UNIVERSITY OF LINCOLN

SCHOOL OF ENGINEERING

DATA MODELLING AND SIMULATION

Lecture 4: Partial Differential equations

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“Since Newton, mankind has come to realize that the laws of physics are always expressed in the language of differential equations.”

- Steven Strogatz

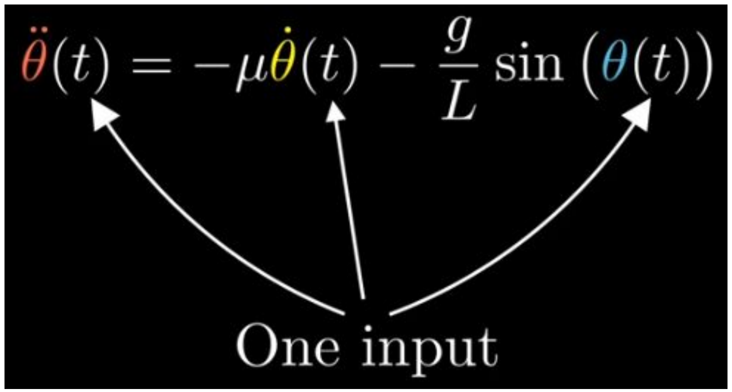
Contents:

Differential vs Partial Differential equation (PD)

Derivative of a function with respect
to the one variable

$$\frac{dy}{dx} = ky(t)$$

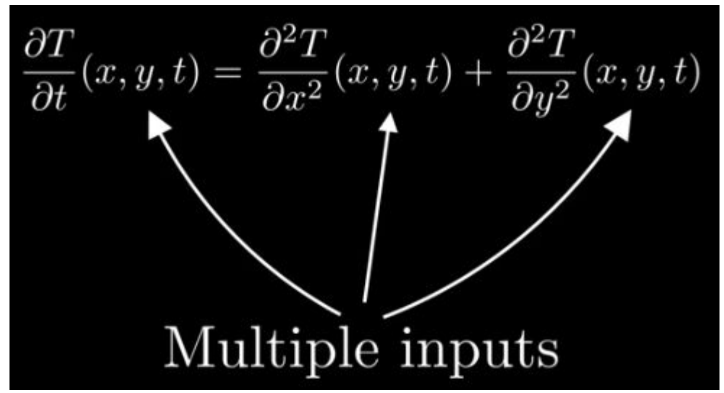
y depends only on one independent variable t



Derivative of a function with respect
to the more than one variable

$$\frac{\delta^2 f}{\delta x^2} + \frac{\delta^2 f}{\delta y^2} = 0$$

f depends on two independent variables x and y



Applications:

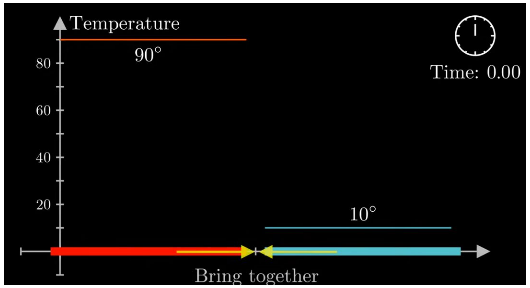
- Heat conduction
- Hydrodynamics
- Aerodynamics etc.

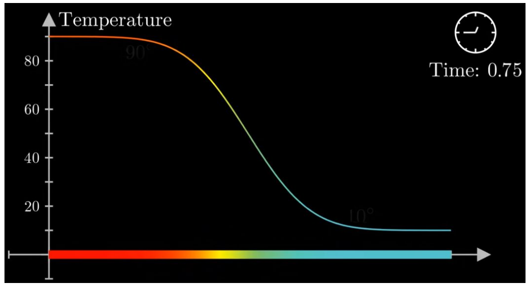
Solutions:

