DATA MODELLING AND SIMULATION

LECTURE 7:

Numerical Methods - I

- Euler's Method

By,
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Euler’s Method, Real-Life Example

<u>Euler’s Method</u>

There are many complex differential equations that we cannot solve analytically. In such cases, we can approximate solutions *numerically*.

By using a tangent line approximation, we can estimate the solution to a differential equation quite well. This is known as **Euler’s Method**.

Leonhard Euler
1707 – 1783

$$y_1 = y_0 + f(t_0, y_0)(t_1 - t_0).$$

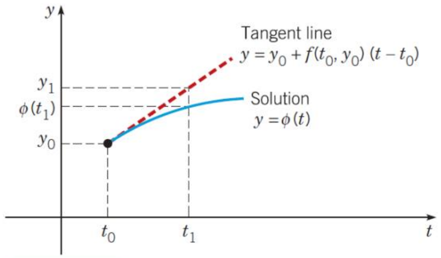


FIGURE 2.7.2 A tangent line approximation of $y' = f(t, y)$ at (t_0, y_0).

Boyce, Diprima, Meade.

Elementary Differential Equations and Boundary Problems

$$\boxed{y_{n+1} = y_n + y'(t_n, y_n)\Delta t}$$

<u>Amazing fact:</u>

Katherine Johnson (1918 – 2020) used Euler’s Method in 1961 at NASA to perform the trajectory analysis that enabled the first human space flight by austronaut Alan Shepard.

<u>Please:</u>

Watch the movie/read the book *Hidden Figures*.

- How would you plan a trajectory from Earth to Moon?
- In real life, one can also use Euler's method to from known aerodynamic coefficients to predicting trajectories. Three degree of freedom (3DOF) models are usually called point mass models, because other than drag acting opposite the velocity vector, they ignore the effects of rigid body motion.

Reference: https://www.physicsforums.com/threads/real-life-application-of-eulers-method-numerical-method.927256/

https://modelinginbiology.github.io/Videos/Katherine-Johnson-And-Eulers-Method

https://clas.sa.ucsb.edu/docs/default-source/tutor-resources-files/05_phase_lines_and_euler_s_method.pdf?sfvrsn=47502afb_0#:~:text=Katherine%20Johnson%20(1918%20%E2%80%93%202020),flight%20by%20austronaut%20Alan%20Shepard

Euler's method

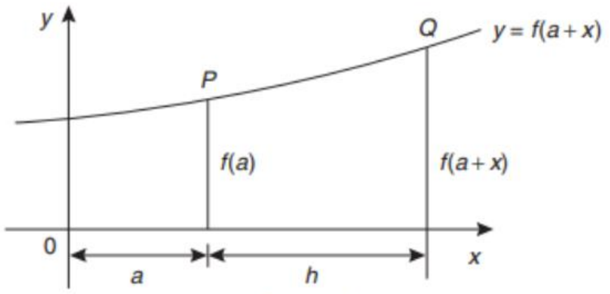


Figure 1

At Point Q at Figure 2:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots$$

Which a statement called Taylor's series and it will be closer looked at in a lecture later on in the module.

For now, if we say h is the interval between the two ordinates $y_0$and $y_1$as shown in Figure 3, and if $f(a) = y_0$ and $y_1 = f(a+h)$, then Euler's method states:

$$f(a+h) = f(a) + hf'(a)$$

i.e.

$$y_1 = y_o + h(y')_0$$

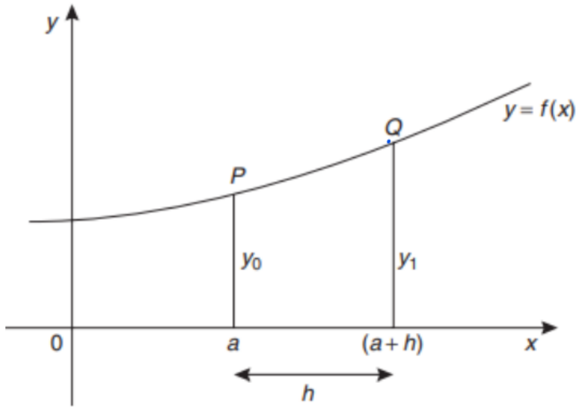


Figure 2

The approximation used with Euler's method is to take only the first two terms of Taylor's series.

