State-Space Control (EGR3032)

Week 1

Mathematical modelling of control systems

Outlines:

Mathematical Modelling

Elements of a mechanical system

Elements of electrical circuits

Examples

Mathematical modelling

The first step one needs to take in order to develop a control system is clearly ***understand*** the system we want to control.

In this lecture we are going to create <u>*dynamic models*</u> to represent *plants* or *processes* for which a control system will be developed.

we use two sets of models for <u>*mechanical*</u> and <u>*electrical*</u> systems and eventually we'll be able to combine.

By the end of this lecture, you should be able to obtain the <u>*differential equations*</u> of mechanical and electrical systems.

Deriving reasonable mathematical models is the most important part of the entire analysis of control systems

aim to model dynamic systems in mathematical terms and analyse their dynamic characteristics.

A mathematical model of a dynamic system is defined as ***a set of equations*** that represents the *dynamics* of the system accurately, or at least fairly well.

The dynamics of many systems

- mechanical
- electrical
- thermal
- economic
- biological
- …

Described as

differential equations

Obtained by

using physical laws governing a particular system

Newton's laws for mechanical systems

Kirchhoff's laws for electrical systems

…

Represented by

a set of linear constant-coefficient differential equations in this course

Linear time-invariant (LTI)

The *differential equations* describing the *dynamic performance* of a physical system are obtained by utilizing the <u>physical laws</u> of the process

Mechanical systems → Newton's laws

Electrical systems → Kirchhoff's laws

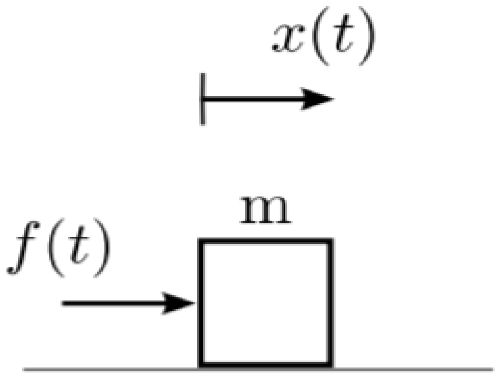
Elements of a mechanical system

let's start our study with a mechanical system. In any mechanical system there will be a mass. If you apply a force to a mass that will make the mass accelerate.
If you have a mass called m and you apply a force $f(t)$ to the mass that will result in the acceleration and the displacement of the mass.

$\rightarrow$ **Mass**: The quantity of matter in a body

$\rightarrow$ **Inertia**: Tendency to resist changes in state of motion

Idealization: Rigid body



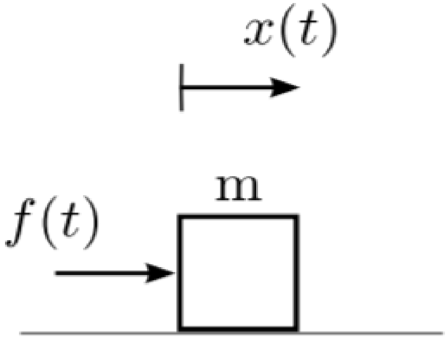
If F is the force applied to the mass and x is the displacement, we can now create a relation between these two quantities by applying *Newton's law*:

The sum of all forces equals to the mass times the acceleration:

$$\sum F = ma$$

In terms of displacement, we can now write the sum of all forces which in this case is simply $f(t)$ equals to the mass times the second derivative of displacement.

$$f(t) = m\frac{d^2x}{dt^2}$$



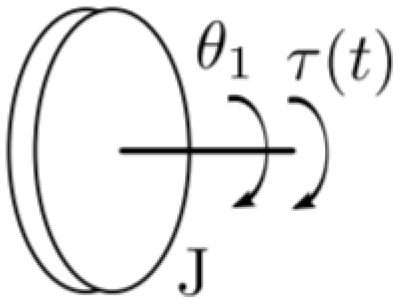
We can show a derivative with a dot and a second derivative by a double dot so you can write that:

$$f(t) = m\ddot{x}$$

Now let’s consider a similar system but instead of a linear system let's consider a rotational system

Instead of applying a force we apply a torque τ and instead of having a linear displacement we have an angular displacement θ_1

- The same way we had the mass that would oppose any changes in motion in the linear system, in the rotational system we have the moment of inertia J

