2	0
o_u	0	0	0	0	1	0	0	0	0	2
$o_{\overrightarrow{sv}}$	0	0	2	0	0	1	0	0	0	0
$o_{\overrightarrow{st}}$	0	0	0	2	0	0	1	0	0	0
$o_{\overrightarrow{vu}}$	0	0	0	0	2	0	0	1	0	0
$o_{\overrightarrow{tv}}$	0	0	2	0	0	0	0	0	1	0
$o_{\overrightarrow{ut}}$	0	0	0	2	0	0	0	0	0	1

(P1) the utilitarian social welfare of the initial endowment is exactly $|V| + |A|$;

(P2) the vertex agent a_s can only participate in a trade with a_c where o_s and o_c are swapped;

(P3) the first swap must happen between agents a_s and a_c, because all other agents value object o_c as 0;

(P4) for a vertex object o_v, among all vertex agents, only vertex agent a_v values it as 1 and all other vertex agents value it as 0;

(P5) for a vertex object o_v, among all arc agents, only arc agents whose arc starts from vertex v value it as 2, and all other arc agents value it as 0;

(P6) for an arc object o_e, among all vertex agents, only vertex agents which is the endpoint of edge e value it as 2, and all other vertex agents value it as 0;

(P7) for an arc object o_e, among all arc agents, only arc agent a_e values it as 1 and all other arc agents value it as 0.

From (P4), (P5), (P6) and (P7), we can see that

(P8) when a vertex object o_v is held by the center agent a_c, the next swap can only be a trade between the center agent a_c and an arc agent a_e with e being an arc starting from vertex v;

(P9) when an arc object o_e is held by the center agent a_c, the next swap can only be a trade between the center agent a_c and the vertex agent a_v with v being the ending point of arc e.

Based on the above properties, we prove the following results.

Lemma 24 *Instance I has a reachable assignment such that the total value of all agents is at least $3|V| + |A| - 1$ if and only if there is a directed Hamiltonian path starting from vertex s in the directed graph D.*

Proof Assume that there is a directed Hamiltonian path $v_1 v_2 v_3 \ldots, v_{|V|}$ starting from s in the directed graph D, where $v_1 = s$ and the arc from v_i to v_{i+1} is denoted by e_i. We show that there is a satisfying sequence of swaps in I. Note that each swap in a star happens between the center agent and a leaf agent. So a sequence of swaps can be specified by a sequence of leaf agents. So we will use a sequence of leaf agents to denote a sequence of swaps. In fact, we will show that $(a_{v_1}, a_{e_1}, a_{v_2}, a_{e_2}, \ldots, a_{e_{|V|-1}}, a_{v_{|V|}})$ is a satisfying sequence of swaps in I.

It is easy to verify that the first two swaps can be executed. We can prove by induction that, for each $i > 1$, before the swap between a_c and a_{v_i} (resp., a_{e_i}), agent a_c holds object $o_{e_{i-1}}$ (resp., $o_{v_{i-1}}$) and agent a_{v_i} holds object o_{v_i} (resp., agent a_{e_i} holds object o_{e_i}). According to (P4), (P5), (P6), (P7) and the fact that the center agent a_c has no difference among all the objects, we know that the sequence of swaps $(a_{v_1}, a_{e_1}, a_{v_2}, a_{e_2}, \ldots, a_{e_{|V|-1}}, a_{v_{|V|}})$ can be executed. Furthermore, for each swap in the sequence, the leaf agent participating in the swap will increase its value by 1. There are $2|V| - 1$ swaps and the initial endowment has a value of $|V| + |A|$. So in the final assignment, the total value is $|V| + |A| + (2|V| - 1) = 3|V| + |A| - 1$.

Next, we assume that there is a reachable assignment σ with the total value of agents at least $3|V| + |A| - 1$ in I. Based on this assumption, we construct a Hamiltonian path starting from s in D. We let a sequence of leaf agents $\pi = (a_{x_1}, a_{x_2}, \ldots, a_{x_l})$ to denote the sequence of swaps from the initial endowment to the final assignment σ, where we assume that $x_i \neq x_{i+1}$ for all i since two consecutive swaps happened between the same pair of agents mean doing nothing and they can be deleted. We will show that π is a sequence of alternating elements between vertices and arcs in D.

By (P3), we know that x_1 is s. We consider two cases: all vertex agents in π are different or not. We first consider the case that all vertex agents in π are different. For this case, it is easy to see that all arc agents in π are also different by the value functions of the agents. Now all agents in π are different. By (P8) and (P9), we know that the sequence of vertices and arcs in D corresponding to π is a directed path starting from s.

