# Week 3 Slides

Recap from last week

- Plant models are usually based on differential equations, for example
  $$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = u$$
  which is a second-order differential equation is standard form.
- Compared to a mass-spring-damper system we identified the natural frequency and the damping ratio
  $$\omega_n = \sqrt{k/m}$$
  $$\zeta = \frac{c}{2\sqrt{mk}}$$
- A transfer function is easily found from a differential equation by using the s-operator, e.g. for the standard second-order equation
  $$G(s) = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- There is a transfer function block in Simulink – very convenient for modelling the plant before ‘**closing the loop**’ in the block diagram
- The integrator block is still useful when we need to use initial conditions –transfer functions **always assumes zero initial conditions**.
- We can also close the loop directly using the rule:

  CLTF = (forward path TF)/(1+loop TF)

  where for example **loop TF** is the product of all transfer functions in the loop, ignoring the (-) sign from negative feedback.
- We started to talk about P, PI and **PID** controllers

$$e(t) \longrightarrow \boxed{K_p + K_i/s + K_d s} \longrightarrow u(t)$$

We also touched on the following important performance measures for a closed-loop control system:

- **Steady-state** output: should =1 for a unit step $r(t) = 1(t)$ in the refence signal.
- **Overshoot** – peak as a percentage of the final value.
- **Rise time** – roughly how quickly the system responds
- **Settling time** – how quickly the response gets within a few percent of the final value.

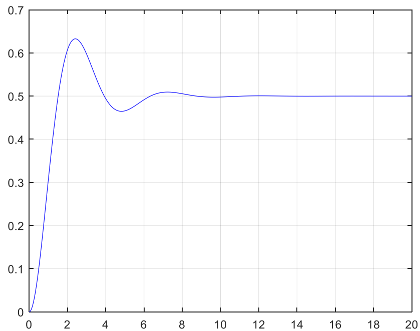
We will analyze these in more detail later.

For now they are important **practical considerations** for the control engineer.

**Steady-State Gain**

This applies to any transfer function – it could be the plant, it could be the whole control system – it just is a ‘thing’ for “SISO LTI”[1] systems.

In the following (closed-loop example) the steady-state gain equals 0.5, which is not good!




[1] Single Input, Single Output Linear Time-Invariant

Fortunately it's really easy to find the steady-state gain of a transfer function.

Take for example the mass-spring-damper system

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F$$

In steady-state the time derivatives become zero so, if $F(t) = F_0\, 1(t)$ we have

$$ky_{ss} = F_0$$

And the steady-state gain is given as the ratio (output)/(input) = $y_{ss}/F_{ss}$

$$ss\ gain\ =\ 1/k$$

So a stiffer spring makes the steady-state (static) deflection smaller – obviously.

In transfer function form,

$$y = G(s)\,F$$

or

$$F(t) \longrightarrow \boxed{\frac{1}{ms^2 + cs + k}} \longrightarrow y(t)$$

We again ignore (set to zero) any derivative terms: $s \to 0$

$$F_{ss} \longrightarrow \boxed{\frac{1}{k}} \longrightarrow y_{ss}$$

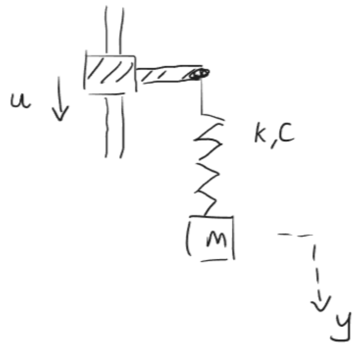
In general

$$ss\ gain\ =\ G(0)$$

Note: steady-state gain is sometimes called the ‘DC gain’

**Example**

Mass-spring (and damper) where the input $u$ is a deflection, not a force.




From basic physics, what do you expect the steady-state gain to be?

Let’s check: the spring compression is given as $u - y$ and the compressive velocity is $\dot{u} - \dot{y}$. Hence

$$m\ddot{y} = k(u - y) + c(\dot{u} - \dot{y})$$

The transfer function is (check this)

$$G(s) = \frac{k + cs}{ms^2 + cs + k}$$

Hence the steady-state gain is $G(0) = 1$, which should agree with our intuition about the system.

**How to solve (anaytically) or simulate (by computer) dynamic systems**
These can be open-loop plant equations or closed-loop control systems.