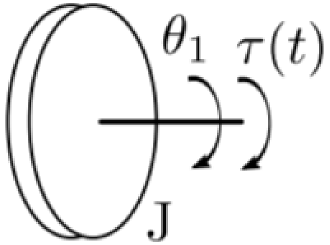


The equation of motion for this system now is the sum of torques is going to be equal the moment of inertia times the angular acceleration α:

$$\sum \tau = J\alpha$$

The sum of all torques acting on this system here is simply:

$$\tau\,(t) = J\frac{d^2\theta_1}{dt^2}$$

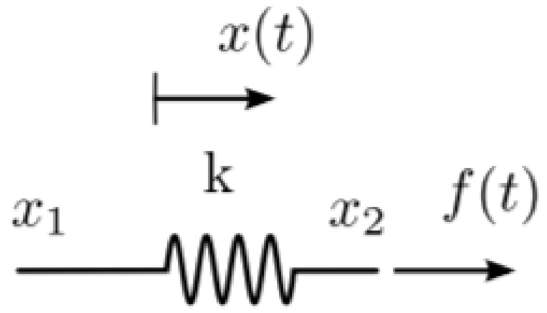


$$\tau(t) = J\ddot{\theta}_1$$

The next mechanical element that we need to study is something which can store energy in the form of potential energy

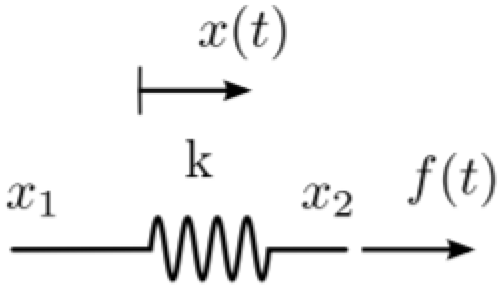
→ **Spring**: Designed to store energy

Idealization: Negligible mass and damping



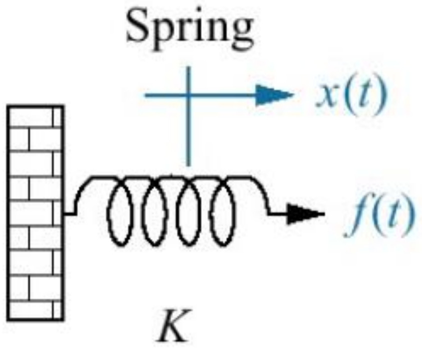
Now we can relate the force developed in the spring and the displacement of the spring.

We can now write that the force developed in that spring is the stiffness constant k times the compression of the spring that is $x_2 - x_1$ where k is the stiffness constant given in newtons per meter $\frac{N}{m}$:



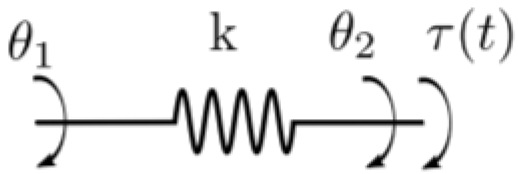
$$f(t) = k\,(x_2 - x_1)$$

If you fix one end of this spring and apply a force to the other end this side here, we have x_2 and this side here we have x_1 which in this case is zero.



If we now look at a torsion spring, we can say that the sum of all torques applied to the spring $\tau(t)$ is equal to the angular stiffness multiplied by the relative angular displacement of the spring that is θ_2 minus θ_1 where k in this case is the angular stiffness that is newton meter per radian:

$$\tau(t) = k\,(\theta_2 - \theta_1)$$

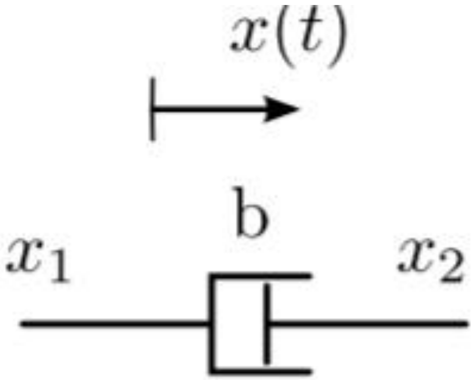


The next element we need is something that can dissipate energy and we are going to use a viscous damper.

We are going to use the same idealization as we did for the spring if you are now dealing with a viscous damper or an element that can dissipate energy, we are going to assume that this damper has no stiffness and has no mass.

