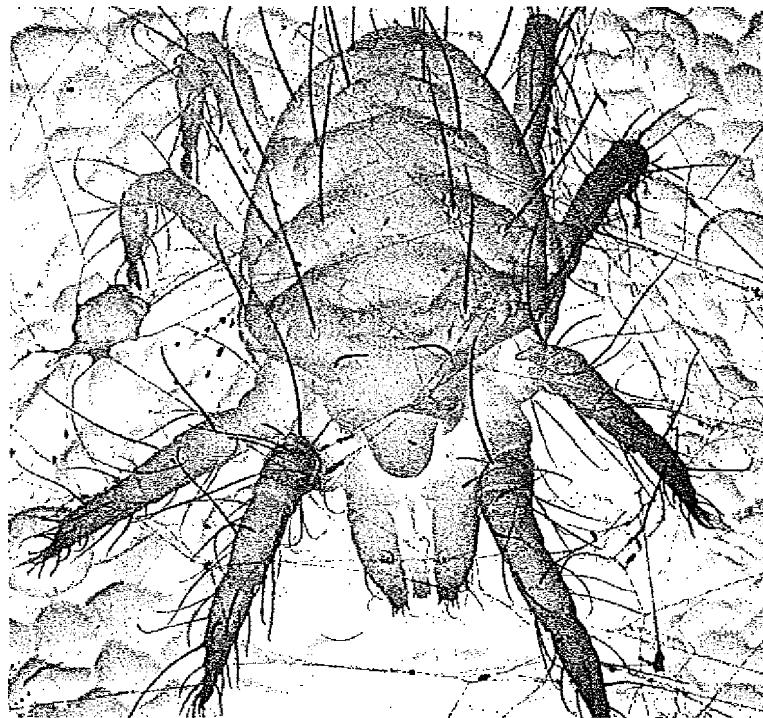


## CHAPTER 3

# Wave Properties of Particles



In a scanning electron microscope, an electron beam that scans a specimen causes secondary electrons to be ejected in numbers that vary with the angle of the surface. A suitable data display suggests the three-dimensional form of the specimen. The high resolution of this image of a red spider mite on a leaf is a consequence of the wave nature of moving electrons.

### 3.1 DE BROGLIE WAVES

A moving body behaves in certain ways as though it has a wave nature

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### 3.8 UNCERTAINTY PRINCIPLE II

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### 3.9 APPLYING THE UNCERTAINTY PRINCIPLE

A useful tool, not just a negative statement

**L**ooking back, it may seem odd that two decades passed between the 1905 discovery of the particle properties of waves and the 1924 speculation that particles might show wave behavior. It is one thing, however, to suggest a revolutionary concept to explain otherwise mysterious data and quite another to suggest an equally revolutionary concept without a strong experimental mandate. The latter is just what Louis de Broglie did in 1924 when he proposed that moving objects have wave as well as particle characteristics. So different was the scientific climate at the time from that around the turn of the century that de Broglie's ideas soon received respectful attention, whereas the earlier quantum theory of light of Planck and Einstein had been largely ignored despite its striking empirical support. The existence of de Broglie waves was experimentally demonstrated by 1927, and the duality principle they represent provided the starting point for Schrödinger's successful development of quantum mechanics in the previous year.

### 3.1 DE BROGLIE WAVES

*A moving body behaves in certain ways as though it has a wave nature*

A photon of light of frequency  $\nu$  has the momentum

$$p = \frac{h\nu}{c} = \frac{h}{\lambda}$$

since  $\lambda\nu = c$ . The wavelength of a photon is therefore specified by its momentum according to the relation

Photon wavelength  $\lambda = \frac{h}{p}$  (3.1)

De Broglie suggested that Eq. (3.1) is a completely general one that applies to material particles as well as to photons. The momentum of a particle of mass  $m$  and velocity  $v$  is  $p = \gamma mv$ , and its de Broglie wavelength is accordingly

De Broglie wavelength  $\lambda = \frac{h}{\gamma mv}$  (3.2)



Louis de Broglie (1892–1987), although coming from a French family long identified with diplomacy and the military and initially a student of history, eventually followed his older brother Maurice in a career in physics. His doctoral thesis in 1924 contained the proposal that moving bodies have wave properties that complement their particle properties: these "seemingly incompatible conceptions can each represent an

aspect of the truth. . . . They may serve in turn to represent the facts without ever entering into direct conflict." Part of de Broglie's inspiration came from Bohr's theory of the hydrogen atom, in which the electron is supposed to follow only certain orbits around the nucleus. "This fact suggested to me the idea that electrons . . . could not be considered simply as particles but that periodicity must be assigned to them also." Two years later Erwin Schrödinger used the concept of de Broglie waves to develop a general theory that he and others applied to explain a wide variety of atomic phenomena. The existence of de Broglie waves was confirmed in diffraction experiments with electron beams in 1927, and in 1929 de Broglie received the Nobel Prize.

The greater the particle's momentum, the shorter its wavelength. In Eq. (3.2)  $\gamma$  is the relativistic factor

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

As in the case of em waves, the wave and particle aspects of moving bodies can never be observed at the same time. We therefore cannot ask which is the "correct" description. All that can be said is that in certain situations a moving body resembles a wave and in others it resembles a particle. Which set of properties is most conspicuous depends on how its de Broglie wavelength compares with its dimensions and the dimensions of whatever it interacts with.

### Example 3.1

Find the de Broglie wavelengths of (a) a 46-g golf ball with a velocity of 30 m/s, and (b) an electron with a velocity of  $10^7$  m/s.

#### Solution

(a) Since  $v \ll c$ , we can let  $\gamma = 1$ . Hence

$$\lambda = \frac{h}{mv} = \frac{6.63 \times 10^{-34} \text{ J} \cdot \text{s}}{(0.046 \text{ kg})(30 \text{ m/s})} = 4.8 \times 10^{-34} \text{ m}$$

The wavelength of the golf ball is so small compared with its dimensions that we would not expect to find any wave aspects in its behavior.

(b) Again  $v \ll c$ , so with  $m = 9.1 \times 10^{-31}$  kg, we have

$$\lambda = \frac{h}{mv} = \frac{6.63 \times 10^{-34} \text{ J} \cdot \text{s}}{(9.1 \times 10^{-31} \text{ kg})(10^7 \text{ m/s})} = 7.3 \times 10^{-11} \text{ m}$$

The dimensions of atoms are comparable with this figure—the radius of the hydrogen atom, for instance, is  $5.3 \times 10^{-11}$  m. It is therefore not surprising that the wave character of moving electrons is the key to understanding atomic structure and behavior.

### Example 3.2

Find the kinetic energy of a proton whose de Broglie wavelength is 1.000 fm =  $1.000 \times 10^{-15}$  m, which is roughly the proton diameter.

#### Solution

A relativistic calculation is needed unless  $pc$  for the proton is much smaller than the proton rest energy of  $E_0 = 0.938$  GeV. To find out, we use Eq. (3.2) to determine  $pc$ :

$$pc = (\gamma mv)c = \frac{hc}{\lambda} = \frac{(4.136 \times 10^{-15} \text{ eV} \cdot \text{s})(2.998 \times 10^8 \text{ m/s})}{1.000 \times 10^{-15} \text{ m}} = 1.240 \times 10^9 \text{ eV} \\ = 1.2410 \text{ GeV}$$

Since  $pc > E_0$  a relativistic calculation is required. From Eq. (1.24) the total energy of the proton is

$$E = \sqrt{E_0^2 + p^2c^2} = \sqrt{(0.938 \text{ GeV})^2 + (1.2340 \text{ GeV})^2} = 1.555 \text{ GeV}$$

The corresponding kinetic energy is

$$KE = E - E_0 = (1.555 - 0.938) \text{ GeV} = 0.617 \text{ GeV} = 617 \text{ MeV}$$

De Broglie had no direct experimental evidence to support his conjecture. However, he was able to show that it accounted in a natural way for the energy quantization—the restriction to certain specific energy values—that Bohr had had to postulate in his 1913 model of the hydrogen atom. (This model is discussed in Chap. 4.) Within a few years Eq. (3.2) was verified by experiments involving the diffraction of electrons by crystals. Before we consider one of these experiments, let us look into the question of what kind of wave phenomenon is involved in the matter waves of de Broglie.

## 3.2 WAVES OF WHAT?

### *Waves of probability*

In water waves, the quantity that varies periodically is the height of the water surface. In sound waves, it is pressure. In light waves, electric and magnetic fields vary. What is it that varies in the case of matter waves?

The quantity whose variations make up matter waves is called the wave function, symbol  $\Psi$  (the Greek letter psi). The value of the wave function associated with a moving body at the particular point  $x, y, z$  in space at the time  $t$  is related to the likelihood of finding the body there at the time.



Max Born (1882–1970) grew up in Breslau, then a German city but today part of Poland, and received a doctorate in applied mathematics at Göttingen in 1907. Soon afterward he decided to concentrate on physics, and was back in Göttingen in 1909 as a lecturer. There he worked on various aspects of the theory of crystal lattices, his "central interest" to which he often returned in later years. In 1915, at

Planck's recommendation, Born became professor of physics in Berlin where, among his other activities, he played piano to Einstein's violin. After army service in World War I and a period at Frankfurt University, Born was again in Göttingen, now as professor of physics. There a remarkable center of theoretical physics developed under his leadership: Heisenberg and Pauli were among his assistants and Fermi, Dirac, Wigner, and Goeppert were among those who worked with him, just to name future Nobel Prize winners. In those days, Born wrote, "There was complete freedom of teaching and learning in German universities, with no class examinations, and no control of students. The University just offered lectures and the student had to decide for himself which he wished to attend."

Born was a pioneer in going from "the bright realm of classical physics into the still dark and unexplored underworld of the new quantum mechanics," he was the first to use the latter term. From Born came the basic concept that the wave function  $\Psi$  of a particle is related to the probability of finding it. He began with an idea of Einstein, who "sought to make the duality of particles (light quanta or photons) and waves comprehensible by interpreting the square of the optical wave amplitude as probability density for the occurrence of photons. This idea could at once be extended to the  $\Psi$ -function:  $|\Psi|^2$  must represent the probability density for electrons (or other particles). To assert this was easy; but how was it to be proved? For this purpose atomic scattering processes suggested themselves." Born's development of the quantum theory of atomic scattering (collisions of atoms with various particles) not only verified his "new way of thinking about the phenomena of nature" but also founded an important branch of theoretical physics.

Born left Germany in 1933 at the start of the Nazi period, like so many other scientists. He became a British subject and was associated with Cambridge and then Edinburgh universities until he retired in 1953. Finding the Scottish climate harsh and wishing to contribute to the democratization of postwar Germany, Born spent the rest of his life in Bad Pyrmont, a town near Göttingen. His textbooks on modern physics and on optics were standard works on these subjects for many years.

The wave function  $\Psi$  itself, however, has no direct physical significance. There is a simple reason why  $\Psi$  cannot be interpreted in terms of an experiment. The probability that something be in a certain place at a given time must lie between 0 (the object is definitely not there) and 1 (the object is definitely there). An intermediate probability, say 0.2, means that there is a 20% chance of finding the object. But the amplitude of a wave can be negative as well as positive, and a negative probability, say -0.2, is meaningless. Hence  $\Psi$  by itself cannot be an observable quantity.

This objection does not apply to  $|\Psi|^2$ , the square of the absolute value of the wave function, which is known as **probability density**:

The probability of experimentally finding the body described by the wave function  $\Psi$  at the point  $x, y, z$ , at the time  $t$  is proportional to the value of  $|\Psi|^2$  there at  $t$ .

A large value of  $|\Psi|^2$  means the strong possibility of the body's presence, while a small value of  $|\Psi|^2$  means the slight possibility of its presence. As long as  $|\Psi|^2$  is not actually 0 somewhere, however, there is a definite chance, however small, of detecting it there. This interpretation was first made by Max Born in 1926.

There is a big difference between the probability of an event and the event itself. Although we can speak of the wave function  $\Psi$  that describes a particle as being spread out in space, this does not mean that the particle itself is thus spread out. When an experiment is performed to detect electrons, for instance, a whole electron is either found at a certain time and place or it is not; there is no such thing as a 20 percent of an electron. However, it is entirely possible for there to be a 20 percent chance that the electron be found at that time and place, and it is this likelihood that is specified by  $|\Psi|^2$ .

W. L. Bragg, the pioneer in x-ray diffraction, gave this loose but vivid interpretation: "The dividing line between the wave and particle nature of matter and radiation is the moment 'now.' As this moment steadily advances through time it coagulates a wavy future into a particle past. . . . Everything in the future is a wave, everything in the past is a particle." If "the moment 'now'" is understood to be the time a measurement is performed, this is a reasonable way to think about the situation. (The philosopher Søren Kierkegaard may have been anticipating this aspect of modern physics when he wrote, "Life can only be understood backwards, but it must be lived forwards.")

Alternatively, if an experiment involves a great many identical objects all described by the same wave function  $\Psi$ , the *actual density* (number per unit volume) of objects at  $x, y, z$  at the time  $t$  is proportional to the corresponding value of  $|\Psi|^2$ . It is instructive to compare the connection between  $\Psi$  and the density of particles it describes with the connection discussed in Sec. 2.4 between the electric field  $E$  of an electromagnetic wave and the density  $N$  of photons associated with the wave.

While the wavelength of the de Broglie waves associated with a moving body is given by the simple formula  $\lambda = h/\gamma mv$ , to find their amplitude  $\Psi$  as a function of position and time is often difficult. How to calculate  $\Psi$  is discussed in Chap. 5 and the ideas developed there are applied to the structure of the atom in Chap. 6. Until then we can assume that we know as much about  $\Psi$  as each situation requires.

### 3.3 DESCRIBING A WAVE

#### *A general formula for waves*

How fast do de Broglie waves travel? Since we associate a de Broglie wave with a moving body, we expect that this wave has the same velocity as that of the body. Let us see if this is true.

If we call the de Broglie wave velocity  $v_p$ , we can apply the usual formula

$$v_p = v\lambda$$

to find  $v_p$ . The wavelength  $\lambda$  is simply the de Broglie wavelength  $\lambda = h/\gamma mv$ . To find the frequency, we equate the quantum expression  $E = \hbar\nu$  with the relativistic formula for total energy  $E = \gamma mc^2$  to obtain

$$\begin{aligned} \hbar\nu &= \gamma mc^2 \\ \nu &= \frac{\gamma mc^2}{h} \end{aligned}$$

The de Broglie wave velocity is therefore

De Broglie phase  
velocity  $v_p = v\lambda = \left(\frac{\gamma mc^2}{h}\right)\left(\frac{h}{\gamma mv}\right) = \frac{c^2}{v}$  (3.3)

Because the particle velocity  $v$  must be less than the velocity of light  $c$ , the de Broglie waves always travel faster than light! In order to understand this unexpected result, we must look into the distinction between **phase velocity** and **group velocity**. (Phase velocity is what we have been calling wave velocity.)

Let us begin by reviewing how waves are described mathematically. For simplicity we consider a string stretched along the  $x$  axis whose vibrations are in the  $y$  direction, as in Fig. 3.1, and are simple harmonic in character. If we choose  $t = 0$  when the displacement  $y$  of the string at  $x = 0$  is a maximum, its displacement at any future time  $t$  at the same place is given by the formula

$$y = A \cos 2\pi\nu t \quad (3.4)$$

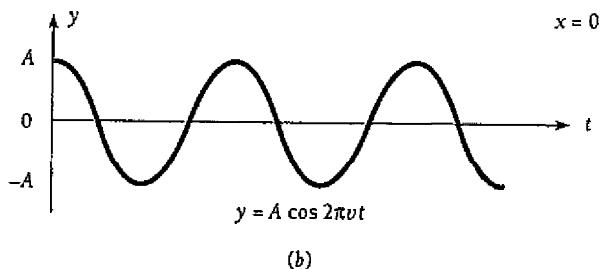
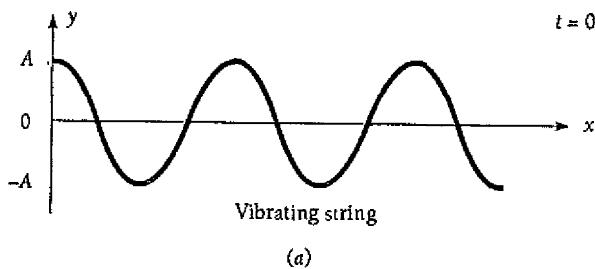


Figure 3.1 (a) The appearance of a wave in a stretched string at a certain time. (b) How the displacement of a point on the string varies with time.

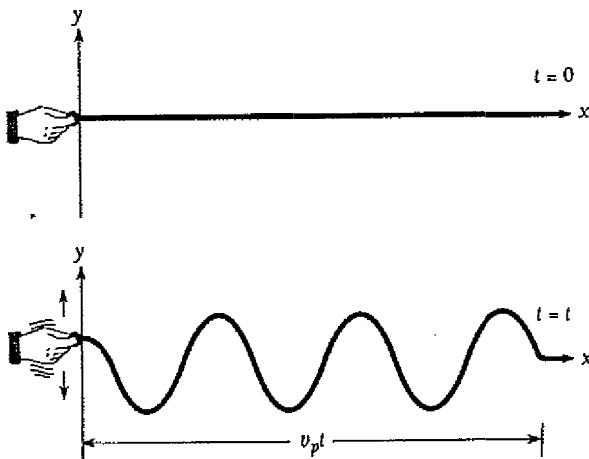


Figure 3.2 Wave propagation.

where  $A$  is the amplitude of the vibrations (that is, their maximum displacement on either side of the  $x$  axis) and  $\nu$  their frequency.

Equation (3.4) tells us what the displacement of a single point on the string is as a function of time  $t$ . A complete description of wave motion in a stretched string, however, should tell us what  $y$  is at any point on the string at any time. What we want is a formula giving  $y$  as a function of both  $x$  and  $t$ .

To obtain such a formula, let us imagine that we shake the string at  $x = 0$  when  $t = 0$ , so that a wave starts to travel down the string in the  $+x$  direction (Fig. 3.2). This wave has some speed  $v_p$  that depends on the properties of the string. The wave travels the distance  $x = v_p t$  in the time  $t$ , so the time interval between the formation of the wave at  $x = 0$  and its arrival at the point  $x$  is  $x/v_p$ . Hence the displacement  $y$  of the string at  $x$  at any time  $t$  is exactly the same as the value of  $y$  at  $x = 0$  at the earlier time  $t - x/v_p$ . By simply replacing  $t$  in Eq. (3.4) with  $t - x/v_p$ , then, we have the desired formula giving  $y$  in terms of both  $x$  and  $t$ :

$$\text{Wave formula} \quad y = A \cos 2\pi\nu \left( t - \frac{x}{v_p} \right) \quad (3.5)$$

As a check, we note that Eq. (3.5) reduces to Eq. (3.4) at  $x = 0$ .

Equation (3.5) may be rewritten

$$y = A \cos 2\pi \left( \nu t - \frac{\nu x}{v_p} \right)$$

Since the wave speed  $v_p$  is given by  $v_p = \nu\lambda$  we have

$$\text{Wave formula} \quad y = A \cos 2\pi \left( \nu t - \frac{x}{\lambda} \right) \quad (3.6)$$

Equation (3.6) is often more convenient to use than Eq. (3.5).

Perhaps the most widely used description of a wave, however, is still another form of Eq. (3.5). The quantities angular frequency  $\omega$  and wave number  $k$  are defined by the formulas

Angular frequency  $\omega = 2\pi\nu$  (3.7)

Wave number  $k = \frac{2\pi}{\lambda} = \frac{\omega}{v_p}$  (3.8)

The unit of  $\omega$  is the radian per second and that of  $k$  is the radian per meter. Angular frequency gets its name from uniform circular motion, where a particle that moves around a circle  $\nu$  times per second sweeps out  $2\pi\nu$  rad/s. The wave number is equal to the number of radians corresponding to a wave train 1 m long, since there are  $2\pi$  rad in one complete wave.

In terms of  $\omega$  and  $k$ , Eq. (3.5) becomes

Wave formula  $y = A \cos(\omega t - kx)$  (3.9)

In three dimensions  $k$  becomes a vector  $\mathbf{k}$  normal to the wave fronts and  $x$  is replaced by the radius vector  $\mathbf{r}$ . The scalar product  $\mathbf{k} \cdot \mathbf{r}$  is then used instead of  $kx$  in Eq. (3.9).

### 3.4 PHASE AND GROUP VELOCITIES

*'A group of waves need not have the same velocity as the waves themselves'*

The amplitude of the de Broglie waves that correspond to a moving body reflects the probability that it will be found at a particular place at a particular time. It is clear that de Broglie waves cannot be represented simply by a formula resembling Eq. (3.9), which describes an indefinite series of waves all with the same amplitude  $A$ . Instead, we expect the wave representation of a moving body to correspond to a wave packet, or **wave group**, like that shown in Fig. 3.3, whose waves have amplitudes upon which the likelihood of detecting the body depends.

A familiar example of how wave groups come into being is the case of beats. When two sound waves of the same amplitude but of slightly different frequencies are produced simultaneously, the sound we hear has a frequency equal to the average of the two original frequencies and its amplitude rises and falls periodically. The amplitude fluctuations occur as many times per second as the difference between the two original frequencies. If the original sounds have frequencies of, say, 440 and 442 Hz, we will hear a fluctuating sound of frequency 441 Hz with two loudness peaks, called beats, per second. The production of beats is illustrated in Fig. 3.4.

A way to mathematically describe a wave group, then, is in terms of a superposition of individual waves of different wavelengths whose interference with one another results in the variation in amplitude that defines the group shape. If the velocities of the waves are the same, the velocity with which the wave group travels is the common phase velocity. However, if the phase velocity varies with wavelength, the different individual waves do not proceed together. This situation is called dispersion. As a result the wave group has a velocity different from the phase velocities of the waves that make it up. This is the case with de Broglie waves.

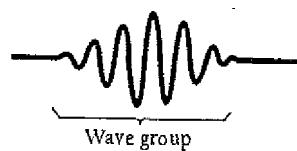


Figure 3.3 A wave group.

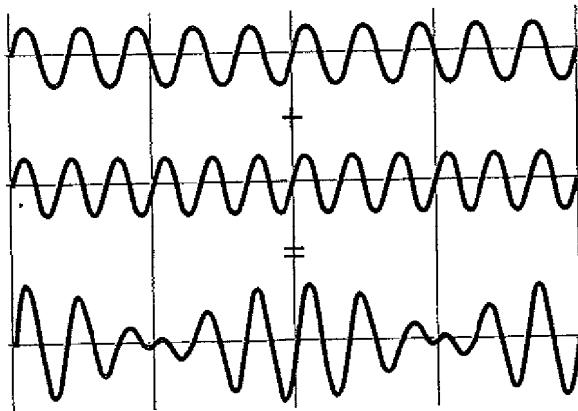


Figure 3.4 Beats are produced by the superposition of two waves with different frequencies.

It is not hard to find the velocity  $v_g$  with which a wave group travels. Let us suppose that the wave group arises from the combination of two waves that have the same amplitude  $A$  but differ by an amount  $\Delta\omega$  in angular frequency and an amount  $\Delta k$  in wave number. We may represent the original waves by the formulas

$$\begin{aligned}y_1 &= A \cos(\omega t - kx) \\y_2 &= A \cos[(\omega + \Delta\omega)t - (k + \Delta k)x]\end{aligned}$$

The resultant displacement  $y$  at any time  $t$  and any position  $x$  is the sum of  $y_1$  and  $y_2$ . With the help of the identity

$$\cos \alpha + \cos \beta = 2 \cos \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta)$$

and the relation

$$\cos(-\theta) = \cos \theta$$

we find that

$$\begin{aligned}y &= y_1 + y_2 \\&= 2A \cos \frac{1}{2}[2\omega t - (2k + \Delta k)x] \cos \frac{1}{2}(\Delta\omega t - \Delta k x)\end{aligned}$$

Since  $\Delta\omega$  and  $\Delta k$  are small compared with  $\omega$  and  $k$  respectively,

$$\begin{aligned}2\omega + \Delta\omega &\approx 2\omega \\2k + \Delta k &\approx 2k\end{aligned}$$

and so

$$\text{Beats } y = 2A \cos(\omega t - kx) \cos \left( \frac{\Delta\omega}{2} t - \frac{\Delta k}{2} x \right) \quad (3.10)$$

Equation (3.10) represents a wave of angular frequency  $\omega$  and wave number  $k$  that has superimposed upon it a modulation of angular frequency  $\frac{1}{2}\Delta\omega$  and of wave number  $\frac{1}{2}\Delta k$ .

The effect of the modulation is thus to produce successive wave groups, as in Fig. 3.4. The phase velocity  $v_p$  is

$$\text{Phase velocity } v_p = \frac{\omega}{k} \quad (3.11)$$

and the velocity  $v_g$  of the wave groups is

$$\text{Group velocity } v_g = \frac{\Delta\omega}{\Delta k} \quad (3.12)$$

When  $\omega$  and  $k$  have continuous spreads instead of the two values in the preceding discussion, the group velocity is instead given by

$$\text{Group velocity } v_g = \frac{d\omega}{dk} \quad (3.13)$$

Depending on how phase velocity varies with wave number in a particular situation, the group velocity may be less or greater than the phase velocities of its member waves. If the phase velocity is the same for all wavelengths, as is true for light waves in empty space, the group and phase velocities are the same.