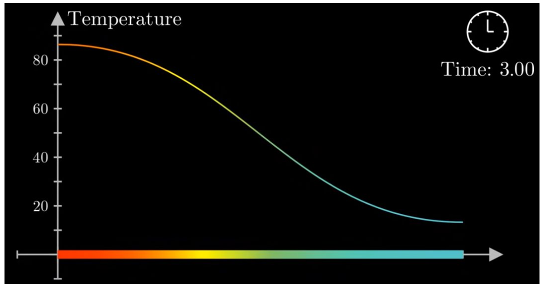
- Difficult to solve (but few techniques are there)
- More difficult types - By Numerical Methods

Differential vs Partial Differential equation (PD)

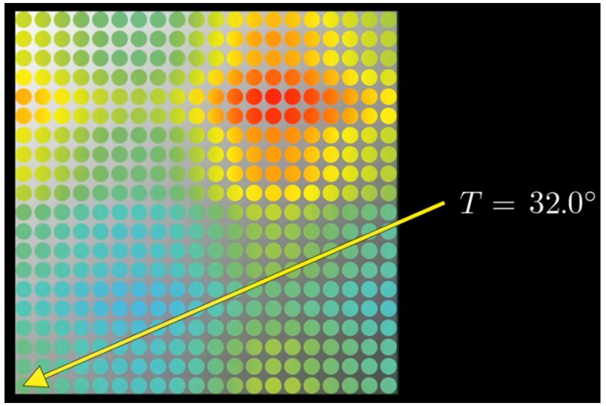
Metal rod

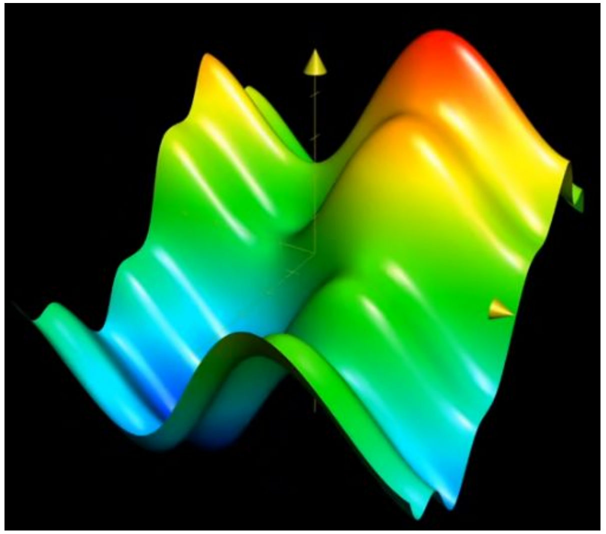






Metal sheet





https://www.youtube.com/watch?v=Iy4S0oi3Yz8

Partial Differential equation (PD) solution

Using Direct Partial Integration

Partial Integration

Integrate $\frac{\partial u}{\partial t} = 5\cos x \sin t$ partially with respect to t.

$$u = 5\cos x \int \sin t \, dt$$

$5\cos x$ term as a constant

$$= (5\cos x)(-\cos t)$$

$$= -5\cos x \cos t + f(x)$$

- Note that we no longer have a constant of integration (+C) but a function of integration (f(x)).

This merely demonstrates that we are aware that there could've been a function of x there prior to differentiation

Partial Differential equation (PD) solution

Partial Integration

Using Direct Partial Integration

Integrate $\frac{\partial^2 u}{\partial x \partial y} = 6x^2 \cos 2y$ partially with respect to y.

Integrating with respect to y and therefore, treating all x terms as constants gives

$$\frac{\partial u}{\partial x} = 6x^2 \int \cos 2y \, dy$$

$$= (6x^2)\left(\frac{1}{2}\sin 2y\right) + f(x)$$

Are we done ?

Now we can go on and integrate with respect to x to obtain an equation in terms of u. This means we must now treat any y terms as constants. This gives,

$$u = \frac{1}{2}\sin 2y \int 6x^2 + f(x) \, dx$$

$$= (2x^3)\left(\frac{1}{2}\sin 2y\right) + xf(x) + g(y)$$

How to determine f(x) and g(y) ?

$$= x^3 \sin 2y + xf(x) + g(y)$$

We can determine the functions f(x) and g(y) using extra information called boundary conditions or initial conditions.