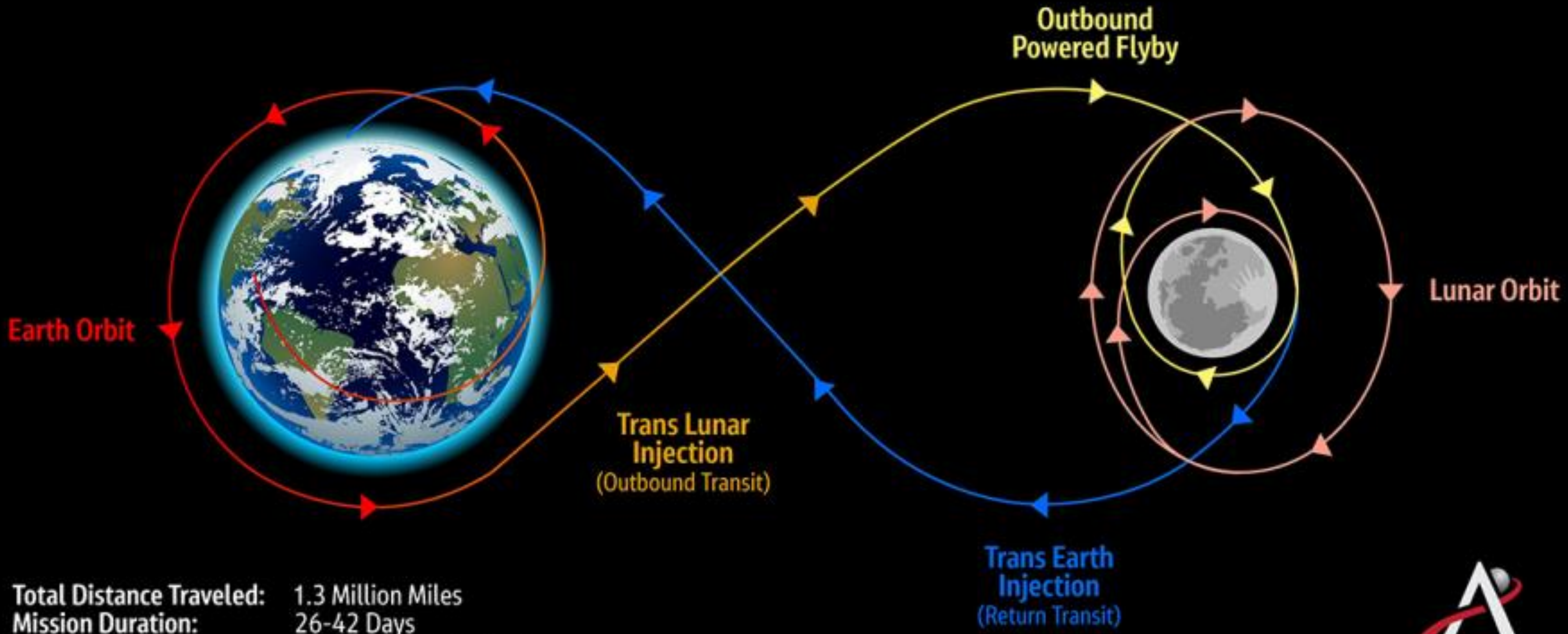
- Identify Inputs:
  - ODE
  - Constraints/initial values (x_0, y_0)
  - Intervals
  - Range
- Use Euler's method:

$$\boxed{y_n = y_{n-1} + h(y')_{n-1}}$$

- Continue finding y_n, y_{n-1}, y_{n-1}', x_n

Artemis I

The first uncrewed, integrated flight test of NASA's Orion spacecraft and Space Launch System rocket, launching from a modernized Kennedy spaceport



Total Distance Traveled:	1.3 Million Miles
Mission Duration:	26-42 Days
Re-Entry Speed:	24,000 mph (Mach 32)
CubeSats Deployed:	13

DATA MODELLING AND SIMULATION

LECTURE 8:

Numerical Methods - II

- Runge-Kutta (RK4) Method

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Euler’s Method, Real-Life Example

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1707 – 1783

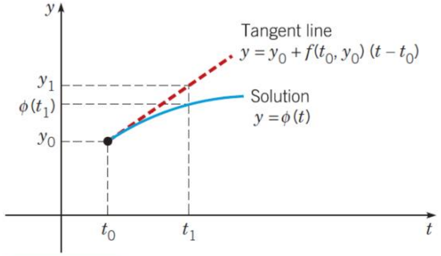

FIGURE 2.7.2 A tangent line approximation of $y' = f(t, y)$ at (t_0, y_0).

