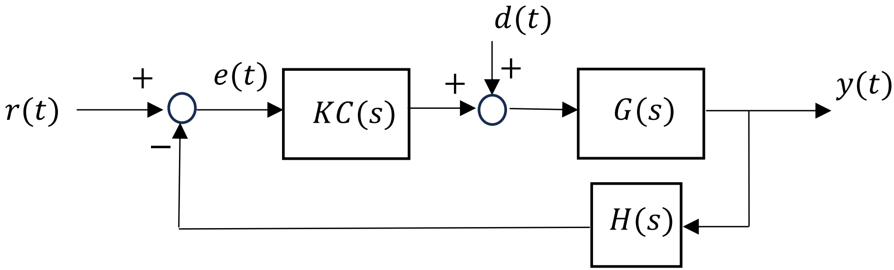
Week 10 Slides – Testing for Stability



Recall some basic ideas about linear systems and stability.

In the above block diagram we have included: plant, controller, adjustable loop gain, sensor dynamics (which is which?)

Negative feedback is normally assumed, we need this in general for stability.

The loop transfer function is the product of all transfer functions in the feedback loop

$$L(s) = KC(s)G(s)H(s)$$

Normally $H(s) \approx 1$, at least at lower frequencies for which the control system is supposed to operate. At higher frequencies beyond which the control system is supposed to perform, $H(s) \to 0$ can be useful, in order to suppress noise.

It is common to ‘mix and match’ any discussion of $H(s)$ etc. with $|H(j\omega)|$ which is what we mean when talking of low and high frequencies.

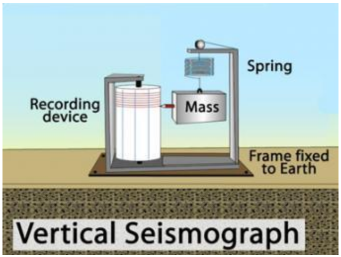
Ideally the <u>loop gain should be very large at low frequencies</u> and fall to zero at higher frequencies. We have seen previously that $G(s)$ contributes to dropoff at higher frequencies (because of poles) and $H(s)$ can also help.

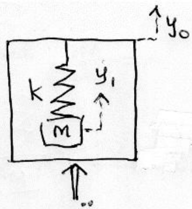
Applying high-gain feedback to noise (meaningless information) is never a good idea – don’t react to noise!

Examples of noise that should be ignored/suppressed

- structural vibrations
- airborne pressure waves (acoustics, wind gusts)
- electromagnetic intererence (from other electronic components)

In any case, sensors are physical systems with 'bandwidth limits', like all physical systems. For example consider the seismometer





INPUT $u = \ddot{y}_0$

OUTPUT spring deflection

$y = y_0 - y_1$

EOM $M\ddot{y}_1 = k \times \text{extension}$

$= k(y_0 - y_1)$

$$m(\ddot{y}_o - \ddot{y}) = ky$$

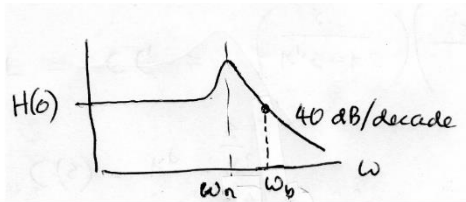
$$m\ddot{y}_o = m\ddot{y} + ky$$

$$u = \ddot{y} + \left(\frac{k}{m}\right)y$$

set $\omega_n = \frac{k}{m}$

$$\ddot{y} + \omega_n^2 y = u$$

$$H(s) = \frac{Y}{U} = \frac{1}{s^2+\omega_n^2} \quad \text{or} \quad \frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}$$



The sensor gives a useful output signal up to the natural frequency, and damping is adjusted to suit the application (ground vibration, structure vibration, vehicle motion sensing).

Sensor **bandwidth** is given by the frequency where $|H(j\omega)| = |H(0)|/\sqrt{2}$ – amplitude is reduced by 3dB (half power point).

