<u>Lecture 4</u> Numerical methods of 1st order differential equations

Where a differential equation and known boundary conditions are given, an approximate solution may be obtained by applying a numerical method. There are a number of such numerical methods available and the simplest of these is called Euler's method.

Euler's method

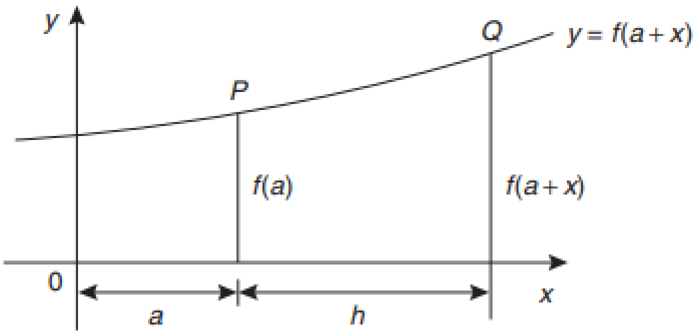

Figure 1

At Point Q at Figure 2:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots$$

Which a statement called Taylor's series and it will be closer looked at in a lecture later on in the module.

For now, if we say h is the interval between the two ordinates $y_0$and $y_1$as shown in Figure 3, and if $f(a) = y_0$ and $y_1 = f(a+h)$, then Euler's method states:

$$f(a+h) = f(a) + hf'(a)$$

i.e. $$y_1 = y_o + h(y')_0$$

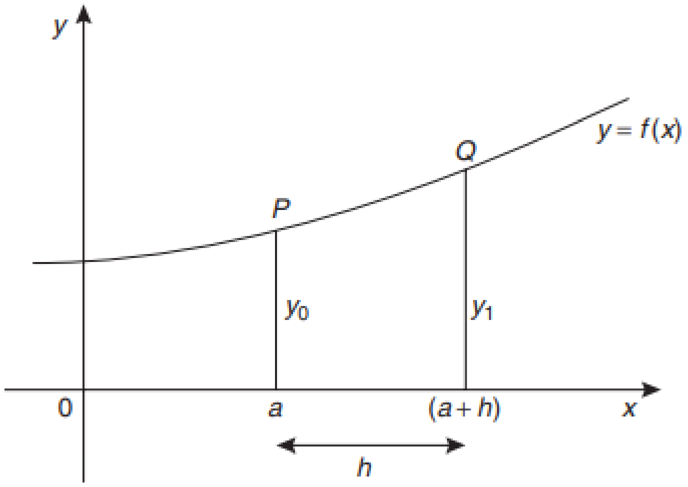

Figure 2

The approximation used with Euler's method is to take only the first two terms of Taylor's series.

Hence if y_0, h and $(y')_0$ are known, y_1, which is an approximate value for the function at Q in Fig. 3, can be calculated. Euler's method is demonstrated in the worked problems following.

Example 1. Obtain a numerical solution of the differential equation

$$\frac{dy}{dx} = 3(1+x) - y$$

given the initial conditions that x = 1 when y = 4, for the range x = 1.0 to x = 2.0 with intervals of 0.2. Draw the graph of the solution

$$\frac{dy}{dx} = y' = 3(1+x) - y$$

With $h = 0.2$, $x_0 = 1$ and $y_0 = 4$, $(y')_0 = 3(1 + 1) - 4 = 2$.

By Euler's method:

$$y_n = y_{n-1} + h(y')_{n-1}$$

Hence

$$y_1 = 4 + (0.2)(2) = 4.4$$

At point Q in Fig. 3, $x_1 = 1.2, y_1 = 4.4$

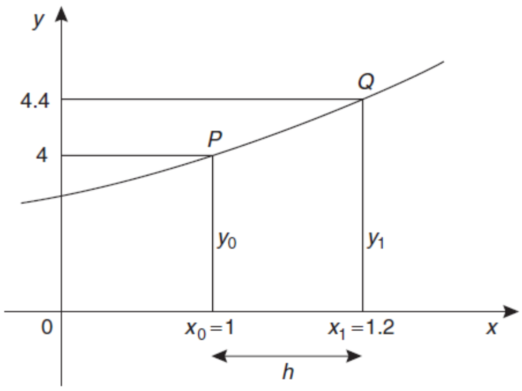

Figure 3

Continuing the calculation:

$$(y')_1 = 3(1 + x_1) - y_1$$

$$(y')_1 = 3(1 + 1.2) - 4.4 = 2.2$$

If the values of x, y and y' found for point Q , are regarded as new starting values of x_0, y_0 , $(y')_0$, the above process can be repeated and values found for the point R shown in Fig. 4

