Question 1

A matrix is a set of number which are laid out in rows and columns which create a rectangular array. The size of a matrix is determined by the number of rows and columns it has. The elements of a matrix are the numbers or expressions within the matrix. Matrix A is shown below:

$$A = \begin{bmatrix} 5 & 9 \\ 7 & 2 \end{bmatrix}$$

Matrix A has 2 rows and 2 columns making it a 2 x 2 square matrix. A square matrix has a main diagonal running from the top left to the bottom right of the matrix. A matrix with ones on the main diagonal and zeros everywhere else in the matrix is called an identity matrix [1], an I is used to denote this type of matrix with a number to the bottom right of the letter to show its size. An example of this is the identity matrix I_2:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

When considering linear transformations, matrices can be used to determine where any vector lands as long as you know where $\hat{\imath}$ and $\hat{\jmath}$ land after the linear transformation. For example, if you have the 2 x 2 matrix show below:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix}$$

The position of $\hat{\imath}$ is equivalent to the vector $\begin{bmatrix} a \\ c \end{bmatrix}$ & the position of $\hat{\jmath}$ is equivalent to the vector $\begin{bmatrix} b \\ d \end{bmatrix}$.

Therefore in order to find the position of $\begin{bmatrix} x \\ y \end{bmatrix}$:

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Which can also be defined as matrix-vector multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

The determinant of a transformation can be calculated using a matrix. The determinant is the factor by which a linear transformation changes an area.

The determinant can be worked out by multiplying the elements of the 2 x 2 matrix together diagonally and subtracting the values received from each other. [1] This is shown in figure 1.

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Question 2

Eigen vectors are special vectors which remain in their own span after a transformation. Eigenvalues are a set of values which are used to transform an Eigenvector. [2] It is important to compute Eigenvalues and Eigenvectors as they allow you to determine breakpoint for any component. For example, figure 2 shows a component from a robot.

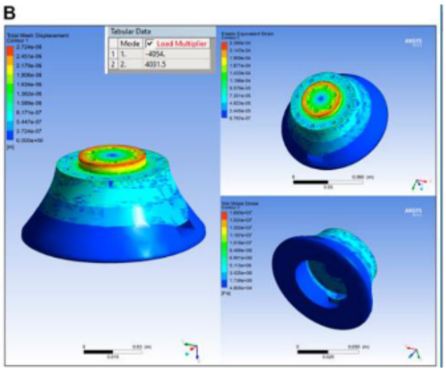


Figure 2 - Buckling of a robot component [3]

If you compute the Eigenvalue and Eigenvector for this component under a force, the Eigenvector determines the direction of the force and the Eigenvalue determines how much this component can be stressed. This can be done using Eigenvalue buckling, this works out how much stress a component can take before breaking which will cause the vector to be disorientated from the original vector, because there will be multiple pieces moving along different vectors. [2]

In calculating this you can calculate the load multiplier for this application. For example if this component was subjected to 100N and through computing the Eigenvalue and Eigenvector, it was determined that the load multiplier was 4054. The maximum force that this component could take along that vector before breaking could be 4054 times 100N so 405,400N.

Eigenvalues can be calculated as shown:

Starting with a 2 x 2 matrix and find the determinant:

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix}\right) = (1-\lambda)(3-\lambda) = 0$$

Find the Eigenvalues:

$$(1-\lambda)(3-\lambda) = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda+1)(\lambda-5) = 0$$

$$\lambda = -1 \qquad Or \qquad \lambda = 5$$

Eigenvectors can be computed by substituting the Eigenvalues in:

$$\begin{bmatrix} 1-5 & 4 \\ 2 & 3-5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Sometimes there can be no eigenvectors as the eigenvectors can be imaginary numbers.

Question 3

a.

1. ***Compute the determinant of*** $A - \lambda I$. With λ subtracted along the diagonal, this determinant starts with λ^n or $-\lambda^n$. It is a polynomial in λ of degree n.
2. ***Find the roots of this polynomial***, by solving $\det(A - \lambda I) = 0$. The n roots are the n eigenvalues of A. They make $A - \lambda I$ singular.
3. For each eigenvalue λ, ***solve*** $(A - \lambda I)x = 0$ ***to find an eigenvector*** x.

b.

i. a – eigenvector as its direction has not changed even after the image is transformed.
ii. b – not eigenvector as the direction of b has changed post transformation.
iii. scaling up in a, so eigen value increased.

c.

Is $\mathbf{v}$ an eigenvector with the corresponding $\lambda = 0$ for the matrix $\mathbf{A}$?

$$\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 6 & 3 \\ -2 & -1 \end{bmatrix}$$