In the same way, the **bandwidth of a control system** is the half-power point of $T(s)$

$$T(s) = \frac{KC(s)G(s)}{1 + KC(s)G(s)H(s)}$$

At lower frequencies (within the bandwidth of closed-loop control) we expect (require) $H(s) \approx 1$ and also want the loop gain to be high

$$L(s) \gg 1$$

In which case

$$T(s) \approx \frac{L(s)}{1 + L(s)} \approx 1$$

so

$$Y(s) = T(s)R(s) \ \approx R(s)$$

In terms of normal time-domain signals, this simply means the output tracks the reference, $y(t) \approx r(t)$ … which is the main purpose of the control system!

As we have seen many times, the loop transfer function contains an adjustable gain and this is increased as far as possible without losing stability or giving poor relative stability.

All relevant closed-loop transfer functions have the same denominator

$$T(s) = \frac{KC(s)G(s)}{1 + L(s)}$$

$$\frac{E(s)}{R(s)} = \frac{1}{1 + L(s)}$$

$$\frac{Y(s)}{D(s)} = \frac{G(s)}{1 + L(s)}$$

and as with root locus and gain/phase margins it is the denominator that determines closed-loop poles and hence stability.

There are two more tests of stability to introduce. The first is based on a frequency response function plot – the Nyquist plot. It conveys the same information as the bode plot but tracks the real and imaginary parts of $L(j\omega)$ rather than gain and phase.

For example, if

$$L(s) = \frac{2s^2 + 5s + 1}{s^2 + 2s + 3}$$

it is easy to check the stability of the open-loop and closed-loop systems

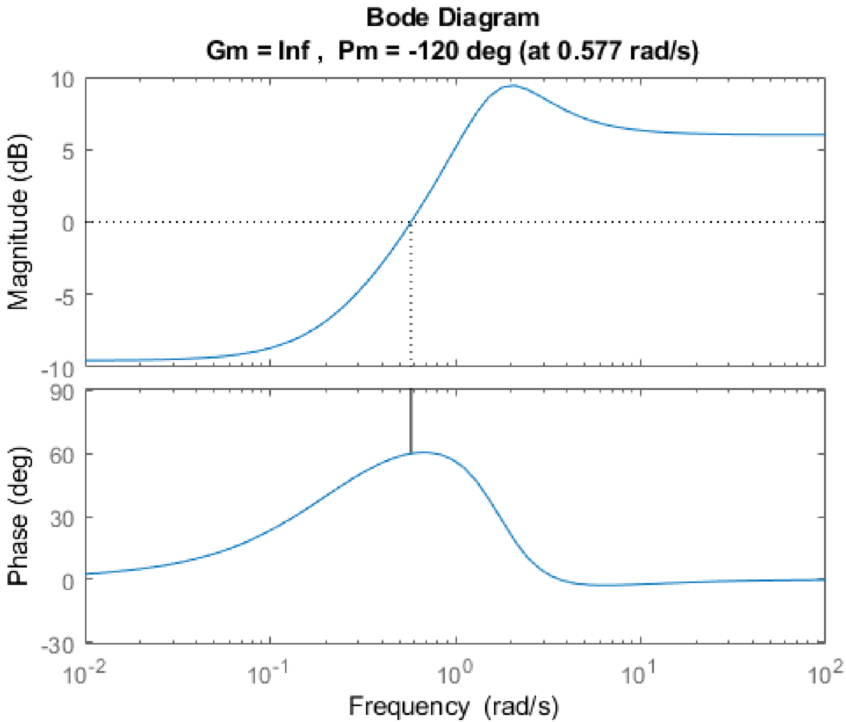
```
L = tf([2 5 1],[1 2 3]);
p0=pole(L);
T=feedback(L,1);
S=1/(1+L);
p1=pole(T);
p2=pole(S);
```

