Controllability
&
Observability

We learn:

- Controllability
- Observability

- Controllability and observability represent two major concepts of modern control system theory.
- These concepts were introduced by R.Kalman in 1960.

In a nutshell:

Controllability:

In order to be able to do whatever we want with the given dynamic system under control input, the system must be controllable.

Observability:

In order to see what is going on inside the system under observation, the system must be observable.

Controllability and observability formal definition:

Controllability: A system is said to be controllable at time if it is possible to transfer the system from any initial state to any other state in finite interval of time by means of an unconstrained control vector.

> A system is completely controllable if the initial state of the system is transferred to any particular state, in a finite time duration, when a controlled input is provided to it.

Observability: A system is said to be observable at time if it is possible to determine the state of the system from the observation of the output over a finite time interval.

> Observability of a control system is the ability of the system to determine the internal states of the system by observing the output in a finite time interval when input is provided to the system.

- Consider the system:

[redacted]

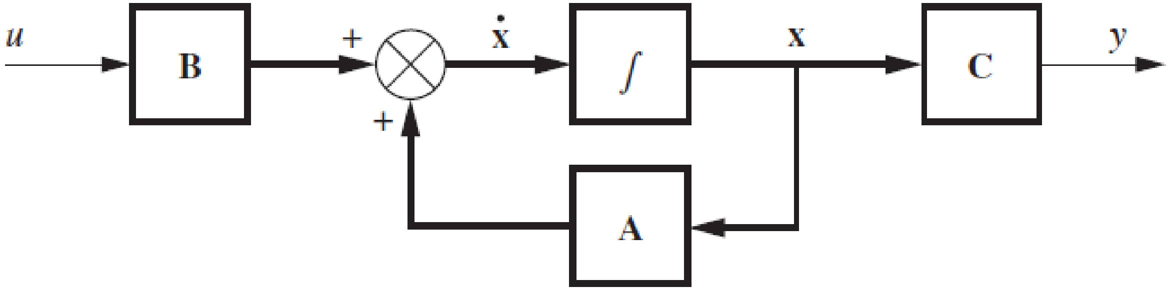
state vector

control signal

n×n   matrix

B :   n×1   matrix

Open-loop system (plant)



Controllability:

[redacted]

Is said to be state controllable at if it is possible to construct an unconstrained control signal that will transfer an initial state to any final state in a finite tie interval

If every state is controllable, then the system is said to be completely state controllable.

How to check controllability:

If the system is completely state controllable, then given any initial state x(0), the rank of the n×n matrix CM must be n.

$$CM = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}_{n \times n}$$

:Controllability matrix

- The rank of **CM** equals the number of linearly independent rows or columns. The rank can be found by finding the highest-order square submatrix that is nonsingular.

Example 1:

Is the system controllable?

$$\dot{X} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} U(t)$$

Solution.

Here **n = 2**

So,

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

And

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Therefore,

$$AB = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence on solving the matrix

$$AB = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$CM = [B \quad AB]_{2\times 2} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$|C\ M| = 1 \neq 0$$

Here determinant of the CM matrix is non-zero.
Also, the rank of CM, **n is 2**.
So, the system is **controllable**

Example 2:

Is the system controllable?

$$\dot{X} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} U(t)$$

Ex7.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

$$\mathbf{C}_M = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ 1 & -2 & 4 \end{bmatrix}$$

.

Since the determinant is not zero, the matrix is nonsingular, and the rank of is 3.

We conclude that the system is controllable since the rank of equals the system order.

Ex8. $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$

$$\mathbf{C}_M = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -2 & 4 \end{bmatrix}$$

Not only is the determinant of this matrix equal to zero, but so is the determinant of any submatrix. Thus, the rank of is 1.

The system is uncontrollable because the rank of is 1, which is less than the order, 3, of the system.

Consider the unforced system described by the equations

state vector

output vector

$\boldsymbol{A}$ n×n matrix

C : m×n matrix

observability

The system is said to be completely observable if every state X(t0) can be determined from the observation of y(t) over a finite time interval, . The system is, therefore, completely observable if every transition of the state eventually affects every element of the output vector.

Note: the concept of observability is useful in solving the problem of reconstructing unmeasurable state variables from measurable state variables in the minimum possible length of time.

How to check observability:

the system is said to be completely observable only under the condition that the rank of the matrix OM is n.

$$OM = \begin{bmatrix} C^T & A^T C^T & (A^T)^2 C^T & \cdots & (A^T)^{n-1} C^T \end{bmatrix}_{mn \times n}$$

Example 3.

The system is observable or not?

