<u>Lectures 1</u>   Eigenvalues and Eigenvectors

In this lecture we will look at how we can calculate eigenvalues and eigenvectors and their meaning. These have many important applications, and particularly in the area of vibration analysis, but generally for solving systems of differential equations.

Some properties of determinants

Before we can discuss eigenvalues, some preliminary results about determinants are required. Consider the simultaneous equations:

$$ax + by = 0$$
$$cx + dy = 0$$

where a, b, c and d are constants. Clearly x = 0 and y = 0 is a solution of these equations. This is called the **trivial solution**. We call solutions other than x = 0 and y = 0 **non-trivial solutions**. Consider 2 cases shown below:

Case 1:

$$5x - 3y = 0$$
$$10x - 2y = 0$$

Case 2:

$$5x - 3y = 0$$
$$10x - 6y = 0$$

Solving for case 1, gives x = 0 and y = 0 as the only possible solution. Thus, the only solution is the trivial solution.

In case 2, both equations are essentially the same and so we only really have one equation to work with:

$$5x - 3y = 0$$

Rearranging gives,

$y = \frac{5}{3}x$

As long as the y value is 5/3 of the x value, the equation is satisfied. For example, $x = 1, y = \frac{5}{3}$; $x = 2, y = \frac{10}{3}$ are both solutions. In general, the solutions have the form $x = t, y = \frac{5}{3}t$ for any value of t.

Now let's return to the system,

$$ax + by = 0$$
$$cx + dy = 0$$

The condition for non-trivial solutions to exist is that ad – bc = 0. Writing the system in matrix form gives,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or AX = 0

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ *and* $0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Note that ad – bc is the determinant of A, so non-trivial solutions exist when the determinant of A is zero.

If $\|A\| = 0$ the system has non-trivial solutions.
If $\|A\| \neq 0$ the system has only the trivial solution.

Eigenvalues

We will explain the meaning of the term eigenvalue by means of an example. Consider the system,

$$2x + y = \lambda x$$
$$3x + 4y = \lambda y$$

where λ is some unknown constant. Clearly these equations can be written in matrix form as,

$$\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$$

or $AX = \lambda X$

We now seek to find values of λ so that the system has non-trivial solutions

As we have a matrix (A) and a constant (λ) on the right-hand side we need to write it in a slightly different way. To help us achieve this we use the 2x2 identity matrix, I. Now $\lambda\begin{pmatrix} x \\ y \end{pmatrix}$ may be expressed as,

$$\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

since multiplying $\begin{pmatrix} x \\ y \end{pmatrix}$ by the identity matrix leaves it unchanged. So λX may be written as λIX. Hence we have,

$AX = \lambda IX$

which can be written as,

$(A - \lambda I)X = 0$

We have seen previously that for AX = 0 to have non-trivial solutions then $|A| = 0$. Hence for,

$(A - \lambda I)X = 0$

to have non-trivial solutions then,

$$|A - \lambda I| = 0$$

Now,

$$A - \lambda I = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{pmatrix}$$

So the condition $|A - \lambda I| = 0$ gives,

$$\begin{vmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{vmatrix} = 0$$

If we now find the determinant of this matrix this gives,

$$(2-\lambda)(4-\lambda) - 3 = 0$$

$$\lambda^2 - 6\lambda + 5 = 0$$

$$(\lambda - 1)(\lambda - 5) = 0$$

So that,

$$\lambda = 1 \quad or \quad \lambda = 5$$

These are the values of λ which cause the system $AX = \lambda X$ to have non-trivial solutions. They are called **eigenvalues**.

Example 1. Find the eigenvalues in the system $\begin{aligned} x + 4y &= \lambda x \\ 2x + 3y &= \lambda y \end{aligned}$

In this case,

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \ X = \begin{pmatrix} x \\ y \end{pmatrix}$$

To have non-trivial solutions we require,

$$|A - \lambda I| = 0$$

This gives,

$$A - \lambda I = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{pmatrix}$$

Hence,

$$|A - \lambda I| = (1-\lambda)(3-\lambda) - 8$$

$$= \lambda^2 - 4\lambda - 5$$

$$= (\lambda + 1)(\lambda - 5) = 0$$

The eigenvalues are therefore, $\lambda = -1$, and $\lambda = 5$.

The process of finding eigenvalues of has been illustrated using a 2x2 matrix. The same process can be applied to a square matrix of any size.

