State-Space Control (EGR3032)

Week 2

State:

The state of a dynamic system is the smallest set of variables (called *state variables*) such that knowledge of these variables at $t = t_0$, together with knowledge of the input for $t \geq t_0$, completely determines the behaviour of the system for any time $t \geq t_0$

The concept of state is by no means limited to physical systems. It is applicable to biological systems, economic systems, social systems, and others.

State variables

The state variables of a dynamic system are the variables making up the smallest set of variables that determine the state of the dynamic system. If at least n variables x_1 x_2 , … , x_n are needed to completely describe the behaviour of a dynamic system, then such n variables are a set of state variables.

Dynamic systems:

- A controlled dynamic system in continuous time can in the simplest case be described by an ordinary differential equation (ODE) on a time interval $[t_{init}\ ,\ t_{fin}]$ by:

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{f}(\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{u}(\boldsymbol{t}), \boldsymbol{t}) \quad \text{, all } t \in [t_{init}\ ,\ t_{fin}]$$

System state: ?

where $t \in \mathbb{R}^{+}$ is the time

$u(t) \in \mathbb{R}^{r}$ are the controls

$x(t) \in \mathbb{R}^{n}$ is the state.

Control input: ?

- The function $f(t)$ is a map from states, controls, and time to the rate of change of the state.
- Due to the explicit time dependence of the function f, this is a time-variant system.

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Dynamic systems:

- We identify dynamic systems with processes that are evolving with time and that can be characterized by states x that allow us to predict the future behaviour of the system.

- We might think of an electric train where the state x consists of the current ***position*** and ***velocity***, and where the control u is the engine power that the train driver can choose at each moment.

- A typical property of a dynamic system is that knowledge of an initial state x_{init} and a control input trajectory $u(t)$ for all $t \in [t_{init}, t_{fin}]$ allows one to determine the whole state trajectory $x(t)$ for $t \in [t_{init}, t_{fin}]$.

- As the motion of a train can very well be modelled by Newton's laws of motion, the usual description of this dynamic system is deterministic and in continuous time and with continuous states.

- A dynamic system can be controlled by a suitable choice of inputs that we denote as controls u

- If the state is not known, we first need to estimate it based on the available measurement information (design observer – week 9/10)

State-Space Equations

a multiple-input, multiple-output system (MIMO) that involves n state variables:

r inputs $u_1(t), u_2(t), \; \dots \; , u_r(t)$

m outputs $y_1(t) \; y_2(t), \; \dots, y_m(t)$

n state variables: $x_1(t), x_2(t), \dots \; , x_n(t)$

System description:

$$\left.\begin{aligned} \dot{x}_1(t) &= f_1(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \\ \dot{x}_2(t) &= f_2(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \\ &\;\;\vdots \\ \dot{x}_n(t) &= f_n(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \end{aligned}\right\} \quad \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{t}) \qquad x \in \mathbb{R}^n \quad u \in \mathbb{R}^r \quad t \in \mathbb{R}^+$$

x: state vector(vector of the state variables)

u: control signal

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The outputs $y_1(t), y_2(t), \dots, y_m(t)$ of the system may be given by

$$\begin{aligned}
y_1(t) &= g_1(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \\
y_2(t) &= g_2(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \\
&\cdot \\
&\cdot \\
&\cdot \\
y_m(t) &= g_m(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t)
\end{aligned}$$

If we define

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix}, \quad \mathbf{f}(\mathbf{x}, \mathbf{u}, t) = \begin{bmatrix} f_1(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \\ f_2(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \\ \cdot \\ \cdot \\ \cdot \\ f_n(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \end{bmatrix},$$

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \cdot \\ \cdot \\ \cdot \\ y_m(t) \end{bmatrix}, \quad \mathbf{g}(\mathbf{x}, \mathbf{u}, t) = \begin{bmatrix} g_1(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \\ g_2(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \\ \cdot \\ \cdot \\ \cdot \\ g_m(x_1, x_2, \dots, x_n; u_1, u_2, \dots, u_r; t) \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ \cdot \\ \cdot \\ u_r(t) \end{bmatrix}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{t})$$