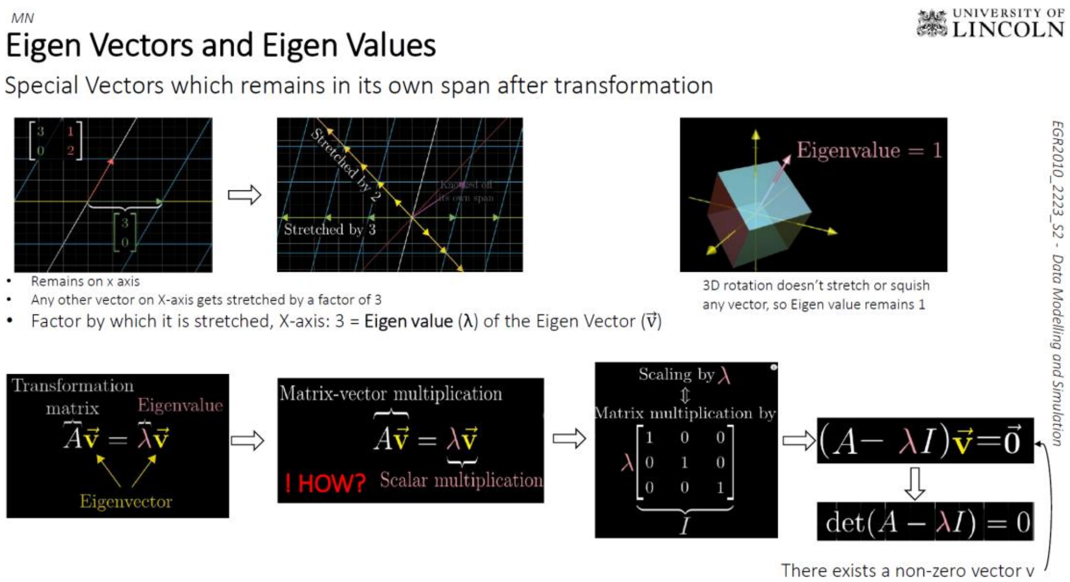
Solution

$$A \cdot \mathbf{v} = \lambda \cdot \mathbf{v}$$

$$\begin{bmatrix} 6 & 3 \\ -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

HELP FROM Presentation:



Question 4

a

Example: Find the eigenvalues and associated eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}.$$

We compute

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} -1-\lambda & 2 \\ 0 & -1-\lambda \end{vmatrix} \\ &= (\lambda+1)^2. \end{aligned}$$

Setting this equal to zero we get that $\lambda = -1$ is a (repeated) eigenvalue. To find any associated eigenvectors we must solve for $\mathbf{x} = (x_1, x_2)$ so that $(\mathbf{A} + \mathbf{I})\mathbf{x} = \mathbf{0}$; that is,

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad x_2 = 0.$$

Thus, the eigenvectors corresponding to the eigenvalue $\lambda = -1$ are the vectors whose second component is zero, which means that we are talking about all scalar multiples of $\mathbf{u} = (1, 0)$.

b

Example: Find the eigenvalues and associated eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{bmatrix}.$$

First we compute $\det(\mathbf{A} - \lambda\mathbf{I})$ via a cofactor expansion along the second column:

$$\begin{aligned} \begin{vmatrix} 7-\lambda & 0 & -3 \\ -9 & -2-\lambda & 3 \\ 18 & 0 & -8-\lambda \end{vmatrix} &= (-2-\lambda)(-1)^4 \begin{vmatrix} 7-\lambda & -3 \\ 18 & -8-\lambda \end{vmatrix} \\ &= -(2+\lambda)[(7-\lambda)(-8-\lambda)+54] \\ &= -(\lambda+2)(\lambda^2+\lambda-2) \\ &= -(\lambda+2)^2(\lambda-1). \end{aligned}$$

Thus $\mathbf{A}$ has two distinct eigenvalues, $\lambda_1 = -2$ and $\lambda_3 = 1$. (Note that we might say $\lambda_2 = -2$, since, as a root, -2 has multiplicity two. This is why we labelled the eigenvalue 1 as λ_3.)

Now, to find the associated eigenvectors, we solve the equation $(\mathbf{A} - \lambda_j\mathbf{I})\mathbf{x} = \mathbf{0}$ for $j = 1, 2, 3$. Using the eigenvalue $\lambda_3 = 1$, we have

$$\begin{aligned} (\mathbf{A} - \mathbf{I})\mathbf{x} &= \begin{bmatrix} 6x_1 - 3x_3 \\ -9x_1 - 3x_2 + 3x_3 \\ 18x_1 - 9x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ &\Rightarrow x_3 = 2x_1 \quad \text{and} \quad x_2 = x_3 - 3x_1 \\ &\Rightarrow x_3 = 2x_1 \quad \text{and} \quad x_2 = -x_1. \end{aligned}$$

So the eigenvectors associated with $\lambda_3 = 1$ are all scalar multiples of

$$\mathbf{u_3} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Now, to find eigenvectors associated with $\lambda_1 = -2$ we solve $(\mathbf{A} + 2\mathbf{I})\mathbf{x} = \mathbf{0}$. We have

$$\begin{aligned} (\mathbf{A} + 2\mathbf{I})\mathbf{x} &= \begin{bmatrix} 9x_1 - 3x_3 \\ -9x_1 + 3x_3 \\ 18x_1 - 6x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ &\Rightarrow \; x_3 = 3x_1. \end{aligned}$$

Something different happened here in that we acquired no information about x_2. In fact, we have found that x_2 can be chosen arbitrarily, and independently of x_1 and x_3 (whereas x_3 cannot be chosen independently of x_1). This allows us to choose two linearly independent eigenvectors associated with the eigenvalue $\lambda = -2$, such as $\mathbf{u_1} = (1, 0, 3)$ and $\mathbf{u_2} = (1, 1, 3)$. It is a fact that all other eigenvectors associated with $\lambda_2 = -2$ are in the span of these two; that is, all others can be written as linear combinations $c_1\mathbf{u_1} + c_2\mathbf{u_2}$ using an appropriate choices of the constants c_1 and c_2.

