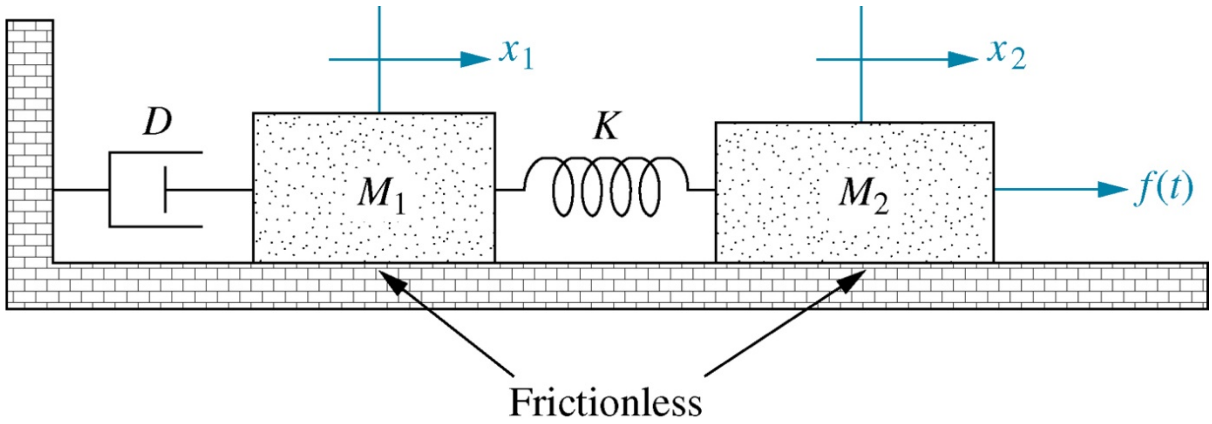
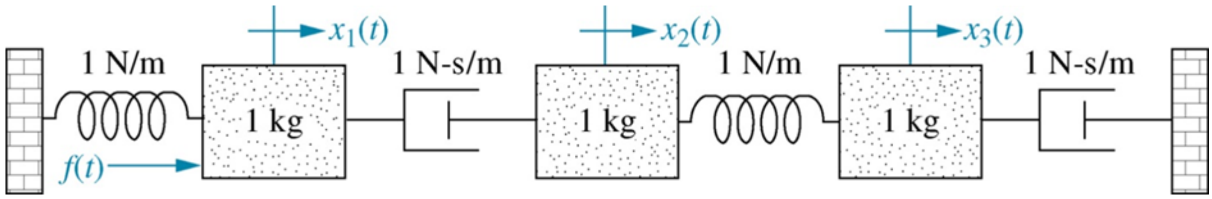
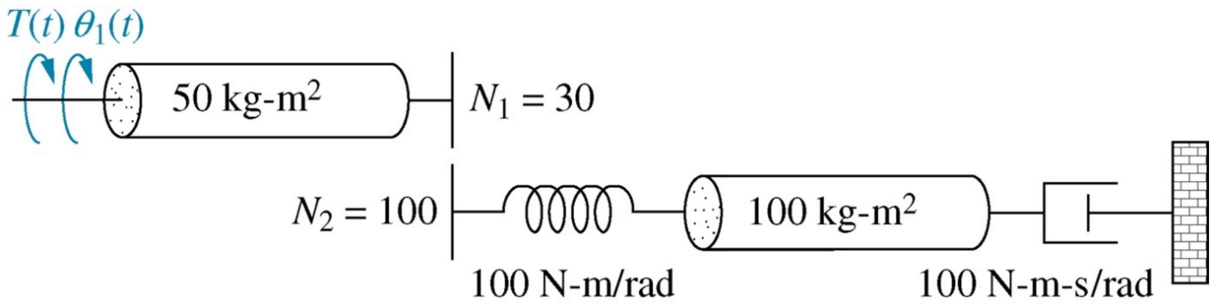
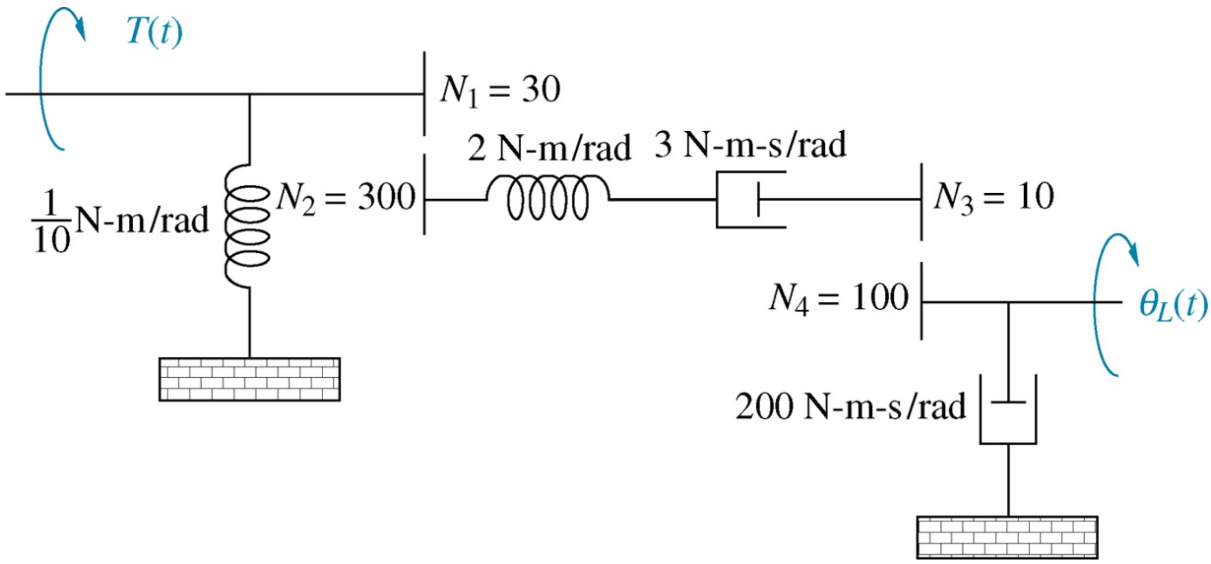
# Tutorial Sheet 1: Constructing System Models

