Lecture 6 - The Fourier Transform

Introduction

Fourier series are only applicable to periodic functions, however we can still decompose a non-periodic function in to its Fourier components – this process is called a Fourier transform. This is no longer expressed as a sum of sine and cosine waves but as an integral. Examples of non-periodic signals include pulse signals and noise signals.

Calculating a Fourier Transform

The Fourier transform of a function f(t) is a function of a new variable ω, which is found from the following formula.

The Fourier transform of f(t) is a function $F(\omega)$ defined by:

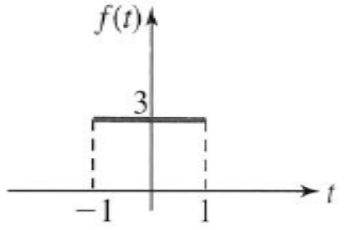
$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

It is frequently the case that when a Fourier transform is calculated the result is a complex function, as you will see in the following examples.

Example 1. Find the Fourier Transform of the function defined by

$$f(t) = \begin{cases} 3 & -1 < t < 1 \\ 0 & otherwise \end{cases}$$

A graph of f(t) is shown below.



We apply the formula for finding the Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

Note that in this case the function is defined to be zero outside the interval $-1 < t < 1$ and so the integral reduces to,

$$F(\omega) = \int_{-1}^{1} 3e^{-j\omega t}\,dt$$

$$= \left[\frac{3e^{-j\omega t}}{-j\omega}\right]_{-1}^{1}$$

$$= \left[\frac{3e^{-j\omega}}{-j\omega} - \frac{3e^{j\omega}}{-j\omega}\right]$$

$$= \frac{3e^{j\omega} - 3e^{-j\omega}}{j\omega}$$

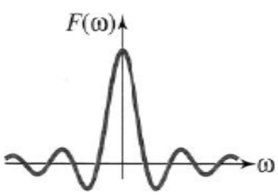
If we now make use of Euler's relations:

$$e^{j\theta} = \cos\theta + j\sin\theta, \qquad e^{-j\theta} = \cos\theta - j\sin\theta$$

We can write the Fourier transform as,

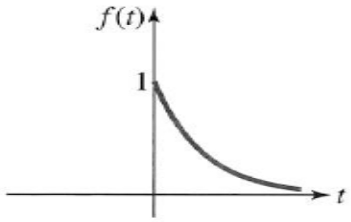
$$F(\omega) = 6\frac{\sin\omega}{\omega}$$

Its graph is shown below. The function $\frac{\sin\omega}{\omega}$ occurs frequently and is often referred to as the sinc function.



Example 2. Find the Fourier transform of the function $f(t) = u(t)e^{-t}$ where $u(t)$ is the unit step function.

The graph of the function is shown below. Note that the function is zero when t is negative.



The Fourier transform of this function is given by,

$$F(\omega) = \int_{-\infty}^{\infty} u(t)e^{-t}e^{-j\omega t}\,dt$$

Because $u(t)e^{-t}$ is zero when t is negative then we can modify the limits accordingly,

$$F(\omega) = \int_{0}^{\infty} u(t)e^{-t}e^{-j\omega t}\,dt$$

Further, since u(t) is equal to 1 for any value of t greater than zero then,

$$F(\omega) = \int_{0}^{\infty} e^{-t}e^{-j\omega t}\,dt$$

Carrying out the integration by combining the two exponential terms,

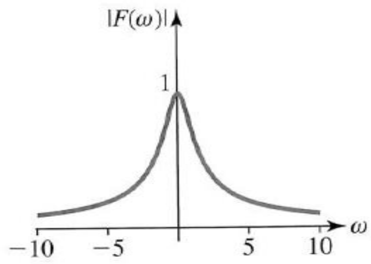
$$F(\omega) = \int_{0}^{\infty} e^{-(1+j\omega)t}\,dt = \left[\frac{e^{-(1+j\omega)t}}{-(1+j\omega)}\right]_{0}^{\infty}$$

Complete the integration by noting that the contribution from the upper limit is zero because e^{-t} tends to zero as t tends to infinity.

$$F(\omega) = \frac{1}{1 + j\omega}$$

Because the Fourier transform is a complex function we cannot immediately plot its graph. However, it is possible to find its modulus and argument, and plot graphs of these against ω. Such plots are called amplitude spectra and phase spectra respectively.

The amplitude spectrum of $f(t) = u(t)e^{-t}$ is the modulus of $\frac{1}{1 + j\omega}$ which equals $\frac{1}{\sqrt{1+\omega^2}}$. A graph of this is shown below.



The phase spectrum is the argument of the $\frac{1}{1 + j\omega}$.

It is usual practice to make use of tables of transforms such as those shown below.

Table of common Fourier transforms.

$f(t)$	$F(\omega)$
$Au(t)e^{-\alpha t}, \alpha > 0$	$\dfrac{A}{\alpha + j\omega}$
$\begin{cases} 1 & -\alpha \le t \le \alpha \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{2 \sin \omega\alpha}{\omega}$
A, constant	$2\pi A\delta(\omega)$
$Au(t)$	$A\left(\pi\delta(\omega) - \dfrac{j}{\omega}\right)$
$\delta(t)$	1
$\delta(t - a)$	$e^{-j\omega a}$
$\cos at$	$\pi[\delta(\omega + a) + \delta(\omega - a)]$
$\sin at$	$\dfrac{\pi}{j}[\delta(\omega - a) - \delta(\omega + a)]$
$e^{-\alpha\lvert t\rvert}, \alpha > 0$	$\dfrac{2\alpha}{\alpha^2 + \omega^2}$