Partial Differential equation (PD) solution

Using Direct Partial Integration

Example 3. Solve the differential equation $\frac{\partial^2 u}{\partial x^2} = 6x^2(2y-1)$ given the boundary conditions that at $x = 0$, $\frac{\partial u}{\partial x} = \sin 2y$ and $u = \cos y$.

First integrating partially with respect to x gives,

$$\frac{\partial u}{\partial x} = 2y - 1\int 6x^2\,dx$$

$$= (2y-1)\frac{6x^3}{3} + f(y)$$

$$= 2x^3(2y-1) + f(y)$$

Now applying the boundary conditions where $\frac{\partial u}{\partial x} = \sin 2y$ at $x = 0$,

$$\sin 2y = 2(0)^3(2y-1) + f(y)$$

This gives,

$$f(y) = \sin 2y$$

And therefore, $\frac{\partial u}{\partial x} = 2x^3(2y-1) + \sin 2y$

Now integrating again with respect to x gives,

$$u = \frac{2x^4}{4}(2y-1) + x\sin 2y + g(y)$$

Applying the boundary condition $u = \cos y$ at $x = 0$ gives,

$$\cos y = \frac{2(0)^4}{4}(2y-1) + (0)\sin 2y + g(y)$$

And therefore,

$$g(y) = \cos y$$

This means the solution of the partial differential equation is,

$$u = \frac{x^4}{2}(2y-1) + x\sin y + \cos y$$

Partial Differential equation (PD) solution

Using Direct Partial Integration

Example 4. Solve the differential equation $\frac{\partial^2 u}{\partial x \partial y} = \cos(x+y)$ given that $\frac{\partial u}{\partial x} = 2$ when $y=0$ and $u = y^2$ when $x=0$.

$$\frac{\partial u}{\partial x} = \int \cos(x+y)\,dy$$

$$= \sin(x+y) + f(x)$$

Now applying the boundary condition $\frac{\partial u}{\partial x} = 2$ when $y=0$ gives,

$$2 = \sin(x+0) + f(x)$$

$$f(x) = 2 - \sin x$$

And therefore,

$$\frac{\partial u}{\partial x} = \sin(x+y) + 2 - \sin x$$

Now integrating partially with respect to x gives,

$$u = \int \sin(x+y) + 2 - \sin x\,dx$$

$$= -\cos(x+y) + 2x + \cos x + f(y)$$

Applying the boundary condition $u = y^2$ when $x=0$ gives,

$$y^2 = -\cos(0+y) + 2(0) + \cos 0 + f(y)$$

$$= 1 - \cos y + f(y)$$

$$f(y) = y^2 - 1 + \cos y$$

The solution of the differential equation is therefore,

$$u = -\cos(x+y) + 2x + \cos x + y^2 - 1 + \cos y$$

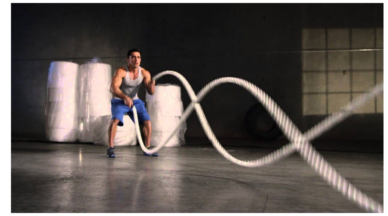
Partial Differential equation (PD)

Important engineering Partial Differential Equations

(a) The **wave equation**, where the equation of motion is given by:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Where $c^2 = \frac{T}{\rho}$, with T being the tension in a string and ρ being the mass/unit length of the string.

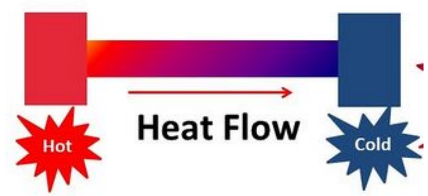


- Waves on a membrane
- Acoustic model for seismic waves
- Sound waves in liquids and gases

(b) The **heat conduction equation** is of the form:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$$

Where $c^2 = \frac{h}{\sigma\rho}$, with h being the thermal conductivity of the material and ρ being the mass/unit length of the material.