The angular frequency and wave number of the de Broglie waves associated with a body of mass  $m$  moving with the velocity  $v$  are

$$\begin{aligned} \omega &= 2\pi\nu = \frac{2\pi\gamma mc^2}{h} \\ \text{Angular frequency of} \\ \text{de Broglie waves} &= \frac{2\pi mc^2}{h\sqrt{1 - v^2/c^2}} \end{aligned} \quad (3.14)$$

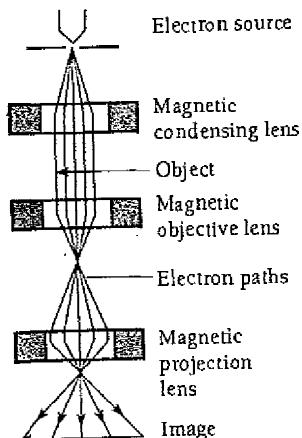
$$\begin{aligned} k &= \frac{2\pi}{\lambda} = \frac{2\pi\gamma mv}{h} \\ \text{Wave number of} \\ \text{de Broglie waves} &= \frac{2\pi mv}{h\sqrt{1 - v^2/c^2}} \end{aligned} \quad (3.15)$$

Both  $\omega$  and  $k$  are functions of the body's velocity  $v$ .

The group velocity  $v_g$  of the de Broglie waves associated with the body is

$$v_g = \frac{d\omega}{dk} = \frac{d\omega/dv}{dk/dv}$$

$$\begin{aligned} \text{Now } \frac{d\omega}{dv} &= \frac{2\pi mv}{h(1 - v^2/c^2)^{3/2}} \\ \frac{dk}{dv} &= \frac{2\pi m}{h(1 - v^2/c^2)^{3/2}} \end{aligned}$$



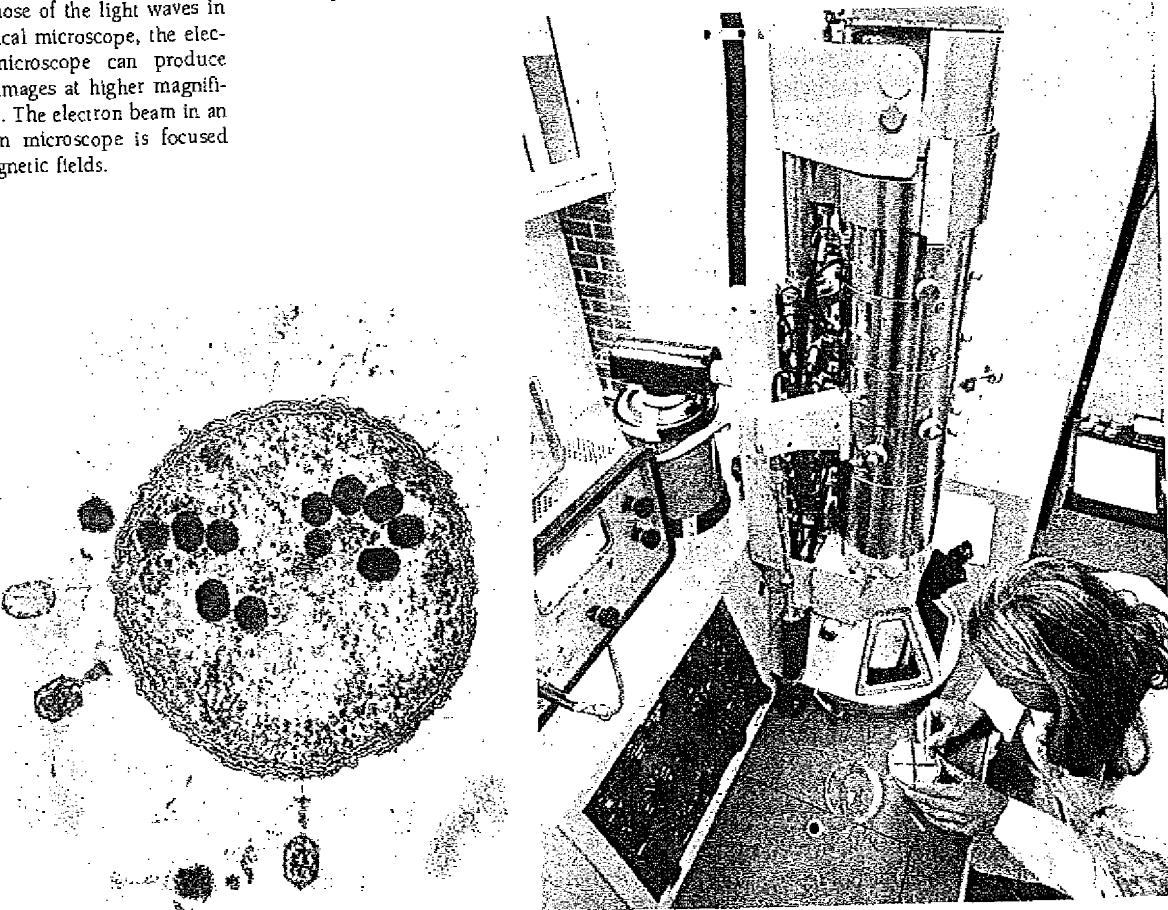
**Figure 3.5** Because the wavelengths of the fast electrons in an electron microscope are shorter than those of the light waves in an optical microscope, the electron microscope can produce sharp images at higher magnifications. The electron beam in an electron microscope is focused by magnetic fields.

### Electron Microscopes

The wave nature of moving electrons is the basis of the electron microscope, the first of which was built in 1932. The resolving power of any optical instrument, which is limited by diffraction, is proportional to the wavelength of whatever is used to illuminate the specimen. In the case of a good microscope that uses visible light, the maximum useful magnification is about 500 $\times$ ; higher magnifications give larger images but do not reveal any more detail. Fast electrons, however, have wavelengths very much shorter than those of visible light and are easily controlled by electric and magnetic fields because of their charge. X-rays also have short wavelengths, but it is not (yet?) possible to focus them adequately.

In an electron microscope, current-carrying coils produce magnetic fields that act as lenses to focus an electron beam on a specimen and then produce an enlarged image on a fluorescent screen or photographic plate (Fig. 3.5). To prevent the beam from being scattered and thereby blurring the image, a thin specimen is used and the entire system is evacuated.

The technology of magnetic "lenses" does not permit the full theoretical resolution of electron waves to be realized in practice. For instance, 100-keV electrons have wavelengths of 0.0037 nm, but the actual resolution they can provide in an electron microscope may be only about 0.1 nm. However, this is still a great improvement on the ~200-nm resolution of an optical microscope. And magnifications of over 1,000,000 $\times$  have been achieved with electron microscopes.



Electron micrograph showing bacteriophage viruses in an *Escherichia coli* bacterium. The bacterium is approximately 1  $\mu\text{m}$  across.

An electron microscope.

and so the group velocity turns out to be

$$\text{De Broglie group velocity} \quad v_g = v \quad (3.16)$$

The de Broglie wave group associated with a moving body travels with the same velocity as the body.

The phase velocity  $v_p$  of de Broglie waves is, as we found earlier,

$$\text{De Broglie phase velocity} \quad v_p = \frac{\omega}{k} = \frac{c^2}{v} \quad (3.3)$$

This exceeds both the velocity of the body  $v$  and the velocity of light  $c$ , since  $v < c$ . However,  $v_p$  has no physical significance because the motion of the wave group, not the motion of the individual waves that make up the group, corresponds to the motion of the body, and  $v_g < c$  as it should be. The fact that  $v_p > c$  for de Broglie waves therefore does not violate special relativity.

### Example 3.3

An electron has a de Broglie wavelength of  $2.00 \text{ pm} = 2.00 \times 10^{-12} \text{ m}$ . Find its kinetic energy and the phase and group velocities of its de Broglie waves.

**Solution .**

(a) The first step is to calculate  $pc$  for the electron, which is

$$pc = \frac{hc}{\lambda} = \frac{(4.136 \times 10^{-15} \text{ eV} \cdot \text{s})(3.00 \times 10^8 \text{ m/s})}{2.00 \times 10^{-12} \text{ m}} = 6.20 \times 10^5 \text{ eV}$$

$$= 620 \text{ keV}$$

The rest energy of the electron is  $E_0 = 511 \text{ keV}$ , so

$$\text{KE} = E - E_0' = \sqrt{E_0^2 + (pc)^2} - E_0 = \sqrt{(511 \text{ keV})^2 + (620 \text{ keV})^2} - 511 \text{ keV}$$

$$= 803 \text{ keV} - 511 \text{ keV} = 292 \text{ keV}$$

(b) The electron velocity can be found from

$$E = \frac{E_0}{\sqrt{1 - v^2/c^2}}$$

to be

$$v = c \sqrt{1 - \frac{E_0^2}{E^2}} = c \sqrt{1 - \left( \frac{511 \text{ keV}}{803 \text{ keV}} \right)^2} = 0.771c$$

Hence the phase and group velocities are respectively

$$v_p = \frac{c^2}{v} = \frac{c^2}{0.771c} = 1.30c$$

$$v_g = v = 0.771c$$

### 3.5 PARTICLE DIFFRACTION

*An experiment that confirms the existence of de Broglie waves*

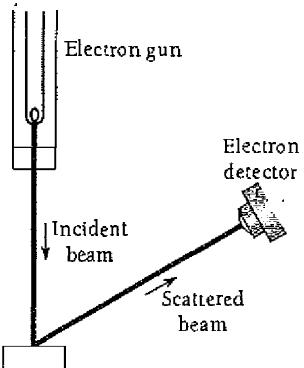


Figure 3.6 The Davisson-Germer experiment.

A wave effect with no analog in the behavior of Newtonian particles is diffraction. In 1927 Clinton Davisson and Lester Germer in the United States and G. P. Thomson in England independently confirmed de Broglie's hypothesis by demonstrating that electron beams are diffracted when they are scattered by the regular atomic arrays of crystals. (All three received Nobel Prizes for their work. J. J. Thomson, G. P.'s father, had earlier won a Nobel Prize for verifying the particle nature of the electron: the wave-particle duality seems to have been the family business.) We shall look at the experiment of Davisson and Germer because its interpretation is more direct.

Davisson and Germer were studying the scattering of electrons from a solid using an apparatus like that sketched in Fig. 3.6. The energy of the electrons in the primary beam, the angle at which they reach the target, and the position of the detector could all be varied. Classical physics predicts that the scattered electrons will emerge in all directions with only a moderate dependence of their intensity on scattering angle and even less on the energy of the primary electrons. Using a block of nickel as the target, Davisson and Germer verified these predictions.

In the midst of their work an accident occurred that allowed air to enter their apparatus and oxidize the metal surface. To reduce the oxide to pure nickel, the target was baked in a hot oven. After this treatment, the target was returned to the apparatus and the measurements resumed.

Now the results were very different. Instead of a continuous variation of scattered electron intensity with angle, distinct maxima and minima were observed whose positions depended upon the electron energy! Typical polar graphs of electron intensity after the accident are shown in Fig. 3.7. The method of plotting is such that the intensity at any angle is proportional to the distance of the curve at that angle from the point of scattering. If the intensity were the same at all scattering angles, the curves would be circles centered on the point of scattering.

Two questions come to mind immediately: What is the reason for this new effect? Why did it not appear until after the nickel target was baked?

De Broglie's hypothesis suggested that electron waves were being diffracted by the target, much as x-rays are diffracted by planes of atoms in a crystal. This idea received

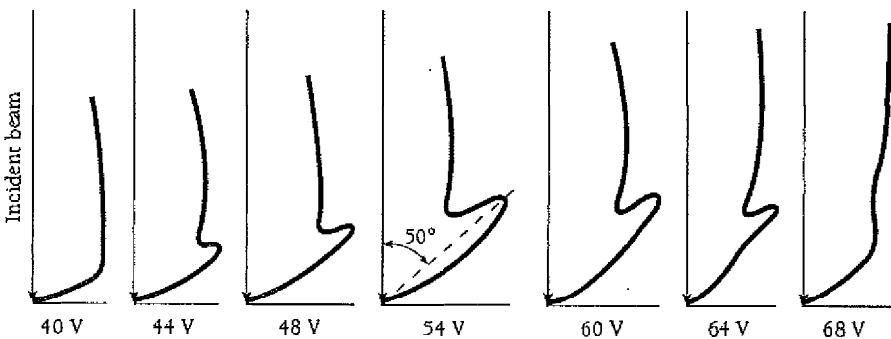


Figure 3.7 Results of the Davisson-Germer experiment, showing how the number of scattered electrons varied with the angle between the incoming beam and the crystal surface. The Bragg planes of atoms in the crystal were not parallel to the crystal surface, so the angles of incidence and scattering relative to one family of these planes were both  $65^\circ$  (see Fig. 3.8).

support when it was realized that heating a block of nickel at high temperature causes the many small individual crystals of which it is normally composed to form into a single large crystal, all of whose atoms are arranged in a regular lattice.

Let us see whether we can verify that de Broglie waves are responsible for the findings of Davisson and Germer. In a particular case, a beam of 54-eV electrons was directed perpendicularly at the nickel target and a sharp maximum in the electron distribution occurred at an angle of  $50^\circ$  with the original beam. The angles of incidence and scattering relative to the family of Bragg planes shown in Fig. 3.8 are both  $65^\circ$ . The spacing of the planes in this family, which can be measured by x-ray diffraction, is 0.091 nm. The Bragg equation for maxima in the diffraction pattern is

$$n\lambda = 2d \sin \theta \quad (2.13)$$

Here  $d = 0.091$  nm and  $\theta = 65^\circ$ . For  $n = 1$  the de Broglie wavelength  $\lambda$  of the diffracted electrons is

$$\lambda = 2d \sin \theta = (2)(0.091 \text{ nm})(\sin 65^\circ) = 0.165 \text{ nm}$$

Now we use de Broglie's formula  $\lambda = h/\gamma mv$  to find the expected wavelength of the electrons. The electron kinetic energy of 54 eV is small compared with its rest energy  $mc^2$  of 0.51 MeV, so we can let  $\gamma = 1$ . Since

$$\text{KE} = \frac{1}{2}mv^2$$

the electron momentum  $mv$  is

$$\begin{aligned} mv &= \sqrt{2m\text{KE}} \\ &= \sqrt{(2)(9.1 \times 10^{-31} \text{ kg})(54 \text{ eV})(1.6 \times 10^{-19} \text{ J/eV})} \\ &= 4.0 \times 10^{-24} \text{ kg} \cdot \text{m/s} \end{aligned}$$

The electron wavelength is therefore

$$\lambda = \frac{h}{mv} = \frac{6.63 \times 10^{-34} \text{ J} \cdot \text{s}}{4.0 \times 10^{-24} \text{ kg} \cdot \text{m/s}} = 1.66 \times 10^{-10} \text{ m} = 0.166 \text{ nm}$$

which agrees well with the observed wavelength of 0.165 nm. The Davisson-Germer experiment thus directly verifies de Broglie's hypothesis of the wave nature of moving bodies.

Analyzing the Davisson-Germer experiment is actually less straightforward than indicated above because the energy of an electron increases when it enters a crystal by an amount equal to the work function of the surface. Hence the electron speeds in the experiment were greater inside the crystal and the de Broglie wavelengths there shorter than the values outside. Another complication arises from interference between waves diffracted by different families of Bragg planes, which restricts the occurrence of maxima to certain combinations of electron energy and angle of incidence rather than merely to any combination that obeys the Bragg equation.

Electrons are not the only bodies whose wave behavior can be demonstrated. The diffraction of neutrons and of whole atoms when scattered by suitable crystals has been observed, and in fact neutron diffraction, like x-ray and electron diffraction, has been used for investigating crystal structures.

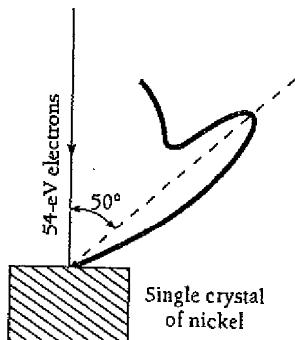
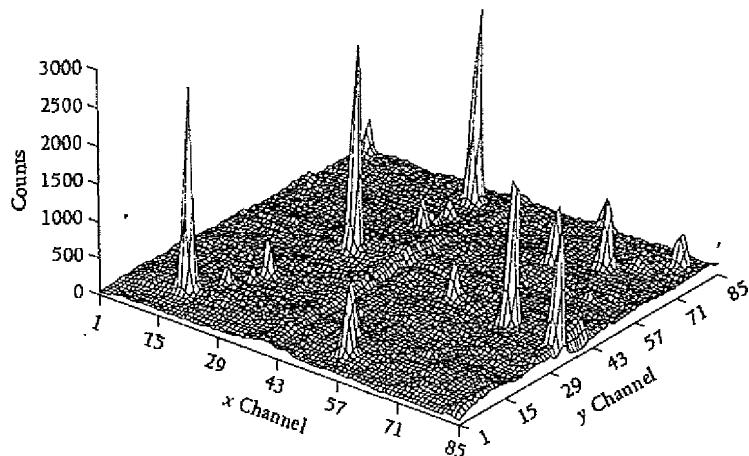


Figure 3.8 The diffraction of the de Broglie waves by the target is responsible for the results of Davisson and Germer.



Neutron diffraction by a quartz crystal. The peaks represent directions in which constructive interference occurred. (Courtesy Frank J. Rotella and Arthur J. Schultz, Argonne National Laboratory)

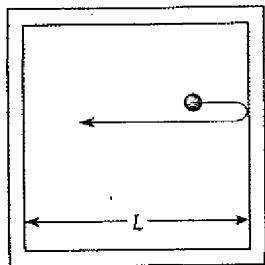


Figure 3.9 A particle confined to a box of width  $L$ . The particle is assumed to move back and forth along a straight line between the walls of the box.

### 3.6 PARTICLE IN A BOX

*Why the energy of a trapped particle is quantized*

The wave nature of a moving particle leads to some remarkable consequences when the particle is restricted to a certain region of space instead of being able to move freely.

The simplest case is that of a particle that bounces back and forth between the walls of a box, as in Fig. 3.9. We shall assume that the walls of the box are infinitely hard, so the particle does not lose energy each time it strikes a wall, and that its velocity is sufficiently small so that we can ignore relativistic considerations. Simple as it is, this model situation requires fairly elaborate mathematics in order to be properly analyzed, as we shall learn in Chap. 5. However, even a relatively crude treatment can reveal the essential results.

From a wave point of view, a particle trapped in a box is like a standing wave in a string stretched between the box's walls. In both cases the wave variable (transverse displacement for the string, wave function  $\Psi$  for the moving particle) must be 0 at the walls, since the waves stop there. The possible de Broglie wavelengths of the particle in the box therefore are determined by the width  $L$  of the box, as in Fig. 3.10. The longest wavelength is specified by  $\lambda = 2L$ , the next by  $\lambda = L$ , then  $\lambda = 2L/3$ , and so forth. The general formula for the permitted wavelengths is

$$\text{De Broglie wavelengths of trapped particle} \quad \lambda_n = \frac{2L}{n} \quad n = 1, 2, 3, \dots \quad (3.17)$$

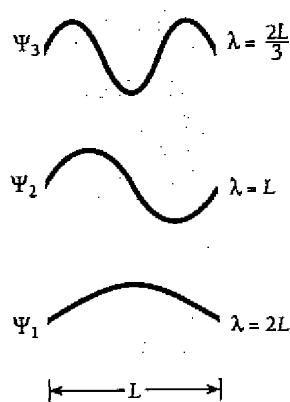


Figure 3.10 Wave functions of a particle trapped in a box  $L$  wide.

Because  $mv = h/\lambda$ , the restrictions on de Broglie wavelength  $\lambda$  imposed by the width of the box are equivalent to limits on the momentum of the particle and, in turn, to limits on its kinetic energy. The kinetic energy of a particle of momentum  $mv$  is

$$KE = \frac{1}{2}mv^2 = \frac{(mv)^2}{2m} = \frac{h^2}{2m\lambda^2}$$

The permitted wavelengths are  $\lambda_n = 2L/n$ , and so, because the particle has no potential energy in this model, the only energies it can have are

Particle in a box       $E_n = \frac{n^2 h^2}{8mL^2} \quad n = 1, 2, 3, \dots$       (3.18)

Each permitted energy is called an **energy level**, and the integer  $n$  that specifies an energy level  $E_n$  is called its **quantum number**.

We can draw three general conclusions from Eq. (3.18). These conclusions apply to any particle confined to a certain region of space (even if the region does not have a well-defined boundary), for instance an atomic electron held captive by the attraction of the positively charged nucleus.

- 1 A trapped particle cannot have an arbitrary energy, as a free particle can. The fact of its confinement leads to restrictions on its wave function that allow the particle to have only certain specific energies and no others. Exactly what these energies are depends on the mass of the particle and on the details of how it is trapped.
- 2 A trapped particle cannot have zero energy. Since the de Broglie wavelength of the particle is  $\lambda = h/mv$ , a speed of  $v = 0$  means an infinite wavelength. But there is no way to reconcile an infinite wavelength with a trapped particle, so such a particle must have at least some kinetic energy. The exclusion of  $E = 0$  for a trapped particle, like the limitation of  $E$  to a set of discrete values, is a result with no counterpart in classical physics, where all non-negative energies, including zero, are allowed.
- 3 Because Planck's constant is so small—only  $6.63 \times 10^{-34} \text{ J} \cdot \text{s}$ —quantization of energy is conspicuous only when  $m$  and  $L$  are also small. This is why we are not aware of energy quantization in our own experience. Two examples will make this clear.

### Example 3.4

An electron is in a box 0.10 nm across, which is the order of magnitude of atomic dimensions. Find its permitted energies.

**Solution**

Here  $m = 9.1 \times 10^{-31} \text{ kg}$  and  $L = 0.10 \text{ nm} = 1.0 \times 10^{-10} \text{ m}$ , so that the permitted electron energies are

$$\begin{aligned} E_n &= \frac{(n^2)(6.63 \times 10^{-34} \text{ J} \cdot \text{s})^2}{(8)(9.1 \times 10^{-31} \text{ kg})(1.0 \times 10^{-10} \text{ m})^2} = 6.0 \times 10^{-18} n^2 \text{ J} \\ &= 38n^2 \text{ eV} \end{aligned}$$

The minimum energy the electron can have is 38 eV, corresponding to  $n = 1$ . The sequence of energy levels continues with  $E_2 = 152 \text{ eV}$ ,  $E_3 = 342 \text{ eV}$ ,  $E_4 = 608 \text{ eV}$ , and so on (Fig. 3.11). If such a box existed, the quantization of a trapped electron's energy would be a prominent feature of the system. (And indeed energy quantization is prominent in the case of an atomic electron.)

### Example 3.5

A 10-g marble is in a box 10 cm across. Find its permitted energies.

**Solution**

With  $m = 10 \text{ g} = 1.0 \times 10^{-2} \text{ kg}$  and  $L = 10 \text{ cm} = 1.0 \times 10^{-1} \text{ m}$ ,

$$\begin{aligned} E_n &= \frac{(n^2)(6.63 \times 10^{-34} \text{ J} \cdot \text{s})^2}{(8)(1.0 \times 10^{-2} \text{ kg})(1.0 \times 10^{-1} \text{ m})^2} \\ &= 5.5 \times 10^{-64} n^2 \text{ J} \end{aligned}$$

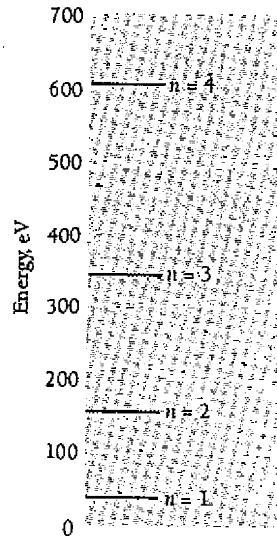


Figure 3.11 Energy levels of an electron confined to a box 0.1 nm wide.

The minimum energy the marble can have is  $5.5 \times 10^{-64}$  J, corresponding to  $n = 1$ . A marble with this kinetic energy has a speed of only  $3.3 \times 10^{-31}$  m/s and therefore cannot be experimentally distinguished from a stationary marble. A reasonable speed a marble might have is, say,  $\frac{1}{3}$  m/s—which corresponds to the energy level of quantum number  $n = 10^{30}!$  The permissible energy levels are so very close together, then, that there is no way to determine whether the marble can take on only those energies predicted by Eq. (3.18) or any energy whatever. Hence in the domain of everyday experience, quantum effects are imperceptible, which accounts for the success of Newtonian mechanics in this domain.

### 3.7 UNCERTAINTY PRINCIPLE 1

*We cannot know the future because we cannot know the present*

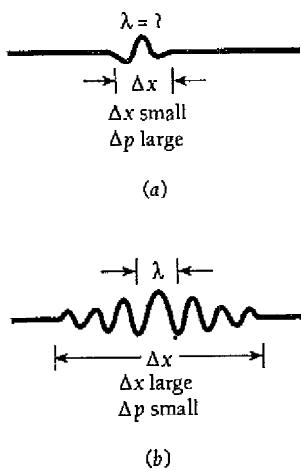


Figure 3.12 (a) A narrow de Broglie wave group. The position of the particle can be precisely determined, but the wavelength (and hence the particle's momentum) cannot be established because there are not enough waves to measure accurately. (b) A wide wave group. Now the wavelength can be precisely determined but not the position of the particle.

To regard a moving particle as a wave group implies that there are fundamental limits to the accuracy with which we can measure such "particle" properties as position and momentum.

To make clear what is involved, let us look at the wave group of Fig. 3.3. The particle that corresponds to this wave group may be located anywhere within the group at a given time. Of course, the probability density  $|\Psi|^2$  is a maximum in the middle of the group, so it is most likely to be found there. Nevertheless, we may still find the particle anywhere that  $|\Psi|^2$  is not actually 0.

The narrower its wave group, the more precisely a particle's position can be specified (Fig. 3.12a). However, the wavelength of the waves in a narrow packet is not well defined; there are not enough waves to measure  $\lambda$  accurately. This means that since  $\lambda = h/\gamma mv$ , the particle's momentum  $\gamma mv$  is not a precise quantity. If we make a series of momentum measurements, we will find a broad range of values.

