Wednesday, May 02, 2012
2:38 PM

Note Title 4/30/2012

Orthogonal matrices

$A^T A = I \quad \leftarrow \quad A$ is an orthogonal matrix

A is the matrix associated with an orthogonal transformation

We care about these, because rotations and reflections are orthogonal transformations.

$$A^T A = I$$

Properties:

① $A^T = A^{-1}$

② $\det A = \pm 1$

<u>Proof</u> if $C = AB$, then $\det C = \det A * \det B$

$\therefore (\det A^T) * \det A = 1$

$(\det A^T) * \det A = 1$

We know that $\det A^T = \det A$

$\therefore \quad (\det A)^2 = 1 \quad \Rightarrow \quad \det A = +1 \leftarrow$ rotation

or

$\det A = -1 \leftarrow$ reflection

(3) Columns of A are orthonormal

$A = \begin{bmatrix} \underset{\sim}{a}_1 & \underset{\sim}{a}_2 \end{bmatrix}$

$A^T = \begin{bmatrix} \underset{\sim}{a}_1^T \\ a_2^T \end{bmatrix}$

and $A^T A = I$

$\therefore \; a_1^T a_1 = 1$

$a_1^T a_2 = 0$

$a_2^T a_2 = 1$

for a 2×2 A

$\begin{bmatrix} a_1^T a_1 & a_1^T a_2 \\ a_2^T a_1 & a_2^T a_2 \end{bmatrix}$

$A = \begin{bmatrix} \underset{\sim}{a_1} & \underset{\sim}{a_2} \end{bmatrix}$ we showed that $a_1^T a_1 = 1$

$a_1^T a_2 = 0$

$a_2^T a_2 = 1$

$\|a_1\| = \|a_2\| = 1$

$a_1^T a_2 = 0$

a_1, a_2 are vectors of length 1 which are mutually perpendicular

In $\mathbb{R}^2$ $\underset{\sim}{a_1} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ $\underset{\sim}{a_2} = ?$ $\|a_2\| = 1$, $a_1^T a_2 = 0$ $\underset{\sim}{a_2} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

perpendicular or $\begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$ one choice $\det = +1$

or

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$ other choice $\det = -1$

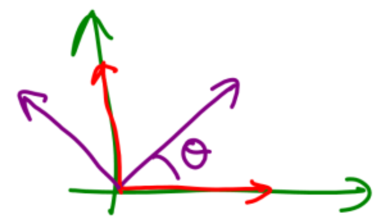
Suppose we have a vector $\begin{bmatrix} x \\ y \end{bmatrix}$, consider $A \begin{bmatrix} x \\ y \end{bmatrix}$, i.e.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

counterclockwise rotation by θ.

Where does the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ get transformed?

$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$



counterclockwise rotation by θ

In 2D we have just two kinds of orthogonal matrices

$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ or $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

det $+1$ det -1

ROTATION REFLECTION

<u>Verify</u> The reflection matrix corresp. to reflection around the line with angle $\theta/2$

In 3D.

Rotation $\begin{bmatrix} \underset{\sim}{a_1} & \underset{\sim}{a_2} & \underset{\sim}{a_3} \end{bmatrix}$

From matrix

$\left.\begin{array}{ll} \cdot\ \underset{\sim}{a_1} & 2 \text{ parameters} \\ \cdot\ \underset{\sim}{a_2} & 1 \text{ parameter} \\ \cdot\ \underset{\sim}{a_3} & 0 \text{ parameter} \end{array}\right\} 3 \text{ parameters}$

By similar reasoning as in the 2D case

$$\|a_1\| = \|a_2\| = \|a_3\| = 1$$

$$a_1^T a_2 = 0 \qquad a_1^T a_3 = 0 \qquad a_2^T a_3 = 0$$

We need three parameters to specify a rotation in 3D

$\rightarrow$ Intuitively rotation in 3D $\rightarrow$ axis (2)

$\rightarrow$ amount of rotation (1)

$2 + 1 = 3$ parameters