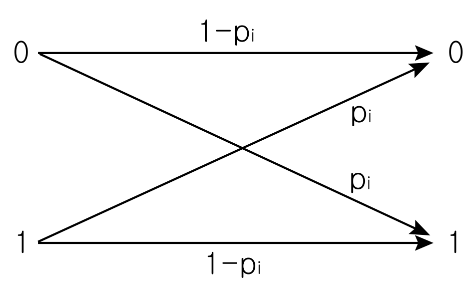
Homework 4
(Due Date: Nov. 13, 2007)

1. Consider a binary symmetric channel with $Y_i = X_i \oplus Z_i$, where $\oplus$ is mod 2 addition, and $X_i, Y_i \in \{0, 1\}$.
   Suppose that $\{Z_i\}$ has constant marginal probabilities $\Pr(Z_i = 1) = p$ and $\Pr(Z_i = 0) = 1 - p$, but that $Z_1, Z_2, \cdots, Z_n$ are not necessarily independent. Let $C = 1 - H(p)$. Show that
   $$\max_{p(x_1, x_2, \cdots, x_n)} I(X_1, X_2, \cdots, X_n; Y_1, Y_2, \cdots, Y_n) \geq nC$$
   Comment on the implications.

2. Consider the channel $Y = X + Z$ (mod 13), where
   $$Z = \begin{cases} 1, & \text{with probability } \frac{1}{3} \\ 2, & \text{with probability } \frac{1}{3} \\ 3, & \text{with probability } \frac{1}{3} \end{cases}$$
   and $X \in \{0, 1, \cdots, 12\}$.
   (a) Find the capacity.
   (b) What is the maximizing $p^*(x)$?

3. Using two channels.
   (a) Consider two discrete memoryless channels $(\mathcal{X}_1, p(y_1|x_1), \mathcal{Y}_1)$ and $(\mathcal{X}_2, p(y_2|x_2), \mathcal{Y}_2)$ with capacities C_1 and C_2 respectively. A new channel $(\mathcal{X}_1 \times \mathcal{X}_2, p(y_1|x_1) \times p(y_2|x_2), \mathcal{Y}_1 \times \mathcal{Y}_2)$ is formed in which $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$, are *simultaneously* sent, resulting in y_1, y_2. Find the capacity of this channel.
   (b) Find the capacity C of the *union* 2 channels $(\mathcal{X}_1, p(y_1|x_1), \mathcal{Y}_1)$ and $(\mathcal{X}_2, p(y_2|x_2), \mathcal{Y}_2)$ where, at each time, one can send a symbol over channel 1 or channel 2 but not both. Assume the output alphabets are distinct and do not intersect. Show $2^C = 2^{C_1} + 2^{C_2}$.

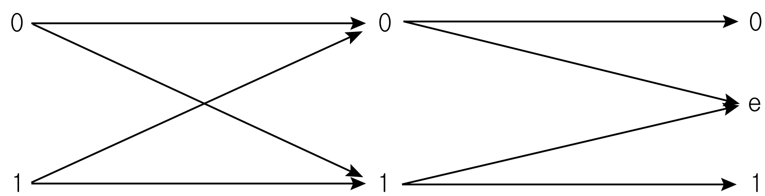
4. Consider a time-varying discrete *memoryless* binary symmetric channel. Let $Y_1, Y_2, \cdots, Y_n$ be conditionally independent given $X_1, X_2, \cdots, X_n$, with conditional distribution given by $p(y^n|x^n) = \prod_{i=1}^n p_i(y_i|x_i)$, as shown below.



   (a) Find $\max_{p(x)} I(X^n; Y^n)$.

(b) We now ask for the capacity for the time invariant version of this problem. Replace each p_i, $1 \le i \le n$, by the average value $\bar{p} = \frac{1}{n}\sum_{j=1}^{n} p_j$, and compare the capacity to part (a).

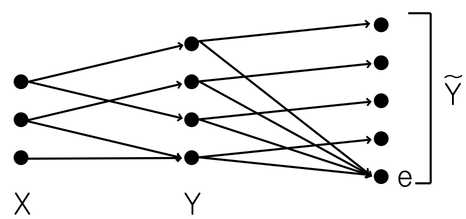
5. Suppose a binary symmetric channel of capacity C_1 is immediately followed by a binary erasure channel of capacity C_2. Find capacity C of the resulting channel.