On the other hand, a wide wave group, such as that in Fig. 3.12b, has a clearly defined wavelength. The momentum that corresponds to this wavelength is therefore a precise quantity, and a series of measurements will give a narrow range of values. But where is the particle located? The width of the group is now too great for us to be able to say exactly where the particle is at a given time.

Thus we have the uncertainty principle:

It is impossible to know both the exact position and exact momentum of an object at the same time.

This principle, which was discovered by Werner Heisenberg in 1927, is one of the most significant of physical laws.

A formal analysis supports the above conclusion and enables us to put it on a quantitative basis. The simplest example of the formation of wave groups is that given in Sec. 3.4, where two wave trains slightly different in angular frequency  $\omega$  and wave number  $k$  were superposed to yield the series of groups shown in Fig. 3.4. A moving body corresponds to a single wave group, not a series of them, but a single wave group can also be thought of in terms of the superposition of trains of harmonic waves. However, an infinite number of wave trains with different frequencies, wave numbers, and amplitudes is required for an isolated group of arbitrary shape, as in Fig. 3.13.

At a certain time  $t$ , the wave group  $\Psi(x)$  can be represented by the Fourier integral

$$\Psi(x) = \int_0^{\infty} g(k) \cos kx dk \quad (3.19)$$

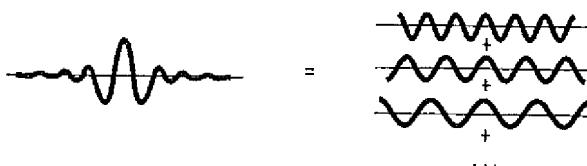


Figure 3.13 An isolated wave group is the result of superposing an infinite number of waves with different wavelengths. The narrower the wave group, the greater the range of wavelengths involved. A narrow de Broglie wave group thus means a well-defined position ( $\Delta x$  smaller) but a poorly defined wavelength and a large uncertainty  $\Delta p$  in the momentum of the particle the group represents. A wide wave group means a more precise momentum but a less precise position.

where the function  $g(k)$  describes how the amplitudes of the waves that contribute to  $\Psi(x)$  vary with wave number  $k$ . This function is called the Fourier transform of  $\Psi(x)$ , and it specifies the wave group just as completely as  $\Psi(x)$  does. Figure 3.14 contains graphs of the Fourier transforms of a pulse and of a wave group. For comparison, the Fourier transform of an infinite train of harmonic waves is also included. There is only a single wave number in this case, of course.

Strictly speaking, the wave numbers needed to represent a wave group extend from  $k = 0$  to  $k = \infty$ , but for a group whose length  $\Delta x$  is finite, the waves whose amplitudes  $g(k)$  are appreciable have wave numbers that lie within a finite interval  $\Delta k$ . As Fig. 3.14 indicates, the narrower the group, the broader the range of wave numbers needed to describe it, and vice versa.

The relationship between the distance  $\Delta x$  and the wave-number spread  $\Delta k$  depends upon the shape of the wave group and upon how  $\Delta x$  and  $\Delta k$  are defined. The minimum value of the product  $\Delta x \Delta k$  occurs when the envelope of the group has the familiar bell shape of a Gaussian function. In this case the Fourier transform happens to be a Gaussian function also. If  $\Delta x$  and  $\Delta k$  are taken as the standard deviations of the respective functions  $\Psi(x)$  and  $g(k)$ , then this minimum value is  $\Delta x \Delta k = \frac{1}{2}$ . Because wave groups in general do not have Gaussian forms, it is more realistic to express the relationship between  $\Delta x$  and  $\Delta k$  as

$$\Delta x \Delta k \geq \frac{1}{2} \quad (3.20)$$

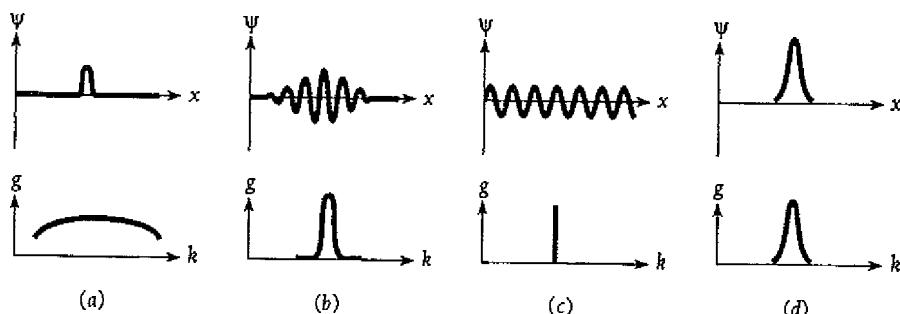


Figure 3.14 The wave functions and Fourier transforms for (a) a pulse, (b) a wave group, (c) a wave train, and (d) a Gaussian distribution. A brief disturbance needs a broader range of frequencies to describe it than a disturbance of greater duration. The Fourier transform of a Gaussian function is also a Gaussian function.

### Gaussian Function

When a set of measurements is made of some quantity  $x$  in which the experimental errors are random, the result is often a Gaussian distribution whose form is the bell-shaped curve shown in Fig. 3.15. The standard deviation  $\sigma$  of the measurements is a measure of the spread of  $x$  values about the mean of  $x_0$ , where  $\sigma$  equals the square root of the average of the squared deviations from  $x_0$ . If  $N$  measurements were made,

$$\text{Standard deviation} \quad \sigma = \sqrt{\frac{1}{N} \sum_{i=1}^N (x_i - x_0)^2}$$

The width of a Gaussian curve at half its maximum value is  $2.35\sigma$ .

The Gaussian function  $f(x)$  that describes the above curve is given by

$$\text{Gaussian function} \quad f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-(x-x_0)^2/2\sigma^2}$$

where  $f(x)$  is the probability that the value  $x$  be found in a particular measurement. Gaussian functions occur elsewhere in physics and mathematics as well. (Gabriel Lippmann had this to say about the Gaussian function: "Experimentalists think that it is a mathematical theorem while mathematicians believe it to be an experimental fact.")

The probability that a measurement lie inside a certain range of  $x$  values, say between  $x_1$  and  $x_2$ , is given by the area of the  $f(x)$  curve between these limits. This area is the integral

$$P_{x_1 x_2} = \int_{x_1}^{x_2} f(x) dx$$

An interesting question is what fraction of a series of measurements has values within a standard deviation of the mean value  $x_0$ . In this case  $x_1 = x_0 - \sigma$  and  $x_2 = x_0 + \sigma$ , and

$$P_{x_0 \pm \sigma} = \int_{x_0 - \sigma}^{x_0 + \sigma} f(x) dx = 0.683$$

Hence 68.3 percent of the measurements fall in this interval, which is shaded in Fig. 3.15. A similar calculation shows that 95.4 percent of the measurements fall within two standard deviations of the mean value.

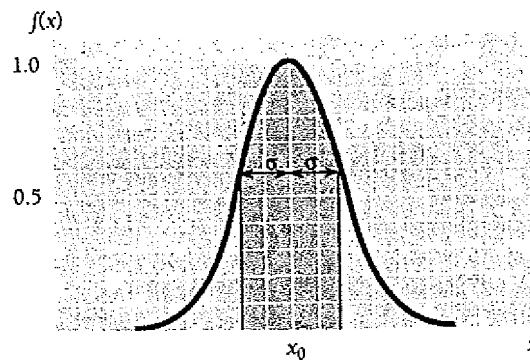


Figure 3.15 A Gaussian distribution. The probability of finding a value of  $x$  is given by the Gaussian function  $f(x)$ . The mean value of  $x$  is  $x_0$ , and the total width of the curve at half its maximum value is  $2.35\sigma$ , where  $\sigma$  is the standard deviation of the distribution. The total probability of finding a value of  $x$  within a standard deviation of  $x_0$  is equal to the shaded area and is 68.3 percent.

The de Broglie wavelength of a particle of momentum  $p$  is  $\lambda = h/p$  and the corresponding wave number is

$$k = \frac{2\pi}{\lambda} = \frac{2\pi p}{h}$$

In terms of wave number the particle's momentum is therefore

$$p = \frac{hk}{2\pi}$$

Hence an uncertainty  $\Delta k$  in the wave number of the de Broglie waves associated with the particle results in an uncertainty  $\Delta p$  in the particle's momentum according to the formula

$$\Delta p = \frac{h \Delta k}{2\pi}$$

Since  $\Delta x \Delta k \geq \frac{1}{2}$ ,  $\Delta k \geq 1/(2\Delta x)$  and

Uncertainty principle 
$$\Delta x \Delta p \geq \frac{h}{4\pi} \quad (3.21)$$

This equation states that the product of the uncertainty  $\Delta x$  in the position of an object at some instant and the uncertainty  $\Delta p$  in its momentum component in the  $x$  direction at the same instant is equal to or greater than  $h/4\pi$ .

If we arrange matters so that  $\Delta x$  is small, corresponding to a narrow wave group, then  $\Delta p$  will be large. If we reduce  $\Delta p$  in some way, a broad wave group is inevitable and  $\Delta x$  will be large.



Werner Heisenberg (1901–1976) was born in Duisberg, Germany, and studied theoretical physics at Munich, where he also became an enthusiastic skier and mountaineer. At Göttingen in 1924 as an assistant to Max Born, Heisenberg became uneasy about mechanical models of the atom: "Any picture of the atom that our imagination is able to invent is for that very reason defective," he later remarked. Instead he conceived an abstract approach using matrix algebra. In 1925, together with Born and Pascual Jordan, Heisenberg developed this approach into a consistent theory of quantum mechanics, but it was so difficult to understand and apply that it had very little impact on physics at the time. Schrödinger's wave formulation of quantum mechanics the following year was much more successful; Schrödinger and others soon showed that the wave and matrix versions of quantum mechanics were mathematically equivalent.

In 1927, working at Bohr's institute in Copenhagen, Heisenberg developed a suggestion by Wolfgang Pauli into the uncertainty principle. Heisenberg initially felt that this principle was a consequence of the disturbances inevitably produced by any

measuring process. Bohr, on the other hand, thought that the basic cause of the uncertainties was the wave-particle duality, so that they were built into the natural world rather than solely the result of measurement. After much argument Heisenberg came around to Bohr's view. (Einstein, always skeptical about quantum mechanics, said after a lecture by Heisenberg on the uncertainty principle: "Marvelous, what ideas the young people have these days. But I don't believe a word of it.") Heisenberg received the Nobel Prize in 1932.

Heisenberg was one of the very few distinguished scientists to remain in Germany during the Nazi period. In World War II he led research there on atomic weapons, but little progress had been made by the war's end. Exactly why remains unclear, although there is no evidence that Heisenberg, as he later claimed, had moral qualms about creating such weapons and more or less deliberately dragged his feet. Heisenberg recognized early that "an explosive of unimaginable consequences" could be developed, and he and his group should have been able to have gotten farther than they did. In fact, alarmed by the news that Heisenberg was working on an atomic bomb, the U.S. government sent the former Boston Red Sox catcher Moe Berg to shoot Heisenberg during a lecture in neutral Switzerland in 1944. Berg, sitting in the second row, found himself uncertain from Heisenberg's remarks about how advanced the German program was, and kept his gun in his pocket.

These uncertainties are due not to inadequate apparatus but to the imprecise character in nature of the quantities involved. Any instrumental or statistical uncertainties that arise during a measurement only increase the product  $\Delta x \Delta p$ . Since we cannot know exactly both where a particle is right now and what its momentum is, we cannot say anything definite about where it will be in the future or how fast it will be moving then. We cannot know the future for sure because we cannot know the present for sure. But our ignorance is not total: we can still say that the particle is more likely to be in one place than another and that its momentum is more likely to have a certain value than another.

### H-Bar

The quantity  $h/2\pi$  appears often in modern physics because it turns out to be the basic unit of angular momentum. It is therefore customary to abbreviate  $h/2\pi$  by the symbol  $\hbar$  ("h-bar"):

$$\hbar = \frac{h}{2\pi} = 1.054 \times 10^{-34} \text{ J} \cdot \text{s}$$

In the remainder of this book  $\hbar$  is used in place of  $h/2\pi$ . In terms of  $\hbar$ , the uncertainty principle becomes

Uncertainty principle	$\Delta x \Delta p \geq \frac{\hbar}{2}$	(3.22)
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### Example 3.6

A measurement establishes the position of a proton with an accuracy of  $\pm 1.00 \times 10^{-11} \text{ m}$ . Find the uncertainty in the proton's position 1.00 s later. Assume  $v \ll c$ .

#### Solution

Let us call the uncertainty in the proton's position  $\Delta x_0$  at the time  $t = 0$ . The uncertainty in its momentum at this time is therefore, from Eq. (3.22),

$$\Delta p \geq \frac{\hbar}{2\Delta x_0}$$

Since  $v \ll c$ , the momentum uncertainty is  $\Delta p = \Delta(mv) = m \Delta v$  and the uncertainty in the proton's velocity is

$$\Delta v = \frac{\Delta p}{m} \geq \frac{\hbar}{2m \Delta x_0}$$

The distance  $x$  the proton covers in the time  $t$  cannot be known more accurately than

$$\Delta x = t \Delta v \geq \frac{\hbar t}{2m \Delta x_0}$$

Hence  $\Delta x$  is inversely proportional to  $\Delta x_0$ : the more we know about the proton's position at  $t = 0$ , the less we know about its later position at  $t > 0$ . The value of  $\Delta x$  at  $t = 1.00 \text{ s}$  is

$$\begin{aligned} \Delta x &\geq \frac{(1.054 \times 10^{-34} \text{ J} \cdot \text{s})(1.00 \text{ s})}{(2)(1.672 \times 10^{-27} \text{ kg})(1.00 \times 10^{-11} \text{ m})} \\ &\geq 3.15 \times 10^3 \text{ m} \end{aligned}$$

This is 3.15 km—nearly 2 mi! What has happened is that the original wave group has spread out to a much wider one (Fig. 3.16). This occurred because the phase velocities of the component waves vary with wave number and a large range of wave numbers must have been present to produce the narrow original wave group. See Fig. 3.14.

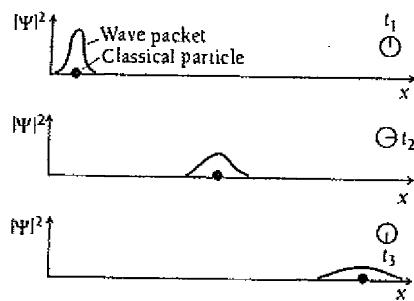


Figure 3.16 The wave packet that corresponds to a moving packet is a composite of many individual waves, as in Fig. 3.13. The phase velocities of the individual waves vary with their wave lengths. As a result, as the particle moves, the wave packet spreads out in space. The narrower the original wavepacket—that is, the more precisely we know its position at that time—the more it spreads out because it is made up of a greater span of waves with different phase velocities.

### 3.8 UNCERTAINTY PRINCIPLE II

*A particle approach gives the same result*

The uncertainty principle can be arrived at from the point of view of the particle properties of waves as well as from the point of view of the wave properties of particles.

We might want to measure the position and momentum of an object at a certain moment. To do so, we must touch it with something that will carry the required information back to us. That is, we must poke it with a stick, shine light on it, or perform some similar act. The measurement process itself thus requires that the object be interfered with in some way. If we consider such interferences in detail, we are led to the same uncertainty principle as before even without taking into account the wave nature of moving bodies.

Suppose we look at an electron using light of wavelength  $\lambda$ , as in Fig. 3.17. Each photon of this light has the momentum  $h/\lambda$ . When one of these photons bounces off the electron (which must happen if we are to "see" the electron), the electron's

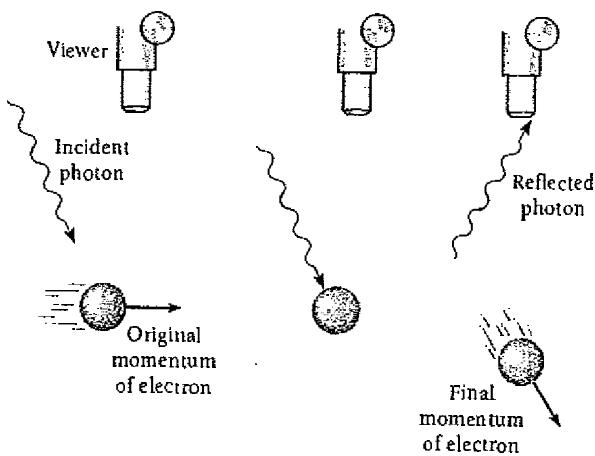


Figure 3.17 An electron cannot be observed without changing its momentum.

original momentum will be changed. The exact amount of the change  $\Delta p$  cannot be predicted, but it will be of the same order of magnitude as the photon momentum  $h/\lambda$ . Hence

$$\Delta p \approx \frac{h}{\lambda} \quad (3.23)$$

The longer the wavelength of the observing photon, the smaller the uncertainty in the electron's momentum.

Because light is a wave phenomenon as well as a particle phenomenon, we cannot expect to determine the electron's location with perfect accuracy regardless of the instrument used. A reasonable estimate of the minimum uncertainty in the measurement might be one photon wavelength, so that

$$\Delta x \geq \lambda \quad (3.24)$$

The shorter the wavelength, the smaller the uncertainty in location. However, if we use light of short wavelength to increase the accuracy of the position measurement, there will be a corresponding decrease in the accuracy of the momentum measurement because the higher photon momentum will disturb the electron's motion to a greater extent. Light of long wavelength will give a more accurate momentum but a less accurate position.

Combining Eqs. (3.23) and (3.24) gives

$$\Delta x \Delta p \geq h \quad (3.25)$$

This result is consistent with Eq. (3.22),  $\Delta x \Delta p \geq \hbar/2$ .

Arguments like the preceding one, although superficially attractive, must be approached with caution. The argument above implies that the electron can possess a definite position and momentum at any instant and that it is the measurement process that introduces the indeterminacy in  $\Delta x \Delta p$ . On the contrary, this indeterminacy is inherent in the nature of a moving body. The justification for the many "derivations" of this kind is first, they show it is impossible to imagine a way around the uncertainty principle; and second, they present a view of the principle that can be appreciated in a more familiar context than that of wave groups.

### 3.9 APPLYING THE UNCERTAINTY PRINCIPLE

*A useful tool, not just a negative statement*

Planck's constant  $\hbar$  is so small that the limitations imposed by the uncertainty principle are significant only in the realm of the atom. On such a scale, however, this principle is of great help in understanding many phenomena. It is worth keeping in mind that the lower limit of  $\hbar/2$  for  $\Delta x \Delta p$  is rarely attained. More usually  $\Delta x \Delta p \geq \hbar$ , or even (as we just saw)  $\Delta x \Delta p \geq h$ .

---

#### Example 3.7

A typical atomic nucleus is about  $5.0 \times 10^{-15}$  m in radius. Use the uncertainty principle to place a lower limit on the energy an electron must have if it is to be part of a nucleus.

**Solution**

Letting  $\Delta x = 5.0 \times 10^{-5}$  m we have

$$\Delta p \geq \frac{\hbar}{2\Delta x} \geq \frac{1.054 \times 10^{-34} \text{ J} \cdot \text{s}}{(2)(5.0 \times 10^{-5} \text{ m})} \geq 1.1 \times 10^{-20} \text{ kg} \cdot \text{m/s}$$

If this is the uncertainty in a nuclear electron's momentum, the momentum  $p$  itself must be at least comparable in magnitude. An electron with such a momentum has a kinetic energy KE many times greater than its rest energy  $mc^2$ . From Eq. (1.24) we see that we can let  $KE = pc$  here to a sufficient degree of accuracy. Therefore

$$KE = pc \geq (1.1 \times 10^{-20} \text{ kg} \cdot \text{m/s})(3.0 \times 10^8 \text{ m/s}) \geq 3.3 \times 10^{-12} \text{ J}$$

Since  $1 \text{ eV} = 1.6 \times 10^{-19} \text{ J}$ , the kinetic energy of an electron must exceed 20 MeV if it is to be inside a nucleus. Experiments show that the electrons emitted by certain unstable nuclei never have more than a small fraction of this energy, from which we conclude that nuclei cannot contain electrons. The electron an unstable nucleus may emit comes into being at the moment the nucleus decays (see Secs. 11.3 and 12.5).

**Example 3.8**

A hydrogen atom is  $5.3 \times 10^{-11}$  m in radius. Use the uncertainty principle to estimate the minimum energy an electron can have in this atom.

**Solution**

Here we find that with  $\Delta x = 5.3 \times 10^{-11}$  m,

$$\Delta p \geq \frac{\hbar}{2\Delta x} \geq 9.9 \times 10^{-25} \text{ kg} \cdot \text{m/s}$$

An electron whose momentum is of this order of magnitude behaves like a classical particle, and its kinetic energy is

$$KE = \frac{p^2}{2m} \geq \frac{(9.9 \times 10^{-25} \text{ kg} \cdot \text{m/s})^2}{(2)(9.1 \times 10^{-31} \text{ kg})} \geq 5.4 \times 10^{-19} \text{ J}$$

which is 3.4 eV. The kinetic energy of an electron in the lowest energy level of a hydrogen atom is actually 13.6 eV.

**Energy and Time**

Another form of the uncertainty principle concerns energy and time. We might wish to measure the energy  $E$  emitted during the time interval  $\Delta t$  in an atomic process. If the energy is in the form of em waves, the limited time available restricts the accuracy with which we can determine the frequency  $\nu$  of the waves. Let us assume that the minimum uncertainty in the number of waves we count in a wave group is one wave. Since the frequency of the waves under study is equal to the number of them we count divided by the time interval, the uncertainty  $\Delta\nu$  in our frequency measurement is

$$\Delta\nu \geq \frac{1}{\Delta t}$$

The corresponding energy uncertainty is

$$\Delta E = \hbar \Delta\nu$$

and so

$$\Delta E \geq \frac{\hbar}{\Delta t} \quad \text{or} \quad \Delta E \Delta t \geq \hbar$$

A more precise calculation based on the nature of wave groups changes this result to

Uncertainties in  $\Delta E \Delta t \geq \frac{\hbar}{2}$  (3.26)  
energy and time

Equation (3.26) states that the product of the uncertainty  $\Delta E$  in an energy measurement and the uncertainty  $\Delta t$  in the time at which the measurement is made is equal to or greater than  $\hbar/2$ . This result can be derived in other ways as well and is a general one not limited to em waves.

### Example 3.9

An "excited" atom gives up its excess energy by emitting a photon of characteristic frequency, as described in Chap. 4. The average period that elapses between the excitation of an atom and the time it radiates is  $1.0 \times 10^{-8}$  s. Find the inherent uncertainty in the frequency of the photon.

#### Solution

The photon energy is uncertain by the amount

$$\Delta E \geq \frac{\hbar}{2\Delta t} \geq \frac{1.054 \times 10^{-34} \text{ J} \cdot \text{s}}{2(1.0 \times 10^{-8} \text{ s})} \geq 5.3 \times 10^{-27} \text{ J}$$

The corresponding uncertainty in the frequency of light is

$$\Delta\nu = \frac{\Delta E}{\hbar} \geq 8 \times 10^6 \text{ Hz}$$

This is the irreducible limit to the accuracy with which we can determine the frequency of the radiation emitted by an atom. As a result, the radiation from a group of excited atoms does not appear with the precise frequency  $\nu$ . For a photon whose frequency is, say,  $5.0 \times 10^{14}$  Hz,  $\Delta\nu/\nu = 1.6 \times 10^{-8}$ . In practice, other phenomena such as the doppler effect contribute more than this to the broadening of spectral lines.

## EXERCISES

It is only the first step that takes the effort. —Marquise du Deffand

### 3.1 De Broglie Waves

1. A photon and a particle have the same wavelength. Can anything be said about how their linear momenta compare? About how the photon's energy compares with the particle's total energy? About how the photon's energy compares with the particle's kinetic energy?
2. Find the de Broglie wavelength of (a) an electron whose speed is  $1.0 \times 10^8$  m/s, and (b) an electron whose speed is  $2.0 \times 10^8$  m/s.
3. Find the de Broglie wavelength of a 1.0-mg grain of sand blown by the wind at a speed of 20 m/s.
4. Find the de Broglie wavelength of the 40-keV electrons used in a certain electron microscope.
5. By what percentage will a nonrelativistic calculation of the de Broglie wavelength of a 100-keV electron be in error?
6. Find the de Broglie wavelength of a 1.00-MeV proton. Is a relativistic calculation needed?
7. The atomic spacing in rock salt, NaCl, is 0.282 nm. Find the kinetic energy (in eV) of a neutron with a de Broglie wavelength of 0.282 nm. Is a relativistic calculation needed? Such neutrons can be used to study crystal structure.
8. Find the kinetic energy of an electron whose de Broglie wavelength is the same as that of a 100-keV x-ray.
9. Green light has a wavelength of about 550 nm. Through what potential difference must an electron be accelerated to have this wavelength?
10. Show that the de Broglie wavelength of a particle of mass  $m$  and kinetic energy  $KE$  is given by

$$\lambda = \frac{hc}{\sqrt{KE(KE + 2mc^2)}}$$

11. Show that if the total energy of a moving particle greatly exceeds its rest energy, its de Broglie wavelength is nearly the same as the wavelength of a photon with the same total energy.
12. (a) Derive a relativistically correct formula that gives the de Broglie wavelength of a charged particle in terms of the potential difference  $V$  through which it has been accelerated.  
(b) What is the nonrelativistic approximation of this formula, valid for  $eV \ll mc^2$ ?

### 3.4 Phase and Group Velocities

13. An electron and a proton have the same velocity. Compare the wavelengths and the phase and group velocities of their de Broglie waves.
14. An electron and a proton have the same kinetic energy. Compare the wavelengths and the phase and group velocities of their de Broglie waves.