State equation

$$\boldsymbol{y}(t) = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, t)$$

output equation

- If vector functions **f** and/or **g** involve time t explicitly, then the system is called a *time varying system*.
- If **f** and/or **g** are linearized about the operating state, then we have the following linearized state equation and output equation:

$$\boxed{\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{B}(\boldsymbol{t})\boldsymbol{u}(\boldsymbol{t})}$$

State equation $A \in \mathbb{R}^{n\times n}$ $B \in \mathbb{R}^{n\times r}$ $u \in \mathbb{R}^{r}$ $x \in \mathbb{R}^{n}$

$$\boxed{\boldsymbol{y} = \boldsymbol{C}\,\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{D}\,\boldsymbol{u}(\boldsymbol{t})}$$

output equation $C \in \mathbb{R}^{m\times n}$ $D \in \mathbb{R}^{m\times r}$ $y \in \mathbb{R}^{m}$

LTI vs LTV dynamical equation for linear systems:

State equation: $\dot{x} = A(t)x(t) + B(t)u(t)$

Output equation: $y = C(t)\,x(t) + D(t)\,u(t)$

Linear Time-varying (LT) dynamical systems

State equation: $\dot{x} = Ax(t) + Bu(t)$

A **and** B **are constant matrices**

Output equation: $y = C\,x(t) + D\,u(t)$

Linear Time-invariant (LTI) dynamical systems

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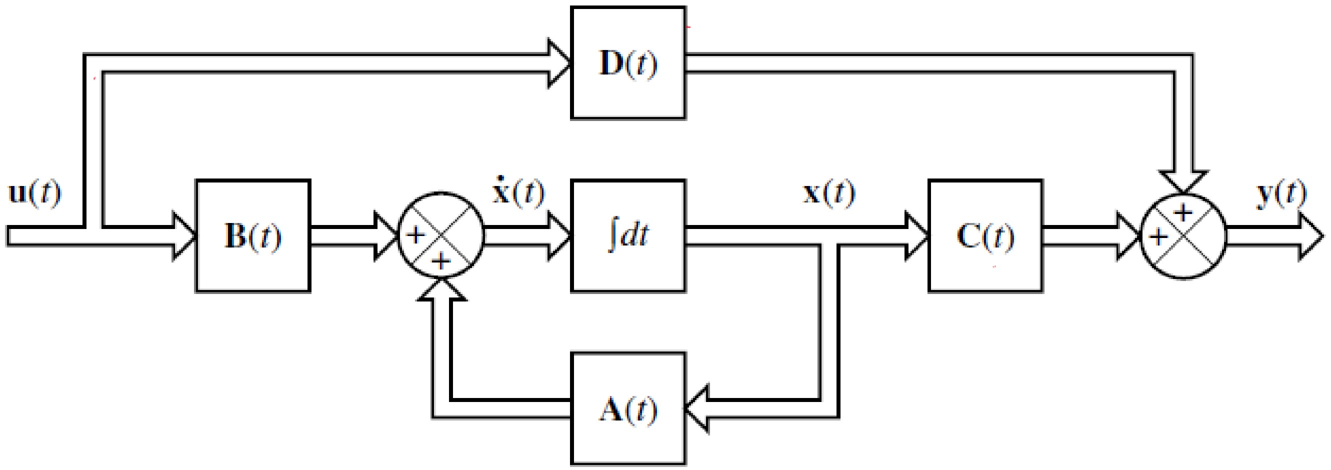
Block diagram of the linear, continuous time control system represented in state space

$$\dot{x} = A(t)x(t) + B(t)u(t)$$

$$y = C(t)\,x(t) + D(t)\,u(t)$$

The outputs of integrators serve as state variables

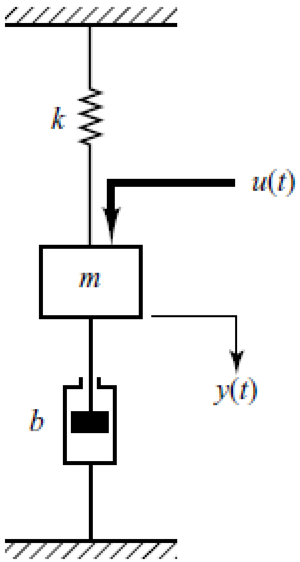


The number of *state variables* to completely define the dynamics of the system is **equal** to the number of *integrators* involved in the system.

