UNIVERSITY OF LINCOLN

First year mathematics for engineers-refresher to get ready for further mathematics for engineers

Differentiation

Interpretation of a Derivative

Introduction

Differentiation is one of the most important processes in engineering mathematics. It is the study of the way in which functions change. The function may represent pressure, stress, volume or some other physical variable. For example, pressure of a vessel may depend upon temperature - as the temperature of the vessel increases, then so does the pressure. Engineers often need to know the rate at which such a variable changes. Rapid rates of change of a variable may indicate that a system is not operating normally and approaching critical values. Alarms may then be triggered.

Rates of change may be positive, negative or zero. A positive rate of change means that the variable is increasing; a negative rate of change means that the variable is decreasing. A zero rate of change means that the variable is not changing.

Often it is not sufficient to describe a rate of change as, for example, 'positive and large' or 'negative and small'. A precise value is needed. Use of differentiation provides a precise value or expression for the rate of change of a function.

Average Rate of Change across an Interval

In many cases a function can have different rates of change at different points on its graph. We begin by defining and then calculating the **average rate of change** of a function across an interval.

Consider x increasing from x_1 to x_2. The change in x is $x_2 - x_1$. As x increases from x_1 to x_2, then y increases from $y(x_1)$ to $y(x_2)$. The change in y is $y(x_2) - y(x_1)$. Then the average rate of change of y across the interval is,

$$\frac{change\,in\,y}{change\,in\,x} = \frac{y(x_2) - y(x_1)}{x_2 - x_1} = \frac{BC}{AC}$$

We can see that $\frac{BC}{AC} = \tan\theta$ which is also the gradient of the straight line or **chord** AB.

Hence we see that the average rate of change across an interval is identical to the gradient of the chord across that interval.

$$\boxed{average\,rate\,of\,change\,of\;y = \frac{change\,in\,y}{change\,in\,x} = gradient\,of\,chord}$$

Example 1 Calculate the average rate of change of $y = x^2$ across the interval (a) $x = 1$ to $x = 4$ and (b) $x = -2$ to $x = 0$.

(a) Change in $x = 4 - 1 = 3$

When $x = 1$, $y = 1^2 = 1$. When $x = 4$, $y = 4^2 = 16$. Hence the change in y is 16 – 1 = 15.

Therefore,

Average rate of change across interval [1, 4] $= \frac{15}{3} = 5$

This means that across the interval [1, 4], on average the y value increases by 5 for every 1 unit increase in x.

(b) Change in $x = 0 - (-2) = 2$

When $x = -2, y = (-2)^2 = 4$. When $x = 0, y = 0^2 = 0$. Hence the change in y is 0 – 4 = -4.

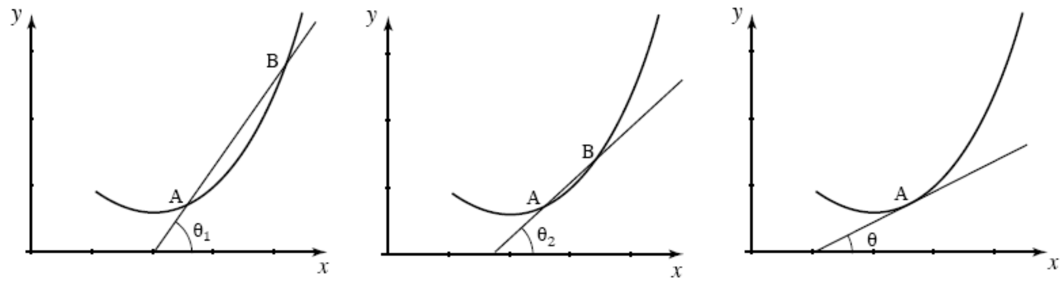
Therefore,

Average rate of change across interval [-2, 0] $= \frac{-4}{2} = -2$

This means that across the interval [-2, 0], on average the y value decreases by 2 for every 1 unit increases in x.

Rate of Change at a Point

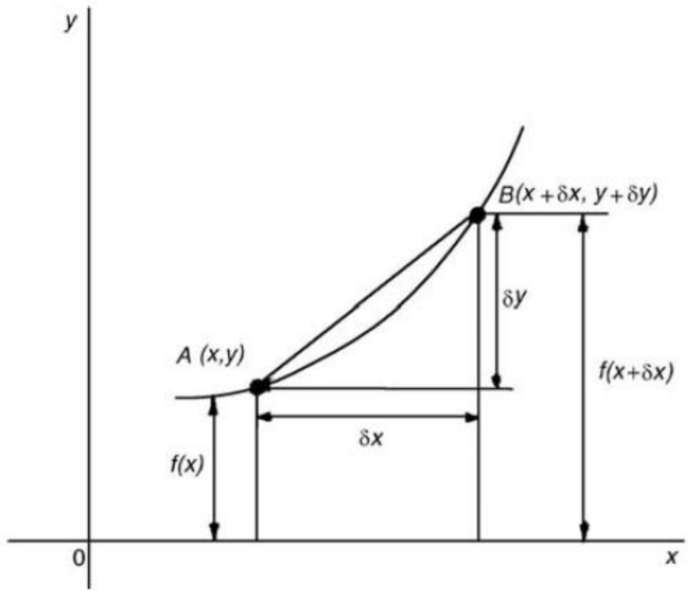
In many cases we often need to know that rate of change of a function at a point and not simply an average rate of change across an interval. Consider the curves below,



In each graph a chord is drawn between two points A and B. On the right-hand graph the two points A and B coincide and the straight line touches only at this point. This is called the tangent to the curve at point A. The gradient of this tangent gives us the rate of change at a point.

Rate of change at a point = gradient of tangent to the curve at that point

Calculating the rate of change of a function at a point by measuring the gradient of a tangent is usually not an accurate method. Consequently we develop an exact, algebraic way of finding rates of change. Consider the function shown below.



Let A be the point on the curve with coordinates $(x, y(x))$. B is a point on the curve near to A. The x coordinate of B is $x + \delta x$. The term δx represents a small change in the x direction. The y coordinate of B is $y(x+\delta x)$. We calculate the gradient of the chord AB as follows:

$$gradient\,of\ AB = \frac{change\,in\,y}{change\,in\,x} = \frac{y(x+\delta x)-y(x)}{x+\delta x-x} = \frac{y(x+\delta x)-y(x)}{\delta x}$$

The change in y , that is $y(x+\delta x)-y(x)$, is also written as δy . So,

$$gradient\,of\ AB = \frac{y(x+\delta x)-y(x)}{\delta x} = \frac{\delta y}{\delta x}$$

The gradient AB gives the average rate of change of y(x) across the small interval from $x\ to\ x+\delta x$. To calculate the rate of change of y(x) at A we require the gradient of the tangent at A.

Consider A as a fixed point and let B move along the curve towards A. At each position of B we can calculate the gradient of the chord AB. As B gets closer to A, the chord AB approximates more closely to the tangent at A. Also, as B approaches A, the distance δx decreases. To find the gradient of the tangent at A we calculate the gradient of the chord AB and let δx get smaller and smaller. We say δx tends to zero and write this as $\delta x \to 0$.

As B approaches A, the x difference between A and B gets smaller, that is $\delta x \to 0$, and likewise the y difference, δy , also gets smaller, so $\delta y \to 0$. However, the gradient of AB, given by the ratio $\frac{\delta y}{\delta x}$, approaches a definite value, called a limit. So we seek the limit of $\frac{\delta y}{\delta x}$ as $\delta x \to 0$. We write this as

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

Rate of change of y = gradient of tangent = $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$

Example 2 Find the rate of change of $y(x)=x^2$

Suppose A is the fixed point with coordinates (x, x^2) as shown below. B is a point on the curve near to A with coordinates $(x+\delta x, (x+\delta x)^2)$. We calculate the gradient of the chord AB.

Change in $x = \delta x$

Change in $y = \delta y$

$$= (x+\delta x)^2 - x^2$$
$$= x^2 + 2x\delta x + (\delta x)^2 - x^2$$
$$= 2x(\delta x) + (\delta x)^2$$

Gradient of chord AB $= \dfrac{\delta y}{\delta x}$

$$= \frac{2x(\delta x) + (\delta x)^2}{\delta x}$$
$$= 2x + \delta x$$

This is the average rate of change of y(x) across the small interval from x to $x+\delta x$. To obtain the gradient of the tangent at A, we let $\delta x \to 0$. Therefore,

Gradient of tangent at A $= \lim_{\delta x \to 0}(2x+\delta x) = 2x$

Hence the rate of change of x^2 is $2x$.

For example, if $x=3$, then A is the point (3, 9) and the rate of change of y at this point is 6.

Using a Table of Derivatives

The rate of change of a function is given by the gradient of a tangent at a particular point. The gradient of a tangent depends upon where the tangent is drawn and thus the gradient of a tangent is a function of x itself.

Rather than calculate the derivative of a function as shown previously, it is common practice to use a table of derivatives. The table below shows some of the common functions used in engineering and their corresponding derivatives.

Common functions and their derivatives
(In this table k, n and c are constants)

Function	Derivative
constant	0
x	1
kx	k
x^n	nx^{n-1}
kx^n	knx^{n-1}
e^x	e^x
e^{kx}	ke^{kx}
$\ln x$	$1/x$
$\ln kx$	$1/x$
$\sin x$	$\cos x$
$\sin kx$	$k\cos kx$
$\sin(kx+c)$	$k\cos(kx+c)$
$\cos x$	$-\sin x$
$\cos kx$	$-k\sin kx$
$\cos(kx+c)$	$-k\sin(kx+c)$
$\tan x$	$\sec^2 x$
$\tan kx$	$k\sec^2 kx$
$\tan(kx+c)$	$k\sec^2(kx+c)$

In the trigonometric functions the angle is in radians.

$$\frac{d}{dx}\sinh x = \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\frac{d}{dx}\cosh x = \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$$

$$\frac{d}{dx}\operatorname{sech} x = -\tanh x \operatorname{sech} x$$

$$\frac{d}{dx}\coth x = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}\operatorname{csch} x = -\coth x \operatorname{csch} x$$

$$\frac{d}{dx}\operatorname{arcsinh} x = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{d}{dx}\operatorname{arccosh} x = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}\operatorname{arctanh} x = \frac{1}{1-x^2}$$

$$\frac{d}{dx}\operatorname{arcsech} x = \frac{-1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}\operatorname{arccoth} x = \frac{1}{1-x^2}$$

$$\frac{d}{dx}\operatorname{arccsch} x = \frac{-1}{|x|\sqrt{1+x^2}}$$

Example 3 Use the table to find $\frac{dy}{dx}$ when y is given by (a) $3x$ (b) $3x^2$ (c) 3 (d) $4x^7$

(a) We note that 3x is of the form kx where k = 3. Using the table above we then have $\frac{dy}{dx} = 3$

(b) We see that $3x^2$ is of the form kx^n, with k = 3 and n = 2. The derivative, knx^{n-1}, is then $6x^1$ or more simply $6x$. So if $y = 3x^2$ then $\frac{dy}{dx} = 6x$

(c) Noting that 3 is a constant we see that $\frac{dy}{dx} = 0$

(d) We see that $4x^7$ is of the form kx^n, with k = 4 and n = 7. The derivative $\frac{dy}{dx} = 28x^6$

Example 4 Find the derivative, $\frac{dy}{dx}$, when y is (a) sin 3x (b) $2e^{3x}$ (c) tan 2x.

(a) Using the rule for $\sin kx$, and taking k = 3, we see that $\frac{dy}{dx} = 3\cos 3x$

(b) Using the rule for e^{kx}, and taking k = 3, we see that $\frac{dy}{dx} = 6e^{3x}$

(c) Using the rule for $\tan kx$, we see that $\frac{dy}{dx} = 2\sec^2 2x$

Higher Derivatives

Using the method shown above allows us to find the derivative of a function. The function $\frac{dy}{dx}$ is more correctly called the **first derivative** of y. By differentiating the first derivative we obtain the **second derivative** $\frac{d^2y}{dx^2}$; by differentiating the second derivative we obtain the **third derivative** $\frac{d^3y}{dx^3}$ and so on.

Example 5 Find the first, second and third derivatives of $y = e^{2x} + x^4$.

The first derivative is,

$$\frac{dy}{dx} = 2e^{2x} + 4x^3$$

The second derivative is,

$$\frac{d^2y}{dx^2} = 4e^{2x} + 12x^2$$

The third derivative is,

$$\frac{d^3y}{dx^3} = 8e^{2x} + 24x$$

Evaluating a Derivative

Engineers may need to find the rate of change of a function at a particular point; that is, find the derivative of a function at a specific point. We do this by finding the derivative of the function, and then evaluating the derivative at the given value of x.