- Fluid mechanics,
- Atmospheric science,
- Climate physics

Partial Differential equation (PD) solutions

Separation of variables

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}}$$

Let $u(x,t) = X(x)T(t)$

$X(x)$ is a function of x only

$T(t)$ is a function of t only

Consider $u = XT$

then $\frac{\partial u}{\partial x} = X'T$ and $\frac{\partial^2 u}{\partial x^2} = X''T$

$\frac{\partial u}{\partial t} = XT'$ and $\frac{\partial^2 u}{\partial t^2} = XT''$

Substituting into the partial differential equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ gives:

$$\boxed{X''T = \frac{1}{c^2}XT''}$$

Separating the variables gives:

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$$

Let $\mu = \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$, where μ is a constant.

The result of this is that we are now left with two ordinary differential equations that we can now solve, viz

$X'' - \mu X = 0$ and $T'' - c^2\mu T = 0$

The solutions to these equations will depend on whether $\mu > 0$, $\mu = 0$ or $\mu < 0$.

Condition	Characteristics	Solution
$\mu = 0$	real and repeated roots	$Ax + B$
$\mu > 0$	real and distinct roots	$Ae^{k_1 x} + Be^{k_2 x}$
$\mu < 0$	complex roots	$e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

Partial Differential equation (PD) solutions

Separation of variables

Condition	Characteristics	Solution
μ = 0	real and repeated roots	Ax + B
μ > 0	real and distinct roots	$Ae^{k_1 x} + Be^{k_2 x}$
μ < 0	complex roots	$e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

Example 1. Find the general solution of the following differential equations:

(a) $X''-4X=0$ (b) $T''+4T=0$

(a) If $X''-4X=0$ then the auxiliary equation is:

$k^2-4=0$

Finding Auxiliary equation

substitute $X = e^{kx}$

From which,

$k=2$ or $k=-2$

The general solution is therefore,

$X = Ae^{2x} + Be^{-2x}$

(b) If $T''+4T=0$ then the auxiliary equation is:

$k^2+4=0$

From which,

$k=2j$ or $k=-2j$

If the auxiliary equation has complex roots, $\alpha+\beta j$ and $\alpha-\beta j$, then the complementary function is

$y_{cf} = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

The general solution is therefore,

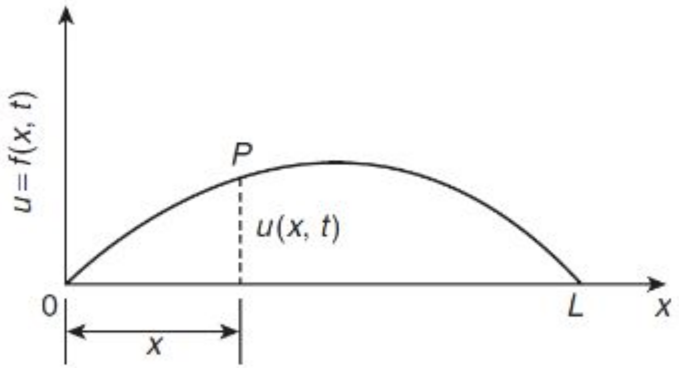
$T = e^0\left(A\cos 2t + B\sin 2t\right)$

$= A\cos 2t + B\sin 2t$

Partial Differential equation (PD)

Wave Equation

- Models the propagation of waves with speed v



- A flexible elastic string stretched between two points at x=0 and x=L with uniform tension T

String is displaced slightly from its initial position = string will vibrate

- A point P on the string depends on its distance from one end, will get displacement u = f(x,t) where x is its distance from O.

