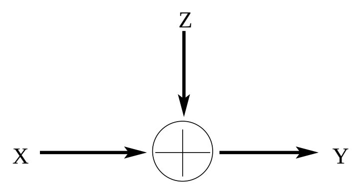
Homework 4
Solutions

1. Find the channel capacity of the following discrete memoryless channel:



where $\Pr\{Z=0\} = \Pr\{Z=a\} = \frac{1}{2}$. The alphabet for x is $\mathbf{X} = \{0,1\}$. Assume that Z is independent of X. Observe that the channel capacity depends on the value of a.

Solution :

$$Y = X + Z \quad X \in \{0,1\}, \quad Z \in \{0,a\}$$

We have to distinguish various cases depending on the values of a.

- $a=0$. In this case, $Y=X$, and $\max I(X;Y) = \max H(X) = 1$. Hence the capacity is 1 bit per transmission.
- $a \neq 0, \pm 1$. In this case, Y has four possible values 0, 1, a, and $1+a$. Knowing Y, we know the X which was sent, and hence $H(X|Y)=0$. Hence $\max I(X;Y) = \max H(X) = 1$, achieved for an uniform distribution on the input X.
- $a=1$. In this case, Y has three possible output values, 0, 1, and 2. The channel is identical to the binary erasure channel with $a=1/2$. The capacity of this channel is 1/2 bit per transmission.
- $a=-1$. This is similar to the case when $a=1$ and the capacity here is also 1/2 bit per transmission.

2. Consider a 26-key typewriter.

   (a) If pushing a key results in printing the associated letter, what is the capacity C in bits?

   (b) Now suppose that pushing a key results in printing that letter or the next (with equal probability). Thus $A \to A$ or B, $\cdots$, $Z \to Z$ or A. What is the capacity?

   (c) What is the highest rate code with block length one that you can find that achieves *zero* probability of error for the channel in part (b).

Solution :

(a) If the typewriter prints out whatever key is struck, then the output Y, is the same as the input X, and

$$C = \max I(X;Y) = \max H(X) = \log 26,$$

attained by a uniform distribution over the letters.

(b) In this case, the output is either equal to the input (with probability $\frac{1}{2}$) or equal to the next letter (with probability $\frac{1}{2}$). Hence $H(Y|X) = \log 2$ independent of the distribution of X, and hence

$$C = \max I(X;Y) = \max H(Y) - \log 2 = \log 26 - \log 2 = \log 13$$

attained for a uniform distribution over the output, which in turn is attained by a uniform distribution on the input.

(c) A simple zero error block length one code is the one that uses every alternate letter, say A,C,E,$\cdots$,W,Y. In this case, none of the codewords will be confused, since A will produce either A or B, C will produce C or D, etc. The rate of this code,

$$R = \frac{\log(\sharp \text{ codewords})}{\text{Block length}} = \frac{\log 13}{1} = \log 13$$

In this case, we can achieve capacity with a simple code with zero error.

3. Consider a binary symmetric channel with $Y_i = X_i \oplus Z_i$, where $\oplus$ is mod 2 addition, and $X_i, Y_i \in \{0,1\}$.
Suppose that $\{Z_i\}$ has constant marginal probabilities $p(Z_i = 1) = p = 1 - p(Z_i = 0)$, but that $Z_1, Z_2, \cdots, Z_n$ are not necessarily independent. Let $C = 1 - H(p)$. Show that

$$\max_{p(x_1,x_2,\cdots,x_n)} I(X_1, X_2, \cdots, X_n; Y_1, Y_2, \cdots, Y_n) \geq nC$$

Comment on the implications.

Solution :
When $X_1, X_2, \cdots, X_n$ are chosen i.i.d. $\sim \text{Bern}(\frac{1}{2})$,

$$\begin{aligned} I(X_1,\cdots,X_n;Y_1,\cdots,Y_n) &= H(X_1,\cdots,X_n) - H(X_1,\cdots,X_n|Y_1,\cdots,Y_n) \\ &= H(X_1,\cdots,X_n) - H(Z_1,\cdots,Z_n|Y_1,\cdots,Y_n) \\ &\geq H(X_1,\cdots,X_n) - H(Z_1,\cdots,Z_n) \\ &\geq H(X_1,\cdots,X_n) - \sum H(Z_i) \\ &= n - nH(p) \end{aligned}$$

