Lecture 7 - Partial Differential Equations 1

Introduction

A Partial Differential Equation (PDE) is an equation that contains one or more partial derivative. For example, $a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}=c$ is a first order PDE since the highest power of the differential is 1. Partial differential equations appear in many areas of engineering such as heat conduction, hydrodynamics and aerodynamics. Such equations are difficult to solve but techniques have been developed for the simpler types. More difficult types can be solved using numerical methods.

To be able to solve partial differential equations knowledge of the following is required:

- Partial integration.
- Partial Differentiation
- The solution of Ordinary Differential Equations.

We have already studied partial differentiation and the solution of ordinary differential equations at level one, so let's look at partial integration here.

Partial Integration

Partial integration is the reverse process of partial differentiation. Consider the following example.

Example 1. Integrate $\frac{\partial u}{\partial t} = 5\cos x \sin t$ partially with respect to t.

Just as we do with partial differentiation, if we are integrating with respect to t, then we treat the $5\cos x$ term as a constant. This gives,

$$u = 5\cos x \int \sin t \, dt$$

$$= (5\cos x)(-\cos t)$$

$$= -5\cos x \cos t + f(x)$$

Note that we no longer have a constant of integration (+C) but a function of integration (f(x)). This is because, in this example we treat any x terms as constants, including functions of x. This merely demonstrates that we are aware that there could've been a function of x there prior to differentiation.

Example 2. Integrate $\frac{\partial^2 u}{\partial x \partial y} = 6x^2 \cos 2y$ partially with respect to y.

Integrating with respect to y and therefore, treating all x terms as constants gives,

$$\frac{\partial u}{\partial x} = 6x^2 \int \cos 2y \, dy$$

$$= (6x^2)\left(\frac{1}{2}\sin 2y\right) + f(x)$$

Now we can go on and integrate with respect to x to obtain an equation in terms of u. This means we must now treat any y terms as constants. This gives,

$$u = \frac{1}{2}\sin 2y \int 6x^2 + f(x) \, dx$$

$$= (2x^3)\left(\frac{1}{2}\sin 2y\right) + xf(x) + g(y)$$

$$= x^3 \sin 2y + xf(x) + g(y)$$

We can determine the functions f(x) and g(y) using extra information called boundary conditions or initial conditions.

Solving Partial Differential Equations using Direct Partial Integration

The simplest form of partial differential equations is those that can be solved by direct partial integration. This is shown in the next example.

Example 3. Solve the differential equation $\frac{\partial^2 u}{\partial x^2} = 6x^2(2y-1)$ given the boundary conditions that at $x = 0$, $\frac{\partial u}{\partial x} = \sin 2y$ and $u = \cos y$.

First integrating partially with respect to x gives,

$$\frac{\partial u}{\partial x} = 2y - 1\int 6x^2\, dx$$

$$= (2y-1)\frac{6x^3}{3} + f(y)$$

$$= 2x^3(2y-1) + f(y)$$

Now applying the boundary conditions where $\frac{\partial u}{\partial x} = \sin 2y$ at $x = 0$,

$$\sin 2y = 2(0)^3(2y-1) + f(y)$$

This gives,

$$f(y) = \sin 2y$$

And therefore,

$$\frac{\partial u}{\partial x} = 2x^3(2y-1) + \sin 2y$$

Now integrating again with respect to x gives,

$$u = \frac{2x^4}{4}(2y-1) + x\sin 2y + g(y)$$

Applying the boundary condition $u = \cos y$ at $x = 0$ gives,

$$\cos y = \frac{2(0)^4}{4}(2y-1) + (0)\sin 2y + g(y)$$

And therefore,

$$g(y) = \cos y$$

This means the solution of the partial differential equation is,

$$u = \frac{x^4}{2}(2y-1) + x\sin y + \cos y$$

Example 4. Solve the differential equation $\frac{\partial^2 u}{\partial x \partial y} = \cos(x+y)$ given that $\frac{\partial u}{\partial x} = 2$ when $y = 0$ and $u = y^2$ when $x = 0$.

Looking at the boundary conditions we know that we need to integrate partially with respect to y. This gives,

$$\frac{\partial u}{\partial x} = \int \cos(x+y)\,dy$$

$$= \sin(x+y) + f(x)$$

Now applying the boundary condition $\frac{\partial u}{\partial x} = 2$ when $y = 0$ gives,

$$2 = \sin(x+0) + f(x)$$

$f(x) = 2 - \sin x$

And therefore,

$$\frac{\partial u}{\partial x} = \sin(x + y) + 2 - \sin x$$

Now integrating partially with respect to x gives,

$$u = \int \sin(x + y) + 2 - \sin x \, dx$$

$$= -\cos(x + y) + 2x + \cos x + f(y)$$

Applying the boundary condition $u = y^2$ when $x = 0$ gives,

$$y^2 = -\cos(0 + y) + 2(0) + \cos 0 + f(y)$$

$$= 1 - \cos y + f(y)$$

$$f(y) = y^2 - 1 + \cos y$$

The solution of the differential equation is therefore,

$$u = -\cos(x + y) + 2x + \cos x + y^2 - 1 + \cos y$$