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Note Title 4/30/2012

Orthogonal transformations / Orthogonal matrices

→ Rotations

→ Reflections

Orthogonal transformations → MATRICES

Suppose we have a point $\underset{\sim}{b}$ represented by a column vector

ψ corresponds to a matrix A

$$\psi(\underset{\sim}{b}) = A\underset{\sim}{b}$$

orthogonal transformation

What can we say about its matrix?

(we call the matrix associated with an orthogonal transformation an orthogonal matrix)

Definition of an orthogonal transformation

$$\underset{\sim}{a} \,,\, \underset{\sim}{b}$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \psi(a) \cdot \psi(b)$$

$$\psi(\underset{\sim}{a}) = A\,a$$

$$\psi(\underset{\sim}{b}) = A\,b$$

where A is the matrix corresponding to ψ

What is the dot product in vector matrix notation?

$$\underset{3\times 1}{\underset{\sim}{a}} \cdot \underset{3\times 1}{\underset{\sim}{b}} = \underbrace{\underset{1\times 3}{a^T}\ \underset{3\times 1}{b}}_{1\times 1} \quad (\text{a number})$$

$\leftarrow$ standard way to get scalar products

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \psi(a) \cdot \psi(b) \qquad \text{definition}$$

vectors $\rightarrow$

$$
\begin{aligned}
a^T b &= (Aa) \cdot (Ab) \\
&= (Aa)^T (Ab) \\
&= a^T A^T A b
\end{aligned}
$$

$$a^T b = a^T (A^T A) b$$

which is true for all choices of $\underset{\sim}{a}$ and $\underset{\sim}{b}$

For simplicity consider $\underset{\sim}{a}, \underset{\sim}{b}$ in $\mathbb{R}^2$ (they are 2×1 column vectors). We can consider $\underset{\sim}{a}, \underset{\sim}{b}$ to be various choices of basis vectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

For example we could have $\underset{\sim}{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\underset{\sim}{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$a^T b = a^T (A^T A) b$$

definition of orthogonal transform

$C = A^T A$

If $\underset{\sim}{a}$, $\underset{\sim}{b}$ are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$a^T b = 1$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Downarrow$$

$$c_{11} \Leftarrow \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$c_{11} = 1$

By this kind of process we can show that

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^T A$$

where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$a^T b = a^T \underbrace{(A^T A)}_{I} b$$

What we have shown is that

$$A^T A = I \leftarrow$$

Sometimes this is used as the definition of an orthogonal matrix

$$A^T A = I$$

$a_{11}^2 + a_{22}^2 = 1$

- $A^T = A^{-1}$
- Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ in 2D

$$A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

$$A^T A = \begin{bmatrix} a_{11}^2 + a_{21}^2 & a_{11} a_{12} + a_{21} a_{22} \\ a_{12} a_{11} + a_{21} a_{22} & a_{11}^2 + a_{21}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$