Hence, the capacity of the channel with memory over n uses of the channel is

$$\begin{aligned} nC^{(n)} = &\max_{p(X_1,\cdots,X_n)} I(X_1,\cdots,X_n;Y_1,\cdots,Y_n) \\ &\geq I(X_1,\cdots,X_n;Y_1,\cdots,Y_n)_{p(x_1,\cdots,x_n)=\text{Bern}(\frac{1}{2})} \\ &\geq n(1 - H(p)) \\ = nC \end{aligned}$$

Hence, channels with memory have higher capacity. The intuitive explanation for this result is that the correlation between the noise decreases the effective noise; one could use the information from the past samples of the noise to combat the present noise.

4. Consider the channel $Y = X + Z \pmod{13}$, where

$$Z = \begin{cases} 1, & \text{with probability } \frac{1}{3} \\ 2, & \text{with probability } \frac{1}{3} \\ 3, & \text{with probability } \frac{1}{3} \end{cases}$$

and $X \in \{0, 1, \cdots, 12\}$.

(a) Find the capacity.
(b) What is the maximizing $p^*(x)$?

Solution :

(a)

$$
\begin{aligned}
C &= \max_{p(x)} I(X;Y) \\
&= \max_{p(x)} H(Y) - H(Y|X) \\
&= \max_{p(x)} H(Y) - \log 3 \\
&= \log 13 - \log 3 \\
&= \log \frac{13}{3}
\end{aligned}
$$

which is attained when Y has a uniform distribution, which occurs (by symmetry) when X has a uniform distribution.

(b) The capacity is achieved by a uniform distribution on the inputs, that is,

$$
p(X = i) = \frac{1}{13} \quad \text{for} \;\; i = 0, 1, \cdots, 12.
$$

5. Using two channels

(a) Consider two discrete memoryless channels $(\mathcal{X}_1, p(y_1|x_1), \mathcal{Y}_1)$ and $(\mathcal{X}_2, p(y_2|x_2), \mathcal{Y}_2)$ with capacities C_1 and C_2 respectively. A new channel $(\mathcal{X}_1 \times \mathcal{X}_2, p(y_1|x_1) \times p(y_2|x_2), \mathcal{Y}_1 \times \mathcal{Y}_2)$ is formed in which $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$, are *simultaneously* sent, resulting in y_1, y_2. Find the capacity of this channel.

(b) Find the capacity C of the union 2 channels $(\mathcal{X}_1, p(y_1|x_1), \mathcal{Y}_1)$ and $(\mathcal{X}_2, p(y_2|x_2), \mathcal{Y}_2)$ where, at each time, one can send a symbol over channel 1 or channel 2 but not both. Assume the output alphabets are distinct and do not intersect. Show $2^C = 2^{C_1} + 2^{C_2}$.

Solution :

(a) To find the capacity of the product channel $(\mathcal{X}_1 \times \mathcal{X}_2, p(y_1, y_2|x_1, x_2), \mathcal{Y}_1 \times \mathcal{Y}_2)$, we have to find the distribution $p(x_1, x_2)$ on the input alphabet $\mathcal{X}_1 \times \mathcal{X}_2$ that maximizes $I(X_1, X_2; Y_1, Y_2)$. Since the transition probabilities are given as $p(y_1, y_2|x_1, x_2) = p(y_1|x_1)p(y_2|x_2)$,

$$
\begin{aligned}
p(x_1, x_2, y_1, y_2) &= p(x_1, x_2)p(y_1, y_2|x_1, x_2) \\
&= p(x_1, x_2)p(y_1|x_1)p(y_2|x_2)
\end{aligned}
$$

Therefore, $Y_1 \rightarrow X_1 \rightarrow X_2 \rightarrow Y_2$ forms a Markov chain and

$$
\begin{aligned}
I(X_1, X_2; Y_1, Y_2) &= H(Y_1, Y_2) - H(Y_1, Y_2|X_1, X_2) \\
&= H(Y_1, Y_2) - H(Y_1|X_1, X_2) - H(Y_2|X_1, X_2) && (1) \\
&= H(Y_1, Y_2) - H(Y_1|X_1) - H(Y_2|X_2) && (2) \\
&\le H(Y_1) + H(Y_2) - H(Y_1|X_1) - H(Y_2|X_2) && (3) \\
&= I(X_1; Y_1) + I(X_2; Y_2)
\end{aligned}
$$