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(t)$$

and

$$Y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Solution.

$$A^T = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

and

$$C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$OM = [C^T \quad A^T C^T]_{2\times 2}$$

So,

$$A^T C^T = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Therefore

$$A^T C^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$OM = [C^T \quad A^T C^T]_{2\times 2} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$|OM| = 2 \neq 0$$

As it is clear that the determinant achieved here is 2, thus is non-zero. Also, the rank here is 2

Exampel 4.

The system is observable or not?

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} U(t)$$

and

$$Y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Solution:

Here

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Thus, the transpose matrix

$$A^T = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

and

$$C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Here we already have C^T. So finding $A^T C^T$

$$A^T C^T = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So,

$$A^T C^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$OM = [C^T \quad A^T C^T]_{2\times 2} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$|OM| = 1 \neq 0$$

As it is clear that the determinant achieved here is 1, thus is non-zero. Also, the rank here is 2

CONT=ctrb(A,B)

OBSER=obsv(A,C)

- The rank of CONT and OBSER respectively determine the controllability and observability of the system.

- If *rank(CONT)* or *rank(OBSER)* is less than n, where n is the order of the system, the system is not controllable or not observable respectively.

- In terms of transfer functions, if the *rank(CONT)* or *rank(OBSER)* is less than n, there is a **cancellation** of terms in the numerator and denominator of the transfer function.
-
- Whether or not the cancellation occurs between the numerator and denominator of the transfer function, we may use the command *minreal(sys)* to see if we get a simplified transfer function.

Example 5.

Examine the controllability and observability of the system defined by:

where

$$A=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \qquad B=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad C=\begin{bmatrix} 5 & 6 & 1 \end{bmatrix}$$

```
A=[0 1 0;0 0 1;-6 -11 -6];
B=[0;0;1];
C=[5 6 1];
D=0;
CONT=ctrb(A,B)
rank(CONT)
OBSER=obsv(A,C)
rank(OBSER)
```

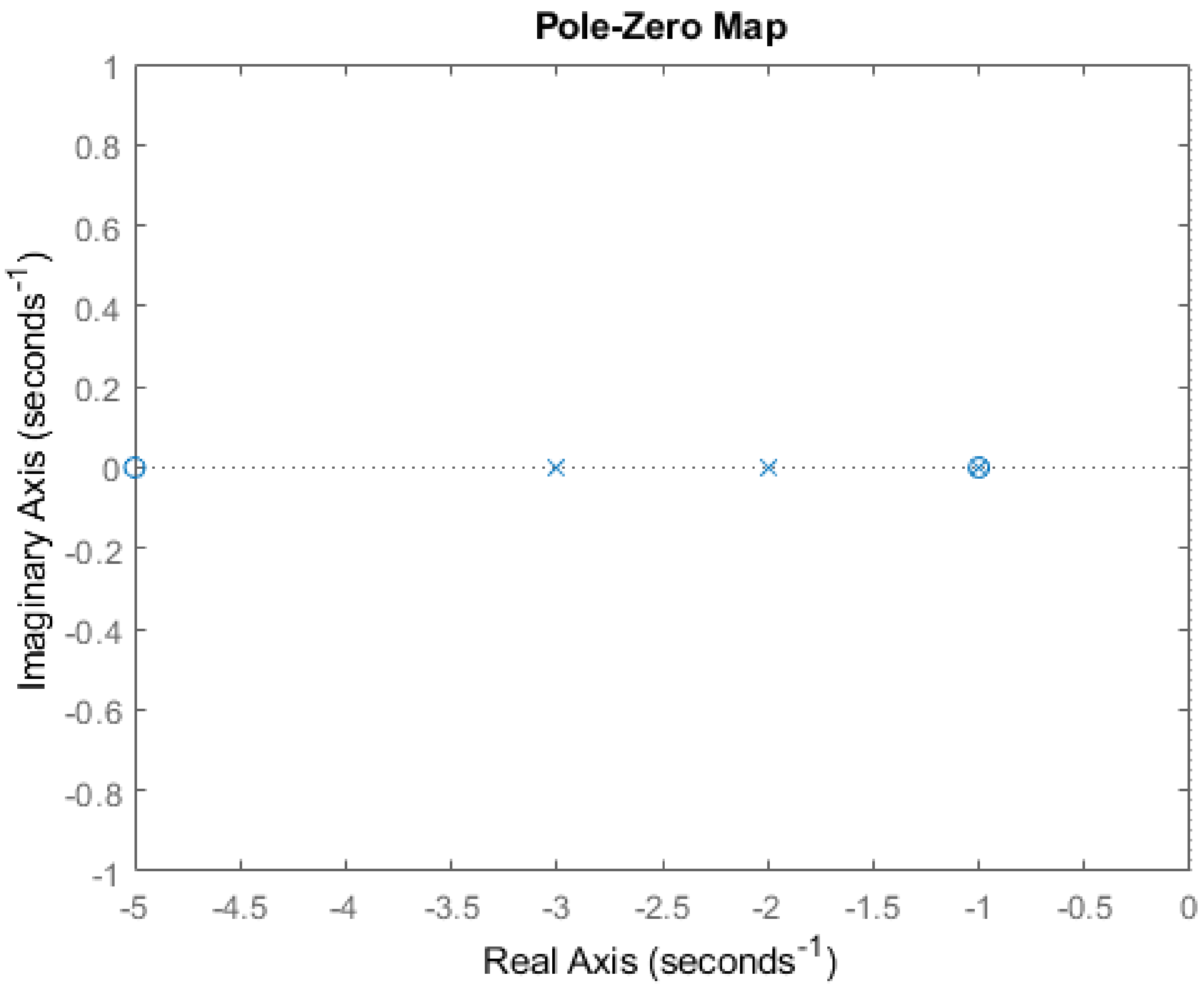
```
CONT =

     0     0     1
     0     1    -6
     1    -6    25


ans =

     3


OBSER =

     5     6     1
    -6    -6     0
     0    -6    -6


ans =

     2
```