Example 2. Find the eigenvalues of A where $A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 3 & 2 & -2 \end{pmatrix}$

We need to calculate $|A - \lambda I|$.

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 2 & 0 \\ -1 & -1-\lambda & 1 \\ 3 & 2 & -2-\lambda \end{pmatrix}$$

and,

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & -1-\lambda & 1 \\ 3 & 2 & -2-\lambda \end{vmatrix}$$

$$= (1-\lambda)[(-1-\lambda)(-2-\lambda) - 2] - 2[-1(-2-\lambda) - 3]$$

This gives,

$$\lambda^3 + 2\lambda^2 - \lambda - 2 = 0$$

Factorising the cubic expression (for example, using the factor theorem) gives the eigenvalues,

λ = -2, -1, 1

Eigenvectors

We have studied the system

AX = λX

and determined the values of λ for which non-trivial solutions exist, called the eigenvalues of the system. For each eigenvalue, there is a non-trivial solution of the system called an **eigenvector**.

Example 1. Find the eigenvectors of AX = λX where $A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$ and $X = \begin{pmatrix} x \\ y \end{pmatrix}$

First, we must find the eigenvalues. This gives,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(2-\lambda) - 3 = 0$$

$$\lambda^2 - 6\lambda + 5 = 0$$

Factorising gives,

λ = 1, λ = 5

Now let's find the eigenvector when λ = 1,

Recalling that AX = λX can be written as $(A - \lambda I)X = 0$. When λ = 1, this gives,

$$\left[\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} - 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Written as individual equations we have,

$$3x + y = 0$$
$$3x + y = 0$$

These are identical equations and so, as long as $y = -3x$ the equation is satisfied. Thus there is an infinite number of solutions such as x = 1, y = -3; x = -5, y = 15, and so on. Generally, we write x = t, y = -3t for any value of t. Thus the eigenvector corresponding to λ = 1 is,

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} t \\ -3t \end{pmatrix}$$

$$= t\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

Now let's find the eigenvector corresponding to the eigenvalue λ = 5,

$$(A - \lambda I)X = 0$$

$$\left[\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Written as individual equations gives,

$$-x + y = 0$$
$$3x - 3y = 0$$

We note that the second equation is just a multiple of the first and so we only have one equation to solve. This gives,

$$-x + y = 0$$
$$y = x$$

So we write x = t, y = t for any value of t.

The eigenvector corresponding to λ = 5 is,

$$X = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Note that both eigenvectors are written with an arbitrary constant, t. This is sometimes assumed to be there and so we can write the eigenvectors as,

$$X = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad and \quad X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The concept of eigenvectors is easily extended to matrices of higher order.

Example 2. Determine the eigenvectors of $\begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 3 & 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

We found the eigenvalues in example 2 of the eigenvalue section of these notes. These were λ = -2, -1, 1. We consider each eigenvalue in turn.

<u>λ = -2</u>

Recalling,

$$(A - \lambda I)X = 0$$

Gives,

$$\left[\begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 3 & 2 & -2 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & 0 \\ -1 & 1 & 1 \\ 3 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We note that the first and second rows are identical and so,

$$3x + 2y = 0$$
$$-x + y + z = 0$$

Solving these equations gives,

$$x = t,\ y = -\frac{3}{2}t,\ z = \frac{5}{2}t$$

Hence, the corresponding eigenvector is,

$$X = t\begin{pmatrix} 1 \\ -\frac{3}{2} \\ \frac{5}{2} \end{pmatrix}$$

<u>λ = -1</u>

We have,

$$\left[\begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 3 & 2 & -2 \end{pmatrix} + 1\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & 0 \\ -1 & 0 & 1 \\ 3 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus we have,

$$2x + 2y = 0$$
$$-x + z = 0$$
$$3x + 2y - z = 0$$

Solving these equations (using Gaussian elimination) gives,

$x = t,\ y = -t,\ z = t$

This gives the eigenvector,

$$X = t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

<u>λ = 1</u>

We have,

$$\left[\begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 3 & 2 & -2 \end{pmatrix} - 1\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & 0 \\ -1 & -2 & 1 \\ 3 & 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus we have,

$$\begin{aligned} 2y &= 0 \\ -x - 2y + z &= 0 \\ 3x + 2y - 3z &= 0 \end{aligned}$$

Solving the equations (by Gaussian elimination) gives,

$x = t,\ y = 0,\ z = t$

Hence the eigenvector is,

$$X = t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

MATLAB Exercise

Try all the above examples in MATLAB using "eig".