Question 1

(a) Answer TRUE or FALSE to the following statements

- A state space model consists of transfer functions and frequency response graphs. **FALSE**

[1 mark]

- The general form of the LTI state equation is $\boldsymbol{y} = \mathbf{C}\boldsymbol{x} + \mathbf{D}u$. **FALSE**

[1 mark]

(b) How many state equations are needed to represent a ninth-order system? **9**

[1 mark]

(c) Consider the system shown in Figure 1. Derive a mathematical model in state space form, showing both state space and output equations.

- Assume a linear spring, and a linear friction model $F_f = \mu\dot{x}$, where μ is a friction coefficient.
- The displacement *x(t)* is output.

[12 marks]

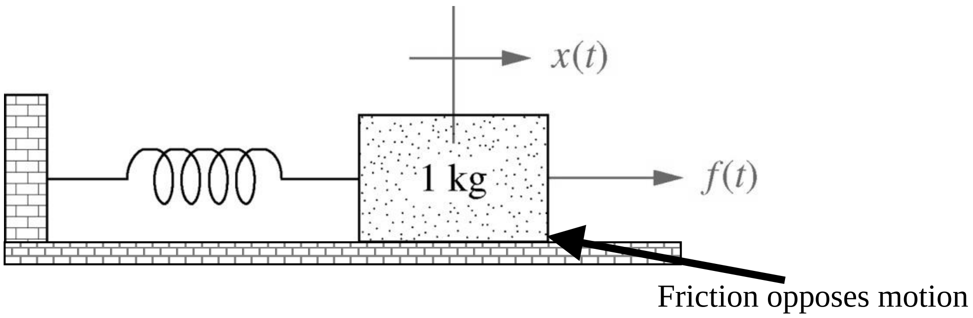
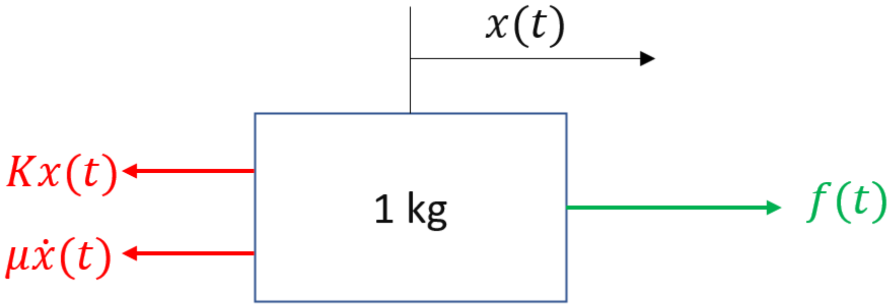


Figure 1: A Simple Mechanical System



$$\sum F_x = m\ddot{x}$$

$$f(t) - Kx(t) - \mu\dot{x}(t) = 1\ddot{x}(t)$$

$$\boxed{\ddot{x}(t) + \mu\dot{x}(t) + Kx(t) = f(t)}$$

2nd order differential equation → we need two states to describe the system behaviour

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$$

Because we have a 2nd order differential equation, we should have 2 first-order differential equations

$$x_1 = x(t) \rightarrow \dot{x}_1 = x_2$$
$$x_2 = \dot{x}(t) \rightarrow \dot{x}_2 = f(t) - Kx_1 - \mu x_2$$

The output is: $y = x(t) = x_1$

The state-space representation would be:

State equation: $\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{\vec{x}}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -K & -\mu \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} \underbrace{f}_{u}$

Output equation: $y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}}$

(d) Assume that friction coefficient μ=*0.1* Nm/s and spring stiffness *K*=200 N/m. Calculate the poles of the system and plot them on the *s*-plane. Is the system stable?

[10 marks]

Answer:

The poles of the system are the same as the eigenvalues of A

$$A = \begin{bmatrix} 0 & 1 \\ -K & -\mu \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -200 & -0.1 \end{bmatrix}$$

For finding the eigenvalues of A:

$$|\lambda I - A| = 0$$

$$\left|\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -200 & -0.1 \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} \lambda & -1 \\ 200 & \lambda + 0.1 \end{bmatrix}\right| = 0$$

$$\lambda^2 + 0.1\lambda + 200 = 0$$

$$\lambda_1 = -0.0500 + 14.1420i$$
$$\lambda_2 = -0.0500 - 14.1420i$$

Note: we can use the Matlab command $\boldsymbol{eig(A)}$ for calculating the eigenvalues of A

Because the real part of the both poles are negative, the system is stable.