We know 2 ways to solve by computer (numerical solution)

- Create a **block diagram** using integrators – we get this from the differential equation or direct from physical principles
- Use one or more **transfer function** blocks – usually from the differential equation of the plant, then add a control loop

And 2 ways to solve analytically:

- Use “**particular integral + complementary function**”
- Use the **Laplace Transform method – new!**

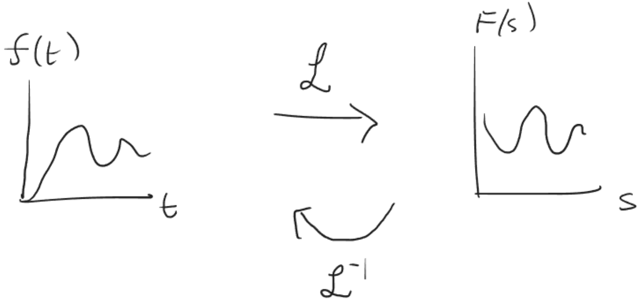
The Laplace transform converts a differential equation into a ‘normal’ algebraic equation.

## Introduction to Laplace Transforms

The idea is to do a translation from one language (time-based signal) into a new language (Laplace signal).

- The time language uses $t$ (seconds)
- The Laplace language uses $s$ (rad/sec)

It's the same $s$ we used before, but now it will be based on solid theory.



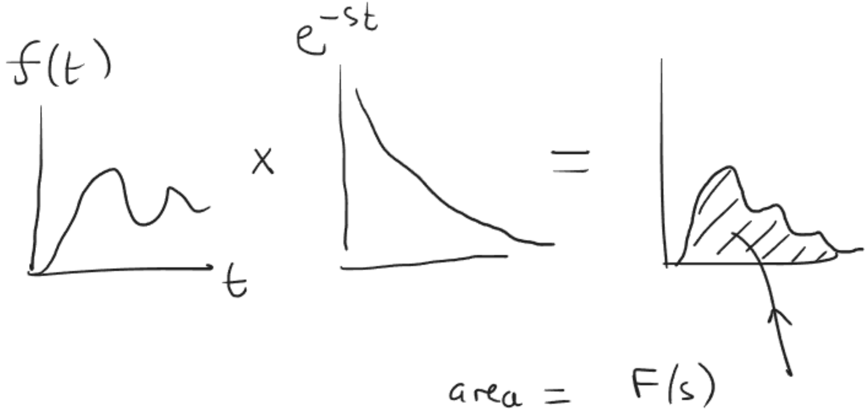

$\mathcal{L}$ is the translator (Laplace transform) and $\mathcal{L}^{-1}$ is how to translate back

For any signal $f(t)$ the Laplace version is defined by

$$F(s) = \int_0^\infty e^{-st} f(t) dt$$

It looks a bit intimidating, but fortunately we don't use this equation much, we simply refer to a 'dictionary' to do the translation – a table of Laplace transforms. This helps with translating back also!

As long as $s$ is a positive number the exponential becomes really small for large values of $t$, so the 'area under the curve' is something finite




We will use standard integration rules to work out a few cases and then refer to the 'dictionary' for other cases.

Two ‘rules’ of integration we need below are: integral of an exponential and integration by parts:

$$\int e^{at} dt = \frac{1}{a} e^{at}$$

$$\int u\, v' dt = \ [uv] - \int u' v dt$$

**Example** – unit step

$$f(t) = 1(t)$$

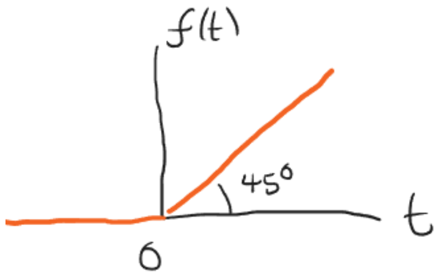
Then

$$F = \int_0^\infty e^{-st} 1\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

The upper limit for $t$ gives zero, the lower limit gives (with two negative signs). Hence $F(s) = 1/s$. This is one entry for the Laplace dictionary.

**Reality check** – this may appear to be the same as our 'integrator block' but note: here talking about **signals**, whereas the integrator is a **system** with inputs and outputs. We will confirm the connection shortly.

**Example** – **unit ramp** function




Here $f(t) = t\ 1(t)$

Since we only start integration from zero we have

$$F = \int_0^\infty e^{-st} t\ dt$$

Using integration-by-parts with $v' = e^{-st}$, we obtain

$$v = \frac{e^{-st}}{-s}$$

and $u(t) = t$ (for $t \geq 0$).

Hence

$$F = \left[t\,\frac{e^{-st}}{-s}\right]_0^\infty - \int 1\,\frac{e^{-st}}{-s}\,dt$$

The square-bracket term is zero at the upper limit because of $e^{-st}$ and at the lower limit because of $t$.

Integrating again for the second term

$$F = -\left[\frac{e^{-st}}{(-s)^2}\right]_0^\infty$$

So

$$F(s) = \frac{1}{s^2}$$

We can similarly calculate (but actually don't!)

$$f(t) = \frac{1}{2}t^2 \quad \rightarrow \quad F(s) = 1/s^3$$

$$f(t) = \frac{1}{3!}t^3 \quad \rightarrow \quad F(s) = 1/s^4$$

…

$$f(t) = \frac{1}{n!}t^n \quad \rightarrow \quad F(s) = 1/s^{n+1}$$

And this works for the case $n = 0$ which is just the step.

