Lecture 2 Solving Systems of ODE's

Now that we have studied eigenvalues and eigenvectors, we can begin to look at how these techniques can be applied to solving systems of differential equations.

Introduction

An equation is a mathematical expression which contains an unknown quantity which we wish to find. A **differential equation** is an equation which contains the derivative of an unknown expression. Examples of some differential equations are shown below,

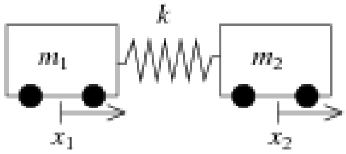
$$\frac{dy}{dx} = x^3, \quad \frac{dv}{dt} + v^2 = 0, \quad \frac{d^2\theta}{dt^2} + \frac{d\theta}{dt} = t$$

Many models of engineering systems involve the rate of change of a quantity. There is thus a need to incorporate derivatives into the mathematical model. Accompanying the differential equation will be one or more conditions that let us obtain a unique solution to a particular problem. Often we solve the differential equation first to obtain a general solution; then we apply the conditions to obtain the unique solution. In a function such as $y = x^2 + 2x$ we say that x is the **independent variable** and y is the **dependent variable** since the value of y depends on the choice we have made for x. When solving a differential equation it is essential that you can identify the dependent and independent variables.

In the differential equation $\frac{dx}{dt} = x + t^3$, x is the dependent variable and t is the independent variable. In the differential equation $\frac{dy}{dx} = y + \cos x$, y is the dependent variable. Note that the dependent variable is always the variable being differentiated i.e. that rate of change of y is dependent upon the value of x.

Often, however, we do not have just one dependent variable and one equation.

As an example application, let us think of a mass and spring system.



Suppose we have one spring with constant k, but two masses m_1 and m_2. We can think of the masses as carts, and we will suppose that they ride along a straight track with no friction. Let x_1 be the displacement of the first cart and x_2 be the displacement of the second cart. That is, we put the two carts somewhere with no tension on the spring, and we mark the position of the first and second cart and call those the zero positions. Then x_1 measures how far the first cart is from its zero position, and x_2 measures how far the second cart is from its zero position. By Hooke's law, the force exerted by the spring on the first cart is $k(x_2 - x_1)$, since $x_2 - x_1$ is how far the string is stretched (or compressed) from the rest position. The force exerted on the second cart is the opposite, thus the same thing with a negative sign. Newton's second law states that force equals mass times acceleration. So the system of equations governing the setup is

$$m_1 x_1'' = k(x_2 - x_1),$$
$$m_2 x_2'' = -k(x_2 - x_1).$$

In this system we cannot solve for the x_1 or x_2 variable separately. That we must solve for both x_1 and x_2 at once is intuitively clear, since where the first cart goes depends exactly on where the second cart goes and vice-versa.

We can use eigenvalues and eigenvectors to solve these types of system.

The general solution of a system of differential equations is:

$$x(t) = a_1 X_1 e^{\lambda_1 t} + a_2 X_2 e^{\lambda_2 t} + \dots\dots\dots + a_n X_n e^{\lambda_n t}$$

Where, n is given by the dimensions of matrix A, a_1 and a_n depend on initial conditions, λ_n are the eigenvalues of matrix A and X_1 and X_n are the eigenvectors corresponding to each of the eigenvalues.

Writing Differential Equations in Matrix Form

The first thing we need to do when solving a system of differential equations is write the equations in matrix form.

Example 1. Write the following system of differential equations in matrix form.

$$x_1' = 4x_1 + x_2$$
$$x_2' = 3x_1 + 2x_2$$

The system, in matrix form is as follows:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Once we have the system in matrix form, we can go to solve using eigenvalues and eigenvectors.

Example 2. Solve the following system of differential equations.

$$x_1' = 4x_1 + x_2$$
$$x_2' = 3x_1 + 2x_2$$

We know from example 1, that the system in matrix form is:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

We call the matrix, $\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$ matrix A. From the general solution shown above, if we can find the eigenvalues and eigenvectors of matrix A, then we have our general solution for this system.