15. Verify the statement in the text that, if the phase velocity is the same for all wavelengths of a certain wave phenomenon (that is, there is no dispersion), the group and phase velocities are the same.
16. The phase velocity of ripples on a liquid surface is  $\sqrt{2\pi S/\lambda\rho}$ , where  $S$  is the surface tension and  $\rho$  the density of the liquid. Find the group velocity of the ripples.
17. The phase velocity of ocean waves is  $\sqrt{g\lambda/2\pi}$ , where  $g$  is the acceleration of gravity. Find the group velocity of ocean waves.
18. Find the phase and group velocities of the de Broglie waves of an electron whose speed is  $0.900c$ .
19. Find the phase and group velocities of the de Broglie waves of an electron whose kinetic energy is 500 keV.
20. Show that the group velocity of a wave is given by  $v_g = d\nu/d(1/\lambda)$ .
21. (a) Show that the phase velocity of the de Broglie waves of a particle of mass  $m$  and de Broglie wavelength  $\lambda$  is given by

$$v_p = c\sqrt{1 + \left(\frac{mc\lambda}{h}\right)^2}$$

(b) Compare the phase and group velocities of an electron whose de Broglie wavelength is exactly  $1 \times 10^{-13}$  m.

22. In his original paper, de Broglie suggested that  $E = h\nu$  and  $p = h/\lambda$ , which hold for electromagnetic waves, are also valid for moving particles. Use these relationships to show that the group velocity  $v_g$  of a de Broglie wave group is given by  $dE/dp$ , and with the help of Eq. (1.24), verify that  $v_g = v$  for a particle of velocity  $v$ .

### 3.5 Particle Diffraction

23. What effect on the scattering angle in the Davisson-Germer experiment does increasing the electron energy have?
24. A beam of neutrons that emerges from a nuclear reactor contains neutrons with a variety of energies. To obtain neutrons with an energy of 0.050 eV, the beam is passed through a crystal whose atomic planes are 0.20 nm apart. At what angles relative to the original beam will the desired neutrons be diffracted?
25. In Sec. 3.5 it was mentioned that the energy of an electron entering a crystal increases, which reduces its de Broglie wavelength. Consider a beam of 54-eV electrons directed at a nickel target. The potential energy of an electron that enters the target changes by 26 eV. (a) Compare the electron speeds outside and inside the target. (b) Compare the respective de Broglie wavelengths.
26. A beam of 50-keV electrons is directed at a crystal and diffracted electrons are found at an angle of  $50^\circ$  relative to the original beam. What is the spacing of the atomic planes of the crystal? A relativistic calculation is needed for  $\lambda$ .

## 3.6 Particle in a Box

27. Obtain an expression for the energy levels (in MeV) of a neutron confined to a one-dimensional box  $1.00 \times 10^{-14}$  m wide. What is the neutron's minimum energy? (The diameter of an atomic nucleus is of this order of magnitude.)
28. The lowest energy possible for a certain particle trapped in a certain box is 1.00 eV. (a) What are the next two higher energies the particle can have? (b) If the particle is an electron, how wide is the box?
29. A proton in a one-dimensional box has an energy of 400 keV in its first excited state. How wide is the box?

## 3.7 Uncertainty Principle I

## 3.8 Uncertainty Principle II

## 3.9 Applying the Uncertainty Principle

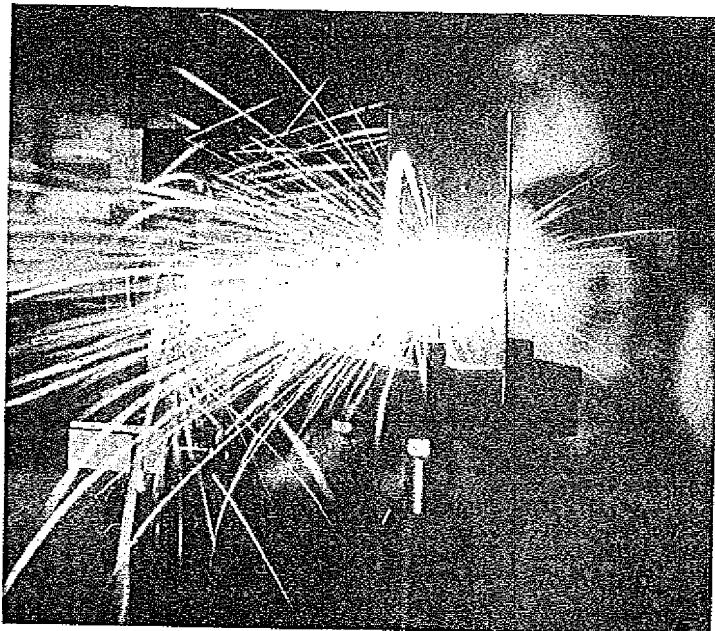
30. Discuss the prohibition of  $E = 0$  for a particle trapped in a box  $L$  wide in terms of the uncertainty principle. How does the minimum momentum of such a particle compare with the momentum uncertainty required by the uncertainty principle if we take  $\Delta x = L$ ?
31. The atoms in a solid possess a certain minimum zero-point energy even at 0 K, while no such restriction holds for the molecules in an ideal gas. Use the uncertainty principle to explain these statements.
32. Compare the uncertainties in the velocities of an electron and a proton confined in a 1.00-nm box.
33. The position and momentum of a 1.00-keV electron are simultaneously determined. If its position is located to within 0.100 nm, what is the percentage of uncertainty in its momentum?
34. (a) How much time is needed to measure the kinetic energy of an electron whose speed is 10.0 m/s with an uncertainty of no more than 0.100 percent? How far will the electron have traveled in this period of time? (b) Make the same calculations

for a 1.00-g insect whose speed is the same. What do these sets of figures indicate?

35. How accurately can the position of a proton with  $v \ll c$  be determined without giving it more than 1.00 keV of kinetic energy?
36. (a) Find the magnitude of the momentum of a particle in a box in its  $n$ th state. (b) The minimum change in the particle's momentum that a measurement can cause corresponds to a change of  $\pm 1$  in the quantum number  $n$ . If  $\Delta x = L$ , show that  $\Delta p \Delta x \geq \hbar/2$ .
37. A marine radar operating at a frequency of 9400 MHz emits groups of electromagnetic waves 0.0800  $\mu s$  in duration. The time needed for the reflections of these groups to return indicates the distance to a target. (a) Find the length of each group and the number of waves it contains. (b) What is the approximate minimum bandwidth (that is, spread of frequencies) the radar receiver must be able to process?
38. An unstable elementary particle called the eta meson has a rest mass of 549 MeV/c<sup>2</sup> and a mean lifetime of  $7.00 \times 10^{-19}$  s. What is the uncertainty in its rest mass?
39. The frequency of oscillation of a harmonic oscillator of mass  $m$  and spring constant  $C$  is  $\nu = \sqrt{C/m}/2\pi$ . The energy of the oscillator is  $E = p^2/2m + Cx^2/2$ , where  $p$  is its momentum when its displacement from the equilibrium position is  $x$ . In classical physics the minimum energy of the oscillator is  $E_{min} = 0$ . Use the uncertainty principle to find an expression for  $E$  in terms of  $x$  only and show that the minimum energy is actually  $E_{min} = \hbar\nu/2$  by setting  $dE/dx = 0$  and solving for  $E_{min}$ .
40. (a) Verify that the uncertainty principle can be expressed in the form  $\Delta L \Delta \theta \geq \hbar/2$ , where  $\Delta L$  is the uncertainty in the angular momentum of a particle and  $\Delta \theta$  is the uncertainty in its angular position. (Hint: Consider a particle of mass  $m$  moving in a circle of radius  $r$  at the speed  $v$ , for which  $L = mur$ .) (b) At what uncertainty in  $L$  will the angular position of a particle become completely indeterminate?

## CHAPTER 4

# Atomic Structure



Solid-state infrared laser cutting 1.6-mm steel sheet. This laser uses an yttrium-aluminum-garnet crystal doped with neodymium. The neodymium is pumped with radiation from small semiconductor lasers, a highly efficient method.

### 4.1 THE NUCLEAR ATOM

An atom is largely empty space

### 4.2 ELECTRON ORBITS

The planetary model of the atom and why it fails

### 4.3 ATOMIC SPECTRA

Each element has a characteristic line spectrum

### 4.4 THE BOHR ATOM

Electron waves in the atom

### 4.5 ENERGY LEVELS AND SPECTRA

A photon is emitted when an electron jumps from one energy level to a lower level

### 4.6 CORRESPONDENCE PRINCIPLE

The greater the quantum number, the closer quantum physics approaches classical physics

### 4.7 NUCLEAR MOTION

The nuclear mass affects the wavelengths of spectral lines

### 4.8 ATOMIC EXCITATION

How atoms absorb and emit energy

### 4.9 THE LASER

How to produce light waves all in step

### APPENDIX: RUTHERFORD SCATTERING

**F**ar in the past people began to suspect that matter, despite appearing continuous, has a definite structure on a microscopic level beyond the direct reach of our senses. This suspicion did not take on a more concrete form until a little over a century and a half ago. Since then the existence of atoms and molecules, the ultimate particles of matter in its common forms, has been amply demonstrated, and their own ultimate particles, electrons, protons, and neutrons, have been identified and studied as well. In this chapter and in others to come our chief concern will be the structure of the atom, since it is this structure that is responsible for nearly all the properties of matter that have shaped the world around us.

Every atom consists of a small nucleus of protons and neutrons with a number of electrons some distance away. It is tempting to think that the electrons circle the nucleus as planets do the sun, but classical electromagnetic theory denies the possibility of stable electron orbits. In an effort to resolve this paradox, Niels Bohr applied quantum ideas to atomic structure in 1913 to obtain a model which, despite its inadequacies and later replacement by a quantum-mechanical description of greater accuracy and usefulness, still remains a convenient mental picture of the atom. Bohr's theory of the hydrogen atom is worth examining both for this reason and because it provides a valuable transition to the more abstract quantum theory of the atom.

## 4.1 THE NUCLEAR ATOM

### *An atom is largely empty space*

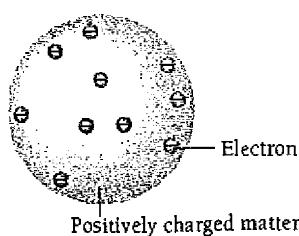


Figure 4.1 The Thomson model of the atom. The Rutherford scattering experiment showed it to be incorrect.

Most scientists of the late nineteenth century accepted the idea that the chemical elements consist of atoms, but they knew almost nothing about the atoms themselves. One clue was the discovery that all atoms contain electrons. Since electrons carry negative charges whereas atoms are neutral, positively charged matter of some kind must be present in atoms. But what kind? And arranged in what way?

One suggestion, made by the British physicist J. J. Thomson in 1898, was that atoms are just positively charged lumps of matter with electrons embedded in them, like raisins in a fruitcake (Fig. 4.1). Because Thomson had played an important role in discovering the electron, his idea was taken seriously. But the real atom turned out to be quite different.

The most direct way to find out what is inside a fruitcake is to poke a finger into it, which is essentially what Hans Geiger and Ernest Marsden did in 1911. At the suggestion of Ernest Rutherford, they used as probes the fast **alpha particles** emitted by certain radioactive elements. Alpha particles are helium atoms that have lost two electrons each, leaving them with a charge of  $+2e$ .

Geiger and Marsden placed a sample of an alpha-emitting substance behind a lead screen with a small hole in it, as in Fig. 4.2, so that a narrow beam of alpha particles was produced. This beam was directed at a thin gold foil. A zinc sulfide screen, which gives off a visible flash of light when struck by an alpha particle, was set on the other side of the foil with a microscope to see the flashes.

It was expected that the alpha particles would go right through the foil with hardly any deflection. This follows from the Thomson model, in which the electric charge inside an atom is assumed to be uniformly spread through its volume. With only weak electric forces exerted on them, alpha particles that pass through a thin foil ought to be deflected only slightly,  $1^\circ$  or less.

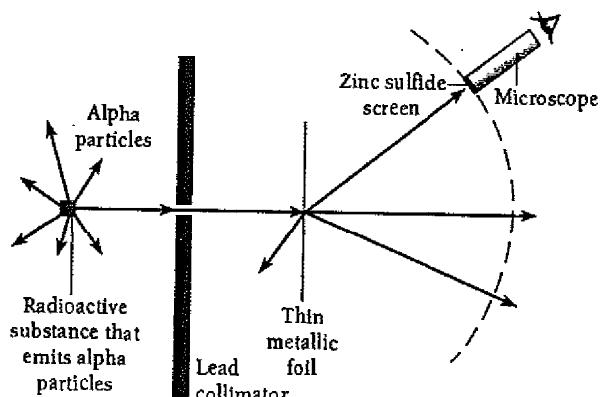


Figure 4.2 The Rutherford scattering experiment.

What Geiger and Marsden actually found was that although most of the alpha particles indeed were not deviated by much, a few were scattered through very large angles. Some were even scattered in the backward direction. As Rutherford remarked, "It was as incredible as if you fired a 15-inch shell at a piece of tissue paper and it came back and hit you."

Alpha particles are relatively heavy (almost 8000 electron masses) and those used in this experiment had high speeds (typically  $2 \times 10^7$  m/s), so it was clear that powerful forces were needed to cause such marked deflections. The only way to



Ernest Rutherford (1871–1937), a native of New Zealand, was on his family's farm digging potatoes when he learned that he had won a scholarship for graduate study at Cambridge University in England. "This is the last potato I will every dig," he said, throwing down his spade. Thirteen years later he received the Nobel Prize in chemistry.

At Cambridge, Rutherford was a research student under J. J. Thomson, who would soon announce the discovery of the electron. Rutherford's own work was on the newly found phenomenon of radioactivity, and he quickly distinguished between alpha and beta particles, two of the emissions of radioactive materials. In 1898 he went to McGill University in Canada, where he found that alpha particles are the nuclei of helium atoms and that the radioactive decay of an element gives rise to another element. Working with the chemist Frederick Soddy and others, Rutherford traced the successive transformations of radioactive elements, such as uranium and radium, until they end up as stable lead.

In 1907 Rutherford returned to England as professor of physics at Manchester, where in 1911 he showed that the nuclear model of the atom was the only one that could explain the observed scattering of alpha particles by thin metal foils. Rutherford's last important discovery, reported in 1919, was the disintegration of nitrogen nuclei when bombarded with alpha particles, the first example of the artificial transmutation of elements into other elements. After other similar experiments, Rutherford suggested that all nuclei contain hydrogen nuclei, which he called protons. He also proposed that a neutral particle was present in nuclei as well.

In 1919 Rutherford became director of the Cavendish Laboratory at Cambridge, where under his stimulus great strides in understanding the nucleus continued to be made. James Chadwick discovered the neutron there in 1932. The Cavendish Laboratory was the site of the first accelerator for producing high-energy particles. With the help of this accelerator, fusion reactions in which light nuclei unite to form heavier nuclei were observed for the first time.

Rutherford was not infallible: only a few years before the discovery of fission and the building of the first nuclear reactor, he dismissed the idea of practical uses for nuclear energy as "moonshine." He died in 1937 of complications of a hernia and was buried near Newton in Westminster Abbey.

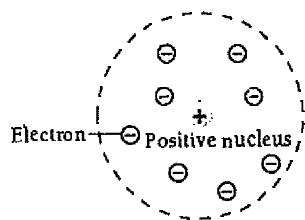


Figure 4.3 The Rutherford model of the atom.

explain the results, Rutherford found, was to picture an atom as being composed of a tiny nucleus in which its positive charge and nearly all its mass are concentrated, with the electrons some distance away (Fig. 4.3). With an atom being largely empty space, it is easy to see why most alpha particles go right through a thin foil. However, when an alpha particle happens to come near a nucleus, the intense electric field there scatters it through a large angle. The atomic electrons, being so light, do not appreciably affect the alpha particles.

The experiments of Geiger and Marsden and later work of a similar kind also supplied information about the nuclei of the atoms that composed the various target foils. The deflection of an alpha particle when it passes near a nucleus depends on the magnitude of the nuclear charge. Comparing the relative scattering of alpha particles by different foils thus provides a way to find the nuclear charges of the atoms involved.

All the atoms of any one element turned out to have the same unique nuclear charge, and this charge increased regularly from element to element in the periodic table. The nuclear charges always turned out to be multiples of  $+e$ ; the number  $Z$  of unit positive charges in the nuclei of an element is today called the atomic number of the element. We know now that protons, each with a charge  $+e$ , provide the charge on a nucleus, so the atomic number of an element is the same as the number of protons in the nuclei of its atoms.

Ordinary matter, then, is mostly empty space. The solid wood of a table, the steel that supports a bridge, the hard rock underfoot, all are simply collections of tiny charged particles comparatively farther away from one another than the sun is from the planets. If all the actual matter, electrons and nuclei, in our bodies could somehow be packed closely together, we would shrivel to specks just visible with a microscope.

### Rutherford Scattering Formula

The formula that Rutherford obtained for alpha particle scattering by a thin foil on the basis of the nuclear model of the atom is

$$\text{Rutherford scattering formula} \quad N(\theta) = \frac{N_i n t Z^2 e^4}{(8\pi\epsilon_0)^2 r^2 KE^2 \sin^4(\theta/2)} \quad (4.1)$$

This formula is derived in the Appendix to this chapter. The symbols in Eq. (4.1) have the following meanings:

$N(\theta)$  = number of alpha particles per unit area that reach the screen at a scattering angle of  $\theta$

$N_i$  = total number of alpha particles that reach the screen

$n$  = number of atoms per unit volume in the foil

$Z$  = atomic number of the foil atoms

$r$  = distance of the screen from the foil

$KE$  = kinetic energy of the alpha particles

$t$  = foil thickness

The predictions of Eq. (4.1) agreed with the measurements of Geiger and Marsden, which supported the hypothesis of the nuclear atom. This is why Rutherford is credited

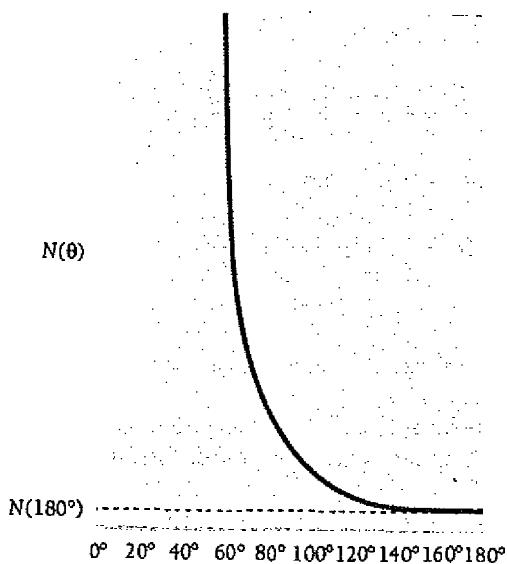


Figure 4.4 Rutherford scattering.  $N(\theta)$  is the number of alpha particles per unit area that reach the screen at a scattering angle of  $\theta$ ;  $N(180^\circ)$  is this number for backward scattering. The experimental findings follow this curve, which is based on the nuclear model of the atom.

with the "discovery" of the nucleus. Because  $N(\theta)$  is inversely proportional to  $\sin^4(\theta/2)$  the variation of  $N(\theta)$  with  $\theta$  is very pronounced (Fig. 4.4): only 0.14 percent of the incident alpha particles are scattered by more than  $1^\circ$ .

### Nuclear Dimensions

In his derivation of Eq. (4.1) Rutherford assumed that the size of a target nucleus is small compared with the minimum distance  $R$  to which incident alpha particles approach the nucleus before being deflected away. Rutherford scattering therefore gives us a way to find an upper limit to nuclear dimensions.

Let us see what the distance of closest approach  $R$  was for the most energetic alpha particles employed in the early experiments. An alpha particle will have its smallest  $R$  when it approaches a nucleus head on, which will be followed by a  $180^\circ$  scattering. At the instant of closest approach the initial kinetic energy  $KE$  of the particle is entirely converted to electric potential energy, and so at that instant

$$KE_{\text{initial}} = PE = \frac{1}{4\pi\epsilon_0} \frac{2Ze^2}{R}$$

since the charge of the alpha particle is  $2e$  and that of the nucleus is  $Ze$ . Hence

$$\text{Distance of closest approach} \quad R = \frac{2Ze^2}{4\pi\epsilon_0 KE_{\text{initial}}} \quad (4.2)$$

The maximum KE found in alpha particles of natural origin is 7.7 MeV, which is  $1.2 \times 10^{-12}$  J. Since  $1/4\pi\epsilon_0 = 9.0 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2$ ,

$$R = \frac{(2)(9.0 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2)(1.6 \times 10^{-19} \text{ C})^2 Z}{1.2 \times 10^{-12} \text{ J}}$$

$$= 3.8 \times 10^{-16} Z \text{ m}$$

The atomic number of gold, a typical foil material, is  $Z = 79$ , so that

$$R (\text{Au}) = 3.0 \times 10^{-14} \text{ m}$$

The radius of the gold nucleus is therefore less than  $3.0 \times 10^{-14}$  m, well under  $10^{-4}$  the radius of the atom as a whole.

In more recent years particles of much higher energies than 7.7 MeV have been artificially accelerated, and it has been found that the Rutherford scattering formula does indeed eventually fail to agree with experiment. These experiments and the information they provide on actual nuclear dimensions are discussed in Chap. 11. The radius of the gold nucleus turns out to be about  $\frac{1}{3}$  of the value of  $R (\text{Au})$  found above.

### Neutron Stars

The density of nuclear matter is about  $2.4 \times 10^{17} \text{ kg/m}^3$ , which is equivalent to 4 billion tons per cubic inch. As discussed in Sec. 9.11, neutron stars are stars whose atoms have been so compressed that most of their protons and electrons have fused into neutrons, which are the most stable form of matter under enormous pressures. The densities of neutron stars are comparable to those of nuclei: a neutron star packs the mass of one or two suns into a sphere only about 10 km in radius. If the earth were this dense, it would fit into a large apartment house.

## 4.2 ELECTRON ORBITS

### *The planetary model of the atom and why it fails*

The Rutherford model of the atom, so convincingly confirmed by experiment, pictures a tiny, massive, positively charged nucleus surrounded at a relatively great distance by enough electrons to render the atom electrically neutral as a whole. The electrons cannot be stationary in this model, because there is nothing that can keep them in place against the electric force pulling them to the nucleus. If the electrons are in motion, however, dynamically stable orbits like those of the planets around the sun are possible (Fig. 4.5).

Let us look at the classical dynamics of the hydrogen atom, whose single electron makes it the simplest of all atoms. We assume a circular electron orbit for convenience, though it might as reasonably be assumed to be elliptical in shape. The centripetal force

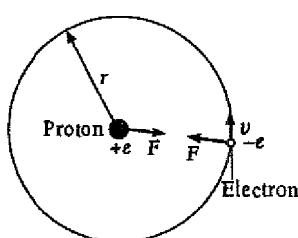


Figure 4.5 Force balance in the hydrogen atom.

$$F_c = \frac{mv^2}{r}$$

holding the electron in an orbit  $r$  from the nucleus is provided by the electric force

$$F_e = \frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2}$$

between them. The condition for a dynamically stable orbit is

$$\begin{aligned} F_c &= F_e \\ \frac{mv^2}{r} &= \frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2} \end{aligned} \quad (4.3)$$

The electron velocity  $v$  is therefore related to its orbit radius  $r$  by the formula

$$\text{Electron velocity } v = \frac{e}{\sqrt{4\pi\epsilon_0 mr}} \quad (4.4)$$

The total energy  $E$  of the electron in a hydrogen atom is the sum of its kinetic and potential energies, which are

$$\text{KE} = \frac{1}{2} mv^2 \quad \text{PE} = -\frac{e^2}{4\pi\epsilon_0 r}$$

(The minus sign follows from the choice of PE = 0 at  $r = \infty$ , that is, when the electron and proton are infinitely far apart.) Hence

$$E = \text{KE} + \text{PE} = \frac{mv^2}{2} - \frac{e^2}{4\pi\epsilon_0 r}$$

Substituting for  $v$  from Eq. (4.4) gives

$$\begin{aligned} E &= \frac{e^2}{8\pi\epsilon_0 r} - \frac{e^2}{4\pi\epsilon_0 r} \\ \text{Total energy of} \\ \text{hydrogen atom} &= -\frac{e^2}{8\pi\epsilon_0 r} \end{aligned} \quad (4.5)$$

The total energy of the electron is negative. This holds for every atomic electron and reflects the fact that it is bound to the nucleus. If  $E$  were greater than zero, an electron would not follow a closed orbit around the nucleus.

Actually, of course, the energy  $E$  is not a property of the electron alone but is a property of the system of electron + nucleus. The effect of the sharing of  $E$  between the electron and the nucleus is considered in Sec. 4.7.

### Example 4.1

Experiments indicate that 13.6 eV is required to separate a hydrogen atom into a proton and an electron; that is, its total energy is  $E = -13.6$  eV. Find the orbital radius and velocity of the electron in a hydrogen atom.

**Solution**

Since  $13.6 \text{ eV} = 2.2 \times 10^{-18} \text{ J}$ , from Eq. (4.5)

$$r = -\frac{e^2}{8\pi\epsilon_0 E} = -\frac{(1.6 \times 10^{-19} \text{ C})^2}{(8\pi)(8.85 \times 10^{-12} \text{ F/m})(-2.2 \times 10^{-18} \text{ J})} \\ = 5.3 \times 10^{-11} \text{ m}$$

An atomic radius of this magnitude agrees with estimates made in other ways. The electron's velocity can be found from Eq. (4.4):

$$v = \frac{e}{\sqrt{4\pi\epsilon_0 mr}} = \frac{1.6 \times 10^{-19} \text{ C}}{\sqrt{(4\pi)(8.85 \times 10^{-12} \text{ F/m})(9.1 \times 10^{-31} \text{ kg})(5.3 \times 10^{-11} \text{ m})}} \\ = 2.2 \times 10^6 \text{ m/s}$$

Since  $v \ll c$ , we can ignore special relativity when considering the hydrogen atom.