Exp 1. Deriving a state equation and output equation.

We assume that the system is **linear**

external force $u(t)$ → input
displacement $y(t)$ of the mass → output



- This system is a single-input, single-output system.

From the diagram, the system equation is

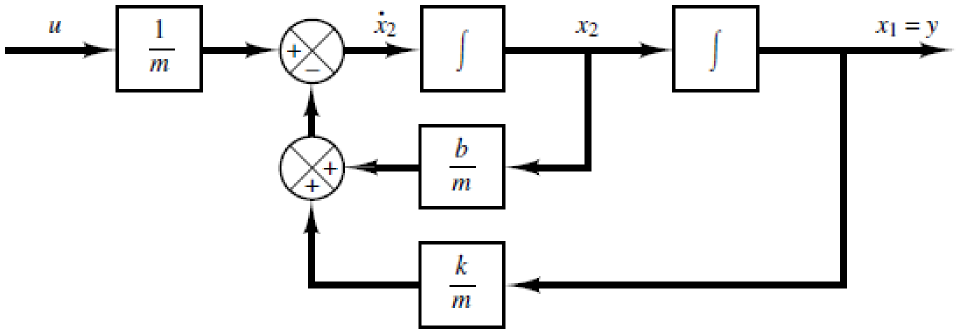
$$m\ddot{y} + b\dot{y} + ky = u$$

We will learn later how to obtain differential equations for the different physical systems

This system is of second order → involves two integrators → state variables $x_1(t)$ and $x_2(t)$

$$x_1(t) = y(t) \quad \Rightarrow \quad \dot{x}_1 = x_2$$

$$x_2(t) = \dot{y}(t) \quad \Rightarrow \quad \dot{x}_2 = \ddot{y} = -\frac{K}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$



The output equation is $y = x_1$

In a vector-matrix form:

State equation: $$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{\boldsymbol{x}}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{K}{m} & -\frac{b}{m} \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\boldsymbol{x}} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{\boldsymbol{B}} \boldsymbol{u}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{B}\boldsymbol{u}(\boldsymbol{t})$$

$$\boldsymbol{y} = \boldsymbol{C}\,\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{D}\,\boldsymbol{u}(\boldsymbol{t})$$

output equation: $$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\boldsymbol{C}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\boldsymbol{x}} \qquad D = 0$$

LINEARIZATION OF NONLINEAR MATHEMATICAL MODELS

Recap Example:

- Additivity $f(x_1 + x_2) = f(x_1) + f(x_2)$
- Homogeneity $f(ax) = af(x)$

Nonlinear Systems:

A system is nonlinear if the ***principle of superposition*** does not apply

- Many physical systems are linear within some range of the variables.
- In general, systems ultimately become nonlinear as the variables are increased without limit.

many electromechanical systems, hydraulic systems, pneumatic systems, and so on, involve ***nonlinear*** relationships among the variables.

Nonlinearity cases:

- The spring-mass-damper system is linear as long as the mass is subjected to small deflections $y(t)$. If $y(t)$ were continually increased, eventually the spring would be overextended and break.

the question of *linearity* and the *range of applicability* must be considered for each system.

- The output of a component may **saturate** for large input signals.

- ***Square-law nonlinearity*** may occur in some components.

