Graph Theory & Network Analysis

CIS 600, Spring 2018

January 30, 2018

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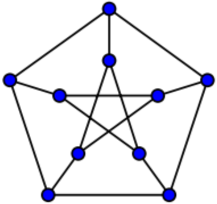

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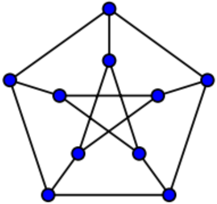

- It is a visual depiction of a graph.

Graphs, defined

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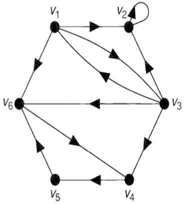
- Example:



(from Graph Theory, by Wilson)

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- Social networks such as Facebook and Twitter.

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- Cost & efficiency of operating a network, e.g. traffic, distribution
- Network analysis is a huge field in its own right.

Assignment: Graphs & Network Analysis

Assignment: read Chapter 2 of *Networks, Crowds & Markets*, and then implement some network analysis in python.
See Blackboard for details.

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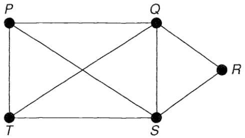
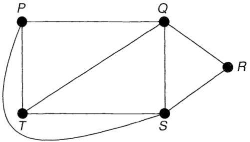
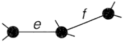
- The *degree* $deg(v)$ is the number of edges incident with v in G.

Examples





adjacent vertices



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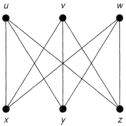

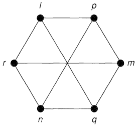

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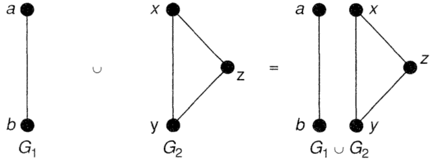

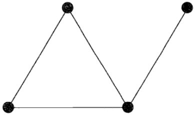

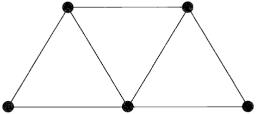

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- Why?

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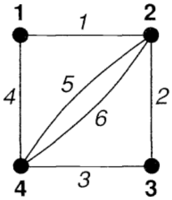
- The *incidence matrix* M of a graph G is the $|V(G)| \times |E(G)|$ matrix such that

$$M_{ij} = \begin{cases} 1 & \text{if node } i \text{ is incident with edge } j \\ 0 & \text{otherwise} \end{cases}$$

Examples



$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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- A *cycle* is a path having $v_0 = v_m$.

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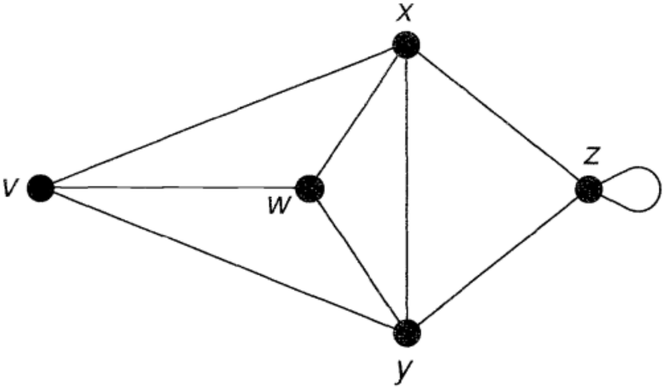

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- G is called *k-connected* or *k-edge connected* if $\lambda(G) \geq k$ or $\kappa(G) \geq k$, respectively.

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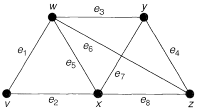
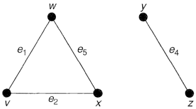
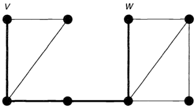
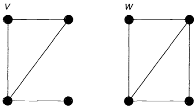






connected



disconnected

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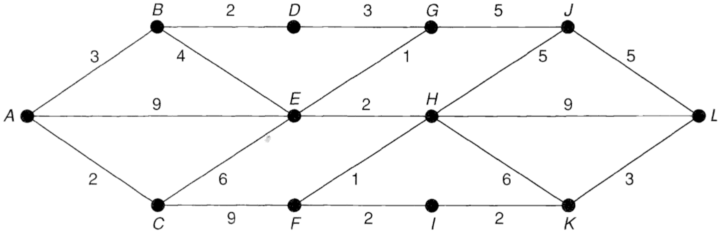
- The pair (G, ω) is then called a *weighted graph*.