c

Find the eigenvalues and eigenvectors of

$$B = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 0 & 1 \\ -2 & 1 & 1 \end{bmatrix}.$$

Solution:

First find the eigenvalues by solving:

$$\begin{aligned} 0 &= \det[B - \lambda I] \\ &= \det \begin{bmatrix} 1-\lambda & 1 & -2 \\ -1 & 0-\lambda & 1 \\ -2 & 1 & 1-\lambda \end{bmatrix} \\ &= (1-\lambda)[-\lambda(1-\lambda) - 1(1)] - 1[-1(1-\lambda) - 1(-2)] - 2[1(-1) - (-2)(-\lambda)] \\ &= (1-\lambda)\left(-\lambda + \lambda^2 - 1\right) - (-1 + \lambda + 2) - 2(-1 - 2\lambda) \\ &= -\lambda + \lambda^2 - 1 + \lambda^2 - \lambda^3 + \lambda + 1 - \lambda - 2 + 2 + 4\lambda \\ &= -\lambda^3 + 2\lambda^2 + 3\lambda. \end{aligned}$$

The characteristic equation is: [7]

$$-\lambda^3 + 2\lambda^2 + 3\lambda = 0.$$

[7] Note that the characteristic equation is a cubic, that is it is a polynomial of degree 3. If we had a 4×4 matrix, the characteristic equation would be a polynomial of degree 4 and so on.

To solve this, we factorize to get:

$$
\begin{aligned}
-\lambda^3 + 2\lambda^2 + 3\lambda &= 0 \\
-\lambda\left(\lambda^2 - 2\lambda - 3\right) &= 0 \\
-\lambda\left(\lambda - 3\right)\left(\lambda + 1\right) &= 0.
\end{aligned}
$$

Hence the eigenvalues of B are: $\lambda_1 = 0,\ \lambda_2 = 3$ and $\lambda_3 = -1$. In many courses you may solve the characteristic equation using a CAS calculator. Check this with your teacher.

We now find the eigenvectors by solving equation (6) for each eigenvalue.

For $\lambda_1 = 0$,we have [8]

[8] Note that $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$$
\begin{aligned}
(B - \lambda I)\, X &= 0. \\
\begin{bmatrix} 1-0 & 1 & -2 \\ -1 & 0-0 & 1 \\ -2 & 1 & 1-0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= 0 \\
\begin{bmatrix} 1 & 1 & -2 \\ -1 & 0 & 1 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= 0 \\
x + y - 2z &= 0 \qquad (8) \\
-x + z &= 0 \qquad (9) \\
-2x + y + z &= 0. \qquad (10)
\end{aligned}
$$

From equation (9)

$$z = x.$$

Substituting this in (8) gives

$$\begin{aligned} x + y - 2x &= 0 \\ x &= y. \end{aligned}$$

Hence the solution is $x = y = z = t$ where $t \in \mathbb{R}$. Hence the eigenvector has the form

$$X = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

So the eigenvector corresponding to $\lambda_1 = 0$ is

$$\begin{aligned} X_1 &= \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \end{aligned}$$

For $\lambda_2 = 3$ we have

$$
\begin{aligned}
(B - \lambda I)\, X &= 0. \\
\begin{bmatrix} 1-3 & 1 & -2 \\ -1 & 0-3 & 1 \\ -2 & 1 & 1-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= 0 \\
\begin{bmatrix} -2 & 1 & -2 \\ -1 & -3 & 1 \\ -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= 0 \\
-2x + y - 2z &= 0 \qquad (11) \\
-x - 3y + z &= 0 \qquad (12) \\
-2x + y - 2z &= 0. \qquad (13)
\end{aligned}
$$

Equations (11) and (13) are identical and so we have to solve (11) and (12). Let $z = t$ where $t \in \mathbb{R}$ then (11) and (12) become

$$
\begin{aligned}
-2x + y &= 2t \qquad (14) \\
-x - 3y &= -t. \qquad (15)
\end{aligned}
$$

Multiplying (15) by 2 and subtracting from (14) gives

$$
\begin{aligned}
7y &= 4t \\
y &= \frac{4}{7}t.
\end{aligned}
$$

Multiplying (14) by 3 and adding to (15) gives

$$
\begin{aligned}
-7x &= 5t \\
x &= -\frac{5}{7}t.
\end{aligned}
$$

Hence the eigenvector has the form

$$
X = t \begin{bmatrix} -\frac{5}{7} \\ \frac{4}{7} \\ 1 \end{bmatrix}.
$$

So the eigenvector corresponding to $\lambda_2 = 3$ is (setting $t = 7$)

$$
\begin{aligned}
X_2 &= \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\
&= \begin{bmatrix} -5 \\ 4 \\ 7 \end{bmatrix}. \qquad (16)
\end{aligned}
$$

Note that we chose $t = 7$ to get rid of the fraction in the eigenvector.
Note also that the eigenvector

$$X = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}$$

is also correct as it is just (16) multiplied by -1.