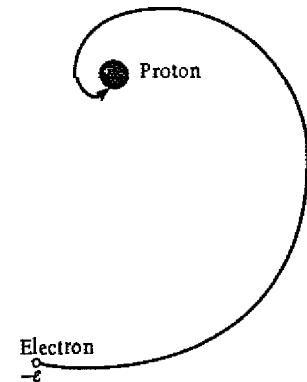


Figure 4.6 An atomic electron should, classically, spiral rapidly into the nucleus as it radiates energy due to its acceleration.

**The Failure of Classical Physics**

The analysis above is a straightforward application of Newton's laws of motion and Coulomb's law of electric force—both pillars of classical physics—and is in accord with the experimental observation that atoms are stable. However, it is not in accord with electromagnetic theory—another pillar of classical physics—which predicts that accelerated electric charges radiate energy in the form of em waves. An electron pursuing a curved path is accelerated and therefore should continuously lose energy, spiraling into the nucleus in a fraction of a second (Fig. 4.6).

But atoms do not collapse. This contradiction further illustrates what we saw in the previous two chapters: The laws of physics that are valid in the macroworld do not always hold true in the microworld of the atom.

**Is Rutherford's Analysis Valid?**

**A**n interesting question comes up at this point. When he derived his scattering formula, Rutherford used the same laws of physics that prove such dismal failures when applied to atomic stability. Might it not be that this formula is not correct and that in reality the atom does not resemble Rutherford's model of a small central nucleus surrounded by distant electrons? This is not a trivial point. It is a curious coincidence that the quantum-mechanical analysis of alpha particle scattering by thin foils yields precisely the same formula that Rutherford found.

To verify that a classical calculation ought to be at least approximately correct, we note that the de Broglie wavelength of an alpha particle whose speed is  $2.0 \times 10^7 \text{ m/s}$  is

$$\lambda = \frac{h}{mv} = \frac{6.63 \times 10^{-34} \text{ J} \cdot \text{s}}{(6.6 \times 10^{-27} \text{ kg})(2.0 \times 10^7 \text{ m/s})} \\ = 5.0 \times 10^{-15} \text{ m}$$

As we saw in Sec. 4.1, the closest an alpha particle with this wavelength ever gets to a gold nucleus is  $3.0 \times 10^{-14} \text{ m}$ , which is six de Broglie wavelengths. It is therefore just reasonable to regard the alpha particle as a classical particle in the interaction. We are correct in thinking of the atom in terms of Rutherford's model, though the dynamics of the atomic electrons—which is another matter—requires a nonclassical approach.

Classical physics fails to provide a meaningful analysis of atomic structure because it approaches nature in terms of "pure" particles and "pure" waves. In reality particles and waves have many properties in common, though the smallness of Planck's constant makes the wave-particle duality imperceptible in the macroworld. The usefulness of classical physics decreases as the scale of the phenomena under study decreases, and we must allow for the particle behavior of waves and the wave behavior of particles to understand the atom. In the rest of this chapter we shall see how the Bohr atomic model, which combines classical and modern notions, accomplishes part of the latter task. Not until we consider the atom from the point of view of quantum mechanics, which makes no compromise with the intuitive notions we pick up in our daily lives, will we find a really successful theory of the atom.

### 4.3 ATOMIC SPECTRA

*Each element has a characteristic line spectrum*

Atomic stability is not the only thing that a successful theory of the atom must account for. The existence of spectral lines is another important aspect of the atom that finds no explanation in classical physics.

We saw in Chap. 2 that condensed matter (solids and liquids) at all temperatures emits em radiation in which all wavelengths are present, though with different intensities. The observed features of this radiation were explained by Planck without reference to exactly how it was produced by the radiating material or to the nature of the material. From this it follows that we are witnessing the collective behavior of a great many interacting atoms rather than the characteristic behavior of the atoms of a particular element.

At the other extreme, the atoms or molecules in a rarefied gas are so far apart on the average that they only interact during occasional collisions. Under these circumstances we would expect any emitted radiation to be characteristic of the particular atoms or molecules present, which turns out to be the case.

When an atomic gas or vapor at somewhat less than atmospheric pressure is suitably "excited," usually by passing an electric current through it, the emitted radiation has a spectrum which contains certain specific wavelengths only. An idealized arrangement for observing such atomic spectra is shown in Fig. 4.7; actual spectrometers use diffraction

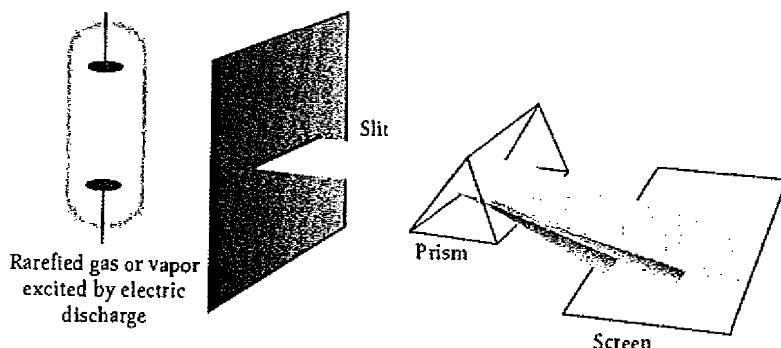


Figure 4.7 An idealized spectrometer.

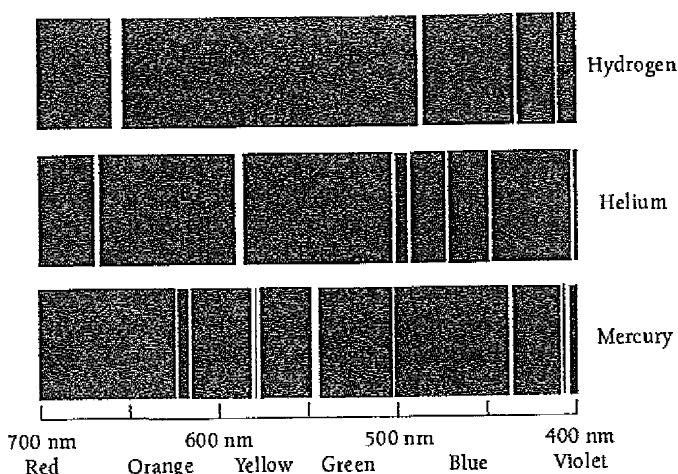
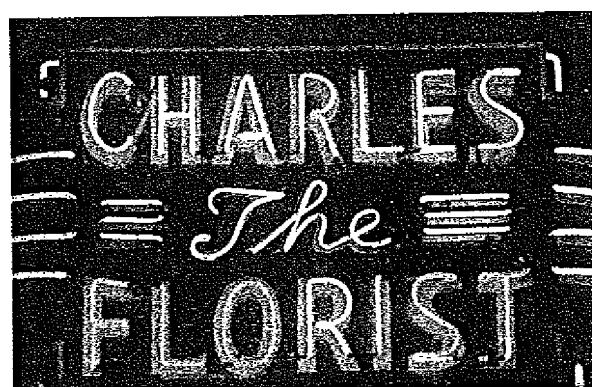


Figure 4.8 Some of the principal lines in the emission spectra of hydrogen, helium, and mercury.



Gas atoms excited by electric currents in these tubes radiate light of wavelengths characteristic of the gas used.

gratings. Figure 4.8 shows the **emission line spectrum** of several elements. Every element displays a unique line spectrum when a sample of it in the vapor phase is excited. Spectroscopy is therefore a useful tool for analyzing the composition of an unknown substance.

When white light is passed through a gas, the gas is found to absorb light of certain of the wavelengths present in its emission spectrum. The resulting **absorption line spectrum** consists of a bright background crossed by dark lines that correspond to the missing wavelengths (Fig. 4.9); emission spectra consist of bright lines on a dark background. The spectrum of sunlight has dark lines in it because the luminous part of the



Figure 4.9 The dark lines in the absorption spectrum of an element correspond to bright lines in its emission spectrum.

sun, which radiates very nearly like a blackbody heated to 5800 K, is surrounded by an envelope of cooler gas that absorbs light of certain wavelengths only. Most other stars have spectra of this kind.

The number, intensity, and exact wavelengths of the lines in the spectrum of an element depend upon temperature, pressure, the presence of electric and magnetic fields, and the motion of the source. It is possible to tell by examining its spectrum not only what elements are present in a light source but much about their physical state. An astronomer, for example, can establish from the spectrum of a star which elements its atmosphere contains, whether they are ionized, and whether the star is moving toward or away from the earth.

### Spectral Series

A century ago the wavelengths in the spectrum of an element were found to fall into sets called **spectral series**. The first such series was discovered by J. J. Balmer in 1885 in the course of a study of the visible part of the hydrogen spectrum. Figure 4.10 shows the **Balmer series**. The line with the longest wavelength, 656.3 nm, is designated  $H_{\alpha}$ , the next, whose wavelength is 486.3 nm, is designated  $H_{\beta}$ , and so on. As the wave-length decreases, the lines are found closer together and weaker in intensity until the series limit at 364.6 nm is reached, beyond which there are no further separate lines but only a faint continuous spectrum. Balmer's formula for the wavelengths of this series is

$$\text{Balmer} \quad \frac{1}{\lambda} = R \left( \frac{1}{2^2} - \frac{1}{n^2} \right) \quad n = 3, 4, 5, \dots \quad (4.6)$$

The quantity  $R$ , known as the **Rydberg constant**, has the value

$$\text{Rydberg constant} \quad R = 1.097 \times 10^7 \text{ m}^{-1} = 0.01097 \text{ nm}^{-1}$$

The  $H_{\alpha}$  line corresponds to  $n = 3$ , the  $H_{\beta}$  line to  $n = 4$ , and so on. The series limit corresponds to  $n = \infty$ , so that it occurs at a wavelength of  $4/R$ , in agreement with experiment.

The Balmer series contains wavelengths in the visible portion of the hydrogen spectrum. The spectral lines of hydrogen in the ultraviolet and infrared regions fall into several other series. In the ultraviolet the **Lyman series** contains the wavelengths given by the formula

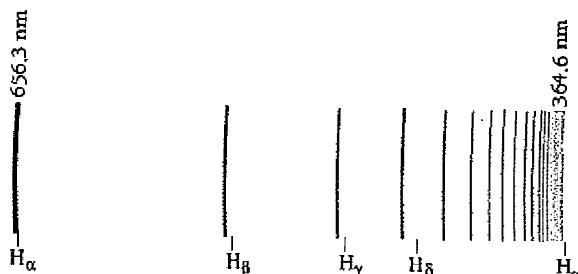


Figure 4.10 The Balmer series of hydrogen. The  $H_{\alpha}$  line is red, the  $H_{\beta}$  line is blue, the  $H_{\gamma}$  and  $H_{\delta}$  lines are violet, and the other lines are in the near ultraviolet.

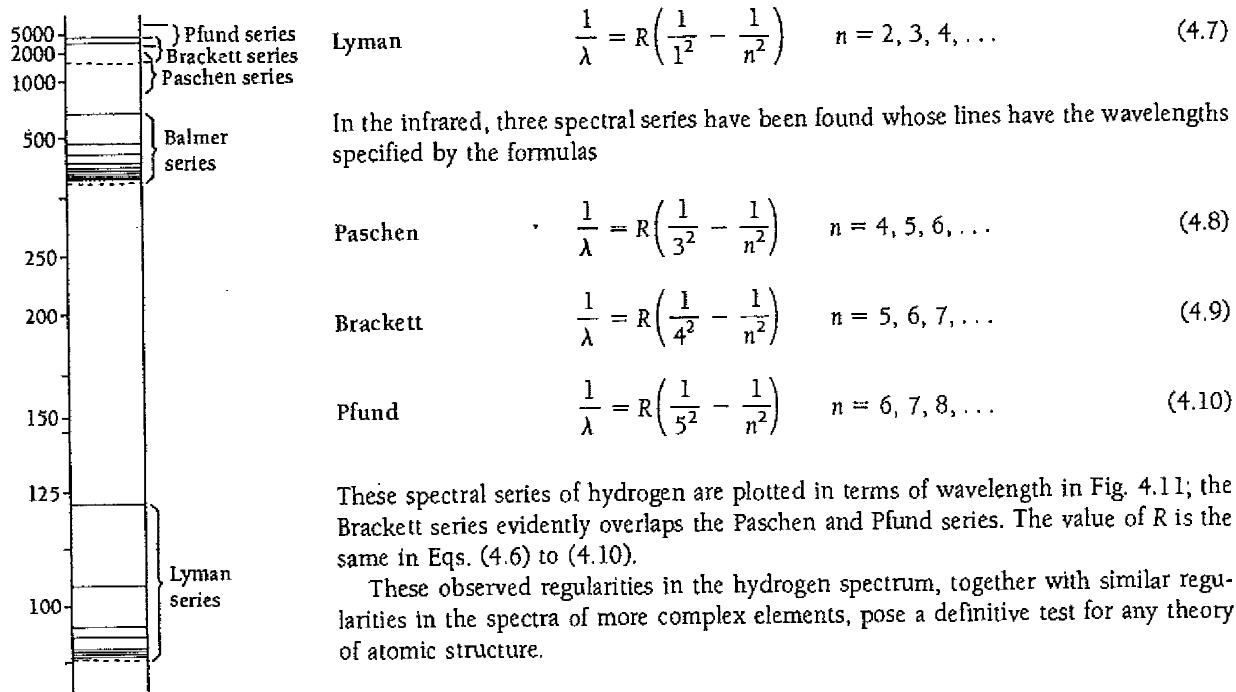


Figure 4.11 The spectral series of hydrogen. The wavelengths in each series are related by simple formulas.

#### 4.4 THE BOHR ATOM

##### *Electron waves in the atom*

The first theory of the atom to meet with any success was put forward in 1913 by Niels Bohr. The concept of matter waves leads in a natural way to this theory, as de Broglie found, and this is the route that will be followed here. Bohr himself used a different approach, since de Broglie's work came a decade later, which makes his achievement all the more remarkable. The results are exactly the same, however.

We start by examining the wave behavior of an electron in orbit around a hydrogen nucleus. (In this chapter, since the electron velocities are much smaller than  $c$ , we will assume that  $\gamma = 1$  and for simplicity omit  $\gamma$  from the various equations.) The de Broglie wavelength of this electron is

$$\lambda = \frac{h}{mv}$$

where the electron velocity  $v$  is that given by Eq. (4.4):

$$v = \frac{e}{\sqrt{4\pi\epsilon_0 mr}}$$

Hence

$$\text{Orbital electron wavelength} \quad \lambda = \frac{h}{e} \sqrt{\frac{4\pi\epsilon_0 r}{m}} \quad (4.11)$$

By substituting  $5.3 \times 10^{-11}$  m for the radius  $r$  of the electron orbit (see Example 4.1), we find the electron wavelength to be

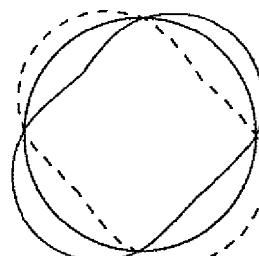
$$\lambda = \frac{6.63 \times 10^{-34} \text{ J} \cdot \text{s}}{1.6 \times 10^{-19} \text{ C}} \sqrt{\frac{(4\pi)(8.85 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2)(5.3 \times 10^{-11} \text{ m})}{9.1 \times 10^{-31} \text{ kg}}} \\ = 33 \times 10^{-11} \text{ m}$$

This wavelength is exactly the same as the circumference of the electron orbit,

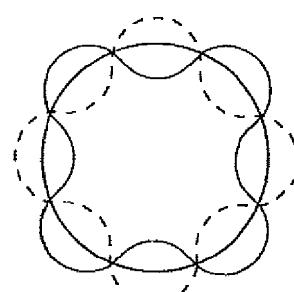
$$2\pi r = 33 \times 10^{-11} \text{ m}$$

The orbit of the electron in a hydrogen atom corresponds to one complete electron wave joined on itself (Fig. 4.12)!

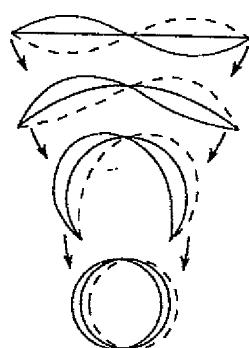
The fact that the electron orbit in a hydrogen atom is one electron wavelength in circumference provides the clue we need to construct a theory of the atom. If we consider the vibrations of a wire loop (Fig. 4.13), we find that their wavelengths always fit an integral number of times into the loop's circumference so that each wave joins smoothly with the next. If the wire were perfectly elastic, these vibrations would continue indefinitely. Why are these the only vibrations possible in a wire loop? If a fractional number of wavelengths is placed around a loop, as in Fig. 4.14, destructive



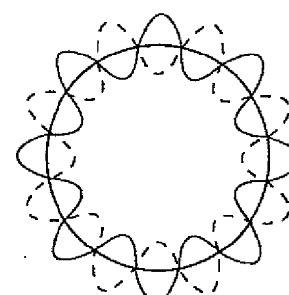
Circumference = 2 wavelengths



Circumference = 4 wavelengths



— Electron path  
— De Broglie electron wave



Circumference = 8 wavelengths

Figure 4.12 The orbit of the electron in a hydrogen atom corresponds to a complete electron de Broglie wave joined on itself.

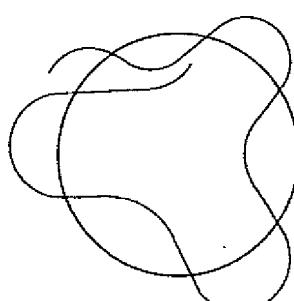


Figure 4.14 A fractional number of wavelengths cannot persist because destructive interference will occur.

Figure 4.13 Some modes of vibration of a wire loop. In each case a whole number of wavelengths fit into the circumference of the loop.



Niels Bohr (1885–1962) was born and spent most of his life in Copenhagen, Denmark. After receiving his doctorate at the university there in 1911, Bohr went to England to broaden his scientific horizons. At Rutherford's laboratory in Manchester, Bohr was introduced to the just-discovered nuclear model of the atom, which was in conflict with the existing principles of physics. Bohr realized

that it was "hopeless" to try to make sense of the atom in the framework of classical physics alone, and he felt that the quantum theory of light must somehow be the key to understanding atomic structure.

Back in Copenhagen in 1913, a friend suggested to Bohr that Balmer's formula for one set of the spectral lines of hydrogen might be relevant to his quest. "As soon as I saw Balmer's formula the whole thing was immediately clear to me," Bohr said later. To construct his theory, Bohr began with two revolutionary ideas. The first was that an atomic electron can circle its nucleus only in certain orbits, and the other was that an atom emits or absorbs a photon of light when an electron jumps from one permitted orbit to another.

What is the condition for a permitted orbit? To find out, Bohr used as a guide what became known as the correspondence principle: When quantum numbers are very large, quantum effects should not be conspicuous, and the quantum theory must then give the same results as classical physics. Applying this principle showed that the electron in a permitted orbit must have an angular momentum that is a multiple

of  $\hbar = h/2\pi$ . A decade later Louis de Broglie explained this quantization of angular momentum in terms of the wave nature of a moving electron.

Bohr was able to account for all the spectral series of hydrogen, not just the Balmer series, but the publication of the theory aroused great controversy. Einstein, an enthusiastic supporter of the theory (which "appeared to me like a miracle—and appears to me as a miracle even today," he wrote many years later), nevertheless commented on its bold mix of classical and quantum concepts, "One ought to be ashamed of the successes [of the theory] because they have been earned according to the Jesuit maxim, 'Let not thy left hand know what the other doeth.'" Other noted physicists were more deeply disturbed: Otto Stern and Max von Laue said they would quit physics if Bohr were right. (They later changed their minds.) Bohr and others tried to extend his model to many-electron atoms with occasional success—for instance, the correct prediction of the properties of the then-unknown element hafnium—but real progress had to wait for Wolfgang Pauli's exclusion principle of 1925.

In 1916 Bohr returned to Rutherford's laboratory, where he stayed until 1919. Then an Institute of Theoretical Physics was created for him in Copenhagen, and he directed it until his death. The institute was a magnet for quantum theoreticians from all over the world, who were stimulated by the exchange of ideas at regular meetings there. Bohr received the Nobel Prize in 1922. His last important work came in 1939, when he used an analogy between a large nucleus and a liquid drop to explain why nuclear fission, which had just been discovered, occurs in certain nuclei but not in others. During World War II Bohr contributed to the development of the atomic bomb at Los Alamos, New Mexico. After the war, Bohr returned to Copenhagen, where he died in 1962.

interference will occur as the waves travel around the loop, and the vibrations will die out rapidly.

By considering the behavior of electron waves in the hydrogen atom as analogous to the vibrations of a wire loop, then, we can say that

An electron can circle a nucleus only if its orbit contains an integral number of de Broglie wavelengths.

This statement combines both the particle and wave characters of the electron since the electron wavelength depends upon the orbital velocity needed to balance the pull of the nucleus. To be sure, the analogy between an atomic electron and the standing waves of Fig. 4.13 is hardly the last word on the subject, but it represents an illuminating step along the path to the more profound and comprehensive, but also more abstract, quantum-mechanical theory of the atom.

It is easy to express the condition that an electron orbit contain an integral number of de Broglie wavelengths. The circumference of a circular orbit of radius  $r$  is  $2\pi r$ , and so the condition for orbit stability is

Condition for orbit stability  $n\lambda = 2\pi r_n \quad n = 1, 2, 3, \dots$  (4.12)

where  $r_n$  designates the radius of the orbit that contain  $n$  wavelengths. The integer  $n$  is called the quantum number of the orbit. Substituting for  $\lambda$ , the electron wavelength given by Eq. (4.11), yields

$$\frac{nh}{e} \sqrt{\frac{4\pi\epsilon_0 r_n}{m}} = 2\pi r_n$$

and so the possible electron orbits are those whose radii are given by

Orbital radii in Bohr atom  $r_n = \frac{n^2 h^2 \epsilon_0}{\pi m e^2} \quad n = 1, 2, 3, \dots$  (4.13)

The radius of the innermost orbit is customarily called the **Bohr radius** of the hydrogen atom and is denoted by the symbol  $a_0$ :

Bohr radius  $a_0 = r_1 = 5.292 \times 10^{-11} \text{ m}$

The other radii are given in terms of  $a_0$  by the formula

$$r_n = n^2 a_0 \quad (4.14)$$

## 4.5 ENERGY LEVELS AND SPECTRA

*A photon is emitted when an electron jumps from one energy level to a lower level*

The various permitted orbits involve different electron energies. The electron energy  $E_n$  is given in terms of the orbit radius  $r_n$  by Eq. (4.5) as

$$E_n = -\frac{e^2}{8\pi\epsilon_0 r_n}$$

Substituting for  $r_n$  from Eq. (4.13), we see that

Energy levels  $E_n = -\frac{me^4}{8\epsilon_0^2 h^2} \left(\frac{1}{n^2}\right) = \frac{E_1}{n^2} \quad n = 1, 2, 3, \dots$  (4.15)

$$E_1 = -2.18 \times 10^{-18} \text{ J} = -13.6 \text{ eV}$$

The energies specified by Eq. (4.15) are called the **energy levels** of the hydrogen atom and are plotted in Fig. 4.15. These levels are all negative, which signifies that the electron does not have enough energy to escape from the nucleus. An atomic electron can have only these energies and no others. An analogy might be a person on a ladder, who can stand only on its steps and not in between.

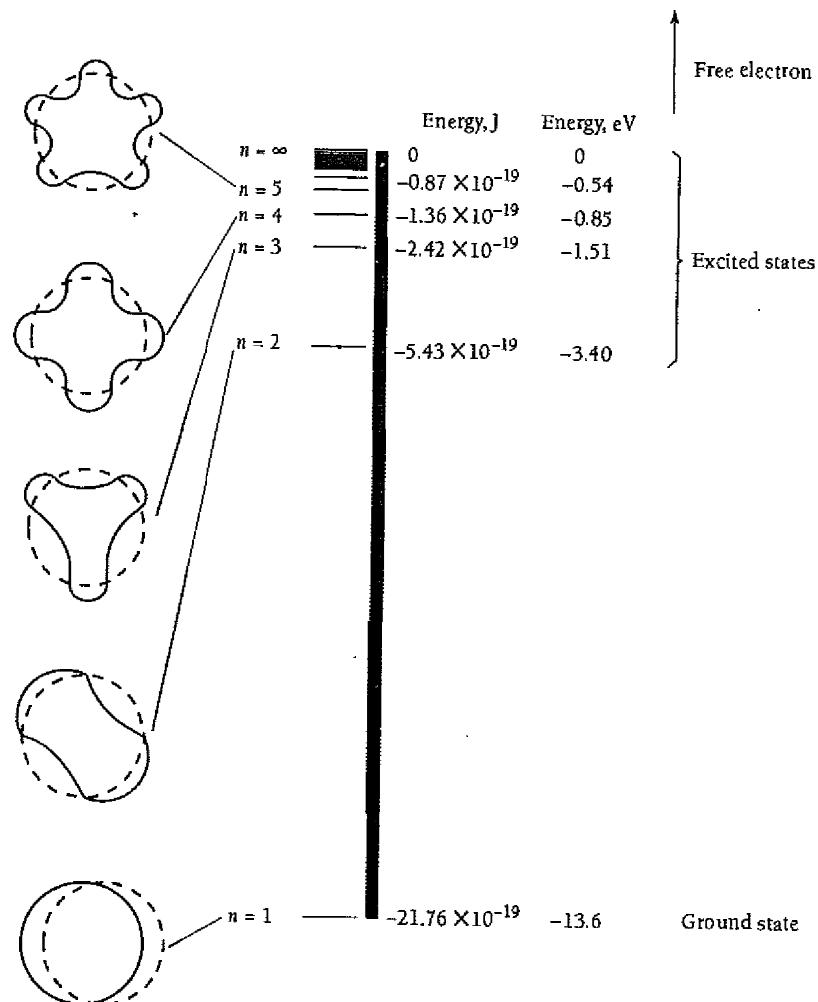


Figure 4.15 Energy levels of the hydrogen atom.

