UNIVERSITY OF LINCOLN

SCHOOL OF ENGINEERING

DATA MODELLING AND SIMULATION

Lecture 9: Heat Equation

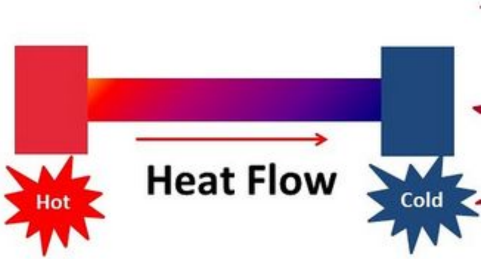
By,
Dr Mithun Poozhiyil

Wave equation and heat equation
(Partial Differential Equations)

(a) The **wave equation**, where the equation of motion is given by:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Where $c^2 = \frac{T}{\rho}$, with T being the tension in a string and ρ being the mass/unit length of the string.



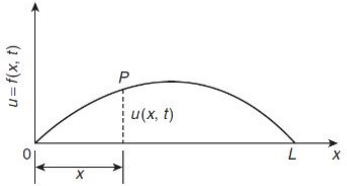
(b) The **heat conduction equation** is of the form:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$$

Where $c^2 = \frac{h}{\sigma\rho}$, with h being the thermal conductivity of the material and ρ being the mass/unit length of the material.

Procedure

1. Identify clearly the initial and boundary conditions.
2. Assume a solution of the form $u = XT$ and express the equations in terms of X and T and their derivatives.
3. Separate the variables by transposing the equation and equate each side to a constant, say μ; two separate equations are obtained, one in x and one in t.
4. Let $\mu = -p^2$ to give an oscillatory solution.



5. The two solutions are of the form:

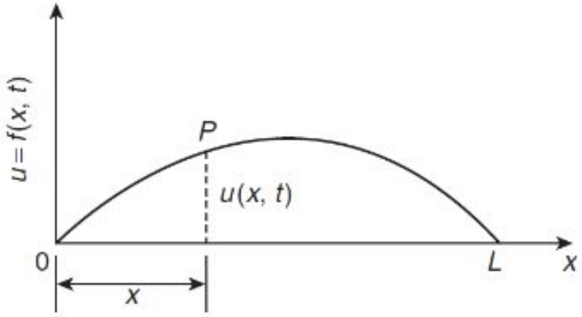
   $X = A\cos px + B\sin px$ and $\cancel{T = A\cos pt + B\sin pt}$ **Using fourier series**

   ~~Then~~ $\cancel{u(x,t) = \{(A\cos px + B\sin px)\}\{(C\cos pt + D\sin pt)\}}$
6. Apply the boundary conditions to determine constants A and B.
7. Determine the general solution as an infinite sum.
8. Apply the remaining boundary and initial conditions and determine the coefficients A_n and B_n using Fourier series techniques.

The Heat Conduction Equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$$

right hand side contains a first partial derivative instead of the second.



Conduction of heat in a uniform bar depends on

- Initial distribution of temperature
- physical properties of the material

- Consider such a bar shown below where the bar extends from x = 0 to x = L.
- The temperature of the ends of the bar is maintained at zero
- The initial temperature distribution along the bar is defined by f(x).

The Heat Conduction Equation

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}}$$

1 As with the wave equation we assume a solution u = XT. Then,

$$\frac{\partial u}{\partial x} = X'T, \quad \frac{\partial^2 u}{\partial x^2} = X''T \quad \text{and} \quad \frac{\partial u}{\partial t} = XT'$$

2 Substituting into the heat equation gives,

$$X''T = \frac{1}{c^2}XT'$$

3 Separating the variables gives,

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = \mu$$

Let $\mu = -p^2$

$$-p^2 = \frac{X''}{X} \text{ then } X'' = -p^2 X \text{ or } X'' + p^2 X = 0$$

The solution of this is,

$$X = A\cos px + B\sin px$$

4 Similarly,

$$-p^2 = \frac{1}{c^2}\frac{T'}{T} \text{ then } \frac{T'}{T} = -p^2c^2$$

integrating with respect to t gives:

$$\int \frac{T'}{T}dt = \int -p^2c^2 dt$$

From which, $\ln T = -p^2c^2t + c_1$

Rearranging for T gives,

$$T = e^{-p^2c^2t}e^{c_1}$$

(where $k = e^{c_1}$)

$$= ke^{-p^2c^2t}$$

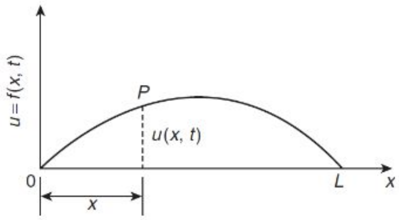
Our general solution now becomes,

$$\boxed{u(x,t) = (A\cos px + B\sin px)ke^{-p^2c^2t}}$$

The boundary conditions can be expressed as:

$$\left.\begin{array}{l} u(0,t) = 0 \\ u(L,t) = 0 \end{array}\right\} \text{for all values of } t \geq 0$$

$u(x,0) = f(x)$ for $0 \leq x \leq L$



1. Identify clearly the initial and boundary conditions.
2. Assume a solution of the form $u = XT$ and express the equations in terms of X and T and their derivatives.
3. Separate the variables by transposing the equation and equate each side to a constant, say μ; two separate equations are obtained, one in x and one in t.
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5. The two solutions are of the form:
   $X = A\cos px + B\sin px$ and $\cancel{T = A\cos pt + B\sin pt}$
   ~~Then~~ $\cancel{u(x,t) = \{(A\cos px + B\sin px)\}\{(C\cos pt + D\sin pt)\}}$
6. Apply the boundary conditions to determine constants A and B.
7. Determine the general solution as an infinite sum.
8. Apply the remaining boundary and initial conditions and determine the coefficients A_n and B_n using Fourier series techniques.

