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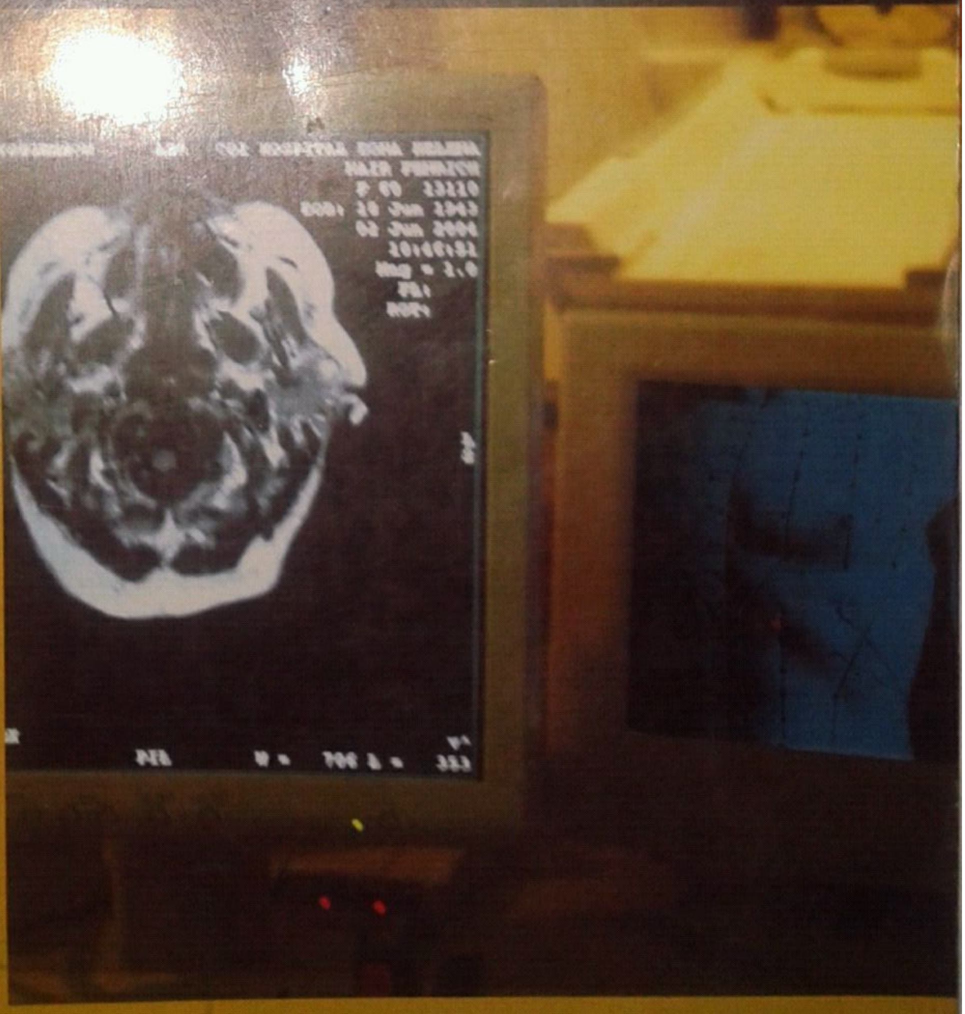
Electrical Engineering Series

Biomedical Signal Processing

PRINCIPLES and TECHNIQUES



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Chapter One

Discrete-Time Signals and Systems

Learning Objectives

- Mathematical representation of discrete-time signals and systems
- Characterization, classification and time-domain representation of equally spaced time sequences
- Important differences between continuous and discrete signals, especially sinusoidal signals
- Classification of systems based on linearity, shift invariance, causality and stability
- Realization of the important class of discrete, linear and shift-invariant systems, through discrete delay elements, modulators and adders.

MOTIVATION

It is a common understanding that a signal in some way conveys information. It is sometimes generated directly by the original information source while at other times it may have to be indirectly obtained or accessed. The purpose could be to learn more about the source such as its structure or functioning. However, the signal available may not directly yield the desired information or it may not be in a desired form, for example, when we may wish to understand the visual processing mechanism of the brain. Here we may present the eye with a flash of light and monitor the activity of the brain by means of electrodes located on the scalp. But the required information related to the visual activity of the brain is buried in the signal which is normally due to the other activities of the brain. Special procedures need to be applied to the signal so as to enhance and make available the relevant information. The task of signal processing is to provide the necessary tools and techniques for this very purpose.

Typically biomedical signals are continuous-time signals. However, it is increasingly becoming more attractive to process continuous-time signals by discrete-time signal processing methods. Towards achieving this objective, continuous-time signals are first converted into discrete-time signals by periodic sampling. Thus arise discrete - time signals or sequences.

In this context it may be noted that discrete signals may also arise by accumulating a variable over a period of time; examples being rainfall data which is usually accumulated over a period such as a day or month, the yield from a chemical process over a period of hours/days, common stock closing prices daily, sunspot numbers yearly, income of a person daily/monthly/yearly and so on.

To understand the theory of digital signal processing and the design of discrete-time systems that implement the various techniques, one needs to know the mathematical representation, characterization and classification of discrete-time signals and systems in the time domain. This is the subject matter of this chapter.

It is a common understanding that **signal** is a quantity which in some manner conveys information about the state of a physical system and / or its functioning. And the task of signal processing is to extract useful information contained in the signal and make it available in a desired form. The method employed to extract this information is dependent on the nature of the signal and the kind of information contained within. Broadly speaking, one may say that the activity encompasses both analytical representation and obtaining the desired information of signals. The representation is mostly mathematical while the extraction may involve analysis as well as development of algorithms. With the aid of basis functions it is possible for a signal to be represented in the domain of its original occurrence or in a transformed domain. Similarly the process of information extraction may be done in either domains.

This chapter is concerned with the time-domain representation of signals and systems and in particular discrete-time signals and systems.

1.1 CHARACTERIZATION, CLASSIFICATION AND TIME-DOMAIN REPRESENTATION OF DISCRETE-TIME SIGNALS

Given the wide variety of signals that occur in everyday life, one observes that most of them are quantities which fluctuate with time. Therefore, it is natural and convenient to think of signals and time functions interchangeably, even if this means occasionally introducing time dependence rather artificially. Nevertheless, the techniques used for dealing with time functions are usually applicable with functions of other variables as for example in image processing where the image is a function of spatial coordinates.

As has been mentioned earlier we are seized with the problem of representation of signals. A familiar representation, in fact one that is deeply ingrained by usage, is the **graph** of a function. It is basically a collection of all ordered pairs of numbers $\{t, x(t)\}$, very often displayed in rectangular coordinates as shown in Fig. 1.1.