The lowest energy level  $E_1$  is called the **ground state** of the atom, and the higher levels  $E_2, E_3, E_4, \dots$  are called **excited states**. As the quantum number  $n$  increases, the corresponding energy  $E_n$  approaches closer to 0. In the limit of  $n = \infty$ ,  $E_{\infty} = 0$  and the electron is no longer bound to the nucleus to form an atom. A positive energy for a nucleus-electron combination means that the electron is free and has no quantum conditions to fulfill; such a combination does not constitute an atom, of course.

The work needed to remove an electron from an atom in its ground state is called its **ionization energy**. The ionization energy is accordingly equal to  $-E_1$ , the energy that must be provided to raise an electron from its ground state to an energy of  $E = 0$ , when it is free. In the case of hydrogen, the ionization energy is 13.6 eV since the ground-state energy of the hydrogen atom is  $-13.6$  eV. Figure 7.10 shows the ionization energies of the elements.

**Example 4.2**

An electron collides with a hydrogen atom in its ground state and excites it to a state of  $n = 3$ . How much energy was given to the hydrogen atom in this inelastic (KE not conserved) collision?

**Solution**

From Eq. (4.15) the energy change of a hydrogen atom that goes from an initial state of quantum number  $n_i$  to a final state of quantum number  $n_f$  is

$$\Delta E = E_f - E_i = \frac{E_1}{n_f^2} - \frac{E_1}{n_i^2} = E_1 \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right)$$

Here  $n_i = 1$ ,  $n_f = 3$ , and  $E_1 = -13.6$  eV, so

$$\Delta E = -13.6 \left( \frac{1}{3^2} - \frac{1}{1^2} \right) \text{eV} = 12.1 \text{ eV}$$

**Example 4.3**

Hydrogen atoms in states of high quantum number have been created in the laboratory and observed in space. They are called Rydberg atoms. (a) Find the quantum number of the Bohr orbit in a hydrogen atom whose radius is 0.0100 mm. (b) What is the energy of a hydrogen atom in this state?

**Solution**

(a) From Eq. (4.14) with  $r_n = 1.00 \times 10^{-5}$  m,

$$n = \sqrt{\frac{r_n}{a_0}} = \sqrt{\frac{1.00 \times 10^{-5} \text{ m}}{5.29 \times 10^{-11} \text{ m}}} = 435$$

(b) From Eq. (4.15),

$$E_n = \frac{E_1}{n^2} = \frac{-13.6 \text{ eV}}{(435)^2} = -7.19 \times 10^{-5} \text{ eV}$$

Rydberg atoms are obviously extremely fragile and are easily ionized, which is why they are found in nature only in the near-vacuum of space. The spectra of Rydberg atoms range down to radio frequencies and their existence was established from radio telescope data.

**Origin of Line Spectra**

We must now confront the equations developed above with experiment. An especially striking observation is that atoms exhibit line spectra in both emission and absorption. Do such spectra follow from our model?

The presence of discrete energy levels in the hydrogen atom suggests the connection. Let us suppose that when an electron in an excited state drops to a lower state, the lost energy is emitted as a single photon of light. According to our model, electrons cannot exist in an atom except in certain specific energy levels. The jump of an electron from one level to another, with the difference in energy between the levels being given off all at once in a photon rather than in some more gradual manner, fits in well with this model.

## Quantization in the Atomic World

**S**equences of energy levels are characteristic of all atoms, not just those of hydrogen. As in the case of a particle in a box, the confinement of an electron to a region of space leads to restrictions on its possible wave functions that in turn limit the possible energies to well-defined values only. The existence of atomic energy levels is a further example of the quantization, or graininess, of physical quantities on a microscopic scale.

In the world of our daily lives, matter, electric charge, energy, and so forth appear to be continuous. In the world of the atom, in contrast, matter is composed of elementary particles that have definite rest masses, charge always comes in multiples of  $+e$  or  $-e$ , electromagnetic waves of frequency  $\nu$  appear as streams of photons each with the energy  $h\nu$ , and stable systems of particles, such as atoms, can possess only certain energies. As we shall find, other quantities in nature are also quantized, and this quantization enters into every aspect of how electrons, protons, and neutrons interact to endow the matter around us (and of which we consist) with its familiar properties.

If the quantum number of the initial (higher-energy) state is  $n_i$  and the quantum number of the final (lower-energy) state is  $n_f$ , we are asserting that

$$\text{Initial energy} - \text{final energy} = \text{photon energy}$$

$$E_i - E_f = h\nu \quad (4.16)$$

where  $\nu$  is the frequency of the emitted photon. From Eq. (4.15) we have

$$E_i - E_f = E_1 \left( \frac{1}{n_i^2} - \frac{1}{n_f^2} \right) = -E_1 \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right)$$

We recall that  $E_1$  is a negative quantity ( $-13.6$  eV, in fact), so  $-E_1$  is a positive quantity. The frequency of the photon released in this transition is therefore

$$\nu = \frac{E_i - E_f}{h} = -\frac{E_1}{h} \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right) \quad (4.17)$$

Since  $\lambda = c/\nu$ ,  $1/\lambda = \nu/c$  and

$$\text{Hydrogen spectrum} \quad \frac{1}{\lambda} = -\frac{E_1}{ch} \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right) \quad (4.18)$$

Equation (4.18) states that the radiation emitted by excited hydrogen atoms should contain certain wavelengths only. These wavelengths, furthermore, fall into definite sequences that depend upon the quantum number  $n_f$  of the final energy level of the electron (Fig. 4.16). Since  $n_i > n_f$  in each case, in order that there be an excess of energy to be given off as a photon, the calculated formulas for the first five series are

$$\text{Lyman} \quad n_f = 1: \quad \frac{1}{\lambda} = -\frac{E_1}{ch} \left( \frac{1}{1^2} - \frac{1}{n_i^2} \right) \quad n = 2, 3, 4, \dots$$

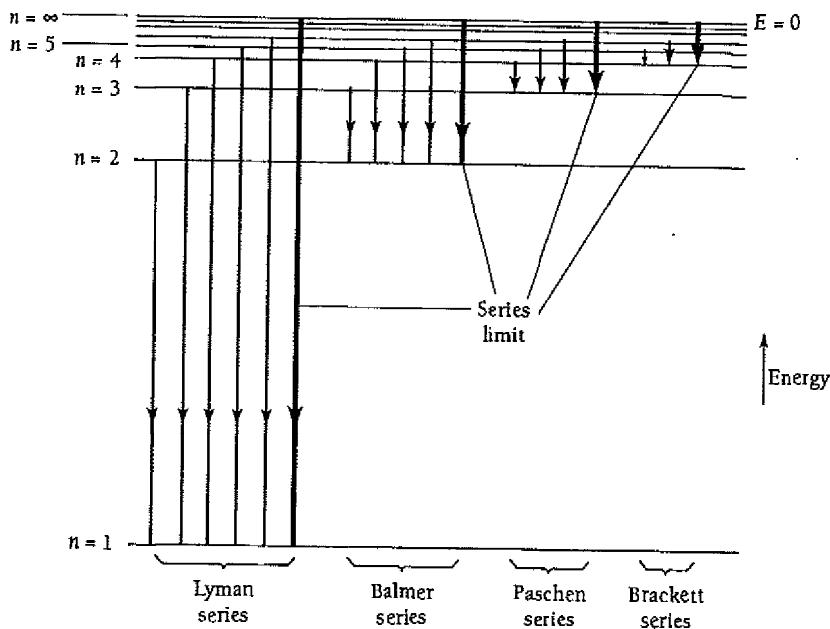


Figure 4.16 Spectral lines originate in transitions between energy levels. Shown are the spectral series of hydrogen. When  $n = \infty$ , the electron is free.

$$\text{Balmer} \quad n_f = 2: \quad \frac{1}{\lambda} = -\frac{E_1}{ch} \left( \frac{1}{2^2} - \frac{1}{n^2} \right) \quad n = 3, 4, 5, \dots$$

$$\text{Paschen} \quad n_f = 3: \quad \frac{1}{\lambda} = -\frac{E_1}{ch} \left( \frac{1}{3^2} - \frac{1}{n^2} \right) \quad n = 4, 5, 6, \dots$$

$$\text{Brackett} \quad n_f = 4: \quad \frac{1}{\lambda} = -\frac{E_1}{ch} \left( \frac{1}{4^2} - \frac{1}{n^2} \right) \quad n = 5, 6, 7, \dots$$

$$\text{Pfund} \quad n_f = 5: \quad \frac{1}{\lambda} = -\frac{E_1}{ch} \left( \frac{1}{5^2} - \frac{1}{n^2} \right) \quad n = 6, 7, 8, \dots$$

These sequences are identical in form with the empirical spectral series discussed earlier. The Lyman series corresponds to  $n_f = 1$ ; the Balmer series corresponds to  $n_f = 2$ ; the Paschen series corresponds to  $n_f = 3$ ; the Brackett series corresponds to  $n_f = 4$ ; and the Pfund series corresponds to  $n_f = 5$ .

Our final step is to compare the value of the constant term in the above equations with that of the Rydberg constant in Eqs. (4.6) to (4.10). The value of the constant term is

$$\begin{aligned} -\frac{E_1}{ch} &= \frac{me^4}{8\epsilon_0^2\hbar^3} \\ &= \frac{(9.109 \times 10^{-31} \text{ kg})(1.602 \times 10^{-19} \text{ C})^4}{(8)(8.854 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2)(2.998 \times 10^8 \text{ m/s})(6.626 \times 10^{-34} \text{ J} \cdot \text{s})^3} \\ &= 1.097 \times 10^7 \text{ m}^{-1} \end{aligned}$$

which is indeed the same as R. Bohr's model of the hydrogen atom is therefore in accord with the spectral data.

### Example 4.4

Find the longest wavelength present in the Balmer series of hydrogen, corresponding to the  $H_{\alpha}$  line.

**Solution**

In the Balmer series the quantum number of the final state is  $n_f = 2$ . The longest wavelength in this series corresponds to the smallest energy difference between energy levels. Hence the initial state must be  $n_i = 3$  and

$$\frac{1}{\lambda} = R \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right) = R \left( \frac{1}{2^2} - \frac{1}{3^2} \right) = 0.139R$$

$$\lambda = \frac{1}{0.139R} = \frac{1}{0.139(1.097 \times 10^7 \text{ m}^{-1})} = 6.56 \times 10^{-7} \text{ m} = 656 \text{ nm}$$

This wavelength is near the red end of the visible spectrum.

## 4.6 CORRESPONDENCE PRINCIPLE

*The greater the quantum number, the closer quantum physics approaches classical physics*

Quantum physics, so different from classical physics in the microworld beyond reach of our senses, must nevertheless give the same results as classical physics in the macroworld where experiments show that the latter is valid. We have already seen that this basic requirement is true for the wave theory of moving bodies. We shall now find that it is also true for Bohr's model of the hydrogen atom.

According to electromagnetic theory, an electron moving in a circular orbit radiates em waves whose frequencies are equal to its frequency of revolution and to harmonics (that is, integral multiples) of that frequency. In a hydrogen atom the electron's speed is

$$v = \frac{e}{\sqrt{4\pi\epsilon_0 mr}}$$

according to Eq. (4.4), where  $r$  is the radius of its orbit. Hence the frequency of revolution  $f$  of the electron is

$$f = \frac{\text{electron speed}}{\text{orbit circumference}} = \frac{v}{2\pi r} = \frac{e}{2\pi\sqrt{4\pi\epsilon_0 mr^3}}$$

The radius  $r_n$  of a stable orbit is given in terms of its quantum number  $n$  by Eq. (4.13) as

$$r_n = \frac{n^2 h^2 \epsilon_0}{\pi m e^2}$$

and so the frequency of revolution is

$$\text{Frequency of revolution } f = \frac{me^4}{8\epsilon_0^2 h^3} \left( \frac{2}{n^3} \right) = \frac{-E_1}{h} \left( \frac{2}{n^3} \right) \quad (4.19)$$

### Example 4.5

(a) Find the frequencies of revolution of electrons in  $n = 1$  and  $n = 2$  Bohr orbits. (b) What is the frequency of the photon emitted when an electron in an  $n = 2$  orbit drops to an  $n = 1$  orbit? (c) An electron typically spends about  $10^{-8}$  s in an excited state before it drops to a lower state by emitting a photon. How many revolutions does an electron in an  $n = 2$  Bohr orbit make in  $1.00 \times 10^{-8}$  s?

**Solution**

(a) From Eq. (4.19),

$$f_1 = \frac{-E_1}{h} \left( \frac{2}{1^3} \right) = \left( \frac{2.18 \times 10^{-18} \text{ J}}{6.63 \times 10^{-34} \text{ J} \cdot \text{s}} \right) (2) = 6.58 \times 10^{15} \text{ rev/s}$$

$$f_2 = \frac{-E_1}{h} \left( \frac{2}{2^3} \right) = \frac{f_1}{8} = 0.823 \times 10^{15} \text{ rev/s}$$

(b) From Eq. (4.17),

$$\nu = \frac{-E_1}{h} \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right) = \left( \frac{2.18 \times 10^{-18} \text{ J}}{6.63 \times 10^{-34} \text{ J} \cdot \text{s}} \right) \left( \frac{1}{1^3} - \frac{1}{2^3} \right) = 2.88 \times 10^{15} \text{ Hz}$$

This frequency is intermediate between  $f_1$  and  $f_2$ .

(c) The number of revolutions the electron makes is

$$N = f_2 \Delta t = (8.23 \times 10^{14} \text{ rev/s})(1.00 \times 10^{-8} \text{ s}) = 8.23 \times 10^6 \text{ rev}$$

The earth takes 8.23 million y to make this many revolutions around the sun.

Under what circumstances should the Bohr atom behave classically? If the electron orbit is so large that we might be able to measure it directly, quantum effects ought not to dominate. An orbit 0.01 mm across, for instance, meets this specification. As we found in Example 4.3, its quantum number is  $n = 435$ .

What does the Bohr theory predict such an atom will radiate? According to Eq. (4.17), a hydrogen atom dropping from the  $n_i$ th energy level to the  $n_f$ th energy level emits a photon whose frequency is

$$\nu = \frac{-E_1}{h} \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right)$$

Let us write  $n$  for the initial quantum number  $n_i$  and  $n - p$  (where  $p = 1, 2, 3, \dots$ ) for the final quantum number  $n_f$ . With this substitution,

$$\nu = \frac{-E_1}{h} \left[ \frac{1}{(n-p)^2} - \frac{1}{n^2} \right] = \frac{-E_1}{h} \left[ \frac{2np - p^2}{n^2(n-p)^2} \right]$$

When  $n_i$  and  $n_f$  are both very large,  $n$  is much greater than  $p$ , and

$$2np - p^2 \approx 2np$$

$$(n-p)^2 \approx n^2$$

so that

$$\text{Frequency of photon} \quad \nu = \frac{-E_1}{\hbar} \left( \frac{2p}{n^3} \right) \quad (4.20)$$

When  $p = 1$ , the frequency  $\nu$  of the radiation is exactly the same as the frequency of rotation  $f$  of the orbital electron given in Eq. (4.19). Multiples of this frequency are radiated when  $p = 2, 3, 4, \dots$ . Hence both quantum and classical pictures of the hydrogen atom make the same predictions in the limit of very large quantum numbers. When  $n = 2$ , Eq. (4.19) predicts a radiation frequency that differs from that given by Eq. (4.20) by almost 300 percent. When  $n = 10,000$ , the discrepancy is only about 0.01 percent.

The requirement that quantum physics give the same results as classical physics in the limit of large quantum numbers was called by Bohr the **correspondence principle**. It has played an important role in the development of the quantum theory of matter.

Bohr himself used the correspondence principle in reverse, so to speak, to look for the condition for orbit stability. Starting from Eq. (4.19) he was able to show that stable orbits must have electron orbital angular momenta of

$$\text{Condition for orbital stability} \quad mvr = \frac{n\hbar}{2\pi} \quad n = 1, 2, 3, \dots \quad (4.21)$$

Since the de Broglie electron wavelength is  $\lambda = h/mv$ , Eq. (4.21) is the same as Eq. (4.12),  $n\lambda = 2\pi r$ , which states that an electron orbit must contain an integral number of wavelengths.

## 4.7 NUCLEAR MOTION

### *The nuclear mass affects the wavelengths of spectral lines*

Thus far we have been assuming that the hydrogen nucleus (a proton) remains stationary while the orbital electron revolves around it. What must actually happen, of course, is that both nucleus and electron revolve around their common center of mass, which is very close to the nucleus because the nuclear mass is much greater than that of the electron (Fig. 4.17). A system of this kind is equivalent to a single particle of mass  $m'$  that revolves around the position of the heavier particle. (This equivalence is

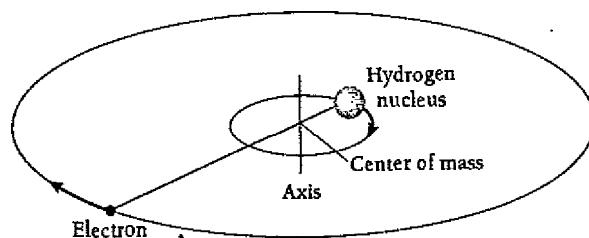


Figure 4.17 Both the electron and nucleus of a hydrogen atom revolve around a common center of mass (not to scale!).

demonstrated in Sec. 8.6.) If  $m$  is the electron mass and  $M$  the nuclear mass, then  $m'$  is given by

$$\text{Reduced mass} \quad m' = \frac{mM}{m + M} \quad (4.22)$$

The quantity  $m'$  is called the **reduced mass** of the electron because its value is less than  $m$ .

To take into account the motion of the nucleus in the hydrogen atom, then, all we need do is replace the electron with a particle of mass  $m'$ . The energy levels of the atom then become

$$\begin{array}{l} \text{Energy levels} \\ \text{corrected for} \\ \text{nuclear motion} \end{array} \quad E'_n = -\frac{m'e^4}{8\epsilon_0^2 h^2} \left( \frac{1}{n^2} \right) = \left( \frac{m'}{m} \right) \left( \frac{E_1}{n^2} \right) \quad (4.23)$$

Owing to motion of the nucleus, all the energy levels of hydrogen are changed by the fraction

$$\frac{m'}{m} = \frac{M}{M + m} = 0.99945$$

This represents an increase of 0.055 percent because the energies  $E_n$ , being smaller in absolute value, are therefore less negative.

The use of Eq. (4.23) in place of (4.15) removes a small but definite discrepancy between the predicted wavelengths of the spectral lines of hydrogen and the measured ones. The value of the Rydberg constant  $R$  to eight significant figures without correcting for nuclear motion is  $1.0973731 \times 10^7 \text{ m}^{-1}$ ; the correction lowers it to  $1.0967758 \times 10^7 \text{ m}^{-1}$ .

The notion of reduced mass played an important part in the discovery of deuterium, a variety of hydrogen whose atomic mass is almost exactly double that of ordinary hydrogen because its nucleus contains a neutron as well as a proton. About one hydrogen atom in 6000 is a deuterium atom. Because of the greater nuclear mass, the spectral lines of deuterium are all shifted slightly to wavelengths shorter than the corresponding ones of ordinary hydrogen. Thus the  $H_\alpha$  line of deuterium, which arises from a transition from the  $n = 3$  to the  $n = 2$  energy level, occurs at a wavelength of 656.1 nm, whereas the  $H_\alpha$  line of hydrogen occurs at 656.3 nm. This difference in wavelength was responsible for the identification of deuterium in 1932 by the American chemist Harold Urey.

### Example 4.6

A positronium "atom" is a system that consists of a positron and an electron that orbit each other. Compare the wavelengths of the spectral lines of positronium with those of ordinary hydrogen.

#### Solution

Here the two particles have the same mass  $m$ , so the reduced mass is

$$m' = \frac{mM}{m + M} = \frac{m^2}{2m} = \frac{m}{2}$$

where  $m$  is the electron mass. From Eq. (4.23) the energy levels of a positronium "atom" are

$$E'_n = \left( \frac{m'}{m} \right) \frac{E_1}{n^2} \approx \frac{E_1}{2n^2}$$

This means that the Rydberg constant—the constant term in Eq. (4.18)—for positronium is half as large as it is for ordinary hydrogen. As a result the wavelengths in the positronium spectral lines are all twice those of the corresponding lines in the hydrogen spectrum.

### Example 4.7

A muon is an unstable elementary particle whose mass is  $207m_e$  and whose charge is either  $+e$  or  $-e$ . A negative muon ( $\mu^-$ ) can be captured by a nucleus to form a muonic atom. (a) A proton captures a  $\mu^-$ . Find the radius of the first Bohr orbit of this atom. (b) Find the ionization energy of the atom.

#### Solution

(a) Here  $m = 207m_e$  and  $M = 1836m_e$ , so the reduced mass is

$$m' = \frac{mM}{m + M} = \frac{(207m_e)(1836m_e)}{207m_e + 1836m_e} = 186m_e$$

According to Eq. (4.13) the orbit radius corresponding to  $n = 1$  is

$$r_1 = \frac{\hbar^2 \epsilon_0}{\pi m_e e^2}$$

where  $r_1 = a_0 = 5.29 \times 10^{-11}$  m. Hence the radius  $r'$  that corresponds to the reduced mass  $m'$  is

$$r'_1 = \left( \frac{m}{m'} \right) r_1 = \left( \frac{m_e}{186m_e} \right) a_0 = 2.85 \times 10^{-13}$$
 m

The muon is 186 times closer to the proton than an electron would be, so a muonic hydrogen atom is much smaller than an ordinary hydrogen atom.

(b) From Eq. (4.23) we have, with  $n = 1$  and  $E_1 = -13.6$  eV,

$$E'_1 = \left( \frac{m'}{m} \right) E_1 = 186E_1 = -2.53 \times 10^3$$
 eV = -2.53 keV

The ionization energy is therefore 2.53 keV, 186 times that for an ordinary hydrogen atom.

## 4.8 ATOMIC EXCITATION

*How atoms absorb and emit energy*

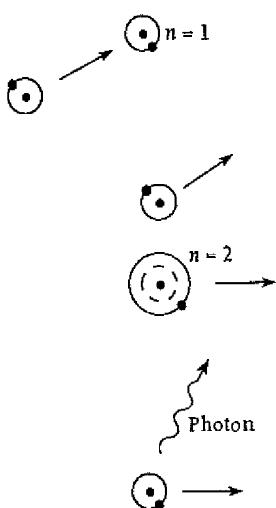
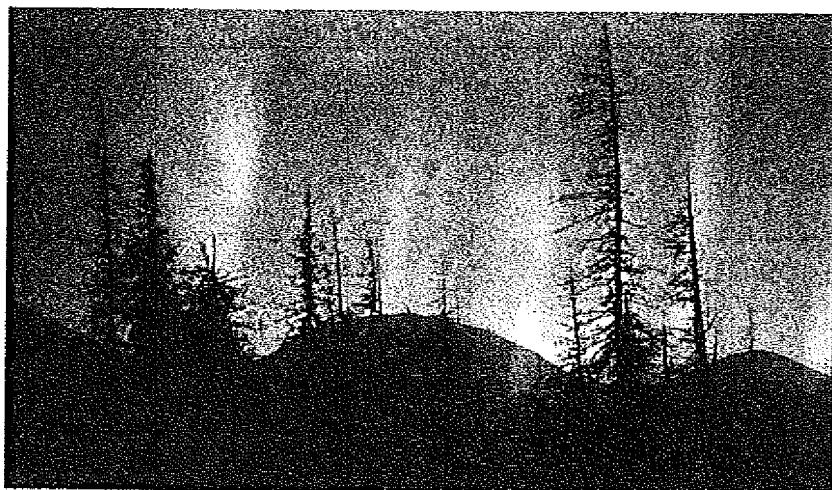


Figure 4.18 Excitation by collision. Some of the available energy is absorbed by one of the atoms, which goes into an excited energy state. The atom then emits a photon in returning to its ground (normal) state.

There are two main ways in which an atom can be excited to an energy above its ground state and thereby become able to radiate. One of these ways is by a collision with another particle in which part of their joint kinetic energy is absorbed by the atom. Such an excited atom will return to its ground state in an average of  $10^{-8}$  s by emitting one or more photons (Fig. 4.18).

To produce a luminous discharge in a rarefied gas, an electric field is established that accelerates electrons and atomic ions until their kinetic energies are sufficient to



Auroras are caused by streams of fast protons and electrons from the sun that excite atoms in the upper atmosphere. The green hues of an auroral display come from oxygen, and the reds originate in both oxygen and nitrogen. This aurora occurred in Alaska.

excite atoms they collide with. Because energy transfer is a maximum when the colliding particles have the same mass (see Fig. 12.22), the electrons in such a discharge are more effective than the ions in providing energy to atomic electrons. Neon signs and mercury-vapor lamps are familiar examples of how a strong electric field applied between electrodes in a gas-filled tube leads to the emission of the characteristic spectral radiation of that gas, which happens to be reddish light in the case of neon and bluish light in the case of mercury vapor.

Another excitation mechanism is involved when an atom absorbs a photon of light whose energy is just the right amount to raise the atom to a higher energy level. For example, a photon of wavelength 121.7 nm is emitted when a hydrogen atom in the  $n = 2$  state drops to the  $n = 1$  state. Absorbing a photon of wavelength 121.7 nm by a hydrogen atom initially in the  $n = 1$  state will therefore bring it up to the  $n = 2$  state (Fig. 4.19). This process explains the origin of absorption spectra.