The Heat Conduction Equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$$

general solution $\boxed{u(x,t) = (A\cos px + B\sin px)ke^{-p^2c^2t}}$

1 $u(x,t) = (P\cos px + Q\sin px)e^{-p^2c^2t}$ (where P = Ak and Q = Bk)

2 Applying the boundary conditions $u(0,t) = 0$ gives,

$$0 = (P\cos p(0) + Q\sin p(0))e^{-p^2c^2t}$$

$$0 = Pe^{-p^2c^2t}$$

From which P = 0.

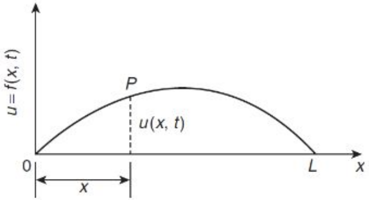
Therefore,

$$u(x,t) = Q\sin px e^{-p^2c^2t}$$

The boundary conditions can be expressed as:

$$\left.\begin{matrix} u(0,t) = 0 \\ u(L,t) = 0 \end{matrix}\right\} \text{for all values of } t \geq 0$$

$u(x,0) = f(x)$ for $0 \leq x \leq L$



3 Now applying the boundary condition $u(L,t) = 0$,

$$0 = Q\sin pLe^{-p^2c^2t}$$

Since Q cannot equal zero then $\sin pL = 0$ from which, $pL = n\pi$ or $p = \frac{n\pi}{L}$ where n = 1, 2, 3,....

There are therefore many values of u(x, t). Thus in general,

$$u(x,t) = \sum_{n=1}^{\infty}\left(Q_n e^{-p^2c^2t}\sin\frac{n\pi x}{L}\right)$$

Applying the remaining boundary condition, that when $t = 0, u(x,t) = f(x)$ for $0 \leq x \leq L$ gives,

$$f(x) = \sum_{n=1}^{\infty}\left(Q_n \sin\frac{n\pi x}{L}\right)$$

From Fourier series, $Q_n = 2 \times$mean value of $f(x)\sin\frac{n\pi x}{L}$ from x to L. Hence,

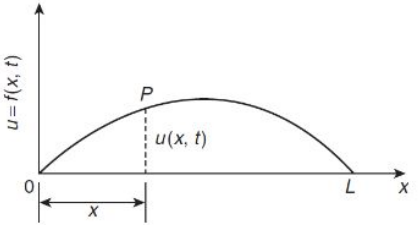
$$Q_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

Thus, the final solution is,

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left\{\left(\int_0^L f(x)\sin\frac{n\pi x}{L}dx\right)e^{-p^2c^2t}\sin\frac{n\pi x}{L}\right\}$$

Example: A metal bar, insulated along its sides, is 1m long. It is initially at room temperature of 15ºC and at time t = 0, the ends are placed into ice at 0ºC.

Find an expression for the temperature at a point P at a distance x metres from one end at any time t seconds after t = 0



$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$$

First let's determine the initial and boundary conditions,

$u(0,t) = 0,\ u(1,t) = 0$ and $u(x,0) = 15$

Assuming a solution of the form u = XT, then

$$X = A\cos px + B\sin px$$
$$T = ke^{-p^2c^2t}$$

The general solution is therefore,

$$u(x,t) = (P\cos px + Q\sin px)e^{-p^2c^2t}$$

$u(0,t) = 0$ thus $0 = Pe^{-p^2c^2t}$

From which,

P = 0 and $0 = (Q\sin p)e^{-p^2c^2t}$

Since Q cannot be zero, $\sin p = 0$ from which $p = n\pi$ where n = 1, 2, 3,.....

The more general solution is therefore,

$$u(x,t) = \sum_{n=1}^{\infty}\left(Q_n e^{-p^2c^2t}\sin n\pi x\right)$$

The final initial condition given was that at t = 0, u = 15, i.e. u(x, 0) = f(x) = 15.

Therefore,

$$15 = \sum_{n=1}^{\infty}(Q_n \sin n\pi x)$$

From Fourier coefficients,

$Q_n = 2\times$ mean value of $15\sin n\pi x$ from x = 0 to x = 1,

i.e. $Q_n = \frac{2}{1}\int_0^1 15\sin n\pi x\,dx$

Performing the integration gives,

$$30\left[-\frac{\cos n\pi x}{n\pi}\right]_0^1$$

$$= -\frac{30}{n\pi}[\cos n\pi - \cos 0]$$

$$= \frac{30}{n\pi}(1 - \cos n\pi)$$

$= 0$ (when n is even) and $\frac{60}{n\pi}$ (when n is odd)

The required solution is therefore,

$$\boxed{\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty}\left(Q_n e^{-p^2c^2t}\sin n\pi x\right) \\ &= \frac{60}{\pi}\sum_{n(odd)=1}^{\infty}\frac{1}{n}(\sin n\pi x)e^{-n^2\pi^2c^2t} \end{aligned}}$$