two

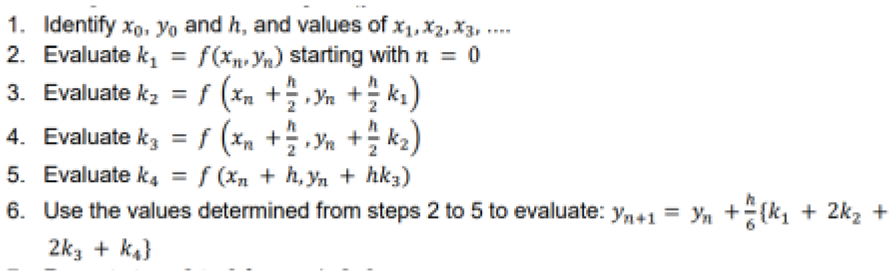
3. Determine the value of $y(0.3)$ using three different numerical methods for the initial value problem $y' = x - y$ and initial condition $x_0 = 0, y_0 = 0$ with h=0.1:

x

$\Rightarrow y' = x - y$

$x_0 = 0 \ , \ y_0 = 0 \ , \ h = 0.1$

a) Runge-Kutta method



1. Identify x_0, y_0 and h, and values of $x_1, x_2, x_3, \ldots$
2. Evaluate $k_1 = f(x_n, y_n)$ starting with $n = 0$
3. Evaluate $k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} k_1\right)$
4. Evaluate $k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} k_2\right)$
5. Evaluate $k_4 = f(x_n + h, y_n + hk_3)$
6. Use the values determined from steps 2 to 5 to evaluate: $y_{n+1} = y_n + \frac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$

b) Euler method

a) Runge-Kutta → $y' = x - y$ $\quad \underline{x_0 = 0} \quad y_0 = 0, \ \underline{h = 0.1}$

$\underbrace{y(0.3)}$ = ? ⇒

n	X	y	K_1	K_2	K_3	K_4
0	0 ↓+h	0 ↓h				
1	$x_1 = 0.1$ → ↓h	$y(0.1)$ ↓h				
2	$x_2 = 0.2$ ↓ y_2 →	$y(0.2)$ ↓				
3	→ $x_3 = 0.3$	$\underbrace{y(0.3)}$ → final				

$$\Rightarrow \overset{\downarrow}{y_{n+1}} = y_n + \frac{h}{6}\left(\overset{\checkmark}{K_1} + 2\overset{\checkmark}{K_2} + 2\overset{\checkmark}{K_3} + \overset{\checkmark}{K_4}\right)$$

if we find $y_3 = y(0.3)$ ⇒ solved

$$K_1 = f(x_n, y_n) \Longrightarrow K_1 = \overset{\to\, x - y}{y'}\underbrace{(x_n, y_n)}_{\to\, y \text{ Function}}$$

$$\downarrow$$

$$K_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} K_1\right) \Longrightarrow y'\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} k_1\right)$$

$$=$$

$$K_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} K_2\right) \Longrightarrow y'\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} K_2\right)$$

$$K_4 = f(x_n + h, y_n + h K_3)$$

$\underline{y_3} \Rightarrow$ target

$\hookrightarrow y_{n+1} = y_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$

$$y_3 = y_{\underset{n \swarrow}{2}+1} = \overset{?\,NO}{\underline{\underline{y_2}}} + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$y_2 = y_{\underset{n \swarrow}{1}+1} = \underset{?}{\underline{y_1}} + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$y_1 = y_{0+1} = \overset{\checkmark}{\underline{y_0}} + \frac{h}{6}\left(\overset{x}{k_1} + 2\overset{x}{k_2} + 2\overset{x}{k_3} + \overset{x}{k_4}\right)$$

$$y_1 = \underline{y_0} + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \qquad n = 0$$

$x_0 = 0$

$y_0 = 0 \qquad y' = x - y \qquad n = 0$

$$k_1 = y'(x_n, y_n) \Rightarrow k_1 = \overset{x-y}{y'}(\overset{0}{x_0}, \overset{0}{y_0}) = 0 - 0 \Rightarrow k_1 = 0$$

$$k_2 = y'\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}\overset{0}{k_1}\right) = y'\left(x_0 + \frac{0.1}{2}, y_0 + \frac{0.1}{2}\times 0\right) = y'(0 + 0.05, 0 + 0)$$

$$\Rightarrow k_2 = y'(\underset{x}{0.05}, \underset{y}{0}) \Rightarrow k_2 = 0.05 - 0 \Rightarrow k_2 = 0.05$$

$$k_3 = y'\left(\underset{x_0 + \frac{h}{2}}{x_n + \frac{h}{2}}, \underbrace{y_n + \frac{h}{2}\times k_2}_{y_0 + \frac{h}{2}\times k_2}\right) = y'\left(\overset{0}{x_0} + \frac{h}{2}, \overset{0}{y_0} + \frac{h}{2}\overset{0.05}{k_2}\right)$$