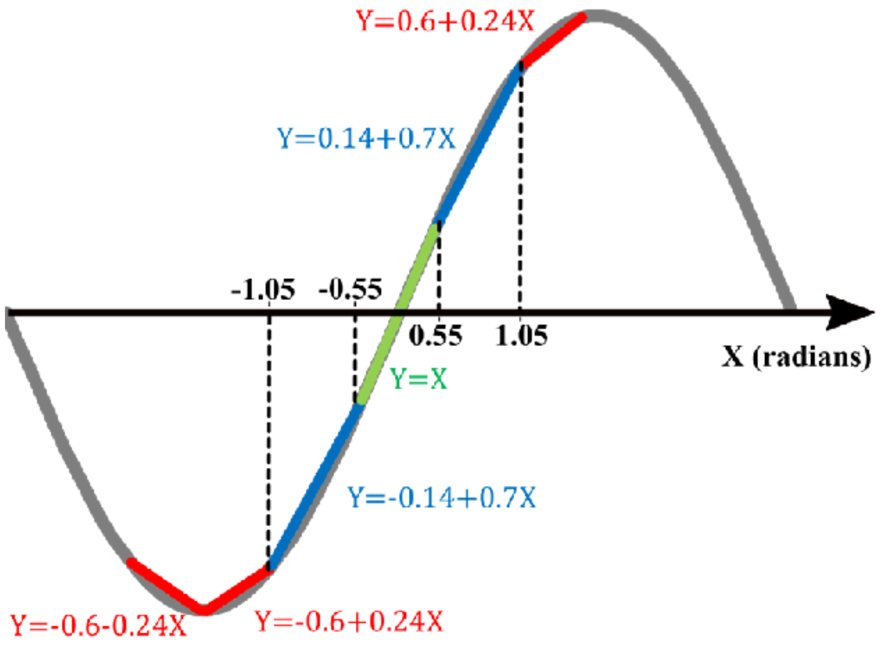
For instance, dampers used in physical systems may be linear for low-velocity operations but may become nonlinear at high velocities, and the damping force may become proportional to the square of the operating velocity.

Linearization of Nonlinear Systems

If the system operates around an equilibrium point and if the signals involved are small signals, then it is possible to approximate the nonlinear system by a linear system

Such a linear system is equivalent to the nonlinear system considered within a limited operating range

Such a linearized model (linear, time-invariant model) is very important in control engineering



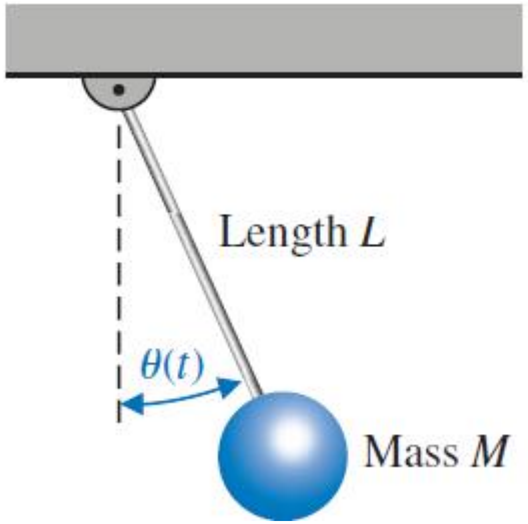
Linear Approximation of Nonlinear Mathematical Models.

To obtain a linear mathematical model for a nonlinear system, we assume that the variables deviate **only slightly** from some operating condition.

Consider a system whose input is $x(t)$ and output is $y(t)$.
The relationship between $y(t)$ and $x(t)$ is given by:

$$y = f(x) \qquad (*)$$

Example: Pendulum oscillator model

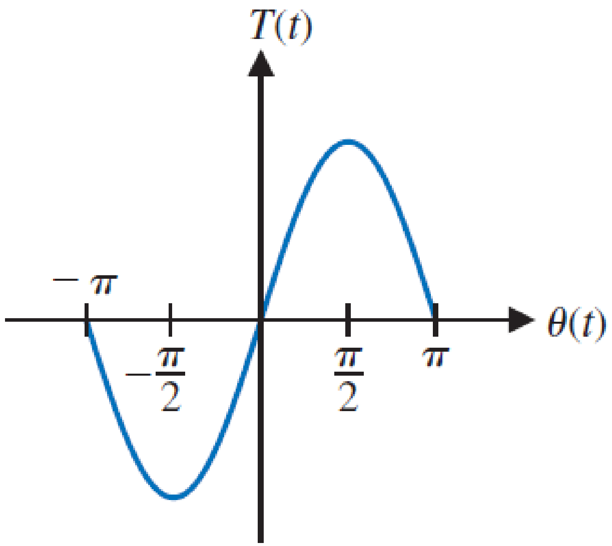


The torque on the mass is $$T(t) = MgL\sin\theta(t)$$

where g is the gravity constant.