For $\lambda_3 = -1$ we have

$$(B - \lambda I)\,X = 0.$$

$$\begin{bmatrix} 1-(-1) & 1 & -2 \\ -1 & 0-(-1) & 1 \\ -2 & 1 & 1-(-1) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & 1 & -2 \\ -1 & 1 & 1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$2x + y - 2z = 0 \qquad (17)$$
$$-x + y + z = 0 \qquad (18)$$
$$-2x + y + 2z = 0. \qquad (19)$$

Adding (17) and (19) we get

$$2y = 0$$
$$y = 0.$$

Substituting this into (18) gives

$$-x + y + z = 0$$
$$x = z.$$

Let $z = t$, $t \in \mathbb{R}$ then the eigenvector has the form

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Hence the eigenvector to $\lambda_3 = -1$ is

$$X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

To summarise, the eigenvalues of B are $\lambda_1 = 0$, $\lambda_2 = 3$ and $\lambda_3 = -1$.
The corresponding eigenvectors are

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; X_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} \text{ and } X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

respectively.

Question 5

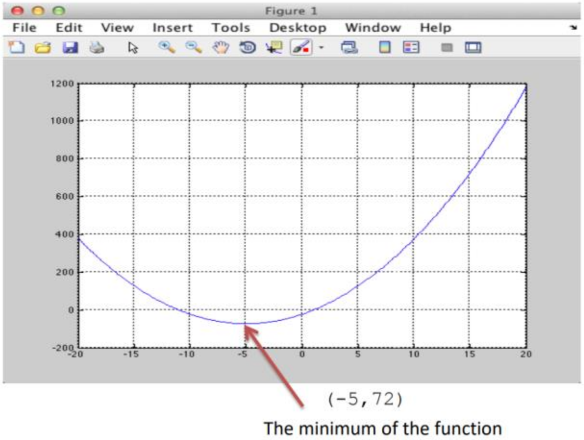
```
function f = mysimplefunc(x)

f = 2*x.^2 + 20.*x -22;
```

```
x_min =

    -5


y =

   -72
```

```
clear
clc
close all

x = -20:1:20;
f = mysimplefunc(x);
plot(x, f)
grid

x_min = fminbnd(@mysimplefunc, -20, 20)

y = mysimplefunc(x_min)
```

Optimization

Example: $y(x) = 2x^2 + 20x - 22$



We have that:

$$\frac{dy}{dx} = 4x + 20$$

Minimum when:

$$\frac{dy}{dx} = 0$$

This gives:

$$4x + 20 = 0$$

$$x = -5$$

fminbnd

Find minimum of single-variable function on fixed interval

Syntax

```
x = fminbnd(fun,x1,x2)
x = fminbnd(fun,x1,x2,options)
x = fminbnd(problem)
[x,fval] = fminbnd(___)
[x,fval,exitflag] = fminbnd(___)
[x,fval,exitflag,output] = fminbnd(___)
```

Description

`fminbnd` is a one-dimensional minimizer that finds a minimum for a problem specified by

$$\min_x f(x) \text{ such that } x_1 < x < x_2.$$

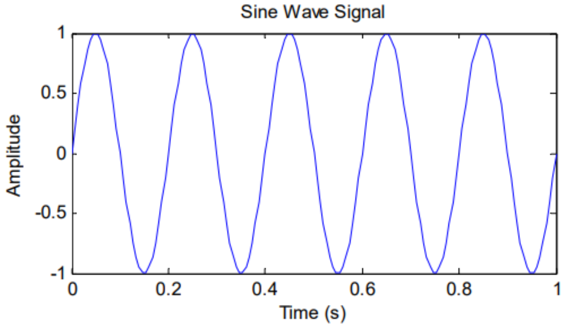
x, x_1, and x_2 are finite scalars, and $f(x)$ is a function that returns a scalar.

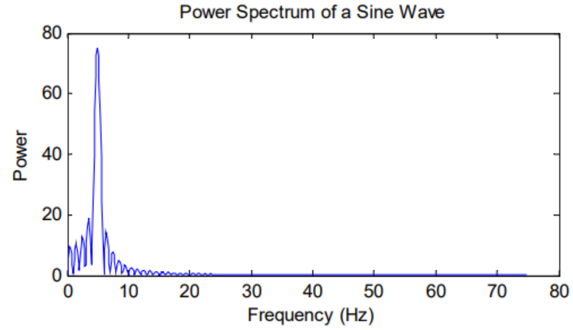
`x = fminbnd(fun,x1,x2)` returns a value `x` that is a local minimizer of the scalar valued function that is described in `fun` in the interval `x1 < x < x2`.