$\rightarrow$ **Viscous damper**: Designed to dissipate energy

Idealization: negligible mass and stiffness



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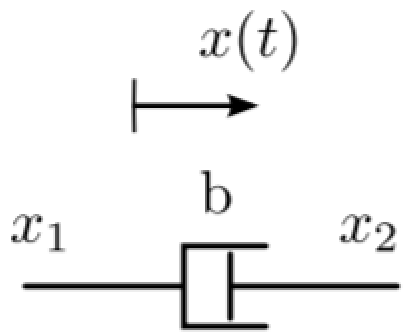
We can say that the force developed in this element is proportional to the speed of both ends of that element. The proportionality constant is b that is the viscous friction coefficient.

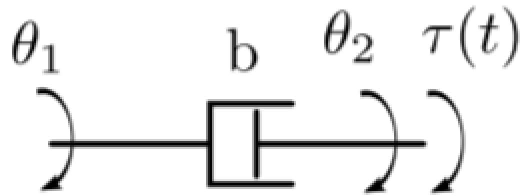
$$f(t) = b\,(\dot{x}_2 - \dot{x}_1)$$

According to our notation again $\dot{x}$ is simply the speed: $\dot{x} = \dfrac{dx}{dt} = v$



For a translational system:

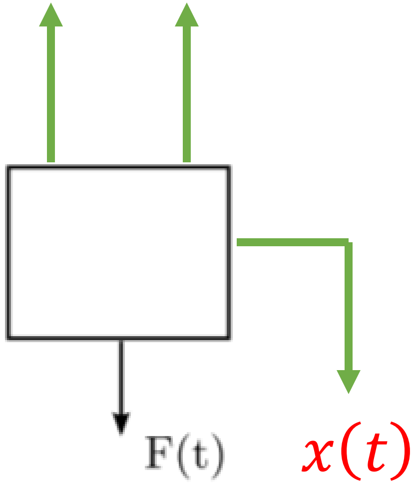
$$\tau(t) = b\,(\dot{\theta}_2 - \dot{\theta}_1)$$

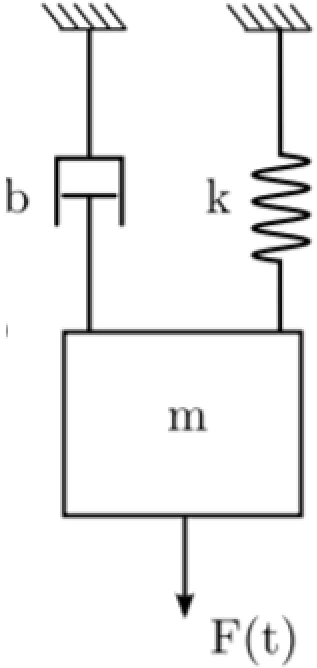


Example

Find the equation of motion for the spring-mass-damper system.

If you want now to find the equations of motion for this mass spring damper system let's start with a free body diagram.



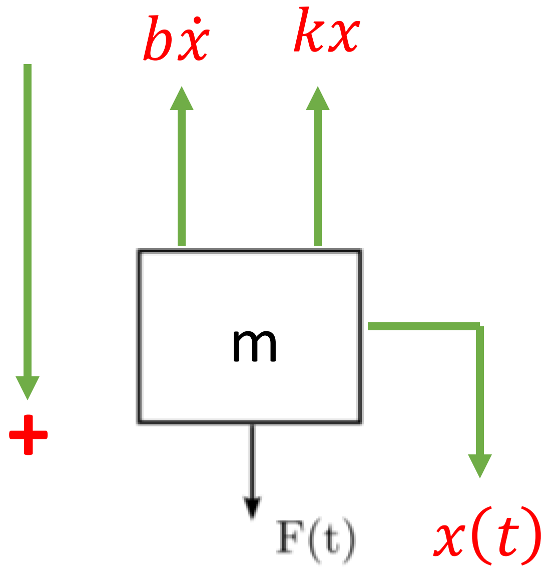


Here we have a mass, we have one external force applied to it that is $F(t)$.

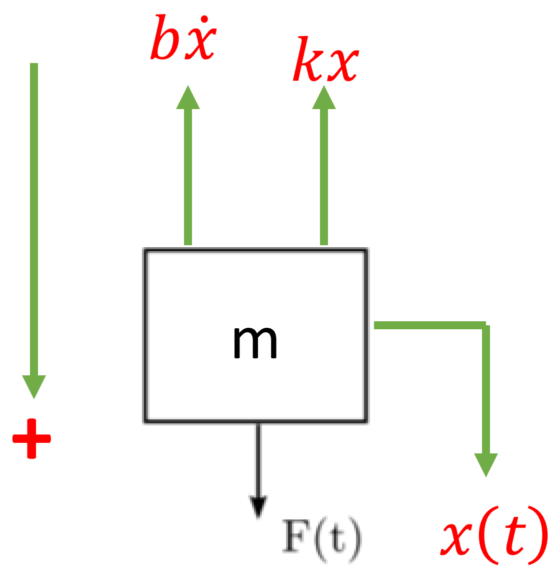
If the mass moves downwards then both the damper and the spring will apply a force against motion and that a force will then be upwards

