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Chapter 2 Solutions

2.1.1 Bases and linear independence

Exercise 2.1: (Linear dependence: example) Show that (1,-1), (1,2) and (2,1) are linearly dependent.

Recall a set of non-zero vectors $|\psi_i\rangle, \dots |\psi_n\rangle$ are linearly dependent if there exists a set of complex numbers $a_1, \dots, a_n$ with $a_i \neq 0$ for at least one value of i, such that

$$a_1|v_1\rangle + a_2|v_2\rangle + \cdots + a_n|v_n\rangle = 0. \tag{1}$$

We can form a matrix consisting of our vector set as

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}. \tag{2}$$

Now, we simply perform a number of elementary row operations to reduce it to reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}. \tag{3}$$

Since some columns do not contain any leading entries, then the system has nontrivial solutions, so some of the values may be nonzero. Hence, our set of vectors $\{(1,-1),(1,2),(2,1)\}$ are linearly dependent.

As an aside, we can note that the dimension of the space $\mathbb{R}^2$ is 2, and hence only a set containing 2 or fewer vectors can possibly be linear independent. In our case, our set consists of 3 vectors and therefore must be linearly dependent.

2.1.2 Linear operators and matrices

Exercise 2.2: (Matrix representations: example) Suppose V is a vector space with basis vectors $|0\rangle$ and $|1\rangle$, and A is a linear operator from V to V such that $A|0\rangle = |1\rangle$ and $A|1\rangle = |0\rangle$. Give a matrix representation for A, with respect to the input basis $|0\rangle, |1\rangle$ and the output basis $|0\rangle, |1\rangle$. Find input and output bases which give rise to a different matrix representation of A.

The matrix representation for the linear operator A that takes $A|0\rangle = |1\rangle$ and $A|1\rangle = |0\rangle$ is defined as

$$X \equiv \sigma_x \equiv \sigma_1 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{4}$$

A second spanning set for the vector space $\mathbb{C}^2$ is the set

$$|v_1\rangle \equiv \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}; \quad |v_2\rangle \equiv \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}. \tag{5}$$

Note that we can write an arbitrary vector $|v\rangle = (|0\rangle, |1\rangle)$ as a linear combination

$$|v\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}|v_1\rangle + \frac{|0\rangle - |1\rangle}{\sqrt{2}}|v_2\rangle. \tag{6}$$

Therefore, it can be verified that the matrix representation of some linear operator A taking $A|v_1\rangle = |v_2\rangle$ and $A|v_2\rangle = |v_1\rangle$ is defined as

$$\frac{Z}{\sqrt{2}} \equiv \frac{\sigma_z}{\sqrt{2}} \equiv \frac{\sigma_3}{\sqrt{2}} \equiv \begin{bmatrix}\frac{1}{\sqrt{2}} & 0\\ 0 & -\frac{1}{\sqrt{2}}\end{bmatrix}. \tag{7}$$

Exercise 2.3: (Matrix representation for operator products) Suppose A is a linear operator from vector space V to vector space W, and B is a linear operator from vector space W to vector space X. Let $|v_i\rangle, |w_j\rangle$, and $|x_k\rangle$ be bases for the vector spaces V, W, and X respectively. Show that the matrix representation for the linear transformation BA is the matrix product of the matrix representations for B and A, with respect to the appropriate bases.

Exercise 2.4: (Matrix representation for identity) Show that the identity operator on a vector space V has a matrix representation which is one along the diagonal and zero everywhere else, if the matrix representation is taken with respect to the same input and output bases. This matrix is known as the *identity matrix*.

2.1.4 Inner products

Exercise 2.5: Verify that $(\cdot,\cdot)$ just defined is an inner product on $\mathbb{C}^n$.

A function $(\cdot,\cdot)$ from $V \times V$ to $\mathbb{C}$ is an inner product if it satisfies the requirements that:

1. $(\cdot,\cdot)$ is linear in the second argument,

$$\left(|v\rangle, \sum_i \lambda_i |w_i\rangle\right) = \sum \lambda_i(|v\rangle, |w_i\rangle). \tag{8}$$

2. $(|v\rangle, |w\rangle) = (|w\rangle, |v\rangle)^*$.
3. $(|v\rangle, |v\rangle) \geq 0$ with equality if and only if $|v\rangle = 0$.

Proof. We will now prove each one of these

1. $(|v\rangle, \sum_k \lambda_k |w_k\rangle) = \sum_i \sum_k \lambda_k v_i w_i^k = \sum_k \lambda_k \sum_i v_i w_i^k = \sum_k \lambda_k(|v\rangle, |w_k\rangle)$

2. $(|v\rangle, |w\rangle) = \left(\sum_i v_i^* w_i\right) = \left(\sum_i w_i^* v_i\right)^* = (|w\rangle, |v\rangle)^*.$

3. $(|v\rangle, |v\rangle) = \sum_i v_i^* v_i = [v_1^* \dots v_n^*][v_1 \dots v_n]^T = [0 \dots 0][0 \dots 0]^T = 0$

$\square$

Exercise 2.6: Show that any inner product $(\cdot, \cdot)$ is conjugate-linear in the first argument,

$$\left(\sum_i \lambda_i |w_i\rangle, |v\rangle\right) = \sum_i \lambda_i^* (|w_i\rangle, |v\rangle). \tag{9}$$

Exercise 2.7: Verify that $|w\rangle \equiv (1, 1)$ and $|v\rangle \equiv (1, \text{-}1)$ are orthogonal. What are the normalized forms of these vectors?

