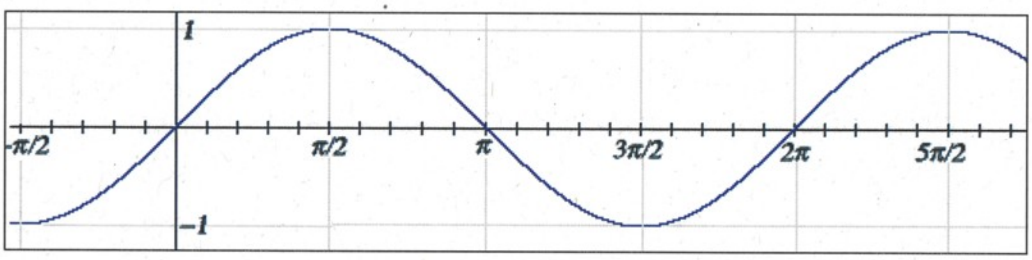
Lecture 5 - Introduction to the Fourier Series

Introduction

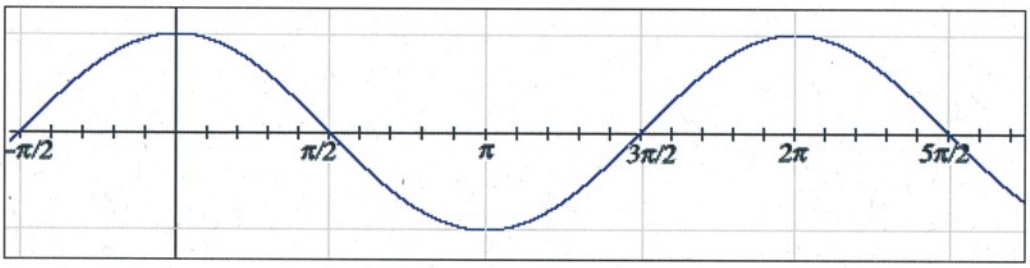
Fourier series provides a method of analysing periodic functions and breaking them down in to their constituent components. For example, the representation of musical sounds can be broken down in to a sum of waves of various frequencies.
The Taylor series is based on the idea that you can write a general function as an infinite series of powers. The idea of Fourier series is that you write a periodic function as an infinite series of sines and cosines. Before we begin to look at the Fourier series representation of a periodic function, let's remind ourselves of some of the properties of a periodic function.

Periodic Functions

The functions $y = \sin\theta$ and $y = \cos\theta$ are shown below.



$y = \sin\theta$



$y = \cos\theta$

The graphs of these functions look like waves where the angle θ is measured in either degrees or radians.

Voltages and currents in electrical circuits or mechanical oscillations usually vary with time, t. Hence we consider sine and cosine waves in which the independent variable is t. For example, consider the sine wave $y = \sin t$. From the graph above, we can see that as t increases from 0 seconds to 2π seconds, one complete cycle is produced.

Amplitude of a Period Function

The amplitude, A, of a wave is the highest value that a wave reaches during one complete cycle. The amplitude of $y = A\cos t$ is A and similarly, the amplitude of the wave $y = A\sin t$ is A. In other words, the value that appears in front of the sine or cosine term is the amplitude of the wave.

Angular Frequency of a Periodic Function

If we consider the wave $y = \sin \omega t$. We call ω the **angular frequency** of the wave. The units of ω are radians per second. This means that ωt has the units of radians. For example, $y = 5\cos 3t$ has an angular frequency of 3 radians per second and an amplitude of 5. We can also express it in the form $y = \sin 2\pi ft$ where f is the frequency in hertz.

The Period of a Wave

The time that a wave takes to complete one cycle is called the **period** of the wave. It is closely related to the angular frequency of the wave. The period, T, of a wave is given by the formula,

$$T = \frac{2\pi}{\omega}$$

Example 1 State the period of the function $y = 3\sin 2t$

$\omega = 2$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$

The Frequency of a Wave

The frequency, f, of a wave is the number of cycles that the wave completes in 1 second. Frequency is measured in Hertz (Hz) where 1 Hz is one cycle per second. The frequency, f, is given by the formula,

$$f = \frac{\omega}{2\pi}$$

Note that,

$$T = \frac{2\pi}{\omega} \quad \text{and} \quad f = \frac{\omega}{2\pi}$$

Therefore,

$$f = \frac{1}{T}$$

Example 2 State the period and frequency of the wave $y = 2\sin 4t$

$\omega = 4$ angular frequency in radians.

