Assignment 4 solns.

1. Express a two qubit system, such that the likelihood to be $|0\rangle$ is 64% times for the first bit and 36% for the second bit [Hint: for $|xx>$; $|x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, each qubit has a likelihood of 50% to be $|0\rangle$]. How many different states are possible for such a system and what are the probabilities for each [Hint: for $|xx>$ above, there are four possible states, each with a probability of 25%].

Ans. The amplitudes are the square roots of probabilities; ($\sqrt{64} = 8$; $\sqrt{36} = 6$)

qubit	prob($\|0\rangle$)	prob($\|1\rangle$)	amplitude($\|0\rangle$)	amplitude($\|1\rangle$)
$\|x\rangle$	0.64	0.36	0.8	0.6
$\|y\rangle$	0.36	0.64	0.6	0.8

Thus the expression for the two qubit system is:

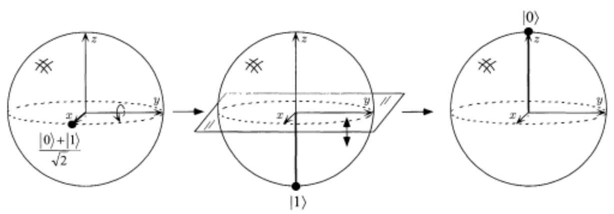
$$|xy>;\ |x\rangle = 0.8|0\rangle + 0.6|1\rangle);\ |y\rangle = 0.6|0\rangle + 0.8|1\rangle$$

2. Show that two consecutive operations of the Hadamard logic gate leaves a qubit unchanged (in its original form).

Ans. $\hat{\mathcal{H}}^2 \begin{pmatrix} \alpha|0\rangle \\ \beta|1\rangle \end{pmatrix} = \hat{\mathcal{H}}.\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha|0\rangle \\ \beta|1\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} . \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha|0\rangle \\ \beta|1\rangle \end{pmatrix} =$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha|0\rangle \\ \beta|1\rangle \end{pmatrix} = \begin{pmatrix} \alpha|0\rangle \\ \beta|1\rangle \end{pmatrix}$

3. Consider a qubit as a vector in the Bloch sphere. Show that a Hadamard operation is equivalent to three rotational transformations.

Ans. Fig. 1.4 in Nielsen-Chuang's text shows it:



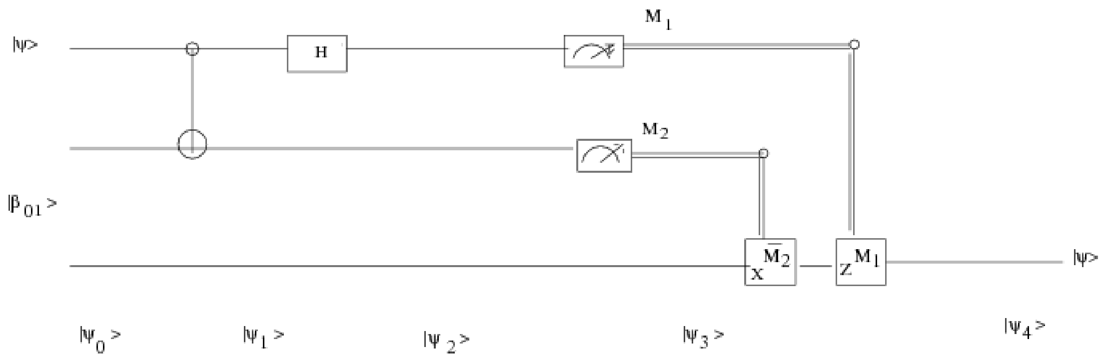
The Hadamard operation is equivalent to rotation around the $y-$ axis followed by reflection across the $x-y$ plane. Shown in the picture is the transformation of the qubit $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ to the qubit

$|0\rangle$. It could be made simpler as a single rotation by $\frac{\pi}{2}$ around the $y-$axis in the clockwise direction. Every rotation can be decomposed to any arbitrary number of rotations.

In general a unitary operation is equivalent to three rotations.

