# Runge-Kutta method

The formula for the fourth order Runge-Kutta method (RK4) is given below. Consider the problem

$$\begin{cases} y' = f(t, y) \\ y(t_0) = \alpha \end{cases}$$

Define $h$ to be the time step size and $t_i = t_0 + ih$. Then the following formula

$$w_0 = \alpha$$

$$k_1 = hf(t_i,\, w_i)$$

$$k_2 = hf\left(t_i + \frac{h}{2},\, w_i + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(t_i + \frac{h}{2},\, w_i + \frac{k_2}{2}\right)$$

$$k_4 = hf(t_i + h,\, w_i + k_3)$$

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

computes an approximate solution, that is $w_i \approx y(t_i)$.

Let us look at an example:

$$\begin{cases} y' = y - t^2 + 1 \\ y(0) = 0.5 \end{cases}$$

The exact solution for this problem is $y = t^2 + 2t + 1 - \frac{1}{2}e^t$, and we are interested in the value of $y$ for $0 \le t \le 2$.

**1. We first solve this problem using RK4 with $h = 0.5$.** From $t = 0$ to $t = 2$ with step size $h = 0.5$, it takes 4 steps: $t_0 = 0$, $t_1 = 0.5$, $t_2 = 1$, $t_3 = 1.5$, $t_4 = 2$.

Step 0 $t_0 = 0$, $w_0 = 0.5$.

Step 1 $t_1 = 0.5$

$$\begin{aligned}
k_1 &= hf(t_0, w_0) = 0.5f(0, 0.5) = 0.75 \\
k_2 &= hf(t_0 + h/2, w_0 + k_1/2) = 0.5f(0.25, 0.875) = 0.90625 \\
K_3 &= hf(t_0 + h/2, w_0 + k_2/2) = 0.5f(0.25, 0.953125) = 0.9453125 \\
K_4 &= hf(t_0 + h, w_0 + K_3) = 0.5f(0.5, 1.4453125) = 1.09765625 \\
w_1 &= w_0 + (k_1 + 2k_2 + 2k_3 + k_4)/6 = 1.425130208333333
\end{aligned}$$

Step 2 $t_2 = 1$

$$\begin{aligned}
k_1 &= hf(t_1, w_1) = 0.5f(0.5, 1.425130208333333) = 1.087565104166667 \\
k_2 &= hf(t_1 + h/2, w_1 + k_1/2) = 0.5f(0.75, 1.968912760416667) = 1.203206380208333 \\
K_3 &= hf(t_1 + h/2, w_1 + k_2/2) = 0.5f(0.75, 2.0267333984375) = 1.23211669921875 \\
K_4 &= hf(t_1 + h, w_1 + K_3) = 0.5f(1, 2.657246907552083) = 1.328623453776042 \\
w_2 &= w_1 + (k_1 + 2k_2 + 2k_3 + k_4)/6 = 2.639602661132812
\end{aligned}$$

Step 3 $t_3 = 1.5$

$$k_1 = hf(t_2, w_2) = 0.5f(1, 2.639602661132812) = 1.319801330566406$$
$$k_2 = hf(t_2 + h/2, w_2 + k_1/2) = 0.5f(1.25, 3.299503326416016) = 1.368501663208008$$
$$K_3 = hf(t_2 + h/2, w_2 + k_2/2) = 0.5f(1.25, 3.323853492736816) = 1.380676746368408$$
$$K_4 = hf(t_2 + h, w_2 + K_3) = 0.5f(1.5, 4.020279407501221) = 1.385139703750610$$
$$w_3 = w_2 + (k_1 + 2k_2 + 2k_3 + k_4)/6 = 4.006818970044454$$

Step 4 $t_4 = 2$

$$k_1 = hf(t_3, w_3) = 0.5f(1.5, 4.006818970044454) = 1.378409485022227$$
$$k_2 = hf(t_3 + h/2, w_3 + k_1/2) = 0.5f(1.75, 4.696023712555567) = 1.316761856277783$$
$$K_3 = hf(t_3 + h/2, w_3 + k_2/2) = 0.5f(1.75, 4.665199898183346) = 1.301349949091673$$
$$K_4 = hf(t_3 + h, w_3 + K_3) = 0.5f(2, 5.308168919136127) = 1.154084459568063$$
$$w_4 = w_3 + (k_1 + 2k_2 + 2k_3 + k_4)/6 = 5.301605229265987$$

Now let's compare what we got with the exact solution

| $t_i$ | Exact solution $y(t_i)$ | Numerical solution $w_i$ | Error $\lvert w_i - y(t_i)\rvert$ |
|---|---|---|---|
| 0.0 | 0.5 | 0.5 | 0 |
| 0.5 | 1.425639364649936 | 1.425130208333333 | 0.000509156316603 |
| 1.0 | 2.640859085770477 | 2.639602661132812 | 0.001256424637665 |
| 1.5 | 4.009155464830968 | 4.006818970044454 | 0.002336494786515 |
| 2.0 | 5.305471950534675 | 5.301605229265987 | 0.003866721268688 |