Example 6 Find the value of the derivative of $y = 3x^2$ where $x = 4$. Interpret your result.

We have $y = 3x^2$ and so $\frac{dy}{dx} = 6x$. We now evaluate the derivative.

When $x = 4$, $\frac{dy}{dx} = 6(4) = 24$

The derivative is positive when $x = 4$ and so y is increasing at this point. Thus when $x = 4$, y is increasing at a rate of 24 vertical units per horizontal unit.

Example 7 Find the rate of change of current, i(t), given by $i(t) = 3e^{-t} + 2 \quad t \geq 0$ when $t = 0.7$ seconds

In this case $\frac{di}{dt} = -3e^{-t}$

When $t = 0.7$ seconds,

$$\frac{di}{dt} = -3e^{-0.7} = -1.4898$$

The derivative is negative and so we know that $i(t)$ is decreasing when $t = 0.7$. Thus, when $t = 0.7$, the current is decreasing at a rate of 1.49 As^{-1}

The Product Rule

The product rule helps us to differentiate a product of functions. Consider the function $y(x)$, where $y(x)$ is the product of two functions, $u(x)$ and $v(x)$, that is

$$y(x) = u(x)v(x)$$

For example, if $y(x) = x^2 \sin x$ then $u(x) = x^2$ and $v(x) = \sin x$. The product rule states:

If	$y(x) = u(x)v(x)$
then	$\dfrac{dy}{dx} = \dfrac{du}{dx}v + u\dfrac{dv}{dx}$

Example 8 Find $\dfrac{dy}{dx}$ where $y = x^2 \sin x$.

In this case $u = x^2$ and $v = \sin x$. Hence,

$$\frac{du}{dx} = 2x \quad \text{and} \quad \frac{dv}{dx} = \cos x$$

Applying the product rule we have,

$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$$

$$= 2x(\sin x) + x^2(\cos x)$$

$$= x(2\sin x + x\cos x)$$

Example 9 Find $\frac{dy}{dx}$ where $y = e^x \cos x$.

In this case $u = e^x$ and $v = \cos x$. Therefore,

$$\frac{du}{dx} = e^x \quad \text{and} \quad \frac{dv}{dx} = -\sin x$$

Applying the product rule gives,

$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$$

$$= e^x(\cos x) + e^x(-\sin x)$$

$$= e^x(\cos x - \sin x)$$

The Quotient Rule

The quotient rule shows us how to differentiate a quotient of functions, for example,

$$\frac{\sin x}{x}, \quad \frac{t^2 - 1}{t^2 + 1}, \quad \frac{e^z + z}{\cos z}$$

The quotient rule states that,

If $$y(x) = \frac{u(x)}{v(x)}$$

then $$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example 10 Find y' given $y = \dfrac{\sin x}{x}$.

In this case $u = \sin x$ and $v = x$. Therefore,

$$\frac{du}{dx} = \cos x \quad \text{and} \quad \frac{dv}{dx} = 1$$

Applying the quotient rule gives,

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{x\cos x - \sin x(1)}{x^2}$$

$$= \frac{x\cos x - \sin x}{x^2}$$

Example 11 Find $\dfrac{dy}{dx}$ given $y = \dfrac{t^3}{t+1}$.

In this case $u = t^3$ and $v = t + 1$. Therefore,

$$\frac{du}{dx} = 3t^2 \quad \text{and} \quad \frac{dv}{dx} = 1$$

Applying the quotient rule gives,

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{(t+1)3t^2 - t^3(1)}{(t+1)^2}$$

$$= \frac{t^2(2t+3)}{(t+1)^2}$$

The Chain Rule

It is often easier to make a substitution before differentiating:

If	y is a function of x
then	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

This is also known as the **'function of a function'** rule.

Example 12 Differentiate $y = 3\cos(5x^2 + 2)$

Let $u = 5x^2 + 2$ then $y = 3\cos u$

Hence $\dfrac{du}{dx} = 10x$ and $\dfrac{dy}{du} = -3\sin u$

Using the chain rule gives,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (-3\sin u)(10x) = -30x\sin u$$

Rewriting u as $5x^2 + 2$ gives,

$$-30x\sin(5x^2 + 2)$$

Example 13 Find the derivative of $y = (4t^3 - 3t)^6$

Let $u = 4t^3 - 3t$ then $y = u^6$

Therefore,

$\dfrac{du}{dt} = 12t^2 - 3$ and $\dfrac{dy}{du} = 6u^5$

Applying the chain rule gives,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (6u^5)(12t^2 - 3)$$

Rewriting u as $(4t^3 - 3t)$ gives,

$$\frac{dy}{dt} = 6(4t^3 - 3t)^5(12t^2 - 3)$$

Implicit, Parametric and Partial Differentiation

Implicit Differentiation

Most of the functions we have looked at so far are of the form y = f(x) or can be rearranged into this form. For example, $\mathrm{y} = \mathrm{x}^2 + 3$, $\mathrm{y} = \mathrm{sinx}$ and $\mathrm{y} = \mathrm{e}^{2\mathrm{x}} + 3$. When a function is expressed solely in terms of x we say that y is expressed explicitly in terms of x. In this section we will look at functions which cannot be rearranged into this form or types that when rearranged look very messy. For example,

$x^2 - y^3 + sinx - cosy = 1$, $\sin(\mathrm{x} + \mathrm{y}) + \mathrm{e}^{\mathrm{x}} + \mathrm{e}^{-\mathrm{y}} = \mathrm{x}^3 + \mathrm{y}^3$ or $\mathrm{x}^2 + \mathrm{y}^2 = 4$.

In these cases we say that y is expressed implicitly in terms of x. If y is expressed explicitly in terms of x then $\frac{\mathrm{dy}}{\mathrm{dx}}$ will be expressed explicitly in terms of x. If y is expressed implicitly in terms of x then $\frac{\mathrm{dy}}{\mathrm{dx}}$ will be expressed in terms of x and y.

Differentiating f(y) with respect to x

If y is expressed implicitly in terms of x and we want to find $\frac{\mathrm{dy}}{\mathrm{dx}}$ then we frequently need to differentiate a function of y with respect to x, that is, find $\frac{d}{dx}(f(y))$. In order to do this we use the chain rule, which may be stated as.

$$\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx}$$

Example 14 Find $\frac{d}{dx}(y^3)$

Using the chain rule,

$$\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx}$$

Let $u = y^3$ and so $\frac{du}{dy} = 3y^2$

Hence,

$$\frac{d}{dx}(y^3) = (3y^2) \times \frac{dy}{dx}$$

Example 15 Differentiate $2y^4$ with respect to x

Let u = $2y^4$ and so $\frac{du}{dy} = 8y^3$

Hence,

$$\frac{d}{dx}(2y^4) = (8y^3) \times \frac{dy}{dx}$$

A simple rule for differentiating an implicit function is summarised below:

$$\frac{d}{dx}(f(y)) = \frac{d}{dy}(f(y)) \times \frac{dy}{dx}$$

An implicit function such as $3x^2 + y^2 - 5x + y = 2$, can be differentiated term by term with respect to x.

Example 16 Find $\frac{dy}{dx}$ given $2y^2 - 5x^4 - 2 - 7y^3 = 0$

Each term is differentiated with respect to x giving,

$$\frac{d}{dx}(2y^2) - \frac{d}{dx}(5x^4) - \frac{d}{dx}(2) - \frac{d}{dx}(7y^3) = \frac{d}{dx}(0)$$

Using the rule,

$$\boxed{\frac{d}{dx}(f(y)) = \frac{d}{dy}(f(y)) \times \frac{dy}{dx}}$$

Gives,

$$4y\frac{dy}{dx} - 20x^3 - 0 - 21y^2\frac{dy}{dx} = 0$$

Rearranging gives,

$$(4y - 21y^2)\frac{dy}{dx} = 20x^3$$

$$\frac{dy}{dx} = \frac{20x^3}{(4y - 21y^2)}$$

Example 17 Find $\dfrac{dy}{dx}$ given $x^2 + xy = 4$

As this is an implicit function, we differentiate each term with respect to x. This gives,

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) = \frac{d}{dx}(4)$$

In this case, x^2 differentiates to 2x and 4 differentiates to 0 but what about the xy term. For this term we must use the product rule,

If	$y(x) = u(x)v(x)$
then	$\dfrac{dy}{dx} = \dfrac{du}{dx}v + u\dfrac{dv}{dx}$

Let $u = x$ and $v = y$ and so $\dfrac{du}{dx} = 1$ and $\dfrac{dv}{dx} = 1\dfrac{dy}{dx}$

Therefore,

$$\frac{d}{dx}(xy) = 1(y) + x\left(1\frac{dy}{dx}\right)$$

$$\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$$

The full solution then becomes,

$$2x + y + x\frac{dy}{dx} = 0$$

Rearranging gives,

$$\frac{dy}{dx} = \frac{-2x - y}{x}$$

Parametric Differentiation

Some relationships between two quantities or variables are so complicated that we sometimes introduce a third quantity or variable in order to make things easier to handle. Instead of one equation relating say, x and y , we have two equations, one relating x with the parameter , and one relating y with the parameter . For example, $y = \cos 2t$, $x = \sin t$. In this case, any value of t will produce a pair of values for x and y, which if necessary could be plotted to provide one point on the curve $y = f(x)$.

In mathematics this third quantity is called a parameter and the two expressions for x and y are called parametric equations. Some examples of common parametric equations are shown below.

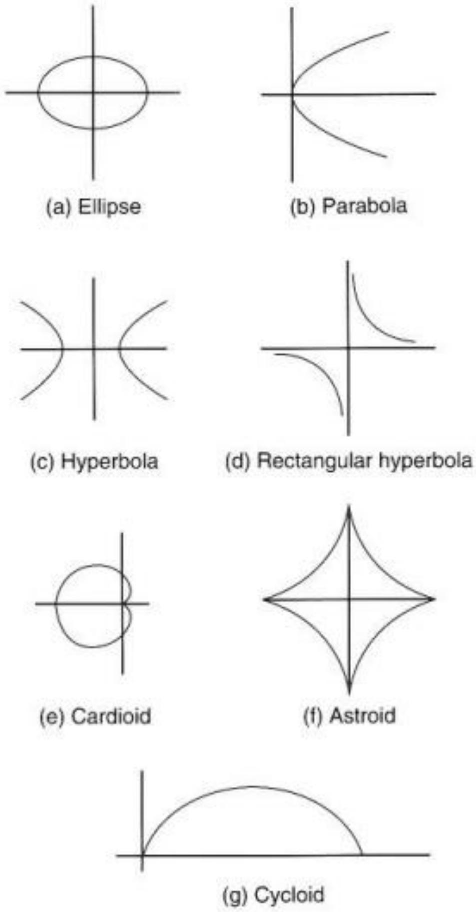


Figure 45.1

(a)	Ellipse	$x = a\cos\theta,\ y = b\sin\theta$
(b)	Parabola	$x = at^2,\ y = 2at$
(c)	Hyperbola	$x = a\sec\theta,\ y = b\tan\theta$
(d)	Rectangular hyperbola	$x = ct,\ y = \frac{c}{t}$

Sometimes, however, we may wish to find the differential coefficient of the function with respect to x. In this section, we will look at how we go about doing this.

Example 18 Find $\frac{dy}{dx}$ when $y = \cos 2t$ and $x = \sin t$

From $y = \cos 2t$, we get $\frac{dy}{dt} = -2\sin 2t$

From $x = \sin t$, we get $\frac{dx}{dt} = \cos t$

So how do we find $\frac{dy}{dx}$?

Simple, we use the chain rule which states,

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{dy/dt}{dx/dt}$$

Applying this gives,

$$\frac{dy}{dx} = \frac{-2\sin 2t}{\cos t}$$

Example 19 Find $\frac{dy}{dx}$ when $y = \sin t + t^2$ and $x = e^t + t$

From $y = \sin t + t^2$, we get $\frac{dy}{dt} = \cos t + 2t$

From $x = e^t + t$, we get $\frac{dx}{dt} = e^t + 1$

Applying the rule,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Gives,

$$\frac{dy}{dx} = \frac{\cos t + 2t}{e^t + 1}$$

Partial Differentiation

Suppose you want to forecast the weather this weekend in Lincoln. You construct a formula for the temperature as a function of several variables, each of which is not entirely predictable. Now you would like to see how your weather forecast would change as one particular environmental factor changes, holding all the other factors constant. To do this investigation, you would use the concept of a **partial derivative**.

Let the temperature T depend on variables x and y, that is $T = f(x, y)$. The rate of change of f with respect to x (holding y constant) is called the partial derivative of T with respect to x denoted by $\frac{\partial T}{\partial x}$. Similarly, the rate of change of T with respect to y (holding x constant) is called the partial derivative of T with respect to y denoted by $\frac{\partial T}{\partial y}$. Notice the new type of 'delta', ∂, meaning partial derivative.