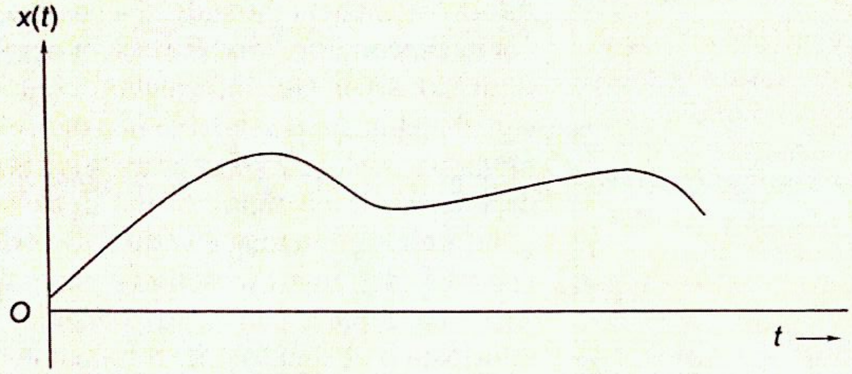

Fig. 1.1

Graph of a signal

Theoretically speaking $x(t)$ is simply a number which is the value of the function at the time instant t. However, because of a popular usage we call $x(t)$ a function, when in reality it is a rule by which values of t and x are paired. Some authors reserve the notation $x(\cdot)$ for the function and $x(t)$ for the number. While the graphical representation appears to be convenient, nevertheless, from the standpoint of a system designer it is unmanageable simply because it comprises too many individual points. Hence the need for other techniques of signal representation.

One such technique is representing a signal by a subset of its graph, i.e. the value that x assumes at a set of instants that are equally spaced in time. The value that a signal assumes at any given value of the independent variable(s) is called **amplitude**. The variation of the amplitude with respect to the independent variable(s) is called **waveform**. The independent variable for one-dimensional signals is usually labeled as time. If the independent variable is continuous, then the signal is referred to as a continuous-time signal or briefly a **continuous signal**.

In this context it may be noted that the amplitude of a continuous signal need not be continuous. However, it is a function of a variable which varies continuously that is usually represented by the symbol, t.

If the independent variable takes on a discrete set of values, then the signal is said to be a discrete-time signal or a **discrete signal**. The independent variable of discrete signals is represented by the symbol n. Since the signal is defined at only discrete instants, it gives rise to a sequence of numbers.

A continuous signal with an amplitude that is continuous is often referred to as an **analog signal**. An example of such a signal is an ECG signal. A discrete signal with amplitudes that are discrete-valued is called a **digital signal**. An example of such a signal is the input to or an output from a digital computer.

A discrete signal with amplitudes that are continuous is called a **sampled data signal**. The output of a switched capacitor is an example of such a signal.

In view of the above definitions it follows that a digital signal is a quantized sampled-data signal (see Fig. 1.2b).

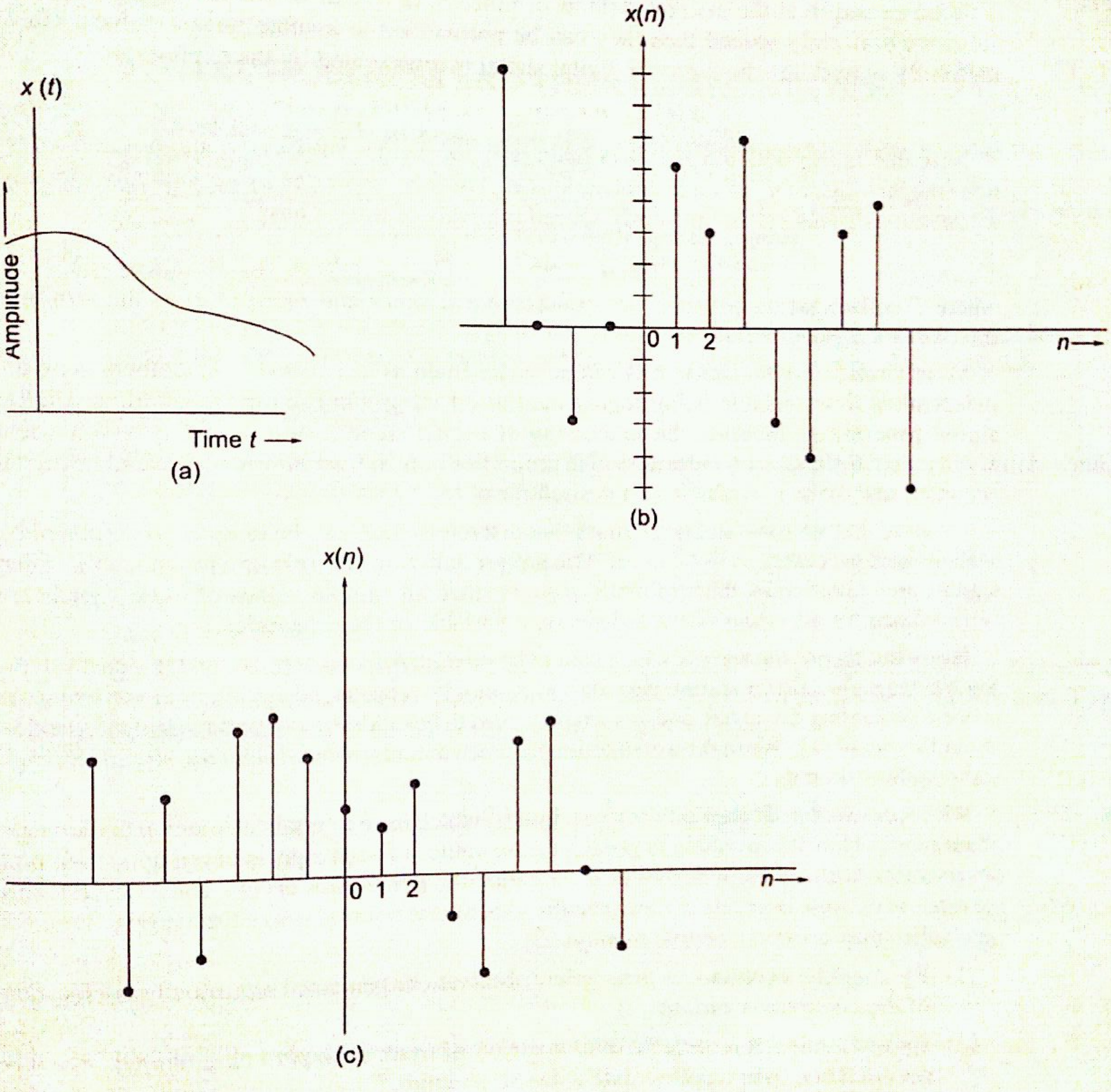

Fig. 1.2

The three kinds of signals that one comes across in signal processing. (a) A continuous-time signal (b) A digital signal (c) A sampled data signal

The emphasis in this book is on one-dimensional digital signals. The functional dependence of such signals is represented mathematically by $x(n)$. However, some authors prefer the notation $x(n)$ for digital signals and $x(nT)$ for discrete signals[1]. Each member of $x(n)$ or $x(nT)$ is called a **sample**. If the discrete instants of time, n, or nT, at which the sample assumes a value are uniformly spaced then they can be normalized to assume integer values. Thus a uniformly spaced one-dimensional digital signal is represented by the expression:

$$x(n); \quad n = -\infty \ldots -2, -1, 0, 1, 2, \ldots \infty \tag{1.1}$$

Note that in this notation $x(n)$ is defined only for integer values of n and not defined for non-integer value of n. In some applications $x(n)$ may be generated by periodically sampling a continuous signal $x(t)$ at uniformly spaced intervals of time. Thus,

$$x(n) = x(t)\big|_{t=nT} = x(nT); \quad -\infty, \ldots -1, 0, 1, \ldots \infty \tag{1.2}$$

where T is the spacing between two consecutive samples and referred to as the sampling interval or sampling period.

