Question 1

1 a) $AX = \lambda X \quad |A - \lambda I| X = 0$

$$\left(\left|\begin{bmatrix}5 & 6\\ 2 & 1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right|\right)\begin{bmatrix}x\\ y\end{bmatrix} = 0$$

$$\left(\left|\begin{matrix}5-\lambda & 6\\ 2 & 1-\lambda\end{matrix}\right|\right)\begin{bmatrix}x\\ y\end{bmatrix} = 0$$

$\lambda = 7;$

$$\begin{bmatrix}5-7 & 6\\ 2 & 1-7\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = 0$$

$-2x + 6y = 0$

$2x - 6y = 0$

when $x = 3y$ equation is satisfied.

Eigenvector, $x = t \quad y = \frac{t}{3}$

$$X = \begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}t\\ t/3\end{bmatrix} = t\begin{bmatrix}1\\ 1/3\end{bmatrix}$$

ans.

$\lambda = -1$

$$\begin{bmatrix}5--1 & 6\\ 2 & 1--1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = 0$$

$6x + 6y = 0$

$2x + 2y = 0$

when $x = -y$, equation is satisfied

Eigenvector: $x = t \quad y = -t$

$$X = \begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}t\\ -t\end{bmatrix} = t\begin{bmatrix}1\\ -1\end{bmatrix}$$

ans.

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b) Same A matrix as in part a) so, same Eigenvalues and Eigenvectors

$$\left(\left|\begin{bmatrix}5 & 6\\ 2 & 1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right|\right)\begin{bmatrix}x\\ y\end{bmatrix} = 0$$

$$\left(\left|\begin{matrix}5-\lambda & 6\\ 2 & 1-\lambda\end{matrix}\right|\right)\begin{bmatrix}x\\ y\end{bmatrix} = 0$$

$\lambda = 7;$

$$\begin{bmatrix}5-7 & 6\\ 2 & 1-7\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = 0$$

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$$\begin{bmatrix}5--1 & 6\\ 2 & 1--1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = 0$$

$6x + 6y = 0$

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when $x = -y$, equation is satisfied

Eigenvector: $x = t \quad y = -t$

$$X = \begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}t\\ -t\end{bmatrix} = t\begin{bmatrix}1\\ -1\end{bmatrix}$$

G.S. $x(t) = a_1\begin{bmatrix}1\\ 1/3\end{bmatrix}e^{7t} + a_2\begin{bmatrix}1\\ -1\end{bmatrix}e^{-t}$

$x_1(t) = a_1(1)e^{7t} + a_2(1)e^{-t}$

$x_2(t) = a_1(1/3)e^{7t} + a_2(-1)e^{-t}$

ans.

$x_1(t) = a_1(1)e^{7t} + a_2(1)e^{-t} \qquad x(0) = \begin{bmatrix}3\\ 1\end{bmatrix}$

$x_2(t) = a_1(1/3)e^{7t} + a_2(-1)e^{-t}$

$x_1(0) = a_1 + a_2 = 3$ ①

$x_2(0) = \frac{a_1}{3} - a_2 = 1$ ②

②; $a_1 = 3 + 3a_2$

a_1 in ①; $(3+3a_2) + a_2 = 3$

$4a_2 = 0$

$a_2 = 0$

a_2 in ①; $a_1 = 3$

so P.S. : $x(t) = 3\begin{bmatrix}1\\ 1/3\end{bmatrix}e^{7t}$

$x_1(t) = 3e^{7t}$

$x_2(t) = e^{7t}$

ans.

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c) $CX = \lambda X$ ①

$C(CX) = C(\lambda X) \quad \times C$

$C(CX) = \lambda(CX)$

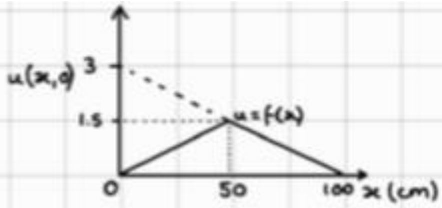
$C(CX) = \lambda(\lambda X)$ ① $CX = \lambda X$

$C^2X = \lambda^2 X$

so λ^2 is an eigenvalue of C^2 ans.