## Section 1 – Find the LTI state space equation for the following systems.

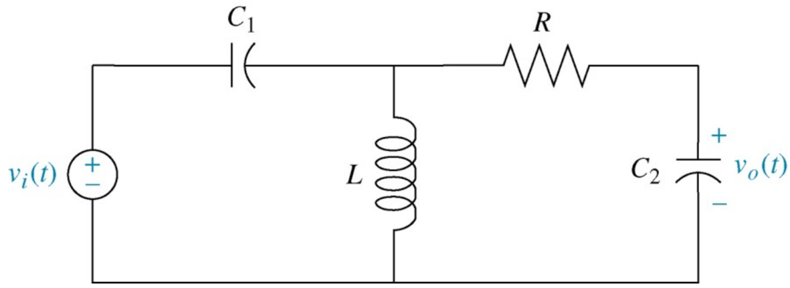
1. 


2. 

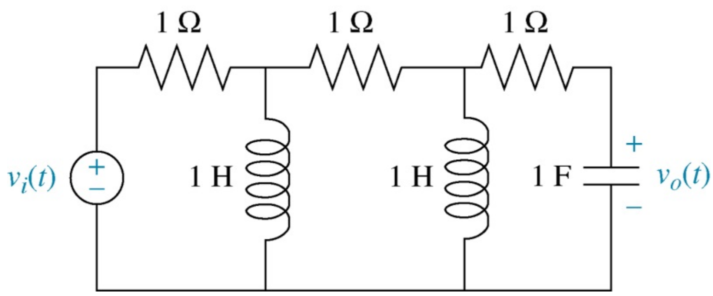

3. 

4. 