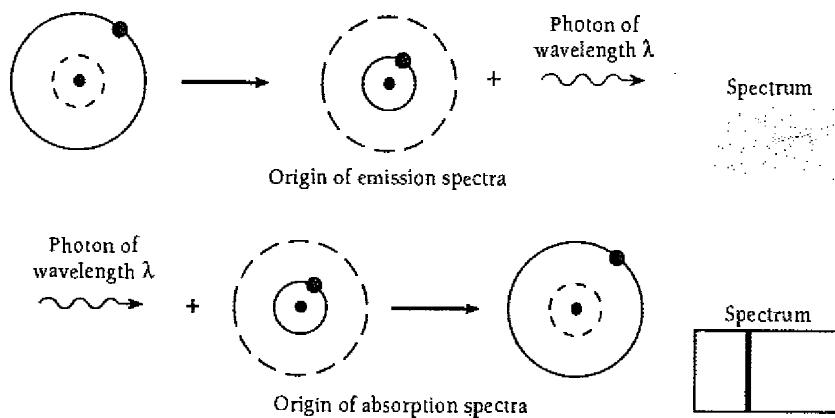


Figure 4.19 How emission and absorption spectral lines originate.

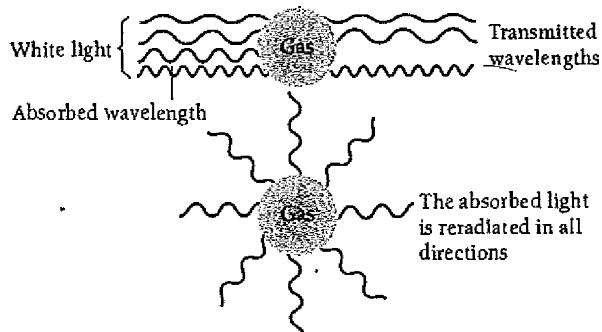


Figure 4.20 The dark lines in an absorption spectrum are never totally dark.

When white light, which contains all wavelengths, is passed through hydrogen gas, photons of those wavelengths that correspond to transitions between energy levels are absorbed. The resulting excited hydrogen atoms reradiate their excitation energy almost at once, but these photons come off in random directions with only a few in the same direction as the original beam of white light (Fig. 4.20). The dark lines in an absorption spectrum are therefore never completely black but only appear so by contrast with the bright background. We expect the lines in the absorption spectrum of any element to coincide with those in its emission spectrum that represent transitions to the ground state, which agrees with observation (see Fig. 4.9).

### Franck-Hertz Experiment

Atomic spectra are not the only way to investigate energy levels inside atoms. A series of experiments based on excitation by collision was performed by James Franck and Gustav Hertz (a nephew of Heinrich Hertz) starting in 1914. These experiments demonstrated that atomic energy levels indeed exist and, furthermore, that the ones found in this way are the same as those suggested by line spectra.

Franck and Hertz bombarded the vapors of various elements with electrons of known energy, using an apparatus like that shown in Fig. 4.21. A small potential difference  $V_0$  between the grid and collecting plate prevents electrons having energies less than a certain minimum from contributing to the current  $I$  through the ammeter. As the accelerating potential  $V$  is increased, more and more electrons arrive at the plate and  $I$  rises (Fig. 4.22).

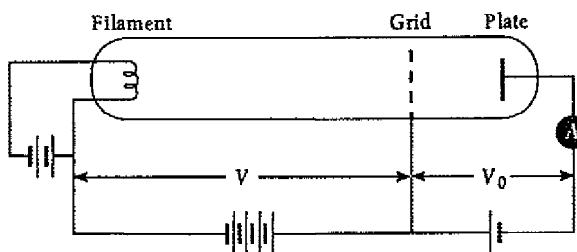


Figure 4.21 Apparatus for the Franck-Hertz experiment.

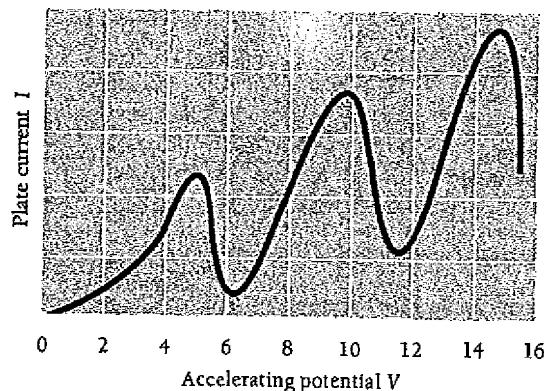


Figure 4.22 Results of the Franck-Hertz experiment, showing critical potentials in mercury vapor.

If KE is conserved when an electron collides with one of the atoms in the vapor, the electron merely bounces off in a new direction. Because an atom is much heavier than an electron, the electron loses almost no KE in the process. After a certain critical energy is reached, however, the plate current drops abruptly. This suggests that an electron colliding with one of the atoms gives up some or all of its KE to excite the atom to an energy level above its ground state. Such a collision is called inelastic, in contrast to an elastic collision in which KE is conserved. The critical electron energy equals the energy needed to raise the atom to its lowest excited state.

Then, as the accelerating potential  $V$  is raised further, the plate current again increases, since the electrons now have enough energy left to reach the plate after undergoing an inelastic collision on the way. Eventually another sharp drop in plate current occurs, which arises from the excitation of the same energy level in other atoms by the electrons. As Fig. 4.22 shows, a series of critical potentials for a given atomic vapor is obtained. Thus the higher potentials result from two or more inelastic collisions and are multiples of the lowest one.

To check that the critical potentials were due to atomic energy levels, Franck and Hertz observed the emission spectra of vapors during electron bombardment. In the case of mercury vapor, for example, they found that a minimum electron energy of 4.9 eV was required to excite the 253.6-nm spectral line of mercury—and a photon of 253.6-nm light has an energy of just 4.9 eV. The Franck-Hertz experiments were performed shortly after Bohr announced his theory of the hydrogen atom, and they independently confirmed his basic ideas.

## 4.9 THE LASER

### How to produce light waves all in step

The laser is a device that produces a light beam with some remarkable properties:

- 1 The light is very nearly monochromatic.
- 2 The light is coherent, with the waves all exactly in phase with one another (Fig. 4.23).

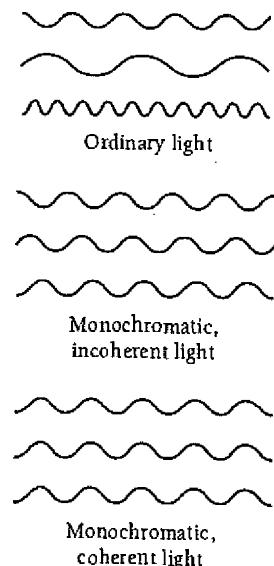


Figure 4.23 A laser produces a beam of light whose waves all have the same frequency (monochromatic) and are in phase with one another (coherent). The beam is also well collimated and so spreads out very little, even over long distances.

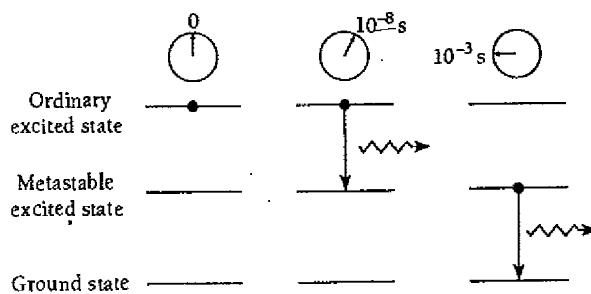


Figure 4.24 An atom can exist in a metastable energy level for a longer time before radiating than it can in an ordinary energy level.

3 A laser beam diverges hardly at all. Such a beam sent from the earth to a mirror left on the moon by the Apollo 11 expedition remained narrow enough to be detected on its return to the earth, a total distance of over three-quarters of a million kilometers. A light beam produced by any other means would have spread out too much for this to be done.

4 The beam is extremely intense, more intense by far than the light from any other source. To achieve an energy density equal to that in some laser beams, a hot object would have to be at a temperature of  $10^{30}$  K.

The last two of these properties follow from the second of them.

The term *laser* stands for light amplification by stimulated emission of radiation. The key to the laser is the presence in many atoms of one or more excited energy levels whose lifetimes may be  $10^{-3}$  s or more instead of the usual  $10^{-8}$  s. Such relatively long-lived states are called **metastable** (temporarily stable); see Fig. 4.24.

Three kinds of transition involving electromagnetic radiation are possible between two energy levels,  $E_0$  and  $E_1$ , in an atom (Fig. 4.25). If the atom is initially in the lower state  $E_0$ , it can be raised to  $E_1$  by absorbing a photon of energy  $E_1 - E_0 = h\nu$ . This process is called **stimulated absorption**. If the atom is initially in the upper state  $E_1$ , it can drop to  $E_0$  by emitting a photon of energy  $h\nu$ . This is **spontaneous emission**.

Einstein, in 1917, was the first to point out a third possibility, **stimulated emission**, in which an incident photon of energy  $h\nu$  causes a transition from  $E_1$  to  $E_0$ . In stimulated emission, the radiated light waves are exactly in phase with the incident ones, so the result is an enhanced beam of coherent light. Einstein showed that stimulated emission has the same probability as stimulated absorption (see Sec. 9.7). That is, a photon of energy  $h\nu$  incident on an atom in the upper

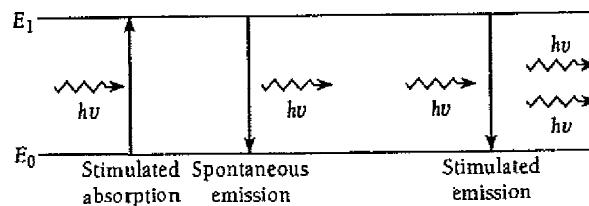


Figure 4.25 Transitions between two energy levels in an atom can occur by stimulated absorption, spontaneous emission, and stimulated emission.



Charles H. Townes (1915– ) was born in Greenville, South Carolina, and attended Furman University there. After graduate study at Duke University and the California Institute of Technology, he spent 1939 to 1947 at the Bell Telephone Laboratories designing radar-controlled bombing systems. Townes then joined the physics department of Columbia University.

In 1951, while sitting on a park bench, the idea for the maser (microwave amplification by stimulated emission of radiation) occurred to him as a way to produce high-intensity microwaves, and in 1953 the first maser began operating. In this device ammonia ( $\text{NH}_3$ ) molecules were raised to an excited vibrational state and then fed into a resonant cavity where, as in a laser, stimulated emission produced a cascade of photons of identical wavelength, here 1.25 cm in the microwave part of the spectrum. "Atomic clocks" of great accuracy are based on this concept, and solid-state maser amplifiers are used in such applications as radioastronomy.

In 1958 Townes and Arthur Schawlow attracted much attention with a paper showing that a similar scheme ought to be possible at optical wavelengths. Slightly earlier Gordon Gould, then a graduate student at Columbia, had come to the same conclusion, but did not publish his calculations at once since that would prevent securing a patent. Gould tried to develop the laser—his term—in private industry, but the Defense Department classified as secret the project (and his original notebooks) and denied him clearance to work on it. Finally, twenty years later, Gould succeeded in establishing his priority and received two patents on the laser, and still later, a third. The first working laser was built by Theodore Maiman at Hughes Research Laboratories in 1960. In 1964 Townes, along with two Russian laser pioneers, Aleksander Prokhorov and Nikolai Basov, was awarded a Nobel Prize. In 1981 Schawlow shared a Nobel Prize for precision spectroscopy using lasers.

Soon after its invention, the laser was spoken of as a "solution looking for a problem" because few applications were then known for it. Today, of course, lasers are widely employed for a variety of purposes.

state  $E_1$  has the same likelihood of causing the emission of another photon of energy  $h\nu$  as its likelihood of being absorbed if it is incident on an atom in the lower state  $E_0$ .

Stimulated emission involves no novel concepts. An analogy is a harmonic oscillator, for instance a pendulum, which has a sinusoidal force applied to it whose period is the same as its natural period of vibration. If the applied force is exactly in phase with the pendulum swings, the amplitude of the swings increases. This corresponds to stimulated absorption. However, if the applied force is  $180^\circ$  out of phase with the pendulum swings, the amplitude of the swings decreases. This corresponds to stimulated emission.

A three-level laser, the simplest kind, uses an assembly of atoms (or molecules) that have a metastable state  $h\nu$  in energy above the ground state and a still higher excited state that decays to the metastable state (Fig. 4.26). What we want is more atoms in the metastable state than in the ground state. If we can arrange this and then shine light of frequency  $\nu$  on the assembly, there will be more stimulated emissions from atoms in the metastable state than stimulated absorptions by atoms in the ground state. The result will be an amplification of the original light. This is the concept that underlies the operation of the laser.

The term **population inversion** describes an assembly of atoms in which the majority are in energy levels above the ground state; normally the ground state is occupied to the greatest extent.

A number of ways exist to produce a population inversion. One of them, called **optical pumping**, is illustrated in Fig. 4.27. Here an external light source is used some of whose photons have the right frequency to raise ground-state atoms to the excited state that decays spontaneously to the desired metastable state.

Why are three levels needed? Suppose there are only two levels, a metastable state  $h\nu$  above the ground state. The more photons of frequency  $\nu$  we pump into the assembly

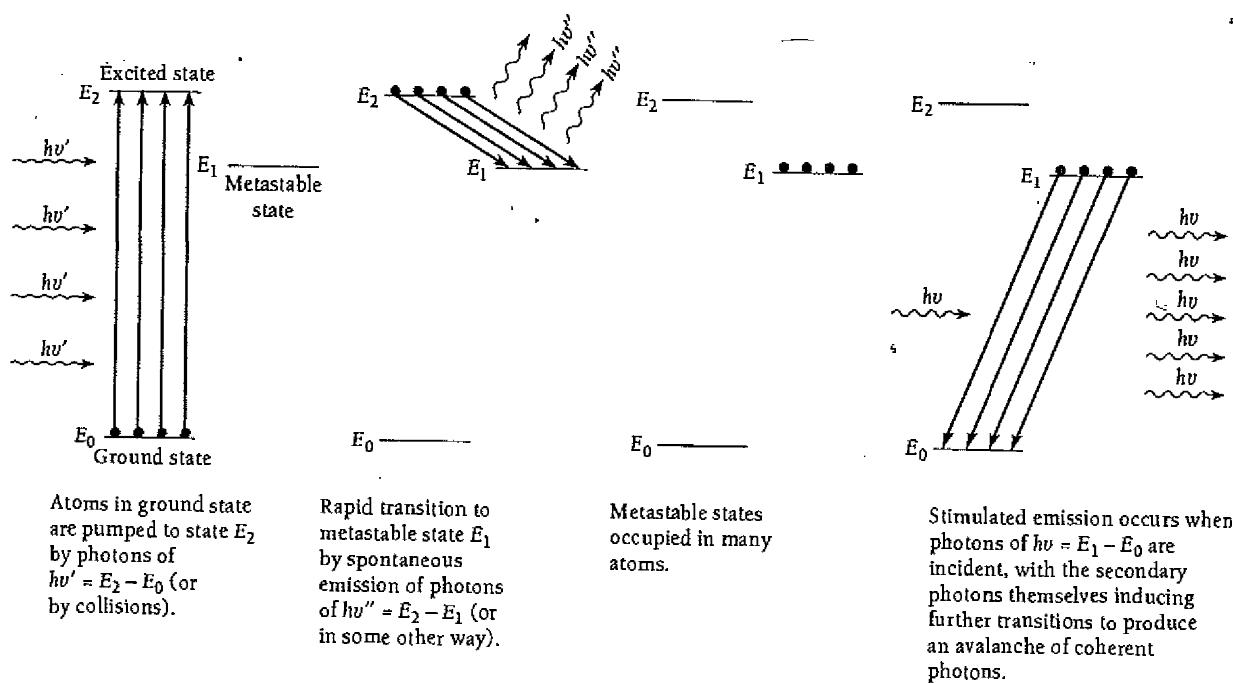
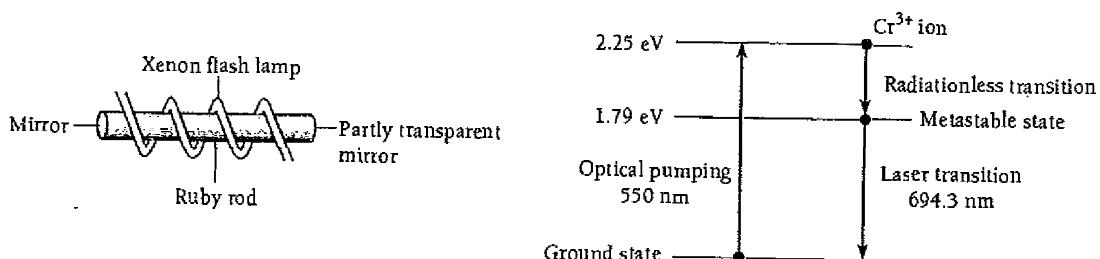


Figure 4.26 The principle of the laser.

of atoms, the more upward transitions there will be from the ground state to the metastable state. However, at the same time the pumping will stimulate downward transitions from the metastable state to the ground state. When half the atoms are in each state, the rate of stimulated emissions will equal the rate of stimulated absorptions, so the assembly cannot ever have more than half its atoms in the metastable state. In this situation laser amplification cannot occur. A population inversion is only possible when the stimulated absorptions are to a higher energy level than the metastable one from which the stimulated emission takes place, which prevents the pumping from depopulating the metastable state.

In a three-level laser, more than half the atoms must be in the metastable state for stimulated induced emission to predominate. This is not the case for a four-level laser.

Figure 4.27 The ruby laser. In order for stimulated emission to exceed stimulated absorption, more than half the  $\text{Cr}^{3+}$  ions in the ruby rod must be in the metastable state. This laser produces a pulse of red light after each flash of the lamp.

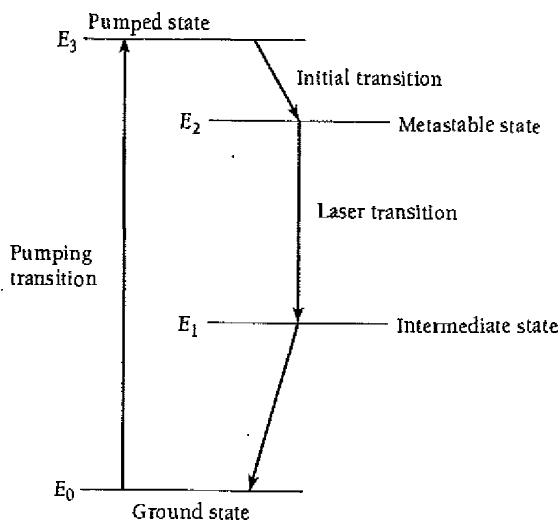
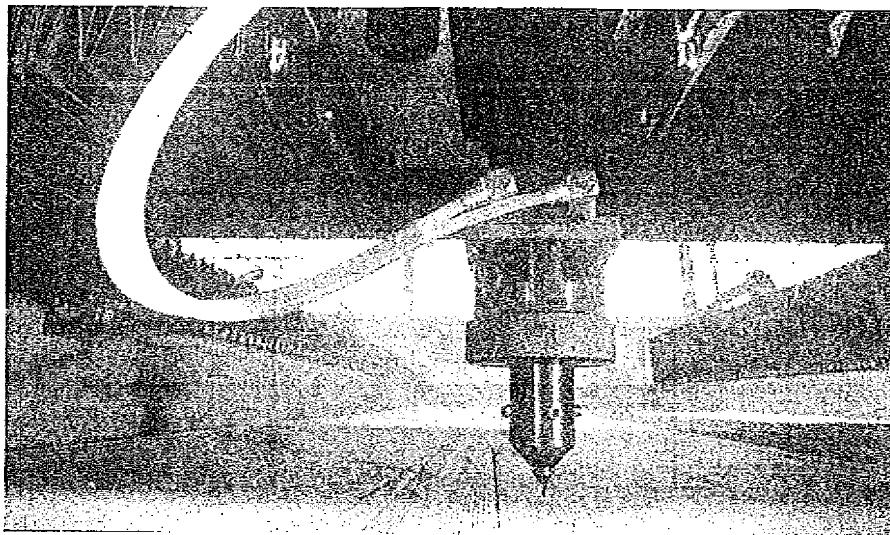


Figure 4.28 A four-level laser.

As in Fig. 4.28, the laser transition from the metastable state ends at an unstable intermediate state rather than at the ground state. Because the intermediate state decays rapidly to the ground state, very few atoms are in the intermediate state. Hence even a modest amount of pumping is enough to populate the metastable state to a greater extent than the intermediate state, as required for laser amplification.

### Practical Lasers

The first successful laser, the ruby laser, is based on the three energy levels in the chromium ion  $\text{Cr}^{3+}$  shown in Fig. 4.27. A ruby is a crystal of aluminum oxide,  $\text{Al}_2\text{O}_3$ ,



A robot arm carries a laser for cutting fabric in a clothing factory.

in which some of the  $\text{Al}^{3+}$  ions are replaced by  $\text{Cr}^{3+}$  ions, which are responsible for the red color. A  $\text{Cr}^{3+}$  ion has a metastable level whose lifetime is about 0.003 s. In the ruby laser, a xenon flash lamp excites the  $\text{Cr}^{3+}$  ions to a level of higher energy from which they fall to the metastable level by losing energy to other ions in the crystal. Photons from the spontaneous decay of some  $\text{Cr}^{3+}$  ions are reflected back and forth between the mirrored ends of the ruby rod, stimulating other excited  $\text{Cr}^{3+}$  ions to radiate. After a few microseconds the result is a large pulse of monochromatic, coherent red light from the partly transparent end of the rod.

The rod's length is made precisely an integral number of half-wavelengths long, so the radiation trapped in it forms an optical standing wave. Since the stimulated emissions are induced by the standing wave, their waves are all in step with it.

The common helium-neon gas laser achieves a population inversion in a different way. A mixture of about 10 parts of helium and 1 part of neon at a low pressure ( $\sim 1$  torr) is placed in a glass tube that has parallel mirrors, one of them partly transparent, at both ends. The spacing of the mirrors is again (as in all lasers) equal to an integral number of half-wavelengths of the laser light. An electric discharge is produced in the gas by means of electrodes outside the tube connected to a source of high-frequency alternating current, and collisions with electrons from the discharge excite He and Ne atoms to metastable states respectively 20.61 and 20.66 eV above their ground states (Fig. 4.29). Some of the excited He atoms transfer their energy to ground-state Ne atoms in collisions, with the 0.05 eV of additional energy being provided by the kinetic energy of the atoms. The purpose of the He atoms is thus to help achieve a population inversion in the Ne atoms.

The laser transition in Ne is from the metastable state at 20.66 eV to an excited state at 18.70 eV, with the emission of a 632.8-nm photon. Then another photon is spontaneously emitted in a transition to a lower metastable state; this transition yields only incoherent light. The remaining excitation energy is lost in collisions with the tube walls. Because the electron impacts that excite the He and Ne atoms occur all the time, unlike the pulsed excitation from the xenon flash lamp in a ruby laser, a He-Ne laser operates continuously. This is the laser whose narrow red beam is used in supermarkets to read bar codes. In a He-Ne laser, only a tiny

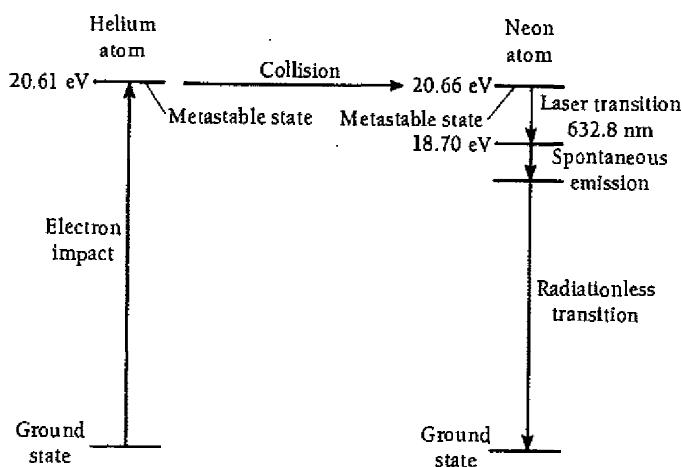


Figure 4.29 The helium-neon laser. In a four-level laser such as this, continuous operation is possible. Helium-neon lasers are commonly used to read bar codes.

### Chirped Pulse Amplification

The most powerful lasers are pulsed, which produces phenomenal outputs for very short periods. The petawatt ( $10^{15}$  W) threshold was crossed in 1996 with pulses less than a trillionth of a second long—not all that much energy per pulse, but at a rate of delivery over 1000 times that of the entire electrical grid of the United States. An ingenious method called chirped pulse amplification made this possible without the laser apparatus itself being destroyed in the process. What was done was to start with a low-power laser pulse that was quite short, only 0.1 picosecond ( $10^{-13}$  s). Because the pulse was short, it consisted of a large span of wavelengths, as discussed in Sec. 3.7 (see Figs. 3.13 and 3.14). A diffraction grating then spread out the light into different paths according to wavelength, which stretched the pulse to 3 nanoseconds ( $3 \times 10^{-9}$  s), 30,000 times longer. The result was to decrease the peak power so that laser amplifiers could boost the energy of each beam. Finally the amplified beams, each of slightly different wavelength, were recombined by another grating to produce a pulse less than 0.5 picoseconds long whose power was 1.3 petawatts.

fraction (one in millions) of the atoms present participates in the laser process at any moment.

Many other types of laser have been devised. A number of them employ molecules rather than atoms. Chemical lasers are based on the production by chemical reactions of molecules in metastable excited states. Such lasers are efficient and can be very powerful: one chemical laser, in which hydrogen and fluorine combine to form hydrogen fluoride, has generated an infrared beam of over 2 MW. Dye lasers use dye molecules whose energy levels are so close together that they can "lase" over a virtually continuous range of wavelengths (see Sec. 8.7). A dye laser can be tuned to any desired wavelength in its range. Nd:YAG lasers, which use the glassy solid yttrium aluminum garnet with neodymium as an impurity, are helpful in surgery because they seal small blood vessels while cutting through tissue by vaporizing water in the path of their beams. Powerful carbon dioxide gas lasers with outputs up to many kilowatts are used industrially for the precise cutting of almost any material, including steel, and for welding.

