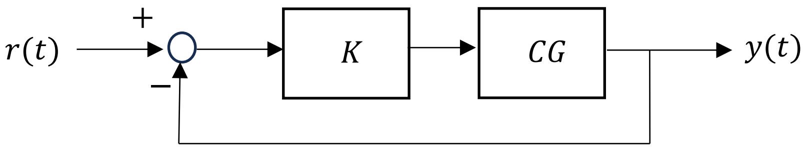
Week 9 Slides – Frequency response and stability



Recall

$$T(s) = \frac{KC(s)G(s)}{1 + KC(s)G(s)}$$

The poles of T should be in a suitable region of the complex plane, and root locus helps find the suitable gain, using **rlocus(C*G)**

If C and G are type zero, and thinking about a step input as reference, we want a high gain to reduce steady-state error

$$e_{ss} = \frac{1}{1 + K_p}$$

with $K_p = KC(0)G(0)$ found from the loop transfer function. Root locus tells us what happens as K increases and whether the controller will work

Frequency response – and especially the Bode plot – can also determine stability. As with root locus we find use the loop transfer function

$$KC(j\omega)G(j\omega)$$

With frequency response can determine stability margins and even deal with time delays.

Recall, if the input is $u(t) = e^{j\omega t}$ the output is $y(t) = He^{j\omega t}$ where for any particular input frequency $|H|$ is the gain and $\angle H = e^{j\phi}$.

As the frequency varies the complex number H will change giving the frequency response function

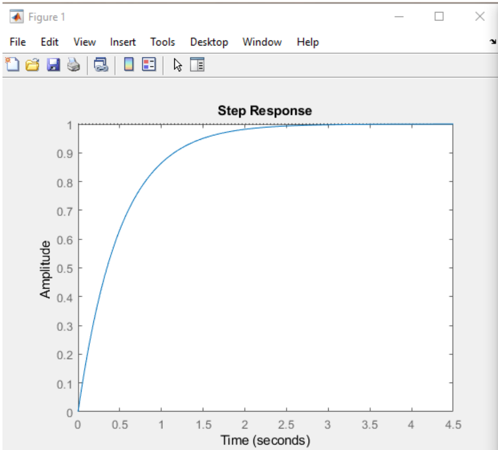
$$H(\omega) = G(s) \text{ with } s = j\omega$$

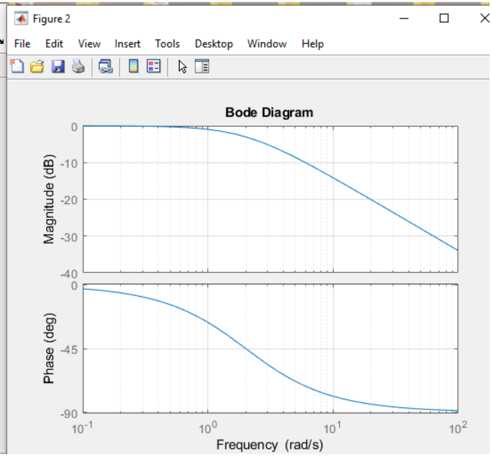
It's easy to see from examples that

- poles reduce the gain as frequency increases
- poles give phase lag
- the 'corner frequency' – where things change most is at $\omega_n = |p|$
- zeros do the opposite

Example – first order transfer function with time constant T

$$G(s) = \frac{1}{1 + Ts}$$






```
s=tf('s');
T=0.5;
G=1/(1+T*s);
step(G)
figure
bode(G),grid
```

time constant T = 0.5 sec

pole $p = -1/T = -2$

$\sigma = \omega_n = 2$, with

$t_s = \frac{3}{\sigma} =$ 1.5 sec (also equals 3*T)