Now we can write the sum of all forces acting on this mass equals to mass times acceleration or mass times $\ddot{x}$

$$\sum F = ma = m\ddot{x}$$



$$\sum F = ma = m\ddot{x}$$



We have three forces acting on this mass now the first one is $F(t)$ is positive because it points downward, we have the viscous friction that is negative $b\,\dot{x}$ minus kx the spring force again negative because it points up and this is equal to $m\ddot{x}$.

$$F(t) - b\,\dot{x} - kx = m\ddot{x}$$

We can now rearrange this equation:

$$F(t) = m\ddot{x} + b\,\dot{x} + kx$$

This is the differential equation that describes the motion of this mass when subjected to a force $F(t)$

Elements of electrical circuits

Now let's move on to electric elements: **Resistor Capacitor Inductor**

Let's start with resistor. If a current i passes through that resistor that creates a voltage drop v.

$$v = v_1 - v_2 = Ri$$

The voltage drop across a resistor is given by Ohm’s law, which states that the voltage drop across a resistor is equal to the product of the current through the resistor and its resistance.
Resistors absorb energy from the system.

$\rightarrow$ **Resistor**: Resistance against electric current

Idealization: No inductance or capacitance

The voltage drop across the capacitor is also v_1 minus v_2.
we can now write that this voltage drop is:

$$v = v_1 - v_2 = \frac{1}{C}\int idt \longrightarrow \boxed{i = C\frac{dv}{dt}}$$

→ **Capacitor**: Stores energy in an electric field

Idealization: No inductance or resistance

The voltage drop across an inductor is given by Faraday's law, which is written

$$v_L = L\frac{di}{dt}$$

This equation states that the voltage drop across an inductor is equal to the product of the inductance and the time rate of increase of current.

$\rightarrow$ **Inductor**: Stores energy in a magnetic field

Idealization: No capacitance or resistance

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The equations for an electric circuit obey Kirchhoff 's laws, which state the following:

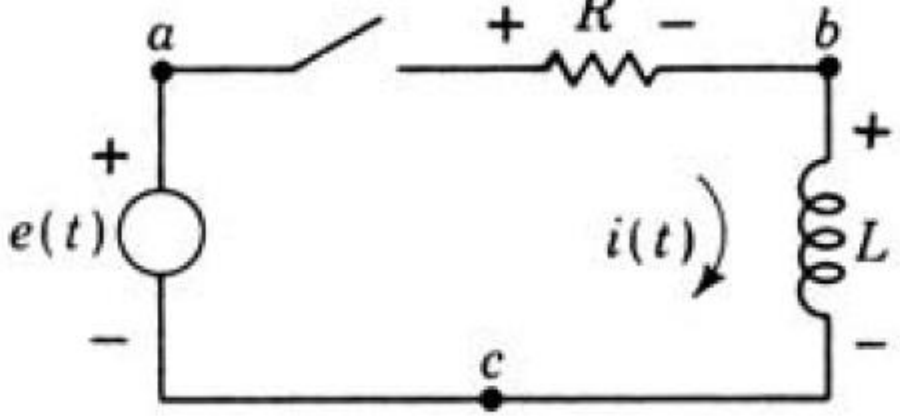
KVL: The algebraic sum of the potential differences around a closed path equals zero

In traversing any closed loop, the sum of the voltage rises equals the sum of the voltage drops.

KCL: The algebraic sum of the currents entering (or leaving) a node is equal to zero

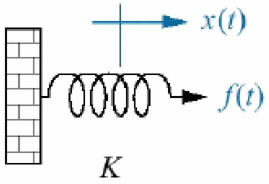
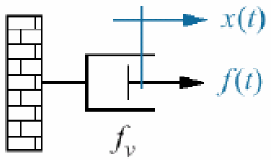
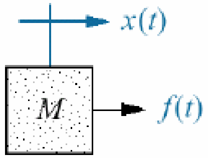
the sum of the currents entering the junction equals the sum of the currents leaving the junction

The junctions of any two elements are called nodes.



