Lecture 3 Stability Analysis of ODEs

Introduction to Nonlinear System of ODEs

So far we examined systems that can be written in the form,

$$\boldsymbol{x}'' = [\boldsymbol{A}](\boldsymbol{x}(\boldsymbol{t}))$$

Which are called linear systems. But in case that the system cannot be written in such a form such as:

$$\boldsymbol{x}'(\boldsymbol{t}) = [\boldsymbol{A}]\boldsymbol{x}(\boldsymbol{t}) + (\boldsymbol{B}_{\boldsymbol{i}}(\boldsymbol{x}(\boldsymbol{t}))$$

In this case there are no explicit forms of solutions, it depends on the particular form of the system whether or not it is possible to solve it. So let's look at some examples:

Example 1. Write the following system in matrix form:

$$\dot{y}_1(t) = 5y_1 - 2y_2 + 4y_1y_2$$

$$\dot{y}_2(t) = -3y_1 + 2y_2 - {y_2}^2$$

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 4y_1y_2 \\ y_2^2 \end{pmatrix}$$

Non-linear part

Example 2. Write the following system in matrix form:

$$\dot{y}_1(t) = 3y_1 + 2y_2 - \sin(y_1)$$

$$\dot{y}_2(t) = 4y_1 - 5y_2 + \cos(y_2)$$

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} -\sin(y_1) \\ \cos(y_2) \end{pmatrix}$$

Non-linear part

In Examples 1 and 2 there is no direct solution and we examine specific characteristics of the solution:

-fixed, stationary or equilibrium points e.g. points when the system is at rest

-periodic orbits of the solution e.g. Nonlinear Normal Modes

-stability of points

-curves that lead to infinity

Fixed points

Example 3. Find the fixed points of the following system:

$$\dot{y_1} = 5y_1 - 2y_2 + 4y_1y_2$$
$$\dot{y_2} = -3y_1 + 2y_2 - y_2^2$$

Subject to the initial conditions

$$\begin{pmatrix}\dot{y_1}\\ \dot{y_2}\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$5y_1 - 2y_2 + 4y_1y_2 = 0 \quad (1)$$
$$-3y_1 + 2y_2 - y_2^2 = 0 \quad (2)$$

Rearrange (1)

$$5y_1 + 4y_1y_2 = 2y_2$$
$$y_1(5 + 4y_2) = 2y_2$$
$$y_1 = \frac{2y_2}{5 + 4y_2}$$

Sub into (2)

$$-3\left(\frac{2y_2}{5 + 4y_2}\right) + 2y_2 - y_2 = 0$$
$$-6y_2 + 10y_2 + 8y_2^2 - 5y_2^2 - 4y_2^3 = 0$$
$$-4y_2^3 + 3y_2^2 + 4y_2 = 0$$
$$-4y_2\,(y_2^2 - \frac{3}{4}y_2 - 1) = 0$$

So we have three y_2 solutions and three y_1 solutions:

$y_1 = 0$	$y_1 = 1.44$	$y_1 = -0.69$
$y_2 = 0$	$y_2 = 0.27$	$y_2 = -0.62$

Stability Analysis of ODEs

Equilibria are not always stable. Since stable and unstable equilibria play quite different roles in the dynamics of a system, it is useful to be able to classify equilibrium points based on their stability. Suppose that we have a set of autonomous ordinary differential equations, written in vector form:

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$$

Suppose that x^* is an equilibrium point. By definition, $f(x^*) = 0$. Now suppose that we take a multivariate Taylor expansion of the right-hand side of our differential equation:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}^*) + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}^*} (\mathbf{x} - \mathbf{x}^*) + \dots \\ &= \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}^*} (\mathbf{x} - \mathbf{x}^*) + \dots\end{aligned}$$

The partial derivative in the above equation is to be interpreted as the **Jacobian matrix.** If the components of the state vector x are $(x_1, x_2, \dots, x_n)$ and the components of the rate vector f are $(f_1, f_2, \dots, f_n)$, then the Jacobian is

