state

- State definition
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- **State-Space Representation of nth-Order Systems of Linear Differential Equations in which the Forcing Function Involves Derivative Terms**
- **Transformation from State Space Representation to Transfer Function with MATLAB**

State:

The state of a dynamic system is the smallest set of variables (called *state variables*) such that knowledge of these variables at , together with knowledge of the input for , completely determines the behaviour of the system for any time

The concept of state is by no means limited to physical systems. It is applicable to biological systems, economic systems, social systems, and others.

State variables

The state variables of a dynamic system are the variables making up the smallest set of variables that determine the state of the dynamic system. If at least variables are needed to completely describe the behaviour of a dynamic system, then such variables are a set of state variables.

Dynamic systems:

- A controlled dynamic system in continuous time can in the simplest case be described by an ordinary differential equation (ODE) on a time interval by:

$$\dot{\boldsymbol{x}}(\boldsymbol{t})=\boldsymbol{f}\left(\boldsymbol{x}(\boldsymbol{t}),\boldsymbol{u}(\boldsymbol{t}),\boldsymbol{t}\right),\text{all}\quad t\in\left[t_{init},t_{fin}\right]$$

where is the time

 are the controls

is the state.

- The function f is a map from states, controls, and time to the rate of change of the state.
- Due to the explicit time dependence of the function f, this is a time-variant system.

Dynamic systems:

- We identify dynamic systems with processes that are evolving with time and that can be characterized by states that allow us to predict the future behaviour of the system.

- We might think of an electric train where the state consists of the current position and velocity, and where the control is the engine power that the train driver can choose at each moment.
- A typical property of a dynamic system is that knowledge of an initial state and a control input trajectory for all allows one to determine the whole state trajectory for .
- As the motion of a train can very well be modelled by Newton's laws of motion, the usual description of this dynamic system is deterministic and in continuous time and with continuous states.
- A dynamic system can be controlled by a suitable choice of inputs that we denote as controls
- If the state is not known, we first need to estimate it based on the available measurement information (design observer – week 9/10)

State-Space Equations

a multiple-input, multiple-output system (MIMO) that involves state variables:

inputs

outputs

state variables:

System description:

$$\begin{aligned}
\dot{x}_1(t) &= f_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\
\dot{x}_2(t) &= f_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\
&\cdot \\
&\cdot \\
&\cdot \\
\dot{x}_n(t) &= f_n(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t)
\end{aligned}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{t})$$

: state vector(vector of the state variables)

: control signal

The outputs of the system may be given by

$$
\begin{aligned}
y_1(t) &= g_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\
y_2(t) &= g_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\
&\cdot \\
&\cdot \\
&\cdot \\
y_m(t) &= g_m(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t)
\end{aligned}
$$

If we define

$$
\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix}, \quad
\mathbf{f}(\mathbf{x}, \mathbf{u}, t) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ f_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ \cdot \\ \cdot \\ \cdot \\ f_n(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \end{bmatrix},
$$

$$
\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \cdot \\ \cdot \\ \cdot \\ y_m(t) \end{bmatrix}, \quad
\mathbf{g}(\mathbf{x}, \mathbf{u}, t) = \begin{bmatrix} g_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ g_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ \cdot \\ \cdot \\ \cdot \\ g_m(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \end{bmatrix}, \quad
\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ \cdot \\ \cdot \\ u_r(t) \end{bmatrix}
$$

$$\dot{\mathbf{x}} = \boldsymbol{f}(\mathbf{x}, \boldsymbol{u}, \boldsymbol{t})$$

State equation

$$\boldsymbol{y}(t) = \boldsymbol{g}(\mathbf{x}, \boldsymbol{u}, t)$$

output equation

- If vector functions **f** and/or **g** involve time explicitly, then the system is called a *time varying system*.
- If **f** and/or **g** are linearized about the operating state, then we have the following linearized state equation and output equation:

$$\boxed{\dot{\mathbf{x}} = \boldsymbol{A}(\boldsymbol{t})\mathbf{x}(\boldsymbol{t}) + \boldsymbol{B}(\boldsymbol{t})\boldsymbol{u}(\boldsymbol{t})}$$

State equation

$$\boxed{\boldsymbol{y} = \boldsymbol{C}\mathbf{x}(\boldsymbol{t}) + \boldsymbol{D}\boldsymbol{u}(\boldsymbol{t})}$$

output equation

$C \in \mathbb{R}^{m\times n}$ $D \in \mathbb{R}^{m\times r}$ $y \in \mathbb{R}^{m}$

LTI vs LTV dynamical equation for linear systems:

State equation: $\boxed{\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{B}(\boldsymbol{t})\boldsymbol{u}(\boldsymbol{t})}$

Output equation: $\boxed{\boldsymbol{y} = \boldsymbol{C}(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{D}(\boldsymbol{t})\boldsymbol{u}(\boldsymbol{t})}$