Example 20 Consider the area of the curved surface of a cylinder denoted by $A = 2\pi rh$. Find $\frac{\partial A}{\partial h}$ and $\frac{\partial A}{\partial r}$

We know that the area of the curved surface of a cylinder is determined by its radius, r, and its height, h. If we keep r constant and increase the height h, the area will increase. The rate at which it increases as h increases can be found by the partial derivative $\frac{\partial A}{\partial h}$. If we keep h constant and increase r then the rate at which the area changes can be found by the partial derivative $\frac{\partial A}{\partial r}$.

To find $\frac{\partial A}{\partial h}$ we differentiate the expression for A with respect to h, keeping all other symbols constant.

To find $\frac{\partial A}{\partial r}$ we differentiate the expression for A with respect to r, keeping all other symbols constant.

$A = 2\pi rh$

Therefore,

$$\frac{\partial A}{\partial h} = 2\pi r \quad \text{and} \quad \frac{\partial A}{\partial r} = 2\pi h$$

Example 21 Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ given $z = x^3 + y^3 - 2x^2y$

Differentiating the function $z(x, y)$ with respect x gives,

$$\frac{\partial z}{\partial x} = 3x^2 + 0 - 4xy = 3x^2 - 4xy$$

In this case,

Partial derivative w.r.t. x of x^3 is $3x^2$
Partial derivative w.r.t. x of y^3 is 0 (y^3 is a constant term)
Partial derivative w.r.t. x of $-2x^2y$ is $-4xy$ (y is a constant factor)

Differentiating the function $z(x, y)$ with respect y gives,

$$\frac{\partial z}{\partial y} = 0 + 3y^2 - 2x^2 = 3y^2 - 2x^2$$

Partial derivative w.r.t. y of x^3 is 0 (x^3 is a constant term)
Partial derivative w.r.t. y of y^3 is $3y^2$
Partial derivative w.r.t. y of $-2x^2y$ is $-2x^2$ ($-2x^2$ is a constant factor)

Note that the normal product, quotient and chain rules still apply to partial differentiation.

Example 22 Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ given $z = (2x - y)(x + 3y)$

As this is a product, then the usual product rule applies except that we keep y constant when finding $\frac{\partial z}{\partial x}$ and x constant when finding $\frac{\partial z}{\partial y}$.

First let's find $\frac{\partial z}{\partial x}$.

Let $u = 2x - y$ and $v = x + 3y$

$\frac{\partial u}{\partial x} = 2 - 0 = 2$ and $\frac{\partial v}{\partial x} = 1 + 0 = 1$

Using the product rule,

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x}v + u\frac{\partial v}{\partial x}$$

Gives,

$$\frac{\partial z}{\partial x} = 2(x + 3y) + 1(2x - y)$$

$$= 2x + 6y + 2x - y$$

$$= 4x + 5y$$

Now let's find $\frac{\partial z}{\partial y}$

Again, let $u = 2x - y$ and $v = x + 3y$

$\frac{\partial u}{\partial y} = 0 - 1$ and $\frac{\partial v}{\partial y} = 0 + 3$

Applying the product rule,

$$\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y}v + u\frac{\partial v}{\partial y}$$

Gives,

$$\frac{\partial z}{\partial y} = -1(x + 3y) + 3(2x - y)$$

$$= -x - 3y + 6x - 3y$$

$$= 5x - 6y$$

Example 23 Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ given $z = \frac{2x - y}{x + y}$

In this case, we need to apply the quotient rule. Let's find $\frac{\partial z}{\partial x}$ first.

Let $u = 2x - y$ and $v = x + y$.

$\frac{\partial u}{\partial x} = 2 - 0 = 2$ and $\frac{\partial v}{\partial x} = 1 + 0 = 1$

Applying the quotient rule,

$$\frac{\partial z}{\partial x} = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$$

Gives,

$$\frac{\partial z}{\partial x} = \frac{2(x+y)-1(2x-y)}{(x+y)^2}$$

$$= \frac{2x+2y-2x+y}{(x+y)^2}$$

$$= \frac{3y}{(x+y)^2}$$

Now let's find $\frac{\partial z}{\partial y}$,

Again, let $u = 2x - y$ and $v = x + y$.

$\frac{\partial u}{\partial y} = 0 - 1 = -1$ and $\frac{\partial v}{\partial y} = 0 + 1 = 1$

Applying the quotient rule,

$$\frac{\partial z}{\partial y} = \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}$$

Gives,

$$\frac{\partial z}{\partial y} = \frac{-1(x+y)-1(2x-y)}{(x+y)^2}$$

$$= \frac{-x-y-2x+y}{(x+y)^2}$$

$$= \frac{-3x}{(x+y)^2}$$

Example 24 Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ given $z = \sin(3x + 2y)$

In this case, we need to use the chain rule. Let's find $\frac{\partial z}{\partial x}$ first.

Let $u = 3x + 2y$ and $z = \sin u$

$\frac{\partial u}{\partial x} = 3 + 0 = 3$ and $\frac{\partial z}{\partial u} = \cos u$

Applying the chain rule,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \times \frac{\partial u}{\partial x}$$

Gives,

$$\frac{\partial z}{\partial x} = \cos u \times 3$$

$$= 3 \times \cos(3x + 2y)$$

$$= 3\cos(3x + 2y)$$

Now let's find $\frac{\partial z}{\partial y}$,

Again, let $u = 3x + 2y$ and $z = \sin u$

$\frac{\partial u}{\partial y} = 0 + 2 = 2$ and $\frac{\partial z}{\partial u} = \cos u$

Applying the chain rule,

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \times \frac{\partial u}{\partial y}$$

Gives,

$$\frac{\partial z}{\partial y} = \cos u \times 2$$

$$= 2 \times \cos(3x + 2y)$$

$$= 2\cos(3x + 2y)$$

Applications of Differentiation

Now we have looked at some of the theory and techniques of differentiation we will look at some of its applications. These include: rates of change, velocity and acceleration and maxima and minima.

Rates of Change

If a quantity y depends on and varies with a quantity x then the rate of change of y with respect to x is $\frac{dy}{dx}$. For example, the rate of change of temperature θ with time is $\frac{d\theta}{dt}$. The rate of change of pressure p with height h is $\frac{dp}{dh}$. A rate of change with respect to time is normally just called 'the rate of change' and the 'with respect to time' is assumed.

Example 25 The length, l metres, of a metal rod at temperature $\theta°C$ is given by $l = 1 + 0.00005\theta + 0.0000004\theta^2$. Determine the rate of change of length, in mm/°C, when the temperature is 100°C.

The rate of change of length means $\frac{dl}{d\theta}$.

The formula for length is given as,

$$l = 1 + 0.00005\theta + 0.0000004\theta^2$$

Differentiating this function gives,

$$\frac{dl}{d\theta} = 0.00005 + 0.0000008\theta$$

When $\theta = 100°C$,

$$\frac{dl}{d\theta} = 0.00005 + 0.0000008(100)$$

$$= 0.00013 m / ^\circ C$$

$$= 0.13 mm / ^\circ C$$

Example 26 Newton's law of cooling is given by $\theta = \theta_0 e^{-kt}$, where θ_0 is the initial temperature of the body, i.e. when time $t = 0$, and $\theta ^\circ C$ is the temperature at time t seconds. Determine the rate of change of temperature after 40 seconds, given that $\theta_0 = 16^\circ C$ and $k = 0.03$.

The rate of change of temperature is $\dfrac{d\theta}{dt}$.

The temperature is given by,

$$\theta = \theta_0 e^{-kt}$$

Differentiating this function with respect to time t gives,

$$\frac{d\theta}{dt} = (-k)(\theta_0)e^{-kt}$$

When $\theta_0 = 16^\circ C$, $k = 0.03$ and $t = 40s$,

$$\frac{d\theta}{dt} = (-0.03)(16)e^{-0.03 \times 40}$$

$$= -0.145^\circ C / s$$

Example 27 The displacement s cm of the end of a stiff spring is given by $s = ae^{-kt}\sin 2\pi ft$. Determine the velocity of the end of the spring after 1 second, if $a = 2, k = 0.9$ and $f = 5$.

Velocity is the rate of change of displacement $\frac{ds}{dt}$.

Differentiating $s = ae^{-kt}\sin 2\pi ft$ using the product rule gives,

$$\frac{ds}{dt} = (ae^{-kt})(2\pi f\cos 2\pi ft) + (\sin 2\pi ft)(-ake^{-kt})$$

When $a = 2$, $k = 0.9$, $f = 5$ and $t = 1$,

Velocity, $v = (2e^{-0.9})(2\pi 5\cos 2\pi 5) + (\sin 2\pi 5)(-2)(0.9)e^{-0.9}$

Velocity $= 25.55 cm/s$

Velocity and Acceleration

Velocity is the speed of a particle and its direction of motion (therefore velocity is a vector quantity, whereas speed is a scalar quantity).
When the velocity (speed) of a moving object is increasing we say that the object is accelerating. If the velocity decreases it is said to be decelerating. Acceleration is therefore the rate of change of velocity (change in velocity / time) and is measured in m/s^2.
If we know the equation that relates displacement and time, we can find the velocity by differentiating the function and the acceleration by differentiating the function again i.e. by finding the second derivative.

Example 28 The distance, x metres, moved by a car in a time, t seconds, is given by $x = 3t^3 - 2t^2 + 4t - 1$. Determine the velocity and acceleration when (a) t = 0 and (b) t = 1.5 seconds.

The rate of change of displacement, i.e. the **velocity**, of the car is found by differentiating the function. This gives

$$v = \frac{dx}{dt} = 9t^2 - 4t + 4 \text{ ms}^{-1}$$

The **acceleration** of the car can be found by finding the second derivative of the function, i.e. by differentiating again. This gives,

$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = 18t - 4$ ms^{-2}

(a) When $t = 0$,

Velocity $= 9(0)^2 - 4(0) + 4 = 4$ ms^{-2}

Acceleration $= 18(0) - 4 = -4$ ms^{-2} (decelerating)

(b) When $t = 1.5$,

Velocity $= 9(1.5)^2 - 4(1.5) + 4 = 18.25$ ms^{-1}

Acceleration $= 18(1.5) - 4 = 23$ ms^{-2}

Example 29 The displacement x cm of the slide valve of an engine is given by $x = 2.2\cos 5\pi t + 3.6\sin 5\pi t$. Find the velocity of the valve (in m/s) when $t = 30$ ms.

The displacement of the valve is given by $x = 2.2\cos 5\pi t + 3.6\sin 5\pi t$ and so the rate of change of displacement i.e. velocity, is given by,

$$v = \frac{dx}{dt} = (2.2)(-5\pi)\sin 5\pi t + (3.6)(5\pi)\cos 5\pi t$$

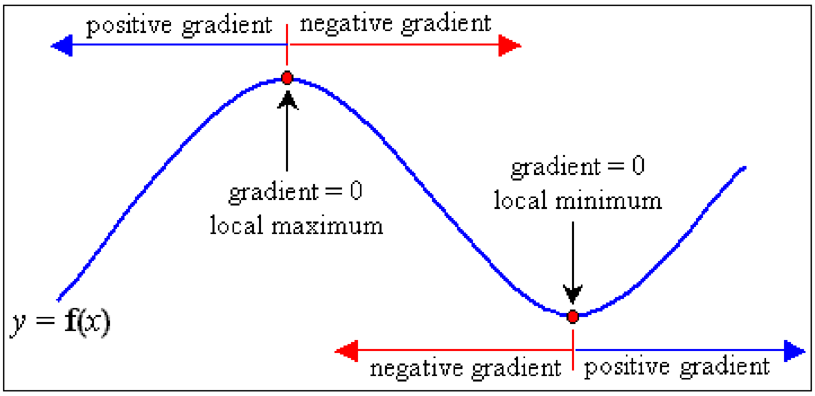
$$= 34.7\, cm/s$$

$$= 0.347\, m/s$$

<u>Maxima and Minima</u>

Turning Points

If we consider the curve below we notice that, the rate of change or gradient of the curve changes from a positive to a negative at the 'crest' of the curve. At this point, the rate of change of the curve $\frac{dy}{dx} = 0$. This is known as a **maximum turning point**. We also notice that the curve changes from a negative to a positive gradient at the 'trough' of the curve. Again, at this point the rate of change of the curve $\frac{dy}{dx} = 0$. This is known as a **minimum turning point**. It is possible to have a turning point where the gradient on either side of it is the same. Such a point is known as a **point of inflexion**.



Example 30 Find the maximum and minimum turning points of the curve $y = x^3 - 3x + 5$.