Proof. First, verifying that the $|w\rangle$ and $|v\rangle$ are orthogonal, we simply check whether their inner product is zero, i.e. $\langle w|v\rangle = 0$?

$$\langle w|v\rangle = (1\ 1)(1\ \text{-}1)^T = 0 \tag{10}$$

Hence vectors $|w\rangle$ and $|v\rangle$ are orthogonal. The normalized forms of the some vector $|v\rangle$ can be found by $|v\rangle / || \, |v\rangle \, ||$. So

$$\begin{aligned} \langle w|w\rangle &= (1\ 1)(1\ 1)^T = 2 & (11) \\ \langle v|v\rangle &= (1\ \text{-}1)(1\ \text{-}1)^T = 2 & (12) \end{aligned}$$

Hence,

$$\begin{aligned} \frac{|w\rangle}{|| \, |w\rangle \, ||} &= \frac{|w\rangle}{||2||} = \frac{1}{\sqrt{2}} & (13) \\ \frac{|v\rangle}{|| \, |v\rangle \, ||} &= \frac{|v\rangle}{||2||} = \frac{1}{\sqrt{2}} & (14) \end{aligned}$$

$\square$

Exercise 2.8: Prove that the Gram-Schmidt procedure produces an orthonormal basis for V.

Exercise 2.9: (Pauli operators and the outer product) The Pauli matrices (given below) can be considered as operators with respect to an orthonormal basis $|0\rangle, |1\rangle$ for a two-dimensional Hilbert space. Express each of the Pauli operators in the outer product notation.

The Pauli-matrices are defined as

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{15}$$

Exercise 2.10: Suppose $|v_i\rangle$ is an orthonormal basis for an inner product space V. What is the matrix representation for the operator $|v_j\rangle \langle v_k|$, with respect to the $|v_i\rangle$ basis?

2.1.5 Eigenvectors and eigenvalues

Exercise 2.11: (Eigendecomposition of the Pauli matrices) Find the eigenvectors, eigenvalues, and diagonal representations of the Pauli matrices X, Y, and Z.

- Pauli-X Matrix:

$$X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{16}$$

Eigenvalues:

First we construct the matrix $X - \lambda I$:

$$X - \lambda I \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \tag{17}$$

Next, we compute the determinant

$$\text{Det}\,|X - \lambda I| = \text{Det} \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = (-\lambda)(-\lambda) - (1)(1) = \lambda^2 - 1.$$

Setting the equation $\lambda^2 - 1 = 0$, we solve for λ and obtain the eigenvalues $\{-1, 1\}$.

Eigenvectors:

- Pauli-Y Matrix:

$$Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{18}$$

Eigenvalues:

Constructing matrix $Y - \lambda I$:

$$Y - \lambda I \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & -i \\ i & -\lambda \end{bmatrix} \tag{19}$$

Next, we compute the determinant

$$\text{Det}\,|Y - \lambda I| = \text{Det}\begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = (-\lambda)(-\lambda) - (-i)(i) = \lambda^2 + 1.$$

Setting the equation $\lambda^2 + 1 = 0$, we solve for λ and obtain the complex eigenvalues $\{-i, i\}$.

Eigenvectors:

- Pauli-Z Matrix:

$$Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{20}$$

Eigenvalues:

We construct the matrix $Z - \lambda I$:

$$Z - \lambda I \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 \\ 0 & -1+\lambda \end{bmatrix} \tag{21}$$

Next, we compute the determinant

$$\text{Det}\,|Z - \lambda I| = \text{Det}\begin{vmatrix} 1-\lambda & 0 \\ 0 & -1+\lambda \end{vmatrix} = (1-\lambda)(-1+\lambda) - 0 = \lambda^2 - 1.$$

Setting the equation $\lambda^2 - 1 = 0$, we solve for λ and obtain the eigenvalues $\{-1, 1\}$.

Eigenvectors:

Exercise 2.12: Prove that the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{22}$$

is not diagonalizable.

Proof. Using the theorem of *spectral decomposition*, we know that an operator is a normal operator if and only if it is diagonalizable. Conversely, any diagonalizable operator is normal. Hence, we need to check if our matrix is normal or not. An operator A is said to be *normal* if $AA^\dagger = A^\dagger A$ Therefore, let us calculate the prior and find out if the two quantities are equal or not.

$$AA^\dagger = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = A^\dagger A \tag{23}$$

□

2.1.6 Adjoints and Hermitian Operators

Exercise 2.13: If $|w\rangle$ and $|v\rangle$ are any two vectors, show that $(|w\rangle\langle v|)^\dagger = |v\rangle\langle w|$.

Proof. Noting that $|v\rangle^\dagger = \langle v|$:

$$(|w\rangle\langle v|)^\dagger = (|w\rangle^\dagger \langle v|^\dagger) = (\langle w| |v\rangle) = |v\rangle\langle w| \tag{24}$$

□

Exercise 2.14: (Anti-linearity of the adjoint) Show that the adjoint operation is anti-linear

$$\left(\sum_i a_i A_i\right)^\dagger = \sum_i a_i^* A_i^\dagger. \tag{25}$$

Exercise 2.15: Show that $(A^\dagger)^\dagger = A$.