Linear Time-varying (LT) dynamical systems

State equation:

and are constant matrices

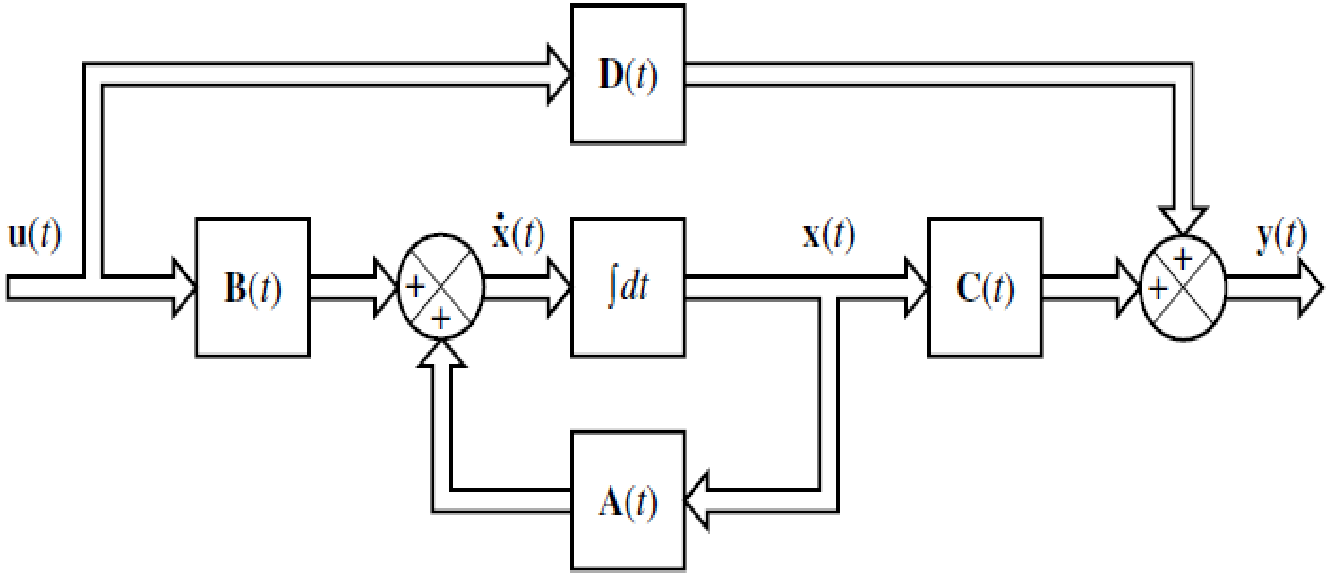
Output equation: $\boxed{\boldsymbol{y} = \boldsymbol{C}\,\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{D}\,\boldsymbol{u}(\boldsymbol{t})}$

Linear Time-invariant (LTI) dynamical systems

Block diagram of the linear, continuous time control system represented in state space

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\,\mathbf{x}(t) + \boldsymbol{B}(t)\,\boldsymbol{u}(t)$$

$$\boldsymbol{y} = \boldsymbol{C}(t)\,\mathbf{x}(t) + \boldsymbol{D}(t)\,\boldsymbol{u}(t)$$



The outputs of integrators serve as state variables

The number of *state variables* to completely define the dynamics of the system is **equal** to the number of *integrators* involved in the system.

Correlation Between Transfer Functions and State-Space Equations.

Transfer function

In Laplace format

$$y(s) = G(s)u(s)$$

In time format

[redacted]

Transfer-function expression of the system in terms of **A**, **B**, **C**, and D.

State-space (modern) control [redacted] Classical control [redacted]

$$\dot{x} = Ax(t) + Bu(t) \overset{\mathcal{L}}{\Rightarrow} sX(s) - \underbrace{\cancel{x(0)}}_{\mathbf{0}} = AX(s) + BU(s)$$

$$y = Cx(t) + Du(t) \overset{\mathcal{L}}{\Rightarrow} Y(s) = CX(s) + DU(s)$$

the transfer function was previously defined as the ratio of the Laplace transform of the output to the Laplace transform of the input when the initial conditions were zero

$$sX(s) = AX(s) + BU(s)$$

$$\boxed{(sI - A)X(s) = BU(s)}$$

By premultiplying to both sides of

$$Y(s) = CX(s) + DU(s) \longrightarrow Y(s) = \{C(sI - A)^{-1}B + D\}U(s)$$

$$\boxed{\boldsymbol{G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D}}$$

Transfer-function expression of MIMO system in terms of **A**, **B**, **C**, and D.

Outputs

inputs

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}_{m \times 1} \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}_{r \times 1}$$

$$\boxed{\boldsymbol{G}(\boldsymbol{s})_{m \times r} = \frac{\boldsymbol{Y}(\boldsymbol{s})_{m \times 1}}{\boldsymbol{U}(\boldsymbol{s})_{r \times 1}} = \boldsymbol{C}_{m \times \boldsymbol{n}} (\boldsymbol{sI} - \boldsymbol{A})^{-\boldsymbol{1}}_{n \times \boldsymbol{n}} \boldsymbol{B}_{n \times \boldsymbol{r}} + \boldsymbol{D}_{m \times \boldsymbol{r}}}$$

Exp. Consider the system define

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

This system involves two inputs and two outputs

Four transfer functions $\dfrac{Y_1(s)}{U_1(s)} \quad \dfrac{Y_2(s)}{U_1(s)} \quad \dfrac{Y_1(s)}{U_2(s)} \quad \dfrac{Y_2(s)}{U_2(s)}$

Answer.

$$\frac{Y_1(s)}{U_1(s)} = \frac{s+4}{s^2+4s+25}, \qquad \frac{Y_2(s)}{U_1(s)} = \frac{-25}{s^2+4s+25}$$

$$\frac{Y_1(s)}{U_2(s)} = \frac{s+5}{s^2+4s+25}, \qquad \frac{Y_2(s)}{U_2(s)} = \frac{s-25}{s^2+4s+25}$$

State-Space Representation of *n*th-Order Systems of Linear Differential Equations in which the Forcing Function Does Not Involve Derivative Terms.

$$\overset{(n)}{y} + a_1\overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = u$$