T(t)
N1 = 30
2 N-m/rad
3 N-m-s/rad
1/10 N-m/rad
N2 = 300
N3 = 10
N4 = 100
θL(t)
200 N-m-s/rad


5. 

C1
R
vi(t)
L
C2
vo(t)

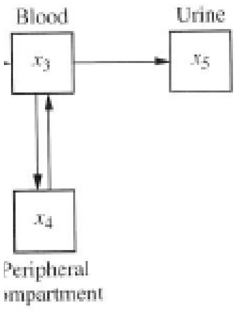

6. 

1 Ω
1 Ω
1 Ω
vi(t)
1 H
1 H
1 F
vo(t)

## Section 2 – Case Study

# Pharmaceutical Drug Absorption

An advantage of state-space representation over the transfer function representation is the ability to focus on component parts of a system and write $n$ simultaneous, first-order differential equations rather than attempt to represent the system as a single, $n$th-order differential equation, as we have done with the transfer function. Also, multiple-input, multiple-output systems can be conveniently represented in state space. This case study demonstrates both of these concepts.




l drug-level

**PROBLEM:** In the pharmaceutical industry we want to describe the distribution of a drug in the body. A simple model divides the process into compartments: the dosage, the absorption site, the blood, the peripheral compartment, and the urine. The rate of change of the amount of a drug in a compartment is equal to the input flow rate diminished by the output flow rate. Figure 3.16 summarizes the system. Here each $x_i$ is the amount of drug in that particular compartment (*Lordi, 1972*). Represent the system in state space, where the outputs are the amounts of drug in each compartment.

**SOLUTION:** The flow rate of the drug into any given compartment is proportional to the concentration of the drug in the previous compartment, and the flow rate out of a given compartment is proportional to the concentration of the drug in its own compartment.

We now write the flow rate for each compartment. The dosage is released to the absorption site at a rate proportional to the dosage concentration, or

$$\frac{dx_1}{dt} = -K_1x_1 \tag{3.99}$$

The flow into the absorption site is proportional to the concentration of the drug at the dosage site. The flow from the absorption site into the blood is proportional to the concentration of the drug at the absorption site. Hence,

$$\frac{dx_2}{dt} = K_1x_1 - K_2x_2 \tag{3.100}$$

Similarly, the net flow rate into the blood and peripheral compartment is

$$\frac{dx_3}{dt} = K_2x_2 - K_3x_3 + K_4x_4 - K_5x_3 \tag{3.101}$$

$$\frac{dx_4}{dt} = K_5x_3 - K_4x_4 \tag{3.102}$$

where $(K_4x_4 - K_5x_3)$ is the net flow rate into the blood from the peripheral compartment. Finally, the amount of the drug in the urine is increased as the blood releases the drug to the urine at a rate proportional to the concentration of the drug in the blood. Thus,

$$\frac{dx_5}{dt} = K_3x_3 \tag{3.103}$$

Equations (3.99) through (3.103) are the state equations. The output equation is a vector that contains each of the amounts, $x_i$. Thus, in vector-matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} -K_1 & 0 & 0 & 0 & 0 \\ K_1 & -K_2 & 0 & 0 & 0 \\ 0 & K_2 & -(K_3+K_5) & K_4 & 0 \\ 0 & 0 & K_5 & -K_4 & 0 \\ 0 & 0 & K_3 & 0 & 0 \end{bmatrix} \mathbf{x} \tag{3.104a}$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x} \tag{3.104b}$$

You may wonder how there can be a solution to these equations if there is no input. In Chapter 4, when we study how to solve the state equations, we will see that initial conditions will yield solutions without forcing functions. For this problem, an initial condition on the amount of dosage, $x_1$, will generate drug quantities in all other compartments.

**CHALLENGE:** We now give you a problem to test your knowledge of this chapter's objectives. The problem concerns the storage of water in aquifers. The principles are similar to those used to model pharmaceutical drug absorption.