where (1) and (2) follow from Markovity and (3) is met with equality of X_1 and X_2 are independent and hence Y_1 and Y_2 are independent. Therefore

$$\begin{aligned} C &= \max_{p(x_1,x_2)} I(X_1, X_2; Y_1, Y_2) \\ &\le \max_{p(x_1,x_2)} I(X_1; Y_1) + \max_{p(x_1,x_2)} I(X_2; Y_2) \\ &= \max_{p(x_1)} I(X_1; Y_1) + \max_{p(x_2)} I(X_2; Y_2) \\ &= C_1 + C_2 \end{aligned}$$

with equality iff $p(x_1, x_2) = p^*(x_1)p^*(x_2)$ and $p^*(x_1)$ and $p^*(x_2)$ are the distributions that maximize C_1 and C_2 respectively.

(b) Let

$$\theta = \begin{cases} 1, & \text{if the signal is sent over the channel 1} \\ 2, & \text{if the signal is sent over the channel 2} \end{cases}$$

Consider the following communication scheme: The sender chooses between two channels according to Bern(α) coin flip. Then the channel input is $X = (\theta, X_\theta)$.
Since the output alphabets $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are disjoint, θ is a function of Y,i.e. $X \to Y \to \theta$. Therefore,

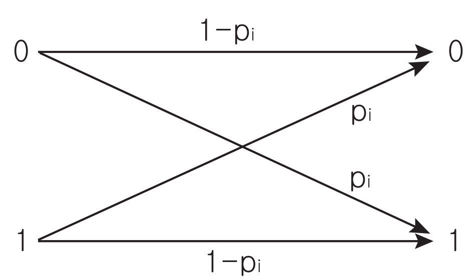
$$\begin{aligned} I(X;Y) &= I(X; Y, \theta) \\ &= I(X_\theta, \theta; Y, \theta) \\ &= I(\theta; Y, \theta) + I(X_\theta; Y, \theta|\theta) \\ &= I(\theta; Y, \theta) + I(X_\theta; Y|\theta) \\ &= H(\theta) + \alpha I(X_\theta; Y|\theta = 1) + (1-\alpha) I(X_\theta; Y|\theta = 2) \\ &= H(\alpha) + \alpha I(X_1; Y_1) + (1-\alpha) I(X_2; Y_2) \end{aligned}$$

Thus, it follows that

$$C = \sup_\alpha \{H(\alpha) + \alpha C_1 + (1-\alpha) C_2\}$$

which is a strictly concave function on α. Hence, the maximum exists and by elementary calculus, one can easily show $C = \log_2(2^{C_1} + 2^{C_2})$, which is attained with $\alpha = 2^{C_1}/(2^{C_1} + 2^{C_2})$.

6. Consider a time-varying discrete *memoryless* binary symmetric channel. Let $Y_1, Y_2, \cdots, Y_n$ be conditionally independent given $X_1, X_2, \cdots, X_n$, with conditional distribution given by $p(y^n|x^n) = \prod_{i=1}^n p_i(y_i|x_i)$, as shown below.



(a) Find $\max_{p(x)} I(X^n; Y^n)$.

(b) We now ask for the capacity for the time invariant version of this problem. Replace each p_i, $1 \le i \le n$, by the average value $\bar{p} = \frac{1}{n}\sum_{j=1}^{n} p_j$, and compare the capacity to part (a).

Solution :

(a)

$$\begin{aligned} I(X^n;Y^n) &= H(Y^n) - \sum_{i=1}^{n} H(Y_i|X_i) \\ &\le \sum_{i=1}^{n} H(Y_i) - \sum_{i=1}^{n} H(Y_i|X_i) \\ &\le \sum_{i=1}^{n} (1 - H(p_i)) \end{aligned}$$

with equality if $X_1, \cdots, X_n$ are chosen i.i.d. $\sim$ Bern($\frac{1}{2}$). Hence

$$\max_{p(x)} I(X_1, \cdots, X_n; Y_1, \cdots, Y_n) = \sum_{i=1}^{n} (1 - H(p_i))$$

(b) Since $H(p)$ is concave on p, by Jensen's inequality,

$$\frac{1}{n}\sum_{i=1}^{n} H(p_i) \le H\left(\frac{1}{n}\sum_{i=1}^{n} p_i\right) = H(\bar{p})$$