Question 6

a

Example 1: Sine Wave

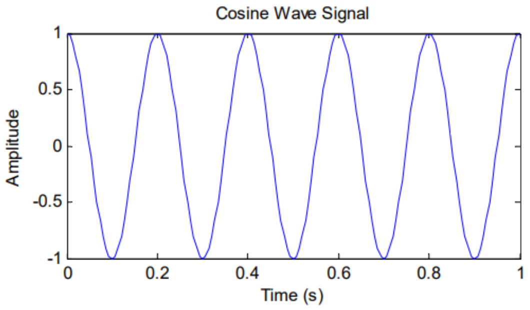


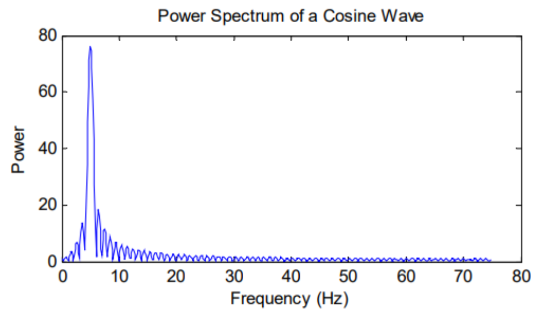



```
Fs = 150; % Sampling frequency
t = 0:1/Fs:1; % Time vector of 1 second
f = 5; % Create a sine wave of f Hz.
x = sin(2*pi*t*f);
nfft = 1024; % Length of FFT
% Take fft, padding with zeros so that length(X)
is equal to nfft
X = fft(x,nfft);
% FFT is symmetric, throw away second half
X = X(1:nfft/2);
% Take the magnitude of fft of x
mx = abs(X);
% Frequency vector
f = (0:nfft/2-1)*Fs/nfft;
% Generate the plot, title and labels.
figure(1);
plot(t,x);
title('Sine Wave Signal');
xlabel('Time (s)');
ylabel('Amplitude');
figure(2);
plot(f,mx);
title('Power Spectrum of a Sine Wave');
xlabel('Frequency (Hz)');
ylabel('Power');
```

a.

Example 2: Cosine Wave

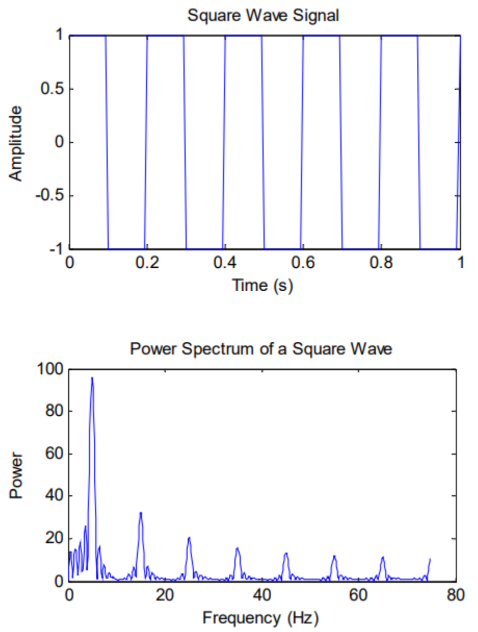





```
Fs = 150; % Sampling frequency
t = 0:1/Fs:1; % Time vector of 1 second
f = 5; % Create a sine wave of f Hz.
x = cos(2*pi*t*f);
nfft = 1024; % Length of FFT
% Take fft, padding with zeros so that length(X) is
equal to nfft
X = fft(x,nfft);
% FFT is symmetric, throw away second half
X = X(1:nfft/2);
% Take the magnitude of fft of x
mx = abs(X);
% Frequency vector
f = (0:nfft/2-1)*Fs/nfft;
% Generate the plot, title and labels.
figure(1);
plot(t,x);
title('Sine Wave Signal');
xlabel('Time (s)');
ylabel('Amplitude');
figure(2);
plot(f,mx);
title('Power Spectrum of a Sine Wave');
xlabel('Frequency (Hz)');
ylabel('Power');
```

b.

Example 4: Square Wave



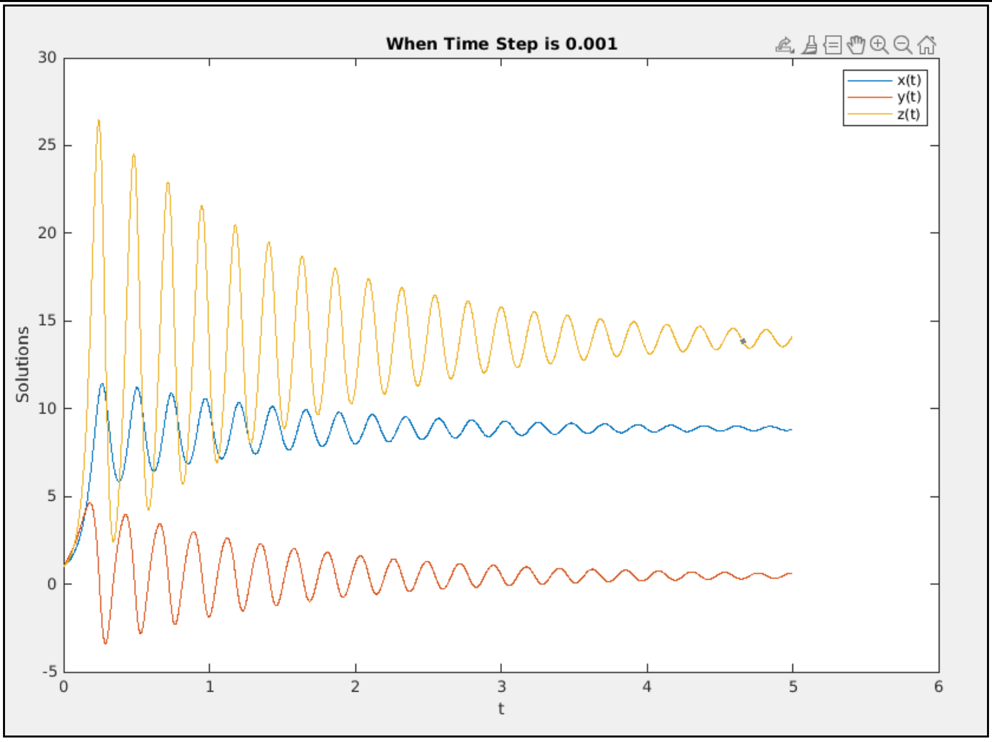

```
Fs = 150; % Sampling frequency
t = 0:1/Fs:1; % Time vector of 1 second
f = 5; % Create a sine wave of f Hz.
x = square(2*pi*t*f);
nfft = 1024; % Length of FFT
% Take fft, padding with zeros so that length(X)
equal to nfft
X = fft(x,nfft);
% FFT is symmetric, throw away second half
X = X(1:nfft/2);
% Take the magnitude of fft of x
mx = abs(X);
% Frequency vector
f = (0:nfft/2-1)*Fs/nfft;
% Generate the plot, title and labels.
figure(1);
plot(t,x);
title('Square Wave Signal');
xlabel('Time (s)');
ylabel('Amplitude');
figure(2);
plot(f,mx);
title('Power Spectrum of a Square Wave');
xlabel('Frequency (Hz)');
ylabel('Power');
```