Question 2

a) 1. B.C. $\left.\begin{array}{l} u(0,t) \\ u(100,t) \end{array}\right\}$ Fixed end points



$$u(x,0) = \frac{3}{50} \quad 0 \leq x \leq 50$$

$$= -\frac{3}{50} \quad 50 \leq x \leq 100$$

2. Assume $u = XT$, $\frac{\partial u}{\partial x} = X'T$, $\frac{\partial^2 u}{\partial x^2} = X''T$, $\frac{\partial u}{\partial t} = XT'$, $\frac{\partial^2 u}{\partial t^2} = XT''$

Wave equation = $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\partial t$, $X''T = \frac{1}{9}XT''$

3. Separate: $\frac{X''}{X} = \frac{1}{9}\frac{T''}{T}$, $\mu = \frac{X''}{X}$, $\mu = \frac{1}{9}\frac{T''}{T}$

$X'' - \mu X = 0$ & $T'' - 9\mu T = 0$

4. $\mu = -p^2$ $\quad X'' + p^2 X = 0$ & $T'' + 9p^2 T = 0$

$k^2 + p^2 = 0 \qquad k^2 + 9p^2 = 0$

$k = pj \qquad k = 3pj$

5. $X = A\cos px + B\sin px$ & $T = C\cos 3pt + D\sin 3pt$

G.S. $u(x,t) = (A\cos px + B\sin px)(C\cos 3pt + D\sin 3pt)$

6. $x = 0$ $u = 0$ so $0 = A(C\cos 3pt + D\sin 3pt)$

$\therefore A = 0$

so $u(x,t) = B\sin px(C\cos 3pt + D\sin 3pt)$, $B \neq 0$

at $x = 100$ $u = 0$ $\quad 0 = B\sin 100p(C\cos 3pt + D\sin 3pt)$,

$B \neq 0$ so $\sin 100p = 0$

$$p = \frac{n\pi}{100}$$

7. p in $u(x,t)$: $u = B\sin\left(\frac{n\pi}{100}x\right)\left(C\cos\left(\frac{n\pi t}{100}\right) + D\sin\left(\frac{n\pi t}{100}\right)\right)$

or, $u_n(x,t) = \sum_{n=1}^{\infty}\left[\sin\left(\frac{n\pi x}{100}\right)\left(A_n\cos\left(\frac{n\pi t}{100}\right) + B_n\sin\left(\frac{n\pi t}{100}\right)\right)\right]$ /ans.

where $A_n = BC$ & $B_n = BD$ /ans.

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b) $\frac{\partial^2 u}{\partial x \partial y} = \cos x \cos y$ B.C. $x = \frac{\partial u}{\partial x}$ $y = \pi$, $u = 2\cos(y)$ $x = \pi$

$\frac{\partial u}{\partial x} = \cos x \int \cos y \; dy$

$= \cos x \sin y + f(x)$

B.C. $x = \frac{\partial u}{\partial x}$ $y = \pi$

$x = \cos x \sin \pi + f(x)$

$f(x) = x$

$\frac{\partial u}{\partial x} = \cos x \sin y + x$

$u = \sin y \int \cos x \; dx + \int x \; dx$

$= \sin y \sin x + \frac{x^2}{2} + f(y)$

B.C. $u = 2\cos(y)$ $x = \pi$

$2\cos(y) = \sin y \sin \pi + \frac{\pi^2}{2} + f(y)$

$2\cos(y) = \frac{\pi^2}{2} + f(y)$

$f(y) = 2\cos(y) + \frac{\pi^2}{2}$

$u = \sin x \sin y + \frac{x^2}{2} + 2\cos y + \frac{\pi^2}{2}$ /ans.

c) $\frac{d^2\theta}{dt^2} + 2\frac{d\theta}{dt} + \theta = 4$ B.C. $t=0$ $\theta=0$, $t=0$ $\frac{d\theta}{dt}=0$

aux: $m^2 + 2m + 1 = 0$

$(m+1)(m+1) = 0$

$m = -1$, $m = -1$

C.F. : $u = (At + B)e^{-t}$

Let P.I., $v = k$ $(D^2 + 2D + 1) = 4$

$k = 4$ $k = v$ so $v = 4$

P.S. $\theta = u + v$

$= (At + B)e^{-t} + 4$

B.C. $t=0$, $\theta = 0$

$0 = B + 4$

$B = -4$

$$\frac{d\theta}{dt} = (At + B)(-e^{-t}) + Ae^{-t}$$

B.C. $t=0$ $\frac{d\theta}{dt} = 0$

$0 = -B + A$

$A = B$

$A = -4$

$\therefore$ P.S.: $\theta = (-4t - 4)e^{-t} + 4$

$\theta = 4 - 4(t+1)e^{-t}$ /ans.