Exercise 2.16: Show that any projector P satisfies the equation $P^2 = P$.

Recall that

$$P \equiv \sum_{i=1}^{k} |i\rangle\langle i|. \tag{26}$$

is a projector onto some subspace.Also recall that a set of vectors $|i\rangle$ with index i is orthonormal if each vector is a unit vector, and distinct vectors in the set are orthogonal, specifically $\langle i|j\rangle = \delta_{ij}$ where i and j are both chosen from the index set. Hence

Proof.

$$P^2 = \sum_{i=j=1}^{k} \langle i| \langle i|j\rangle |j\rangle = \sum_{i=j=1}^{k} |i\rangle \delta_{ij} \langle j| = \sum_{i=1} |i\rangle\langle i| = P. \tag{27}$$

□

Exercise 2.17: Show that a normal matrix is Hermitian if and only if it has real eigenvalues.

An operator A is normal if it satisfies $AA^\dagger = A^\dagger A$ and is Hermitian if $A = A^\dagger$.

Exercise 2.18: Show that all eigenvalues of a unitary matrix have modulus 1, that is, can be written in the form $e^{i\theta}$ for some real θ.

Exercise 2.19: (Pauli matrices: Hermitian and unitary) Show that the Pauli matrices are Hermitian and unitary.

Recall the definition for unitarity is $U^\dagger U = UU^\dagger = I$ and for a Hermitian operator is $A = A^\dagger$. The Pauli matrices are given by

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{28}$$

Proceeding with basic matrix multiplication, we first validate the unitarity condition:

$$(\sigma_x^\dagger)(\sigma_x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{29}$$

$$(\sigma_y^\dagger)(\sigma_y) = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{30}$$

$$(\sigma_z^\dagger)(\sigma_z) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{31}$$

Now, verify that each of the Pauli matrices are Hermitian

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \equiv \sigma_x^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{32}$$

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \equiv \sigma_y^\dagger = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{33}$$

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \equiv \sigma_z^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{34}$$

Hence the Pauli matrices are both unitary and Hermitian.

Exercise 2.20: (Basis changes) Suppose A' and A'' are matrix representations of an operator A on a vector space V with respect to two different orthonormal bases, $|v_i\rangle$ and $|w_i\rangle$. Then the elements of A' and A'' are $A'_{ij} = \langle v_i| A |v_j\rangle$ and $A''_{ij} = \langle w_i| A |w_j\rangle$. Characterize the relationship between A' and A''.

2.1.7 Tensor products

Exercise 2.26: Let $|\psi\rangle = (|0\rangle + |1\rangle / \sqrt{2}$. Write out $|\psi\rangle^{\otimes 2}$ and $|\psi\rangle^{\otimes 3}$ explictly, both in terms of tensor products like $|0\rangle |1\rangle$, and using the Kronecker product.

Note that,

$$|\psi\rangle^{\otimes 2} = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) \quad (35)$$

$$|\psi\rangle^{\otimes 3} = \frac{1}{2\sqrt{2}}\left(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle\right). \quad (36)$$

Explicitly writing $|\psi\rangle^{\otimes 2}$ and $|\psi\rangle^{\otimes 3}$ out yields

$$|\psi\rangle^{\otimes 2} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$|\psi\rangle^{\otimes 3} = \frac{1}{2\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

.

Exercise 2.27: Calculate the matrix representation of the tensor products of the Pauli operators (a) X and Z; (b) I and X; (c) X and I. Is the tensor product commutative?

(a) $\underline{X \otimes Z}$:

$$X \otimes Z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

(b) $\underline{I \otimes X}$:

$$I \otimes X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(c) $\underline{X \otimes I}$:

$$X \otimes I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The tensor product is not commutative. For example, consider $X \otimes Z$ and $Z \otimes X$

$$X \otimes Z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & \text{-}1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{-}1 \\ 0 & 0 & \text{-}1 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{-}1 \\ 0 & 0 & \text{-}1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \text{-}1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = Z \otimes X$$

Exercise 2.28: Show that the transpose, complex conjugation, and adjoint operations distribute over the tensor product,

$$(A \otimes B)^* = A^* \otimes B^*;\ (A \otimes B)^T = A^T \otimes B^T;\ (A \otimes B)^\dagger = A^\dagger \otimes B^\dagger. \tag{37}$$

- $\underline{(A \otimes B)^* = A^* \otimes B^*}$:

Proof.

$$(A \otimes B)^* = \left(\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \otimes \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix} \right)^* \tag{38}$$

$$= \begin{bmatrix} a_{1,1} \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix} & a_{1,2} \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix} & \cdots & a_{1,n} \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix} \\ a_{2,1} \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix} & a_{2,2} \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix} & \cdots & a_{2,n} \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix} & a_{m,2} \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix} & \cdots & a_{m,n} \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix} \end{bmatrix}^* \tag{39}$$

$$
= \begin{bmatrix}
a_{1,1}^* \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix}^* & a_{1,2}^* \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix}^* & \cdots & a_{1,n}^* \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix}^* \\
a_{2,1}^* \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix}^* & a_{2,2}^* \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix}^* & \cdots & a_{2,n}^* \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix}^* \\
\vdots & \vdots & \ddots & \vdots \\
a_{m,1}^* \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix}^* & a_{m,2}^* \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix}^* & \cdots & a_{m,n}^* \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix}^*
\end{bmatrix} \tag{40}
$$