To reiterate, a digital signal $x(n)$ can be represented as a sequence[2] of numbers, with the independent time variable being represented as an integer in the range $-\infty$ to ∞. **Digital signal processing** involves the processing of digital signals by a digital system to yield another digital signal with more desirable properties or to extract information from the digital sequence and make it available in a desired form.

Observe that we have so far assumed that discrete signals can be uniquely determined by well defined processes such as an analytical representation or a rule or a lookup table. Such signals are classified as **deterministic signals** since all sample values of these signals are well defined for all values of the independent variable or the time index.

However, there exist signals which cannot be predicted ahead based upon their past values. Such signals are called **random signals** and cannot be reproduced at will, not even using the process generating the signal, and as such they need to be modeled using **statistical** information about the signal [1]. We make a distinction between a random and **stochastic signal** although some authors do not.

We define **stochastic signals** as those (signals) which have a certain element or randomness about them. Thus it is possible to predict future values of such signals based upon their past observations with a certain degree of error called the **prediction error**. If the observations are made at discrete intervals of time then the sequence is referred to as a **time series**. Discrete time series may arise in a couple of ways [2]

(1) By sampling a continuous time series (observations generated sequentially as a function of the continuous variable, t).

(2) By accumulating a variable over a period of time; example being rainfall, which is accumulated over a period, like a day or a month.

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[1] Unless otherwise stated we will assume that all discrete signals are digital signals. As such the words digital and discrete are used interchangeably throughout the text book.

[2] The notation $x(n)$ is used to indicated the entire sequence (i.e. with $-\infty \le n \le \infty$) or on occasions just to denote the nth sample $x(n)$. The context will clarify the exact meaning.

The discrete-time random signal consists of an infinitely large collection called an **ensemble** of discrete-time sequences represented by the symbol $\{X(n)\}$. One particular member of this set or collection, $\{x(n)\}$, is called a realization of the random process. Consequently, at any given time instant n, the observed value $x(n)$, is the value assumed by the random variable $X(n)$.

1.2 TYPICAL SEQUENCES AND THEIR REPRESENTATION

We shall now give a brief explanation of several typical sequences that play important roles in the analysis and design of digital-time systems. For instance, it is possible to represent an arbitrary signal in terms of these basic sequences. Also, it is possible to characterize discrete systems on the basis of their response to these special signals.

1.2.1 Some Typical Sequences

Signals that typify the special signals mentioned are the unit sample sequence, the unit step sequence, the exponential sequence and the single frequency (sinusoidal) sequence.

Unit Sample Sequence

The unit sample sequence, sometimes called the unit pulse, and often called the discrete-time impulse is defined by the relation

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases} \tag{1.3}$$

and shown in Fig. 1.3

The discrete-time impulse should not be compared with the impulse function $\delta(t)$ of the real continuous variable, t. The function $\delta(t)$ is usually called the **Dirac delta function**. This function is defined to be zero everywhere except $t = 0$, and

$$\int_a^b \delta(t)dt = 1, \text{ if and only if } a < t < b. \tag{1.4}$$

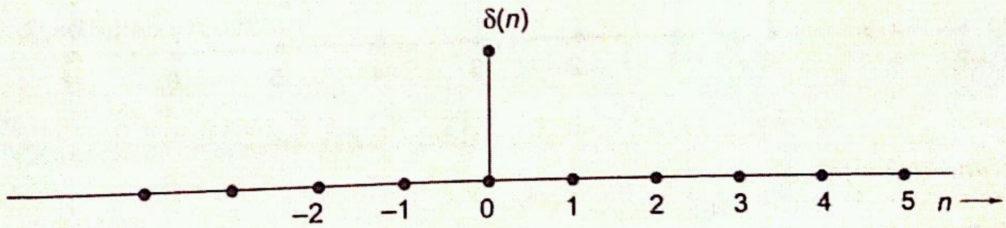


Fig. 1.3

The unit sample sequence

A unit sample sequence that is shifted by m samples is given by the equation

$$\delta(n-m) = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases} \tag{1.5}$$

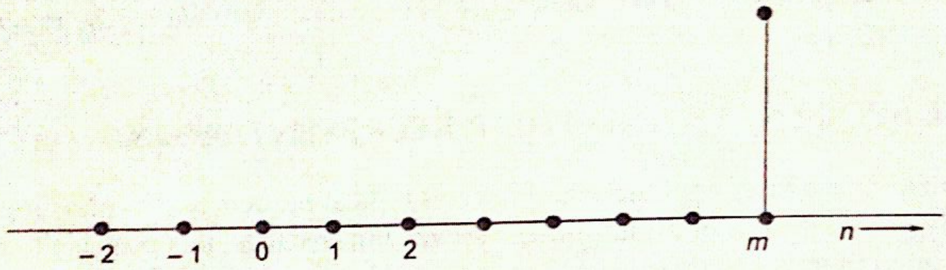


Fig. 1.4

A shifted unit sample sequence

The importance of a unit sample sequence can never be over emphasized. For instance, any arbitrary sequence can always be expressed as a linear sum of weighted and time shifted unit sequences. Also it is possible to completely characterize in the time-domain a certain class of discrete-time systems by their response to a unit sample sequence. And once the impulse response is known, the response of the class to any arbitrary input can be obtained.

Unit Step Sequence

The second special signal is the unit step sequence shown in Fig. 1.5 and defined by the equation

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & \text{elsewhere} \end{cases} \tag{1.6}$$

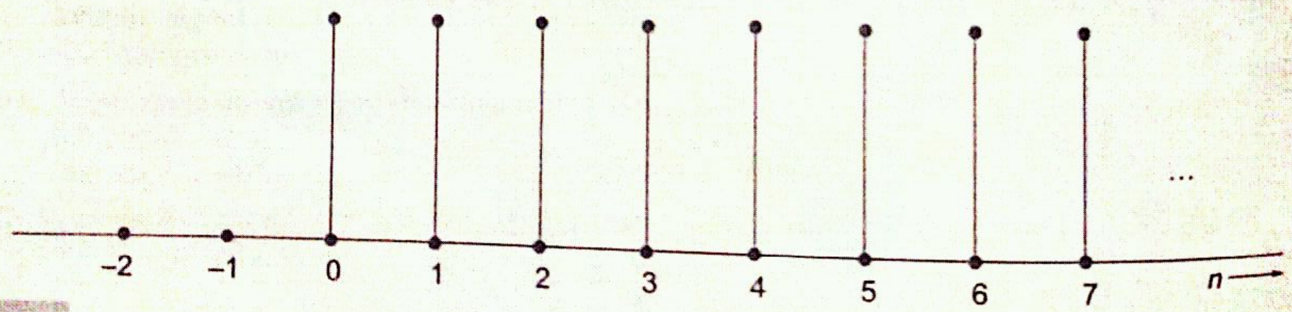


Fig. 1.5

A unit step sequence

The unit step sequence shifted by m samples is given by the equation

$$u(n-m) = \begin{cases} 1, & n \geq m \\ 0, & \text{elsewhere} \end{cases} \tag{1.7}$$