Question 3

a) $\dot{y}_1 = 4y_1 + 3y_2 - y_1 y_2^2$ assuming rest conditions: $\dot{y}_1 = 0$, $\dot{y}_2 = 0$

$\dot{y}_2 = 6y_1 - y_2 - y_2^3$

① $4y_1 + 3y_2 - y_1 y_2^2 = 0$

② $6y_1 - y_2 - y_2^3 = 0$

$6y_1 = y_2^3 - y_2$

$y_1 = \dfrac{y_2^3 - y_2}{6}$ ③

③ in ①: $4\left(\dfrac{y_2^3 - y_2}{6}\right) + 3y_2 - \left(\dfrac{y_2^3 - y_2}{6}\right) y_2^2 = 0$

$\frac{2}{3}y_2^3 - \frac{2}{3}y_2 + 3y_2 - \frac{1}{6}y_2^5 + \frac{1}{6}y_2^3 = 0$

$\frac{5}{6}y_2^3 + \frac{7}{3}y_2 - \frac{1}{6}y_2^5 = 0$

$\frac{1}{6}y_2(-y_2^4 + 5y_2^2 + 14) = 0$

$\frac{1}{6}y_2(y_2^2 - 7)(y_2^2 + 2) = 0$

$y_2 = -\sqrt{7}$, $y_2 = 0$, $y_2 = \sqrt{7}$

y_2 in ③: $(-\sqrt{7}, -\sqrt{7})$, $(0,0)$, $(\sqrt{7}, \sqrt{7})$

↳ so $(0,0)$ is fixed point

ans.

$$J = \begin{bmatrix} \frac{\partial \dot{y}_1}{\partial y_1} & \frac{\partial \dot{y}_1}{\partial y_2} \\ \frac{\partial \dot{y}_2}{\partial y_1} & \frac{\partial \dot{y}_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 4 - y_2^2 & 3 - 2y_1 y_2 \\ 6 & -1 - 3y_2^2 \end{bmatrix} \quad J_{(0,0)} = \begin{bmatrix} 4 & 3 \\ 6 & -1 \end{bmatrix} = A$$

$|A - \lambda I| = 0$

$$\left| \begin{bmatrix} 4 & 3 \\ 6 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 4-\lambda & 3 \\ 6 & -1-\lambda \end{vmatrix} = 0 = (4-\lambda)(-1-\lambda) - (3)(6)$$

$= \lambda^2 - 3\lambda - 4 - 18$

$= \lambda^2 - 3\lambda - 22$

$\lambda_1 = \dfrac{3 + \sqrt{97}}{2}$ $\quad \lambda_2 = \dfrac{3 - \sqrt{97}}{2}$

$\lambda_1 = 6.42$ $\quad \lambda_2 = -3.42$

1 positive real root so, system is unstable at $(0,0)$

ans.

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b) $\dfrac{\partial y}{\partial x} = 2y - 4x$ $\quad x_0 = 0 \quad y_0 = 1 \quad h = 0.5$

$x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$, $x_4 = 2$

1. $x_0 = 0 \quad y_0 = 1 \quad h = 0.5$

2. $k_1 = f(x_n, y_n)$

$k_1 = 2(1) - 4(0)$

$k_1 = 2.000000$

3. $k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$

$k_2 = 2(1.5) - 4(0.25)$

$k_2 = 2.000000$

4. $k_3 = f(x_n + \frac{h}{2}, y_n \frac{h}{2}k_2)$

$k_3 = 2(1.5) - 4(0.25)$

$k_3 = 2.000000$

5. $k_4 = f(x_n + h, y_n + hk_3)$

$k_4 = 2(2) - 4(0.5)$

$k_4 = 2.000000$

6. $y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1 + \frac{0.5}{6}(2 + 2(2) + 2(2) + 2)$

$y_1 = 2.000000$ ans.

$n = 1$

2. $k_1 = f(x_n, y_n)$

$k_1 = 2(2) - 4(0.5)$

$k_1 = 2.000000$

3. $k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$

$k_2 = 2(2.5) - 4(0.75)$

$k_2 = 2.000000$

4. $k_3 = f(x_n + \frac{h}{2}, y_n \frac{h}{2}k_2)$

$k_3 = 2(2.5) - 4(0.75)$

$k_3 = 2.000000$

5. $k_4 = f(x_n + h, y_n + hk_3)$

$k_4 = 2(3) - 4(1)$

$k_4 = 2.000000$

6. $y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

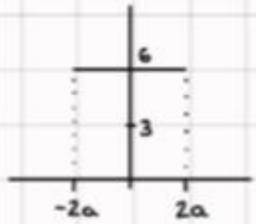
$y_2 = 2 + \frac{0.5}{6}(2 + 2(2) + 2(2) + 2)$

$y_2 = 3.000000$ ans.