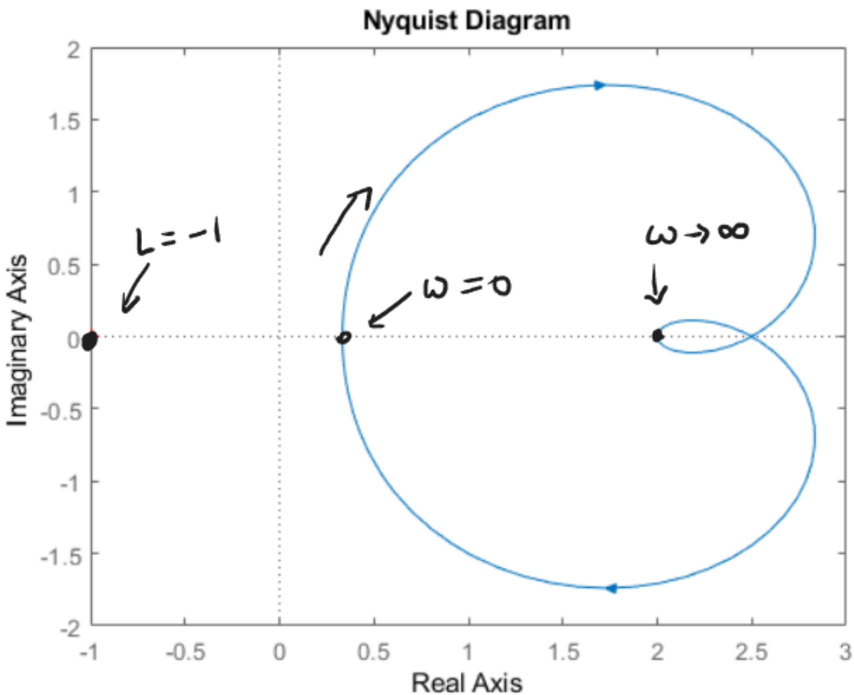
```
>> [p0,p1,p2]

ans =

  -1.0000 + 1.4142i  -1.3333 + 0.0000i  -1.3333 + 0.0000i
  -1.0000 - 1.4142i  -1.0000 + 0.0000i  -1.0000 + 0.0000i
```

Note: all closed loop systems have the same poles, based on $1 + L = 0$.

However the bode plot is not clear in this case.

Using **margin(L)** suggests the phase margin is negative so it might be unstable. In this kind of situation it is better to use the **Nyquist plot** than the bode plot, **nyquist(L)**


Bode Diagram
Gm = Inf , Pm = -120 deg (at 0.577 rad/s)
Magnitude (dB)
10
5
0
-5
-10
Phase (deg)
90
60
30
0
-30
10-2
10-1
100
101
102
Frequency (rad/s)


Nyquist Diagram
Imaginary Axis
Real Axis
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1
-0.5
0
0.5
1
1.5
2
2.5
3
L = -1
ω = 0
ω → ∞

Provided the open-loop system is itself stable (true for all examples we have seen so far!) the **condition for stability** is that the Nyquist plot should not encircle the point -1.

The Nyquist plot just plots the real and imaginary parts of $L(j\omega)$ as the frequency changes.

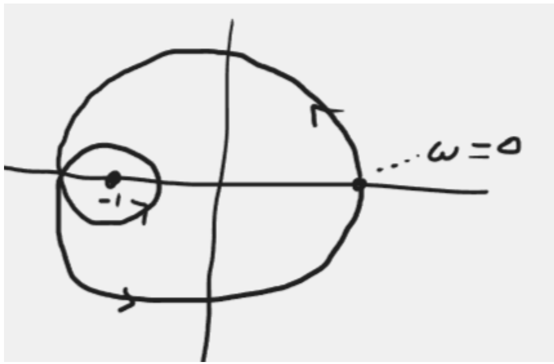
The point -1 is important because the characteristic equation is of course $1 + L = 0$ or $L = -1$.