The equilibrium condition for the mass is $\theta\,(0) = \theta_0 = 0°$.

The nonlinear relation between $T(t)$ and $\theta(t)$



$$T(t) - T_0 \cong MgL \frac{\partial \sin\theta}{\partial\theta}\bigg|_{\theta(t)=\theta_0} (\theta(t) - \theta_0)$$

$$T(0) = T_0 MgLsin(0) = 0$$

$$\boxed{T(t) = MgL\,\theta(t)}$$

This approximation is reasonably accurate for $-\pi/4 \leq \theta \leq \pi/\ 4$

Taylor series

- The linearization procedure to be presented in the following is based on the expansion of nonlinear function into a ***Taylor series*** about the operating point and the retention of only the linear term.

If the normal operating condition corresponds to $\bar{x}$, $\bar{y}$ then Equation $(*)$ may be expanded into a Taylor series about this point as follows:

$$y = f(x) = f(\bar{x}) + \frac{df}{dx}(x - \bar{x}) + \frac{1}{2!}\frac{d^2 f}{dx^2}(x - \bar{x})^2 + \cdots \qquad (**)$$

where the derivatives $\frac{df}{dx}$, $\frac{d^2 f}{dx^2}$, ... are evaluated at $x = \bar{x}$

Hint:

- Because we neglect higher-order terms of the Taylor series expansion, these neglected terms must be small enough; that is, the variables deviate only slightly from the operating condition. (Otherwise, the result will be inaccurate.)

- If the variation $x - \bar{x}$ is small, we may neglect the higher-order terms in $x - \bar{x}$

Then Equation $(**)$ may be written as

$$\boxed{y = \bar{y} + K(x - \bar{x})} \qquad (***)$$

$$\bar{y} = f(\bar{x}) \qquad K = \left.\frac{df}{dx}\right|_{x=\bar{x}}$$

Equation (2–44) may be rewritten as

$$\boxed{y - \bar{y} = K(x - \bar{x})}$$

gives a linear mathematical model for the nonlinear system given by Equation $(*)$ near the operating point $x = \bar{x}$ and $y = \bar{y}$

which indicates that $y - \bar{y}$ is proportional to $x - \bar{x}$

Problem 1

Linearize $f(x) = x^2$ using Taylor series about $x_0 = 2$

$$f(\bar{x}) = \bar{x}^2$$

$$\left.\frac{df}{dx}\right|_{x=\bar{x}} = 2\bar{x}$$

$$\left.\frac{d^2f}{dx^2}\right|_{x=\bar{x}} = 2$$

Taylor series:

$$y = f(x) = f(\bar{x}) + \frac{df}{dx}(x-\bar{x}) + \frac{1}{2!}\frac{d^2f}{dx^2}(x-\bar{x})^2 + \cdots = \bar{x}^2 + 2\bar{x}(x-\bar{x}) + (x-\bar{x})^2$$

$$\left.f(x)\right|_{\bar{x}=2} = \underbrace{4}_{\text{Zero order}} + \underbrace{4(x-2)}_{\text{1st order}} + \underbrace{(x-2)^2}_{\text{2nd order}}$$