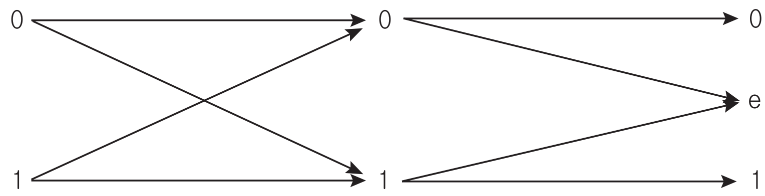
i.e.,

$$\sum_{i=1}^{n} H(p_i) \le nH(\bar{p})$$

Hence,

$$\begin{aligned} C_{\text{time-varying}} &= \sum_{i=1}^{n} (1 - H(p_i)) \\ &= n - \sum_{i=1}^{n} H(p_i) \\ &\ge n - nH(\bar{p}) \\ &= \sum_{i=1}^{n} (1 - H(\bar{p})) \\ &= C_{\text{time invariant}} \end{aligned}$$

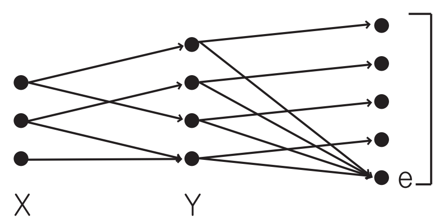
7. Suppose a binary symmetric channel of capacity C_1 is immediately followed by a binary erasure channel of capacity C_2. Find capacity C of the resulting channel.



Now consider an arbitrary discrete memoryless channel $(\mathcal{X}, p(y|x), \mathcal{Y})$ followed by a binary erasure channel , resulting in an output

$$\tilde{Y} = \begin{cases} Y, & \text{with probability } 1-\alpha \\ e, & \text{with probability } \alpha \end{cases}$$

where e denotes erasure. Thus the output $\mathcal{Y}$ is erased with probability α What is the capacity of this channel?



Solution :

(a) Let $C_1 = 1 - H(p)$ be the capacity of the BSC with parameter p, and $C_2 = 1 - \alpha$ be the capacity of the BEC with parameter α. Let $\tilde{Y}$ denote the output of the cascaded channel, and Y the output of the BSC. Then, the transition rule for the cascaded channel is simply

$$p(\tilde{y}|x) = \sum_{y=0,1} p(\tilde{y}|y)p(y|x)$$

for each $(x, \tilde{y})$ pair.
Let $X \sim \text{Bern}(\pi)$ denote the input to the channel. Then,

$$H(\tilde{Y}) = H((1-\alpha)(\pi(1-p) + p(1-\pi)), \alpha, (1-\alpha)(p\pi + (1-p)(1-\pi)))$$

and also

$$\begin{aligned} H(\tilde{Y}|X=0) &= H((1-\alpha)(1-p), \alpha, (1-\alpha)p) \\ H(\tilde{Y}|X=1) &= H((1-\alpha)p, \alpha, (1-\alpha)(1-p)) = H(\tilde{Y}|X=0) \end{aligned}$$

Therefore,

$$\begin{aligned} C &= \max_{p(x)} I(X;\tilde{Y}) \\ &= \max_{p(x)} \{H(\tilde{Y}) - H(\tilde{Y}|X)\} \\ &= \max_{p(x)} \{H(\tilde{Y})\} - H(\tilde{Y}|X) \\ &= \max_{p(x)} \{H((1-\alpha)(\pi(1-p) + p(1-\pi)), \alpha, (1-\alpha)(p\pi + (1-p)(1-\pi)))\} \\ &\quad -H((1-\alpha)(1-p), \alpha, (1-\alpha)p) \end{aligned} \tag{4}$$

Note that the maximum value of $H(\tilde{Y})$ occurs when $\pi = 1/2$ by the concavity and symmetry of $H(\cdot)$. (We can check this also by differentiating (4) with respect to π.)
Substituting the value $\pi = 1/2$ in the expression for the capacity yields

$$\begin{aligned} C &= H((1-\alpha)/2, \alpha, (1-\alpha)/2) - H((1-p)(1-\alpha), \alpha, p(1-\alpha)) \\ &= (1-\alpha)(1 + p\log p + (1-p)\log(1-p)) \\ &= C_1 C_2 \end{aligned}$$