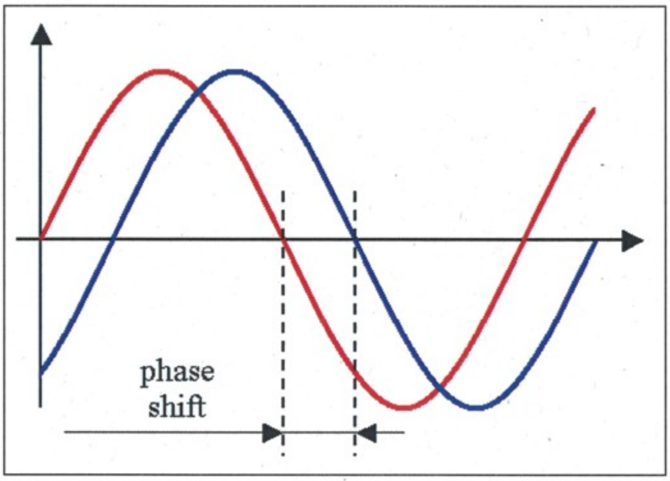
period $T = \frac{2\pi}{4} = \frac{\pi}{2}$

frequency in Hz

$$f = \frac{1}{T} = \frac{2}{\pi}$$

Phase and Time Displacement of a Wave

We will now look at waves of the form $y = A\sin(\omega t + \alpha)$ and $y = A\cos(\omega t + \alpha)$. Introducing the α has the effect of moving the wave to either the left or right.



We see from the waveforms above that the peaks of the waves are reached at different times. This difference in time is known as the **time displacement**. The wave which reaches its peak value first is said to be **leading** the other wave by the time displacement. The wave that reaches its peak second is said to be **lagging** the other wave by its time displacement.

The angle, α, is called the **phase angle** or **phase**. We call $\frac{\alpha}{\omega}$ the **time displacement** of the wave.

Example 3 State the phase and time displacement of the wave $y = \sin(2t + 1)$

phase $\alpha = 1$

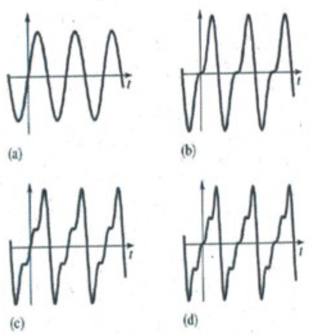
time displacement $\frac{\alpha}{\omega} = \frac{1}{2}$

The Fourier Series

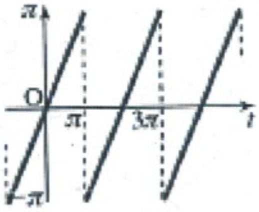
Suppose we have a periodic function f(t). Under certain conditions it can be expressed as the sum of an infinite number of sine and/or cosine functions. This infinite sum is known as a Fourier series.
For example, consider the graphs shown below which show the following:

(a) $y = 2\sin t$ (b) $y = 2\sin t - \sin 2t$ (c) $y = 2\sin t - \sin 2t + \frac{2}{3}\sin 3t$

(d) $y = 2\sin t - \sin 2t + \frac{2}{3}\sin 3t - \frac{1}{2}\sin 4t$



As more and more sine terms are added, the graph appears to resemble more and more closely the periodic saw-tooth waveform shown below.



This infinite series of sine functions is the Fourier series of the saw-tooth waveform. Notice that the Fourier series is built up of sine functions of increasing frequencies, and with decreasing amplitudes. In this series of lectures you will learn how to calculate the appropriate frequencies and amplitudes for yourself, so that, given any periodic function, you will be able to calculate its corresponding Fourier series. In most cases we will also need to include sine and cosine functions in order to construct the Fourier series, unless the periodic function being considered takes rather special forms.