$\underline{n = 0}$

$$K_3 = y'\left(0 + 0.05,\ 0 + \underbrace{\overset{h/2}{(0.05)} \times \overset{k_2}{0.05}}\right) = y'\left(\underbrace{0.05}_{x},\ \underbrace{0.0025}_{y}\right)$$

$$\Longrightarrow K_3 = 0.05 - 0.0025 = 0.0475$$

$$\Longrightarrow K_4 = y'\left(x_n + h,\ y_n + h \times K_3\right) = y'\left(\overset{0}{x_0} + \overset{0.1}{h},\ \overset{0}{y_0} + h\overset{\uparrow}{K_3}\right)$$

$$\Longrightarrow K_4 = y'\left(0 + 0.1,\ 0 + (0.1) \times (0.0475)\right)$$

$$K_4 = y'\left(\underbrace{0.1}_{x},\ \underbrace{0.00475}_{y}\right) = 0.1 - 0.00475$$

$$\Longrightarrow K_4 = 0.09525$$

$$y_1 = y_0 + \frac{h}{6}\left(\underline{K_1} + 2\underline{\underline{K_2}} + 2K_3 + \underline{\underline{K_4}}\right)$$

$$\hookrightarrow y_1 = 0 + \frac{0.1}{6}\left(0 + 2(0.05) + 2(0.0475) + 0.09525\right)$$

$$\Rightarrow \underline{\underline{y_1 = 0.004644}}$$

$y_2 = ?$ $\Longrightarrow$ $y_{\boxed{2}} = y_1 + \frac{h}{6}(\underline{k_1} + 2\underline{k_2} + 2\underline{k_3} + \underline{k_4})$

$n = ?$ $\Longrightarrow$ $\overset{n}{1} + 1$ $\Longrightarrow$ $n = 1$

? what is the value of x_1

$k_1 = y'(x_n, y_n) \Longrightarrow k_1 = y'(x_1, y_1)$?

x_1 = 0.1 based on the description of table

$k_1 = \overset{x-y}{y'}(x_1, y_1) = y'(\underbrace{0.1}_{x}, \underbrace{0.004644}_{y}) = 0.1 - 0.004644$

$\Longrightarrow k_1 = 0.095356$

$$K_2 = y'\left(x_{\underset{\underset{1}{\downarrow}}{n}} + \frac{h}{2},\ y_n + \frac{h}{2} K_1\right)$$

$\downarrow$

$$K_2 = y'\left(\underline{x_1} + \frac{0.1}{2},\ \underline{y_1} + \frac{0.1}{2} K_1\right) = y'\left(0.1 + 0.05,\ 0.004644 + 0.05 \times K_1\right)$$

$$\Rightarrow K_2 = y'\left(\underbrace{0.15}_{x},\ \underbrace{0.004644 + 0.05 \times (0.095356)}_{y}\right)$$

$$\Rightarrow K_2 = 0.140588$$

$h = 1$

$$K_3 = y'\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} k_2\right)$$

$\downarrow$

$$k_3 = y'\left(x_{\underline{\underline{1}}} + \frac{h}{2}, y_{\underline{\underline{1}}} + \frac{h}{2} \underline{\underline{k_2}}\right)$$

$$\Rightarrow k_3 = y'(0.1 + 0.05, 0.004644 + (0.05)(0.140588))$$

$$\Rightarrow k_3 = y'(\underbrace{0.15}_{x}, \underbrace{0.004644 + (0.05)(0.140588)}_{y})$$

$$\Rightarrow K_3 = 0.138326$$

$$K_4 = y'(x_n + h , y_n + h K_3)$$

$$\downarrow$$

$$K_4 = y'(\underline{\underline{x_1}} + \underline{\underline{h}} , \underline{\underline{y_1}} + h K_3)$$

$$\downarrow$$

$$K_4 = y'(0.1 + 0.1 , 0.004644 + (0.1)(0.138326))$$

$$K_4 = y'(\underbrace{0.2}_{x} , \underbrace{0.004644 + (0.1)(0.138326)}_{y})$$

$$y' = x - y$$

$$\Longrightarrow \qquad K_4 = 0.181523$$

$$y_2 = y_{\underline{\underline{1}}} + \frac{h}{6}\left(k_{\underline{\underline{1}}} + 2k_{\underline{\underline{2}}} + 2k_3 + k_4\right)$$

$\downarrow$

$$y_2 = 0.004644 + \frac{6.1}{6}\left(0.095356 + 2(0.140588) + 2(0.138326) + 0.181523\right)$$

$$\Longrightarrow \underline{\underline{y_2}} = \underline{0.017999}$$

$\Longrightarrow$

$y_3 = ?$ $\Rightarrow$ $\underline{y_3} = y_{\boxed{2}} + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