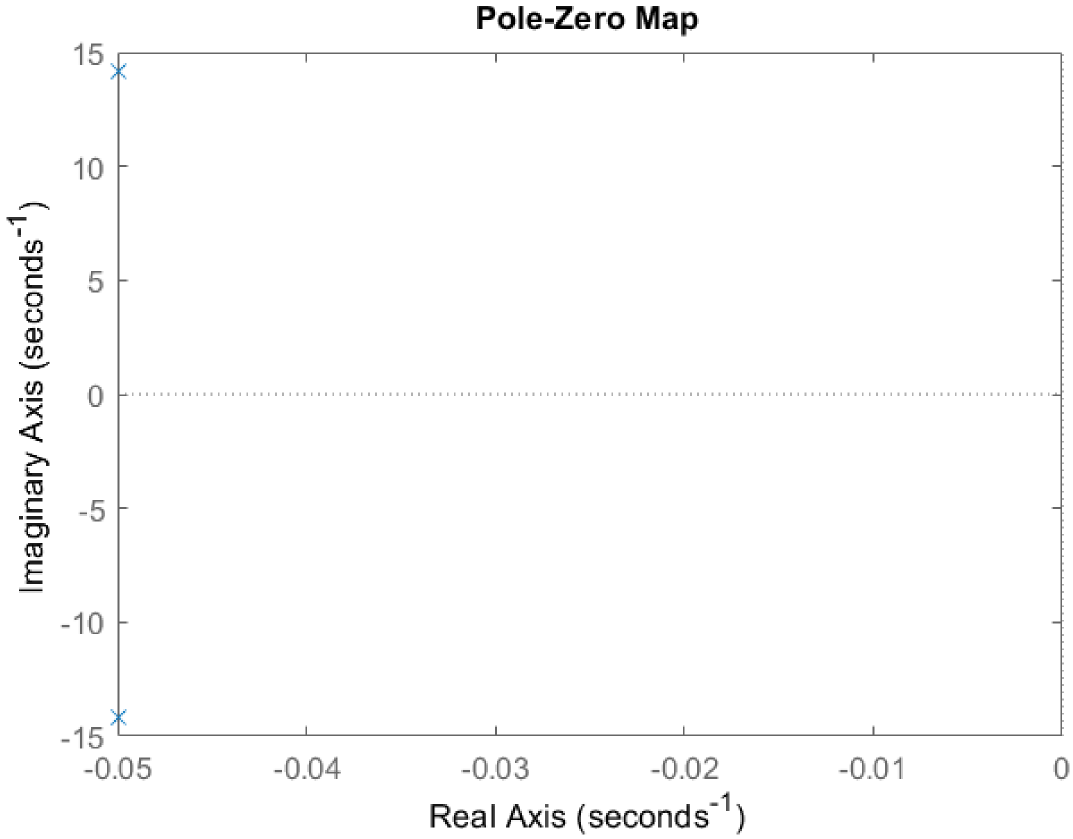
```
% System Modelling
A = [0 1;
   -200 -0.1];
B = [0; 1];
C = [1 0];    %Assume outputting displacement
D = 0;

sys_ol = ss(A,B,C,D);   %Open loop system
pol = eig(A);

%plots
close all
figure;
step(sys_ol);

figure;
```

```
pzmap(sys_ol);    %Pole zero map
```



Question 2

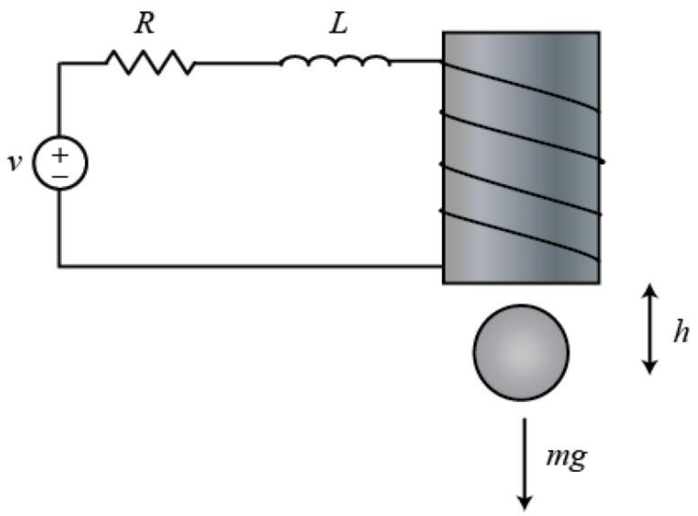


Figure 2: Diagram of a Magnetically Suspended Ball

(a) You are given the following model for a Magnetically Suspended Ball (shown in Figure 3).

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 980 & -2.8 \\ 0 & -100 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 100 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Where the states are displacement $x_1 = h$ and current $x_2 = i$. Control input is the voltage v(t). You are asked to design a controller for the system.

(i) check the stability of the open-loop system.