(b) For the cascade of an arbitrary discrete memoryless channel (with capacity C) with the erasure channel (with the erasure probability α), we will show that

$$I(X;\tilde{Y}) = (1-\alpha)I(X;Y) \tag{5}$$

Then, by taking suprema of both sides over all input distributions $p(x)$, we can conclude the capacity of the cascaded channel is $(1-\alpha)C$.
Proof of (5):
Let

$$E = \begin{cases} 1, & \tilde{Y} = e \\ 0, & \tilde{Y} = Y \end{cases}$$

Then, since E is a function of Y,

$$\begin{aligned} H(\tilde{Y}) &= H(\tilde{Y}, E) \\ &= H(E) + H(\tilde{Y}|E) \\ &= H(\alpha) + \alpha H(\tilde{Y}|E=1) + (1-\alpha)H(\tilde{Y}|E=0) \\ &= H(\alpha) + (1-\alpha)H(Y), \end{aligned}$$

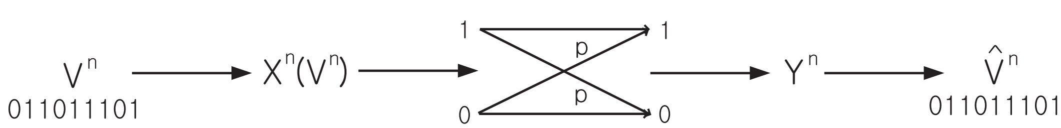
where the last equality comes directly from the construction of E. Similarly,

$$\begin{aligned} H(\tilde{Y}|X) &= H(\tilde{Y}, E|X) \\ &= H(E|X) + H(\tilde{Y}|X, E) \\ &= H(E) + \alpha H(\tilde{Y}|X, E=1) + (1-\alpha)H(\tilde{Y}|X, E=0) \\ &= H(\alpha) + (1-\alpha)H(Y|X), \end{aligned}$$

whence

$$I(X;\tilde{Y}) = H(\tilde{Y}) - H(\tilde{Y}|X) = (1-\alpha)I(X;Y)$$

8. We wish to encode a Bernoulli(α) process $V_1, V_2, \cdots$ for transmission over a binary symmetric channel with error probability p.



Find conditions on α and p so that the probability of error $p(\hat{V}^n \neq V^n)$ can be made to go to zero as $n \to \infty$.

Solution :
Suppose we want to send a binary i.i.d. Bern(α) source over a binary symmetric channel with error

probability p. By the source-channel separation theorem, in order to achieve the probability of error that vanishes asymptotically, i.e. $P(\hat{V}^n \neq V^n) \to 0$, we need the entropy of the source to be less than the capacity of the channel. Hence,

$$H(\alpha) + H(p) < 1,$$

or, equivalently,

$$\alpha^\alpha (1-\alpha)^{1-\alpha} p^p (1-p)^{1-p} < \frac{1}{2}.$$

9. Let (X_i, Y_i, Z_i) be i.i.d. according to $p(x, y, z)$. We will say that (x^n, y^n, z^n) is jointly typical [written $(x^n, y^n, z^n) \in A_\epsilon^{(n)}$] if

   - $2^{-n(H(X)+\epsilon)} \leq p(x^n) \leq 2^{-n(H(X)-\epsilon)}$
   - $2^{-n(H(Y)+\epsilon)} \leq p(y^n) \leq 2^{-n(H(Y)-\epsilon)}$
   - $2^{-n(H(Z)+\epsilon)} \leq p(z^n) \leq 2^{-n(H(Z)-\epsilon)}$
   - $2^{-n(H(X,Y)+\epsilon)} \leq p(x^n, y^n) \leq 2^{-n(H(X,Y)-\epsilon)}$
   - $2^{-n(H(X,Z)+\epsilon)} \leq p(x^n, z^n) \leq 2^{-n(H(X,Z)-\epsilon)}$
   - $2^{-n(H(Y,Z)+\epsilon)} \leq p(y^n, z^n) \leq 2^{-n(H(Y,Z)-\epsilon)}$
   - $2^{-n(H(X,Y,Z)+\epsilon)} \leq p(x^n, y^n, z^n) \leq 2^{-n(H(X,Y,Z)-\epsilon)}$

   Now suppose that $(\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n)$ is drawn according to $p(x^n)p(y^n)p(z^n)$. Thus, $\tilde{X}^n$, $\tilde{Y}^n$, $\tilde{Z}^n$ have the same marginals as $p(x^n, y^n, z^n)$ but are independent. Find (bounds on) $\Pr\{(\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n) \in A_\epsilon^{(n)}\}$ in terms of the entropies $H(X)$, $H(Y)$, $H(Z)$, $H(X,Y)$, $H(X,Z)$, $H(Y,Z)$ and $H(X,Y,Z)$.