As this is a cubic polynomial we know that there must be a maximum and minimum turning point. We also know that the rate of change of a curve at a turning point $\frac{dy}{dx}$ is zero. Therefore,

$$\frac{dy}{dx} = 3x^2 - 3 = 0$$

Now we need to solve this equation to find the value of x when $\frac{dy}{dx} = 0$.

This gives,

$$3x^2 = 3$$

$$x^2 = 1$$

Therefore,

$$x = 1 \text{ or } -1$$

This means that there is a turning point at x = 1 and x = -1. Now we need to find the y co-ordinate of these turning points.

When x = 1,

$y = (1)^3 - 3(1) + 5 = 3$

When x = -1,

$y = (-1)^3 - 3(-1) + 5 = 7$

Therefore,

(1, 3) and **(-1, 7)** are the co-ordinates of the turning points.

Now we must determine which of the turning points is a maximum and which is a minimum.

(i) Let's consider the point (1, 3).

If x is slightly less than 1 i.e. 0.9 then,

$$\frac{dy}{dx} = 3(0.9)^2 - 3 = -0.57$$

This means that when x is 0.9 the gradient of the curve is **negative**.

If x is slightly more than 1 i.e. 1.1 then,

$$\frac{dy}{dx} = 3(1.1)^2 - 3 = 0.63$$

This means that the gradient of the curve when x is 1.1 is **positive.**

We now know that the curve is going from a negative gradient to a positive gradient at x = 1 and so, this is a **minimum turning point**.

A more simple way of determining whether the turning point is a maximum or minimum is to find the second derivative of $y = x^3 - 3x + 5$.

$$\frac{d^2y}{dx^2} = 6x$$

When x = 1 then $\frac{d^2y}{dx^2} = 6$ and positive indicating a **minimum turning point**.

(ii) Now let's consider the point (-1, 7)

If x is slightly less than -1 i.e. -1.1 then,

$$\frac{dy}{dx} = 3(-1.1)^2 - 3 = 0.63$$

This means that when x is -1.1 the gradient of the curve is **positive**.

If x is slightly more than -1 i.e. -0.9 then,

$$\frac{dy}{dx} = 3(-0.9)^2 - 3 = -0.57$$

This means that when x is -0.9 the gradient of the curve is **negative**.

Therefore, the curve is going from a positive to a negative gradient at x = -1 hence this is a **maximum turning point**.

Using the alternative method,

When x = -1,

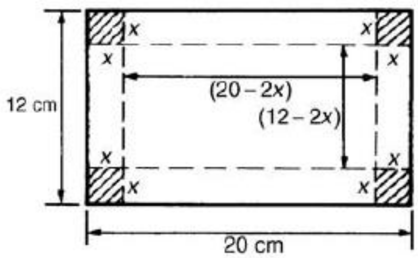
$\frac{d^2y}{dx^2} = -6$ and negative indicating a **maximum turning point**.

Practical Applications of Maxima and Minima

There are many practical problems involving maximum and minimum values which occur in engineering. Usually, an equation has to be formed from given data, and rearranged where necessary, so that it contains only one variable. We will look at some examples of these problems here.

Example 31 A rectangular sheet of metal having dimensions 20cm and 12cm has squares removed from each corner and the sides bent upwards to form an open box. Determine the maximum possible volume of the open box.

Consider the diagram below.



The volume of the box is given by,

$$V = l \times w \times h$$

Length $= (20 - 2x)$ cm, Width $= (12 - 2x)$ cm and Height $= x$ cm

Therefore,

$$V = (20 - 2x)(12 - 2x)(x)$$

$$= 240x - 64x^2 + 4x^3$$

Rearranging gives,

$$V = 4x^3 - 64x^2 + 240x$$

For a turning point,

$$\frac{dV}{dx} = 12x^2 - 128x + 240 = 0$$

We now need to solve the quadratic in order to find the values of x. Using the quadratic formula gives,

$$x = \frac{32 \pm \sqrt{(-32)^2 - 4(3)(60)}}{2(3)}$$

$x = 8.239$ cm or $x = 2.427$ cm

As the width of the box is $(12 - 2x)$ cm then $x = 8.239$ cm is not possible and so $x = 2.427$ cm.

Now we must determine whether this is a maximum or minimum value. Finding the second derivative gives,

$$\frac{d^2V}{dx^2} = 24x - 128$$

When $x = 2.427$, $\frac{d^2V}{dx^2}$ is negative and therefore, a maximum value.

Now we know the value of x we can find the dimensions of the box. These are,

Length $= 20 - 2(2.427) = 15.146$ cm

Width $= 12 - 2(2.427) = 7.146$ cm

Height $= 2.427$ cm

Maximum volume $= (15.146)(7.146)(2.427) = 262.7\ \text{cm}^3$

Example 32 The speed, v , of a car (in m/s) is related to time t s by the equation $v = 3 + 12t - 3t^2$. Determine the maximum speed of the car in km/h.

For a turning point,

$$\frac{dv}{dt} = 12 - 6t = 0$$

Solving the equation $12 - 6t = 0$ gives,

$12 = 6t$ and $t = 2s$

Substituting this value of $t = 2s$ in to the equation for the velocity gives,
$v = 3 + 12(2) - 3(2)^2 = 15$ m/s

Now we must determine whether this is a maximum or minimum value. The second derivative gives,

$$\frac{d^2v}{dt^2} = -6$$

As the second derivative is negative then this must be a maximum value i.e. 15 m/s is the maximum velocity of the car.

Converting this in to km/h gives,

$$v = 15 \times \frac{60 \times 60}{1000} = 54 \text{ km/h}$$

Integration

Differentiation in Reverse

If we differentiate the function $y = x^2$ we get $\frac{dy}{dx} = 2x$. Integration reverses this process and we say that the integral of $2x$ is x^2. The only problem with this is that there are lots more functions we can differentiate to give $2x$. For example,

$x^2 + 4, \quad x^2 - 10, \quad x^2 + 0.3$ ………………………

All of these functions have the same derivative, $2x$, because the constant term in each goes to zero when differentiated. Consequently, when we reverse the process we have no idea what the constant term might have been. Because of this we include in our answer an unknown constant, c, called the **constant of integration**. We state that the integral of $2x$ is $x^2 + c$.

The symbol for integration is $\int$ known as an **integral sign**. Formally we write,

$$\int 2x\, dx = x^2 + c$$

The function being integrated i.e. $2x$, is called the **integrand**. The dx simply means the function is being integrated with respect to x. Integrals of this sort are called **indefinite integrals** and when dealing with this type of integral we must include a constant of integration in the answer.

Table of Integrals

In order to integrate some common functions we can use a table of integrals as shown below.

Function $f(x)$	Indefinite integral $\int f(x)\mathrm{d}x$	
constant, k	$kx + c$	
x	$\dfrac{x^2}{2} + c$	
x^2	$\dfrac{x^3}{3} + c$	
x^n	$\dfrac{x^{n+1}}{n+1} + c$	$n \neq -1$
$x^{-1} = \dfrac{1}{x}$	$\ln\lvert x\rvert + c$	
$\sin x$	$-\cos x + c$	
$\cos x$	$\sin x + c$	
$\sin kx$	$\dfrac{-\cos kx}{k} + c$	
$\cos kx$	$\dfrac{\sin kx}{k} + c$	
$\tan kx$	$\frac{1}{k}\ln\lvert\sec kx\rvert + \mathrm{c}$	
$\sec kx$	$\frac{1}{k}\ln\lvert\sec kx + \tan kx\rvert + c$	
e^x	$\mathrm{e}^x + c$	
e^{-x}	$-\mathrm{e}^{-x} + c$	
e^{kx}	$\dfrac{\mathrm{e}^{kx}}{k} + c$	
$\cosh kx$	$\frac{1}{k}\sinh kx + c$	
$\sinh kx$	$\frac{1}{k}\cosh kx + c$	
$\dfrac{1}{x^2 + a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a} + c$	$a > 0$
$\dfrac{1}{x^2 - a^2}$	$\frac{1}{2a}\ln\dfrac{x-a}{x+a} + c$	$\lvert x\rvert > a > 0$
$\dfrac{1}{a^2 - x^2}$	$\frac{1}{2a}\ln\dfrac{a+x}{a-x} + c$	$\lvert x\rvert < a$
$\dfrac{1}{\sqrt{x^2 + a^2}}$	$\sinh^{-1}\left(\frac{x}{a}\right) + c$	$a > 0$
$\dfrac{1}{\sqrt{x^2 - a^2}}$	$\cosh^{-1}\left(\frac{x}{a}\right) + c$	$x \geq a > 0$
$\dfrac{1}{\sqrt{x^2 + k}}$	$\ln(x + \sqrt{x^2 + k}) + c$	
$\dfrac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right) + c$	$-a \leq x \leq a$

Example 33

(a) Use the table of integrals to find $\int x^7 dx$

(b) Check the result by differentiating the answer

(a) From the table we can see that,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

In words, this means to integrate a power of x , increase the power by one, and divide the result by the new power. In this case we get,

$$\int x^7 dx = \frac{x^8}{8} + c$$

(b) The answer can be differentiated as a check.

$$\frac{d}{dx}\left(\frac{x^8}{8} + c\right) = \frac{1}{8} \times 8x^7 = x^7$$

Example 34 Find $\int \sin\frac{x}{2} dx$

Note that $\frac{x}{2}$ is equivalent to $\frac{1}{2}x$. Use the table of integrals with $k = \frac{1}{2}$. Therefore,

$$\int \sin\frac{x}{2} dx = -\frac{\cos\frac{x}{2}}{\frac{1}{2}} + c = -2\cos\frac{x}{2} + c$$

Rules of Integration

In order to integrate a wider range of functions other than those given in the table we can make use of the following rules.

The integral of $kf(x)$ where k is a constant

A constant factor in an integral can be moved outside the integral sign.

Example 35 Find $\int 11x^2 dx$

$$\int 11x^2 dx = 11\int x^2 dx$$

$$= 11\left(\frac{x^3}{3} + c\right)$$

$$= \frac{11x^3}{3} + c$$

The integral of $f(x) \pm g(x)$

When we need to integrate the sum or difference of two functions, we integrate each term separately.

Example 36 Find $\int (x^3 + \sin x)dx$

$$\int (x^3 + \sin x)dx = \int x^3 dx + \int \sin x \, dx$$

$$= \frac{x^4}{4} - \cos x + c$$

Definite Integrals

In the previous section we looked at integrals which were indefinite. In this section we will look at definite integrals. The result of finding an indefinite integral is usually a function plus a constant of integration. With definite integrals the result will be a definite answer, usually a number, with no constant of integration. Definite integrals have many applications, for example in finding areas bounded by curves, and finding volumes of solids. We will look at these applications in a later unit.

Evaluating Definite Integrals

Definite integrals can be recognised by numbers written to the upper and lower of the integral sign. The quantity

$$\int_a^b f(x)dx$$

is called the definite integral of $f(x)$ from a to b. The numbers a and b are known as the lower and upper limits of the integral.

Example 37 Find $\int_1^4 x^2 dx$

First of all we perform the integration of x^2 in the normal way. However, to show we are dealing with a definite integral, the result is shown in square brackets with the limits of integration written on the right bracket. This is shown as follows,

$$\int_1^4 x^2 dx = \left[\frac{x^3}{3} + c\right]_1^4$$

We then evaluate the quantity in square brackets by first letting x equal the value at the upper limit and then by letting x equal the value at the lower limit. The difference between these resulting values is then found. This gives,

$$\left[\frac{x^3}{3}+c\right]_1^4 = \left(\frac{4^3}{3}+c\right)-\left(\frac{1^3}{3}+c\right)$$

$$=\frac{64}{3}-\frac{1}{3}=21$$

Example 38 Find $\int_1^2 (x^2+1)dx$

First we perform the integration. This gives,

$$\left[\frac{x^3}{3}+x\right]_1^2$$

Now we must evaluate the integral using the upper and lower limits. This gives,

$$\left(\frac{8}{3}+2\right)-\left(\frac{1}{3}+1\right)=3.333 \text{ (to 3 dp)}$$

Integration by Parts

The technique known as integration by parts is used to integrate a product of two functions, for example $\int x^2e^{2x}$ and $\int_0^1 x^4\cos 2x$. Note that as this is just a product of two functions we can write the terms either way round i.e. $\int x^2e^{2x}$ can be written as $\int e^{2x}x^2$.