Gain falls at “20 dB per decade”, up to 90° phase lag

In general

$$H(\omega) = \frac{1}{1 + j\omega T} = \frac{1 - j\omega T}{1 + \omega^2 T^2}$$

$$|H| = \frac{1}{\sqrt{1 + \omega^2 T^2}}$$

$$\angle H = -\arctan \omega T$$

so the phase lag increases up to 90° as $\omega \to \infty$. More simply

$$G(s) = \frac{1}{1 + Ts}$$

so

- for small ω: $G(j\omega) \approx 1$
- for large ω: $G(j\omega) \approx \frac{1}{j\omega T} = -j(T^{-1})\omega^{-1}$ … phase is $-90°$

gain is $k\omega^{-1} \Longrightarrow 20\log_{10}|H| = 20\log_{10} k - \boldsymbol{20\log_{10}\omega}$
… slope is -20 on the log-log scale of the bode plot.

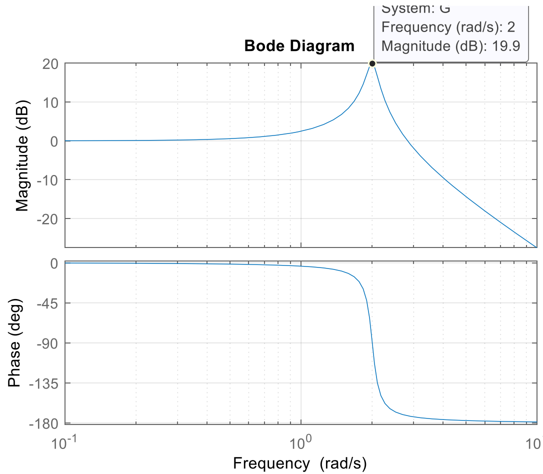
Example Second-order system with low damping - makes the phase and gain shifts very clear (OLTF of course)

$$G = \frac{4.01}{s^2 + 0.2s + 4.01}$$

Poles are at $-0.1 \pm\ 2j$ so very close to the imaginary axis

$$\omega_n \cong 2$$

After ω_n there is a roughly a 180 degree phase lag (90 degrees per pole) and. For large s (large omega) $G \approx k/s^2$ so the gain falls off at 40dB per decade (20 dB per pole). The pattern is clear – more poles, more phase lag, faster dropoff in gain.



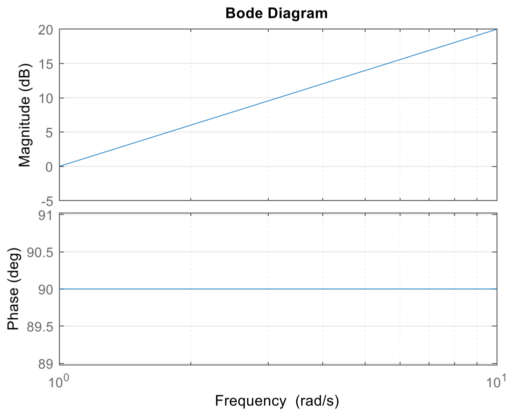
Zeros have the opposite effect – increasing gain and phase lead.

Example Pure differentiator and "lead compensator"

$$G(s) = s$$

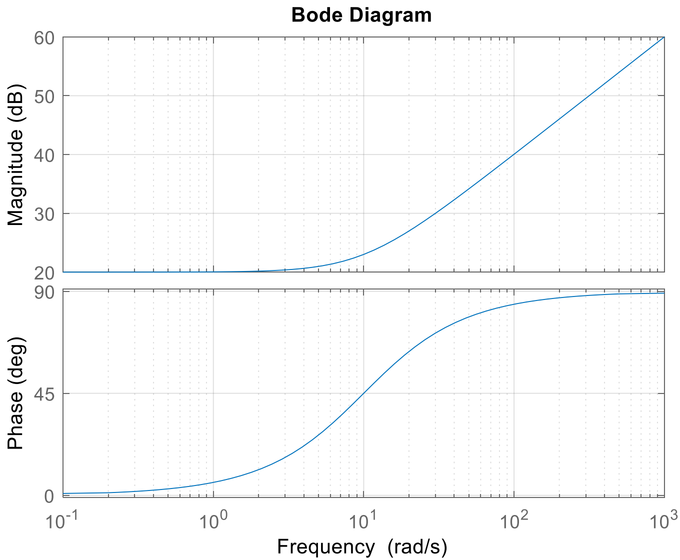
It's easy to check that there is a constant phase lead of 90 degrees and the gain keeps increasing at +20 dB per decade.

