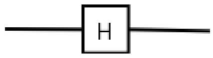
Qubit Gates

Schematic symbols are used to denote unitary transforms useful in the design of the quantum circuits.

For example, Hadamard gate 

denotes the unitary transformation

$$\begin{pmatrix} |\xi\rangle \\ |\eta\rangle \end{pmatrix} \Longleftarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix}$$

where

$$|\xi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}; \quad |\eta\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Here, $|0\rangle$ and $|1\rangle$ denote orthogonal quantum-mechanical states.e.g Energy states labelled by a quantum number, n with values 0 and 1 respectively. The inputs are pure states (called classical bits) and outputs $|\xi\rangle$ and $|\eta\rangle$ are superposed states.
One can consider the unitary transformation as an operation. The Hadamard operator, $\hat{H}$ can be defined as

$$\hat{H}|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

and

$$\hat{H}|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Notice

$$H_{00} = \langle 0|\hat{H}|0\rangle = \left\langle \frac{\langle 0| + \langle 1|}{\sqrt{2}} \middle| \hat{H} \middle| \frac{|0\rangle + |1\rangle}{\sqrt{2}} \right\rangle$$

$$= \frac{1}{2}\Big[\langle 0|\hat{H}|\hat{0}\rangle + \langle 0|\hat{H}|1\rangle + \langle 1|\hat{H}|\hat{0}\rangle + \langle 1|\hat{H}|1\rangle\Big]$$

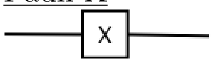
$$= \frac{1}{2}\left[\frac{\langle 0|0\rangle + \langle 0|1\rangle}{\sqrt{2}} + \frac{\langle 0|0\rangle - \langle 0|1\rangle}{\sqrt{2}} + \frac{\langle 1|0\rangle + \langle 1|1\rangle}{\sqrt{2}} + \frac{\langle 1|0\rangle - \langle 1|1\rangle}{\sqrt{2}}\right]$$

$$= \frac{1}{2}\left[\frac{1+0}{\sqrt{2}} + \frac{1-0}{\sqrt{2}} + \frac{0+1}{\sqrt{2}} + \frac{0-1}{\sqrt{2}}\right]$$

$$= \frac{1}{2}\frac{2}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

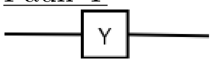
We have taken $|0\rangle$ and $|1\rangle$ to be orthonormal states.
similarly, $H_{01} = \frac{1}{\sqrt{2}} = H_{10}$ and $H_{11} = -\frac{1}{\sqrt{2}}$
Thus, matrix representation of the Hadamard operator in the basis $|0\rangle$ and $|1\rangle$ is $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
Some more examples of qubit gates : Matrix representation corresponding to quantum-mechanical operator

Pauli X

 $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

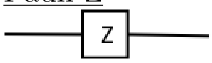
$\hat{\sigma}_x \equiv \frac{2}{\hbar}\hat{S}_x$ (in the basis $|\alpha\rangle$ and $|\beta\rangle$)
where $|\alpha\rangle \equiv |m_s = \frac{1}{2}$ and $|\beta\rangle \equiv |m_s = -\frac{1}{2}$

Pauli Y

 $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\hat{\sigma}_y \equiv \frac{2}{\hbar}\hat{S}_y$ (in the form $|\alpha\rangle$ and $|\beta\rangle$)

Pauli Z

 $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\hat{\sigma}_z \equiv \frac{2}{\hbar}\hat{S}_z$ (in the basis $|\alpha\rangle$ and $|\beta\rangle$)
The change in a qubit passing through such a gate is described in terms of a unitary matrix.
The transformation matrix corresponding to a Z gate is easy to obtain.

$$\mathrm{S_z} = \begin{pmatrix} \langle\alpha|S_z|\alpha\rangle & \langle\alpha|S_z|\beta\rangle \\ \langle\beta|S_z|\alpha\rangle & \langle\beta|S_z|\beta\rangle \end{pmatrix} = \frac{\boldsymbol{\hbar}}{\mathbf{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We will follow the procedure below to obtain the matrices corresponding to the X-gate, Y-gate:
Define operators $\hat{S_+}$ and $\hat{S_-}$ as : $\hat{S_+}|\alpha\rangle = 0$; $\hat{S_+}|\beta\rangle = \hbar|\alpha\rangle$
and $\hat{S_-}|\alpha\rangle = \hbar|\beta\rangle$; $\hat{S_-}|\beta\rangle = 0$ (please note that there was an error in the original question, there was a factor of $\frac{1}{2}$ icluded by mistake.
$\hat{S_+}$ and $\hat{S_-}$ are the raising and lowering operators for the spin states $|\alpha\rangle$ and $|\beta\rangle$ with the quantum number for the z-component of the spin angula r momentum $+\frac{1}{2}$ and $-\frac{1}{2}$ respectively. (Notice that the raising operator can only raise $|\beta\rangle$ to $|\alpha\rangle$ and likewise the lowering operator can only lower $|\alpha\rangle$ to $|\beta\rangle$. Any other operation gives the result 0).
Build the matrices $\mathrm{S}_\pm$ as $\begin{pmatrix} S_{\pm_{\alpha\alpha}} & S_{\pm_{\alpha\beta}} \\ S_{\pm_{\beta\alpha}} & S_{\pm_{\beta\beta}} \end{pmatrix}$
Here it is:

$$\mathrm{S}_+ = \begin{pmatrix} \langle\alpha|S_+|\alpha\rangle & \langle\alpha|S_+|\beta\rangle \\ \langle\beta|S_+|\alpha\rangle & \langle\beta|S_+|\beta\rangle \end{pmatrix} = \boldsymbol{\hbar} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and $\mathrm{S}_{-} = \begin{pmatrix} \langle\alpha|S_{-}|\alpha\rangle & \langle\alpha|S_{-}|\beta\rangle \\ \langle\beta|S_{-}|\alpha\rangle & \langle\beta|S_{-}|\beta\rangle \end{pmatrix} = \boldsymbol{\hbar}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Now obtain $\mathrm{S}_{\mathrm{X}} = \frac{\mathrm{S}_{+}+\mathrm{S}_{-}}{\mathbf{2}} = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\hbar}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

and $\mathrm{S}_{\mathrm{Y}} = \frac{\mathrm{S}_{+}-\mathrm{S}_{-}}{\mathbf{2i}} = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\hbar}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

The Pauli X matrix $\sigma_{\mathrm{X}} = \frac{\mathbf{2}}{\boldsymbol{\hbar}}\mathrm{S}_{\mathrm{X}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

and Pauli Y matrix $\sigma_{\mathrm{Y}} = \frac{\mathbf{2}}{\boldsymbol{\hbar}}\mathrm{S}_{\mathrm{Y}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

are the matrices corresponding to the X-gate, Y-gate respectively.