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$\vdots$$
$$x_n = \overset{(n-1)}{y}$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n$$
$$\dot{x}_n = -a_n x_1 - \cdots - a_1 x_n + u$$

or

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

Note that the state-space representation for the transfer function system

$$\frac{Y(s)}{U(s)} = \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

is given also by

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = u$$

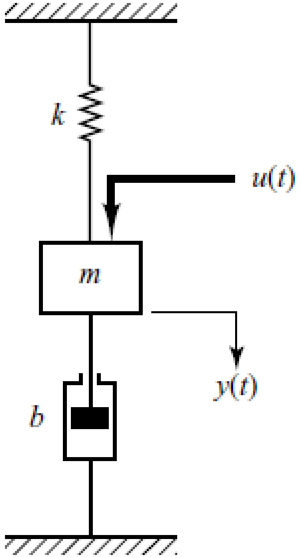
$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\vdots$$

$$\dot{x}_n = u - a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n$$

Exp 1. Deriving a state equation and output equation.



We assume that the system is **linear**

external force → input
displacement of the mass → output

- This system is a single-input, single-output system.

From the diagram, the system equation is

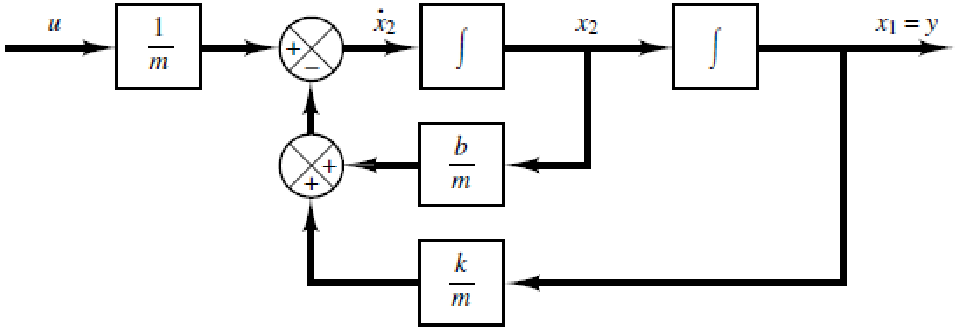
$$m\ddot{y}+b\dot{y}+ky=u$$

We will learn later how to obtain differential equations for the different physical systems

This system is of second order ➡ involves two integrators ➡ state variables and

$$x_1(t) = y(t) \quad \Rightarrow \quad \dot{x}_1 = x_2$$

$$x_2(t) = \dot{y}(t) \quad \Rightarrow \quad \dot{x}_2 = \ddot{y} = -\frac{K}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$



The output equation is $y = x_1$

In a vector-matrix form:

State equation:

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{K}{m} & -\frac{b}{m} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{\mathbf{B}} \boldsymbol{u}$$

output equation:

$$D = 0$$

Exp 2. obtain the transfer function for the system from the state-space equations of **Exp 1**.

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

$$= [1 \quad 0]\left\{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix}\right\}^{-1}\begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} + 0$$

$$= [1 \quad 0]\begin{bmatrix} s & -1 \\ \dfrac{k}{m} & s + \dfrac{b}{m} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix}$$

Note that $\begin{bmatrix} s & -1 \\ \dfrac{k}{m} & s + \dfrac{b}{m} \end{bmatrix}^{-1} = \dfrac{1}{s^2 + \dfrac{b}{m}s + \dfrac{k}{m}}\begin{bmatrix} s + \dfrac{b}{m} & 1 \\ -\dfrac{k}{m} & s \end{bmatrix}$

$$G(s) = [1 \quad 0]\frac{1}{s^2 + \dfrac{b}{m}s + \dfrac{k}{m}}\begin{bmatrix} s + \dfrac{b}{m} & 1 \\ -\dfrac{k}{m} & s \end{bmatrix}\begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix}$$

$$= \frac{1}{ms^2 + bs + k}$$

State-Space Representation of nth-Order Systems of Linear Differential Equations in which the Forcing Function Involves Derivative Terms.

Consider the differential equation system that involves derivatives of the forcing function

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u \qquad (1)$$

- The main problem in defining the state variables for this case lies in the derivative terms of the input
- The state variables must be chosen such that they will eliminate the derivatives of in the state equation.

Solution.

define the following variables as a set of state

$$x_1 = y - \beta_0 u$$

$$x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u$$

$$x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u$$

$$\vdots$$

$$x_n = \overset{(n-1)}{y} - \beta_0 \overset{(n-1)}{u} - \beta_1 \overset{(n-2)}{u} - \cdots - \beta_{n-2} \dot{u} - \beta_{n-1} u = \dot{x}_{n-1} - \beta_{n-1} u$$

where are determined from

$$\beta_0 = b_0$$

$$\beta_1 = b_1 - a_1 \beta_0$$

$$\beta_2 = b_2 - a_1 \beta_1 - a_2 \beta_0$$

$$\beta_3 = b_3 - a_1 \beta_2 - a_2 \beta_1 - a_3 \beta_0$$

$$\vdots$$

$$\beta_{n-1} = b_{n-1} - a_1 \beta_{n-2} - \cdots - a_{n-2} \beta_1 - a_{n-1} \beta_0$$

With the present choice of state variables, we obtain

$$\dot{x}_1 = x_2 + \beta_1 u$$
$$\dot{x}_2 = x_3 + \beta_2 u$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n + \beta_{n-1} u$$
$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + \beta_n u$$

where β_n is given by

$$\beta_n = b_n - a_1\beta_{n-1} - \cdots - a_{n-1}\beta_1 - a_{n-1}\beta_0$$

vector-matrix equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_{n-1} \\ \beta_n \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \beta_0 u$$

or

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x} + Du$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_{n-1} \\ \beta_n \end{bmatrix}, \qquad \mathbf{C} = [1 \quad 0 \quad \cdots \quad 0], \qquad D = \beta_0 = b_0$$

Matrices **A** and **C** are exactly the same as

The derivatives on the right-hand side of Equation affect only the elements of the **B** matrix.