Question 7

Refer:
https://www.youtube.com/watch?v=B9cSAmrgz4s&ab_channel=MATLABStarter

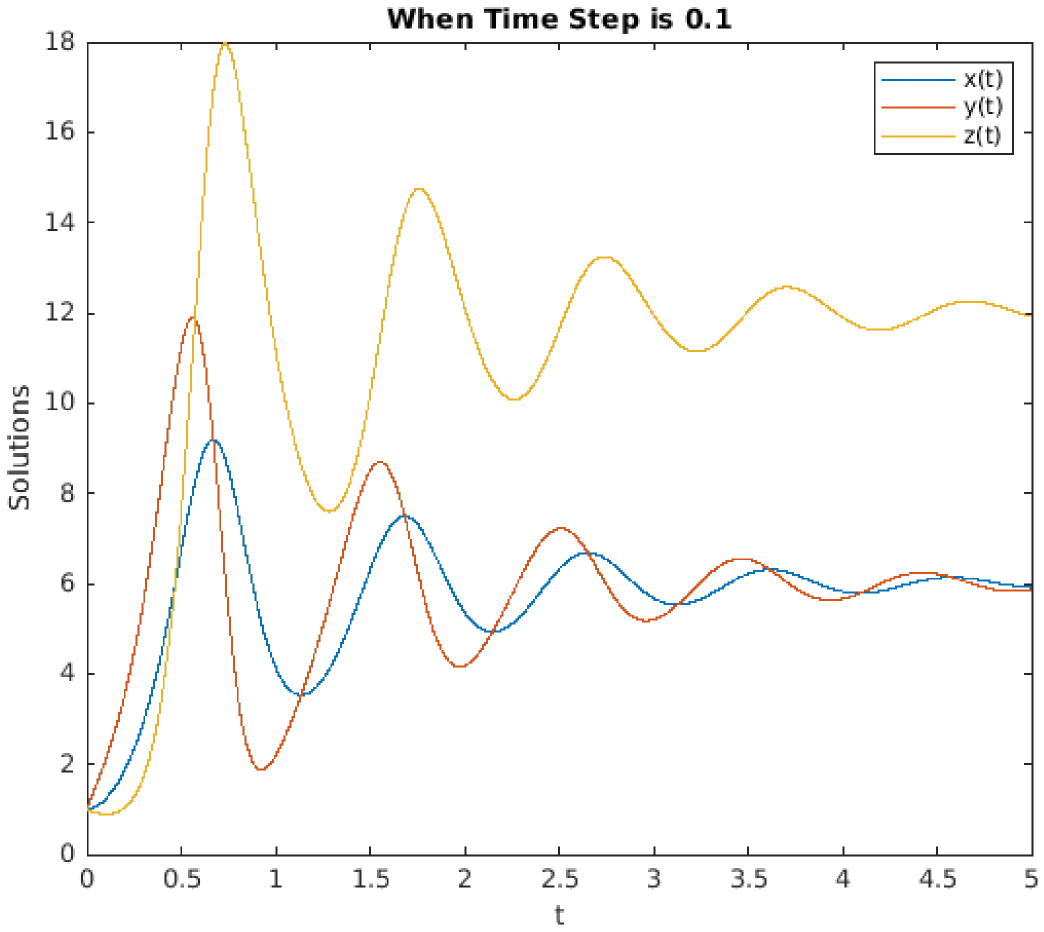
Question 8a

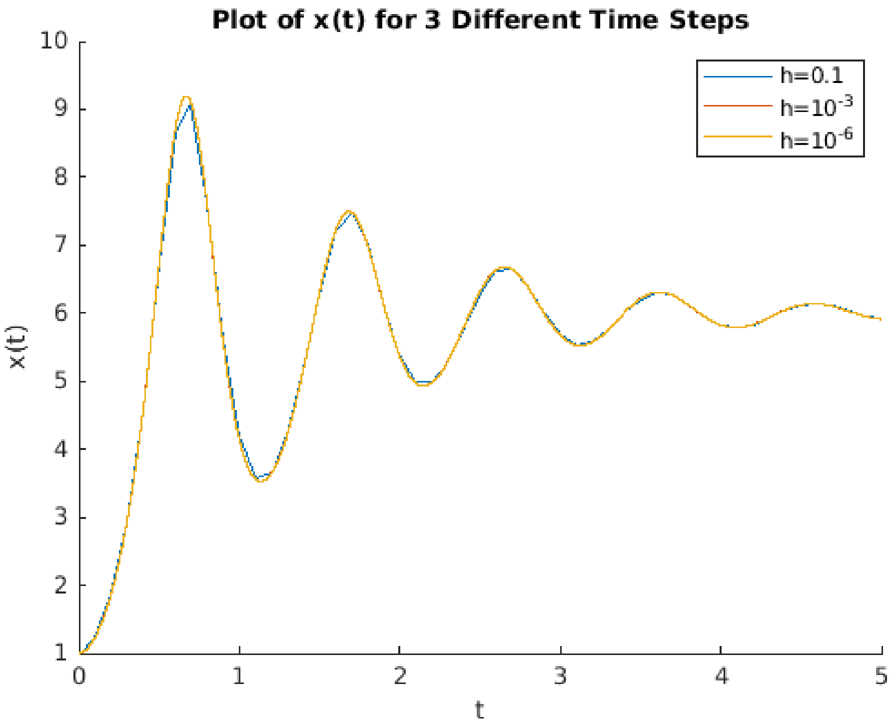
```
close all;

clc;
format long;

% Defining the three differential equations of the problem
f = @(t,y) [-10*y(1)+10*y(2)+6*y(3); 14*y(1)-y(1)*y(3); -3*y(3)+10*y(1)*y(2)];
[x,y] = explicit_euler(f,[0,5],[1;1;1],0.001); % Calling Euler function
plot(x,y);
title('When Time Step is 0.001');
legend('x(t)', 'y(t)', 'z(t)', 'Location', 'NorthEast')
xlabel('t')
ylabel('Solutions')


function [x, y] = explicit_euler( f, xRange, y_initial, h )
% This function uses Euler’s explicit method to solve the ODE
% dv/dt=f(t,v); x refers to independent and y refers to dependent variables
% f defines the differential equation of the problem
% xRange = [x1, x2] where the solution is sought on
% y_initial = column vector of initial values for y at x1
% numSteps = number of equally-sized steps to take from x1 to x2
% x = row vector of values of x
% y = matrix whose k-th column is the approximate solution at x(k)
x(1) = xRange(1);
numSteps = ( xRange(2) - xRange(1) ) /h ;
y(:,1) = y_initial(:);
for k = 1 : numSteps
x(k + 1) = x(k) + h;
y(:,k+1) = y(:,k) + h * f( x(k), y(:,k) );
end
end
```



Question 8b

```
clc;
format long;

% Defining the three differential equations of the problem
f=@(t,y) [-5*y(1)+5*y(2), 14*y(1)-2*y(2)-y(1)*y(3), -3*y(3)+y(1)*y(2)];
[x,y]=runge_kutta_4(f,[0,5],[1,1,1],0.001); % Calling RK4 function defined below