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Now define $x = x - x^*$. Taking a derivative of this definition, we get $\dot{\delta x} = \dot{x}$. If δx is small, then only the first term in Taylor equation above is significant since the higher terms involve powers of our small displacement from equilibrium. If we want to know how trajectories behave near the equilibrium point, e.g. whether they move toward or away from the equilibrium point, it should therefore be good enough to keep just this term. Then we have:

$$\dot{\delta x} = J^* \delta x$$

Where J^* is the Jacobian evaluated at the equilibrium point. The matrix J^* is a constant, so this is just a linear differential equation. According to the theory of linear differential equations, the solution can be written as a superposition of terms of the form $e^{\lambda_j t}$ where $\{\lambda j\}$ is the set of eigenvalues of the Jacobian is. The eigenvalues of the Jacobian are, in general, complex numbers. Let $\lambda_j = \mu_j + i\nu_j$, where μ_j and ν_j are, respectively, the real and imaginary parts of the eigenvalue. Each of the exponential terms in the expansion can therefore be written:

$$e^{\lambda_j t} = e^{\mu_j t} e^{i\nu_j t}$$

The complex exponential in turn can be written

$$e^{i\nu_j t} = cos(\nu_j t) + isin(\nu_j t)$$

The complex part of the eigenvalue therefore only contributes an oscillatory component to the solution. It's the real part that matters: If $\mu_j > 0$ for any j, $e^{\mu_j t}$ grows with time, which means that trajectories will tend to move away from the equilibrium point. This leads us to a very important theorem:

Theorem An equilibrium point x^* of the differential equation is stable if all the eigenvalues of J^*, the Jacobian evaluated at x^*, have negative real parts. The equilibrium point is unstable if at least one of the eigenvalues has a positive real part.

Example 4. Examine the stability of fixed points of the following system:

$$\dot{y_1} = 5y_1 - 2y_2 + 4y_1y_2$$
$$\dot{y_2} = -3y_1 + 2y_2 - y_2^2$$

Subject to the initial conditions

$$\begin{pmatrix} \dot{y_1} \\ \dot{y_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Jacobian matric is $\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$

Therefore each equation needs to be partially differentiated with respect to y_1 and y_2:

$$J = \begin{vmatrix} \frac{\partial \dot{y}_1}{y_1} & \frac{\partial \dot{y}_1}{y_2} \\ \frac{\partial \dot{y}_2}{y_1} & \frac{\partial \dot{y}_2}{y_2} \end{vmatrix} = \begin{vmatrix} 5 + 4y_2 & -2 + 4y_1 \\ -3 & 2 - 2y_2 \end{vmatrix}$$

i) Substitute $(0{,}0)$ fixed point from Example 3.

$$\begin{vmatrix} 5 + 4(0) & -2 + 4(0) \\ -3 & 2 - 2(0) \end{vmatrix} = \begin{vmatrix} 5 & -2 \\ -3 & 2 \end{vmatrix}$$

Find eigenvalues:

$$\begin{bmatrix} 5 & -2 \\ -3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 5 - \lambda & -2 \\ -3 & 2 - \lambda \end{vmatrix} = 0$$
$$(5 - \lambda)(2 - \lambda) - 6 = 0$$
$$\lambda^2 - 7\lambda + 4 = 0$$
$$\lambda_1 = 0.63 \qquad \lambda_2 = 6.37$$

Real and positive → UNSABLE

ii) Substitute $(1.44, 0.27)$ fixed point from Example 3 and follow same steps as in i) to find eigenvalues:

$$\lambda^2 - 9.88\lambda - 12.25 = 0$$
$$\lambda_1 = 10.99 \qquad \lambda_2 = -1.11$$

Real and positive → UNSABLE

iii) Substitute $(-0.69, -0.62)$ fixed point from Example 3 and follow same steps as in i) to find eigenvalues:

$$\lambda^2 - 5.47\lambda - 6.23 = 0$$
$$\lambda_1 = 6.52 \qquad \lambda_2 = -0.91$$

Real and positive → UNSABLE

Note:

Stability of hyperbolic (non-zero and real) points:
a) If all real parts of the eigenvalues are negative then the fixed point is stable
b) If one or more eigenvalues have positive real part then the fixed point is unstable.

Stability of nonhyperbolic points
a) If one or more eigenvalues of J have positive real part then the fixed point is unstable
b) If all eigenvalues are nonzero and purely imaginary then it is centre, and it is marginally stable.
c) If some are zero and the rest with only negative and zero real parts then it needs to include more terms in Taylor series expansion to examine stability. In such case there are more advance techniques e.g. Normal Forms, Catastrophe Theory, Direct Lyapunov method etc.