4. In the teleportation problem discussed in the class, if Alice and Bob had shared the Bell state $|\beta_{01}\rangle$ initially, (instead of the $|\beta_{00}\rangle$), how will that change the circuit required for teleportation?

Ans. It is similar to the model discussed in the text for the case when Alice and Bob had shared the Bell state $|\beta_{00}\rangle$. The difference is that Bob has to pass his bit through $X^{\bar{M}_2}Y^{M_1}$ as opposed to $X^{M_2}Y^{M_1}$.



The state to be teleported is $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where$\alpha$and β are unknown amplitudes. The (three qubit system) state input into the circuit$|\psi_0\rangle$is

$$|\psi_0\rangle = |\psi\rangle|\beta_{01}\rangle$$

$$= \frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|01\rangle + |10\rangle) + \beta|1\rangle(|01\rangle + |10\rangle)\right]$$

where we use the convention that the first two qubits (on the left) belong to Alice , and the third qubit to Bob. As we explained previously,Alice's second qubit and Bob's qubit start out in an EPR state.Alice sends her qubits through a CNOT gate, obtaining

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left[\alpha|0\rangle\left(|01\rangle + |10\rangle\right) + \beta|1\rangle\left(|11\rangle + |00\rangle\right)\right]$$

she then sends the first qubit through a Hadamard gate, obtaining

$$|\psi_2\rangle = \frac{1}{2}\left[\alpha\left(|0\rangle + |1\rangle\right)\left(|01\rangle + |10\rangle\right) + \beta\left(|0\rangle - |1\rangle\right)\left(|11\rangle + |00\rangle\right)\right]$$

This state may be re-written in the following way,simply by regrouping terms:

$$|\psi_2\rangle = \frac{1}{2}\left[|00\rangle\left(\alpha|1\rangle + \beta|0\rangle\right) + |01\rangle\left(\alpha|0\rangle + \beta|1\rangle\right) + |10\rangle\left(\alpha|1\rangle - \beta|0\rangle\right) + |11\rangle\left(\alpha|0\rangle - \beta|1\rangle\right)\right]$$

This expression naturally breaks down into four terms. The first term has Alice's qubits in the state$|00\rangle$,and Bob's qubit in the state $\alpha|1\rangle + \beta|0\rangle$- which is the original state $|\psi\rangle$.If Alice performs a measurement and obtains the result 00 then Bob's system will be in the state $|\psi\rangle$.Similarly,from the previous expression we can read off Bob's post-measurement state,given the result of Alice's measurement:

$$00 \rightarrow |\psi_3\left(00\right)\rangle \equiv \left[\alpha|1\rangle + \beta|0\rangle\right]$$

$$01 \rightarrow |\psi_3\left(01\right)\rangle \equiv \left[\alpha|0\rangle + \beta|1\rangle\right]$$

$$10 \rightarrow |\psi_3\left(10\right)\rangle \equiv \left[\alpha|1\rangle - \beta|0\rangle\right]$$

$$11 \rightarrow |\psi_3\left(11\right)\rangle \equiv \left[\alpha|0\rangle - \beta|1\rangle\right]$$

Depending on Alice's measurement outcome, Bob's qubit will end up in one of these four possible states. Of course, to know which state it is in, Bob must be told the result of Alice's measurement. Once Bob has learned the measurement outcome, Bob can 'fix up' his state, recovering $|\psi\rangle$, by applying the appropriate quantum gate. For example, in the case where the measurements yields 01, Bob doesn't need to do anything. If the measurement is 00 then Bob can fix up his state by applying 00 the X gate.If the measurement is 11 then Bob can fix up his state by applying Z gate. If the measurement is 10 then Bob can fix up his state by applying first an X and then a Z gate. Summing up, Bob needs to apply the transformation $Z^{M_1}X^{\bar{M}_2}$ (note how time goes from left to right in circuit diagrams, but in matrix products terms on the *right* happen *first*) to his qubit, and he will recover the state$|\psi\rangle$.

5. How will you generate a classical NAND gate using the quantum logic gates that you have learned so far.

Ans. See Figs. 1.14 and 1.15 in Nielsen-Chuang's text.