figure(2)
plot(x,y);
title('When Time Step is 0.1');
legend('x(t)', 'y(t)', 'z(t)', 'Location', 'NorthEast')
xlabel('t')
ylabel('Solutions')
figure(3);
hold on;

for h=[0.1, 10^-3, 10^-6] % Implementing 3 different time steps
[x,y]=runge_kutta_4(f,[0,5],[1,1,1],h);
plot(x,y(:,1));
end

title('Plot of x(t) for 3 Different Time Steps');
legend('h=0.1', 'h=10^{-3}', 'h=10^{-6}');
xlabel('t')
ylabel('x(t)')

% Fourth order Runge-Kutta method
function [x,y]=runge_kutta_4(f,tspan,y0,h)
x = tspan(1):h:tspan(2); % Calculates upto y(3)
y = zeros(length(x),3);
y(1,:) = y0; % Initial Conditions
for i=1:(length(x)-1)
k_1 = f(x(i),y(i,:));
k_2 = f(x(i)+0.5*h,y(i,:)+0.5*h*k_1);
k_3 = f((x(i)+0.5*h),(y(i,:)+0.5*h*k_2));
k_4 = f((x(i)+h),(y(i,:)+k_3*h));
y(i+1,:) = y(i,:) + (1/6)*(k_1+2*k_2+2*k_3+k_4)*h;
end

end
```


When Time Step is 0.1
Solutions
t
x(t)
y(t)
z(t)
0
2
4
6
8
10
12
14
16
18
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5


Plot of x(t) for 3 Different Time Steps
x(t)
t
h=0.1
h=10^-3
h=10^-6
1
2
3
4
5
6
7
8
9
10
0
1
2
3
4
5

Question 9

Since $\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 1$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = x(1-2y)$ or $y' = x(1-2y)$

If initially $x_0 = 0$ and $y_0 = 1$, (and $h = 0.2$), then $(y')_0 = 0(1-2) = 0$

Line 1 in the table below is completed with $x = 0$, $y = 1$ and $(y')_0 = 0$

x	y	$(y')_0$
0	**1**	0
0.2	**1**	–0.2
0.4	**0.96**	–0.368
0.6	**0.8864**	–0.46368
0.8	**0.793664**	–0.469824
1.0	**0.699692**	