Tiny semiconductor lasers by the million process and transmit information today. (How such lasers work is described in Chap. 10.) In a compact disk player, a semiconductor laser beam is focused to a spot a micrometer ( $10^{-6}$  m) across to read data coded as pits that appear as dark spots on a reflective disk 12 cm in diameter. A compact disk can store over 600 megabytes of digital data, about 1000 times as much as the floppy disks used in personal computers. If the stored data is digitized music, the playing time can be over an hour.

Semiconductor lasers are ideal for fiber-optic transmission lines in which the electric signals that would normally be sent along copper wires are first converted into a series of pulses according to a standard code. Lasers then turn the pulses into flashes of infrared light that travel along thin (5–50  $\mu\text{m}$  diameter) glass fibers and at the other end are changed back into electric signals. Over a million telephone conversations can be carried by a single fiber; by contrast, no more than 32 conversations can be carried at the same time by a pair of wires. Telephone fiber-optic systems today link many cities and exchanges within cities everywhere, and fiber-optic cables span the world's seas and oceans.

## APPENDIX TO CHAPTER 4

# Rutherford Scattering

**R**utherford's model of the atom was accepted because he was able to arrive at a formula to describe the scattering of alpha particles by thin foils on the basis of this model that agreed with the experimental results. He began by assuming that the alpha particle and the nucleus it interacts with are both small enough to be considered as point masses and charges; that the repulsive electric force between alpha particle and nucleus (which are both positively charged) is the only one acting; and that the nucleus is so massive compared with the alpha particle that it does not move during their interaction. Let us see how these assumptions lead to Eq. (4.1).

### Scattering Angle

Owing to the variation of the electric force with  $1/r^2$ , where  $r$  is the instantaneous separation between alpha particle and nucleus, the alpha particle's path is a hyperbola with the nucleus at the outer focus (Fig. 4.30). The impact parameter  $b$  is the minimum distance to which the alpha particle would approach the nucleus if there were no force between them, and the scattering angle  $\theta$  is the angle between the asymptotic direction of approach of the alpha particle and the asymptotic direction in which it recedes. Our first task is to find a relationship between  $b$  and  $\theta$ .

As a result of the impulse  $\int F dt$  given it by the nucleus, the momentum of the alpha particle changes by  $\Delta p$  from the initial value  $p_1$  to the final value  $p_2$ . That is,

$$\Delta p = p_2 - p_1 = \int F dt \quad (4.24)$$

Because the nucleus remains stationary during the passage of the alpha particle, by hypothesis, the alpha-particle kinetic energy is the same before and after the scattering. Hence the magnitude of its momentum is also the same before and after, and

$$p_1 = p_2 = mv$$

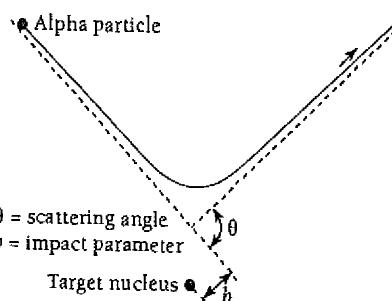


Figure 4.30 Rutherford scattering.

Here  $v$  is the alpha-particle velocity far from the nucleus.

From Fig. 4.31 we see that according to the law of sines,

$$\frac{\Delta p}{\sin \theta} = \frac{mv}{\sin \frac{\pi - \theta}{2}}$$

Since

$$\sin \frac{1}{2}(\pi - \theta) = \cos \frac{\theta}{2}$$

and

$$\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$$

we have for the magnitude of the momentum change

$$\Delta p = 2mv \sin \frac{\theta}{2} \quad (4.25)$$

Because the impulse  $\int F dt$  is in the same direction as the momentum change  $\Delta p$ , its magnitude is

$$|\int F dt| = \int F \cos \phi dt \quad (4.26)$$

where  $\phi$  is the instantaneous angle between  $F$  and  $\Delta p$  along the path of the alpha particle. Inserting Eqs. (4.25) and (4.26) in Eq. (4.24),

$$2mv \sin \frac{\theta}{2} = \int_{-\infty}^{\infty} F \cos \phi dt$$

To change the variable on the right-hand side from  $t$  to  $\phi$ , we note that the limits of integration will change to  $-\frac{1}{2}(\pi - \theta)$  and  $+\frac{1}{2}(\pi - \theta)$ , corresponding to  $\phi$  at  $t = -\infty$  and  $t = \infty$  respectively, and so

$$2mv \sin \frac{\theta}{2} = \int_{-(\pi-\theta)/2}^{+(\pi-\theta)/2} F \cos \phi \frac{dt}{d\phi} d\phi \quad (4.27)$$

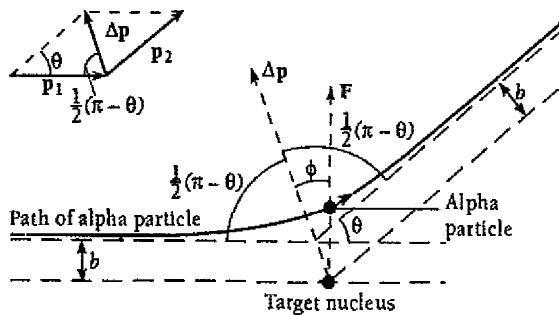


Figure 4.31 Geometrical relationships in Rutherford scattering.

The quantity  $d\phi/dt$  is just the angular velocity  $\omega$  of the alpha particle about the nucleus (this is evident from Fig. 4.31).

The electric force exerted by the nucleus on the alpha particle acts along the radius vector joining them, so there is no torque on the alpha particle and its angular momentum  $m\omega r^2$  is constant. Hence

$$m\omega r^2 = \text{constant} = mr^2 \frac{d\phi}{dt} = mvb$$

from which we obtain

$$\frac{dt}{d\phi} = \frac{r^2}{vb}$$

Substituting this expression for  $dt/d\phi$  in Eq. (4.27) gives

$$2mv^2 b \sin \frac{\theta}{2} = \int_{-(\pi-\theta)/2}^{+(\pi-\theta)/2} Fr^2 \cos \phi \, d\phi \quad (4.28)$$

As we recall,  $F$  is the electric force exerted by the nucleus on the alpha particle. The charge on the nucleus is  $Ze$ , corresponding to the atomic number  $Z$ , and that on the alpha particle is  $2e$ . Therefore

$$F = \frac{1}{4\pi\epsilon_0} \frac{2Ze^2}{r^2}$$

and  $\frac{4\pi\epsilon_0 mv^2 b}{Ze^2} \sin \frac{\theta}{2} = \int_{-(\pi-\theta)/2}^{+(\pi-\theta)/2} \cos \phi \, d\phi = 2 \cos \frac{\theta}{2}$

The scattering angle  $\theta$  is related to the impact parameter  $b$  by the equation

$$\cot \frac{\theta}{2} = \frac{2\pi\epsilon_0 mv^2}{Ze^2} b$$

It is more convenient to specify the alpha-particle energy KE instead of its mass and velocity separately; with this substitution,

Scattering angle  $\cot \frac{\theta}{2} = \frac{4\pi\epsilon_0 KE}{Ze^2} b \quad (4.29)$

Figure 4.32 is a schematic representation of Eq. (4.29); the rapid decrease in  $\theta$  as  $b$  increases is evident. A very near miss is required for a substantial deflection.

### Rutherford Scattering Formula

Equation (4.29) cannot be directly confronted with experiment because there is no way of measuring the impact parameter corresponding to a particular observed scattering angle. An indirect strategy is required.

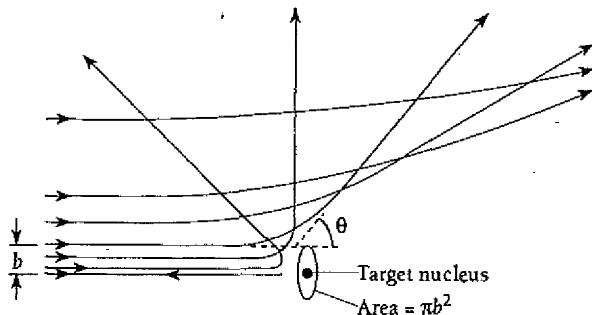


Figure 4.32 The scattering angle decreases with increasing impact parameter.

Our first step is to note that all alpha particles approaching a target nucleus with an impact parameter from 0 to  $b$  will be scattered through an angle of  $\theta$  or more, where  $\theta$  is given in terms of  $b$  by Eq. (4.29). This means that an alpha particle that is initially directed anywhere within the area  $\pi b^2$  around a nucleus will be scattered through  $\theta$  or more (Fig. 4.32). The area  $\pi b^2$  is accordingly called the **cross section** for the interaction. The general symbol for cross section is  $\sigma$ , and so here

$$\text{Cross section} \quad \sigma = \pi b^2 \quad (4.30)$$

Of course, the incident alpha particle is actually scattered before it reaches the immediate vicinity of the nucleus and hence does not necessarily pass within a distance  $b$  of it.

Now we consider a foil of thickness  $t$  that contains  $n$  atoms per unit volume. The number of target nuclei per unit area is  $nt$ , and an alpha-particle beam incident upon an area  $A$  therefore encounters  $ntA$  nuclei. The aggregate cross section for scatterings of  $\theta$  or more is the number of target nuclei  $ntA$  multiplied by the cross section  $\sigma$  for such scattering per nucleus, or  $ntA\sigma$ . Hence the fraction  $f$  of incident alpha particles scattered by  $\theta$  or more is the ratio between the aggregate cross section  $ntA\sigma$  for such scattering and the total target area  $A$ . That is,

$$\begin{aligned} f &= \frac{\text{alpha particles scattered by } \theta \text{ or more}}{\text{incident alpha particles}} \\ &= \frac{\text{aggregate cross section}}{\text{target area}} = \frac{ntA\sigma}{A} \\ &= nt\pi b^2 \end{aligned}$$

Substituting for  $b$  from Eq. (4.30),

$$f = \pi n t \left( \frac{Z e^2}{4 \pi \epsilon_0 K E} \right)^2 \cot^2 \frac{\theta}{2} \quad (4.31)$$

In this calculation it was assumed that the foil is sufficiently thin so that the cross sections of adjacent nuclei do not overlap and that a scattered alpha particle receives its entire deflection from an encounter with a single nucleus.

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**Example 4.8**

Find the fraction of a beam of 7.7-MeV alpha particles that is scattered through angles of more than  $45^\circ$  when incident upon a gold foil  $3 \times 10^{-7}$  m thick. These values are typical of the alpha-particle energies and foil thicknesses used by Geiger and Marsden. For comparison, a human hair is about  $10^{-4}$  m in diameter.

**Solution**

We begin by finding  $n$ , the number of gold atoms per unit volume in the foil, from the relationship

$$n = \frac{\text{atoms}}{\text{m}^3} = \frac{\text{mass/m}^3}{\text{mass/atom}}$$

Since the density of gold is  $1.93 \times 10^4$  kg/m<sup>3</sup>, its atomic mass is 197 u, and 1 u =  $1.66 \times 10^{-27}$  kg, we have

$$\begin{aligned} n &= \frac{1.93 \times 10^4 \text{ kg/m}^3}{(197 \text{ u/atom})(1.66 \times 10^{-27} \text{ kg/u})} \\ &= 5.90 \times 10^{28} \text{ atoms/m}^3 \end{aligned}$$

The atomic number  $Z$  of gold is 79, a kinetic energy of 7.7 MeV is equal to  $1.23 \times 10^{-12}$  J, and  $\theta = 45^\circ$ ; from these figures we find that

$$f = 7 \times 10^{-5}$$

of the incident alpha particles are scattered through  $45^\circ$  or more—only 0.007 percent! A foil this thin is quite transparent to alpha particles.

---

In an actual experiment, a detector measures alpha particles scattered between  $\theta$  and  $\theta + d\theta$ , as in Fig. 4.33. The fraction of incident alpha particles so scattered is found by differentiating Eq. (4.31) with respect to  $\theta$ , which gives

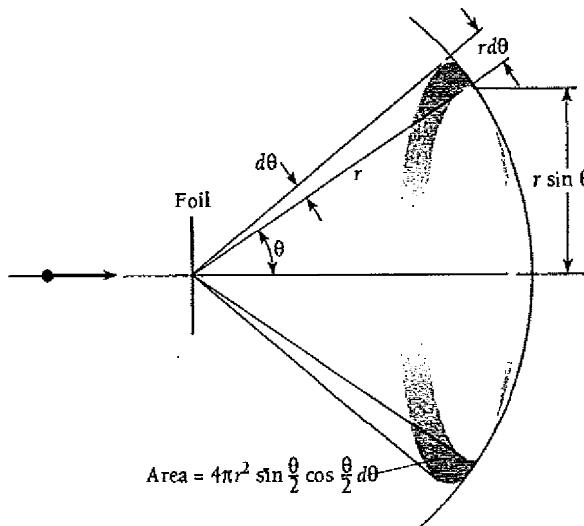


Figure 4.33 In the Rutherford experiment, particles are detected that have been scattered between  $\theta$  and  $\theta + d\theta$ .

$$df = -\pi nt \left( \frac{Ze^2}{4\pi\epsilon_0 KE} \right)^2 \cot \frac{\theta}{2} \csc^2 \frac{\theta}{2} d\theta \quad (4.32)$$

The minus sign expresses the fact that  $f$  decreases with increasing  $\theta$ .

As we saw in Fig. 4.2, Geiger and Marsden placed a fluorescent screen a distance  $r$  from the foil and the scattered alpha particles were detected by means of the scintillations they caused. Those alpha particles scattered between  $\theta$  and  $\theta + d\theta$  reached a zone of a sphere of radius  $r$  whose width is  $r d\theta$ . The zone radius itself is  $r \sin \theta$ , and so the area  $dS$  of the screen struck by these particles is

$$\begin{aligned} dS &= (2\pi r \sin \theta)(r d\theta) = 2\pi r^2 \sin \theta d\theta \\ &= 4\pi r^2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta \end{aligned}$$

If a total of  $N_i$  alpha particles strike the foil during the course of the experiment, the number scattered into  $d\theta$  at  $\theta$  is  $N_i df$ . The number  $N(\theta)$  per unit area striking the screen at  $\theta$ , which is the quantity actually measured, is

$$\begin{aligned} N(\theta) &= \frac{N_i |df|}{dS} = \frac{\frac{N_i \pi n t \left( \frac{Ze^2}{4\pi\epsilon_0 KE} \right)^2 \cot \frac{\theta}{2} \csc^2 \frac{\theta}{2} d\theta}{2}}{4\pi r^2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta} \\ \text{Rutherford} \quad \text{scattering formula} \quad N(\theta) &= \frac{N_i n t Z^2 e^4}{(8\pi\epsilon_0)^2 r^2 KE^2 \sin^4(\theta/2)} \end{aligned} \quad (4.1)$$

Equation (4.1) is the Rutherford scattering formula. Figure 4.4 shows how  $N(\theta)$  varies with  $\theta$ .

It isn't that they can't see the solution. It is that they can't see the problem. —Gilbert Chesterton

#### 4.1 The Nuclear Atom

1. The great majority of alpha particles pass through gases and thin metal foils with no deflections. To what conclusion about atomic structure does this observation lead?
2. The electric field intensity at a distance  $r$  from the center of a uniformly charged sphere of radius  $R$  and total charge  $Q$  is  $Qr/4\pi\epsilon_0 R^3$  when  $r < R$ . Such a sphere corresponds to the Thomson model of the atom. Show that an electron in this sphere executes simple harmonic motion about its center and derive a formula for the frequency of this motion. Evaluate the

frequency of the electron oscillations for the case of the hydrogen atom and compare it with the frequencies of the spectral lines of hydrogen.

3. Determine the distance of closest approach of 1.00-MeV protons incident on gold nuclei.

#### 4.2 Electron Orbits

4. Find the frequency of revolution of the electron in the classical model of the hydrogen atom. In what region of the spectrum are electromagnetic waves of this frequency?

#### 4.3 Atomic Spectra

5. What is the shortest wavelength present in the Brackett series of spectral lines?
6. What is the shortest wavelength present in the Paschen series of spectral lines?

#### 4.4 The Bohr Atom

7. In the Bohr model, the electron is in constant motion. How can such an electron have a negative amount of energy?
8. Lacking de Broglie's hypothesis to guide his thinking, Bohr arrived at his model by postulating that the angular momentum of an orbital electron must be an integral multiple of  $\hbar$ . Show that this postulate leads to Eq. (4.13).
9. The fine structure constant is defined as  $\alpha = e^2/2\epsilon_0hc$ . This quantity got its name because it first appeared in a theory by the German physicist Arnold Sommerfeld that tried to explain the fine structure in spectral lines (multiple lines close together instead of single lines) by assuming that elliptical as well as circular orbits are possible in the Bohr model. Sommerfeld's approach was on the wrong track, but  $\alpha$  has nevertheless turned out to be a useful quantity in atomic physics. (a) Show that  $\alpha = v_1/c$ , where  $v_1$  is the velocity of the electron in the ground state of the Bohr atom. (b) Show that the value of  $\alpha$  is very close to  $1/137$  and is a pure number with no dimensions. Because the magnetic behavior of a moving charge depends on its velocity, the small value of  $\alpha$  is representative of the relative magnitudes of the magnetic and electric aspects of electron behavior in an atom. (c) Show that  $\alpha a_0 = \lambda_C/2\pi$ , where  $a_0$  is the radius of the ground-state Bohr orbit and  $\lambda_C$  is the Compton wavelength of the electron.
10. An electron at rest is released far away from a proton, toward which it moves. (a) Show that the de Broglie wavelength of the electron is proportional to  $\sqrt{r}$ , where  $r$  is the distance of the electron from the proton. (b) Find the wavelength of the electron when it is  $a_0$  from the proton. How does this compare with the wavelength of an electron in a ground-state Bohr orbit? (c) In order for the electron to be captured by the proton to form a ground-state hydrogen atom, energy must be lost by the system. How much energy?
11. Find the quantum number that characterizes the earth's orbit around the sun. The earth's mass is  $6.0 \times 10^{24}$  kg, its orbital radius is  $1.5 \times 10^{11}$  m, and its orbital speed is  $3.0 \times 10^4$  m/s.
12. Suppose a proton and an electron were held together in a hydrogen atom by gravitational forces only. Find the formula for the energy levels of such an atom, the radius of its ground-state Bohr orbit, and its ionization energy in eV.
13. Compare the uncertainty in the momentum of an electron confined to a region of linear dimension  $a_0$  with the momentum of an electron in a ground-state Bohr orbit.

#### 4.5 Energy Levels and Spectra

14. When radiation with a continuous spectrum is passed through a volume of hydrogen gas whose atoms are all in the ground state, which spectral series will be present in the resulting absorption spectrum?

15. What effect would you expect the rapid random motion of the atoms of an excited gas to have on the spectral lines they produce?
16. A beam of 13.0-eV electrons is used to bombard gaseous hydrogen. What series of wavelengths will be emitted?
17. A proton and an electron, both at rest initially, combine to form a hydrogen atom in the ground state. A single photon is emitted in this process. What is its wavelength?
18. How many different wavelengths would appear in the spectrum of hydrogen atoms initially in the  $n = 5$  state?
19. Find the wavelength of the spectral line that corresponds to a transition in hydrogen from the  $n = 10$  state to the ground state. In what part of the spectrum is this?
20. Find the wavelength of the spectral line that corresponds to a transition in hydrogen from the  $n = 6$  state to the  $n = 3$  state. In what part of the spectrum is this?
21. A beam of electrons bombards a sample of hydrogen. Through what potential difference must the electrons have been accelerated if the first line of the Balmer series is to be emitted?
22. How much energy is required to remove an electron in the  $n = 2$  state from a hydrogen atom?
23. The longest wavelength in the Lyman series is 121.5 nm and the shortest wavelength in the Balmer series is 364.6 nm. Use the figures to find the longest wavelength of light that could ionize hydrogen.
24. The longest wavelength in the Lyman series is 121.5 nm. Use this wavelength together with the values of  $c$  and  $\hbar$  to find the ionization energy of hydrogen.
25. An excited hydrogen atom emits a photon of wavelength  $\lambda$  in returning to the ground state. (a) Derive a formula that gives the quantum number of the initial excited state in terms of  $\lambda$  and  $R$ . (b) Use this formula to find  $n_i$  for a 102.55-nm photon.
26. An excited atom of mass  $m$  and initial speed  $v$  emits a photon in its direction of motion. If  $v \ll c$ , use the requirement that linear momentum and energy must both be conserved to show that the frequency of the photon is higher by  $\Delta\nu/\nu \approx v/c$  than it would have been if the atom had been at rest. (See also Exercise 16 of Chap. 1.)
27. When an excited atom emits a photon, the linear momentum of the photon must be balanced by the recoil momentum of the atom. As a result, some of the excitation energy of the atom goes into the kinetic energy of its recoil. (a) Modify Eq. (4.16) to include this effect. (b) Find the ratio between the recoil energy and the photon energy for the  $n = 3 \rightarrow n = 2$  transition in hydrogen, for which  $E_f - E_i = 1.9$  eV. Is the effect a major one? A nonrelativistic calculation is sufficient here.

#### 4.6 Correspondence Principle

28. Of the following quantities, which increase and which decrease in the Bohr model as  $n$  increases? Frequency of revolution, electron speed, electron wavelength, angular momentum, potential energy, kinetic energy, total energy.

29. Show that the frequency of the photon emitted by a hydrogen atom in going from the level  $n + 1$  to the level  $n$  is always intermediate between the frequencies of revolution of the electron in the respective orbits.

#### 4.7 Nuclear Motion

30. An antiproton has the mass of a proton but a charge of  $-e$ . If a proton and an antiproton orbited each other, how far apart would they be in the ground state of such a system? Why might you think such a system could not occur?
31. A  $\mu^-$  muon is in the  $n = 2$  state of a muonic atom whose nucleus is a proton. Find the wavelength of the photon emitted when the muonic atom drops to its ground state. In what part of the spectrum is this wavelength?
32. Compare the ionization energy in positronium with that in hydrogen.
33. A mixture of ordinary hydrogen and tritium, a hydrogen isotope whose nucleus is approximately 3 times more massive than ordinary hydrogen, is excited and its spectrum observed. How far apart in wavelength will the  $H_\alpha$  lines of the two kinds of hydrogen be?
34. Find the radius and speed of an electron in the ground state of doubly ionized lithium and compare them with the radius and speed of the electron in the ground state of the hydrogen atom. ( $\text{Li}^{++}$  has a nuclear charge of  $3e$ .)
35. (a) Derive a formula for the energy levels of a hydrogenic atom, which is an ion such as  $\text{He}^+$  or  $\text{Li}^{2+}$  whose nuclear charge is  $+Ze$  and which contains a single electron.  
 (b) Sketch the energy levels of the  $\text{He}^+$  ion and compare them with the energy levels of the H atom. (c) An electron joins a bare helium nucleus to form a  $\text{He}^+$  ion. Find the wavelength of the photon emitted in this process if the electron is assumed to have had no kinetic energy when it combined with the nucleus.

#### 4.9 The Laser

36. For laser action to occur, the medium used must have at least three energy levels. What must be the nature of each of these levels? Why is three the minimum number?
37. A certain ruby laser emits 1.00-J pulses of light whose wavelength is 694 nm. What is the minimum number of  $\text{Cr}^{3+}$  ions in the ruby?

38. Steam at  $100^\circ\text{C}$  can be thought of as an excited state of water at  $100^\circ\text{C}$ . Suppose that a laser could be built based upon the transition from steam to water, with the energy lost per molecule of steam appearing as a photon. What would the frequency of such a photon be? To what region of the spectrum does this correspond? The heat of vaporization of water is 2260 kJ/kg and its molar mass is 18.02 kg/kmol.

#### Appendix: Rutherford Scattering

39. The Rutherford scattering formula fails to agree with the data at very small scattering angles. Can you think of a reason?
40. Show that the probability for a 2.0-MeV proton to be scattered by more than a given angle when it passes through a thin foil is the same as that for a 4.0-MeV alpha particle.
41. A 5.0-MeV alpha particle approaches a gold nucleus with an impact parameter of  $2.6 \times 10^{-13}$  m. Through what angle will it be scattered?
42. What is the impact parameter of a 5.0-MeV alpha particle scattered by  $10^\circ$  when it approaches a gold nucleus?
43. What fraction of a beam of 7.7-MeV alpha particles incident upon a gold foil  $3.0 \times 10^{-7}$  m thick is scattered by less than  $1^\circ$ ?
44. What fraction of a beam of 7.7-MeV alpha particles incident upon a gold foil  $3.0 \times 10^{-7}$  m thick is scattered by  $90^\circ$  or more?
45. Show that twice as many alpha particles are scattered by a foil through angles between  $60^\circ$  and  $90^\circ$  as are scattered through angles of  $90^\circ$  or more.
46. A beam of 8.3-MeV alpha particles is directed at an aluminum foil. It is found that the Rutherford scattering formula ceases to be obeyed at scattering angles exceeding about  $60^\circ$ . If the alpha-particle radius is assumed small enough to neglect here, find the radius of the aluminum nucleus.
47. In special relativity, a photon can be thought of as having a "mass" of  $m = E_\nu/c^2$ . This suggests that we can treat a photon that passes near the sun in the same way as Rutherford treated an alpha particle that passes near a nucleus, with an attractive gravitational force replacing the repulsive electrical force. Adapt Eq. (4.29) to this situation and find the angle of deflection  $\theta$  for a photon that passes  $b = R_{\text{sun}}$  from the center of the sun. The mass and radius of the sun are respectively  $2.0 \times 10^{30}$  kg and  $7.0 \times 10^8$  m. In fact, general relativity shows that this result is exactly half the actual deflection, a conclusion supported by observations made during solar eclipses as mentioned in Sec. 1.10.