Question 4

a) $p(t) = \begin{cases} 6 & -2a < t < 2a \\ 0 & \text{otherwise} \end{cases}$

$T = 4a$



$$f(\omega) = \int_{-2a}^{2a} 6e^{-j\omega t}\, dt$$

$$= \left[\frac{6e^{-j\omega t}}{-j\omega} \right]_{-2a}^{2a}$$

$$= \left(\frac{6e^{-2j\omega a}}{-j\omega} \right) - \left(\frac{6e^{2j\omega a}}{-j\omega} \right)$$

Eulers: $e^{j\theta} = \cos\theta + j\sin\theta$

$e^{-j\theta} = \cos\theta - j\sin\theta$

$$= \frac{6(\cos 2\omega a + j\sin 2\omega a) + 6(\cos 2\omega a - j\sin 2\omega a)}{j\omega}$$

$$= \frac{12 j \sin 2\omega a}{j\omega}$$

$= \frac{12 \sin 2\omega a}{\omega}$, aka sync function

ans.

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b) $z = x^3 - 6x^2 - 8y^2$

$\frac{\partial z}{\partial x} = 3x^2 - 12x$ $\quad \frac{\partial^2 z}{\partial x \partial y} = 0$ $\quad \frac{\partial z}{\partial y} = -16y$ $\quad \frac{\partial^2 z}{\partial y \partial x} = 0$

$\frac{\partial^2 z}{\partial x^2} = 6x - 12$ $\quad \frac{\partial^2 z}{\partial y^2} = -16$

For stationary points: $\frac{\partial z}{\partial x} = 0$ & $\frac{\partial z}{\partial y} = 0$

① $3x^2 - 12x = 0$ ② $-16y = 0$

$3x(x-4) = 0$ $\quad y = 0$

$x = 0,\ x = 4$ $\quad (0,0),\ (4,0)$

Point	$\frac{\partial^2 f}{\partial x^2}$	$\frac{\partial^2 f}{\partial y^2}$	$\frac{\partial^2 f}{\partial x \partial y}$	$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$	Nature
(0,0)	-12	-16	0	192, positive	maxima at (0, 0)
(4,0)	12	-16	0	-192, negative	saddle at (4, 0)

ans.

c) $z = x^3 - x + y^3 - y$ $\quad \frac{\partial z}{\partial x} = 3x^2 - 1$ $\quad \frac{\partial^2 z}{\partial x \partial y} = 0$ $\quad \frac{\partial z}{\partial y} = 3y^2 - 1$ $\quad \frac{\partial^2 z}{\partial y \partial x} = 0$

$\frac{\partial^2 z}{\partial x^2} = 6x$ $\quad \frac{\partial^2 z}{\partial y^2} = 6y$

For stationary points : $\frac{\partial z}{\partial x} = 0$ & $\frac{\partial z}{\partial y} = 0$

① $3x^2 - 1 = 0$ $\quad$ ② $3y^2 - 1 = 0$

$x^2 = \frac{1}{3}$ $\quad$ $y^2 = \frac{1}{3}$

$x = \pm \frac{\sqrt{3}}{3}$ $\quad$ $y = \pm \frac{\sqrt{3}}{3}$

$$H = \begin{bmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\ \frac{\partial^2 z}{\partial y \partial x} & \frac{\partial^2 z}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix}$$

Stationary points $\left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right), \left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right), \left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right), \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$

$|H| = (6x)(6y) - (0)$
$= 36xy$

For $\left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right)$: $\quad |H| = 36\left(-\frac{\sqrt{3}}{3}\right)\left(-\frac{\sqrt{3}}{3}\right)$ $\quad \frac{\partial^2 z}{\partial x^2} = 6\left(-\frac{\sqrt{3}}{3}\right)$

$= 12$ $\quad = -2\sqrt{3}$

as $|H| > 0$ & $\frac{\partial^2 z}{\partial x^2} < 0$, $\left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right)$ is a maxima
ans.

For $\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$: $\quad |H| = 36\left(-\frac{\sqrt{3}}{3}\right)\left(\frac{\sqrt{3}}{3}\right)$

$= -12$

as $|H| < 0$, $\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$ is a saddle point
ans.

For $\left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right)$: $\quad |H| = 36\left(\frac{\sqrt{3}}{3}\right)\left(-\frac{\sqrt{3}}{3}\right)$

$= -12$

as $|H| < 0$, $\left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right)$ is a saddle point
ans.

For $\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$: $\quad |H| = 36\left(\frac{\sqrt{3}}{3}\right)\left(\frac{\sqrt{3}}{3}\right)$ $\quad \frac{\partial^2 z}{\partial x^2} = 6\left(\frac{\sqrt{3}}{3}\right)$

$= 12$ $\quad = 2\sqrt{3}$

as $|H| > 0$ & $\frac{\partial^2 z}{\partial x^2} > 0$, $\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$ is a minima
ans.