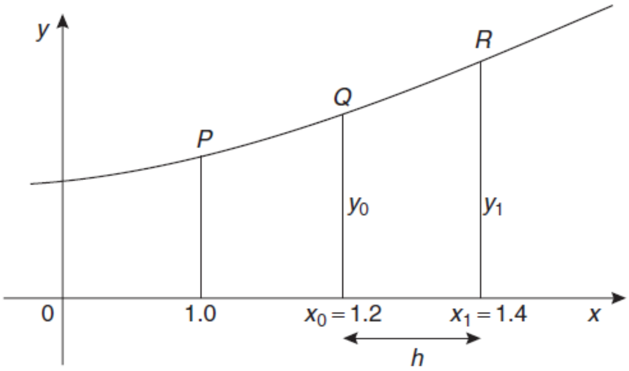

Figure 4

Thus at point *R*,

$$y_2 = 4.4 + (0.2)(2.2) = 4.84$$

When $x_2 = 1.4$ and $y_2 = 4.84$,

$$(y')_2 = 3(1 + 1.4) - 4.84 = 2.36$$

This step by step Euler's method can be continued and it is easiest to list the results in a table:

	x_0	y_0	$(y')_0$
1.	1	4	2
2.	1.2	4.4	2.2
3.	1.4	4.84	2.36
4.	1.6	5.312	2.488
5.	1.8	5.8096	2.5904
6.	2.0	6.32768	

For line 4, where $x_3 = 1.6$:

$$y_3 = 4.84 + (0.2)(2.36) = 5.312$$

$$(y')_3 = 3(1 + 1.6) - 5.312 = 2.488$$

For line 5, where $x_4 = 1.8$:

$$y_4 = 5.312 + (0.2)(2.488) = 5.8096$$

$$(y')_4 = 3(1 + 1.8) - 5.8096 = 2.5904$$

For line 6, where $x_5 = 2.0$:

$$y_5 = 5.8096 + (0.2)(2.5904) = 6.32768$$

(As the range is 1.0 to 2.0 there is no need to calculate $(y')_0$ in line 6).

The particular solution is given by the value of *y* against *x*.

A graph of the solution of

$$\frac{dy}{dx} = 3(1 + x) - y$$

with the initial conditions x = 1 when y = 4 is shown in Fig. 5.

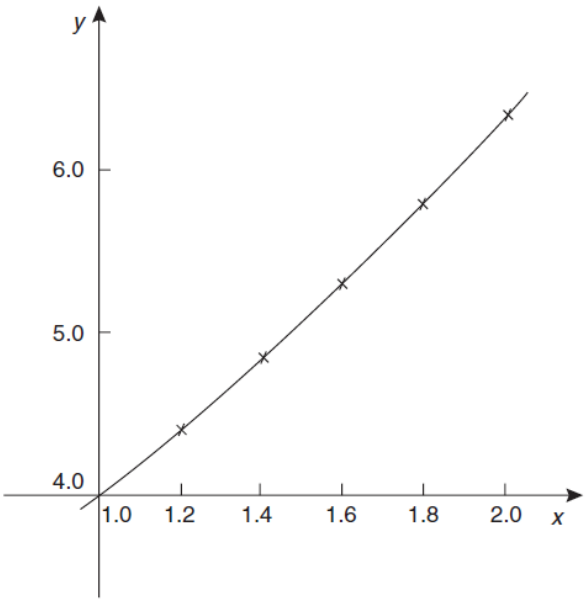

Figure 5

The Runge-Kutta method

The Runge-Kutta method for solving first order differential equations is widely used and provides a high degree of accuracy. Again, as with the two previous methods, the Runge-Kutta method is a step-by-step process where results are tabulated for a range of values of x. Although several intermediate calculations are needed at each stage, the method is fairly straightforward. The 7 step procedure for the Runge-Kutta method, without proof, is as follows:

To solve the differential equation $\frac{dy}{dx} = f(x, y)$ given the initial condition $y = y_0$ at $x = x0$ for a range of values of $x = x_0(h)x_n$:

1. Identify x_0, y_0 and h, and values of x_1, x_2, x_3, ...
2. Evaluate $k_1 = f(x_n, y_n)$ starting with $n = 0$
3. Evaluate $k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} k_1\right)$
4. Evaluate $k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} k_2\right)$
5. Evaluate $k_4 = f(x_n + h, y_n + hk_3)$
6. Use the values determined from steps 2 to 5 to evaluate: $y_{n+1} = y_n + \frac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$
7. Repeat steps 2 to 6 for n = 1, 2, 3, ...

Thus, step 1 is given, and steps 2 to 5 are intermediate steps leading to step 6. It is usually most convenient to construct a table of values. The Runge-Kutta method is demonstrated in the following worked problems.