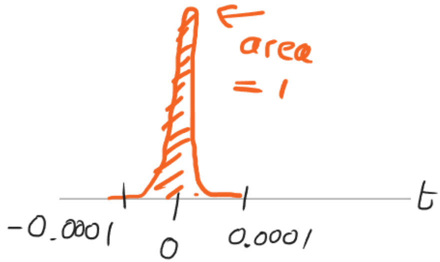
**Little Table of Laplace Transforms**

| $f(t)$ | $F(s)$ |
|---|---|
| Impulse function $\delta(t)$ | $1$ |
| Step function, $u(t)$ | $\frac{1}{s}$ |
| $e^{-at}$ | $\frac{1}{s+a}$ |
| $\sin \omega t$ | $\frac{\omega}{s^2+\omega^2}$ |
| $\cos \omega t$ | $\frac{s}{s^2+\omega^2}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ |

Of course you can easily find bigger tables of Laplace transforms online, e.g. https://web.stanford.edu/~boyd/ee102/laplace-table.pdf - see page 2 in particular. Pages 1 and 3 give more information which may be of interest.

**Tutorial Questions**

1. Calculate $F(s)$ for $f(t) = e^{-at}$ where $a$ is a positive constant. Check with the table above.
2. The unit impulse function $\delta(t)$ is a kind of concentrated pulse that looks a little like this. The integration should start just before $t = 0$ and 'during the pulse' the term $e^{-st}$ hardly changes (since $t$ hardly changes). Find the Laplace transform of this function.




3. Calculate $F(s)$ for $f(t) = e^{j\omega t}$ treating $j\omega$ like a constant. The calculation is similar to question 1.
4. Noting that $e^{j\omega t} = \cos\omega t + j\sin\omega t$, try to derive $F(s)$ for $\cos\omega t$ (real part) and $\sin\omega t$ (imaginary part).
5. Use integration by parts to find the Laplace transform of $\frac{df}{dt}$, assuming $F(s)$ is the transform of $f(t)$.

**Link to the s-operator**

We use the answer for question 5: the Laplace transform of $\frac{df}{dt}$ is

$$sF(s) - f(0)$$

Note: in case there is a jump in $f(t)$ at $t = 0$ it's necessary to evaluate $f(0)$ just before that jump. So we write

$$sF(s) - f(0^-)$$

But in any case, **if the initial condition is zero …**

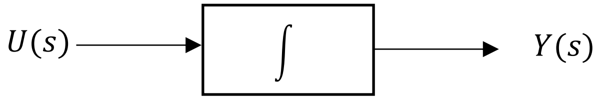
**differentiation of the time signal is the same as multiplying the Laplace signal by s**

This explains the link to the s-variable we used before

And now the transfer function can be theoretically defined

**The transfer function of an LTI system is the Laplace transform of the output divided by the Laplace transform of the input, assuming zero initial conditions.**

E.g. for an integrator system we can now derive the transfer function




If $u(t)$ is the impulse function (transform = 1) the output will be the unit step (think about it!). Hence

$$G(s) = \mathcal{L}(y) \div \mathcal{L}(u) = \frac{1}{s} \div 1 = \frac{1}{s}$$

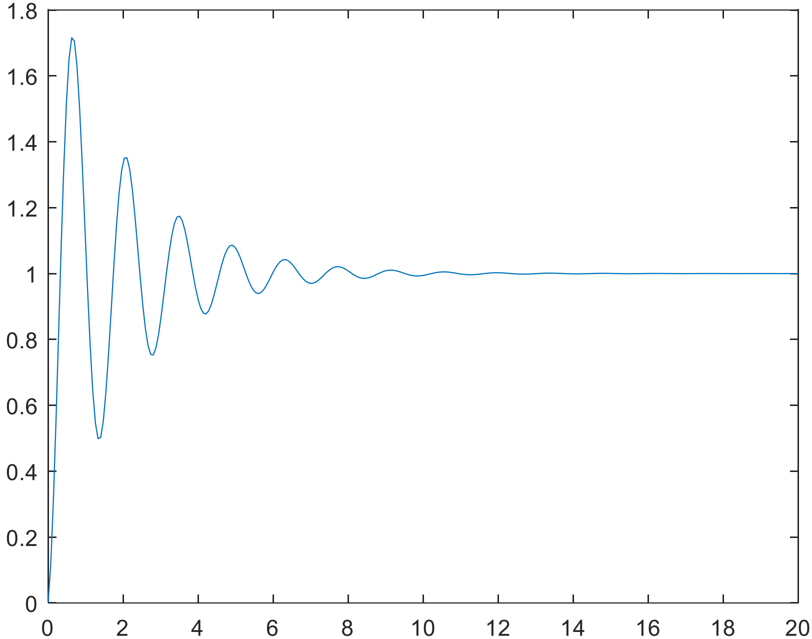
We now have a good theory for ‘s=derivative’ and ‘1/s = integrator’.

