Practice Questions and Solutions

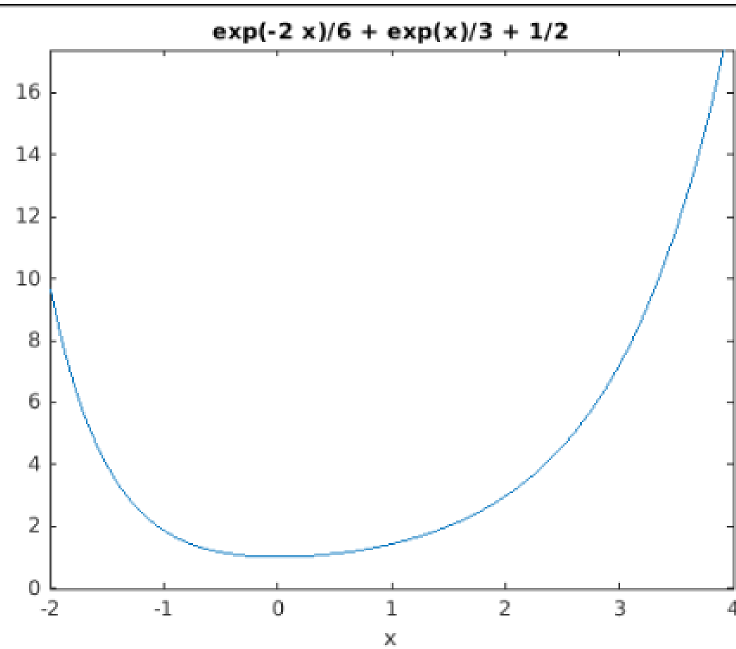
Question 1

Solving diff equation using matlab command 'dsolve' and find the minimum value from the plot.

$$\frac{\partial^2 y}{\partial x^2} + 2\frac{\partial y}{\partial x} = e^x, y(0) = 1, y'(0) = 0$$

Answer:

```
clear all
close all
sol = dsolve('D2y+2*Dy=exp(x)','y(0)=1','Dy(0)=0','x')
ezplot(sol,[-2,4])
```

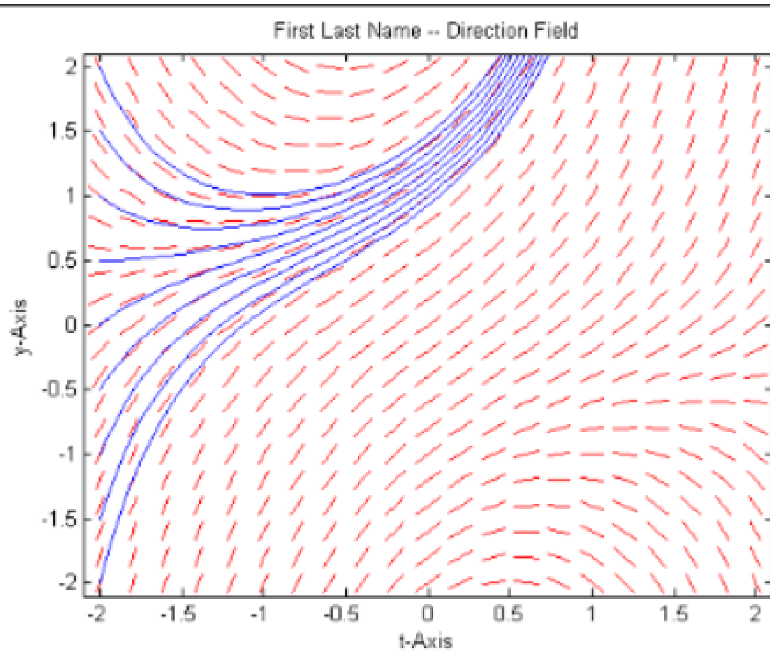


Question 2

A system is described by $\frac{dy}{dx} = 1 + xy$ for an application under investigation. Find the solution behavior of the system in matlab using ode45 between (-2,2) with initial conditions between (-1 ,1). Show the plots of the direction field and solutions.