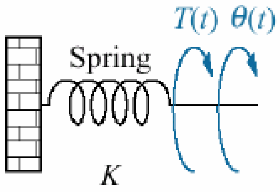
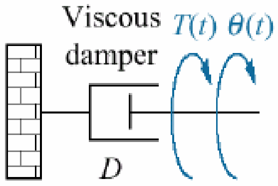
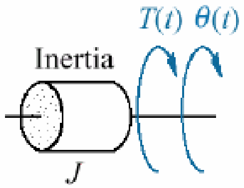
Symbol	Quantity	Units
e or v	Voltage	Volts
i	Current	Amperes
L	Inductance	Henrys
C	Capacitance	Farads
R	Resistance	Ohms

Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K\int_0^t v(\tau)\,d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \dfrac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\dfrac{dv(t)}{dt}$	$f(t) = M\dfrac{d^2x(t)}{dt^2}$	Ms^2

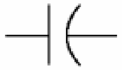
Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m (newton-seconds/meter), M = kg (kilograms = newton-seconds²/meter).

Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

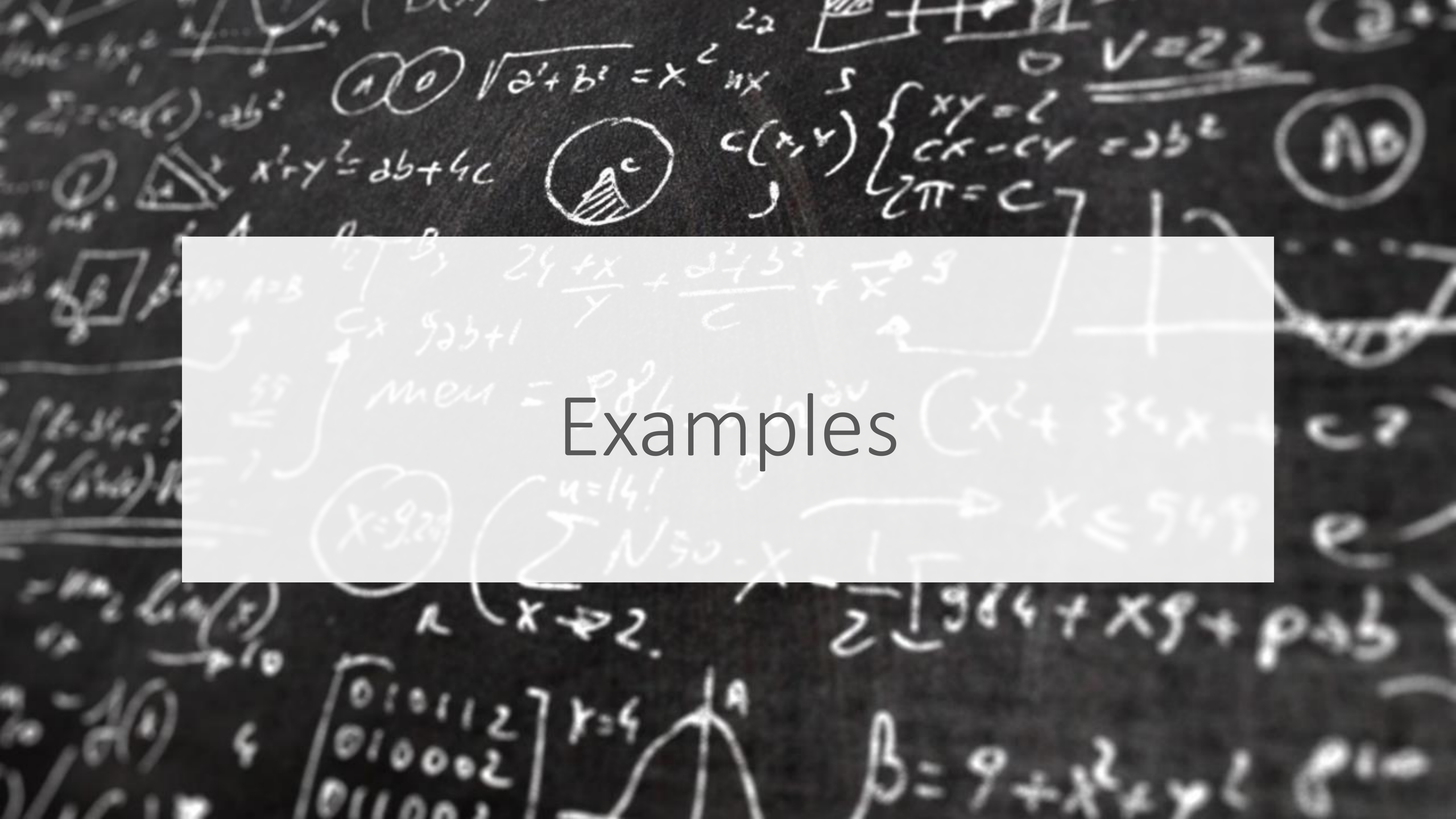
Component	Torque-angular velocity	Torque-angular displacement	Impedance $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)\,d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\dfrac{d\theta(t)}{dt}$	Ds
	$T(t) = J\dfrac{d\omega(t)}{dt}$	$T(t) = J\dfrac{d^2\theta(t)}{dt^2}$	Js^2