✓ ✓ ✓ ✓

target

$n = 2$

$x_2 = 0.2$ from table

$x - y$

$K_4 = y'(x_n, y_n) = y'(\underbrace{x_2}_{x}, \underbrace{y_2}_{y}) = 0.2 - 0.017999$

$K_4 = 0.182001$

$$k_2 = y'\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} k_1\right)$$

$\downarrow$

$$k_2 = y'\left(x_2 + \frac{0.1}{2}, \underline{\underline{y_2}} + \frac{h}{2} k_1\right)$$

$\downarrow$

$$k_2 = y'\left(\underbrace{0.2 + 0.05}, 0.017999 + (0.05) \times 0.182001\right)$$

$\downarrow$

$$k_2 = y'\left(\underbrace{0.25}_{x}, \underbrace{0.027099}_{y}\right) = 0.25 - 0.027099$$

$$\Longrightarrow k_2 = 0.222900$$

$$K_3 = y'\left(x_2 + \frac{h}{2}, \; y_2 + \frac{h}{2} k_2\right)$$

$\downarrow$

$$K_3 = y'\left[0.2 + 0.05, \; 0.017999 + 0.05 \times (0.222900)\right]$$

$\downarrow$

$$K_3 = y'\left(\underbrace{0.25}_{x}, \; \underbrace{0.029144}_{y}\right) = 0.25 - 0.029144$$

$\downarrow$

$$\Longrightarrow \quad K_3 = 0.220856$$

$$K_4 = y'(x_n + h, y_n + h\,k_3)$$

$\downarrow n=2$

$$K_4 = y'(x_2 + h, \underline{\underline{y_2}} + h\,\underline{k_3})$$

$\downarrow$

$$K_4 = y'(0.2 + 0.1, 0.017999 + (0.1)(0.220856))$$

$\downarrow$

$$K_4 = y'(\underbrace{0.3}_{x}, \underbrace{0.040084}_{y}) = 0.3 - 0.040084$$

$$\Longrightarrow K_4 = 0.259915$$

$$y_3 = \underset{\checkmark}{\underline{\underline{y_2}}} + \frac{h}{6}\left(\underline{\underline{k_1}} + 2k_2 + 2k_3 + k_4\right)$$

$$\downarrow$$

$$y_3 = 0.017999 + \frac{0.1}{6}\Big(0.182001 + 2[0.222900) + 2(0.220856) + 0.259915\Big)$$

$$\Longrightarrow y_3 = 0.039269$$

Euler method $\Rightarrow$ $y' = x - y$, $x_0 = 0$, $h = 0.1$

$y_0 = 0$

↓

$y(0.3)$

↓

$$y_{n+1} = y_n + h \times y'(x_n, y_n)$$

n	x	y	y'
0	x_0 0	0	
1	$x_1 = 0.1$	$y(0.1)$	
2	$x_2 = 0.2$	$y(0.2)$	
3	$x_3 = 0.3$	$y(0.3)$	

$$\underset{?}{\underline{y(0.3)}} = y_3 = y_{\underset{n}{2}+1} = \underset{\underset{y(0.2)}{\downarrow}}{\underline{y_2}} + h \times y'(x_2, y_2)$$

$$y_2 = y_{1+1} = \underset{\downarrow}{\underline{y_1}} + h \times y'(x_1, y_1)$$

$n=1$

$$y_1 = y_{0+1} = \overset{\checkmark}{y_0} + \overset{\checkmark}{h} \times \overset{x-y}{y'}(x_0, y_0)$$

$n=0$

$$y_1 = y_{\underset{n=0 \Leftarrow}{0}+1} = y_0 + h\,\underbrace{y'(x_0, y_0)}$$

$$\hookrightarrow y' = x - y = \overset{\overset{x_0}{\downarrow}}{0} - \overset{\overset{y_0}{\downarrow}}{0}$$

$$\Longrightarrow y_0 = 0 , \quad x_0 = 0 , \quad h = 0.1$$

$$\Longrightarrow y_1 = 0 + 0.1 \times (0 - 0) = 0 \quad \Longrightarrow y_1 = 0$$

$$y_2 = y_{\underset{\swarrow\, n=1}{1+1}} = \underline{y}_1 + \underset{\swarrow\, 0.1}{h} \times y'(\underset{\swarrow\, 0.1}{x_1}, \underset{\swarrow\, =0}{y_1})$$

$\swarrow$ based on the table

$$\Rightarrow y_2 = 0 + (0.1) \times (0.1 - 0) = 0 + 0.01 \Rightarrow$$

$$y_2 = 0.01$$

next and final step

$$y_3 = y_{2+1} = y_2 + h \times y'(x_2, y_2)$$

$\rightarrow 0.01$

$n = 2$

0.1

0.2

from table

$$\Rightarrow y_3 = 0.01 + (0.1)(\underbrace{0.2 - 0.01}_{0.19}) = 0.01 + 0.019$$

$$\Rightarrow y_3 = 0.029$$