Answer:

```
close;
clear;
f = @(x,y) 1+x*y;
dirfield(f,-2:0.2:2,-2:0.2:2);
title('First Last Name -- Direction Field');
xlabel('x-Axis');
ylabel('y-Axis');
hold on
for y0=-2:0.5:2
    [xs,ys] = ode45(f,[-2,2],y0);
    plot(xs,ys);
    pause(1);
end
```



Question 3

Given the following Rosenbrock's banana function:

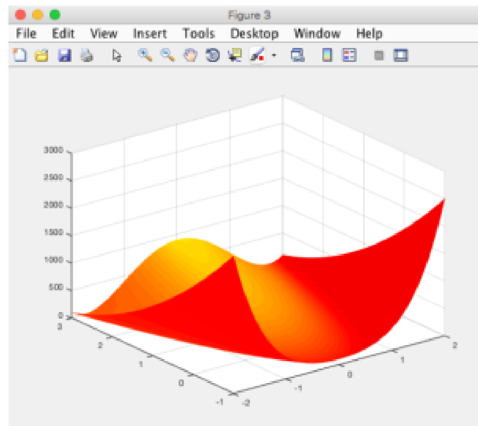
$$f(x, y) = (1 - x)^2 + 100(y - x^2)^2$$

a. Plot the function (Use meshgrid, surf commands)
b. Find the minimum for function (Use fminsearch command)

Answer:

We plot the Banana function:

```
clear,clc

[x,y] = meshgrid(-2:0.1:2, -1:0.1:3);

f = (1-x).^2 + 100.*(y-x.^2).^2;

figure(1)
surf(x,y,f)

figure(2)
mesh(x,y,f)


figure(3)
surfl(x,y,f)
shading interp;
colormap(hot);
```



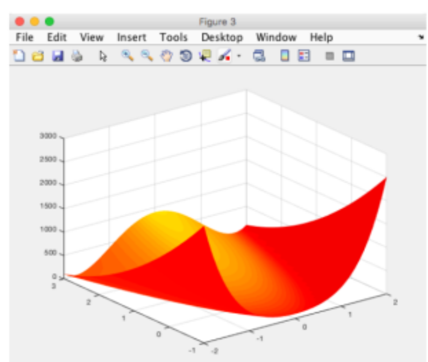

```
function f = bananafunc(x)

f = (1-x(1)).^2 + 100.*(x(2)-x(1).^2).^2;
```

```
[x,fval] = fminsearch(@bananafunc, [-1.2;1])
```

From MATLAB we get:

```
x =      1.0000     1.0000

fval =    8.1777e-10
```

Which is correct



Question 4

Solve $\dfrac{\partial^2 u}{\partial x \partial y} = 8\,\mathrm{e}^y \sin 2x$ given that at $y = 0$, $\dfrac{\partial u}{\partial x} = \sin x$, and at $x = \dfrac{\pi}{2}$, $u = 2y^2$

Answer:

Since $\dfrac{\partial^2 u}{\partial x \partial y} = 8\,\mathrm{e}^y \sin 2x$ then integrating partially with respect to y gives:

$$\frac{\partial u}{\partial x} = \int 8\,\mathrm{e}^y \sin 2x \,\mathrm{d}\,y = 8\,\mathrm{e}^y \sin 2x + f(x)$$

From the boundary conditions, $\dfrac{\partial u}{\partial x} = \sin x$ when $y = 0$, hence

$\sin x = 8\mathrm{e}^0 \sin 2x + f(x)$ from which, $f(x) = \sin x - 8 \sin 2x$

i.e. $\dfrac{\partial u}{\partial x} = 8\,\mathrm{e}^y \sin 2x + \sin x - 8 \sin 2x$

Integrating partially with respect to x gives:

$$u = \int [8\,\mathrm{e}^y \sin 2x + \sin x - 8 \sin 2x]\,\mathrm{d}\,x = -4\,\mathrm{e}^y \cos 2x - \cos x + 4 \cos 2x + f(y)$$

From the boundary conditions, $u = 2\,y^2$ when $x = \dfrac{\pi}{2}$, hence

$2\,y^2 = -4\mathrm{e}^y \cos \pi - 0 + 4 \cos \pi + f(y)$

$= 4e^y - 4 + f(y)$ from which, $f(y) = 2\,y^2 - 4\mathrm{e}^y + 4$

Hence, the solution of $\dfrac{\partial^2 u}{\partial x \partial y} = 8\,\mathrm{e}^y \sin 2x$ is given by:

$$u = -4\,\mathrm{e}^y \cos 2x - \cos x + 4 \cos 2x + 2\,y^2 - 4\mathrm{e}^y + 4$$