$$
= \begin{bmatrix} a_{1,1}^* & a_{1,2}^* & \cdots & a_{1,n}^* \\ a_{2,1}^* & a_{2,2}^* & \cdots & a_{2,n}^* \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}^* & a_{m,2}^* & \cdots & a_{m,n}^* \end{bmatrix} \otimes \begin{bmatrix} b_{1,1}^* & b_{1,2}^* & \cdots & b_{1,n}^* \\ b_{2,1}^* & b_{2,2}^* & \cdots & b_{2,n}^* \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1}^* & b_{m,2}^* & \cdots & b_{m,n}^* \end{bmatrix} = A^* \otimes B^* \tag{41}
$$

□

- $\underline{(A \otimes B)^T = A^T \otimes B^T}$

  Proof. □

- $\underline{(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger}$

  Proof. □

Exercise 2.29: Show that the tensor product of two unitary operators is unitary.

Proof. Recall an operator A is unitary if and only if $A^\dagger A = I$. Hence assuming A and B are two unitary operators,

$$
(A \otimes B)^\dagger (A \otimes B) = (A^\dagger \otimes B^\dagger)(A \otimes B) = (A^\dagger A) \otimes (B^\dagger B) = I \otimes I. \tag{42}
$$

Thus if A and B are unitary, so is their tensor product, $A \otimes B$. □

Exercise 2.30: Show that the tensor product of two Hermitian operators is Hermitian.

Proof. Recall an operator A is Hermitian if $A = A^\dagger$. Hence assuming A and B are two Hermitian operators,

$$(A \otimes B)^\dagger = (A^\dagger \otimes B^\dagger) = (A \otimes B) \tag{43}$$

Thus, given two Hermitian operators A and B, their tensor product $A \otimes B$ is Hermitian as well. $\square$

Exercise 2.31: Show that the tensor product of two positive operators is positive.

Exercise 2.32: Show that the tensor product of two projectors is projector.

Exercise 2.33: The Hadamard operator on one qubit may be written as

$$H = \frac{1}{\sqrt{2}} \left[(|0\rangle + |1\rangle) \langle 0| + (|0\rangle - |1\rangle) \langle 1| \right]. \tag{44}$$

Show explicitly that the Hadamard transform on n qubits, $H^{\otimes n}$, may be written as

$$H^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x,y} (-1)^{x \cdot y} |x\rangle \langle y| . \tag{45}$$

Write out an explicit matrix representation for $H^{\otimes 2}$.

The Hadamard operator on n-qubits as stated in the question can be written as

$$H^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x,y} (-1)^{x \cdot y} |x\rangle \langle y| . \tag{46}$$

Considering just a single qubit, we can write the above as

$$H = \frac{1}{\sqrt{2^n}} \sum_{x_1,y_1} (-1)^{x_1 \cdot y_1} |x_1\rangle \langle y_1| . \tag{47}$$

An explicit matrix representation for $H^{\otimes 2}$ can be written as follows

$$H^{\otimes 2} = H \otimes H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{48}$$

2.1.8 Operator functions

Exercise 2.34: Find the square root and logarithm of the matrix

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \tag{49}$$

Our first step is to find the spectral decomposition of A. Since A is an $N \times N$ square matrix with N linearly independent eigenvectors, (q_i with $i \in \{1, \dots N\}$), the spectral decomposition of A can be factorized as

$$A = Q\Lambda Q^{-1} \tag{50}$$

where Q is the square $N \times N$ matrix whose i^{th} column is the eigenvector q_i of A and Λ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues. Computing the eigenvalues and eigenvectors of A can be performed as follows

<u>Eigenvalues:</u>

First we construct the matrix $A - \lambda I$:

$$A - \lambda I \equiv \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 4-\lambda & 3 \\ 3 & 4-\lambda \end{bmatrix} \tag{51}$$

Next, we compute the determinant

$$\text{Det}\,|A - \lambda I| = \text{Det}\begin{vmatrix} 4-\lambda & 3 \\ 3 & 4-\lambda \end{vmatrix} = (4-\lambda)(4-\lambda) - (3)(3) = \lambda^2 - 8\lambda + 7$$

Setting the equation $\lambda^2 - 8\lambda + 7$, we solve for λ and obtain the eigenvalues $\{-1, 1\}$.

<u>Eigenvectors:</u>

Now we find the eigenvectors that correspond to each eigenvalue. We label them as $\{|\phi_1\rangle, |\phi_2\rangle\}$. For the first eigenvalue, which is $\lambda_1 = -1$, the eigenvalue equation is given by $A|\phi_1\rangle = |\phi_i\rangle$. Letting eigenvector $|\phi_1\rangle$ contain arbitrary components, we have

$$|\phi_1\rangle = \begin{bmatrix} a \\ b \end{bmatrix} \tag{52}$$

So we obtain

$$A|\phi_1\rangle = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = -1 \cdot \begin{bmatrix} a \\ b \end{bmatrix} \tag{53}$$

Carrying out matrix multiplication yields

$$4a + 3b = a \tag{54}$$
$$3a + 4b = b \tag{55}$$

Solving the above equation we get that $-a = b$, thus our eigenvector for λ_1 is $\begin{bmatrix} \text{-}1 \\ 1 \end{bmatrix}$