[5 marks]

Answer: Calculate the eigen values of A:

$$|\lambda I - A| = 0$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 980 & -2.8 \\ 0 & -100 \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} \lambda - 980 & 2.8 \\ 0 & \lambda + 100 \end{bmatrix} \right| = 0$$

$$(\lambda - 980)(\lambda + 100) - (2.8)(0) = 0$$

$$\lambda_1 = 980$$
$$\lambda_2 = -100$$

The open-loop system is ***unstable*** because it has a positive pole

(ii) Show whether the system is controllable.

[3 marks]

Answer: The condition for controllability is that the controllability matrix Q_c has full rank. In other words, $|Q_c| \neq 0$

$$Q_c = [B|AB]$$

$$Q_c = \begin{bmatrix} 0 & -280 \\ 100 & -10000 \end{bmatrix}$$

$$|Q_c| = 28000 \neq 0$$

The system is controllable and we can design a controller.

(iii) Design a regulator with the following characteristic equation, designed to hold the ball at the reference height h=0:

$$s^2 + 150s + 9000 = 0$$

[10 marks]

Answer:

We want to design the state-feedback control law for this system. For obtaining the state-feedback matrix gains, the characteristic equation must be the same as the given desired characteristic equation. In other words:

$$|sI - A_c| = s^2 + 150s + 9000$$

Here,

$$A_c = A - BK$$
$$K = [k_1 \quad k_2]$$

$$|sI - A + BK| = s^2 + 150s + 9000$$

$$\left|\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 980 & -2.8 \\ 0 & -100 \end{bmatrix} + \begin{bmatrix} 0 \\ 100 \end{bmatrix} [k_1 \quad k_2]\right| = s^2 + 150s + 9000$$

$$\left|\begin{bmatrix} s - 980 & 2.8 \\ 100k_1 & s + 100 + 100k_2 \end{bmatrix}\right| = s^2 + 150s + 9000$$

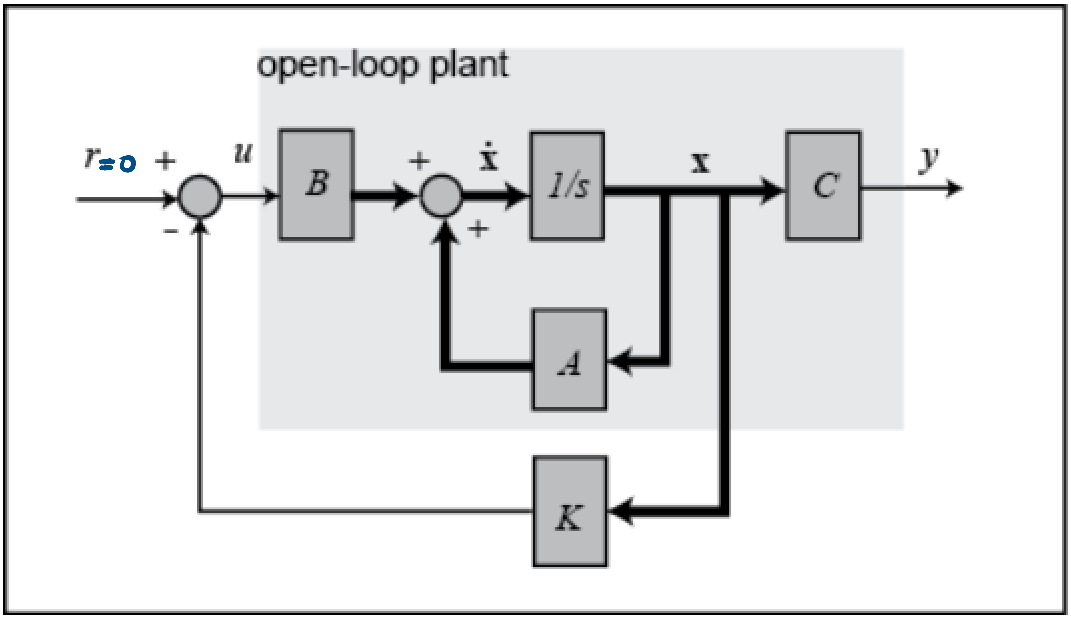
$$s^2 + (100k_2 - 880)s + (-280k_1 - 98000k_2 - 98000)$$
$$= s^2 + 150s + 9000$$

$$100k_2 - 880 = 150 \rightarrow k_2 = 10.3$$
$$-280k_1 - 98000k_2 - 98000 = 9000 \rightarrow k_1 = -3987.1$$

$$K = [-3987.1 \quad 10.3]$$

(b) Sketch the model as a Block Diagram, including the state feedback controller.

[7 marks]



Question 3

A System contains a 'v^2' damper, shown in **Error! Reference source not found.**. I nspecting the free body diagram and summing forces gives:

$$M\ddot{x} + f_D(t) = f(t)$$

Where damping force $f_D(t) = D\dot{x}^2(t)$, $D = 0.1$ Nm/s, and applied force $f(t) = 10 + \delta f(t)$

(a) Explain why this model is considered to be nonlinear.