Note that the state-space representation for the transfer function

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

$$\frac{Y(s)}{U(s)}=\frac{s}{s^3+14\,s^2+56\,s+160} \quad \Longrightarrow \quad \dddot{y}+14\,\ddot{y}+56\,\dot{y}+160\,y=\dot{u}$$

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u$$

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \beta_1 u \\
\dot{x}_2 &= x_3 + \beta_2 u \\
&\;\;\vdots \\
\dot{x}_{n-1} &= x_n + \beta_{n-1} u \\
\dot{x}_n &= -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + \beta_n u
\end{aligned}
$$

where β_n is given by

$$\beta_n = b_n - a_1\beta_{n-1} - \cdots - a_{n-1}\beta_1 - a_{n-1}\beta_0$$

Transformation from State Space Representation to Transfer Function with MATLAB

To obtain the transfer function from state-space equations, use the following command:

$$\boxed{[num,den] = ss2tf(A,B,C,D,iu)}$$

must be specified for systems with <u>more than one input</u>.
For example, if the system has three inputs then must be either 1, 2, or 3, where 1 implies , 2 implies , and 3 implies .

- If the system has only one input:

```
[num,den] = ss2tf(A,B,C,D)
```

or

```
[num,den] = ss2tf(A,B,C,D,1)
```

Exp. Obtain the transfer function of the system defined by the following state-space equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -25 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 25 \\ -120 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

```
A = [0   1   0;   0   0   1;   -5   -25   -5];
B = [0; 25; -120];
C = [1   0     0];
D = [0];
[num,den] = ss2tf(A,B,C,D)

num =

   0   0.0000   25.0000  5.0000

den

   1.0000   5.0000   25.0000  5.0000

% ***** The same result can be obtained by entering the following command: *****

[num,den] = ss2tf(A,B,C,D,1)

num =

   0   0.0000   25.0000  5.0000

den =

   1.0000   5.0000   25.0000  5.0000
```

$$\frac{Y(s)}{U(s)} = \frac{25s + 5}{s^3 + 5s^2 + 25s + 5}$$

Exp. Obtain the transfer function of the system defined by the following state-space equations by using MATLAB

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5.008 & -25.1026 & -5.03247 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 25.04 \\ -121.005 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \frac{25.04s + 5.008}{s^3 + 5.0325s^2 + 25.1026s + 5.008}$$

Exp. Consider the system defin $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

This system involves two inputs and two outputs

Four transfer functions $\frac{Y_1(s)}{U_1(s)} \quad \frac{Y_2(s)}{U_1(s)} \quad \frac{Y_1(s)}{U_2(s)} \quad \frac{Y_2(s)}{U_2(s)}$

- When considering input , we assume that input is zero and vice versa.

Answer.

$$\frac{Y_1(s)}{U_1(s)} = \frac{s+4}{s^2+4s+25}, \qquad \frac{Y_2(s)}{U_1(s)} = \frac{-25}{s^2+4s+25}$$

$$\frac{Y_1(s)}{U_2(s)} = \frac{s+5}{s^2+4s+25}, \qquad \frac{Y_2(s)}{U_2(s)} = \frac{s-25}{s^2+4s+25}$$

```
A = [0  1;-25  -4];
B = [1  1;0  1];
C = [1  0;0  1];
D = [0  0;0  0];
[NUM,den] = ss2tf(A,B,C,D,1)

NUM =

     0    1     4
     0    0   -25

den =

     1   4   25

[NUM,den] = ss2tf(A,B,C,D,2)

NUM =

     0   1.0000     5.0000
     0   1.0000   -25.0000

den =

     1    4    25
```

$$\frac{Y_1(s)}{U_1(s)} = \frac{s+4}{s^2+4s+25}, \qquad \frac{Y_2(s)}{U_1(s)} = \frac{-25}{s^2+4s+25}$$

$$\frac{Y_1(s)}{U_2(s)} = \frac{s+5}{s^2+4s+25}, \qquad \frac{Y_2(s)}{U_2(s)} = \frac{s-25}{s^2+4s+25}$$

state space to Transfer function format:

Transfer function to state space format:

Is it possible? How?

Yes

$$G(s)=\frac{b_0 s^n+b_1 s^{n-1}+b_2 s^{n-2}+\cdots+b_n}{s^n+a_1 s^{n-1}+a_2 s^{n-2}+\cdots+a_n}$$

known $\boxed{A,B,C,D}$ *unknown*

order of the system

Note: The system under discussion is SISO, so we only have one transfer function

Note: The order of the system is the same as the number of state variables of the system

State equations from transfer function (Realization):

$$G(s) \xrightarrow{\text{Realization}} A, B, C, D$$

Note: We have infinite number of realizations (state-space representations) for the system ()

There are some specific forms called ***Canonical forms (Direct Realizations):***

Because they derive directly from the expanded polynomial form of the transfer functions