Note: in **block diagrams** we don't care if the signal is written $u(t)$ or $U(s)$ … it still represents the same physical signal.

Clearly all the s-operator work we did before remains valid.

Even better, Matlab understands the s-variable … for example the following script simulates the step response for the displacement-controlled mass-spring-damper

```
clear
close all
m=1;c=1;k=20;
s=tf('s');
G=(k+c*s)/(m*s^2+c*s+k);
[y,t]=step(G,20);
plot(t,y)
```


1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
2
4
6
8
10
12
14
16
18
20

It's an under-damped response (in fact the damping ratio is just 0.224 – you can check this).

The command "`s=tf('s')`" is understood by Matlab as "s is the Laplace operator" and therefore it understands the construction of G as a **symbolic calculation**, not just as a single number.

Matlab even displays G(s) is a simple form

```
G =

     s + 20
  ------------
  s^2 + s + 20
```

Using this format is a really quick and convenient method for dealing with transfer functions (though Simulink is generally better when we have focus on the block diagram and feedback loops).

**Solving differential equations with Laplace transforms**

We now have the theory to find analytical solutions using the new method.

<u>Example</u>

$$\frac{dy}{dt} + y = u$$

with $u(t) = 1(t)$ and $y(0) = 5$. Converting to Laplace form ("take Laplace transforms of each term")

$$sY(s) - y(0) + Y(s) = \frac{1}{s}$$

so

$$(s+1)Y(s) - 5 = \frac{1}{s}$$

and

$$Y(s) = \frac{1}{s(s+1)} + \frac{5}{s+1}$$

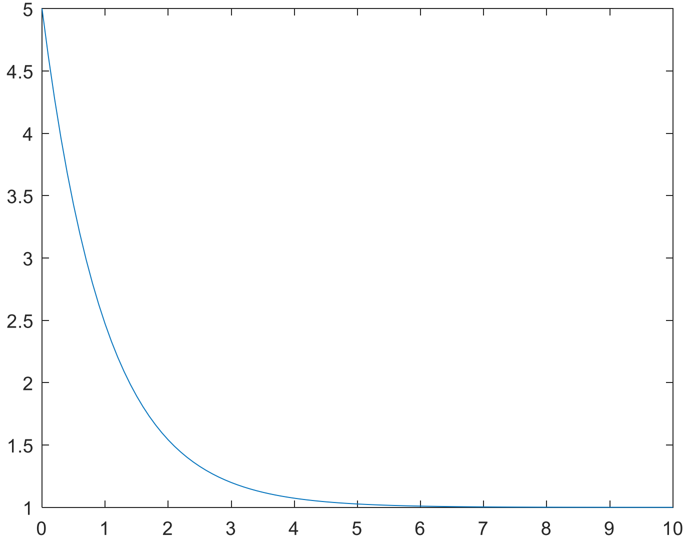
Using partial fractions we find

$$Y(s) = \frac{1}{s} - \frac{1}{(s+1)} + \frac{5}{s+1} = \frac{1}{s} + \frac{4}{(s+1)}$$

And then from the table of Laplace transforms (translating back)

$$y(t) = 1 + 4e^{-t}$$

here is the plot



Let's recap:

- Laplace transforms are defined by an integration formula converting $f(t) \leftrightarrow F(s)$
- The Laplace variable $s$ acts like differentiation but we sometimes have to take initial conditions into account
- The Laplace transform is **linear**, so that

$$\mathcal{L}\big(f_1(t) + f_2(t)\big) = F_1(s) + F_2(s)$$

and

$$\mathcal{L}\big(cf(t)\big) = cF(s)$$

**Tutorial Questions** continued

6. For the above example, $\dot{y} + y = u$, now assuming zero initial conditions, use the Laplace method to calculate (i) the impulse response ($u = \delta(t)$), (ii) the step response ($u = 1(t)$) and (iii) the ramp response ($u = t$). Plot the solutions in Matlab.
7. Recompute the plots from question 6 using the `s=tf('s')` method. Note that as well as the command **step**, Matlab has a command **impulse**. There is no command “ramp” so you need to be a bit creative to get that result.
8. For the second-order system

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 5y = u$$

(i) find  the transfer function

(ii) determine the steady-state gain for this transfer function

(iii) use Laplace to find the unit step response, assuming zero initial conditions. Plot the result.

(iv) repeat (iii) but with $y(0) = 1$, $\dot{y}(0) = 2$.