[3 marks]

Answer: Because we have a nonlinear term ($\dot{x}^2$) in the system described by the second-order differential equation.

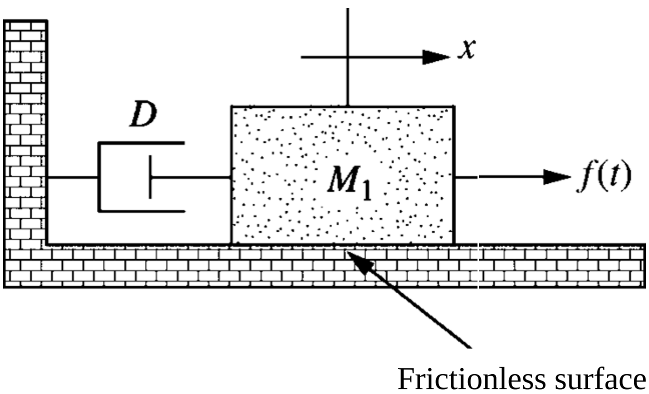


Figure 3: Nonlinear Translational Mechanical System

(b) Linearise the model about the equilibrium point.

[10 marks]

Answer:

$$f_D = D\dot{x}^2$$

$$M_1\ddot{x} + D\dot{x}^2 = f(t) \qquad (1)$$

(1) is a nonlinear 2nd order differential equation. We want to linearize around equilibrium point. So the equilibrium point would be our operating point (o) for linearization.

In the equilibrium point $\ddot{x_o} = 0$, so

$$f(t)_o = D\dot{x}_o^2 \qquad (2)$$

$f(t)_o$ is the static force to have the static equilibrium condition.

Now linearize the nonlinear term $(\dot{x}^2)$ around the operating point using Taylor Series Expansion:

$$\dot{x}^2 \approx \dot{x}_o^2 + 2\dot{x}_o(\dot{x} - \dot{x}_o)$$

By defining $\boldsymbol{\dot{\acute{x}} = \dot{x} - \dot{x}_o, \acute{f} = f - f_o}$

And also considering $\ddot{\acute{x}} = \ddot{x}$

the equation (1) would be:

$$M_1\ddot{\acute{x}} + D\left(\dot{x}_o^2 + 2\dot{x}_o\dot{\acute{x}}\right) = \acute{f} + f_o \qquad (3)$$

Substituting (2) into (3)

$$M_1\ddot{\acute{x}} + D\,\dot{x}_o^2 + 2D\dot{x}_o\dot{\acute{x}} = \acute{f} + D\dot{x}_o^2$$

Or simply

$$M_1\ddot{\acute{x}} + 2D\dot{x}_o\dot{\acute{x}} = \acute{f}$$

Which is a *linear* 2nd order differential equation describing the motion of the system around the equilibrium (operating) point.

(c) Write the linearised model in LTI state space format

[6 marks]

Answer:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \acute{x} \\ \dot{\acute{x}} \end{bmatrix}$$

Because we have a 2nd order differential equation, we should have 2 first-order differential equations

$$x_1 = \acute{x} \rightarrow \dot{x}_1 = x_2$$
$$x_2 = \dot{\acute{x}} \rightarrow \dot{x}_2 = \frac{1}{M_1}(\acute{f} - 2D\dot{x}_o x_2)$$

State equation: $\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\frac{2D\dot{x}_o}{M_1} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M_1} \end{bmatrix}}_{B} \underbrace{\acute{f}}_{u}$

In this question: $D = 0.1$ and $f(t) = 10 + \delta f(t)$

Where $\acute{f} = \delta f(t) = f(t) - 10$

From (2):

$$f(t)_o = D\dot{x}_o^2$$

Or

$$10 = 0.1\dot{x}_o^2$$
$$\dot{x}_o^2 = 100 \rightarrow \dot{x}_o = 10$$

So the state equation would be like:

State equation: $\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\frac{2}{M_1} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M_1} \end{bmatrix}}_{B} \underbrace{\acute{f}}_{u} \qquad (*)$

(i) Can this linearised model be used to assess the system when relatively small forces are applied, causing small deflections of ± 5mm from the equilibrium point? Explain your answer.

[3 marks]

Answer: This linearized model is only representative of the system in the small vicinity of the equilibrium (operating) point. So yes, for the small excursions around the operating point the linearized model is representative of the whole system.

(ii) Can this linearised model be used to assess the system when a large impact is applied, causing the damper to fully compress and extend? Explain your answer.

[3 marks]

Answer: This linearized model $(*)$ cannot describe the motion of the system when a large force applied to the system. The linearized model is only valid for excursions around 10 N.