Square Root and Logarithm of A:

Since the real eigenvalues of A are $\{1,7\}$ and the eigenvectors of 1 is $\{-1,1\}$ and of 7 is $\{1,1\}$, then noting our decomposition formula of $A = Q\Lambda Q^{-1}$, we can write A as the following factorization.

$$A \equiv \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \tag{56}$$

Therefore, the square root of matrix A is

$$\sqrt{A} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{7} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \tag{57}$$

and the logarithm of matrix A is

$$\log(A) = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \log(1) & 0 \\ 0 & \log(7) \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \tag{58}$$

Exercise 2.35: (Exponential of the Pauli matrices) Let $\vec{v}$ be any real, three-dimensional unit vector and θ a real number. Prove that

$$exp(i\theta \vec{v} \cdot \vec{\sigma}) = \cos(\theta)I + i\sin(\theta) \vec{v} \cdot \vec{\sigma}, \tag{59}$$

where $\vec{v} \cdot \vec{\sigma} \equiv \sum_{i=1}^{3} v_i\sigma_i$. This exercise is generalized in Problem 2.1

Exercise 2.36: Show that the Pauli matrices except for I have trace zero.

The trace is simply defined as the sum of the diagonals of a matrix.

$$tr(\sigma_x) = 0 + 0 = 0 \tag{60}$$
$$tr(\sigma_y) = 0 + 0 = 0 \tag{61}$$
$$tr(\sigma_z) = 1 + \text{-}1 = 0 \tag{62}$$
$$tr(I) = 1 + 1 = 2 \tag{63}$$

Exercise 2.37: (Cyclic property of the trace) If A and B are two linear operators show that

$$tr(AB) = tr(BA). \tag{64}$$

Proof. $tr(AB)^T = tr(A^T B^T) = tr((BA)^T) = tr(BA)$. □

Exercise 2.38: (Linearity of the trace) If A and B are two linear operators, show that

$$tr(A+B) = tr(A) + tr(B) \tag{65}$$

and if z is an arbitrary complex number show that

$$tr(zA) = ztr(A). \tag{66}$$

Recall that trace of an operator A is defined as the sum of its diagonal elements.

$$tr(A) \equiv \sum_i A_{ii} \tag{67}$$

First show that

$$tr(A+B) = tr(A) + tr(B) \tag{68}$$

Proof.

$$tr(A+B) = \sum_i (A_{ii} + B_{ii}) = \sum_i (A_{ii}) + \sum_j (B_{ii}) = tr(A) + tr(B). \tag{69}$$

□

Next, it is shown that

$$tr(zA) = ztr(A). \tag{70}$$

Proof.

$$tr(zA) = \sum zA_{ii} = z\sum_i A_{ii} = ztr(A). \tag{71}$$

□

Exercise 2.39: (The Hilbert-Schmidt inner product on operators) The set L_V of linear operators on a Hilbert space V is obviously a vector space - the sum of two linear operators is a linear operator, zA is a linear operator if A is a linear operator and z is a complex number, and there is a zero element 0. An important additional result is that the vector space L_V can be given a natural inner product structure, turning it into a Hilbert space.

1. Show that the function $(\cdot,\cdot)$ on $L_V \times L_V$ defined by

$$(A, B) \equiv tr(A^\dagger B) \tag{72}$$

   is an inner product function. The inner product is known as the *Hilbert-Schmidt* or *trace* inner product.
2. If V has d dimensions show that L_V has dimension d^2.
3. Find an orthonormal basis of Hermitian matrices for the Hilbert space L_V.

The commutator and anti-commutator

Exercise 2.40: (Commutation relations for the Pauli matrices) Verify the commutation relations

$$[X, Y] = 2iZ; [Y, Z] = 2iX; [Z, X] = 2iY. \tag{73}$$

Recall that the commutator between two operators A and B is defined to be

$$[A, B] \equiv AB - BA. \tag{74}$$

We shall use this identity to verify the following claims.

- $[X, Y] = 2iZ$

$$[X, Y] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} = 2i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{75}$$

- $[Y, Z] = 2iX$

$$[Y, Z] = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} = 2i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{76}$$

- $[Z, X] = 2iY$

$$[Z, X] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = 2i \begin{bmatrix} 0 & -i \\ 0 & i \end{bmatrix} \tag{77}$$

Exercise 2.41: (Anti-commutation relations for Pauli matrices) Verify the anti-commutation relations

$$\{\sigma_i, \sigma_j\} = 0 \tag{78}$$

where $i \neq j$ are both chosen from the set 1,2,3. Also verify that (i = 0,1,2,3).

$$\sigma_i^2 = I. \tag{79}$$

Recall that the anti-commutator between two operators A and B is defined by

$$\{A, B\} = AB + BA. \tag{80}$$

Thus, verifiying each $\{\sigma_i, \sigma_j\} = 0$ for $i \neq j$ can be checked as follows.