It is possible to represent the unit step sequence in terms of the unit sample sequence. Thus

$$u(n) = \sum_{n=-\infty}^{\infty} \delta(m) \tag{1.8}$$

and conversely

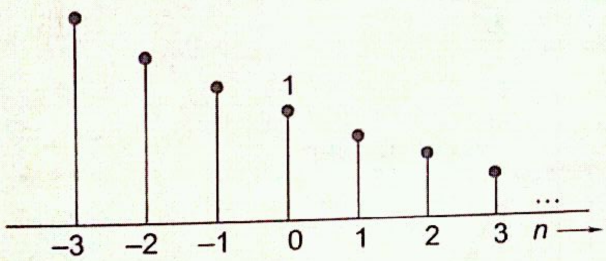
$$\delta(n) = u(n) - u(n-1) \tag{1.9}$$

The Exponential Sequence

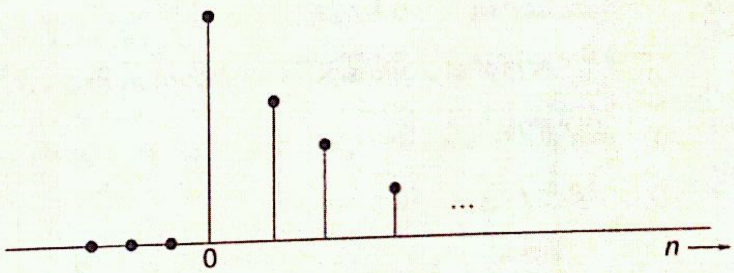
Another basic sequence is the exponential defined by the equation

$$x(n) = K \cdot a^n ; \qquad -\infty < n < \infty \tag{1.10}$$

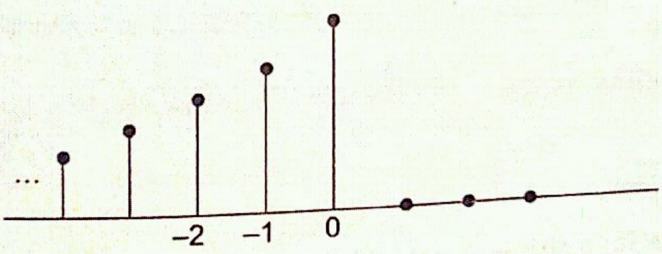
where K and a are arbitrary (possibly complex) constants. Sequences defined by the expressions $Ka^n u(n)$ and $Kb^n u(-n)$ are referred to as one-sided exponentials or truncated exponentials. Hence, $Ka^n u(n)$ is a right-sided and $Kb^n u(-n)$ is a left-sided exponential.



(a) A real exponential



(b) A right-sided exponential



(c) A left-sided exponential

Fig. 1.6 *Demonstration of exponential signals*

Figure 1.6(a) gives an example of an exponential where K and, a are real while Figs 1.6(b) and 1.6(c) are examples of a right-sided and a left-sided exponential.

Single Frequency Sequences [1]

Another commonly used sequence is $Ke^{j\omega_0 n}$ which is said to be a single frequency sequence. This is an exponential sequence with $a = e^{j\omega_0}$ in Eq. 1.10. Here ω_0 is real, and can have either a positive or negative sign. It is informally referred to as a sinusoid with frequency ω_0. However, a sequence of the form

$$x(n) = A\cos(\omega_0 n + \phi) \tag{1.11}$$

is a true sinusoid where A, ω_0 and ϕ are real numbers.

The parameters A, ω_0 and ϕ are called the amplitude, the angular frequency and the phase respectively, of the sinusoidal sequence, $x(n)$.

Note that,

$$\cos(\omega_0 n + \varphi) = 0.5e^{j(\omega_0 n + \varphi)} + 0.5e^{-j(\omega_0 n + \varphi)} \tag{1.12}$$

contains two frequencies that is ω_0 and $-\omega_0$. As such, it is not a single frequency signal [1].

The sinusoid of Eq. 1.11 can also be written as

$$x(n) = x_{in}(n) + x_{qu}(n) \tag{1.13}$$

where $x_{in}(n)$ and $x_{qu}(n)$ are, respectively, the in-phase and the quadrature components of $x(n)$, and are given by the equations

$$x_{in}(n) = A\cos\phi\,\cos(\omega_0 n) \tag{1.14}$$

$$x_{qu}(n) = -A\sin\phi\,\sin(\omega_0 n) \tag{1.15}$$

A plot of a sinusoid is as shown in Fig. 1.7.

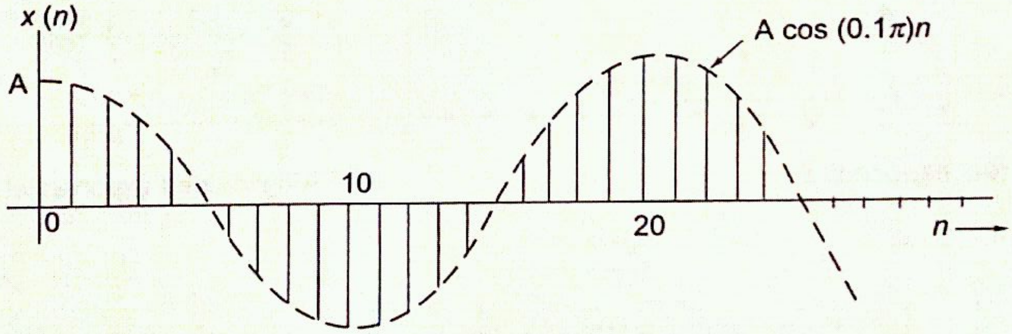


Fig. 1.7

A discrete time sinusoid

A point to be noted in discussing discrete sinusoids is that they are periodic, if and only if, the angular frequency ω_0, is a rational multiple of 2π that is $\omega_0 = \frac{2\pi n}{N}$ where n and N are positive integers or equivalently $\frac{\omega_0}{2\pi} = \frac{n}{N}$ is a rational number.

The smallest possible value of N satisfying the condition is the fundamental period of the sequence.

As an example consider the signal

$$x(n) = \cos(\omega_0 n + \phi) \tag{1.16}$$

and its periodic extension

$$x(n + N) = \cos[\omega_0(n + N) + \phi] \tag{1.17}$$

where N is a positive integer.