Question 5

An elastic string is stretched between two points 40 cm apart. Its centre point is displaced 1.5 cm from its position of rest at right-angles to the original direction of the string and then released with zero velocity. Determine the subsequent motion u(x, t) by applying the wave equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \text{ with } c^2 = 9$$

Answer:

1. The boundary and initial conditions given are:

$$\left.\begin{array}{l} u(0,t) = 0 \\ u(40,t) = 0 \end{array}\right\}$$

$$u(x, 0) = f(x) = \frac{1.5}{20}x \qquad 0 \le x \le 20$$

$$= -\frac{1.5}{20}x + 3 = \frac{60 - 1.5x}{20} \qquad 20 \le x \le 40$$

$$\left[\frac{\partial u}{\partial t}\right]_{t=0} = 0 \text{ i.e. zero initial velocity}$$

2. Assuming a solution $u = XT$, where X is a function of x only, and T is a function of t only,

then $\frac{\partial u}{\partial x} = X'T$ and $\frac{\partial^2 u}{\partial x^2} = X''T$ and $\frac{\partial u}{\partial y} = XT'$ and $\frac{\partial^2 u}{\partial y^2} = XT''$

Substituting into the partial differential equation, $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

gives: $X''T = \frac{1}{c^2}XT''$ i.e. $X''T = \frac{1}{9}XT''$ since $c^2 = 9$

3. Separating the variables gives: $\frac{X''}{X} = \frac{T''}{9T}$

Let constant, $\mu = \frac{X''}{X} = \frac{T''}{9T}$ then $\mu = \frac{X''}{X}$ and $\mu = \frac{T''}{9T}$

from which, $X'' - \mu X = 0$ and $T'' - 9\mu T = 0$

4. Letting $\mu = -p^2$ to give an oscillatory solution gives

$X'' + p^2 X = 0$ and the auxiliary equation is: $m^2 + p^2 = 0$ from which, $m = \sqrt{-p^2} = \pm jp$

and $T'' + 9p^2 T = 0$ and the auxiliary equation is:

$$m^2 + 9p^2 = 0 \text{ from which, } m = \sqrt{-9p^2} = \pm j3p$$

5. Solving each equation gives: $X = A\cos px + B\sin px$ and $T = C\cos 3pt + D\sin 3pt$

   Thus, $u(x, t) = \{A\cos px + B\sin px\}\{C\cos 3pt + D\sin 3pt\}$

6. Applying the boundary conditions to determine constants A and B gives:

   (i) $u(0, t) = 0$, hence $0 = A\{C\cos 3pt + D\sin 3pt\}$ from which we conclude that $A = 0$

   Therefore, $u(x, t) = B\sin px\,\{C\cos 3pt + D\sin 3pt\}$ (1)

   (ii) $u(40, t) = 0$, hence $0 = B\sin 40p\{C\cos 3pt + D\sin 3pt\}$

   $B \neq 0$ hence $\sin 40p = 0$ from which, $40p = n\pi$ and $p = \dfrac{n\pi}{40}$

7. Substituting in equation (1) gives: $u(x, t) = B\sin\dfrac{n\pi x}{40}\left\{C\cos\dfrac{3n\pi t}{40} + D\sin\dfrac{3n\pi t}{40}\right\}$

   or, more generally, $u_n(x,t) = \sum_{n=1}^{\infty}\sin\dfrac{n\pi x}{40}\left\{A_n\cos\dfrac{3n\pi t}{40} + B_n\sin\dfrac{3n\pi t}{40}\right\}$ (2)

   where $A_n = BC$ and $B_n = BD$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{40}\left[\int_0^{20}\left(\frac{1.5}{20}x\right)\sin\frac{n\pi x}{40}\,dx + \int_{20}^{40}\left(\frac{60-1.5x}{20}\right)\sin\frac{n\pi x}{40}\,dx\right]$$

Each integral is determined using integration by parts (see Chapter 68, page 739) with the result:

$$A_n = \frac{(8)(1.5)}{n^2\pi^2}\sin\frac{n\pi}{2} = \frac{12}{n^2\pi^2}\sin\frac{n\pi}{2}$$

From equation (9), page 890 of textbook, $B_n = \dfrac{2}{cn\pi}\int_0^L g(x)\sin\dfrac{n\pi x}{L}\,dx$