Boyce, Diprima, Meade.

Elementary Differential Equations and Boundary Problems

$$\boxed{y_{n+1} = y_n + y'(t_n, y_n)\Delta t}$$

i.e. $y_1 = y_o + h(y')_0$

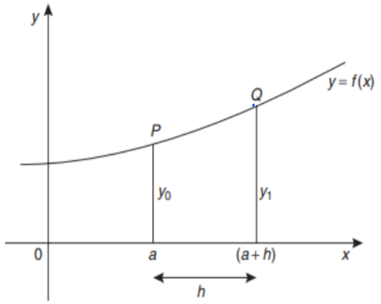

Figure 2

The approximation used with Euler’s method is to take only the first two terms of Taylor’s series.

- Identify Inputs:
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- Use Euler’s method:

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- Continue finding y_n, y_{n-1}, y_{n-1}', x_n

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Runge-Kutta Method, Real-Life Example

In numerical analysis, the **Runge–Kutta methods** (English: /ˈrʊŋəˈkʊtɑː/ listen) *RUUNG-ə-KUUT-tah*[1]) are a family of implicit and explicit iterative methods, which include the Euler method, used in temporal discretization for the approximate solutions of simultaneous nonlinear equations.[2] These methods were developed around 1900 by the German mathematicians Carl Runge and Wilhelm Kutta.

"Runge-Kutta methods are used widely in many types of research mainly in fluid dynamics and mechanics for better solutions of the fluids. Other real-life application of Runge-Kutta method is simulation and games."

"When sending a satellite to another planet, it is often necessary to make a course correction mid-way."

The Runge-Kutta method

The Runge-Kutta method for solving first order differential equations is widely used and provides a high degree of accuracy. Again, as with the two previous methods, the Runge-Kutta method is a step-by-step process where results are tabulated for a range of values of x. Although several intermediate calculations are needed at each stage, the method is fairly straightforward. The 7 step procedure for the Runge-Kutta method, without proof, is as follows:

To solve the differential equation $\frac{dy}{dx} = f(x, y)$ given the initial condition $y = y_0$ at $x = x0$ for a range of values of $x = x_0(h)x_n$:

1. Identify x_0, y_0 and h, and values of $x_1, x_2, x_3, \ldots$
2. Evaluate $k_1 = f(x_n, y_n)$ starting with $n = 0$
3. Evaluate $k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} k_1\right)$
4. Evaluate $k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} k_2\right)$
5. Evaluate $k_4 = f(x_n + h, y_n + hk_3)$
6. Use the values determined from steps 2 to 5 to evaluate: $y_{n+1} = y_n + \frac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$
7. Repeat steps 2 to 6 for n = 1, 2, 3, …

Euler Vs Runge-Kutta

- Identify Inputs:
  - ODE
  - Constraints/initial values (x_0, y_0)
  - Intervals h
  - Range
- Use Euler's method:

$$\boxed{y_n = y_{n-1} + h(y')_{n-1}}$$

- Continue finding y_n, y_{n-1}, y_{n-1}', x_n

1. Identify x_0, y_0 and h, and values of x_1, x_2, x_3, ...
2. Evaluate $k_1 = f(x_n, y_n)$ starting with $n = 0$
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7. Repeat steps 2 to 6 for n = 1, 2, 3, ...

Thus, step 1 is given, and steps 2 to 5 are intermediate steps leading to step 6. It is usually most convenient to construct a table of values. The Runge-Kutta method is demonstrated in the following worked problems.

Runge-Kutta Example

Example 2. Use the Runge-Kutta method to solve the differential equation:

$$\frac{dy}{dx} = y - x$$

in the range 0(0.1)0.5, given the initial conditions that at $x = 0, y = 2$

$\boxed{x = x_0(h)x_n}$

Using the above procedure:

1. $x_0 = 0, y_0 = 2$ and since $h = 0.1$, and the range is from $x = 0$ to $x = 0.5$, then $x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, x_4 = 0.4$, and $x_5 = 0.5$

Let $n = 0$ to determine y_1:

2.

$$k_1 = f(x_0, y_0) = f(0, 2);$$

Since

$$\frac{dy}{dx} = y - x$$

$$k_1 = f(0, 2) = 2 - 0 = \mathbf{2}$$

3.