$$\approx 4 + 4(x-2)$$

$$= 4x - 4$$

$$\left.f(x)\right|_{\bar{x}=2} \approx 4x - 4$$

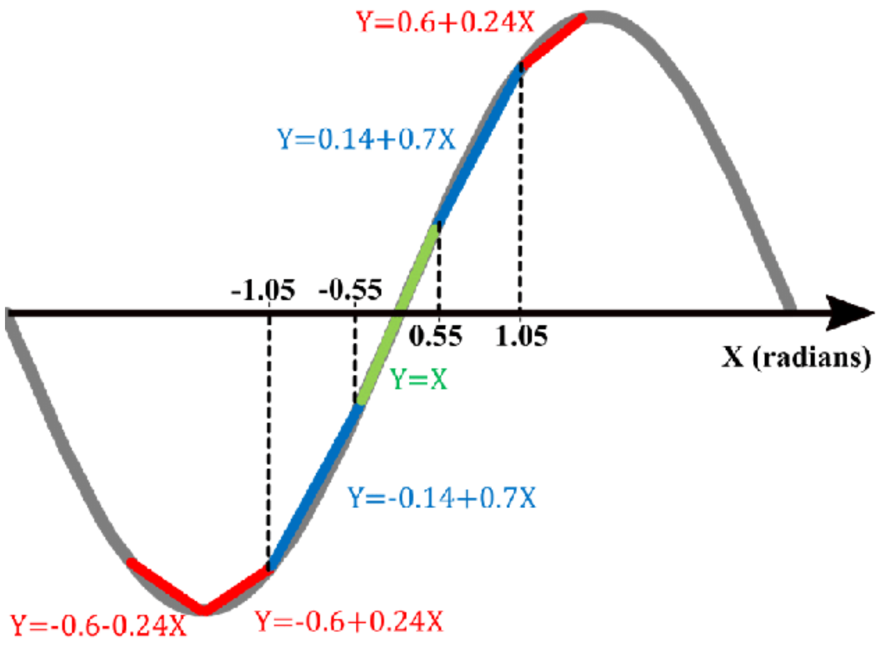
Jacobian Matrix

- In a *small neighbourhood* about each of the equilibrium points, a *nonlinear* system *behaves like* a *linear* system.

- The states can therefore be written as **x** = **x0** + **x***, where **x*** represents the perturbation or *state deviation* from the equilibrium point **x0**.

- Each of the elements of **f(x)** can be expanded in a Taylor series about one of the equilibrium points **x0**.

- Assuming that **x*** is restricted to a small neighbourhood of the equilibrium point, the higher-order terms in the Taylor series may be neglected.



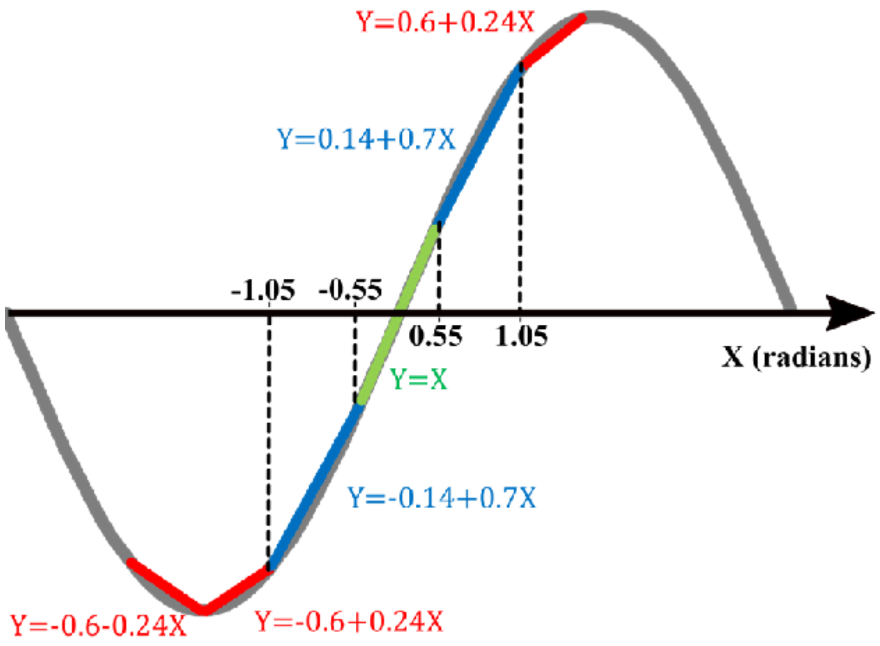
Jacobian Matrix

- The resulting linear variational state equation is

$$\dot{\mathbf{x}}^* = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\mathbf{x}=\mathbf{x_0}} \Rightarrow \quad \dot{\mathbf{x}}^* = \mathbf{J_x x}^* + \mathbf{J}_u \mathbf{u}^*$$