Instability is not dependent on L hitting the point -1, but encircling it.

Applying an extra loop gain (as in root locus) is equivalent to magnifying the plot, moving points away from $(0, 0)$. In the previous plot, however much we magnify it will never encircle the point $(-1, 0)$.

It's possible to have a stable closed-loop system **even when the plant itself is unstable**. Examples?

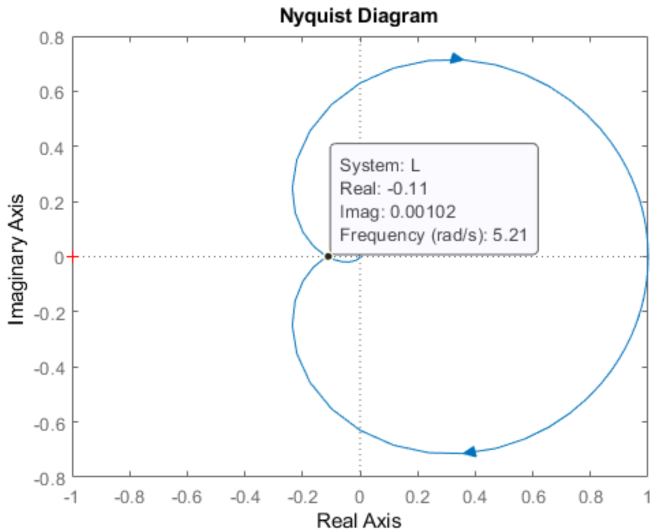
If $L(s)$ has n unstable poles (poles in the right-half plane) then the condition for closed-loop stability is there should be n anticlocwise encirclements of -1. For example, if $L(s)$ has one unstable pole, the following Nyquist plot would confirm stability in closed-loop.



The Nyquist stability condition is essentially the same as Bode, but it works in all cases.

Unlike root locus, Nyquist does not tell us exactly <u>where</u> the closed-loop poles will be, but simply says if they are stable or not.

Gain and phase margins can also be found from the Nyquist plot (though not as convenient as **bode**). For example in the case $L = \frac{24}{(s+2)(s+3)(s+4)}$ ...



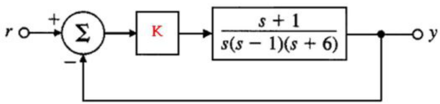
The data tip shows a point where the gain is 0.11. Expanding the plot by a factor (1/0.11) it will be on the point of encircling the critical point -1. So the gain margin is approximately 1/0.11 = 9 (approx).

For phase margin it's not so clear, but the overall distance of the plot from -1 is an excellent measure of stability.

Tutorial Questions

1. Plot the bode diagram for an accelerometer with natural frequency ω_n = 100 rad/sec and damping ratio $\zeta = 0.6$. Estimate the bandwidth of this sensor.
2. The accelerometer of question 1 operates below the natural frequency of the sensor. However, in measuring ground movement in earthquakes it is also useful to measure displacement of the ground. Revise the analysis shown above for the case where u = displacment $u = y_0(t)$. Find the transfer function when ω_n = 1 rad/sec and damping ratio $\zeta = 0.6$. Comment on the bandwidth in this case.

3. Using the example from above, $L = \frac{24}{(s+2)(s+3)(s+4)}$, use Matlab to draw both the Nyquist and Bode plots. Use data tips for frequencies $\omega = 1, 10$ rad/sec to find the value of $L(j\omega)$ in each case – check they agree.
4. For the system of problem 3, find the gain margin using: (i) the Nyquist plot – as in the slides above, (ii) the bode plot, (iii) the Matlab command **margin**. Check they all agree.
5. Use the Nyquist to investigate the stability of the following system, which has one unstable pole. Set $K = 1$ in the first instance, but then vary K until the it correctly encircles the point -1.



Routh-Hurwitz Stability Condition

This is a method that can be performed without Matlab! It checks stability without finding the actual poles, directly from the coefficients of the characteristic equation, without solving it.