We also have the following claims by the fact that each vertex and arc agent appears at most once in π. For each swap in π, if it is a trade between a_c and a vertex agent a_v, then the object hold by agent a_v changes from the initially endowed object o_v to an arc object o_e, where agent a_v must value object o_e as 2. Thus, the value of agent a_v increases by 1. For each swap in π, if it is a trade between a_c and an arc agent a_e, then the object hold by agent a_e changes from the initially endowed object o_e to a vertex object o_v, where agent a_e must value object o_v as 2. Thus, the value of agent a_e increases by 1. In any case, each swap will increase the total value by exactly 1. By (P1), the utilitarian social welfare of the initial endowment is $|V| + |A|$. So there are exactly $3|V| + |A| - 1 - (|V| + |A|) = 2|V| - 1$ swaps in π. The length of π is $2|V| - 1$ and then π contains $|V|$ vertices and $|V| - 1$ arcs, where all vertices are different. This can only be a Hamiltonian path.

Next, we consider the case that some vertex agents appear at least twice in π. Let π' be the shortest subsequence of π starting from the beginning agent a_s such that a vertex agent appears twice in π'. Let a_v be the last agent in π'. By the choice of π', we know that each agent in π' except the last one appears at most once in π'. By (P8) and (P9), we know that π' is corresponding to a directed path starting from s in D, where only the last vertex v in the path appears twice. By (P2), we know that v cannot be s. Thus, there is an arc agent a_{e_1} before the first appearance of a_v in π'. Let a_{e_2} be the agent before the second appearance of a_v in π', i.e., the last but one agent in π'. We can see that before the second swap between a_c and a_v, center agent a_c holds object o_{e_2} and vertex agent a_v holds object o_{e_1}, where both e_1 and e_2 are arcs with the ending point being v. After this swap, center agent a_c will hold object o_{e_1} and vertex agent a_v will hold object o_{e_2}.

From then on, only vertex agent a_v can make a trade with center agent a_c. We have assumed that there are no two consecutive swaps that happened between the same pair of agents. Thus the second a_v must be the last agent in π. Furthermore, in the last swap between a_c and a_v, the value of a_v will not increase. By the above analysis for the first case, we know that $\pi'' = (a_{x_1}, a_{x_2}, \ldots, a_{x_{l-2}})$ is corresponding to a simple directed path, where $a_{x_{l-2}}$ is a vertex agent. In π, the total value of the agents can increase by at most 1 during the last but one swap between a_c and $a_{x_{l-1}}$, and can increase by 0 during the last swap between a_c and a_{x_l}. There is an odd number of agents in π'' since vertex agents and arc agents appear alternately in it and both the first and the last agents are vertex agents. Each swap in π'' can increase the total value by 1. Since the total value of the last assignment is at least $3|V| + |A| - 1$ and the initial value is $|V| + |A|$, we know that the length of π'' is at least $2|V| - 1$, which implies that π'' is corresponding to a Hamiltonian path.   □

Lemma 24 implies that

Theorem 6 *Finding a reachable assignment that maximizes the utilitarian social welfare is NP-hard even when the network is a star.*

6 Conclusion and Discussion

In this paper, we investigate some problems about how to obtain certain assignments from an initial endowment via a sequence of rational trades between two agents under two different preferences: ties are allowed (weak preferences) or not (strict preferences). We assume that the agents are embedded in a social network and only neighbors are allowed to exchange their objects. This is a reasonable assumption since it well simulates the situation in real life. To understand the computational complexity of the problem, we consider different network structures. See Table 1 in the introduction for a survey of the results.

For OBJECT REACHABILITY, now we know it can be solved in polynomial time in paths. Recently, it was shown in [21] that it can also be solved in polynomial time in cycles, which implies the problem is polynomial-time solvable in graphs with maximum degree 2. Will the problem become NP-hard in graphs with maximum degree 3? How about the computational complexity of the problem in bounded-degree trees? Finding more social structures under which the problem is polynomial-time solvable will be helpful for us to understand the nature of the problem.

In this paper, we regard the preferences of agents as independent. In real life, the preferences of agents may be similar, i.e., some objects are preferred by most agents. Will the problem become easier if the preferences have some properties like single peakedness or single crossingness?

In our model, we only consider trades between two agents. Trades among three or more agents also exist, such as in kidney exchange, a trade among three agents is common and a trade among nine agents (a 9-cycle kidney exchange) was performed successfully in 2015 at two San Francisco hospitals [26]. General trades among an arbitrary number of agents have been considered in [20]. It would be interesting to further reveal more properties of the model under general trades. In kidney exchange, each agent is allowed to participate in at most one trade. In our model,

an agent can participate in an arbitrary number of trades. Another direction for further study is to consider the model with an upper bound on the number of trades for each agent. It has been proved in [27] that OBJECT REACHABILITY is NP-hard in graphs with maximum degree 4 even if the maximum number of swaps that each agent can participate is 2.

We believe it is worthy to further study more object allocation models under exchange operations, especially for problems coming from real-life applications.

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