The integration by parts formula states:

For indefinite integrals:

$$\int u\left(\frac{dv}{dx}\right)dx = uv-\int v\left(\frac{du}{dx}\right)dx$$

For definite integrals:

$$\int u\left(\frac{dv}{dx}\right)dx = [uv]_a^b - \int_a^b v\left(\frac{du}{dx}\right)dx$$

Example 39 Find $\int x \sin x \, dx$

First, we let $u = x$ and $\frac{dv}{dx} = \sin x$. Therefore,

$\frac{du}{dx} = 1$ and $v = \int \sin x \, dx = -\cos x$

Applying the formula we get,

$$\int u\left(\frac{dv}{dx}\right)dx = uv - \int v\left(\frac{du}{dx}\right)dx$$

$$= x(-\cos x) - \int (-\cos x).1dx$$

$$= -x\cos x + \int \cos x \, dx$$

$$= -x\cos x + \sin x + c$$

Example 40 Find $\int (5x+1)\cos 2x \, dx$

Let $u = 5x + 1$ and $\frac{dv}{dx} = \cos 2x$. Therefore,

$\frac{du}{dx} = 5$ and $v = \int \cos 2x \, dx = \frac{\sin 2x}{2}$

Applying the formula we get,

$$\frac{(5x+1)\sin 2x}{2} - \int \frac{\sin 2x}{2} 5dx$$

$$= \frac{(5x+1)\sin 2x}{2} + \frac{5}{4}\cos 2x + c$$

Sometimes it is necessary to apply the formula more than once.

Example 41 Find $\int_0^2 x^2 e^x dx$

In this case $u = x^2$ and $\frac{dv}{dx} = e^x$. Therefore,

$\frac{du}{dx} = 2x$ and $v = e^x$

Applying the formula we get,

$$\int_0^2 x^2 e^x dx = \left[x^2 e^x\right]_0^2 - \int_0^2 2xe^x dx$$

$$= 4e^2 - 2\int_0^2 xe^x dx$$

The remaining integral must now be integrated by parts. This gives,

$$\int x^2 e^x dx = 4e^2 - 2\left[e^2 + 1\right]$$

$$= 2e^2 - 2$$

$$= 12.778\ (3\ dp)$$

Differential Equations

In this lecture we will begin to look at types of equations that involve derivatives known as **differential equations**. We will look at some of the basic concepts relating to these types of equations and begin to look at some of the techniques involved in finding their solutions.

Introduction

An equation is a mathematical expression which contains an unknown quantity which we wish to find. A **differential equation** is an equation which contains the derivative of an unknown expression. Examples of some differential equations are shown below,

$$\frac{dy}{dx} = x^3, \quad \frac{dv}{dt} + v^2 = 0, \quad \frac{d^2\theta}{dt^2} + \frac{d\theta}{dt} = t$$

Many models of engineering systems involve the rate of change of a quantity. There is thus a need to incorporate derivatives into the mathematical model. Accompanying the differential equation will be one or more conditions that let us obtain a unique solution to a particular problem. Often we solve the differential equation first to obtain a general solution; then we apply the conditions to obtain the unique solution. As there are different techniques associated with solving different sorts of differential equations it is important to be able to identify some its features. In this section we will look at some of the terminology associated with differential equations which then allows these features to be described.

Basic Definitions

Dependent and independent variables

In a function such as $y = x^2 + 2x$ we say that x is the **independent variable** and y is the **dependent variable** since the value of y depends on the choice we have made for x. When solving a differential equation it is essential that you can identify the dependent and independent variables.

In the differential equation $\frac{dx}{dt} = x + t^3$, x is the dependent variable and t is the independent variable. In the differential equation $\frac{dy}{dx} = y + \cos x$, y is the dependent variable. Note that the dependent variable is always the variable being differentiated i.e. that rate of change of y is dependent upon the value of x.

Order of a Differential Equation

The **order** of a differential equation is the highest order of the highest derivative that appears in the equations. Thus in the equation $\frac{dy}{dx} + y^2 = 0$ the highest derivative is the first derivative $\frac{dy}{dx}$ and so, this is called a **first-order differential equation**.

In the equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} = y$ the highest derivative is the second derivative and so, this is called a **second-order differential equation**.

Linear and Non-linear Differential Equations

A differential equation is said to be **linear** if the dependent variable and its derivatives occur to the first power only and if there are no products involving the dependent variable and/or its derivatives. There should be no non-linear functions of the dependent variable, such as sine, exponential, etc. A differential equation which is not linear is said to be **non-linear**. The linearity of a differential equation is not affected by the presence of non-linear terms involving the independent variable. The equations $\frac{d^2y}{dx^2} - y = e^x$ and $\frac{dy}{dx} + y = e^x$ are **linear** but the equations $\left(\frac{dy}{dx}\right)^2 = y$ and $y\frac{dy}{dx} = x$ are **non-linear**. The former because the derivative of the dependent variable is raised to the power 2, and the latter because of the product $y\frac{dy}{dx}$.

It is very important to know whether and equation is linear or non-linear because non-linear equations can be much harder to solve.

Constant-coefficient Linear Equations

A differential equation has **constant coefficients** if the coefficients of the dependent variable and its derivatives are constants. The differential equation $5\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 8y = x^2$ **is** constant coefficient whereas the equation $x\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 4y = 0$ **is not** as the coefficient of the second derivative of the dependent variable is not a constant.

Forming Differential Equations

Often, the hardest part of dealing with differential equations is not finding their solution but forming an equation that models the system in question. Although we will look at this in much more detail in a separate lecture we will introduce this technique briefly here.

Example 42 Consider a liquid system. The tank has a constant cross-sectional area A. The liquid can flow out from the tank through a valve near the base. As it does so, the height, or **head**, h, of liquid in the tank will reduce. Let q stand for the rate at which the liquid flows out of the tank. Under certain conditions the rate of outflow is proportional to the head, so that $q = kh$ where k is a constant of proportionality. Form a differential equation for this system.

First, let's form an expression for the volume V of the liquid in the tank at any time. This gives,

$$V = A \times h$$

Because liquid is flowing out of the tank the volume of liquid changes. Specifically the rate at which the volume changes,

Volume change = rate of flow in – rate of flow out

The is the **law of conservation of mass**. The rate of change of volume is $\frac{dV}{dt}$. There is no flow into the tank and liquid flows out at a rate q. Hence,

$$\frac{dV}{dt} = -q$$

But $V = Ah$ and A is constant, so the rate of change of volume is $A\frac{dh}{dt}$. Therefore,

$$A\frac{dh}{dt} = -q$$

We are also given that $q = kh$ and so,

$$A\frac{dh}{dt} = -kh$$

This is a first-order differential equation with dependent variable h and independent variable t. It is linear and has constant coefficients. The unknown function which we seek is h(t). We will look at how we can solve this equation in a later unit.

The Solution of a Differential Equation

Given a differential equation such as $\frac{dy}{dx} = y$ a solution is found when we have obtained an expression for y in terms of x, which can be substituted into both sides of the equation to make both sides equal. A solution is an expression which satisfies the differential equation. Note that a solution is an expression of the dependent variable in terms of the independent variable.

Example 43 Show that $y = 5e^{2x}$ is a solution of the equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2y$

In order to prove this, first we must write expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

$\frac{dy}{dx} = 10e^{2x}$ and $\frac{d^2y}{dx^2} = 20e^{2x}$

Now substituting these expressions into the differential equation and noting that $y = 5e^{2x}$ gives,

$$20e^{2x} - 10e^{2x} = 2(5e^{2x})$$

$$10e^{2x} = 10e^{2x}$$

We can conclude from this that $y = 5e^{2x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2y$

There are actually many different expressions which could satisfy the differential equation given. For example, $y = 6e^{2x}$ and $y = 2e^{2x}$ would also satisfy the equation. A solution from which all the possible solutions can be found is called a **general solution**.

The general solution of the equation $\frac{dy}{dx} = y$ is $y = Ce^{x}$ where C is called an arbitrary constant. By choosing different values of C different solutions are obtained.

Conditions

To determine a value for the constant *C* we must be given more information in the form of a **condition**. For example, if we are told that at $x = 0$, $y = 4$ then from $y = Ce^x$ we have,

$$4 = Ce^0$$

$$C = 4$$

Therefore $y = 4e^x$ is the solution of the differential equation which additionally satisfies the condition $y(0) = 4$. This is called a **particular solution**.

When the solution of a differential equation is sought, and a condition is to be satisfied at the leftmost point of the interval of interest, such a condition is called an **initial condition**. The problem of solving a differential equation subject to an initial condition is often referred to as an **initial value problem**.

Example 44 Show that $x(t) = 2e^{3t}$ is a solution of the initial value problem

$$\frac{dx}{dt} - 2x = 2e^{3t}, \quad x(0) = 2$$

First, we must check that $x(t) = 2e^{3t}$ satisfies the differential equation.

$$\frac{dx}{dt} = 6e^{3t}$$

Given that $x(t) = 2e^{3t}$ then,

$$6e^{3t} - 2(2e^{3t}) = 2e^{3t}$$

$$2e^{3t} = 2e^{3t}$$

Thus, $x(t) = 2e^{3t}$ satisfies the differential equation $\frac{dx}{dt} - 2x = 2e^{3t}$

Now we must check that it satisfies the initial condition $x(0) = 2$.

$x(0) = 2e^{3\times 0} = 2$

Hence, the initial condition is satisfied.

Solving a Differential Equation by Direct Integration

When a differential equation has a particular form it is easy to solve it by integration. When $\frac{dy}{dx}$ is equal to a function of x only, such as $\frac{dy}{dx} = 4x^3$, $\frac{dy}{dx} = 5\sin 3x$ etc., then y is given by,

$y = \int f(x)dx$

If $\frac{dy}{dx} = f(x)$ then $y = \int f(x)dx$

Example 45 Obtain the general solution of the differential equation $\frac{dy}{dx} = \cos x + \sin x$

As the right-hand side of the equation is a function of x only then we can solve by direct integration. This gives,

$y = \int \cos x + \sin x \, dx$

$= \sin x - \cos x + c$

This is the general solution. Note that it contains one arbitrary constant.

The same technique can be applied to second-order differential equations of the form $\frac{d^2y}{dx^2} = f(x)$

Example 46 Obtain the general solution of the equation $\frac{d^2y}{dx^2} = 5e^{2x}$.

As the right-hand side of the equation is a function of x only then we can solve by direct integration. Integrating once gives,

$$\frac{dy}{dx} = \int 5e^{2x}\,dx$$

$$\frac{dy}{dx} = \frac{5e^{2x}}{2} + A$$

We must integrate again to find y. This gives,

$$y = \int\left(\frac{5e^{2x}}{2} + A\right)dx$$

$$= \frac{5e^{2x}}{4} + Ax + B$$

This is the general solution. Note that it contains two arbitrary constants. This is because it is a second order differential equation.

If the equation is not of this form then we must apply different techniques. We will look at these in a later unit.

Example 47 The vertical motion of a projectile such as a ball travelling vertically under the action of gravity can be described by the differential equation $\frac{d^2y}{dt^2} = -g$, where g is a constant called the acceleration due to gravity. The dependent variable y is the vertical displacement of the projectile, and the independent variable is time t. This equation is a statement of Newton's second law of motion. Solving the equation gives y in terms of t, that is the displacement as a function of time.

(a) Find the general solution of the differential equation.

(b) Apply the initial conditions $y = 0$ at $t = 0$, $\frac{dy}{dt} = v_0$ at $t = 0$, in order to find a particular solution.

(a) As we have a second order equation we must integrate twice. This gives,

$$\frac{dy}{dt} = -gt + A$$

$$y = \frac{-gt^2}{2} + At + B$$

This is the general solution.

(b) Applying the initial condition $y = 0$ when $t = 0$ to obtain a value for B gives,

$$B = 0$$

Applying the initial condition $\frac{dy}{dt} = v_0$ when $t = 0$ gives,

$$\frac{dy}{dt} = -gt + A = v_0$$

When $t = 0$,

$$\frac{dy}{dt} = A = v_0$$

Therefore,

$$A = v_0$$

The particular solution is,

$$y = -\frac{gt^2}{2} + v_0 t$$

Introduction to Separation of Variables

Separation of variable is a technique commonly used to solve first-order differential equations. It gets its name due to the fact that we rearrange the equation to be solved in such a way that all terms involving the dependent variable (say y) appear on one side of the equation and all terms involving the independent variable (say x) appear on the other side of the equation. Once we have done this then we use integration to complete the solution. It is not possible, however, to rearrange all differential equations in this way and so this technique is not always appropriate. Also, it is not always possible to perform the integration even if the variables are separable. In this section we will look at how to decide whether the method is appropriate, and how to apply it in such cases.