- $\{\sigma_1, \sigma_2\} = 0$

$$\{\sigma_1, \sigma_2\} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0 \tag{81}$$

- $\{\sigma_1, \sigma_3\} = 0$

$$\{\sigma_1, \sigma_3\} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0 \tag{82}$$

- $\{\sigma_2, \sigma_1\} = 0$

$$\{\sigma_2, \sigma_1\} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = 0 \tag{83}$$

- $\{\sigma_2, \sigma_3\} = 0$

$$\{\sigma_2, \sigma_3\} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = 0 \tag{84}$$

- $\{\sigma_3, \sigma_1\} = 0$

$$\{\sigma_3, \sigma_1\} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 0 \tag{85}$$

- $\{\sigma_3, \sigma_2\} = 0$

$$\{\sigma_3, \sigma_2\} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 0 \tag{86}$$

Next, we need to verify that $\sigma_i^2 = I$ for $(i = 0{,}1{,}2{,}3)$

- $\sigma_0^2 = I$

$$\sigma_0^2 \equiv I^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{87}$$

- $\sigma_1^2 = I$

$$\sigma_1^2 \equiv X^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{88}$$

- $\sigma_2^2 = I$

$$\sigma_2^2 \equiv Y^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{89}$$

- $\sigma_3^2 = I$

$$\sigma_3^2 \equiv Z^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{90}$$

Exercise 2.42: Verify that

$$AB = \frac{[A,B] + \{A,B\}}{2}. \tag{91}$$

Proof.

$$AB = \frac{[A,B] + \{A,B\}}{2} = \frac{AB - BA + AB + BA}{2} = \frac{2AB}{2} = AB. \tag{92}$$

□

Exercise 2.43: Show that for $j,k = 1,2,3$,

$$\sigma_j \sigma_k = \delta_{jk} I + i \sum_{l=1}^{3} \epsilon_{jkl} \sigma_l. \tag{93}$$

Exercise 2.44: Suppose $[A,B] = 0, \{A,B\} = 0$, and A is invertible. Show that B must be 0.

Exercise 2.45: Show that $[A,B]^\dagger = [B^\dagger, A^\dagger]$.

Proof. The key to this proof is to realize that $(AB)^\dagger = B^\dagger A^\dagger$.

$$[B^\dagger, A^\dagger] = ((BA)^\dagger - (AB)^\dagger) = (A^\dagger B^\dagger - B^\dagger A^\dagger) = (AB - BA)^\dagger = [A,B]^\dagger \tag{94}$$

□

Exercise 2.46: Show that [A,B] = -[B,A].

Proof. A simple application of the definition of the commutator for two operators A and B is defined as $[A,B] \equiv AB - BA$.

$$-[B,A] = -(BA - AB) = (AB - BA) = [A,B]. \tag{95}$$

□

Exercise 2.47: Suppose A and B are Hermitian. Show that $i[A, B]$ is Hermitian.

Proof.

$$ \tag{96} $$

□

The polar and singular value decompositions

Exercise 2.48: What is the polar decomposition of a positive matrix P? Of a unitary matrix U? Or a Hermitian matrix, H?

Exercise 2.49: Express the polar decomposition of a normal matrix in the outer product representation.

Exercise 2.50: Find the left and right polar decompositions of the matrix

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{97}$$

The left polar decomposition of some arbitrary operator A is denoted by the expression $A = UJ$, where $J \equiv \sqrt{A^\dagger A}$. The right polar decomposition of some arbitrary operator A is denoted by the expression $A = KU$ where $K \equiv \sqrt{AA^\dagger}$.

First finding the left polar decomposition:

Evolution

Exercise 2.51: Verify that the Hadamard gate H is unitary

Recall that a unitary operator is defined as $U^\dagger U = UU^\dagger = I_n$. So we can check if $H^\dagger H = HH^\dagger = I$.

$$H^\dagger H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \tag{98}$$

Exercise 2.52: Verify that $H^2 = I$.

$$H^2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{99}$$

Exercise 2.53: What are the eigenvalues and eigenvectors of H?

$$H \equiv \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \tag{100}$$

<u>Eigenvalues:</u>

We construct the matrix $H - \lambda I$:

$$H - \lambda I \equiv \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} - \lambda & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} - \lambda \end{bmatrix} \tag{101}$$

Next, we compute the determinant

$$\text{Det}\,|H - \lambda I| = \text{Det} \begin{vmatrix} \frac{1}{\sqrt{2}} - \lambda & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} - \lambda \end{vmatrix} = \left(\frac{1}{\sqrt{2}} - \lambda\right)\left(-\frac{1}{\sqrt{2}} - \lambda\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \lambda^2 - 1 = 0$$

Setting the equation $\lambda^2 - 1 = 0$, we solve for λ and obtain the eigenvalues $\{-1, 1\}$.

<u>Eigenvectors:</u>

Exercise 2.54: Suppose A and B are commuting Hermitian operators. Prove that exp(A)exp(B) = exp(A+B). (Hint: Use the results of Section 2.1.9.)

Exercise 2.55: Prove that $U(t_1, t_2)$ defined as

$$U(t_1, t_2) \equiv \exp\left[\frac{-iH(t_2 - t_1)}{\hbar}\right] \tag{102}$$

is unitary.

2.2.3 Quantum measurement

Exercise 2.56: Use the spectral decomposition to show that $K \equiv -i \log(U)$ is Hermitian for any unitary U, and thus $U = \exp(iK)$ for some Hermitian K.

Exercise 2.57: (Cascaded measurements are single measurements) Suppose $\{L_l\}$ and $\{M_m\}$ are two sets of measurement operators. Show that a measurement defined by the measurement operators $\{L_l\}$ following by a measurement defined by the measurement operators $\{M_m\}$ is physically equivalent to a single measurement defined by measurement operators $\{N_{lm}\}$ with representation $N_{lm} \equiv M_m L_l$.

