Week 3 Slides

Recap from last week

- Laplace transform as the ‘official’ way to use the s-operator for differentiation.
- From differential equation:

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = u$$

to transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

assuming zero initial conditions
- Solving differential equations with Laplace and including initial conditions

For the second-order system (Week 3, tutorial question 8)

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 5y = u$$

the transfer function was

$$G(s) = \frac{1}{s^2 + 6s + 5} = \frac{1}{(s+1)(s+5)}$$

and then Laplace transforms were used to get a solution to the step response of the form (with zero initial conditions)

$$Y(s) = \frac{1}{(s+1)(s+5)s} = \frac{a}{s+1} + \frac{b}{s+5} + \frac{c}{s}$$

and hence

$$y(t) = ae^{-t} + be^{-5t} + c\,1(t).$$

- Non-zero initial conditions affect only the values of a, b, c
- The input (impulse, step, ramp, sinus, …) only affects the third term, the **first two exponentials are always included** in the output signal. They come from the **transfer function denominator**.
- In general a transfer function has a numerator polyomial $N(s)$ and a denominator polynomial $D(s)$. These exponentials come from the denominator.

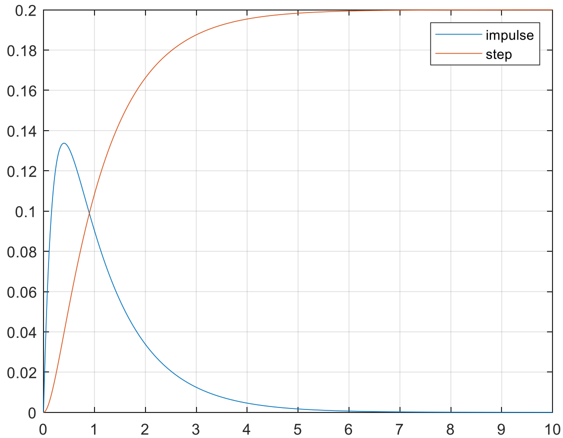
$$\frac{1}{s^2 + 6s + 5} = \frac{N(s)}{D(s)}$$

- The exponentials e^{-t} and e^{-5t} come from the values of s for which the denominator is zero, $D(s) = 0$ has two solutions $s = -1, -5$.

POLES AND ZEROS

- The solutions of $D(s) = 0$ are called the **poles of G** and they have a major effect on the time response, including the stability of the system.
- If for example $D = s^2 + 4s - 5$, the poles are $s = 1, -5$ and the exponentials become e^t and e^{-5t}. The growing exponential makes the system unstable.
- The equation $D(s) = 0$ is given a special name, the **characteristic equation** for G. It's solutions are sometimes called its roots, or "characteristic roots".
- Solutions of $N(s) = 0$ are called the **zeros of G**.

For the impulse and step responses (of the original stable system) the exponentials are the only 'interesting' part of the response. In the next plot we see the Matlab results



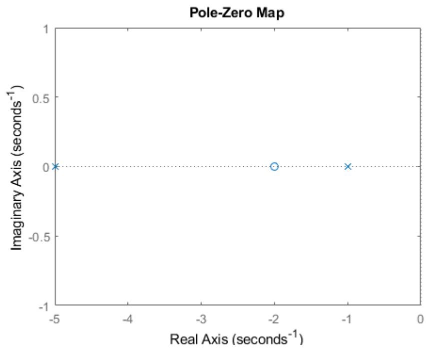
In particular the settling time is the same for both curves. And settling is **dominated by the slowest pole** ('dominant pole'). E.g.,

$$y(t) = ae^{-t} + be^{-5t} + c \;\approx ae^{-t} + c$$

since e^{-5t} tends to zero much faster than e^{-t}.