Separating the Variables

We can apply this technique when a differential equation takes the form,

$$\frac{dy}{dx} = f(x)\,g(y)$$

Note that the right hand side of the equation is a product of a function of x and a function of y . Examples of such equations are:

$$\frac{dy}{dx} = x^2y^3, \qquad \frac{dy}{dx} = y^2 \sin x, \qquad \frac{dv}{dr} = \frac{v}{r^2}$$

Not all first-order differential equations can be written in this form. For example, it is not possible to write $\frac{dy}{dx} = x^2 + y^2$ in this form.

In general, the solution of the equation

$$\frac{dy}{dx} = f(x)\,g(y)$$

is found from

$$\int \frac{1}{g(y)}\,dy = \int f(x)dx$$

In other words, we can divide through by $g(y)$ to separate the variables and then integrate both sides to find the solution.

Example 48 Solve the first-order differential equation $\frac{dy}{dx} = 3ye^x$ by separation of variables.

Note that the right-hand side is of the form $f(x)g(y)$ where $f(x) = e^x$ and $g(y) = 3y$. Dividing both sides by $g(y)$ gives,

$$\frac{1}{3y}\frac{dy}{dx} = e^x$$

We now must integrate both sides of the equation with respect to x,

$$\int \frac{1}{3y}\frac{dy}{dx}dx = \int e^x \, dx$$

$$= \int \frac{1}{3y}dy = \int e^x \, dx$$

Performing the two integrations gives,

$$\frac{1}{3}\ln y = e^x + C$$

Note that we only need to write one constant of integration even though there should technically be two. We now have a relationship between y and x. This is the general solution. In this case we could rearrange this to give y explicitly if required.

Example 49 Solve the differential equation $\frac{dy}{dx} = \frac{3x^2}{y}$ using separation of variables and find the particular solution which satisfies $y(0) = -1$

The equation is in the form $f(x)g(y)$ where $f(x) = 3x^2$ and $g(y) = \frac{1}{y}$. Separating the variables gives,

$$y\frac{dy}{dx} = 3x^2$$

Integrating both sides gives,

$$\int y\frac{dy}{dx}dx = \int 3x^2\,dx$$

$$\int y\,dy = \int 3x^2\,dx$$

$$\frac{y^2}{2} = x^3 + C$$

We now have a relationship between y and x, i.e. the general solution. We can rearrange this in terms of y explicitly if required giving,

$$y = \sqrt{2x^3 + 2C}$$

Applying the condition $y(0) = -1$ gives,

$$-1 = \sqrt{2 \times 0^3 + 2C}$$

$$-1 = \sqrt{2C}$$

$$(-1)^2 = 2C$$

$$2C = 1$$

The particular solution is therefore,

$$y = \pm\sqrt{2x^3 + 1}$$

Solving second order Differential Equations

The solution to these types of equations is achieved in stages.

1) Find the complementary function, y_{cf}
2) Find the particular integral, y_{pi}
3) The general solution is the sum of the complementary function and the particular integral: $y = y_{cf} + y_{pi}$

Complementary Function

The general form of a second-order linear equation which has constant coefficients is

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

where a, b and c are constants. An example of such an equation is

$$4\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 6y = e^x \sin x$$

When f(x) is not zero then the equation is said to be **non-homogeneous.** If we replace the function f(x) with zero then this is called the **homogeneous** form of the original equation. The homogeneous form is found by ignoring the term which is independent of y, or its derivatives. The general solution of the homogeneous form of an equation is called the **complementary function** and will always contain two arbitrary constants.

If $y_1(x)$ and $y_2(x)$ are any two (linearly independent) solutions of a linear, homogeneous second-order differential equation then the general solution $y_{cf}(x)$ is

$$y_{cf}(x) = Ay_1(x) + By_2(x)$$

where A and B are constants.

If we can find two independent solutions of a homogeneous differential equation, we can form the complementary function by simply adding constant multiples of the two solutions.

Example 50

(a) Verify that $y_1 = e^{4x}$ and $y_2 = e^{2x}$ both satisfy the constant-coefficient homogeneous equation,

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$$

(b) Write down the general solution of this equation and so form the complementary function.

(a) Consider $y_1 = e^{4x}$.

$$\frac{dy}{dx} = 4e^{4x} \text{ and } \frac{d^2y}{dx^2} = 16e^{4x}$$

Substituting these into the equation gives,

$$16e^{4x} - 6(4e^{4x}) + 8(e^{4x})$$

This simplifies to zero and so is indeed a solution.

Consider $y_2 = e^{2x}$

$$\frac{dy}{dx} = 2e^{2x} \text{ and } \frac{d^2y}{dx^2} = 4e^{2x}$$

Substituting these into the equation gives,

$$4e^{2x} - 6(2e^{2x}) + 8(e^{2x})$$

Again, this simplifies to zero and so is another solution.

(b) Now we have two independent solutions of the equation: these are $y_1 = e^{4x}$ and $y_2 = e^{2x}$. They are linearly independent because e^{2x} is not a multiple of e^{4x}. The general solution and complementary function is therefore,

$$y_{cf} = Ae^{4x} + Be^{2x}$$

Example 51

Substitute $y = e^{kx}$, where k is a constant, into the equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$$

in order to find the values of k which make $y = e^{kx}$ a solution. Hence state the general solution.

Substituting $y = e^{kx}$ we find,

$$\frac{dy}{dx} = ke^{kx} \quad \text{and} \quad \frac{d^2y}{dx^2} = k^2e^{kx}$$

Substituting these into the equation gives,

$$k^2e^{kx} - ke^{kx} - 6e^{kx} = 0$$

$$= \left(k^2 - k - 6\right)e^{kx} = 0$$

Since e^{kx} can never be zero it follows that,

$$k^2 - k - 6 = 0$$

From which $k = 3$ or $k = -2$.

We have, therefore, found two linearly independent solutions:

$$y_1 = e^{3x} \quad \text{and} \quad y_2 = e^{-2x}$$

The general solution is,

$$y_{cf} = Ae^{3x} + Be^{-2x}$$

This is the complementary function for the given differential equation. The equation for determining k, i.e. $k^2 - k - 6 = 0$, is called the **auxiliary equation**.

The auxiliary equation of,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

is,

$$ak^2 + bk + c = 0$$

The solutions of the auxiliary equation give values of k which make $y = e^{kx}$ a solution of the differential equation.

Note that the auxiliary equation is a quadratic equation and so has two roots. Depending on the values of a, b and c, in some cases the roots will be complex. The procedure for dealing with such cases is demonstrated in the following example.

Example 52 Find the general solution of $\dfrac{d^2y}{dx^2} + 4y = 0$

First, write down the auxiliary equation. This is,

$k^2 + 4 = 0$

From this,

$k^2 = -4$

$k = \pm 2j$

Note we now have complex roots giving the two solutions of the differential equation,

$y = e^{2jx}$ and $y = e^{-2jx}$

The general solution can now be written as,

$$y_{cf} = Ae^{2jx} + Be^{-2jx}$$

However, it is usual to rewrite the solution in a different form. Using Euler's relations we can write,

$$e^{2jx} = \cos 2x + j\sin 2x \quad \text{and} \quad e^{-2jx} = \cos 2x - j\sin 2x$$

So that,

$$y(x) = A(\cos 2x + j\sin 2x) + B(\cos 2x - j\sin 2x)$$

$$= (A + B)\cos 2x + (Aj - Bj)\sin 2x$$

If we now re-label the constants such that,

$$A + B = C \quad \text{and} \quad Aj - Bj = D$$

Then the general solution can be written in the form,

$$y_{cf} = C\cos 2x + D\sin 2x$$

This is the complementary function.

If the auxiliary equation has complex roots, $\alpha + \beta j$ and $\alpha - \beta j$, then the complementary function is

$$y_{cf} = e^{\alpha x}(C\cos\beta x + D\sin\beta x)$$

Example 53 Find the general solution of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 0$

The auxiliary equation is,

$$k^2 + 2k + 4 = 0$$

The roots of the equation are,

$$k = -1 \pm \sqrt{3}j$$

Noting that $\alpha = -1$ and $\beta = \sqrt{3}$ then the general solution can be written as,

$$y = e^{-x}(C\cos\sqrt{3}x + D\sin\sqrt{3}x)$$

A Particular Integral

The general form of a second-order linear equation which has constant coefficients is

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

where a, b and c are constants. An example of such an equation is

$$4\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 6y = e^x \sin x$$

When f(x) is not zero then the equation is said to be **non-homogeneous**. Any solution of an non-homogeneous differential equation is called a **particular integral** denoted by y_{pi} .

Example 54 Show that $y = \sin 2x$ is a particular integral of $\frac{d^2y}{dx^2} + 7y = 3\sin 2x$

If $y = \sin 2x$ then,

$$\frac{dy}{dx} = 2\cos 2x$$

$$\frac{d^2y}{dx^2} = -4\sin 2x$$

Substituting these into the differential equation gives,

$$\frac{d^2y}{dx^2} + 7y = 3\sin 2x$$

$$-4\sin 2x + 7(\sin 2x) = 3\sin 2x$$

$$3\sin 2x = 3\sin 2x$$

This shows that $y = \sin 2x$ is indeed a solution. This solution is called the particular integral.

Finding a Particular Integral

There are a number of advanced techniques available for finding a particular integral but we shall adopt a simpler strategy. The strategy is actually rather crude and involves trial and error. We try solutions which are of the same form as the inhomogeneous term, $f(x)$, on the right-hand side of the equation. The trial solutions all contain constants which can be adjusted to force the trial solution to be an actual solution of the inhomogeneous equation. As a guide we can use the table below.

Type	*Straightforward cases* Try as particular integral:	*'Snag' cases* Try as particular integral:
(a) $f(x) =$ a constant	$v = k$	$v = kx$ (used when C.F. contains a constant)
(b) $f(x) =$ polynomial (i.e. $f(x) = L + Mx + Nx^2 + \cdots$ where any of the coefficients may be zero)	$v = a + bx + cx^2 + \cdots$	
(c) $f(x) =$ an exponential function (i.e. $f(x) = Ae^{ax}$)	$v = ke^{ax}$	(i) $v = kxe^{ax}$ (used when e^{ax} appears in the C.F.) (ii) $v = kx^2e^{ax}$ (used when e^{ax} **and** xe^{ax} both appear in the C.F.)
(d) $f(x) =$ a sine or cosine function (i.e. $f(x) = a\sin px + b\cos px$, where a or b may be zero)	$v = A\sin px + B\cos px$	$v = x(A\sin px + B\cos px)$ (used when $\sin px$ and/or $\cos px$ appears in the C.F.)
(e) $f(x) =$ a sum e.g. (i) $f(x) = 4x^2 - 3\sin 2x$ (ii) $f(x) = 2 - x + e^{3x}$	(i) $v = ax^2 + bx + c + d\sin 2x + e\cos 2x$ (ii) $v = ax + b + ce^{3x}$	
(f) $f(x) =$ a product e.g. $f(x) = 2e^x\cos 2x$	$v = e^x(A\sin 2x + B\cos 2x)$	

Example 55 Find the particular integral of the equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = e^{2x}$

We will attempt to find the particular integral by trying a function of the same form as that on the right-hand side. In this case we will try $y = ke^{2x}$, where k is a constant that we will now determine.

If $y = ke^{2x}$ then,

$$\frac{dy}{dx} = 2ke^{2x}$$

$$\frac{d^2y}{dx^2} = 4ke^{2x}$$

Substituting these into the original equation gives,

$$4ke^{2x} - 2ke^{2x} - 6ke^{2x} = e^{2x}$$

$$-4ke^{2x} = e^{2x}$$

This means that y will be a solution if k is chosen so that $-4k = 1$, i.e. $k = -\frac{1}{4}$

Therefore, the particular integral is,

$$y_{pi} = -\frac{1}{4}e^{2x}$$

Example 56 Find a particular integral for the equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 3\cos x$

In this case we can try the function $y = A\sin x + B\cos x$, where we need to determine A and B.

If $y = A\sin x + B\cos x$ then,

$$\frac{dy}{dx} = A\cos x - B\sin x$$

$$\frac{d^2y}{dx^2} = -A\sin x - B\cos x$$

Substituting into the differential equation gives,

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 3\cos x$$

$$-A\sin x - B\cos x - 6(A\cos x - B\sin x) + 8(A\sin x + B\cos x) = 3\cos x$$

$$-A\sin x - B\cos x - 6A\cos x + 6B\sin x + 8A\sin x + 8B\cos x = 3\cos x$$

$$7A\sin x + 7B\cos x - 6A\cos x + 6B\sin x = 3\cos x$$

Equating the coefficients of $\sin x$,

$$7A + 6B = 0$$

Equating the coefficients of $\cos x$,

$$7B - 6A = 3$$

Solving these two equations simultaneously gives,

$A = -\frac{18}{85}$ and $B = \frac{21}{85}$

Therefore, a particular integral is,

$$y_{pi} = \frac{21}{85}\cos x - \frac{18}{85}\sin x$$

Finding the General Solution of a Second-Order Inhomogeneous Equation

Now we have looked at how we can identify the complementary function and a particular integral of a differential equation we can look at how we can use this information in finding the general solution. Recall that,

$$y(x) = y_{pi} + y_{cf}$$

Example 57 Find the general solution of $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 10y = 3x^2$

First, we need to find the complementary function. The auxiliary equation is,

$$k^2 + 3k - 10 = 0$$

Solving this equation for k gives,

$k = 2$ and $k = -5$

The complementary function is therefore,

$$y_{cf} = Ae^{2x} + Be^{-5x}$$

Now we need to find a particular integral. Noting that $3x^2$ is a polynomial of degree 2 we can try the function $y = ax^2 + bx + c$.