Note: The following set of symbols and units is used throughout this book: $T(t)$ = N-m (newton-meters), $\theta(t)$ = rad (radians), $\omega(t)$ = rad/s (radians/second), K = N-m/rad (newton-meters/radian), D = N-m-s/rad (newton-meters-seconds/radian), J = kg-m² (kilogram-meters² = newton-meters-seconds²/radian).

Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance	Admittance
 Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

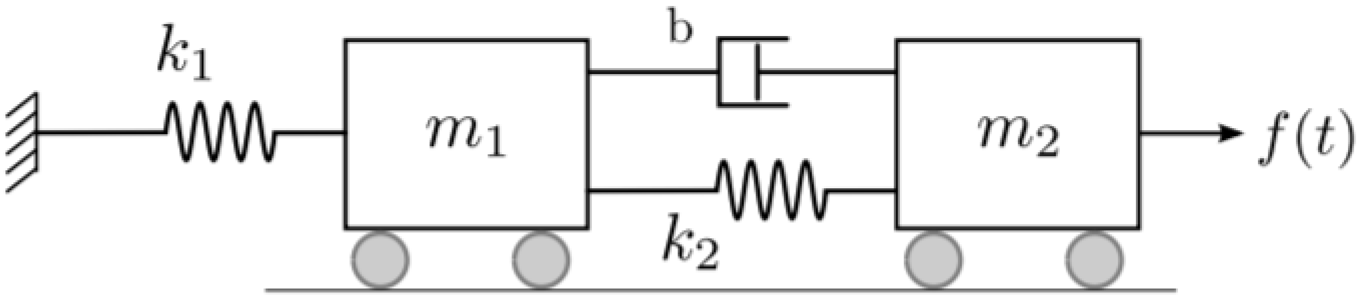
Note: The following set of symbols and units is used throughout this book: $v(t)$ = V (volts), $i(t)$ = A (amps), $q(t)$ = Q (coulombs), C = F (farads), R = Ω (ohms), G = $\mho$ (mhos), L = H (henries).



Examples

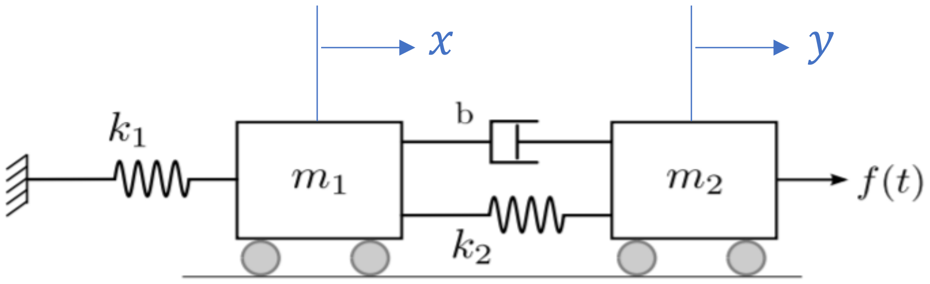
Example 1

Find the differential equations to model the behavior of the system shown.