Calculating a Fourier Series

To find a Fourier series you will need to make use of your knowledge of the integration of trigonometrical functions and integration by parts.
Given a periodic function f(t), with period T, calculate the quantities a_0, a_n and b_n from the following formulae. These quantities are known as Fourier coefficients.

Fourier coefficients

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\frac{2n\pi t}{T}\,dt, \quad n = 1, 2, 3, \ldots\ldots$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin\frac{2n\pi t}{T}\,dt, \quad n = 1, 2, 3, \ldots\ldots$$

When these quantities have been calculated the Fourier series is given by the following:

Fourier series

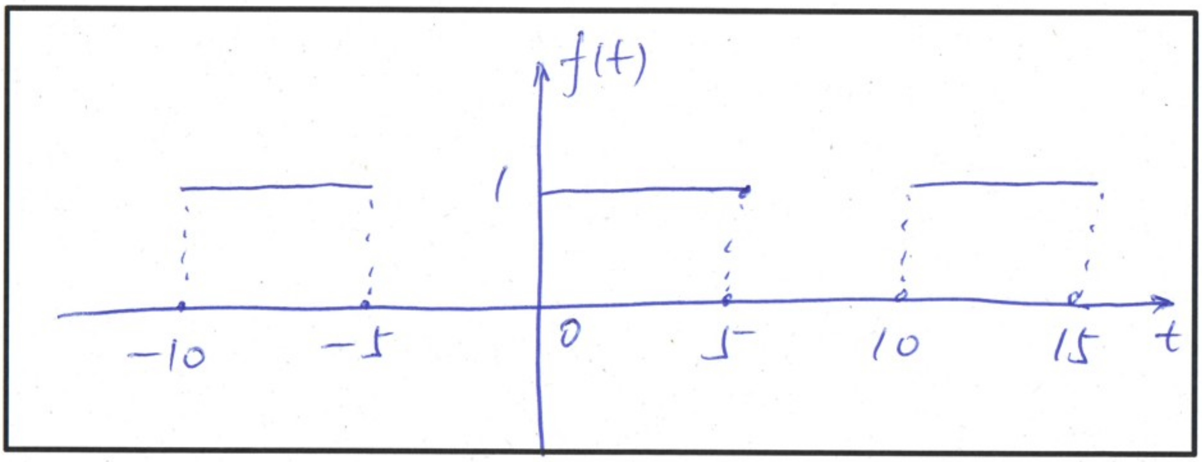
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi t}{T} + b_n\sin\frac{2n\pi t}{T}\right)$$

If we write out the first few terms of the infinite series explicitly we find,

$$f(t) = \frac{a_0}{2} + a_1\cos\frac{2\pi t}{T} + b_1\sin\frac{2\pi t}{T} + a_2\cos\frac{4\pi t}{T} = b_2\sin\frac{4\pi t}{T} + \ldots..$$

In this form, we can see that the Fourier coefficients a_n are the amplitudes of the cosine terms and b_n are the amplitudes of the sine terms in the series.
These formulae can look quite intimidating when first met but we will look carefully at the various stages of the calculation. The first thing you should always do is sketch a graph of the periodic function.

Example 4 Sketch a graph of the periodic function $f(t)=\begin{cases}0 & -5<t<0\\ 1 & 0<t<5\end{cases}$ of period T = 10, from t = -10 to t = 15.



Example 5 For the function $f(t)=\begin{cases}0 & -5<t<0\\ 1 & 0<t<5\end{cases}$ of period T = 10, calculate the Fourier coefficient, a_0.