Canonical forms

- Controllable canonical form
- Observable canonical form

Controllable canonical form

$$G(s)=\frac{b_0 s^n+b_1 s^{n-1}+b_2 s^{n-2}+\cdots+b_n}{s^n+a_1 s^{n-1}+a_2 s^{n-2}+\cdots+a_n}=\frac{Y(s)}{U(s)}$$

$$u^{(i)}=\frac{d^i}{dt^i}(u)$$

$$y^{(j)}=\frac{d^j}{dt^j}(y)$$

$$\boldsymbol{C}=\begin{bmatrix} \boldsymbol{b_n}-\boldsymbol{a_n}\boldsymbol{b_0} & \boldsymbol{b_{n-1}}-\boldsymbol{a_{n-1}}\boldsymbol{b_0} & \cdots & \boldsymbol{b_1}-\boldsymbol{a_1}\boldsymbol{b_0} \end{bmatrix}_{\boldsymbol{1\times n}}$$

$$\boldsymbol{A}=\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1} \\ -a_n & -\boldsymbol{a_{n-1}} & -\boldsymbol{a_{n-2}} & \cdots & \cdots & \cdots & -\boldsymbol{a_1} \end{bmatrix}_{\boldsymbol{n\times n}} \boldsymbol{B}=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{\boldsymbol{n\times 1}}$$

$$D=[b_0]$$

Observable canonical form

$$G(s)=\frac{b_0 s^n+b_1 s^{n-1}+b_2 s^{n-2}+\cdots+b_n}{s^n+a_1 s^{n-1}+a_2 s^{n-2}+\cdots+a_n}=\frac{Y(s)}{U(s)}$$

$$A=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & \cdots & -a_n \\ 1 & 0 & 0 & 0 & \cdots & \cdots & -a_{n-1} \\ 0 & 1 & 0 & 0 & 0 & \cdots & -a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 & -a_1 \end{bmatrix}_{n\times n}$$

$$\boldsymbol{identity\ matrix:(n-1\times n-1)}$$

$$B=\begin{bmatrix} b_n-a_n b_0 \\ b_{n-1}-a_{n-1} b_0 \\ \vdots \\ b_1-a_1 b_0 \end{bmatrix}_{\boldsymbol{n\times 1}} \qquad C=\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}_{\boldsymbol{1\times n}}$$

$$D=[b_0]$$

Relationship between observable and controllable canonical form:

[redacted]

TRANSFORMATION OF MATHEMATICAL MODELS WITH MATLAB

MATLAB is quite useful to transform the system model from transfer function to state space, and vice versa

$$\frac{Y(s)}{U(s)} = \frac{\text{numerator polynomial in } s}{\text{denominator polynomial in } s} = \frac{\text{num}}{\text{den}}$$

$$\boxed{[A,B,C,D] = \text{tf2ss(num,den)}}$$

state-space representation for any system

- It is important to note that the state-space representation for any system is ***<u>not unique</u>***. There are many (infinitely many) state-space representations for the same system.
- The MATLAB command gives one possible such state-space representation.

Exp 2. Consider the transfer-function system

$$\frac{Y(s)}{U(s)}=\frac{s}{(s+10)(s^2+4s+16)}=\frac{s}{s^3+14s^2+56s+160}$$

Obtain the state-space representation for this system.

$$G(s)=\frac{b_0 s^n+b_1 s^{n-1}+b_2 s^{n-2}+\cdots+b_n}{s^n+a_1 s^{n-1}+a_2 s^{n-2}+\cdots+a_n}=\frac{Y(s)}{U(s)}$$

Controllable canonical form:

$$A_c=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix} \qquad B_c=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad C_c=\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \qquad D_c=[0]$$

observable canonical form:

$$A_o=A_c^T \Longrightarrow A_o=\begin{bmatrix} 0 & 0 & -160 \\ 1 & 0 & -56 \\ 0 & 1 & -14 \end{bmatrix} \qquad B_o=C_c^T \Longrightarrow B_o=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C_o=B_c^T \Longrightarrow C_o=\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \qquad D_c=D_o=[0]$$

$$\boldsymbol{A}=\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1} \\ -a_n & -\boldsymbol{a}_{\boldsymbol{n}-\mathbf{1}} & -\boldsymbol{a}_{\boldsymbol{n}-\mathbf{2}} & \cdots & \cdots & \cdots & -\boldsymbol{a}_{\mathbf{1}} \end{bmatrix}_{\boldsymbol{n}\times\boldsymbol{n}}$$

$$\frac{Y(s)}{U(s)}=\frac{s}{(s+10)(s^2+4s+16)}=\frac{s}{s^3+14s^2+56s+160}$$

$$G(s)=\frac{b_0 s^n+b_1 s^{n-1}+b_2 s^{n-2}+\cdots+b_n}{s^n+a_1 s^{n-1}+a_2 s^{n-2}+\cdots+a_n}=\frac{Y(s)}{U(s)}$$

$$\boldsymbol{C}=\begin{bmatrix} \boldsymbol{b}_{\boldsymbol{n}}-\boldsymbol{a}_{\boldsymbol{n}}\boldsymbol{b}_{\boldsymbol{0}} & \boldsymbol{b}_{\boldsymbol{n}-\boldsymbol{1}}-\boldsymbol{a}_{\boldsymbol{n}-\boldsymbol{1}}\boldsymbol{b}_{\boldsymbol{0}} & \cdots & \boldsymbol{b}_{\boldsymbol{1}}-\boldsymbol{a}_{\boldsymbol{1}}\boldsymbol{b}_{\boldsymbol{0}} \end{bmatrix}_{\boldsymbol{1}\times\boldsymbol{n}}$$

$$\boldsymbol{A}=\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1} \\ -a_n & -\boldsymbol{a}_{\boldsymbol{n}-\boldsymbol{1}} & -\boldsymbol{a}_{\boldsymbol{n}-\boldsymbol{2}} & \cdots & \cdots & \cdots & -\boldsymbol{a}_{\boldsymbol{1}} \end{bmatrix}_{\boldsymbol{n}\times\boldsymbol{n}} \boldsymbol{B}=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{\boldsymbol{n}\times\boldsymbol{1}} \qquad D=\begin{bmatrix} b_0 \end{bmatrix}$$

There are many (infinitely many) possible state-space representations for this system.