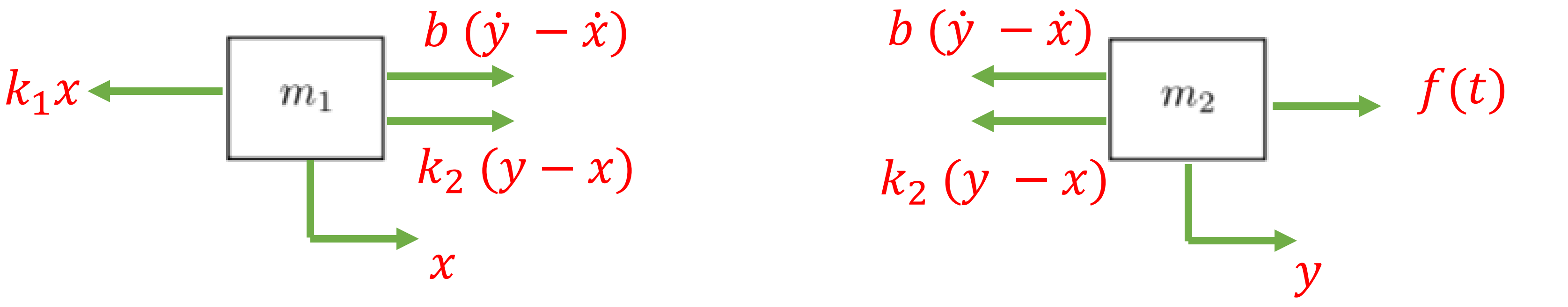
In this system we have two masses connected through a damper and the spring and the mass m_2 is subjected to a force $f(t)$.

Let's create the free body diagram for both masses.

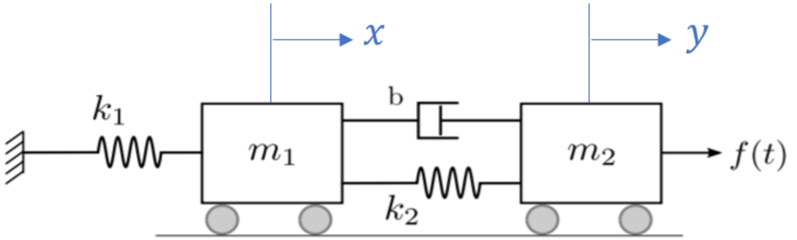


Assuming that a displacement of mass m_1 is x_1 And the displacement of mass m_2 is x_2. The external force f(t) is applied to m_2.

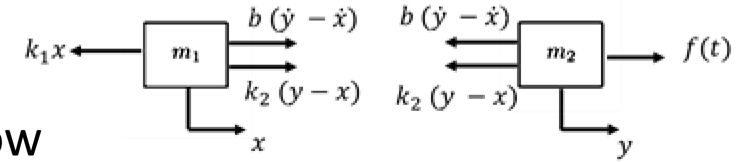
Let’s draw the free diagram of the system



Now that we have all the free body diagrams you can write the equations of motions for masses m_1 and m_2



$$\sum F = ma = m_2\ddot{y} \longrightarrow \mathrm{f}(t) - b\,(\dot{y} - \dot{x}) - k_2\,(y - x) = m_2\ddot{y}$$



If you now look at mass m_1 we are going to have here the same forces but now they go in the opposite direction:

$$\sum F = ma = m_1\ddot{x} \longrightarrow b\,(\dot{y} - \dot{x}) + k_2\,(y - x) - k_1\,x = m_1\ddot{x}$$

Differential Equation form:

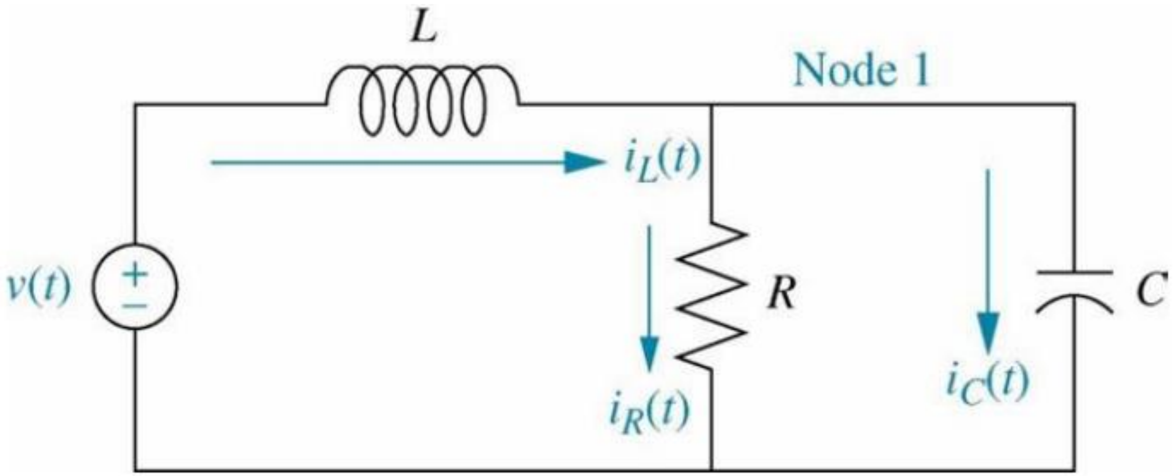
$$\begin{cases} \dot{x}_1 = \dot{x} = x_2 \\ \dot{x}_2 = \ddot{x} = \dfrac{b}{m_1}y_2 - \dfrac{b}{m_1}x_2 + \dfrac{k_2 - k_1}{m_1}x_1 + \dfrac{k_2}{m_1}y_1 \\ \dot{y}_1 = \dot{y} = y_2 \\ \dot{y}_2 = \ddot{y} = -\dfrac{b}{m_2}y_2 + \dfrac{b}{m_2}x_2 - \dfrac{k_2}{m_2}y_1 + \dfrac{k_2}{m_2}x_1 + \dfrac{1}{m_2}f(t) \end{cases}$$