A zero at $s = -a$ gives a corner frequency at $\omega = a$



For example, if $a = 10$ rad/s

$G(s) = s + 10$ ….



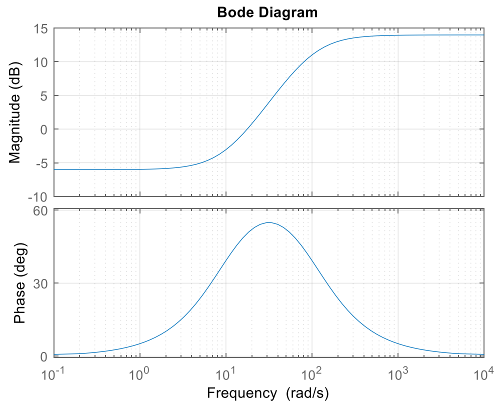
So zeros can be an ‘antidote’ to phase lag from poles!

On the other hand no physical system can increase its gain forever as frequency increases!

The **lead compensator** is like a derivative but it limits the increase in gain at some higher frequency. For example

$$G(s) = 5\frac{s+10}{s+100}$$

has corner frequencies at 10 and 100 and betwee thse it works like a differentiator.

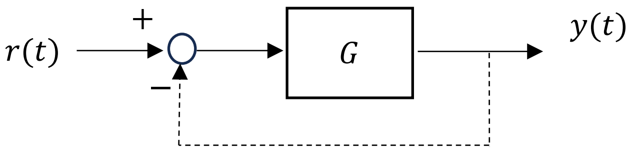


Now the gain stops increasing at higher frequencies.

The **frequency of maximum phase lead** is the mean of the two corner frequencies on the log plot, which equals the geometric mean on a standard plot: wm=sqrt(10*100) = 31.6 rad/sec

The lead compensator creates a phase lead and this can be an antidote to instability in the same way that a zero can be a good influence in the root locus plot.

Gain and phase margins



think: open loop analysis FOR closed loop design

Closed-loop characteristic equation:

$$G(s) = -1$$

Critical stability when the **root locus of $\boldsymbol{G}$ crosses the imaginary axis**. In that case $s = j\omega$, $G = H$ and

$$|H| = 1$$

$$\angle H = 180°$$

In that case, any more gain will make the system unstable.

For a stable system we know (from root locus) that increasing the loop gain will lead to instability. So at the frequency where $\angle H = 180°$ the gain must be less than one

$$|H| = h < 1$$

The **gain margin $\boldsymbol{Gm}$ is defined as the amount of extra gain K needed to make it unstable**

$$Gm = K$$

with $K \times h = 1$. Or in terms of dB

$$Gm = 20\log_{10} K > 0$$

E.g. if $K = 2$ the gain margin is roughly 6dB.

If the gain margin is negative (in dB) the system is unstable and $K < 1$ is needs to be applied to bring the closed-loop system to the critical point.

The idea is similar to Ziegler Nichols but here we 'test' the open loop system and apply a sinusoidal signal in that 'test'.

Again assume the system will be stable, so when $|H| = 1$ the phase lag should be less than $180°$, and the **difference is called the phase margin Gp**.

So if the phase lag is actually $125°$, the phase margin will be $55°$.