Another possible state-space representation (among infinitely many alternatives) is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -14 & -56 & -160 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

MATLAB Program 2–2

```
num = [1    0];
den = [1   14   56   160];
[A,B,C,D] = tf2ss(num,den)

A =

   -14   -56  -160
     1     0     0
     0     1     0

B =

     1
     0
     0

C =

     0     1     0

D =

     0
```

Another way (1):

$$\dddot{y}+14\ddot{y}+56\dot{y}+160y=\dot{u}$$

$$x_1=y \quad \Rightarrow \quad \dot{x}_1=x_2$$

$$x_2=\dot{y} \quad \Rightarrow \quad \dot{x}_2=x_3+u$$

$$x_3=\ddot{y}-u \quad \Rightarrow \quad \dot{x}_3=-160x_1-56x_2-14(x_3+u)$$

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-160 & -56 & -14\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\\1\\-14\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+[0]u$$

Another way

$$\frac{Y(s)}{U(s)}=\frac{s}{s^3+14s^2+56s+160}$$

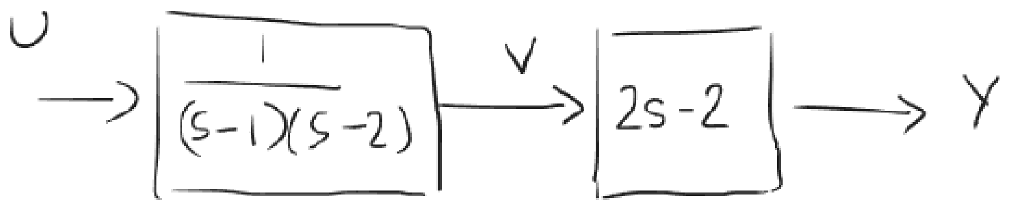
Problem – Find (by hand) the transfer function of the state-space system

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 \end{pmatrix}, D = (0)$$

ANS.

$$G(s) = \frac{2s - 2}{(s - 1)(s - 2)}$$

Example – find (by hand) a state-space form for the above transfer function. The s-term in the numerator can cause a slight problem. One way to approach it is via this block diagram



From the first block we can obtain the state equations as before, defining $x_1 = v, x_2 = \dot{v}$:

$$V = \frac{1}{(s-1)(s-2)}U$$

so

$$(s^2 - 3s + 2)V = U$$

$$\ddot{v} - 3\dot{v} + 2v = u$$

so

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -2x_1 + 3x_2 + u$$

and hence

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The second block tells us $y = -2v + 2\dot{v} = -2x_1 + 2x_2$, so

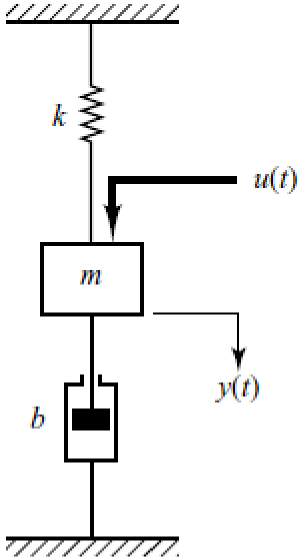
$$C = \begin{pmatrix} -2 & 2 \end{pmatrix}, D = (0)$$

Problem – check (by hand or using Matlab) that this state-space system has the expected transfer function.

Wrap up of approaches to present the system equation in terms of state-space equations

How to obtain state-space representation when you have n-th order differential equation (ODE):

- 1. We have a physical system (e.g. mass-spring-damper system) and we need to obtain the system equation in terms of differential equations (constant-coefficient ODEs)



(constant coefficient ODE)

- 2. Now you have system equation in terms of ODE so the next step is to define the ***state variables***:

$$m\,\ddot{z} + b\,\dot{z} + kz = ¿\,u$$

$$x_1 = z \quad \Rightarrow \quad \dot{x}_1 = x_2$$

$$x_2 = \dot{z} \quad \Rightarrow \quad \dot{x}_2 = \ddot{z} = -\frac{K}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

The idea in state-space representation of a system is to transform
 a n-th order differential equation to n 1st –order differential equation

- 3. The third step is to put the 1^{st}-order state equations in the state-space matrix representation: consider a single-input single-output (SISO) system:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \ddot{z} = -\frac{K}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

State equation:

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{\boldsymbol{x}}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{K}{m} & -\frac{b}{m} \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\boldsymbol{x}} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{\boldsymbol{B}} \boldsymbol{u}$$

Consider measuring the mass displacement

output equation:

$$D = 0$$

In general:

State-Space Representation of *n*th-Order Systems of Linear Differential Equations in which the Forcing Function Does Not Involve Derivative Terms.

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = u$$

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$\vdots$$
$$x_n = \overset{(n-1)}{y}$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n$$
$$\dot{x}_n = -a_n x_1 - \cdots - a_1 x_n + u$$

or

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

Note that the state-space representation for the transfer function system
is given also by

$$\frac{Y(s)}{U(s)} = \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$


What happens if we have
derivative in your input ?