$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1\right)$$

$$k_2 = f\left(0 + \frac{0.1}{2}, 2 + \frac{0.1}{2}(2)\right)$$

$$k_2 = f(0.05, 2.1) = 2.1 - 0.05 = \mathbf{2.05}$$

4.

$$k_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_2\right)$$

$$k_3 = f\left(0 + \frac{0.1}{2}, 2 + \frac{0.1}{2}(2.05)\right)$$

$$k_3 = f(0.05, 2.1025) = 2.1025 - 0.05 = \mathbf{2.0525}$$

5.

$$k_4 = f(x_0 + h, y_0 + h\,k_3)$$

$$k_4 = f(0 + 0.1, 2 + 0.1\,(2.0525))$$

$$k_4 = f(0.1, 2.20525) = 2.20525 - 0.1 = \mathbf{2.10525}$$

1. Identify x_0, y_0 and h, and values of $x_1, x_2, x_3, \ldots$
2. Evaluate $k_1 = f(x_n, y_n)$ starting with $n = 0$
3. Evaluate $k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right)$
4. Evaluate $k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right)$
5. Evaluate $k_4 = f(x_n + h, y_n + hk_3)$
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7. Repeat steps 2 to 6 for n = 1, 2, 3, …

Runge-Kutta Example

6.

$$y_{n+1} = y_n + \frac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

When $n = 0$

$$y_1 = y_0 + \frac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

$$y_1 = 2 + \frac{0.1}{6}\{2 + 2(2.05) + 2(2.0525) + 2.10525\} = 2 + \frac{0.1}{6}\{12.31025\} = \mathbf{2.205171}$$

Let $n = 1$ to determine y_2:

2.

$$k_1 = f(x_1, y_1) = f(0.1, 2.205171);$$

Since

$$\frac{dy}{dx} = y - x$$

$$k_1 = f(0.1, 2.205171) = 2.205171 - 0.1 = \mathbf{2.105171}$$

3.

$$k_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}k_1\right)$$

$$k_2 = f\left(0.1 + \frac{0.1}{2}, 2.205171 + \frac{0.1}{2}(2.105171)\right)$$

$$k_2 = f(0.15, 2.31042955) = 2.31042955 - 0.15 = \mathbf{2.160430}$$

4.

$$k_3 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}k_2\right)$$

$$k_3 = f\left(0.1 + \frac{0.1}{2}, 2.205171 + \frac{0.1}{2}(2.160430)\right)$$

$$k_3 = f(0.15, 2.3131925) = 2.3131925 - 0.15 = \mathbf{2.163193}$$

5.

$$k_4 = f(x_1 + h, y_1 + h\,k_3)$$

$$k_4 = f(0.1 + 0.1, 2.205171 + 0.1(2.163193))$$

$$k_{\cancel{1}} = f(0.2, 2.421490) = 2.421490 - 0.2 = \mathbf{2.221490}$$

4

6.

$$y_{n+1} = y_n + \frac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

When $n = 1$

$$y_2 = y_1 + \frac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

$$y_2 = 2.205171 + \frac{0.1}{6}\{2.105171 + 2(2.160430) + 2(2.163193) + 2.221490\} = \mathbf{2.421403}$$

1. Identify x_0, y_0 and h, and values of $x_1, x_2, x_3, \ldots$
2. Evaluate $k_1 = f(x_n, y_n)$ starting with $n = 0$
3. Evaluate $k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right)$
4. Evaluate $k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right)$
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7. Repeat steps 2 to 6 for n = 1, 2, 3, …

Runge-Kutta Example

In a similar manner y_3, y_4 and y_5 can be calculated and the results are shown in the following table:

n	x_n	k_1	k_2	k_3	k_4	y_n
0	**0**					**2**
1	**0.1**	2.0	2.05	2.0525	2.10525	**2.205171**
2	**0.2**	2.105171	2.160430	2.163193	2.221490	**2.421403**
3	**0.3**	2.221403	2.282473	2.285527	2.349956	**2.649859**
4	**0.4**	2.349859	2.417339	2.420726	2.491932	**2.891824**
5	**0.5**	2.491824	2.566415	2.570145	2.648838	**3.148720**

If we would to use Euler's method on the same question and compare the results, the following would be found:

x	**Euler's method** y	**Runge-Kutta method** y	**Exact value** $y = x + 1 + e^x$
0	2	2	2
0.1	2.2	2.205171	2.205170918
0.2	2.41	2.421403	2.421402758
0.3	2.631	2.649859	2.649858808
0.4	2.8641	2.891824	2.891824698
0.5	3.11051	3.148720	3.148721271

It is seen from the table that the Runge-Kutta method is exact, correct to 5 decimal places.