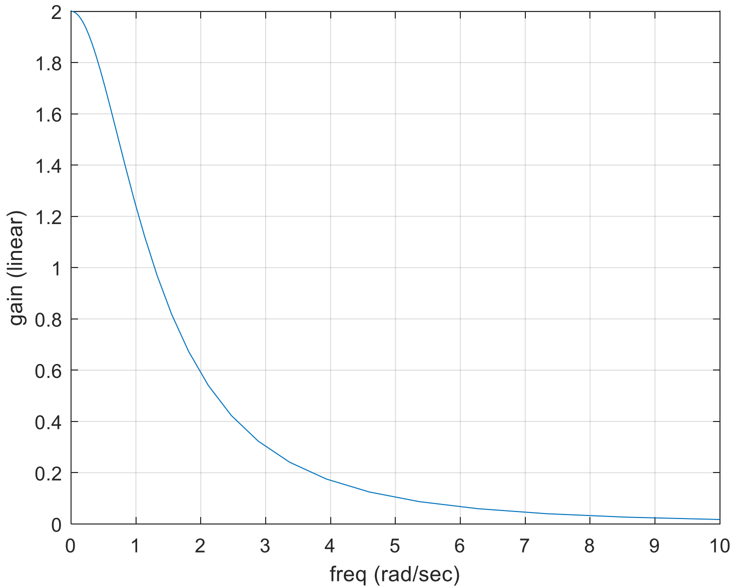
Example – find the gain and phase margins for the transfer function

$$G(s) = \frac{20}{(s+1)(s+2)(s+5)}$$

Using the command bode with {wmin,wmax} = {0,10} ….

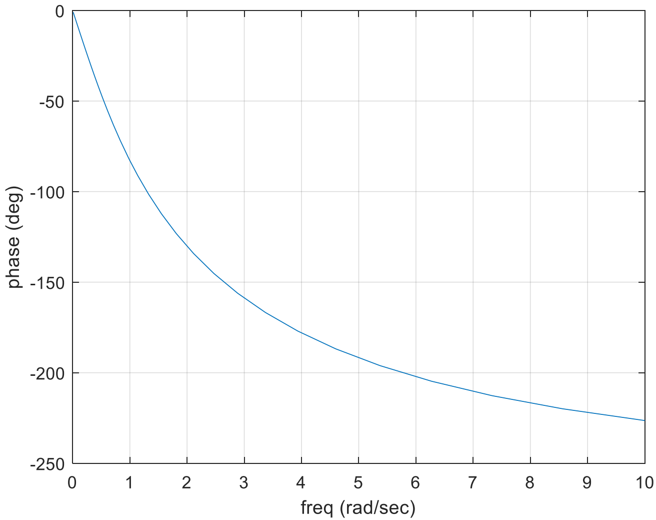
```
clear, close all
s=tf('s');
N=20;
D=(s+1)*(s+2)*(s+5);
G=N/D;
[mag,phase,w]=bode(G,{0,10});
mag=squeeze(mag);phase=squeeze(phase);
figure, plot(w,mag)
grid,xlabel('freq (rad/sec)'),ylabel('gain (linear)')
figure, plot(w,20*log10(mag))
grid,xlabel('freq (rad/sec)'),ylabel('gain (dB)')
figure, plot(w,phase)
grid,xlabel('freq (rad/sec)'),ylabel('phase (deg)')
```

Here we find both the linear and dB forms of the gain plot.



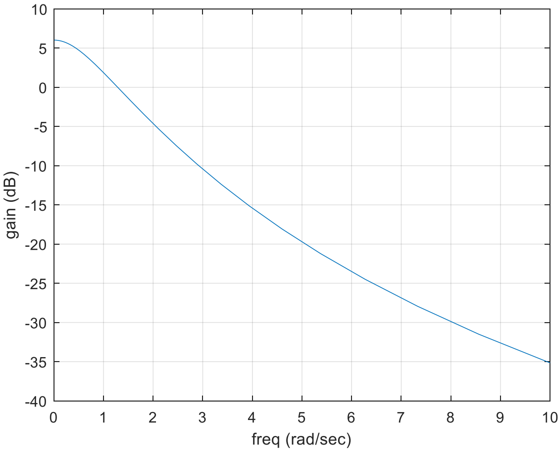
The gain ‘crosses over’ at a frequency of around ω= 1.3 rad/sec.