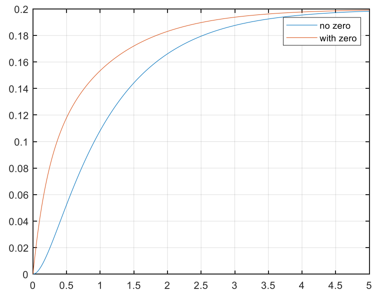
Since the poles of a transfer function are so important, Matlab has a nice little command to plot them (also the zeros). For example (including a zero and keeping the steady-state gain unchanged)

```
%% pole-zero map
clear
close all
s=tf('s');
G=(0.5*s+1)/((s+1)*(s+5));
pzmap(G)
```



The effect of zeros is more subtle than the effect of poles. In the following plot we show the step responses with and without the extra zero term

```
%% effect of a zero on the step response
clear
close all
s=tf('s');
G1=1/((s+1)*(s+5));
G2=(0.5*s+1)/((s+1)*(s+5));
[y1,t1]=step(G1,5);
[y2,t2]=step(G2,5);
plot(t1,y1), grid
hold on
plot(t2,y2)
legend('no zero','with zero')
```



Including a zero makes for a quicker start, and sometimes this can make the response more oscillatory. It can also cause slightly weird behaviour sometimes (see the tutorial questions) but it has less effect than the positions of the poles.

Tutorial Questions

1. Plot in matlab the following functions and comment on the results

$$y_1 = 1 - e^{-t}\,,\;\; y_2 = 1 - e^{-t} + e^{-5t}$$

2. The "5% settling time" is defined as the time it takes for a signal to reach and stay within 5% of it's final value. For a signal e^{-at} this happens (roughly) when $at = 3$, or $t_s = 3/a$.
(i) use this rule to find the settling time t_s for y_1 in question 1.
(ii) check by zooming in on the graph.
(iii) repeat for y_2 (which has two exponentials – so make some assumption and check)

3. Use Matlab to give the pole-zero map for the transfer function $G(s) = \frac{s+5}{s^2+2s+5}$. Check the results by hand and also plot the step-response. What do you notice?

4. For question 3, calculate the settling time, using only the real part of the pole – check with the graph. Does it work?

5. In question 3, remove the s term in the numerator (so there is no zero). How does this affect the step response?

6. Also from question 3, reverse the sign of s in the numerator, so $G(s) = \frac{-s+5}{s^2+2s+5}$. Use Matlab for the pzmap and the step response. Comment on what difference this makes.

In conclusion, locations of poles and zeros affect the response of a system (open-loop or closed loop).

But the poles have the greater effect. Poles are simply the coefficients used in the exponentials that come from using Laplace.

This is still the case when we have complex poles …

Systems with Complex Poles

Complex poles always come about when solving a quadratic (or higher-order) equation. We saw this in question 3 above …

$$G(s) = \frac{s+5}{s^2+2s+5}$$

The characteristic equation is $D(s) = s^2 + 2s + 5 = 0$

- can be re-written $(s+1)^2 + 4 = 0$
- solutions are $s = -1 \pm 2j$
- write these poles as: $p_1 = -1 + 2j, p_2 = -1 - 2j$

Let’s compute the impulse response ($U(s) = 1$) using Laplace

$$Y(s) = G(s) \times 1 = \frac{s+5}{s^2+2s+5} = \frac{a}{s-p_1} + \frac{b}{s-p_2}$$

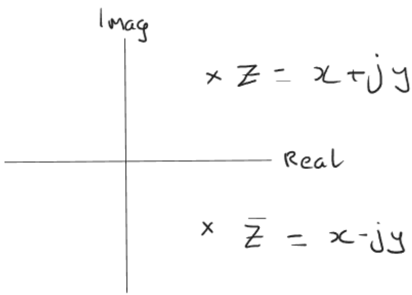
Using the standard partial fractions (but with complex numbers)

$$a = \frac{1-2j}{2}\ , b = \frac{1+2j}{2}$$

Hence

$$y(t) = ae^{p_1 t} + be^{p_2 t}$$

Note that a and b ($b = \bar{a}$) are complex conjugates
So are p_1 and p_2 ($p_2 = \overline{p_1}$)



So the two terms are actually equivalent, of the form

$$(x + jy) + (x - jy) = 2x$$

so twice the real part.