If Eq. 1.17 is, indeed, a periodic extension then

$$x(n) = \cos(\omega_0 n + \phi) = x(n + N) = \cos[\omega_0(n + N) + \phi] \qquad (1.18)$$

The above is possible if and only if

$$\omega_0 N = 2\pi l \qquad (1.19)$$

where l is a positive integer or equivalently

$$\frac{\omega_0}{2\pi} = \frac{l}{N} \qquad (1.20)$$

must be a ratio of two integers, i.e a rational number.

If $\frac{\omega_0}{2\pi}$ is not a rational number then this sequence is aperiodic although it may have a sinusoidal envelope.

An example of an aperiodic sequence is

$$x(n) = \cos\left(\sqrt{5}n + \phi\right) \qquad (1.21)$$

The units of ω_0, the angular frequency, and also that of the phase ϕ are radians.

If n is designated as samples then the units of both ω_0 and ϕ are radians per sample.

Example 1.1 Determine if the following sequences are periodic and if periodic determine their periods.

(i) $x(n) = \sin 0.14\pi$

(ii) $x(n) = \sin 0.68\pi$

Solution For a sequence to be periodic $\frac{\omega_0}{2\pi} = \frac{l}{N}$ = a rational number

(i) We are given that $\omega_0 = 0.14\pi$. Consequently,

$$\therefore \quad \frac{\omega_0}{2\pi} = \frac{0.14\pi}{2\pi} = \frac{14}{200} \text{ is a rational number.}$$

Hence, $\quad x(n) = \sin 0.14\pi$ is periodic.

To determine the fundamental period recall that N must be the smallest possible integer that satisfies the relationship

$$\omega_0 N = 2\pi l$$

or

$$N = \frac{2\pi}{0.14\pi} l = \frac{200}{14} l = \frac{100}{7} l$$

This implies that $l = 7$ and consequently,

$$N = 100$$

Hence, the fundamental period of the sinusoid is 100 sample units.

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(i) $x(n) = \sin 0.14\pi$

(ii) $x(n) = \sin 0.68\pi$

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To determine the fundamental period recall that N must be the smallest possible integer that satisfies the relationship

$$\omega_0 N = 2\pi l$$

or

$$N = \frac{2\pi}{0.14\pi} l = \frac{200}{14} l = \frac{100}{7} l$$

This implies that $l = 7$ and consequently,

$$N = 100$$

Hence, the fundamental period of the sinusoid is 100 sample units.

(ii) We are given that $\omega_0 = 0.68\pi$

For periodicity check if Eq. 1.20 is satisfied. Thus,

$$\frac{\omega_0}{2\pi} = \frac{0.68\pi}{2\pi} = \frac{34}{100} \text{ is a rational number.}$$

Hence, $x(n) = \sin 0.68\pi$ is periodic.

To determine the fundamental period, N, must be the smallest possible integer such that,

$$\omega_0 N = 2\pi l$$

or

$$N = \frac{2\pi}{0.68\pi} \cdot l = \frac{50}{17} \cdot l$$

This implies that $l = 17$ and $N = 50$. Hence, the period of $x(n) = \sin 0.68\pi$ is 50 sample units.

From the above example it is clear that while in the case of discrete sinusoids, ω_0 is constrained to be a rational multiple of 2π, there is no such requirement to be satisfied in the case of continuous-time sinusoids.

There are two other interesting properties of the sinusoidal sequence which are distinct from its counterpart namely the continuous-time sinusoid.

The first is that two discrete sinusoidal sequences are indistinguishable from one another if their frequencies differ by $2l\pi$ where l is an integer.

For this consider the sequences

$$x_1(n) = e^{j\omega_1 n} \tag{1.22}$$

and

$$x_2(n) = e^{j\omega_2 n} \tag{1.23}$$

where

$$\omega_2 = (\omega_1 + 2l\pi), \qquad l = \pm 1, \pm 2, \ldots \tag{1.24}$$

Substitution of Eq. 1.24 in Eq. 1.23 yields

$$x_2(n) = e^{j(\omega_1 + 2\pi l)n} = e^{j\omega_1 n} e^{j2\pi l n} = e^{j\omega_1 n} \tag{1.25}$$

as stated.

This is also true when

$$x_1(n) = \cos(\omega_1 n + \phi) \quad \text{and} \quad x_2(n) = \cos(\omega_2 n + \phi)$$

where $\omega_2 = (\omega_1 + 2l\pi)$ as above.

Note that this is not the case with continuous-time sinusoids,

$$x_1(t) = \cos \omega_1 t \quad \text{and} \quad x_2(t) = \cos(\omega_1 + 2\pi l)t$$

The second property is that, as the angular frequency ω increases from 0 to π, the frequency of oscillation of a discrete sinusoid, increases. However, when ω_0 increases from π to 2π, the frequency of oscillation decreases. As such, frequencies in the neighbourhood of $\omega_0 = 2\pi l$ are usually referred to as low frequencies and those in the neighbourhood of $\omega_0 = \pi(2l + 1)$ called high frequencies.

To appreciate this better consider the following periodic sequence:

$$x(n) = \sin \omega_0 n$$

where $\omega_0 = 0.0001\pi$, $.001\pi$, $.01\pi$, 0.1π, 0.2π, 0.5π, π, 1.2π, 1.5π, 1.8π, 1.9π, 1.99π, 1.999π and 1.9999.

Let us compute the period in each case and ascertain whether what has been stated above is true.

Table 1.1 gives the fundamental period N corresponding to each angular frequency ω_0.

TABLE 1.1

ω_0 =*Angular frequency*	N = *Corresponding fundamental period*
$.0001\pi$	20,000
0.001π	2,000
0.01	200
0.1π	20
0.2π	10
0.5π	4
π	2
1.2π	5
1.5π	4
1.8π	10
1.9π	20
1.99π	2,00
1.999π	2,000
1.9999π	20,000

As is evident from the table that Angular frequencies in the neighbourhood of π have smaller fundamental periods (high frequencies) while those close to 0 and 2π have larger fundamental periods (low frequencies). This validates the statement made above. Having discussed the representation of sequences let us next consider the problem of their classification.

1.3 CLASSIFICATION OF SEQUENCES

As classification enables one to generalize about the nature of objects or physical phenomena, let us classify discrete signals. Below we give some of the often used ones.

Energy and Power Signals (Sequences)

Signals (sequences) may be classified on the basis of their energy where the total energy ξ_x of a sequence $x(n)$ is defined by the equation

$$\xi_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 \tag{1.26}$$

An infinite length sequence with finite sample values may or may not have finite energy i.e. the series given by Eq. 1.26 may or may not converge. We shall illustrate this by of a few examples.

Example 1.2 Consider the infinite sequence, $x_1(n)$ given by the equation :

$$x_1(n) = \frac{1}{n^4}, \quad n = 1, 2, \ldots \infty$$

Solution It can be shown [3] that

$$\xi_{x_1} = \sum_{n=1}^{\infty}\left(\frac{1}{n^4}\right) = \frac{\pi^4}{90}$$

Hence, it is a convergent sequence and the signal $x_1(n)$ has finite energy.