For example, we might have the characteristic equation

$$s^3 + 3s^2 + 7s + 5 = 0$$

Or if there is an unknown coefficient we can't use Matlab directly:

$$s^3 + 3s^2 + 7s + K = 0$$

Here we might want to know the range of values of K for which stability holds.

In general the characteristic equation is written:

$$s^n + a_{n-1}\, s^{n-1} + a_{n-1}\, s^{n-1} + a_{n-2}\, s^{n-2} + \cdots + a_1\, s + a_0 = 0$$

If the leading coefficient is not equal to 1, divide through so it is

Routh-Hurwitz stability conditions

- All coefficents are positive
- [none of them are ‘missing’]
- The first column of the “Routh Array” is positive

And if there are negative terms in that first column, the number of sign changes gives the number of unstable poles, i.e. poles in the right half of the complex plane (RHP).

While we have Matlab to find poles, the method is still useful, especially when the characteristic equation has an unknown coefficient.

Method:

System characteristic equation

$$\boxed{a(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + a_{n-3} s^{n-3} + \cdots + a_1 s + a_0}$$

Formation of Routh Array

$$\begin{array}{ll|ccccc}
\text{Row n} & s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
\text{Row n - 1} & s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
\text{Row n - 2} & s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
\text{Row n - 3} & s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & & & & & & \\
\text{Row 2} & s^2 & * & * & & & \\
\text{Row 1} & s^1 & * & & & & \\
\text{Row 0} & s^0 & * & & & &
\end{array}$$

$$b_1 = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$b_2 = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

$$c_1 = \frac{-1}{b_1} \begin{vmatrix} a_n - 1 & a_{n-3} \\ b_1 & b_2 \end{vmatrix}$$

$$c_2 = \frac{-1}{b_1} \begin{vmatrix} a_n - 1 & a_{n-5} \\ b_1 & b_3 \end{vmatrix}$$

Number of sign changes in the first column = number of RHP poles

It's fairly simple once you see a few examples …

Example 1 $s^3 + 3s^2 + 7s + 5 = 0$

first two rows straight from the characteristic equation …

s^3	1	7
s^2	3	5

the next term is from the determinant (1 x 5 – 7 x 3) = –16 … divide by the value in the bottom left and flip the sign: = + 16/3.

Add a zero to allow for the final term and then complete the table

s^3	1	7
s^2	3	5
s^1	16/3	0
s^0	5	

The first column is all positive so poles are all stable.

Example 2 $s^3 + 3s^2 + 7s + K = 0$

s^3	1	7
s^2	3	K
s^1	(21-K)/3	
s^0	K	

So for stability K must be in the range 0 < K < 21. Can check in Matlab

```
num=[1];den=[1 3 7 21.01];
G=tf(num,den);p=pole(G);
```

gives

```
p: 3x1 complex double =

  -3.0006 + 0.0000i
   0.0003 + 2.6461i
   0.0003 - 2.6461i
```

For K>21 there are two sign changes so a pair of poles in the RHP is correct

For K<0 there is just one sign change, so a real pole in the RHP:

```
num=[1];den=[1 3 7 -0.01];
G=tf(num,den);p=pole(G);
```

```
p: 3x1 complex double =

  -1.5007 + 2.1799i
  -1.5007 - 2.1799i
   0.0014 + 0.0000i
```

which is confirmed.

However, in this case the negative coefficient in the characteristic equation tells us it’s unstable even without calculating the array.

Example 3

$$s^4 + 2s^3 + 5s^2 + 7s + 4 = 0$$

s^4	1	5	4
s^3	2	7	
s^2	3/2	4	
s^1	5/3		
s^0	4		

so the system is stable.