Hence

$$y(t) = 2 \times Real\ (ae^{p_1 t})$$

or “twice the real part”.

To make things simpler, as we know $y(t)$ is a real physical signal we can simply write

$$y(t) = 2ae^{p_1 t}$$

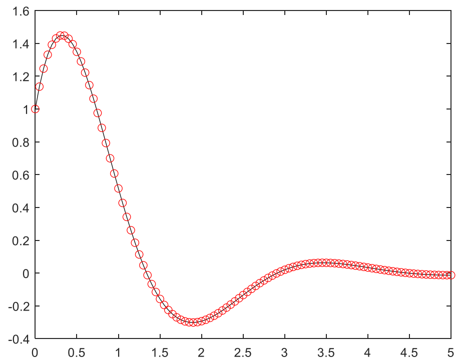
or

$$y = (1 - 2j)e^{(-1+2j)t}$$

We just need to remember that the **physical signal is the real part** of the expression, and it makes all the equations a lot simpler.

As a sanity check, let's compare this with the result from **impulse** in Matlab (here only the main commands are shown, the full set are in the Week 4 m-file)

```
G=(s+5)/(s^2+2*s+5);
[y1,t]=impulse(G,5);
y2=(1-2*j)*exp((-1+2*j)*t);
figure, plot(t,y1,'k'),hold on
plot(t,y2,'ro')
```



The black curve is from 'impulse', the red circles from the formula.

Matlab make a couple of mild compaints – that it only plots the real part of y2, and that we are supposed to write ‘1j’ instead of ‘j’ - since j might be a defined variable.

The benefit is that we can write the solution – a complex exponential – in a compact form.

To again make sure the formula makes complete sense, we should recall

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

and

$$e^{a+b} = e^a e^b$$

Then

$$y = (1 - 2j)e^{(-1+2j)t}$$

This can be reduced to a more comfortable looking form:

$$y = e^{-t}(\cos 2t + 2\sin 2t)$$

The Week 4 Matlab script checks by plotting .

In summary

- the time response always includes terms of the form ae^{pt} where p is a pole of the transfer function
- p may be complex, in which case a will be complex as well.
- Complex poles come in conjugate pairs, but both contain the same information. And it is often convenient just to work with one
- For a signal which may be complex, we always assume that the physical signal is the <u>real part of the expression</u> (and Matlab knows this rule)
- The real and imaginary parts of a both get used, giving cosine and sine terms in the resulting physical signal:

$$Re\{(a + jb)(\cos \omega t + j \sin \omega t)\} = a \cos \omega t - b \sin \omega t$$

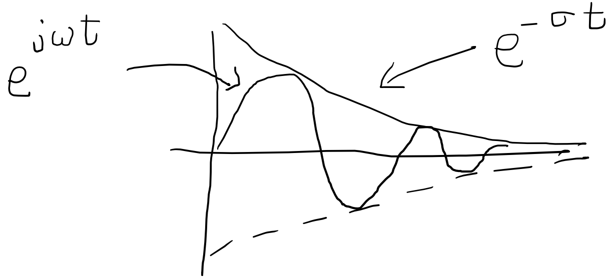
We also see that exponential decay results from the real part of p

In general, for a complex pole (pair of poles)