Underground water supplies, called aquifers, are used in many areas for agricultural, industrial, and residential purposes. An aquifer system consists of a number of interconnected natural storage tanks. Natural water flows through the sand and sandstone of the aquifer system, changing the water levels in the tanks on its way to the sea. A water conservation policy can be established whereby water is pumped between tanks to prevent its loss to the sea.

A model for the aquifer system is shown in Figure 3.17. In this model, the aquifer is represented by three tanks, with water level $h_i$ called the *head*. Each $q_n$ is the

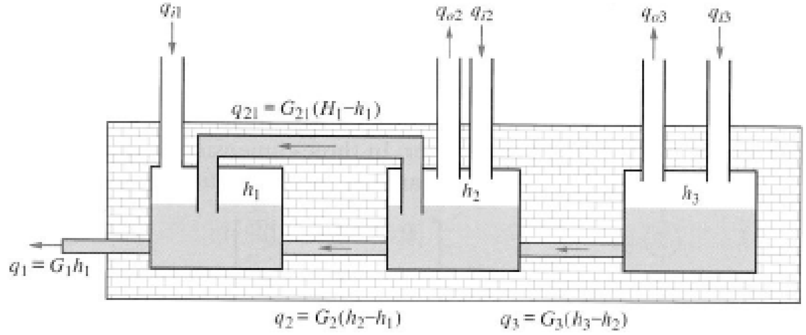



**FIGURE 3.17** Aquifer system model

natural water flow to the sea and is proportional to the difference in head between two adjoining tanks, or $q_n = G_n(h_n - h_{n-1})$, where $G_n$ is a constant of proportionality and the units of $q_n$ are $m^3$/yr.

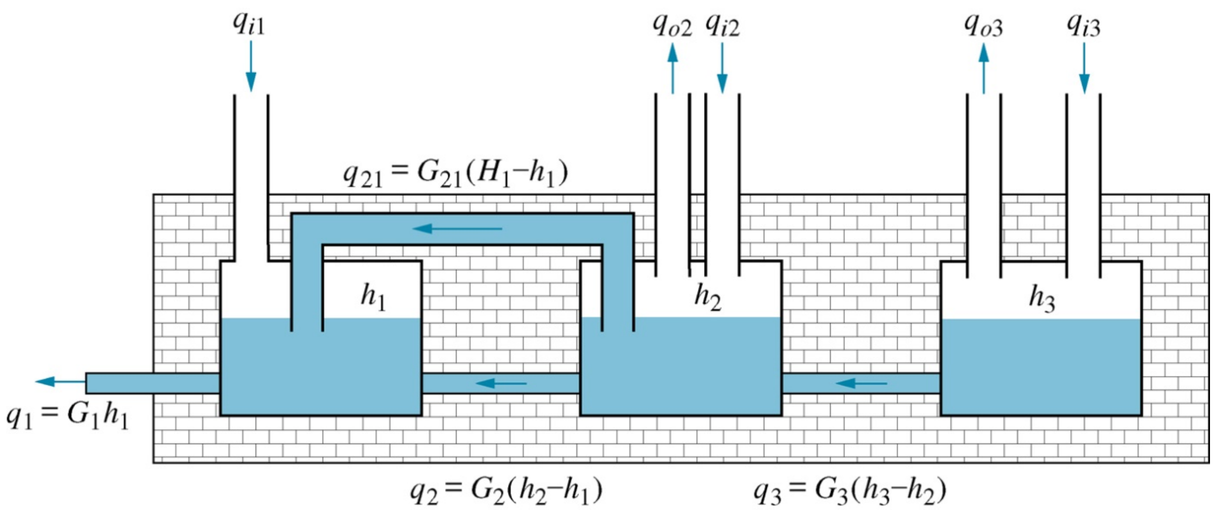
The engineered flow consists of three components, also measured in $m^3$/yr: (1) flow from the tanks for irrigation, industry, and homes, $q_{on}$; (2) replenishing of the tanks from wells, $q_{in}$; and (3) flow, $q_{21}$, created by the water conservation policy to prevent loss to the sea. In this model, water for irrigation and industry will be taken only from Tank 2 and Tank 3. Water conservation will take place only between Tank 1 and Tank 2, as follows. Let $H_1$ be a reference head for Tank 1. If the water level in Tank 1 falls below $H_1$, water will be pumped from Tank 2 to Tank 1 to replenish the head. If $h_1$ is higher than $H_1$, water will be pumped back to Tank 2 to prevent loss to the sea. Calling this *flow for conservation* $q_{21}$, we can say this flow is proportional to the difference between the head of Tank 1, $h_1$, and the reference head, $H_1$, or $q_{21} = G_{21}(H_1 - h_1)$.