Examples



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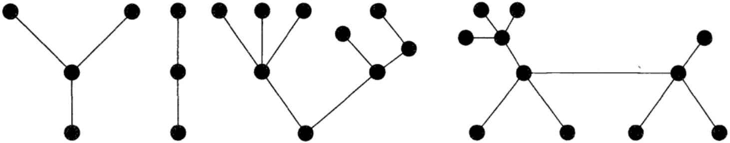
- Many equivalent definitions, e.g.

$$|E(G)| = |V(G)| - 1$$

Examples



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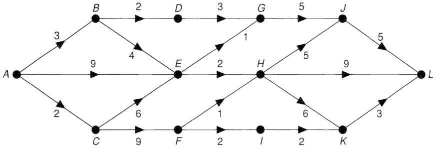
- Traveling Salesman Problem - visit all nodes (optimally). NP-Hard, whereas two above are polynomial-time.
- Max-Flow, Min-Cut - maximize 'flow' between nodes in a directed network.



Centrality - Degree

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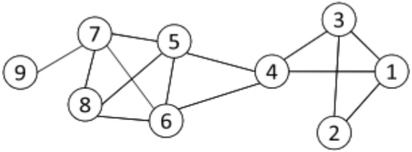
- The *normalized degree centrality* is defined by

$$C'_D(v_i) = \frac{1}{n-1} C_D(v_i)$$

(where n is the number of nodes).

Examples



Node	1	2	3	4	5	6	7	8	9
1	-	1	1	1	0	0	0	0	0
2	1	-	1	0	0	0	0	0	0
3	1	1	-	1	0	0	0	0	0
4	1	0	1	-	1	1	0	0	0
5	0	0	0	1	-	1	1	1	0
6	0	0	0	1	1	-	1	1	0
7	0	0	0	0	1	1	-	1	1
8	0	0	0	0	1	1	1	-	0
9	0	0	0	0	0	0	1	0	-

Node	Degree Centrality	Node	Degree Centrality
1	0.375	6	**0.5**
2	0.25	7	**0.5**
3	0.375	8	0.375
4	**0.5**	9	0.125
5	**0.5**		

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- The distance $g(v_i, v_j)$ between v_i, v_j is the length of a shortest path from v_i to v_j.
- Closeness centrality $C_C(v_i)$ is defined by

$$C_C(v_i) = \frac{n-1}{\sum_{j \neq i} g(v_i, v_j)}$$

Examples

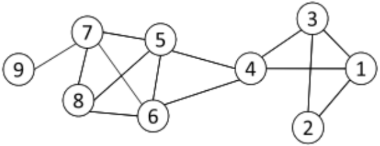

Table 2.1: Pairwise geodesic distance

Node	1	2	3	4	5	6	7	8	9
1	0	1	1	1	2	2	3	3	4
2	1	0	1	2	3	3	4	4	5
3	1	1	0	1	2	2	3	3	4
4	1	2	1	0	1	1	2	2	3
5	2	3	2	1	0	1	1	1	2
6	2	3	2	1	1	0	1	1	2
7	3	4	3	2	1	1	0	1	1
8	3	4	3	2	1	1	1	0	2
9	4	5	4	3	2	2	1	2	0

$$C_C(3) = \frac{9-1}{1+1+1+2+2+3+3+4} = 8/17 = 0.47,$$

$$C_C(4) = \frac{9-1}{1+2+1+1+1+2+2+3} = 8/13 = 0.62.$$

Node 4 is more central than node 3

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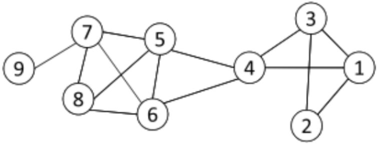
- The *betweenness centrality* $C_B(v_j)$ of v_j is then

$$C_B(v_j) = \sum_{i,k \neq j, i<k} \frac{\sigma_{ik}(v_j)}{\sigma_{ik}}$$

Examples



$$C_B(4) = 15$$

Table 2.2: $\sigma_{st}(4)/\sigma_{st}$

	$s=1$	$s=2$	$s=3$
$t=5$	1/1	2/2	1/1
$t=6$	1/1	2/2	1/1
$t=7$	2/2	4/4	2/2
$t=8$	2/2	4/4	2/2
$t=9$	2/2	4/4	2/2

What's the betweenness centrality for node 5?

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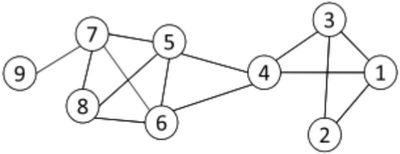
- Let's use `numpy`.

Examples



Node	1	2	3	4	5	6	7	8	9
1	-	1	1	1	0	0	0	0	0
2	1	-	1	0	0	0	0	0	0
3	1	1	-	1	0	0	0	0	0
4	1	0	1	-	1	1	0	0	0
5	0	0	0	1	-	1	1	1	0
6	0	0	0	1	1	-	1	1	0
7	0	0	0	0	1	1	-	1	1
8	0	0	0	0	1	1	1	-	0
9	0	0	0	0	0	0	1	0	-

Node	Eigenvector Centrality	Node	Eigenvector Centrality
1	0.195751798216	6	**0.468084576647**
2	0.111686197298	7	0.409977787365
3	0.195751798216	8	0.384018975728
4	0.37874977785	9	0.116955395176
5	**0.468084576647**		