2.2.5 Projective measurements

Exercise 2.58: Suppose we prepare a quantum system in an eigenstate $|\psi\rangle$ of some observable M, with corresponding eigenvalue m. What is the average observed value of M, and the standard deviation?

The average observed value of M is defined as

$$m = \langle\psi|\, M \,|\psi\rangle \tag{103}$$

where m are the eigenvalues that correspond to the measurement outcome of the observable. The formula for the standard deviation associated to observations of M is given as

$$\begin{aligned} [\triangle(M)]^2 &= \langle (M - \langle M \rangle)^2 \rangle & (104) \\ &= \langle M^2 \rangle - \langle M \rangle^2. & (105) \end{aligned}$$

In the scenario of our problem, the standard deviation will yield zero.

Exercise 2.59: Suppose we have a qubit in the state $|0\rangle$, and we measure the observable X. What is the average value of X? What is the standard deviation of X?

Exercise 2.60: Show that $\vec{v} \cdot \vec{\sigma}$ has eigenvalues ± 1, and that the projectors onto the corresponding eigenspaces are given by $P_{\pm} = (I \pm \vec{v} \cdot \vec{\sigma})/2$.

Exercise 2.61: Calculate the probability of obtaining the result $+1$ for a measuring of $\vec{v} \cdot \vec{\sigma}$, given that the state prior to measurement is $|0\rangle$. What is the state of the system after the measurement if $+1$ is obtained?

2.2.6 POVM measurements

Exercise 2.62: Show that any measurement where the measurement operators and the POVM elements coincide is a projective measurement.

Exercise 2.63: Suppose a measurement is described by measurement operators M_m. Show that there exist unitary operators U_m such that $M_m = U_m\sqrt{E_m}$, where E_m is the POVM associated to the measurement.

Exercise 2.64: Suppose Bob is given a quantum state chose from a set $|\psi_1\rangle, \ldots |\psi_m\rangle$ of linearly independent states. Construct a POVM $\{E_1, E_2, \ldots, E_{m+1}\}$ such that if outcome E_i occurs, $1 \le i \le m$, then Bob knows with certainty that he was given the state $|\psi_i\rangle$ (The POVM must be such that $\langle\psi_i|E_i|\psi_i\rangle > 0$ for each i.)

2.2.8 Composite systems

Exercise 2.65: Express the states $(|0\rangle + |1\rangle)\sqrt{2}$ and $(|0\rangle - |1\rangle)/\sqrt{2}$ in a basis in which they are *not* the same up to a relative phase shift.

Exercise 2.66: Show that the average value of the observable $X_1 Z_2$ for a two qubit system measured in the state $(|00\rangle + |11\rangle)/\sqrt{2}$ is zero.

Exercise 2.67: Suppose V is a Hilbert space with a subspace W. Suppose $U : W \to V$ is a linear operator which preserves inner products, that is, for any $|w_1\rangle$ and $|w_2\rangle$ in W,

$$\langle w_1| U^{\dagger} U |w_2\rangle = \langle w_1|w_2\rangle. \tag{106}$$

Prove that there exists a unitary operator $U' : V \to V$ which *extends* U. That is, $U' |w\rangle] = U |w\rangle$ for all $|w\rangle$ in W, but U' is defined on the entire space V. Usually we omit the prime symbol $'$ and just write U to denote the extension.

Exercise 2.68: Prove that $|\psi\rangle \neq |a\rangle |b\rangle$ for all single qubit states $|a\rangle$ and $|b\rangle$.

2.3 Application: superdense coding

Exercise 2.69: Verify that the Bell basis forms an orthonormal basis for the two qubit state space.

Recall the Bell basis for the two-qubit state space is defined as

$$|\psi_{00}\rangle \to \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{107}$$

$$|\psi_{01}\rangle \to \frac{|00\rangle - |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \tag{108}$$

$$|\psi_{10}\rangle \to \frac{|10\rangle + |01\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \tag{109}$$

$$|\psi_{11}\rangle \rightarrow \frac{|01\rangle - |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ \text{-1} \\ 0 \end{bmatrix} \tag{110}$$

Also recall that a set of arbitrary vectors, A and B, are orthonormal is they are both orthogonal ($\langle A|B\rangle = 0$) and normal ($A^\dagger A = AA^\dagger$ and $B^\dagger B = BB^\dagger$). Thus, checking first for orthogonality among the Bell basis vectors

$$\langle\psi|\psi\rangle = \tag{111}$$

Finally, checking for normality

$$|\psi_{00}\rangle^\dagger |\psi_{00}\rangle = \begin{bmatrix} 0 \\ 1 \\ \text{-1} \\ 0 \end{bmatrix} \tag{112}$$

Exercise 2.70: Suppose E is any positive operator acting on Alice's qubit. Show that $\langle\psi| E \otimes I |\psi\rangle$ *takes the same value* when $|\psi\rangle$ is any of the four Bell states. Suppose some malevolent third party ('Eve') intercepts Alice's qubit on the way to Bob in the superdense coding protocol. Can Eve infer anything about which of the four possible bit string 00, 01, 10, 11 Alice is trying to send? If so, how, or if not, why not?