Finding eigenvalues first gives,

$$\begin{pmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{pmatrix}$$

The characteristic equation is,

$$(4-\lambda)(2-\lambda)-(1\times 3)$$
$$=8-4\lambda-2\lambda+\lambda^2-3$$
$$\lambda^2-6\lambda+5=0$$

Solving gives,

$$\lambda_1=5,\ \lambda_2=1$$

Now finding the eigenvectors that correspond to each eigenvalue noting that

$$(A-\lambda I)X=0,$$

$$\lambda_1=5,$$

$$\begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}-5\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}-\begin{pmatrix}5 & 0\\ 0 & 5\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}-1 & 1\\ 3 & -3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

This gives the equations,

$$-x_1+x_2=0$$
$$3x_1-3x_2=0$$

Solving for x and y gives us our first eigenvector corresponding to our eigenvalue λ_1=5,

$$X_1=\begin{pmatrix}1\\ 1\end{pmatrix}$$

Now finding the eigenvector corresponding to the eigenvalue λ_1=1,

$$\left(\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} - 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Giving the equations,

$$3x_1 + x_2 = 0$$
$$3x_1 + x_2 = 0$$

Solving gives the eigenvector,

$$X_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

Now using the general solution for a system of differential equations,

$$x(t) = a_1 X_1 e^{\lambda_1 t} + a_2 X_2 e^{\lambda_2 t} + ...+ a_n X_n e^{\lambda_n t}$$

Gives,

$$x(t) = a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t} + a_2 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{t}$$

Expanding this gives our general solutions,

$$x_1(t) = a_1(1)e^{5t} + a_2(1)e^{t}$$

$$x_2(t) = a_1(1)e^{5t} + a_2(-3)e^{t}$$

Example 3. Solve the following system of differential equations:

$$x_1' = -2x_1 + x_2$$
$$x_2' = x_1 - 2x_2$$

Subject to the initial conditions $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Putting the system in to matrix form and finding the eigenvalues and eigenvectors, as shown in example 2, gives,

$$\lambda_1 = -1, \ \lambda_2 = -3$$

$$X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

This gives the general solution,

$$x(t) = a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + a_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}$$

Which gives,

$$x_1(t) = a_1 e^{-t} + a_2 e^{-3t}$$
$$x_2(t) = a_1 e^{-t} - a_2 e^{-3t}$$

Given the initial conditions we know that when t = 0,

$$x(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Therefore,

$$x(0) = a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-1\times 0} + a_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3\times 0}$$

$$x(0) = a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Expanding this gives the equations,

$$\begin{aligned} a_1 + a_2 &= 1 \\ a_1 - a_2 &= 2 \end{aligned}$$

Solving gives,

$$\begin{aligned} a_1 &= \frac{3}{2} \\ a_2 &= -\frac{1}{2} \end{aligned}$$

This gives the particular solution,

$$x(t) = \frac{3}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} - \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}$$

Or,

$$\begin{aligned} x_1(t) &= \frac{3}{2} e^{-t} - \frac{1}{2} e^{-3t} \\ x_2(t) &= \frac{3}{2} e^{-t} + \frac{1}{2} e^{-3t} \end{aligned}$$

Solving Linear ODE systems of 2nd order

To solve linear 2nd order differential equation, we write the system as 1st order differential equation

Consider the following:

$$\ddot{x_1} + 5\dot{x_1} + 6\dot{x_2} + 3x_1 + x_2 = 0$$
$$\ddot{x_2} - 6\dot{x_1} - 4\dot{x_2} + 3x_1 - x_2 = 0$$

We can write higher order differential equations as a system with a very simple change of variable. We'll start by defining the following new variables.

$$y_1 = x_1 \qquad y_2 = x_2$$
$$y_3 = \dot{x_1} \qquad y_4 = \dot{x}_2$$

The equation can then be written as a system:

$$\dot{y}_1 = y_3$$
$$\dot{y}_2 = y_4$$
$$\dot{y}_3 = -5y_3 - 6y_4 - 3y_1 - y_2$$
$$\dot{y}_4 = 6y_3 + 4y_4 - 3y_1 + y_2$$

The system can then be solved following the steps from 1st order differential equation.