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt$$

$$= \frac{1}{5}\int_0^5 dt = \frac{1}{5}\times 5$$

$$= 1$$

Example 6 For the function $f(t) = \begin{cases} 0 & -5 < t < 0 \\ 1 & 0 < t < 5 \end{cases}$ of period T = 10, calculate the Fourier coefficients, a_n.

$$a_n = \frac{2}{T}\int_0^T f(t) \cos\frac{2n\pi t}{T}\,dt$$

$$= \frac{1}{5}\int_0^5 \cos\frac{2n\pi t}{10}\,dt$$

$$= \frac{1}{5}\cdot\frac{5}{n\pi} \sin\frac{n\pi t}{5}\Big|_0^5$$

$$= 0$$

Example 7 For the function $f(t)=\begin{cases}0 & -5<t<0\\ 1 & 0<t<5\end{cases}$ of period T = 10, calculate the Fourier coefficients, b_n.

$$b_n = \frac{2}{T}\int_0^T f(t)\sin\frac{2n\pi t}{T}\,dt$$

$$= \frac{1}{5}\int_0^5 \sin\frac{2n\pi t}{10}\,dt$$

$$= -\frac{1}{n\pi}\cos\frac{n\pi t}{5}\Big|_0^5$$

$$= -\frac{1}{n\pi}(\cos n\pi - 1)$$

$$= \begin{cases}\dfrac{2}{n\pi}, & n \text{ odd}\\[2ex] 0, & n \text{ even}\end{cases}$$

Example 8 Combine the results from examples 1 to 4 to write down the Fourier series for $f(t) = \begin{cases} 0 & -5 < t < 0 \\ 1 & 0 < t < 5 \end{cases}$ of period T = 10.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2n\pi t}{T}$$

$$+ \sum_{n=1}^{\infty} b_n \sin \frac{2\pi n t}{T}$$

$$= \frac{1}{2} + \sum_{n \text{ odd}} \frac{2}{n\pi} \sin \frac{n\pi t}{5}$$

Integration over any convenient period

Although the integrals given in the formulae for the Fourier coefficients have limits 0 and T, they can in fact be performed over any complete period. On occasions it may be more convenient to integrate over a different interval, say from $-\frac{T}{2}$ to $\frac{T}{2}$.

Example 9 Sketch a graph of the function $f(t)=\begin{cases}-t & -\pi<t<0\\ 0 & 0<t<\pi\end{cases}$ and find its Fourier series.

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\frac{2n\pi t}{T}\,dt, \quad n = 1,2,3,........$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{0} -t\cos\frac{2n\pi t}{2\pi}\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -t\cos nt\ dt$$

Now using integration by parts as we have a product,

$$= \frac{1}{\pi}\left[-t\frac{\sin nt}{n}\right]_{-\pi}^{0} + \int_{-\pi}^{0}\frac{\sin nt}{n}\,dt$$

$$= \frac{1}{\pi}\left[-\frac{\cos nt}{n^2}\right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left(\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right)$$

This gives,

$$a_1 = -\frac{2}{\pi}$$

$$a_2 = 0$$

$$a_3 = -\frac{2}{9\pi}$$

Now to find b_n, noting that the integrand is a product, use integration by parts in the same way.

$$b_n = \frac{2}{T}\int_0^T f(t)\sin\frac{2n\pi t}{T}\,dt$$

$$= \frac{1}{\pi}\int_{\pi}^{2\pi}(-t)\sin\frac{2n\pi t}{T}\,dt$$

$$= \frac{1}{n\pi}\int_{\pi}^{2\pi} t\, d\cos nt$$

$$= \frac{1}{n\pi}\, t\cos nt\,\Big|_{\pi}^{2\pi}$$

$$- \frac{1}{n^2\pi}\sin nt\,\Big|_{\pi}^{2\pi}$$

$$= \begin{cases} \dfrac{3}{n}, & n \text{ odd} \\ \dfrac{1}{n}, & n \text{ even} \end{cases}$$

If we have a function where its value is not zero within the given domain, e.g. $f(t) = \begin{cases} -t & -\pi < t < 0 \\ t & 0 < t < \pi \end{cases}$ then we simply integrate each part separately and add the result. For instance, our calculation of a_0 for this function would become,

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -t\,dt + \int_{0}^{\pi} t\,dt\right]$$

A similar result would be true for a_n and b_n.

Fourier Series of Odd Functions and Even Functions

When a function is odd, its Fourier series will contain no cosine or constant terms. Consequently $a_n = 0$ for all n.

When a function is even, its Fourier series will contain no sine terms. Consequently $b_n = 0$ for all n.

This will save you considerable time during the calculation of a Fourier series.