Check the phase plot at the gain crossover frequency



The phase lag there is around $100°$ so the phase margin is roughly $80°$.

Now checking for “phase crossover” it’s around 4.1 rad/sec …



It’s around -16 dB, so the gain margin is 16dB.

Or we can go back to the linear scale and see the gain of G is roughly 0.16. Hence the gain margin is 1/0.16 = 6.25. That’s how much ‘extra gain’ we can apply before the system goes unstable.

Of course Matlab knows about gain and phase margins and can give them to us directly – see in the tutorial problems.

Interpretation of Gain and Phase Margins

Roughly speaking we want to ensure the control system is not close to being unstable.

If our design model was a little off, will that make the system unstable?

They are similar to **safety factors** in engineering design. A bridge might be designed with a safety factor of 2, so it will support a load that's double the stated design load.

For gain margin it also tells us if we have room to increase the controller gain without causing problems. Typically we want a factor of at least 2 which equates to 6dB. So if the gain margin is 20dB we expect to increase the gain by 14dB (to improve steady-state error) without causing problems.

For phase margin, this tells us about **resilience to time delays** in the control loop.

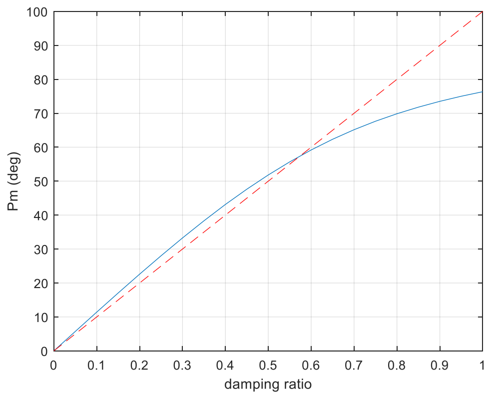
In any case might be looking for at least 55° for good relative stability. Just like the two previous rules of thumb using the number 55, it's just a guideline. The following plot, derived from a second-order transfer function plots phase margin vs. damping ratio.

The red dashed line is the approximation

$$Pm = 100\zeta$$

So if ζ=0.55 the phase margin is expected to be around 55°.

This plot is one of the tutorial questions.

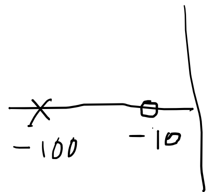


Lead Compensator

We previously considered this, but now we can see a purpose.

Previously

$$G(s) = 5\frac{s+10}{s+100}$$



In general

$$G(s) = K\frac{b(s+a)}{a(s+b)}$$

with zero at $-a$ and pole at $-b$. In the previous example $b/a = 10$ and this ratio determines the maximum phase lead of $55°$. Note the extra factors a and b are used so the steady-state gain equal K.

If instead $b/a = 5$ the maximum phase is $40°$. For a lead compensator the ratio b/a will typically be between thest two values. As before, the frequency of maximum phase is the geometric mean, $\omega = \sqrt{ab}$
The purpose of the lead compensator is **improve the phase margin** in an existing plant or plant+controller.

Design Example using a lead compensator

Starting with a plant

$$G(s) = \frac{1}{s(s+1)}$$

and a constant gain controller K, we want the steady-state error to be less than 10% for a ramp input and a phase margin of at least $45°$.