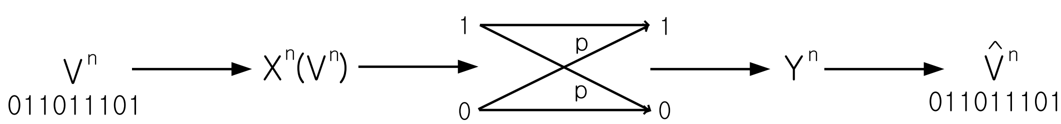
Now consider an arbitrary discrete memoryless channel $(\mathcal{X}, p(y|x), \mathcal{Y})$ followed by a binary erasure channel , resulting in an output

$$\tilde{Y} = \begin{cases} Y, & \text{with probability } 1-\alpha \\ e, & \text{with probability } \alpha \end{cases}$$

where e denotes erasure. Thus the output $\mathcal{Y}$ is erased with probability α What is the capacity of this channel?



6. We wish to encode a Bernoulli(α) process $V_1, V_2, \cdots$ for transmission over a binary symmetric channel with error probability p.



Find conditions on α and p so that the probability of error $p(\hat{V}^n \neq V^n)$ can be made to go to zero as $n \to \infty$.

7. Let (X_i, Y_i, Z_i) be i.i.d. according to $p(x, y, z)$. We will say that (x^n, y^n, z^n) is jointly typical [written $(x^n, y^n, z^n) \in A_\epsilon^{(n)}$] if

- $2^{-n(H(X)+\epsilon)} \le p(x^n) \le 2^{-n(H(X)-\epsilon)}$
- $2^{-n(H(Y)+\epsilon)} \le p(y^n) \le 2^{-n(H(Y)-\epsilon)}$
- $2^{-n(H(Z)+\epsilon)} \le p(z^n) \le 2^{-n(H(Z)-\epsilon)}$
- $2^{-n(H(X,Y)+\epsilon)} \le p(x^n, y^n) \le 2^{-n(H(X,Y)-\epsilon)}$
- $2^{-n(H(X,Z)+\epsilon)} \le p(x^n, z^n) \le 2^{-n(H(X,Z)-\epsilon)}$
- $2^{-n(H(Y,Z)+\epsilon)} \le p(y^n, z^n) \le 2^{-n(H(Y,Z)-\epsilon)}$
- $2^{-n(H(X,Y,Z)+\epsilon)} \le p(x^n, y^n, z^n) \le 2^{-n(H(X,Y,Z)-\epsilon)}$

Now suppose that $(\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n)$ is drawn according to $p(x^n)p(y^n)p(z^n)$. Thus, $\tilde{X}^n$, $\tilde{Y}^n$, $\tilde{Z}^n$ have the same marginals as $p(x^n, y^n, z^n)$ but are independent. Find (bounds on) $\Pr\{(\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n) \in A_\epsilon^{(n)}\}$ in terms of the entropies $H(X)$, $H(Y)$, $H(Z)$, $H(X,Y)$, $H(X,Z)$, $H(Y,Z)$ and $H(X,Y,Z)$.

8. Twenty questions.

   (a) Player A chooses some object in the universe, and player B attempts to identify the object with a series of yes-no questions. Suppose that player B is clever enough to use the code achieving the minimal expected length with respect to player A's distribution. We observe that player B requires an average 38.5 questions to determine the object. Find a rough lower bound to the number of objects in the universe.

   (b) Let X be uniformly distributed over $\{1, 2, \cdots, m\}$. Assume that $m = 2^n$. We ask random questions: Is $X \in S_1$? Is $X \in S_2$? $\cdots$ until only one integer remains. All 2^m subsets S of $\{1, 2, \cdots m\}$ are equally likely.

      i. How many deterministic questions are needed to determine X?

      ii. Without loss of generality, suppose that $X = 1$ is the random object. What is the probability that object 2 yields the same answers as object 1 for k questions?

      iii. What is the expected number of objects in $\{2, 3, \cdots, m\}$ that have the same answers to the questions as those of the correct object 1?