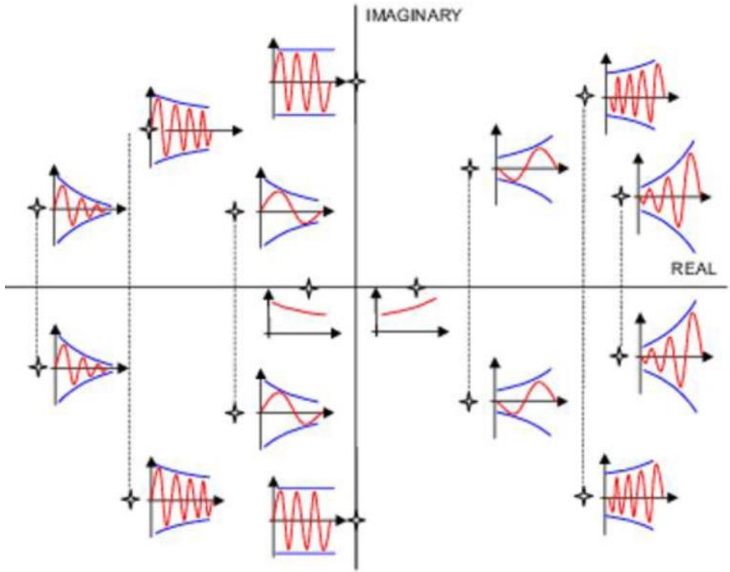
$$p = -\sigma + j\omega$$

$$y(t) = Ae^{pt} = Ae^{-\sigma t}e^{j\omega t}$$

- real part of p gives the exponential decay (if negative)
- settling time (for 5% settling) is $t_s = 3/\sigma$
- imaginary part gives an oscillation



The transfer function poles don't contain enough information to give the exact response of a system, but they indicate the important trends for oscillation and stability.



Finally, for the standard form of the second-order characteristic equation

$$D(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

the poles are located at

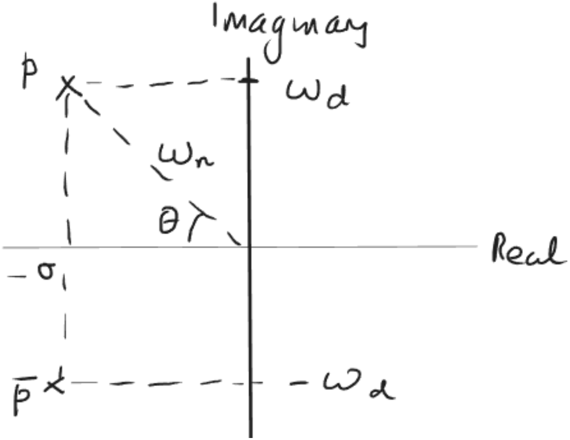
$$-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

Hence

$\sigma = -\zeta\omega_n$ is the decay constant
$\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the oscillation frequency

ω_d corresponds to ω we had above, and is called the damped natural frequency.

The following figure shows the poles, with a bit more information added



$$\boxed{\begin{aligned} \omega_d &= \omega_n\sqrt{1-\zeta^2} \\ \sigma &= \zeta\omega_n \\ \omega_n &= |p| \\ \cos\theta &= \zeta \end{aligned}}$$

It may seem like a lot to take in, but most of it is just notation – remembering the symbols and what they mean.

Tutorial Questions (continued)

7. Using the equation for second-order transfer function poles:

$$p = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$$

   and the sketch above, show that
   (i) the modulus of p equals ω_n
   (ii) the angle of p (represented by θ in the figure) satisfies $\cos\theta = \zeta$

8. Plot the following functions in Matlab, using your own value for ω
   (i) $y = \cos\omega t$
   (ii) $y = e^{j\omega t}$
   (iii) $y = je^{j\omega t}$
   (iv) $y = \frac{1+j}{\sqrt{2}}\ e^{j\omega t}$

   You should see that in each case the oscillation has the same amplitude and frequency. So what changes?

9. Find (by hand) the poles and zeros of the following transfer function

$$G(s) = \frac{3s + 6}{s^3 + 3s^2 + 7s + 5}$$

10. Check the result using **pzmap**. Also check using the functions **pole** and **zero**. (Use **help** or **doc** to find out about these functions) .