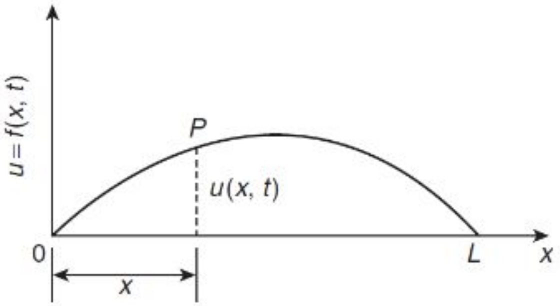
The equation of motion

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}}$$

Condition	Characteristics	Solution
μ = 0	real and repeated roots	Ax + B
μ > 0	real and distinct roots	$Ae^{k_1 x} + Be^{k_2 x}$
μ < 0	complex roots	$e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

oscillatory solution

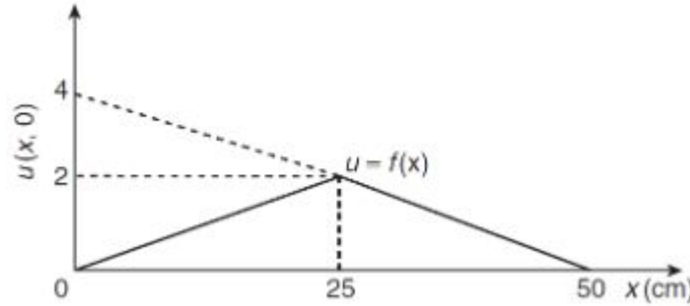
Wave equation solutions



Procedure

1. Identify clearly the initial and boundary conditions.
2. Assume a solution of the form $u = XT$ and express the equations in terms of X and T and their derivatives.
3. Separate the variables by transposing the equation and equate each side to a constant, say μ; two separate equations are obtained, one in x and one in t.
4. Let $\mu = -p^2$ to give an oscillatory solution.
5. The two solutions are of the form:
   $X = A\cos px + B\sin px$ and $T = A\cos pt + B\sin pt$
   Then $u(x,t) = \{(A\cos px + B\sin px)\}\{(C\cos pt + D\sin pt)\}$
6. Apply the boundary conditions to determine constants A and B.
7. Determine the general solution as an infinite sum.
8. Apply the remaining boundary and initial conditions and determine the coefficients A_n and B_n using Fourier series techniques.

Wave equation solutions



Example 3. A stretched string of length 50cm is set oscillating by displacing its midpoint a distance of 2cm from its rest position and releasing it with zero velocity. Solve the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ where $c^2 = 1$, to determine the resulting motion $u(x,t)$.

1. The boundary and initial conditions given are:

$$\left.\begin{array}{l} u(0,t)=0 \\ u(50,t)=0 \end{array}\right\} \text{ i.e. fixed end points}$$

$$u(x,0) = f(x) = \frac{2}{25}x \qquad 0 \le x \le 25$$

$$= -\frac{2}{25}x + 4 \qquad 25 \le x \le 50$$

initial velocity is zero thus, $\frac{\partial u}{\partial t} = 0$

2. Assuming a solution $u = XT$ we get,

$$\frac{\partial u}{\partial x} = X'T \text{ and } \frac{\partial^2 u}{\partial x^2} = X''T$$

$$\frac{\partial u}{\partial t} = XT' \text{ and } \frac{\partial^2 u}{\partial t^2} = XT''$$

Substituting into $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ gives,

$$X''T = \frac{1}{c^2}XT''$$

(since $c^2 = 1$)

$$X''T = XT''$$

3. Separating the variables gives:

$$X'' - \mu X = 0 \quad \text{and} \quad T'' - \mu T = 0$$

4. Letting $\mu = -p^2$ to give an oscillatory solution gives:

$$X'' + p^2 X = 0 \quad \text{and} \quad T'' + p^2 T = 0$$

The auxiliary equation for each is,

$$k^2 + p^2 = 0$$

Which gives,

$$k = \sqrt{-p^2} = \pm pj$$

5. Solving each equation gives,

$$u(x,t) = \{A\cos px + B\sin px\}\{C\cos pt + D\sin pt\}$$

Condition	Characteristics	Solution
μ = 0	real and repeated roots	Ax + B
μ > 0	real and distinct roots	$Ae^{k_1 x} + Be^{k_2 x}$
μ < 0	complex roots	$e^{\alpha x}(A\cos \beta x + B\sin \beta x)$

Wave equation solutions

6. Apply the boundary conditions to determine constants A and B.
7. Determine the general solution as an infinite sum.
8. Apply the remaining boundary and initial conditions and determine the coefficients A_n and B_n using Fourier series techniques.

