

1. Eigenvalues

$$(a) \begin{vmatrix} 5-\lambda & 6 \\ 2 & 1-\lambda \end{vmatrix} = (\lambda-5)(\lambda-1) - 12 = 0$$

$$\lambda^2 - 6\lambda - 7 = 0$$

$$(\lambda-7)(\lambda+1) = 0$$

$$\lambda_1 = 7 \quad \lambda_2 = -1$$

$$(b) \begin{vmatrix} -3-\lambda & 4 \\ -4 & 5-\lambda \end{vmatrix} = (3+\lambda)(\lambda-5) + 16 = 0$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda-1)^2 = 0$$

$$\lambda_1 = \lambda_2 = 1$$

$$(c) \begin{vmatrix} 7-\lambda & -2 \\ 1 & 4-\lambda \end{vmatrix} = (\lambda-7)(\lambda-4) + 2 = 0$$

$$\lambda^2 - 11\lambda + 30 = 0$$

$$(\lambda-5)(\lambda-6) = 0$$

$$\lambda_1 = 5 \quad \lambda_2 = 6$$

$$(d) \begin{vmatrix} 1-\lambda & -1 & 2 \\ -3 & -2-\lambda & 3 \\ 2 & -1 & 1-\lambda \end{vmatrix}$$

$$= (\lambda-1)^2(-2-\lambda) + 6 - 6 - 4(-2-\lambda) + 3(1-\lambda) - 3(1-\lambda)$$

$$= [(\lambda-1)^2 - 4](-2-\lambda) = 0$$

$$\lambda - 1 = \pm 2$$

$$\lambda_1 = 3 \quad \lambda_2 = -1$$

$$\lambda + 2 = 0$$

$$\lambda_3 = -2$$

2. Eigenvectors

$$(a) \quad \lambda_1 = 7 \quad \begin{pmatrix} -2 & 6 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 = 3x_2 \quad \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1 \quad \begin{pmatrix} 6 & 6 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 = -x_2 \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$(b) \quad \lambda_1 = \lambda_2 = 1$$

$$\begin{pmatrix} -4 & 4 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 = x_2 \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(c) \quad \lambda_1 = 5 \quad \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 = x_2 \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_1 = 6 \quad \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 = 2x_2 \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$(d) \quad \lambda_1 = 3 \quad \begin{pmatrix} -2 & -1 & 2 \\ -3 & -5 & 3 \\ 2 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{l} -2x_1 - x_2 + 2x_3 = 0 \\ 2x_1 - x_2 - 2x_3 = 0 \end{array} \quad \left. \begin{array}{l} \\ \end{array} \right\} \oplus \Rightarrow \begin{array}{l} x_2 = 0 \\ x_1 = x_3 \end{array}$$

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1 \quad \begin{pmatrix} 2 & -1 & 2 \\ -3 & -1 & 3 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2x_1 - x_2 + 2x_3 = 0 \quad \textcircled{1}$$

$$-3x_1 - x_2 + 3x_3 = 0 \quad \textcircled{2}$$

$$\textcircled{1} - \textcircled{2} \Rightarrow 5x_1 = x_3$$

$$x_2 = 12x_1$$

$$\begin{pmatrix} 1 \\ 12 \\ 5 \end{pmatrix}$$

$$\lambda_3 = -2$$

$$\begin{pmatrix} 3 & -1 & 2 \\ -3 & 0 & 3 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$3x_1 - x_2 + 2x_3 = 0$$

$$-3x_1 + 3x_3 = 0$$

$$\Rightarrow x_2 = 5x_1$$

$$\Rightarrow x_1 = x_3$$

$$\begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$$