LTI State-Space form:

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{bmatrix}}_{\dot{X}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{k_2 - k_1}{m_1} & -\dfrac{b}{m_1} & \dfrac{k_2}{m_1} & \dfrac{b}{m_1} \\ 0 & 0 & 0 & 1 \\ \dfrac{k_2}{m_2} & \dfrac{b}{m_2} & -\dfrac{k_2}{m_2} & -\dfrac{b}{m_2} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{m_2} \end{bmatrix}}_{B} \underbrace{f(t)}_{U}$$

$$\dot{X} = AX + BU$$

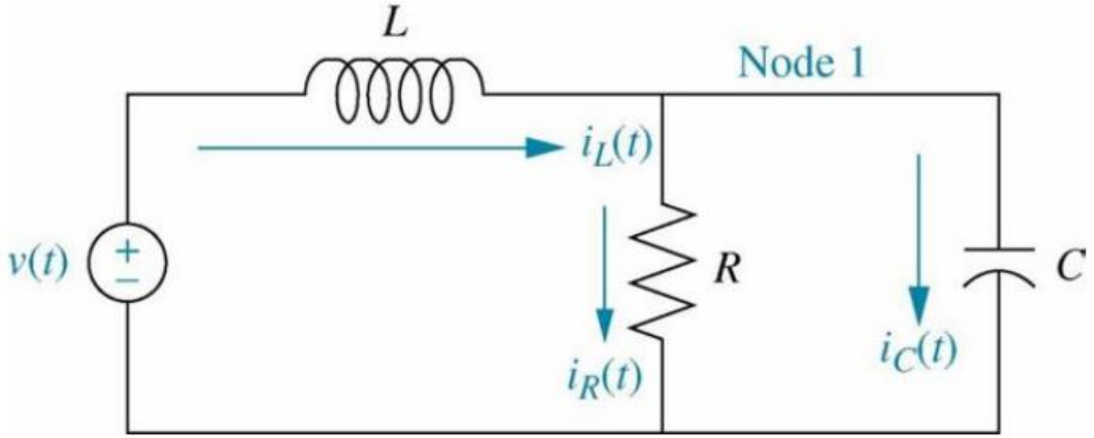
Example 2



- Let’s look at how a simple electrical circuit can be modelled.
- There is a clear input, which in this case is the voltage source $v(t)$.

Hint:

- A linear resistor has the equation: $v = iR$
- A capacitor has the equation: $i = C\frac{dv}{dt}$
- A linear inductor has the equation: $v = I\frac{di}{dt}$



- Using the junction rule, at 'Node 1' the currents can be summed:

$$i_L = i_C + i_R$$

- Using the loop rule, voltages can be summed around the outer loop:

$$v(t) = v_L + v_C$$

- Substituting in the equations give:

$$i_L = C\frac{dv_C}{dt} + \frac{1}{R}v_C$$

$$v(t) = L\frac{di_L}{dt} + \frac{1}{R}v_C$$

- This can be rearranged to give:

$$\begin{cases} \dfrac{dv}{dt} = -\dfrac{1}{RC}v_C + \dfrac{1}{C}i_L \\ \dfrac{di_L}{dt} = -\dfrac{1}{L}v_C + \dfrac{1}{L}v(t) \end{cases}$$

- Written in <u>LTI State-Space form</u>:

$$\dot{X} = AX + BU \Leftarrow \begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{RC} & \dfrac{1}{C} \\ -\dfrac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{L} \end{bmatrix} v(t)$$