   Solution :

$$\begin{aligned}
\Pr\{(\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n) \in A_\epsilon^{(n)}\} &= \sum_{(x^n,y^n,z^n)\in A_\epsilon^{(n)}} p(x^n)p(y^n)p(z^n) \\
&\leq \sum_{(x^n,y^n,z^n)\in A_\epsilon^{(n)}} 2^{-n(H(X)+H(Y)+H(Z)-3\epsilon)} \\
&\leq |A_\epsilon^{(n)}| 2^{-n(H(X)+H(Y)+H(Z)-3\epsilon)} \\
&\leq 2^{n(H(X,Y,Z)+\epsilon)} 2^{-n(H(X)+H(Y)+H(Z)-3\epsilon)} \\
&\leq 2^{n(H(X,Y,Z)-H(X)-H(Y)-H(Z)+4\epsilon)}
\end{aligned}$$

   Also,

$$\begin{aligned}
\Pr\{(\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n) \in A_\epsilon^{(n)}\} &= \sum_{(x^n,y^n,z^n)\in A_\epsilon^{(n)}} p(x^n)p(y^n)p(z^n) \\
&\geq \sum_{(x^n,y^n,z^n)\in A_\epsilon^{(n)}} 2^{-n(H(X)+H(Y)+H(Z)+3\epsilon)} \\
&\geq |A_\epsilon^{(n)}| 2^{-n(H(X)+H(Y)+H(Z)+3\epsilon)} \\
&\geq (1-\epsilon) 2^{n(H(X,Y,Z)-\epsilon)} 2^{-n(H(X)+H(Y)+H(Z)-3\epsilon)} \\
&\geq (1-\epsilon) 2^{n(H(X,Y,Z)-H(X)-H(Y)-H(Z)-4\epsilon)}
\end{aligned}$$

   Note that the upper bound is true for all n, but the lower bound only hold for n large.

10. Twenty questions.

   (a) Player A chooses some object in the universe, and player B attempts to identify the object with a series of yes-no questions. Suppose that player B is clever enough to use the code achieving the minimal expected length with respect to player A's distribution We observe that player B requires an average 38.5 questions to determine the object. Find a rough lower bound to the number of objects in the universe.

   (b) Let X be uniformly distributed over $\{1, 2, \cdots, m\}$. Assume that $m = 2^n$. We ask random questions: Is $X \in S_1$? Is $X \in S_2$? $\cdots$ until only one integer remains. All 2^m subsets S of $\{1, 2, \cdots m\}$ are equally likely.

      i. How many deterministic questions are needed to determine X?

      ii. Without loss of generality, suppose that $X = 1$ is the random object. What is the probability that object 2 yields the same answers as object 1 for k questions?

      iii. What is the expected number of objects in $\{2, 3, \cdots, m\}$ that have the same answers to the questions as those of the correct object 1?

Solution :

(a)

$$37.5 = L^* - 1 < H(X) \leq \log |\mathcal{X}|$$

and hence number of objects in the universe $> 2^{37.5} = 1.94 \times 10^{11}$.

(b) i. Obviously, Huffman codewords for X are all of length n. Hence, with n deterministic questions, we can identify an object out of 2^n candidates.

   ii. Observe that the total number of subsets which include both object 1 and object 2 or neither of them is 2^{m-1}. Hence, the probability that object 2 yields the same answers for k questions as object 1 is $(2^{m-1}/2^m)^k = 2^{-k}$.

   iii. Let

$$1_j = \begin{cases} 1, & \text{object } j \text{ yields the same answers for } k \text{ questions as object 1} \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{for} \quad j = 2, \cdots, m$$

Then

$$\begin{aligned} E[N] &= E\left[\sum_{j=2}^{m} 1_j\right] \\ &= \sum_{j=2}^{m} E[1_j] \\ &= \sum_{j=2}^{m} 2^{-k} \\ &= (m-1)2^{-k} \\ &= (2^n - 1)2^{-k} \end{aligned}$$

where N is the number of objects in $\{2, 3, \cdots, m\}$ with the same answers.