5. Solving each equation gives,

$$u(x,t) = \{A\cos px + B\sin px\}\{C\cos pt + D\sin pt\}$$

6. Applying the boundary conditions to determine A and B gives,

(i) $u = 0$ when $x = 0$, for all values of t, hence $0 = A\{C\cos pt + D\sin pt\}$ from which we conclude that A = 0.

Therefore,

$$u(x,t) = B\sin px\{C\cos pt + D\sin pt\} \qquad \text{Equation (a)}$$

(ii) $u = 0$ when $x = 50$, for all values of t, hence $0 = B\sin 50p\{C\cos pt + D\sin pt\}$ where B cannot be 0 otherwise $u(x,t)$ would also equal 0.
As B and $C\cos pt + D\sin pt$ cannot be zero then $\sin 50p$ must be zero hence,

$$\sin 50p = 0$$

From which,

$$50p = n\pi$$

$$p = \frac{n\pi}{50}$$

7. Substituting into equation (a) above gives:

$$u(x,t) = B\sin\frac{n\pi x}{50}\left\{C\cos\frac{n\pi t}{50} + D\sin\frac{n\pi t}{50}\right\} \qquad u_n(x,t) = f(x) = \sum_{n=1}^{\infty}\left\{\sin\frac{n\pi x}{50}\left(A_n\cos\frac{nc\pi t}{50} + B_n\sin\frac{nc\pi t}{50}\right)\right\}$$

8. Now applying the remaining boundary conditions:

$$u(x,0) = f(x) = \frac{2}{25}x \qquad 0 \le x \le 25$$

$$= -\frac{2}{25}x + 4 \qquad 25 \le x \le 50$$

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx \qquad = \frac{2}{50}\left[\int_0^{25}\left(\frac{2}{25}x\right)\sin\frac{n\pi x}{50}dx + \int_{25}^{50}\left(\frac{100-2x}{25}\right)\sin\frac{n\pi x}{50}dx\right]$$

Each integral is determined using integration by parts with the result:

$$A_n = \frac{16}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Secondly,

$$B_n = \frac{2}{nc\pi}\int_0^l g(x)\sin\frac{n\pi x}{l}dx$$

From the initial condition $\left[\frac{\partial u}{\partial t}\right]_{t=0} = 0 = g(x)$,

$$B_n = 0$$

Substituting into our equation,

$$u_n(x,t) = \sum_{n=1}^{\infty}\left\{\sin\frac{n\pi x}{50}\left(A_n\cos\frac{n\pi t}{50} + B_n\sin\frac{n\pi t}{50}\right)\right\}$$

$$= \sum_{n=1}^{\infty}\left\{\sin\frac{n\pi x}{50}\left(\frac{16}{n^2\pi^2}\sin\frac{n\pi}{2}\cos\frac{n\pi t}{50} + (0)\sin\frac{n\pi t}{50}\right)\right\}$$

Hence, our final solution is,

$$u(x,t) = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi x}{50}\sin\frac{n\pi}{2}\cos\frac{n\pi t}{50}$$

Self study

1. Solutions of partial differential equation
2. Auxiliary equation
3. ToDo

Example 2. Solve the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ given the initial and boundary conditions:

(a) The string is fixed at both ends, i.e. at x = 0 and x = l for all values of t.

(b) The initial deflection of the string is denoted by f(x).

(c) The initial velocity of P is $\left[\frac{\partial u}{\partial t}\right]_{t=0} = g(x)$