Example 1.3 Consider the infinite sequence

$$x_2 n = \frac{1}{(2n+1)^2}; \qquad n = 0, 1, 2 \ldots$$

Solution Again it can be shown [3] that

$$\xi_{x_2} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

Since the sequence converges it has finite energy.

Example 1.4 Consider the infinite sequence $x_3(n)$ given by the equation,

$$x_3(n) = \left(\frac{1}{\sqrt[3]{n}}\right); \qquad n = 1, 2, \ldots \infty$$

Solution $x_3(n)$ is a harmonic series and it can be shown that [3]

$$\xi_{x_3} = \sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}}$$

diverges and hence the signal $x_3(n)$ has infinite energy.

Now let us define the average power of an aperiodic sequence [1] given by the equation,

$$P_x = \underset{k\to\infty}{\mathrm{Lt}}\ \frac{1}{(2k+1)}\sum_{n=-k}^{k}|x(n)|^2 \tag{1.27}$$

The average power of a sequence can be related to its energy over a finite interval $-k \le n \le k$ as,

$$\xi_{x,k} = \sum_{n=-k}^{k}|x(n)|^2 \tag{1.28}$$

Thus,

$$P_x = \underset{k \to \infty}{\mathrm{Lt}} \frac{1}{(2k+1)} \xi_{x,k} \tag{1.29}$$

Let us consider the periodic sequence, which by definition is infinite in extent. The average power of a periodic sequence $x_N(n)$ with the period N is defined by the equation,

$$P_{x_N} = \frac{1}{N} \sum_{n=0}^{N-1} |x_N(n)|^2 \tag{1.30}$$

On the basis of the above definition, signals are classified as energy signals and power signals.

A finite energy signal with zero average power is called an **Energy signal.**

An infinite energy signal with finite average power is called a **Power signal.**

A periodic signal has infinite energy but finite average power and therefore an example of a power signal.

A finite length sequence has finite energy but zero average power and thus serves as an example of an energy signal.

Symmetric Sequences

We next define sequences based upon their symmetry, if any.

A sequence $x(n)$ is defined as being **conjugate-symmetric** if

$$x(n) = x^*(-n) \tag{1.31}$$

where '*' stands for conjugate.

A sequence $x(n)$ is called a **conjugate-anti-symmetric** sequence if

$$x(n) = -x^*(-n) \tag{1.32}$$

Recall that a real sequence satisfies the relationship

$$x(n) = x^*(n) \tag{1.33}$$

As such a real conjugate-symmetric sequence is called an even sequence while a real conjugate-anti symmetric sequence is referred to as an odd sequence.

Note that for a conjugate-anti symmetric sequence $x(n)$, the sample value at $n = 0$ must be purely imaginary. It follows then that for an odd sequence $x(0) = 0$.

Examples of an odd and even sequence are shown in Fig. 1.8.

The importance of symmetric sequences lies in the fact that any complex sequence $x(n)$ can be expressed as a sum of its conjugate symmetric part, $x_{cs}(n)$ and its conjugate anti-symmetric part, $x_{ca}(n)$. Thus,

$$x(n) = x_{cs}(n) + x_{ca}(n) \tag{1.34}$$

where

$$x_{cs}(n) = \frac{1}{2}\left[x(n) + x^*(-n)\right] \tag{1.35}$$

$$x_{ca}(n) = \frac{1}{2}\left[x(n) - x^*(-n)\right] \tag{1.36}$$

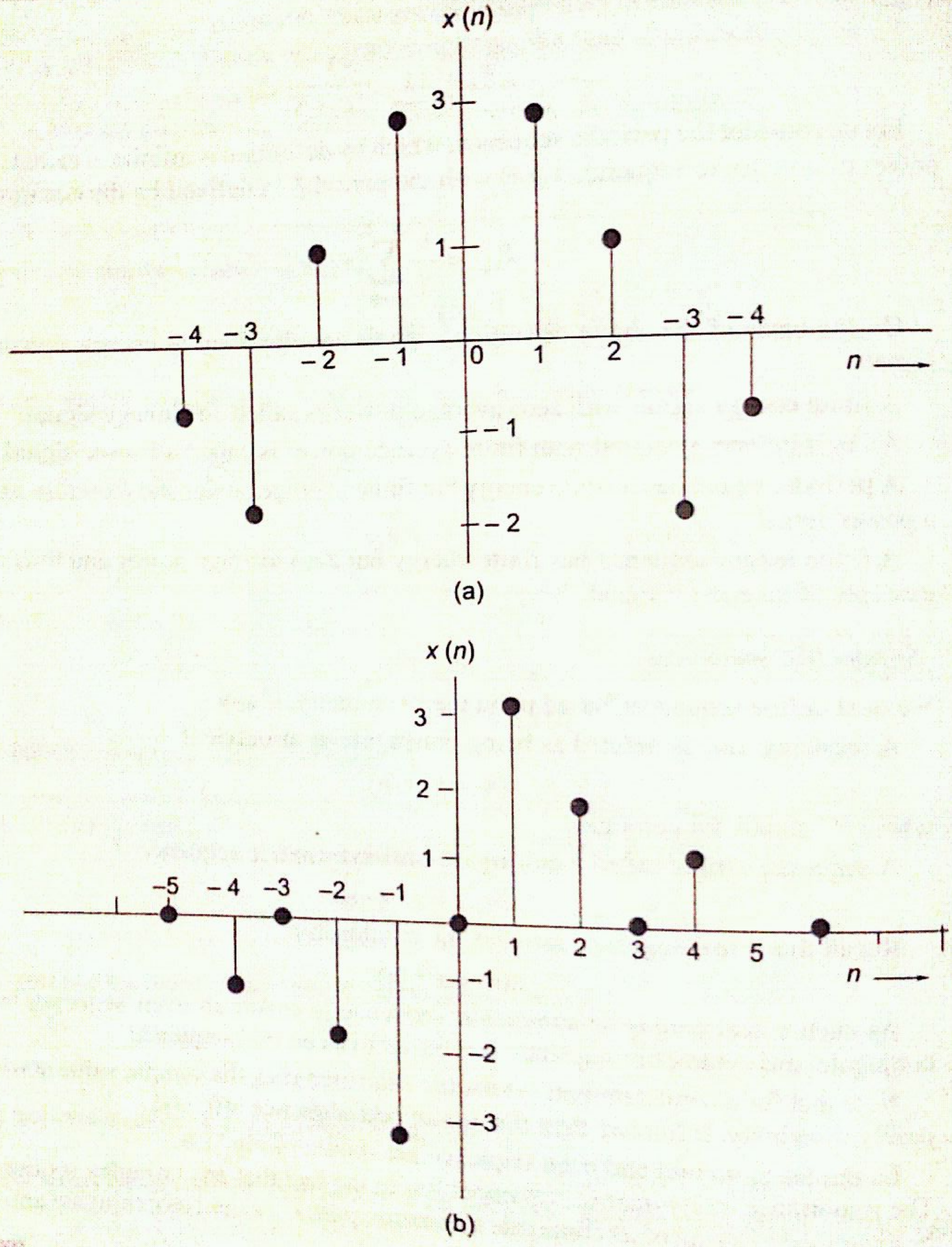

Fig. 1.8

(a) An even sequence (b) An odd sequence

Likewise, any real sequence, $x(n)$, can be expressed as a sum of its even part, x_{re}, and its odd part, x_{ro}. Thus,

$$x(n) = x_{re}(n) + x_{ro}(n) \tag{1.37}$$

where

$$x_{re} = \frac{1}{2}\left[x(n) + x(-n)\right] \quad (1.38)$$

$$x_{ro} = \frac{1}{2}\left[x(n) - x(-n)\right] \quad (1.39)$$

Other types of classification of sequences are as discussed below.