2.4.2 General properties of the density operator

Exercise 2.71: (Criterion to decide if a state is mixed or pure) Let ρ be a density operator. Show that $tr(\rho^2) \leq 1$, with equality if and only if ρ is a pure state.

Proof. A pure state is defined as $\rho = |\psi\rangle\langle\psi|$. From this definition, it can be seen that $\rho^2 = \rho$.

Proof. $\rho^2 = (|\psi\rangle\langle\psi|)(|\psi\rangle\langle\psi|) = |\psi\rangle(\langle\psi|\psi\rangle)\langle\psi| = |\psi\rangle\langle\psi| = \rho$ □

Furthermore, since our state $\rho = \sum_i |\psi_i\rangle\langle\psi_i|$ is our density operator, then

$$tr(\rho) = \sum_i p_i tr(|\psi_i\rangle\langle\psi_i|) = \sum_i p_i = 1. \tag{113}$$

So since $\rho^2 = \rho$, we note that the probabilities in the above equation have $p_i^2 = p_i$ for all i, indicating that $\sum_i p_i^2 = 1$. Hence, $tr(\rho^2) = 1$ if and only if ρ is a pure state. □

Exercise 2.72: (Bloch sphere for mixed states) The Bloch sphere picture for pure states of a single qubit was introduced in Section 1.2. This description has an important generalization to mixed states as follows

1. Show that an arbitrary density matrix for a mixed state qubit may be written as

$$\rho = \frac{I + \vec{r} \cdot \vec{\sigma}|}{2}, \tag{114}$$

   where $\vec{r}$ is a real three-dimensional vector such that $\left\| \vec{r} \right\| \leq 1$. This vector is known as the *Bloch vector* for the state ρ.

2. What is the Bloch vector representation for the state $\rho = I/2$?

3. Show that a state ρ is pure if and only if $\left\| \vec{r} \right\| = 1$.

4. Show that for pure states the description of the Bloch vector we have given coincides with that in Section 1.2.

Exercise 2.73: Let ρ be a density operator. A *minimal ensemble* for ρ is an ensemble $\{p_i, |\psi_i\rangle\}$ containing a number of elements equal to the rank of ρ. Let $|\psi\rangle$ be any state in the support of ρ. (The *support* of a Hermitian operator A is the vector space spanned by the eigenvectors of A with non-zero eigenvalues.) Show that there is a minimal ensemble for ρ that contains $|\psi\rangle$, and moreover that in any such ensemble $|\psi\rangle$ must appear with probability

$$p_i = \frac{1}{\langle \psi_i | \rho^{-1} | \psi_i \rangle}, \tag{115}$$

where ρ^{-1} is defined the be the inverse of ρ, when ρ is considered as an operator acting only on the support of ρ. (This definition removes the problem that ρ may not have an inverse.)

The reduced density operator

Exercise 2.74: Suppose a composite of systems A and B is in the state $|a\rangle |b\rangle$, where $|a\rangle$ is a pure state of system A, and $|b\rangle$ is a pure state of system B. Show that the reduced density operator of system A alone is a pure state.

Exercise 2.75: For each of the four Bell states, find the reduced density operator for each qubit.

2.5 The Schmidt decomposition and purifications

Exercise 2.76: Extend the proof of the Schmidt decomposition to the case where A and B may have state spaces of different dimensionality.

Exercise 2.77: Suppose ABC is a three component quantum system. Show by example that there are quantum states $|\psi\rangle$ of such systems which can not be written in the form

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle |i_C\rangle , \tag{116}$$

where λ_i are real numbers, and $|i_A\rangle , |i_B\rangle , |i_C\rangle$ are orthonormal bases of the respective systems.

Exercise 2.78: Prove that a state $|\psi\rangle$ of a composite system AB is a product state if and only if it has Schmidt number 1. Prove that $|\psi\rangle$ is a product state if and only if ρ^A (and thus ρ^B) are pure states.

Exercise 2.79: Consider a composite system consisting of two qubits. Find the Schmidt decomposition of the states

$$\frac{|00\rangle + |11\rangle}{2}; \quad \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}; \quad \text{and} \quad \frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}}. \tag{117}$$

Exercise 2.80: Suppose $|\psi\rangle$ and $|\varphi\rangle$ are two pure states of a composite quantum system with components A and B, with identical Schmidt coefficients. Show that there are unitary transformations U on system A and V on system B such that $|\psi\rangle = (U \otimes V) |\varphi\rangle$.

Exercise 2.81: (Freedom in purifications) Let $|AR_1\rangle$ and $|AR_2\rangle$ be two purifications of a state ρ^A to a composite system AR. Prove that there exists a unitary transformation U_R acting on system R such that $|AR_1\rangle = (I_A \otimes U_R) |AR_2\rangle$.

Exercise 2.82: Suppose $\{p_i, |\psi_i\rangle\}$ is an ensemble of states generating a density matrix $\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|$ for a quantum system A. Introduce a system R with orthonormal basis $|i\rangle$.

1. Show that $\sum_i \sqrt{p_i} |\psi_i\rangle |i\rangle$ is a purification of ρ.
2. Suppose we measure R is the basis $|i\rangle$, obtaining outcome i. With what probability do we obtain the result i, and what is the corresponding post-measurement state for system A is $|\psi_i\rangle$ with probability p_i.