- This system is controllable, but not observable.
- This suggest that there is a cancellation of terms in the numerator and denominator of the transfer function

```
[num,den]=ss2tf(A,B,C,D)
sys=tf(num,den)
```

```
sys=tf(num,den)

num =

    0     1     6     5

den =

    1.0000    6.0000   11.0000    6.0000

sys =

      s^2 + 6 s + 5
  ----------------------
  s^3 + 6 s^2 + 11 s + 6

Continuous-time transfer function.
```

```
pzmap(sys)
```


Pole-Zero Map
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-5
-4.5
-4
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
Imaginary Axis (seconds^-1)
Real Axis (seconds^-1)

minreal
Minimal realization or pole-zero cancellation

```
sys_min=minreal(sys)
```

```
sys_min =

    s + 5
  -------------
  s^2 + 5 s + 6

Continuous-time transfer function.
```

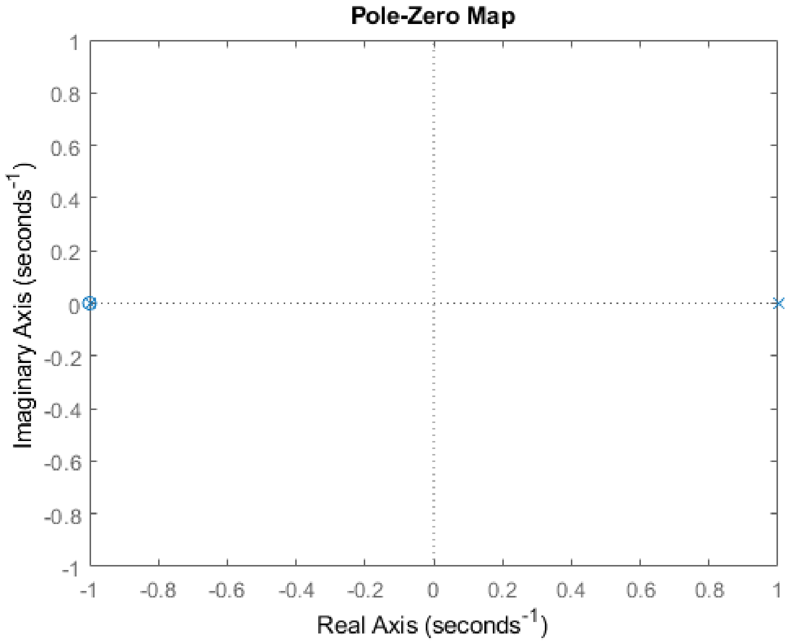
(s+1) terms in the numerator and denominator of the transfer function are cancelled.

Example 6.

- Consider the system defined by.

```
A=[1 0;0 -1];
B=[1;0];
C=[1 1];
D=0;
CONT=ctrb(A,B)
rank(CONT)
OBSER=obsv(A,C)
rank(OBSER)
[num,den]=ss2tf(A,B,C,D)
sys=tf(num,den)
pzmap(sys)
sys_min=minreal(sys)
```

```
CONT =

   1   1
   0   0
ans =

   1

OBSER =

   1   1
   1  -1

ans =

   2

num =

   0   1   1

den =

   1   0  -1

sys =

  s + 1
 -------
 s^2 - 1

Continuous-time transfer function.


sys_min =

  1
 -----
 s - 1

Continuous-time transfer function.
```



PROBLEM: Determine whether the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 3 & -4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u$$

is controllable.

ANSWER: Controllable

```
A=[-1  1  2
    0 -1  5
    0  3 -4]
B=[2;1;1]
Cm=ctrb(A,B)
Rank=rank(Cm)
```