Differentiating twice gives,

$$\frac{dy}{dx} = 2ax + b$$

$$\frac{d^2y}{dx^2} = 2a$$

Substituting into the differential equation gives,

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 10y = 3x^2$$

$$2a + 3(2ax + b) - 10(ax^2 + bx + c) = 3x^2$$

Equating the coefficients of x^2,

$$-10a = 3$$

Equating the coefficients of x,

$$6a - 10b = 0$$

Equating constants,

$$2a + 3b - 10c = 0$$

Solving these to find a, b and c gives,

$a = -\dfrac{3}{10}$, $b = -\dfrac{9}{50}$ and $c = -\dfrac{57}{500}$

The particular integral is therefore,

$$y_{pi} = -\frac{3}{10}x^2 - \frac{9}{50}x - \frac{57}{500}$$

The general solution is,

$$y(x) = y_{pi} + y_{cf}$$

$$y(x) = -\frac{3}{10}x^2 - \frac{9}{50}x - \frac{57}{500} + Ae^{2x} + Be^{-5x}$$

Example 58 Find the particular solution of $y'' + y' - 12y = 4e^{2x}$ which satisfies $y(0) = 7$ and $y'(0) = 0$.

First, we need to find the complementary function. The auxiliary equation is,

$$k^2 + k - 12 = 0$$

Solving this equation gives,

$k = 3$ and $k = -4$.

The complementary function is,

$$y_{cf} = Ae^{3x} + Be^{-4x}$$

Now finding the particular integral,

Trying the function $y = ke^{2x}$ gives,

$$\frac{dy}{dx} = 2ke^{2x}$$

$$\frac{d^2y}{dx^2} = 4ke^{2x}$$

Substituting these expressions into the differential equation gives,

$$y'' + y' - 12y = 4e^{2x}$$

$$4ke^{2x} + 2ke^{2x} - 12ke^{2x} = 4e^{2x}$$

$$-6ke^{2x} = 4e^{2x}$$

Therefore,

$$-6k = 4$$

$$k = -\frac{2}{3}$$

The general solution is therefore,

$$y(x) = Ae^{3x} + Be^{-4x} - \frac{2}{3}e^{2x}$$

The particular solution is now found by applying the boundary conditions.

Applying the condition $y(0) = 7$,

$$7 = A + B - \frac{2}{3}$$

To apply the condition $y'(0) = 0$ we must differentiate the general solution. This gives,

$$\frac{dy}{dx} = 3Ae^{3x} - 4Be^{-4x} - \frac{4}{3}e^{2x}$$

Now applying the condition $y'(0) = 0$,

$$0 = 3A - 4B - \frac{4}{3}$$

Solving these two equations simultaneously gives,

$A = \frac{32}{7}$ and $B = \frac{65}{21}$

The particular solution is therefore,

$$y = \frac{32}{7}e^{3x} + \frac{65}{21}e^{-4x} - \frac{2}{3}e^{2x}$$

Matrices

A matrix is a rectangular array of numbers or expressions. The plural of matrix is matrices. In this lecture we will look at how we add, subtract and multiply matrices. The division of matrices is not defined.

The theory of matrices is particularly useful in the solution of large systems of simultaneous equations with many unknowns. We will look at this in a later lecture.

Introduction

A **matrix** is a rectangular array of numbers or expressions usually enclosed in brackets. For example,

$$\begin{pmatrix} 3 & 2 & 4 \\ 0 & 5 & 6 \end{pmatrix}, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \begin{pmatrix} 2-\lambda & 4 \\ 7 & 5-\lambda \end{pmatrix}$$

are all matrices. We often denote a matrix by a capital letter, for example,

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 0 & 5 & 6 \end{pmatrix}$$

The **size** of a matrix is given by the number of rows and the number of columns. The matrix A above has 2 rows and 3 columns and so is defined as a 2 x 3 (2 by 3) matrix. Note that the number of rows is always stated first.

The individual numbers or expressions in a matrix are called the **elements** and are usually denoted by a small letter. For example, for A, the element in row 1, column 2, is denoted by a_{12}, the element in row 3 column 1 is denoted by a_{31} and so on. So, in matrix A above,

$$a_{11} = 3, a_{12} = 2, a_{13} = 4, a_{21} = 0, a_{22} = 5, a_{23} = 6$$

A **square** matrix has the same number of columns as rows i.e. a 2 x 2 matrix. The **main diagonal** of a square matrix is the diagonal running from top-left to bottom-right of the matrix.

An **identity** matrix is a matrix with ones on the main diagonal and zeros everywhere else. Identity matrices are usually denoted by an I with a subscript to denote its size. The 3 x 3 identity matrix, I_3, is shown below.

$$I_3 = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

The **transpose** of a matrix is obtained by writing rows as columns. The transpose of A is denoted A^T. For example, if

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 0 & 5 & 6 \end{pmatrix}$$

then the transpose of A, A^T, is given by

$$A^T = \begin{pmatrix} 3 & 0 \\ 2 & 5 \\ 4 & 6 \end{pmatrix}$$

The transpose of the sum of two matrices is the same as the sum of the individual matrices transposed:

$$(A + B)^T = A^T + B^T$$

Addition and Subtraction of Matrices

Matrices of the **same** size can be added to and subtracted from one another. To do this, the corresponding elements are added and subtracted.

Example 59 Given $A = \begin{pmatrix} 3 & 1 & -1 \\ 0 & -2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 6 & 1 \\ 10 & -3 & 6 \end{pmatrix}$

find (a) **A + B** and (b) **B – A**

(a) First we must determine whether the matrices are of the same size which in this case they are. We can, therefore, go ahead and add or subtract them.

If $A = \begin{pmatrix} 3 & 1 & -1 \\ 0 & -2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 6 & 1 \\ 10 & -3 & 6 \end{pmatrix}$ then,

$$A+B=\begin{pmatrix}3 & 1 & -1\\ 0 & -2 & 4\end{pmatrix}+\begin{pmatrix}2 & 6 & 1\\ 10 & -3 & 6\end{pmatrix}=\begin{pmatrix}3+2 & 1+6 & -1+1\\ 0+10 & -2-3 & 4+6\end{pmatrix}$$

$$=\begin{pmatrix}5 & 7 & 0\\ 10 & -5 & 10\end{pmatrix}$$

(b) If $A=\begin{pmatrix}3 & 1 & -1\\ 0 & -2 & 4\end{pmatrix}$ and $B=\begin{pmatrix}2 & 6 & 1\\ 10 & -3 & 6\end{pmatrix}$ then,

$$B-A=\begin{pmatrix}2 & 6 & 1\\ 10 & -3 & 6\end{pmatrix}-\begin{pmatrix}3 & 1 & -1\\ 0 & -2 & 4\end{pmatrix}=\begin{pmatrix}2-3 & 6-1 & 1-(-1)\\ 10-0 & -3-(-2) & 6-4\end{pmatrix}$$

$$=\begin{pmatrix}-1 & 5 & 2\\ 10 & -1 & 2\end{pmatrix}$$

<u>Multiplication of a Matrix by a Scalar</u>

Any matrix can be multiplied by a number. To do this, each element of the matrix is multiplied by the number.

Example 60 If $A=\begin{pmatrix}3 & 1 & -1\\ 0 & -2 & 4\end{pmatrix}$ find,

(a) $2A$ (b) $\frac{1}{2}A$ (c) $-A$

(a) $2A=\begin{pmatrix}3\times 2 & 1\times 2 & -1\times 2\\ 0\times 2 & -2\times 2 & 4\times 2\end{pmatrix}=\begin{pmatrix}6 & 2 & -2\\ 0 & -4 & 8\end{pmatrix}$

(b) $\frac{1}{2}A=\frac{1}{2}\begin{pmatrix}3 & 1 & -1\\ 0 & -2 & 4\end{pmatrix}=\begin{pmatrix}1.5 & 0.5 & -0.5\\ 0 & -1 & 2\end{pmatrix}$

(c) $-A=(-1)A=(-1)\begin{pmatrix}3 & 1 & -1\\ 0 & -2 & 4\end{pmatrix}=\begin{pmatrix}-3 & -1 & 1\\ 0 & 2 & -4\end{pmatrix}$

Multiplication of Matrices

Now we will look at how a matrix can be multiplied by another matrix and the conditions under which such a multiplication can take place.

Conditions needed for two matrices to be multiplied together

If A and B are matrices then the product of A and B i.e. A x B, can only exist if the number of columns in A is the same as the number of rows in B. If this is not the case then they cannot be multiplied together and we say that the product does not exist.

Example 61 Given that $A = \begin{pmatrix} 3 & 1 & 4 \\ 5 & 2 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ -6 & 4 \end{pmatrix}$ calculate the product of AB.

We note that A is a 2 x 3 matrix, B is a 3 x 2 matrix and so the product AB can be calculated. As matrix A has 2 rows and matrix B has 2 columns then the result is a 2 x 2 matrix, say C.

$$AB = C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

To find the element c_{11} we use row 1 from A and column 1 from B. The elements in this row and column are multiplied together and added thus:

$$c_{11} = 3(1) + 1(2) + 4(-6) = -19$$

The element c_{12} is calculated by using row 1 from A and column 2 from B.

$$c_{12} = 3(-1) + 1(0) + 4(4) = 13$$

The element c_{21} is calculated by using row 2 from A and column 1 from B.

$$c_{21} = 5(1) + 2(2) - 2(-6) = 21$$

The element c_{22} is calculated by using row 2 from A and column 2 from B.

$$c_{22} = 5(-1) + 2(0) - 2(4) = -13$$

Hence,

$$AB = \begin{pmatrix} -19 & 13 \\ 21 & -13 \end{pmatrix}$$

Determinants

We have looked at how we can add, subtract and multiply matrices. Division of matrices is not defined; however, for some matrices it is possible to calculate an inverse matrix. In some ways, use of the inverse matrix takes the place of division. To calculate an inverse matrix requires knowledge of determinants so we will look at this first. All square matrices possess a determinant. We will look at the simplest square matrices first: 2 x 2 matrices.

Determinant of a 2 x 2 Matrix

Consider the 2 x 2 matrix A where,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then the determinant of A is ad-bc. The determinant of A is denoted by,

$$\det(A), \quad |A| \quad or \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

The determinant of a matrix is no longer a matrix but a single number or expression.

Example 62 Given $A = \begin{pmatrix} 3 & 6 \\ -1 & 1 \end{pmatrix}$ calculate the determinant of (a) A and (b) A^T

(a) $|A| = 3(1) - 6(-1) = 9$

(b) $A^T = \begin{pmatrix} 3 & -1 \\ 6 & 1 \end{pmatrix}$

Hence,

$|A^T| = 3(1) - (-1)6 = 9$

The determinant of a matrix and its transpose are the same.

The Inverse of a 2 x 2 Matrix

Suppose A is a 2 x 2 matrix and $|A| \neq 0$. Then the inverse of A can be found. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then we recall that $|A| = ad - bc$. Now A^{-1} is given by,

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

In other words, to find the inverse of a 2 x 2 matrix we swap elements a and d over and change the signs on the other two.

Example 63 Given $A = \begin{pmatrix} 3 & 4 \\ 2 & 6 \end{pmatrix}$ find A^{-1}.

First, let's find the determinant of A.

$|A| = (3)(6) - (4)(2) = 10$

The inverse of A is therefore,

$$A^{-1} = \frac{1}{10}\begin{pmatrix} 6 & -4 \\ -2 & 3 \end{pmatrix}$$

Note that we can only find the inverse of a square matrix i.e. 2 x 2, 3 x 3, etc..

If the determinant of a matrix is zero then the inverse does not exist.

<u>The Determinant of a 3 x 3 Matrix</u>

In order to understand how to find the determinant of a 3 x 3 matrix we must first introduce the **minor** of an element and the **cofactor** of an element.

The Minor of an Element

Consider a 3 x 3 matrix A where,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

If we choose an element, say a_{11}, and cross out the row and column in which the element appears we are left with a 2 x 2 matrix. The determinant of this matrix is called the **minor** of a_{11}.