(i) Sequences may be classified on the basis of their repetition at regular intervals. Thus a sequence $x(n)$ satisfying the relationship

$$x(n) = x\,(n \pm kN) \quad (1.40)$$

is called a **periodic** sequence with period N where N and k are any positive integers. On the other hand, a sequence that is not periodic is called **aperiodic**. The fundamental period of a periodic signal is the smallest value of N that satisfies Eq. 1.40.

(ii) Yet another classification of sequences is based on "boundedness". A sequence $x(n)$ is said to be **bounded** if there exists a finite values, B such that each of its samples

$$|x(n)| \le B < \infty \text{ for all } n. \quad (1.41)$$

Examples of these are:

(a) $a^n\, u(n)$, $|a| < 1$

(b) $\cos\omega_0 n$ for real ω_0

Observe that an exponential a^n is not a bounded sequence unless $a = 0$ or $|a| < 1$

(iii) Let a sequence $x_D(N)$ be the set of signals which vanishes outside a specified time interval $-N \le n \le N$. Then it is said to be a **duration limited signal**, i.e.

$$x_D(n) \quad \{x : x(n) = 0 \quad \text{for all } |n| > N\} \quad (1.42)$$

(iv) A discrete signal whose spectrum is limited to a portion of the angular frequency region $0 \le |\omega| \le \pi$, is referred to as a **band limited** signal. On the other hand a full band discrete time signal has a spectrum that occupies the whole frequency range $0 \le |\omega| \le \pi$. In this context, it may be mentioned that a low pass discrete time signal has a spectrum that occupies the frequency range $0 \le |\omega| \le \omega_p < \pi$, where ω_p is referred to as the **bandwidth** of the signal.

A bandpass discrete-time signal has a spectrum that occupies the frequency range $0 \le \omega_L \le |\omega| \le \omega_H < \pi$. The term $(\omega_H - \omega_L)$ is called the bandwidth of the signal.

(v) A sequence $x(n)$ is said to be **absolutely summable** if

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty \quad (1.43)$$

(vi) A sequence $x(n)$ is said to be **square summable** if

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty \quad (1.44)$$

If a sequence is square summable, it has finite energy and qualifies to be an energy signal if it also has zero average power.

1.4 BASIC OPERATIONS ON SEQUENCES

Certain basic operations are now defined in respect of discrete sequences which enable the implemention of various digital signal processing algorithms to be described later in the text.

1.4.1 Basic Operations

Let $x(n)$ and $y(n)$ be two known sequences. We define the operation of addition of two sequences as

$$\vartheta_1(n) = x(n) + y(n) \tag{1.45}$$

that is obtained by adding the sample values of each of the sequences individually.

The device that implements the above operation is called an **adder.** Its schematic is as shown in Fig. 1.9.

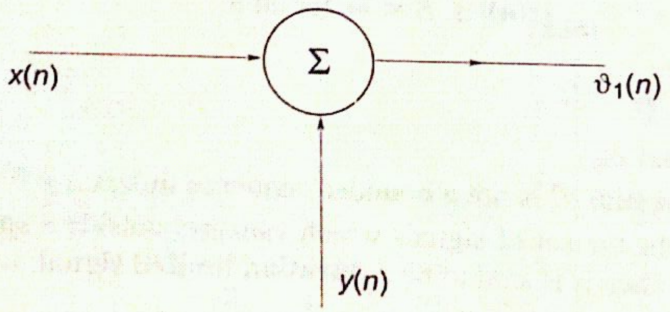

Fig. 1.9

Schematic of an adder

The next basic operation is the scalar multiplication. Here a new sequence is generated by multiplying each sample of a sequence $x(n)$ by a scalar 'C'.

Thus,

$$\vartheta_2(n) = C\,x(n) \tag{1.46}$$

The implementing device for the multiplication operation is called a **multiplicator**; a schematic of which is shown in Fig. 1.10 below.

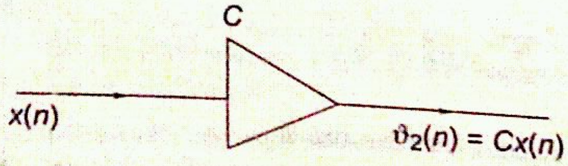

Fig. 1.10

Schematic of a multiplier

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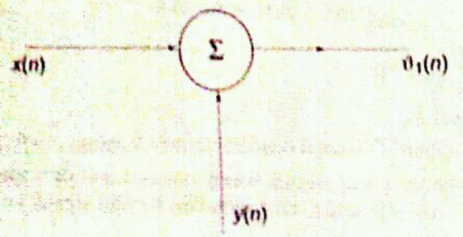

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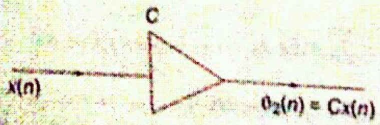


Fig. 1.10

Schematic of a multiplier

The third basic operation is forming the product of two sample sequences. Thus, if $x(n)$ and $y(n)$ are two known sequences, we define the product of these two sequences as

$$\vartheta_3(n) = x(n) \cdot y(n) \tag{1.47}$$

A schematic of the device that implements this operation is shown in Fig. 1.11. In some books it is referred to as a **modulator.**

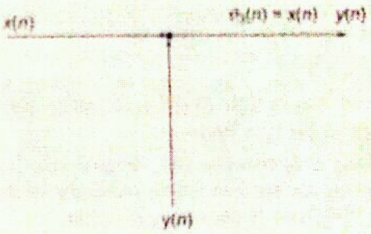


Fig. 1.11

Schematic of a modulator

Another basic operation involving sequences is the time-shifting operation given by the equation

$$\vartheta_4(n) = x(n - N) \tag{1.48}$$

A unit delay operation is illustrated in Fig. 1.12.

Fig. 1.12

A unit delay operator

Sometimes a unit delay operator is represented as shown in the schematic below.

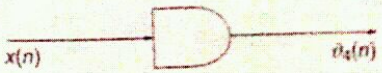


Fig. 1.13

A unit delay operator

If $N < 0$, then the operator is called an advancing operator. A unit advancing operator is illustrated in Fig. 1.14.