The net flow into a tank is proportional to the rate of change of head in each tank. Thus,

$$C_n dh_n/dt = q_{in} - q_{on} + q_{n+1} - q_n + q_{(n+1)n} - q_{n(n-1)}$$

(*Kandel, 1973*).

Represent the aquifer system in state space, where the state variables and the outputs are the heads of each tank.



## Section 3: Aircraft Case Study

The equations governing the motion of an aircraft are a very complicated set of six nonlinear coupled differential equations. However, under certain assumptions, they can be decoupled and linearized into longitudinal and lateral equations. Aircraft pitch is governed by the longitudinal dynamics. The basic coordinate axes and forces acting on an aircraft are shown in the figure given below.

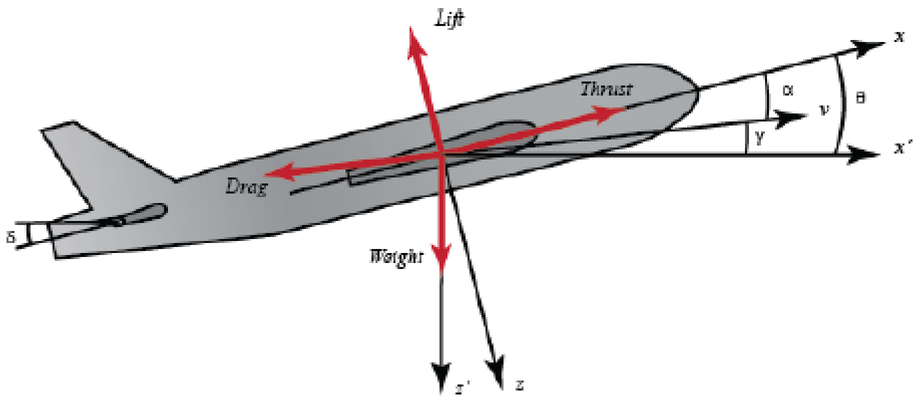



**Figure 1: Basic Coordinates and Forces on a generic aircraft**

We will assume that the aircraft is in steady-cruise at constant altitude and velocity; thus, the thrust, drag, weight and lift forces balance each other in the *x*- and *y*-directions. We will also assume that a change in pitch angle will not change the speed of the aircraft under any circumstance (unrealistic but simplifies the problem a bit). Under these assumptions, the longitudinal equations of motion for the aircraft can be written as follows.

$$\dot{\alpha} = \mu\Omega\sigma\left[-(C_L + C_D)\alpha + \frac{1}{(\mu - C_L)}q - (C_W \sin\gamma)\theta + C_L\right]$$

$$\dot{q} = \frac{\mu\Omega}{2i_{yy}}\left[[C_M - \eta(C_L + C_D)]\alpha + [C_M + \sigma C_M(1 - \mu C_L)]q + (\eta C_W \sin\gamma)\delta\right]$$

$$\dot{\theta} = \Omega q$$

For this system, the input will be the elevator deflection angle and the output will be the pitch angle of the aircraft. Before finding the state-space model, let's plug in some numerical values to simplify the modeling equations shown above:

$$\dot{\alpha} = -0.313\alpha + 56.7q + 0.232\delta$$

$$\dot{q} = -0.0139\alpha - 0.426q + 0.0203\delta$$

$$\dot{\theta} = 56.7q$$

These values are taken from the data from one of Boeing's commercial aircraft.

Write the LTI State Space equation for the aircraft.