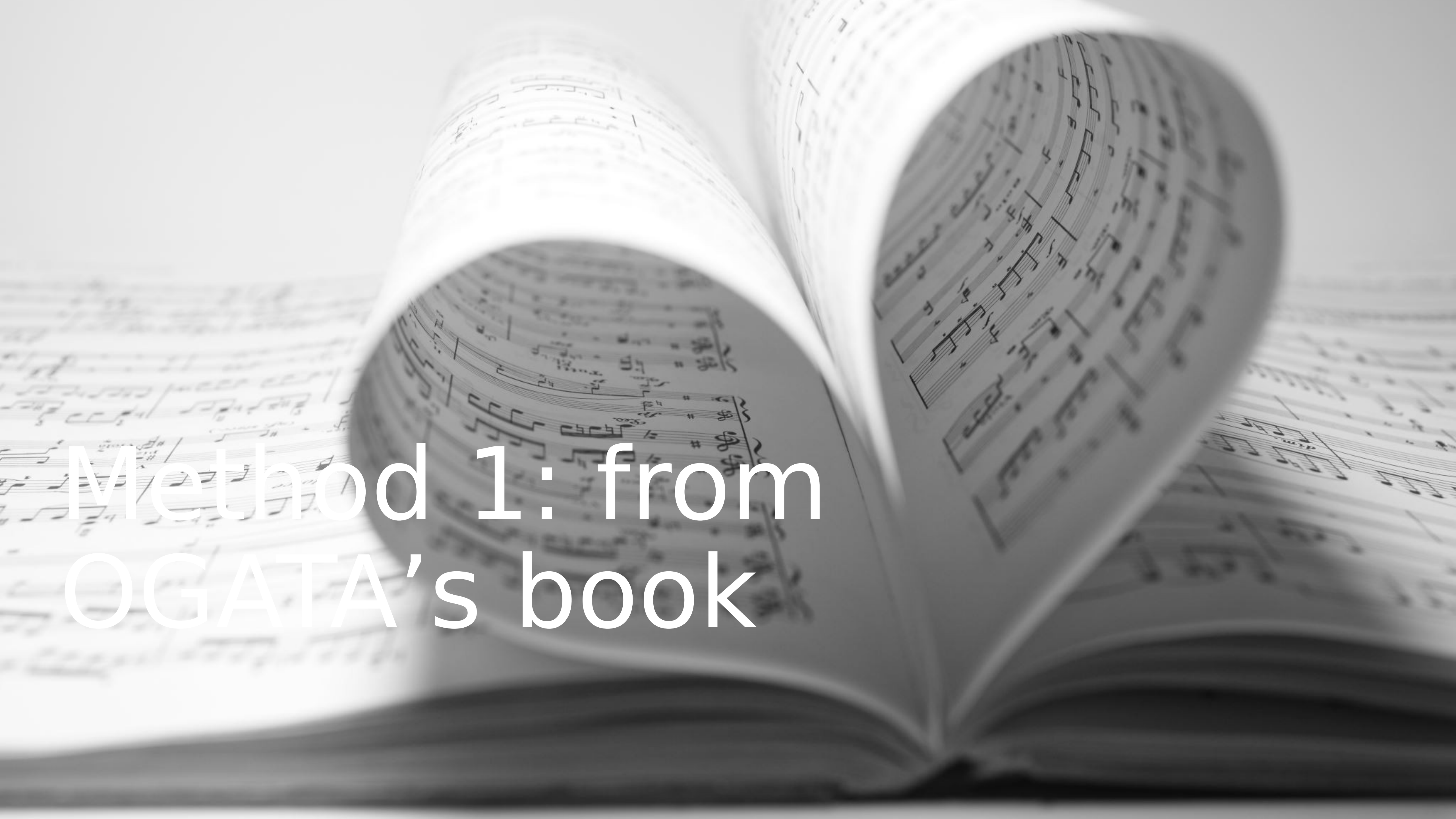
State-Space Representation of nth-Order Systems of Linear Differential Equations in which the Forcing Function Involves Derivative Terms.

Consider the differential equation system that involves derivatives of the forcing function

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u \qquad (1)$$

- The main problem in defining the state variables for this case lies in the derivative terms of the input
- The state variables must be chosen such that they will eliminate the derivatives of in the state equation.

Method 1: from OGATA’s book



Solution.

define the following variables as a set of state

$$x_1 = y - \beta_0 u$$
$$x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u$$
$$x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u$$
$$\vdots$$
$$x_n = \overset{(n-1)}{y} - \beta_0 \overset{(n-1)}{u} - \beta_1 \overset{(n-2)}{u} - \cdots - \beta_{n-2}\dot{u} - \beta_{n-1}u = \dot{x}_{n-1} - \beta_{n-1}u$$

where are determined from

$$\beta_0 = b_0$$
$$\beta_1 = b_1 - a_1\beta_0$$
$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0$$
$$\beta_3 = b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0$$
$$\vdots$$
$$\beta_{n-1} = b_{n-1} - a_1\beta_{n-2} - \cdots - a_{n-2}\beta_1 - a_{n-1}\beta_0$$

With the present choice of state variables, we obtain

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \beta_1 u \\
\dot{x}_2 &= x_3 + \beta_2 u \\
&\ \ \vdots \\
\dot{x}_{n-1} &= x_n + \beta_{n-1} u \\
\dot{x}_n &= -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + \beta_n u
\end{aligned}
$$

where β_n is given by

$$\beta_n = b_n - a_1\beta_{n-1} - \cdots - a_{n-1}\beta_1 - a_{n-1}\beta_0$$

vector-matrix equations

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix}
=
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot \\
0 & 0 & 0 & \cdots & 1 \\
-a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix}
+
\begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_{n-1} \\ \beta_n \end{bmatrix} u
$$

$$
y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \beta_0 u
$$

or

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x} + Du$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_{n-1} \\ \beta_n \end{bmatrix}, \qquad \mathbf{C} = [1 \quad 0 \quad \cdots \quad 0], \qquad D = \beta_0 = b_0$$

Matrices **A** and **C** are exactly the same as

The derivatives on the right-hand side of Equation affect only the elements of the **B** matrix.