We need to use the velocity error constant

$$e_{ss} = 0.1 = \frac{1}{K_v}$$

Hence

$$K_v = 10 = \lim_{s \to 0} sKG = K$$

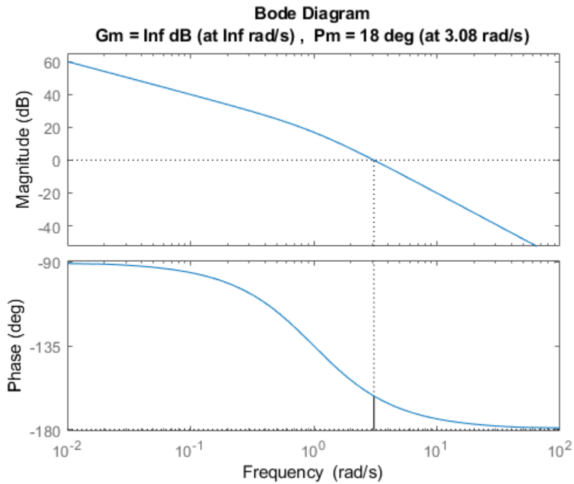
The ‘uncompensated system’ is therefore

$$KG = \frac{10}{s(s+1)}$$

The gain margin is not a problem since it never goes unstable (why exactly is that?)

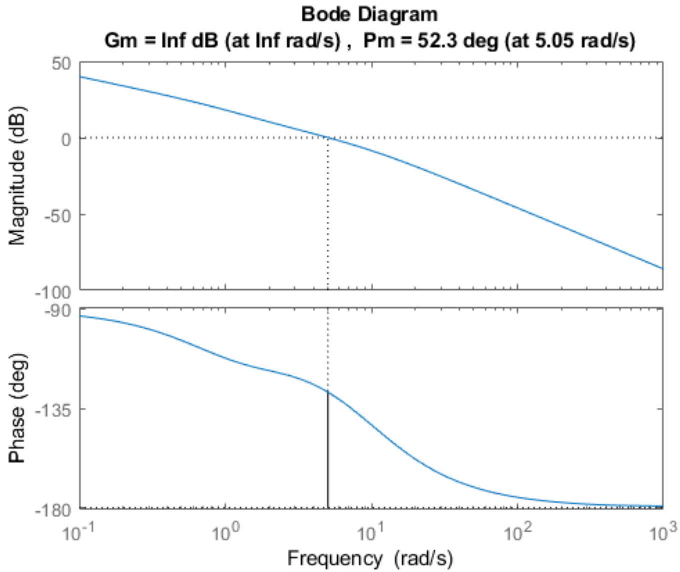
But the phase margin is only $18°$.

We need another $27°$ or so, but it's safer to increase it a bit more: $b/a = 5$ will give $40°$ max phase lead.

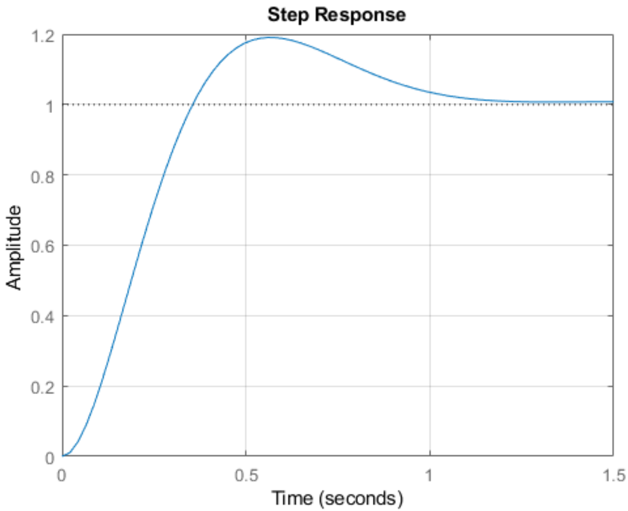


We might want to place this at $\omega = 3$ rad/sec, to improve the phase at that frequency. But the lead compensator will increase the gain there. So let's put the extra $40°$ at ω= 4 rad/sec.

Then $\sqrt{ab} = 4$ and $b/a = 5$. Solving we find $a = 1.8, b = 9$ (approx).



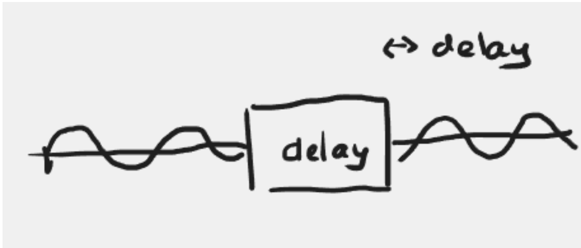
The guess of 4 rad/sec was a bit low (we can iterate on this) but the phase margin is good and hopefully we get a good step response …



Of course, being type 1 there is zero steady-state error, but we can also expect 10% error for a ramp response.