There are two special cases which sometimes occur. The first is when 0 appears in the first row. The method is to put a small value ε in its place and continue with forming the array. Then draw conclusions as ε wiggles around zero, either small positive or small negative ($\varepsilon \to 0^+$ or $\varepsilon \to 0^-$)

Example 4 (where we need to use ε)

$$\alpha(s) = s^4 + 5s^3 + 7s^2 + 5s + 6 = 0$$

s^4	1	7	6
s^3	5	5	
s^2	6	6	
s^1	ε		
s^0	6		

If ε > 0 we have a stable system, while if ε < 0 it goes ustable with two sign changes. As we decrease this small parameter, a pair of poles cross the imaginary axis from the LHP (stable) to the RHP (unstable).

So of course, for ε = 0, there must be two poles on the imaginary axis, which usually indicates a pair of complex poles $s = \pm j\omega$.

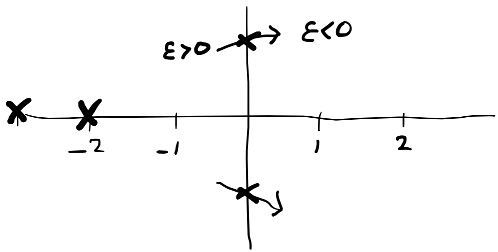
In this example we can check this by re-writing the "characteristic polynomial"

$$\alpha(s) = s^4 + 5s^3 + 7s^2 + 5s + 6$$

as

$$\alpha(s) = (s+3)(s+2)(s^2+1)$$

so the pair of "critically stable" poles are at $\pm j$, so $\omega = 1$. The idea of poles moving around as a parameter changes is something we have seen before!



The second special case is when a whole row in the table becomes zero. In this case we introduce a new polynomial ("auxiliary polynomial") to isolate the problem.

Example 4 (row becomes zero, capture the problem poles via $\alpha_1(s)$)

$$\alpha(s) = s^5 + 6s^4 + 12s^3 + 12s^2 + 11s + 6$$

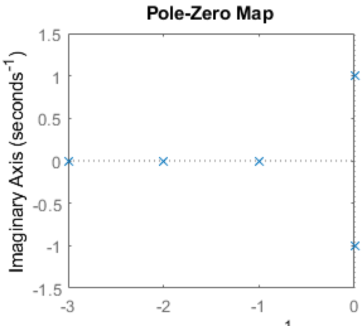
	s^5	1	12	11		
	s^4	6	12	6		
	s^3	10	10	0		
	s^2	6	6		→	$\alpha_1(s) = 6s^2 + 6$
						⇓ differentiate
oops!	$\cancel{s^1}$	$\cancel{0}$	$\cancel{0}$	ignore		
new →	s^1	12	0	use this	←	$\alpha_1'(s) = 12s$
	s^0	6				

The problem poles are given by $6s^2 + 6 = 0$, so two critically stable poles ($s = \pm 1j$).

The remainder of the Routh array is fine, so the system is otherwise stable.

Here we also use pzmap(G) to sketch the pole locations

```
num=[1];den=[1 6 12 12 11 6];
G=tf(num,den);p=pole(G);

p: 5x1 complex double =

  -3.0000 + 0.0000i
  -0.0000 + 1.0000i
  -0.0000 - 1.0000i
  -2.0000 + 0.0000i
  -1.0000 + 0.0000i
```



Apart from the special cases the Routh-Hurwitz method is quite simple to apply.

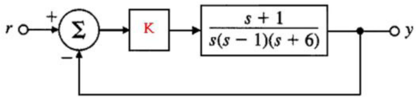
Tutorial Questions (continued)

6. Use the Routh method to check the stability of the following system

$$G(s) = \frac{s+1}{s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4}$$

   Find the poles in Matlab as a check and use pzmap to show the pole (and zero) locations graphically.

7. Check that the characteristic equation for the following closed-loop system is $s^3 + 5s^2 + (K-6)s + K = 0$.



   Show that the system is stable provided $K > 7.5$

8. Check the stability for the following characteristic equation

$$s^3 - 3s + 2 = 0$$

   (use ε in this case)