Example 64 Given $A = \begin{pmatrix} 3 & -1 & 6 \\ 9 & -5 & 2 \\ 0 & 4 & 7 \end{pmatrix}$ calculate the minor of (a) 3, (b) 9, (c) 2

(a) The element 3 appears in row 1 and column 1 and so, deleting this row and column gives the 2 x 2 matrix,

$$\begin{pmatrix} -5 & 2 \\ 4 & 7 \end{pmatrix}$$

The determinant of this matrix is $(-5)(7)-(2)(4)=-43$. The minor of 3 is -43.

(b) The element 9 appears in row 2 and column 1. Deleting these gives the 2 x 2 matrix,

$$\begin{pmatrix} -1 & 6 \\ 4 & 7 \end{pmatrix}$$

The determinant of this matrix is $(-1)(7)-(6)(4)=-31$

The minor of 9 is -31.

(c) The element 2 occurs in row 2 and column 3. Deleting these gives the 2 x 2 matrix,

$$\begin{pmatrix} 3 & -1 \\ 0 & 4 \end{pmatrix}$$

The determinant of this matrix is $(3)(4)-(-1)(0)=12$

The minor of 2 is 12.

The Cofactor of an Element

Closely related to the minor of an element is the **cofactor** of an element. The cofactor of an element is either +1 or -1 times its minor depending on the elements position in the matrix. The following grid of + and – is an easy way to visualise this.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

The + and – are known as the **place signs** of the elements. If the place sign is + then the cofactor and minor are identical. If the place sign is – then the cofactor is –(minor).

Example 65 The matrix A is given as $A = \begin{pmatrix} 3 & -1 & 6 \\ 9 & -5 & 2 \\ 0 & 4 & 7 \end{pmatrix}$.

Calculate the cofactor of (a) 3, (b) 9, (c) 2

(a) The minor of 3 is -43. Since the place sign of this element is + then the cofactor is also -43.
(b) The minor of 9 is -31. The place sign is – so the cofactor is 31.
(c) The minor of 2 is 12. The place sign is – and so the cofactor is -12.

Determinant of a 3 x 3 Matrix

Consider a general 3 x 3 matrix, A

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The determinant of A is found using,

$$|A| = a_{11} \times (its\,cofactor) + a_{12} \times (its\,cofactor) + a_{13} \times (its\,cofactor)$$

Example 66 Calculate $|A|$ where A is given by $A = \begin{pmatrix} 3 & -1 & 6 \\ 9 & -5 & 2 \\ 0 & 4 & 7 \end{pmatrix}$

We have $a_{11} = 3, a_{12} = -1$ and $a_{13} = 6$. In order to find the determinant of the matrix we must first find the cofactors of these elements. The cofactor of 3 is -43 as shown above.

The minor of -1 is $\begin{vmatrix} 9 & 2 \\ 0 & 7 \end{vmatrix} = 63$

The place sign of -1 is - and so its cofactor is -63.

The minor of 6 is $\begin{vmatrix} 9 & -5 \\ 0 & 4 \end{vmatrix} = 36$

The place sign of 6 is + and so its cofactor is 36.

The determinant of A is therefore,

$$|A| = 3(-43) + (-1)(-63) + 6(36) = 150$$

<u>The Inverse of a 3 x 3 Matrix</u>

In order to find the inverse of a 3 x 3 matrix we must proceed as follows:

1. Find the transpose of the matrix A, by interchanging the rows and columns of the matrix.
2. Replace each element of A^T by its cofactor. The resulting matrix is known as the **adjoint** of A, written adj(A).
3. The inverse of A is given by,

$$A^{-1} = \frac{adj\,(A)}{|A|}$$

Example 67 Find the inverse of $A = \begin{pmatrix} 3 & 1 & 0 \\ 5 & 2 & -1 \\ 1 & 4 & -2 \end{pmatrix}$

First, we find the transpose of A by interchanging rows and columns. This gives,

$$A^T = \begin{pmatrix} 3 & 5 & 1 \\ 1 & 2 & 4 \\ 0 & -1 & -2 \end{pmatrix}$$

Each element of A^T is replaced by its cofactor to give the adjoint of A. This gives,

$$adj\,(A) = \begin{pmatrix} 0 & 2 & -1 \\ 9 & -6 & 3 \\ 18 & -11 & 1 \end{pmatrix}$$

Calculating the determinant of A gives $|A| = 9$ and so,

$$A^{-1} = \frac{adj\,(A)}{|A|}$$

$$= \frac{1}{9}\begin{pmatrix} 0 & 2 & -1 \\ 9 & -6 & 3 \\ 18 & -11 & 1 \end{pmatrix}$$

Using Matrices and Determinants to Solve Equations

In this section we will look at two methods involving matrices that allow us to solve a set of simultaneous linear equations. These are: Cramer's rule and the inverse matrix method.

Cramer's Rule

Consider two linear simultaneous equations with two unknowns x and y,

$$a_1x + b_1y = k_1$$
$$a_2x + b_2y = k_2$$

Cramer's rule is a way of finding the solution of equations like these as the ratio of two determinants.

Cramer's rule states,

$$x = \frac{\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Note that the denominator is the same in each case and is simply the determinant made up of the coefficients of x and y. If this denominator is zero then Cramer's rule cannot be applied. In such a case, either a unique solution to the system does not exist or there is no solution.

Note how the numerator of each fraction is formed. To form the numerator for the x fraction we replace the column of a's in $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ by a column of k's. Similarly the numerator for the y fraction is formed by replacing the column of b's by a column of k's.

Example 68 The currents i_1 and i_2 in a simple circuit are connected by the equations,

$$i_1 + i_2 = 3$$
$$2i_1 + 3i_2 = 7$$

Calculate i_1 and i_2 using Cramer's rule.

Using Cramer's rule we have,

$$i_1 = \frac{\begin{vmatrix} 3 & 1 \\ 7 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} = \frac{2}{1} = 2, \qquad i_2 = \frac{\begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} = \frac{1}{1} = 1$$

The currents in the circuit are $i_1 = 2, i_2 = 1$.

We can extend Cramer's rule to systems of linear equations with more than two unknowns. Consider the following linear system with three unknowns, x, y and z.

$$a_1x + b_1y + c_1z = k_1$$
$$a_2x + b_2y + c_2z = k_2$$
$$a_3x + b_3y + c_3z = k_3$$

Cramer's rule states,

$$x = \frac{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Example 69 The voltages v_1, v_2 and v_3 in an electrical circuit are connected by the system,

$$v_1 + 2v_2 + v_3 = 2$$
$$2v_1 - v_2 - 2v_3 = 5$$
$$2v_1 + 2v_2 + 3v_3 = 7$$

Use Cramer's rule to determine the three voltages.

$$v_1 = \frac{\begin{vmatrix} 2 & 2 & 1 \\ 5 & -1 & -2 \\ 7 & 2 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & -2 \\ 2 & 2 & 3 \end{vmatrix}} = \frac{-39}{-13} = 3$$

$$v_2 = \frac{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{13}{-13} = -1$$

$$v_3 = \frac{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{-13}{-13} = 1$$

The voltages are $v_1 = 3$, $v_2 = -1$ and $v_3 = 1$.

<u>Using the Inverse Matrix to Solve Simultaneous Equations</u>

Now we will look at how an inverse matrix can be used to solve a system of linear equations.

Writing Equations in Matrix Form

Consider the simultaneous equations,

$$7x + 2y = 12$$
$$3x + y = 5$$

Recalling how matrices are multiplied we note that the product,

$$\begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

is equivalent to,

$$\begin{pmatrix} 7x + 2y \\ 3x + y \end{pmatrix}$$

Hence we can write the simultaneous equations as,

$$\begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$$

This is the **matrix form** of the simultaneous equations.

Writing $A = \begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad B = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$

we have the standard form

AX = B

Note that the elements of A come from the coefficients of x and y and that the elements of B come from the right-hand sides of the equations.

Solving Equations using the Inverse Matrix Method

Given AX = B, then $X = A^{-1}B$ if A^{-1} exists.

Example 70 Solve $\begin{matrix} 7x + 2y = 12 \\ 3x + y = 5 \end{matrix}$ using the inverse matrix method.

Writing the equations in matrix form gives,

$$\begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$$

Here $A = \begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$

Now we must calculate A^{-1}.

$$A^{-1} = \frac{1}{1}\begin{pmatrix} 1 & -2 \\ -3 & 7 \end{pmatrix}$$

Therefore,

$$X = A^{-1}B$$

$$= \begin{pmatrix} 1 & -2 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} 12 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Now we have,

$$X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

This means that $x = 2$ and $y = -1$

We can extend this to systems with three unknowns using the inverse of a 3 x 3 matrix.

Example 71 The forces f_1, f_2 and f_3 in a simple pulley system are connected by the equations,

$$-f_1 + 3f_2 + 4f_3 = 7$$
$$f_2 - 4f_1 = 7$$
$$f_2 + 3f_1 - f_3 = -9$$

Solve the system for f_1, f_2 and f_3.

First, we must write the equations in matrix form. This gives,

$$\begin{pmatrix} -1 & 3 & 4 \\ -4 & 1 & 0 \\ 3 & 1 & -1 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ -9 \end{pmatrix}$$

Writing $A = \begin{pmatrix} -1 & 3 & 4 \\ -4 & 1 & 0 \\ 3 & 1 & -1 \end{pmatrix}$ we then calculate A^{-1}. This gives,

$$A^{-1} = \frac{1}{39}\begin{pmatrix} 1 & -7 & 4 \\ 4 & 11 & 16 \\ 7 & -10 & -11 \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \frac{1}{39}\begin{pmatrix} 1 & -7 & 4 \\ 4 & 11 & 16 \\ 7 & -10 & -11 \end{pmatrix}\begin{pmatrix} 7 \\ 7 \\ -9 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$$

This means that $f_1 = -2, f_2 = -1$ and $f_3 = 2$.

Elementary Row Operations

In order to solve a system of equations we must first write the augmented matrix that represents the system and then transform it in to row-echelon form. Once in row-echelon form the solution of the system can easily be found. We can transform an augmented matrix in to row-echelon form by using **elementary row operations**. The three elementary operations that we can apply to a system of equations are as follows:

1) Interchange the order of equations.
2) Multiply or divide an equation by a non-zero constant.
3) Add, or subtract, a multiple of one equation to, or from, another equation.

All of these operations will change the system of equations but will leave the solution of the equations unchanged.

Example 72

(a) Write down the augmented matrix of the system

$$2x + 5y = 12$$
$$x + y = 3$$

(b) Carry out the following elementary row operations, each time applying the operation to the most recent augmented matrix:

(i) Interchange the rows
(ii) Subtract 2 x row 1 from row 2
(iii) Divide row 2 by 3

(c) Solve the system

(a) The augmented matrix is,

$$\begin{pmatrix} 2 & 5 & 12 \\ 1 & 1 & 3 \end{pmatrix}$$

(b)

(i) Interchanging the rows gives,

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & 5 & 12 \end{pmatrix}$$

(ii) Subtracting 2 x row 1 from row 2 gives,

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 3 & 6 \end{pmatrix}$$

(ii) Dividing row 2 by 3 gives,

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$

Note that the matrix is now in row-echelon form

(c) From row 2 we get,

$$0x + 1y = 2$$

$$y = 2$$

From row 1 we get,

$$1x + 1y = 3$$

We know that $y = 2$ and so,

$$x = 1$$

Thus, the solution to the original system is $x = 1$ and $y = 2$.

This method is known as **Gaussian elimination**.

Example 73 Use Gaussian elimination to solve $\begin{aligned} 3x - y &= 1 \\ 2x + 3y &= 19 \end{aligned}$

First, we must write the augmented matrix,

$$\begin{pmatrix} 3 & -1 & 1 \\ 2 & 3 & 19 \end{pmatrix}$$

We now apply a series of elementary row operations to transform the augmented matrix in to row-echelon form.

Subtracting row 2 from row 1,

$$\begin{pmatrix} 1 & -4 & -18 \\ 2 & 3 & 19 \end{pmatrix}$$

Subtracting 2 x row 1 from row 2,

$$\begin{pmatrix} 1 & -4 & -18 \\ 0 & 11 & 55 \end{pmatrix}$$

Dividing row 2 by 11,

$$\begin{pmatrix} 1 & -4 & -18 \\ 0 & 1 & 5 \end{pmatrix}$$

The matrix is now in row-echelon form. From row 2 we can see that,

$$y = 5$$

Substituting this in to row 1 gives,

$$1x - 4y = -18$$

$$1x - 4(5) = -18$$

$$x = 2$$