Example 2. Use the Runge-Kutta method to solve the differential equation:

$$\frac{dy}{dx} = y - x$$

in the range 0(0.1)0.5, given the initial conditions that at $x = 0, y = 2$

Using the above procedure:

1. $x_0 = 0, y_0 = 2$ and since $h = 0.1$, and the range is from $x = 0$ to $x = 0.5$, then $x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, x_4 = 0.4$, and $x_5 = 0.5$

Let $n = 0$ to determine y_1:

2.

$$k_1 = f(x_0, y_0) = f(0, 2);$$

Since

$$\frac{dy}{dx} = y - x$$

$$k_1 = f(0, 2) = 2 - 0 = \mathbf{2}$$

3.

$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} k_1\right)$$

$$k_2 = f\left(0 + \frac{0.1}{2}, 2 + \frac{0.1}{2} (2)\right)$$

$$k_2 = f(0.05, 2.1) = 2.1 - 0.05 = \mathbf{2.05}$$

4.

$$k_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} k_2\right)$$
$$k_3 = f\left(0 + \frac{0.1}{2}, 2 + \frac{0.1}{2}(2.05)\right)$$

$$k_3 = f(0.05, 2.1025) = 2.1025 - 0.05 = \mathbf{2.0525}$$

5.

$$k_4 = f(x_0 + h, y_0 + h\, k_3)$$
$$k_4 = f(0 + 0.1, 2 + 0.1(2.0525))$$

$$k_3 = f(0.1, 2.20525) = 2.20525 - 0.1 = \mathbf{2.10525}$$

6.

$$y_{n+1} = y_n + \frac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

When $n = 0$

$$y_1 = y_0 + \frac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

$$y_1 = 2 + \frac{0.1}{6}\{2 + 2(2.05) + 2(2.0525) + 2.10525\} = 2 + \frac{0.1}{6}\{12.31025\} = \mathbf{2.205171}$$

Let $n = 1$ to determine y_2:

2.

$$k_1 = f(x_1, y_1) = f(0.1, 2.205171);$$

Since

$$\frac{dy}{dx} = y - x$$
$$k_1 = f(0.1, 2.205171) = 2.205171 - 0.1 = \mathbf{2.105171}$$

3.

$$k_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} k_1\right)$$
$$k_2 = f\left(0.1 + \frac{0.1}{2}, 2.205171 + \frac{0.1}{2}(2.105171)\right)$$
$$k_2 = f(0.15, 2.31042955) = 2.31042955 - 0.15 = \mathbf{2.160430}$$

4.

$$k_3 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} k_2\right)$$
$$k_3 = f\left(0.1 + \frac{0.1}{2}, 2.205171 + \frac{0.1}{2}(2.160430)\right)$$

$$k_3 = f\,(0.15, 2.3131925) = 2.3131925 - 0.15 = \mathbf{2.163193}$$

5.

$$k_4 = f\,(x_1 + h\,, y_1 + h\,k_3)$$
$$k_4 = f\,(0.1 + 0.1\,, 2.205171 + 0.1\,(2.163193))$$

$$k_3 = f\,(0.2, 2.421490) = 2.421490 - 0.2 = \mathbf{2.221490}$$

6.

$$y_{n+1} = y_n + \frac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

When $n = 1$

$$y_2 = y_1 + \frac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

$$y_2 = 2.205171 + \frac{0.1}{6}\{2.105171 + 2(2.160430) + 2(2.163193) + 2.221490\} = \mathbf{2.421403}$$

In a similar manner y_3, y_4 and y_5 can be calculated and the results are shown in the following table:

n	x_n	k_1	k_2	k_3	k_4	y_n
0	**0**					**2**
1	**0.1**	2.0	2.05	2.0525	2.10525	**2.205171**
2	**0.2**	2.105171	2.160430	2.163193	2.221490	**2.421403**
3	**0.3**	2.221403	2.282473	2.285527	2.349956	**2.649859**
4	**0.4**	2.349859	2.417339	2.420726	2.491932	**2.891824**
5	**0.5**	2.491824	2.566415	2.570145	2.648838	**3.148720**

If we would to use Euler's method on the same question and compare the results, the following would be found:

x	Euler's method y	Runge-Kutta method y	Exact value $y = x + 1 + e^x$
0	2	2	2
0.1	2.2	2.205171	2.205170918
0.2	2.41	2.421403	2.421402758
0.3	2.631	2.649859	2.649858808
0.4	2.8641	2.891824	2.891824698
0.5	3.11051	3.148720	3.148721271

It is seen from the table that the Runge-Kutta method is exact, correct to 5 decimal places.