For line 2, where $x_0 = 0.2$ and $h = 0.2$:

$$y_1 = y_0 + h(y')_0 = 1.0 + (0.2)(0) = 1$$

and $(y')_0 = x_0(1-2y_0) = 0.2(1-2) = -0.2$

For line 3, where $x_0 = 0.4$: $y_1 = y_0 + h(y')_0 = 1.0 + (0.2)(-0.2) = 0.96$

and $(y')_0 = x_0(1-2y_0) = 0.4(1-2(0.96)) = -0.368$

For line 4, where $x_0 = 0.6$: $y_1 = y_0 + h(y')_0 = 0.96 + (0.2)(-0.368) = 0.8864$

and $(y')_0 = x_0(1-2y_0) = 0.6(1-2(0.8864)) = -0.46368$

For line 5, where $x_0 = 0.8$: $y_1 = y_0 + h(y')_0 = 0.8864 + (0.2)(-0.46368) = 0.793664$

and $(y')_0 = x_0(1-2y_0) = 0.8(1-2(0.793664)) = -0.4698624$

For line 6, where $x_0 = 1.0$: $y_1 = y_0 + h(y')_0 = 0.793664 + (0.2)(-0.4698624) = 0.699692$

Hence a numerical solution to the differential equation $\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 1$ is given by the first two columns in the above table.

Question 10

$\frac{\mathrm{d}y}{\mathrm{d}x} = 3 - \frac{y}{x}$

1. $x_0 = 1$, $y_0 = 2$ and since $h = 0.1$, and the range is from $x = 1$ to $x = 1.5$, then

   $x_1 = 1.1$, $x_2 = 1.2$, $x_3 = 1.3$, $x_4 = 1.4$ and $x_5 = 1.5$

Let $n = 0$ to determine y_1:

2. $k_1 = f(x_0, y_0) = f(1, 2)$; since $\frac{\mathrm{d}y}{\mathrm{d}x} = 3 - \frac{y}{x}$, $f(1, 2) = 3 - \frac{2}{1} = 1$

3. $k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1\right) = f\left(1 + \frac{0.1}{2}, 2 + \frac{0.1}{2}(1)\right) = f(1.05, 2.05) = 3 - \frac{2.05}{1.05} = 1.047619$

4. $k_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_2\right) = f\left(1 + \frac{0.1}{2}, 2 + \frac{0.1}{2}(1.047619)\right) = f(1.05, 2.05238095)$

   $= 3 - \frac{2.05238095}{1.05} = 1.045351476$

5. $k_4 = f(x_0 + h, y_0 + hk_3) = f(1 + 0.1, 2 + 0.1(1.045351476)) = f(1.1, 2.1045351476)$

   $= 3 - \frac{2.1045351476}{1.1} = 1.086786229$

6. $y_{n+1} = y_n + \frac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$ and when $n = 0$:

   $y_1 = y_0 + \frac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\} = 2 + \frac{0.1}{6}\{1 + 2(1.047619) + 2(1.045351476) + 1.086786229\}$

   $= 2 + \frac{0.1}{6}\{6.272727181\} = 2.10454545$

n	x_n	y_n
0	**1.0**	**2.0**
1	**1.1**	**2.104545**
2	**1.2**	**2.216667**
3	**1.3**	**2.334615**
4	**1.4**	**2.457143**
5	**1.5**	**2.533333**

Lines 1 and 2 have now been completed in the above table

Let $n = 1$ to determine y_2:

2. $k_1 = f(x_0,\, y_0) = f(1.1,\, 2.104545)$; since $\dfrac{\mathrm{d}\,y}{\mathrm{d}\,x} = 3 - \dfrac{y}{x}$, $f(1.1,\, 2.104545) = 3 - \dfrac{2.104545}{1.1}$

$$= 1.08677727$$

3. $k_2 = f\left(x_0 + \dfrac{h}{2},\, y_0 + \dfrac{h}{2}k_1\right) = f\left(1.1 + \dfrac{0.1}{2},\, 2.104545 + \dfrac{0.1}{2}(1.08677727)\right) = f(1.15,\, 2.15888386)$

$$= 3 - \frac{2.15888386}{1.15} = 1.1227097$$

4. $k_3 = f\left(x_0 + \dfrac{h}{2},\, y_0 + \dfrac{h}{2}k_2\right) = f\left(1.1 + \dfrac{0.1}{2},\, 2.104545 + \dfrac{0.1}{2}(1.1227097)\right) = f(1.15,\, 2.1606805)$

$$= 3 - \frac{2.1606805}{1.15} = 1.1211474$$

5. $k_4 = f(x_0 + h,\, y_0 + hk_3) = f(1.1 + 0.1,\, 2.104545 + 0.1(1.1211474)) = f(1.2,\, 2.2166597)$

$$= 3 - \frac{2.2166597}{1.2} = 1.15278355$$

6. $y_{n+1} = y_n + \dfrac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$ and when $n = 0$:

$$y_1 = y_0 + \frac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

$$= 2.104545 + \frac{0.1}{6}\{1.08677727 + 2(1.1227097) + 2(1.1211474) + 1.15278355\}$$

$$= 2.104545 + \frac{0.1}{6}\{6.72727045\} = 2.216667$$

Line 3 has now been completed in the above table. In a similar manner y_3, y_4 and y_5 can be calculated.