Transfer function of a pure time delay

Also called a “trasportation delay” this is an important transfer function that we didn’t meet so far. In terms of frequency response at input frequency



At frequency ω and with time delay T seconds, the phase lag will be $\phi = \omega T$ radians. The gain equals 1 at all frequencies. Hence

$$H(\omega) = e^{-j\omega T}$$

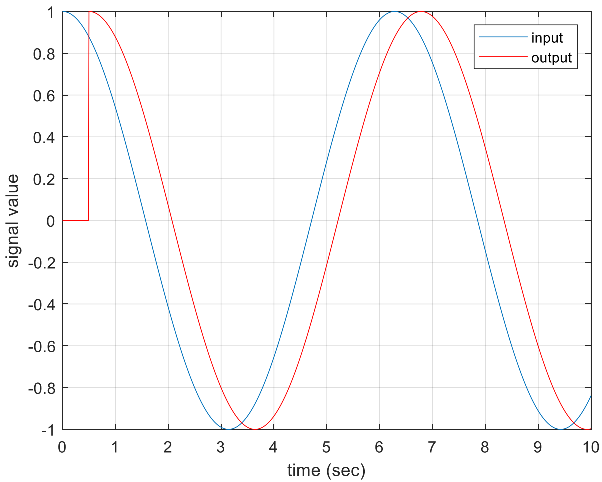
and transfer function

$$G(s) = e^{-sT}$$

Matlab understands this as a time delay, for example a 0.5 second delay can be simulated as follows

```
%% system with time delay exp(-T*s)
clear, close all
s=tf('s');
T=0.5; %delay time
G=exp(-T*s);
t=0:0.01:10;
w=1; %input frequ
u=cos(w*t);
y=lsim(G,u,t);
plot(t,u)
hold on
plot(t,y,'r')
```

giving the following result.



Time delay adds to any phase lag, so phase margin is important when time delays are a possibility, for example when a data network is included in a control loop. Because its gain=1, a time-delay does not affect the gain crossover frequency.

Tutorial Questions

1. A plant has a gain margin of 2.5 dB. What is the (linear) gain that can be applied before making the system unstable.
2. The gain of a controller can by increased by a factor of 4 before it becomes unstable in closed loop. What is the gain margin in dB?
3. A plant has a phase margin of $45°$, which occurs when the frequency is $\omega = 1$ rad/sec. What is the time delay in the feedback loop that will just make the system unstable.
4. Use the function **bode** in Matlab to plot gain and phase for the following transfer functions. In each case find the gain and phase margins by zooming in on the relevant part of the plot.
   (a) $G(s) = \frac{10}{(s+1)^2(s+2)}$ (b) $G(s) = 5\frac{e^{-0.2s}}{s}$
5. Repeat question 4 using the Matlab command **margin**.
6. Given the open-loop transfer function

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s}$$

show that unity feedback produces the standard second-order system. Find the phase margin in case $\zeta = 0.55$. You can select severall different values for ω_n but the answer should be the same – check this.

7. Continuing from question 5, use Matlab to recreate the plot on slide 19, showing the relationship between phase margin and ζ for this standard second order system. [Introduce a for loop with the value of zeta changing].
8. For each of the two systems of question 4, design a lead compensator to increase the phase margin to $45°$.
9. A lag compensator has the familiar form $G(s) = K\frac{s+a}{s+b}$ but now $a > b$. For example $a = 10$ and $b = 1$. Plot the gain and phase and suggest a way that this can be used to reduce steady-state error
10. For the general lead (or lag) compensator, show by differentiation that the phase shift reaches a maximum (or minimum) at the frequency $\omega = \sqrt{ab}$.