$\left[\dfrac{\partial u}{\partial t}\right]_{t=0} = 0 = g(x)$ thus, $B_n = 0$

Substituting into equation (2) gives: $u_n(x,t) = \sum_{n=1}^{\infty}\sin\dfrac{n\pi x}{40}\left\{A_n\cos\dfrac{3n\pi t}{40} + B_n\sin\dfrac{3n\pi t}{40}\right\}$

$$= \sum_{n=1}^{\infty}\sin\frac{n\pi x}{40}\left\{\frac{12}{n^2\pi^2}\sin\frac{n\pi}{2}\cos\frac{3n\pi t}{40} + (0)\sin\frac{n\pi t}{50}\right\}$$

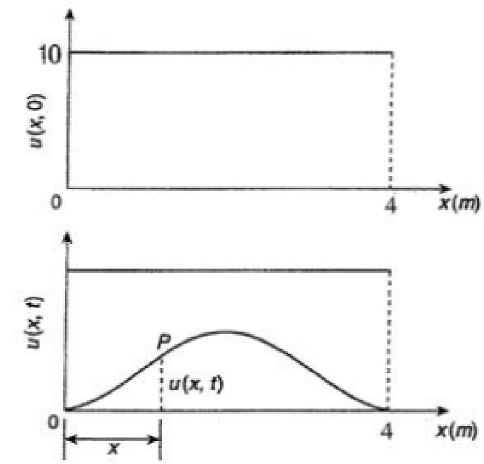
Hence, $u(x, t) = \dfrac{12}{\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{n^2}\sin\dfrac{n\pi}{2}\sin\dfrac{n\pi x}{40}\cos\dfrac{3n\pi t}{40}$

Question 6

A metal bar, insulated along its sides, is 4 m long. It is initially at a temperature of 10°C and at time t = 0, the ends are placed into ice at 0°C. Find an expression for the temperature at a point P at a distance x m from one end at any time t seconds after t = 0

Answer:

The temperature u along the length of the bar is shown in the diagram below



The heat conduction equation is $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$ and the given boundary conditions are:

$$u(0, t) = 0, \quad u(4, t) = 0 \quad \text{and} \quad u(x, 0) = 10$$

Assuming a solution of the form $u = XT$, then, $X = A \cos px + B \sin px$

and $T = k\,e^{-p^2c^2t}$

Thus, the general solution is given by: $u(x, t) = \{P \cos px + Q \sin px\}\, e^{-p^2c^2t}$

$u(0, t) = 0$ thus $0 = P\,e^{-p^2c^2t}$ from which, $P = 0$ and $u(x, t) = \{Q \sin px\}\, e^{-p^2c^2t}$

Also, $u(4, t) = 0$ thus $0 = \{Q \sin 4p\}\, e^{-p^2c^2t}$

Since $Q \neq 0$, $\sin 4p = 0$ from which, $4p = n\pi$ where $n = 1, 2, 3, \ldots$ and $p = \frac{n\pi}{4}$

Hence, $u(x, t) = \sum_{n=1}^{\infty}\left\{Q_n\, e^{-p^2c^2t} \sin\frac{n\pi x}{4}\right\}$

The final initial condition given was that at $t = 0$, $u = 10$, i.e. $u(x, 0) = f(x) = 10$

Hence, $$10 = \sum_{n=1}^{\infty}\left\{Q_n \sin\frac{n\pi x}{4}\right\}$$

where, from Fourier coefficients, $Q_n = 2 \times$ mean value of $10 \sin\frac{n\pi x}{4}$ from $x = 0$ to $x = 4$

i.e. $$Q_n = \frac{2}{4}\int_0^4 10\sin\frac{n\pi x}{4}\,\mathrm{d}x = 5\left[-\frac{\cos\frac{n\pi x}{4}}{\frac{n\pi}{4}}\right]_0^4 = -\frac{20}{n\pi}\left[\cos\frac{4n\pi}{4} - \cos 0\right] = \frac{20}{n\pi}(1 - \cos n\pi)$$

$$= 0 \text{ (when } n \text{ is even) and } \frac{40}{n\pi} \text{ (when } n \text{ is odd)}$$

Hence, the required solution is: $$u(x, t) = \sum_{n=1}^{\infty}\left\{Q_n\, \mathrm{e}^{-p^2c^2t} \sin\frac{n\pi x}{4}\right\}$$

$$= \frac{40}{\pi}\sum_{n(odd)=1}^{\infty}\frac{1}{n}\mathrm{e}^{-\frac{n^2\pi^2c^2t}{16}}\sin\frac{n\pi x}{4}$$