Note that the state-space representation for the transfer function

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

$$\frac{Y(s)}{U(s)} = \frac{s}{s^3 + 14s^2 + 56s + 160} \Longrightarrow \dddot{y} + 14\ddot{y} + 56\dot{y} + 160y = \dot{u}$$

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u$$

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \beta_1 u \\
\dot{x}_2 &= x_3 + \beta_2 u \\
&\;\;\vdots \\
\dot{x}_{n-1} &= x_n + \beta_{n-1} u \\
\dot{x}_n &= -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + \beta_n u
\end{aligned}
$$

where β_n is given by

$$\beta_n = b_n - a_1\beta_{n-1} - \cdots - a_{n-1}\beta_1 - a_{n-1}\beta_0$$

Method 2 and 3: Controllable and observable canonical form (for both cases whether you have derivative of control input or not)

Controllable canonical form

$$u^{(i)} = \frac{d^i}{dt^i}(u)$$

$$y^{(j)} = \frac{d^j}{dt^j}(y)$$

$$C = \left[b_n - a_n b_0 \quad b_{n-1} - a_{n-1} b_0 \quad \cdots \quad b_1 - a_1 b_0 \right]_{1 \times n}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & \cdots & \cdots & -a_1 \end{bmatrix}_{n \times n} B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n \times 1}$$

$$D = \left[b_0 \right]$$

Observable canonical form

$$A=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & \cdots & -a_n \\ 1 & 0 & 0 & 0 & \cdots & \cdots & -a_{n-1} \\ 0 & 1 & 0 & 0 & 0 & \cdots & -a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 & -a_1 \end{bmatrix}_{n\times n}$$

$$\boldsymbol{identity\ matrix:}(\boldsymbol{n-1\times n-1})$$

$$B=\begin{bmatrix} b_n-a_nb_0 \\ b_{n-1}-a_{n-1}b_0 \\ \vdots \\ b_1-a_1b_0 \end{bmatrix}_{\boldsymbol{n\times 1}} \qquad C=\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}_{\boldsymbol{1\times n}}$$

$$D=\begin{bmatrix} b_0 \end{bmatrix}$$

Relationship between observable and controllable canonical form:

[redacted]

How to obtain state-space representation when you have transfer function:

Controllable canonical form

$$G(s)=\frac{b_0 s^n+b_1 s^{n-1}+b_2 s^{n-2}+\cdots+b_n}{s^n+a_1 s^{n-1}+a_2 s^{n-2}+\cdots+a_n}=\frac{Y(s)}{U(s)}$$

$$u^{(i)}=\frac{d^i}{dt^i}(u)$$

$$y^{(j)}=\frac{d^j}{dt^j}(y)$$

$$\boldsymbol{C}=\begin{bmatrix}\boldsymbol{b_n}-\boldsymbol{a_n}\boldsymbol{b_0} & \boldsymbol{b_{n-1}}-\boldsymbol{a_{n-1}}\boldsymbol{b_0} & \cdots & \boldsymbol{b_1}-\boldsymbol{a_1}\boldsymbol{b_0}\end{bmatrix}_{\boldsymbol{1\times n}}$$

$$\boldsymbol{A}=\begin{bmatrix}\mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \cdots & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1}\\ -a_n & -\boldsymbol{a_{n-1}} & -\boldsymbol{a_{n-2}} & \cdots & \cdots & \cdots & -\boldsymbol{a_1}\end{bmatrix}_{\boldsymbol{n\times n}} \boldsymbol{B}=\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}_{\boldsymbol{n\times 1}}$$

$$D=[b_0]$$

Observable canonical form

$$G(s)=\frac{b_0 s^n+b_1 s^{n-1}+b_2 s^{n-2}+\cdots+b_n}{s^n+a_1 s^{n-1}+a_2 s^{n-2}+\cdots+a_n}=\frac{Y(s)}{U(s)}$$

$$A=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & \cdots & -a_n \\ 1 & 0 & 0 & 0 & \cdots & \cdots & -a_{n-1} \\ 0 & 1 & 0 & 0 & 0 & \cdots & -a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 & -a_1 \end{bmatrix}_{n\times n}$$

$$\boldsymbol{identity\ matrix:(n-1\times n-1)}$$

$$B=\begin{bmatrix} b_n-a_n b_0 \\ b_{n-1}-a_{n-1} b_0 \\ \vdots \\ b_1-a_1 b_0 \end{bmatrix}_{\boldsymbol{n\times 1}} \qquad C=\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}_{\boldsymbol{1\times n}}$$

$$D=[b_0]$$

Relationship between observable and controllable canonical form:

Another way

$$\frac{Y(s)}{U(s)}=\frac{s}{s^3+14s^2+56s+160}$$