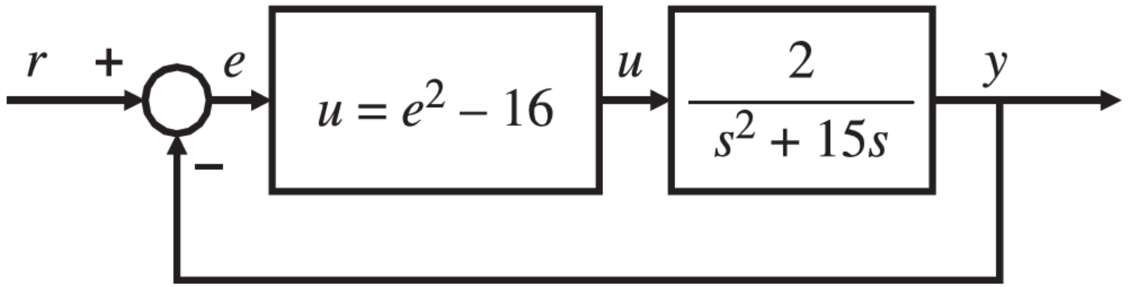
where $\mathbf{J_x} = \partial \mathbf{f} / \partial \mathbf{x}^T$ is called the *Jacobian matrix* and is evaluated at $\mathbf{x_0}$, and $\mathbf{J_u} = \partial \mathbf{f} / \partial \mathbf{u}^T$.

- Note that the linearized equations are applicable only in the neighbourhood of the singular points. Thus, they describe stability *in the small*.



Problem 2

The system in figure below consists of linear and nonlinear parts.



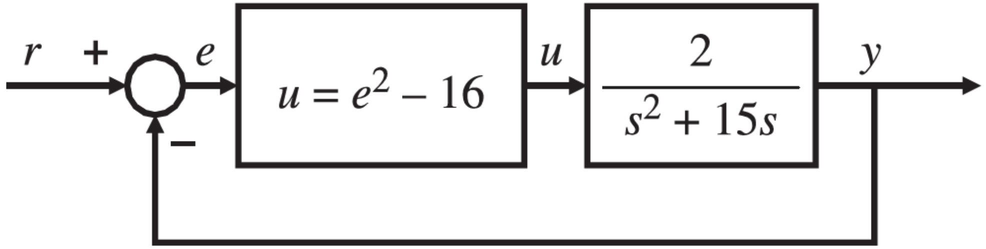
A. Write the differential equation connecting the reference r to the output y.

B. Define the state as $x_1 = y$ and $x_2 = \dot{y}$ and write the state equations.

C. Find the equilibrium points for a constant input $r(t) = 1$.

D. Find the linearization $\{A, B, C, D\}$ around all equilibrium points.

Problem 2



A. Write the differential equation connecting the reference r to the output y.

$$\frac{Y(s)}{U(s)} = \frac{2}{s^2 + 15s}$$

$$(s^2 + 15s)Y(s) = s^2Y(s) + 15sY(s) = 2U(s)$$

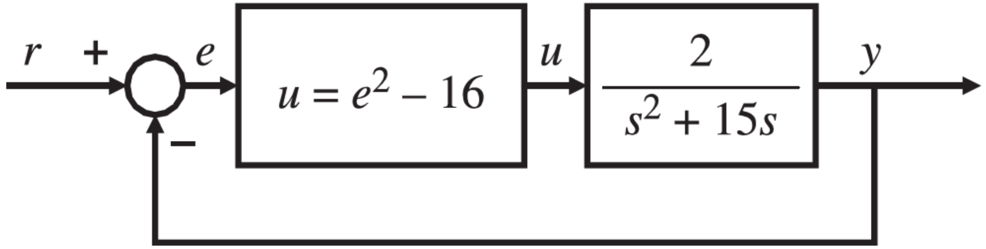
$$\ddot{y}(t) + 15\dot{y}(t) = 2u(t)$$

$$\begin{cases} u = e^2 - 16 \\ e = r - y \end{cases}$$

$$\ddot{y}(t) + 15\dot{y}(t) = 2(e^2(t) - 16) = 2\big(r(t) - y(t)\big)^2 - 32$$

$$\ddot{y}(t) + 15\dot{y}(t) - 2\big(r(t) - y(t)\big)^2 + 32 = 0$$

Problem 2



B. Define the state as $x_1 = y$ and $x_2 = \dot{y}$ and write the state equations.

From previous answer we have:

$$\ddot{y}(t) = -15\dot{y}(t) + 2\big(r(t) - y(t)\big)^2 - 32$$

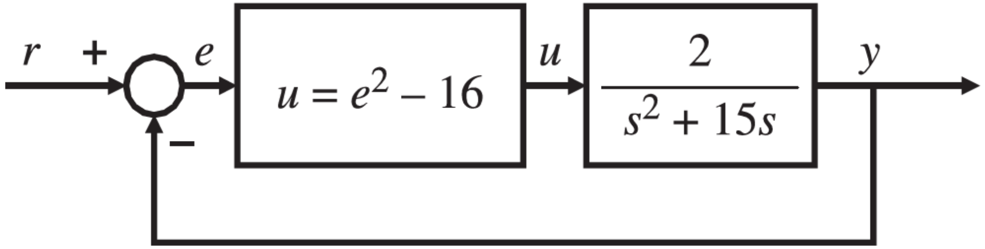
Given the state variable definitions:

$$\begin{cases} x_1 = y \\ x_2 = \dot{y} \end{cases}$$

The state-space equations are:

$$\begin{cases} \dot{x}_1 = \dot{y} = x_2 \\ \dot{x}_2 = \ddot{y} = -15x_2 + 2(r - x_1)^2 - 32 \\ y = x_1 \end{cases}$$

Problem 2



C. Find the equilibrium points for a constant input $r(t) = 1$.

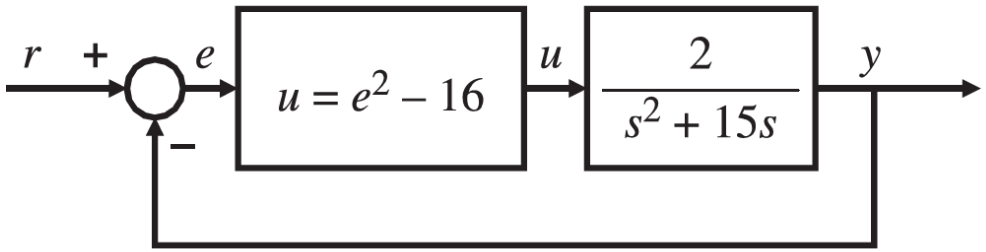
The equilibrium point should satisfy $\dot{x}_1 = \dot{x}_2 = 0$. So

$$\begin{cases} \dot{x}_1 = x_2 = 0 \\ \dot{x}_2 = -15x_2 + 2(r - x_1)^2 - 32 = 0 \end{cases} \Rightarrow \begin{cases} r - x_1 = \pm 4 \\ x_2 = 0 \end{cases}$$

As $r = 1$, $x_1 = -3$ or 5.

Thus, for $r = 1$, the equilibrium point is $x_{e1} = (-3, 0)^T$ and $x_{e2} = (5, 0)^T$.

Problem 2



D. Find the linearization $\{A, B, C, D\}$ around all equilibrium points.

$$\begin{cases} f_1 = x_2 \\ f_2 = -15x_2 + 2(r - x_1)^2 - 32 \\ g = x_1 \end{cases}$$

$$C = \frac{\partial g}{\partial x} = \begin{bmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = \frac{\partial g}{\partial r} = 0$$

$$A = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4(r - x_1) & -15 \end{bmatrix}$$

$$B = \frac{\partial \mathbf{f}}{\partial \mathbf{r}} = \begin{bmatrix} \frac{\partial f_1}{\partial r} \\ \frac{\partial f_2}{\partial r} \end{bmatrix} = \begin{bmatrix} 0 \\ 4(r - x_1) \end{bmatrix}$$

- For the 1st equilibrium point $x_{e1} = (-3, 0)^T$

$$A = \begin{bmatrix} 0 & 1 \\ -16 & -15 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 16 \end{bmatrix}; C = \begin{bmatrix} 1 & 0 \end{bmatrix}; D = 0$$

- For the 2nd equilibrium point $x_{e2} = (5, 0)^T$

$$A = \begin{bmatrix} 0 & 1 \\ 16 & -15 \end{bmatrix}; B = \begin{bmatrix} 0 \\ -16 \end{bmatrix}; C = \begin{bmatrix} 1 & 0 \end{bmatrix}; D = 0$$