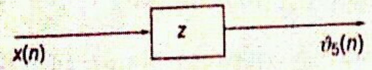


Fig. 1.14

A unit advancing operator

A sequence that is advanced by N units is given by the equation

$$\vartheta_5(n) = x(n + N) \tag{1.49}$$

The operation of addition and modulation can be carried out if the sequences are of the same length and defined for the same range of the time index, n.

But if one still insists on performing these operations on two unequal length sequences, they may be carried out by appending zeros to the smaller length sequence so that both the sequences are of the same length. This is illustrated in the example below.

Example 1.5 Let $x(n) = \{1, 3, 2, -2, 1, 4\}$ and $y(n) = \{4, -2, 8\}$

Solution Since we cannot perform the operations of summation and modulation with unequal length sequences we append zeros to $y(n)$ so that it is of the same length as $x(n)$. Thus, define

$$y_{app} = \{4, -2, 8, 0, 0, 0\}$$

Then,

$$v_1(n) = x(n) + y(n) = \{5, 1, 10, -2, 1, 4\}$$

and

$$v_3(n) = x(n)(y(n) = \{4, -6, 16, 0, 0, 0\}$$

Having understood the various basic mathematical operations that can be carried out on sequences we are now ready to discuss the various system operations that can be performed on them to achieve the desired objectives.

1.5 DISCRETE-TIME SYSTEMS

A **discrete-time system** operates on an input sequence $x(n)$ to generate an output sequence $y(n)$.

Our attention, for most part, will be focused on single-input single-output systems, a schematic of which is shown in Fig. 1.15.

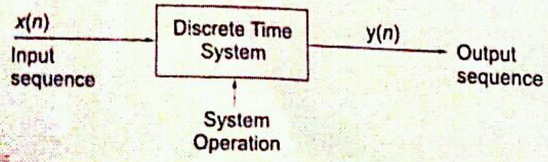


Fig. 1.15

Block schematic of a single-input single-output system

It is assumed that the value of the input sequence, $x(n)$, in the range $-\infty \le n \le \infty$ uniquely determines the output sequence, $y(n)$, in the range $-\infty \le n \le \infty$

Since the objective of this book is limited, we shall restrict our attention to the class of discrete time systems with certain specific characteristics.

Before we begin a formal description of these systems it is to be noted that in real life all discrete signals are digital signals and any operation on such signals results in digital signals. Therefore we shall refer to discrete time systems as digital systems.

1.5.1 Classification of Discrete-Time Systems

Discrete-Time systems may be classified on the basis of their input-output relationship. Of great interest to us in this text are linear systems and shift invariant (time – invariant) systems. We shall begin with **Linear Systems**.

A discrete-time system is said to be linear if given that $y_1(n)$ is the response to an input $x_1(n)$ and $y_2(n)$ is the response to an input $x_2(n)$, then the response to an input

$$x(n) = ax_1(n) + bx_2(n) \tag{1.50}$$

is

$$y(n) = ay_1(n) + by_2(n) \tag{1.51}$$

for every possible $x_1(n)$ and $x_2(n)$ and for every pair of constants a and b. The above may be succinctly put in system theoretic notation as below.

Let T be the discrete-time system that operates on $x(n)$ to yield the response $y(n)$, i.e.

$$y(n) = T[x(n)] \tag{1.52}$$

A discrete-time system is said to be linear if:

$$\begin{aligned} T[ax_1(n) + bx_2(n)] &= aT[x_1(n)] + bT[x_2(n)] \\ &= ay_1(n) + by_2(n) \end{aligned} \tag{1.53}$$

for every possible $x_1(n)$ and $x_2(n)$ and every pair of constants a and b.

Let us illustrate the above property of discrete-time systems through a series of examples.

Example 1.6 Determine if the following systems are linear.

(i) $y(n) = \sum_{n=0}^{N-1} x(n)$

(ii) $y(n) = n^2 x(n)$

(iii) $y(n) = K\,x(-n)$ where a is an arbitrary constant

(iv) $y(n) = y(n-1) + \sum_{n=0}^{N-1} x(n)$

Solution

(i) $y(n) = \sum_{n=0}^{N-1} x(n)$

Here the system operator T is $\sum_{n=0}^{N-1}$ *(Input Sequence)*. For linearity

$$T[ax_1(n) + bx_2(n)] = ay_1(n) + by_2(n) \tag{1.54}$$

L.H.S of Eq. 1.54

$$= \sum_{n=0}^{N-1}[ax_1(n) + bx_2(n)]$$

$$= \sum_{n=0}^{N-1} ax_1(n) + \sum_{n=0}^{N-1} bx_2(n)$$

$$= a\sum_{n=0}^{N-1} x_1(n) + b\sum_{n=0}^{N-1} x_2(n)$$

$$= ay_1(n) + by_2(n) = \text{R.H.S. of Eq. 1.54}$$

Hence the above system is linear.

(ii) $y(n) = n^2 x(n)$

The system operator $T = n^2$ multiplied by the *(Input sequence)*.

For linearity

$$T[ax_1(n) + bx_2(n)] = ay_1(n) + by_2(n) \tag{1.55}$$

L.H.S of Eq. 1.55

$$n^2 \cdot (ax_1(n) + bx_2(n))$$

$$= n^2\, ax_1(n) + n^2 bx_2(n)$$

$$= an^2 x_1(n) + bn^2 x_2(n)$$

$$= ay_1(n) + by_2(n) = \text{R.H.S of Eq. 1.55}$$

Hence the above system is linear.

(iii) $y(n) = K\,x(-n)$

The system operator $T = K$ multiplied by *(Time reversed input sequence)*

For linearity

$$T[ax_1(n) + bx_2(n)] = ay_1(n) + by_2(n) \tag{1.56}$$

L.H.S of Eq. 1.56

$$T[ax_1(n) + bx_2(n)] = K \cdot (ax_1(-n) + bx_2(-n))$$

$$= aK \cdot x_1(-n) + bK \cdot x_2(-n)$$

$$= ay_1(n) + by_2(n) = \text{R.H.S of Eq. 1.56}$$

Hence the above system is linear.

(iv) $y(n) = y(n-1) + \sum_{n=0}^{N-1} x(n)$

Unlike the previous three examples, the above is not an input-output representation. Hence, it is difficult to represent in terms of the system operator T.

However, we will check for linearity as below. The outputs $y_1(n)$ and $y_2(n)$ corresponding to inputs $x_1(n)$ and $x_2(n)$ are respectively given by the equations.

$$y_1(n) = y_1(n-1) + \sum_{n=0}^{N-1} x_1(n)$$

and

$$y_2(n) = y_2(n-1) + \sum_{n=0}^{N-1} x_2(n)$$

Hence the output $y(n)$ for an input $ax_1(n) + bx_2(n)$ is :

$$y(n) = y(n-1) + \sum_{n=0}^{N-1}(ax_1(n) + bx_2(n))$$

$$= y(n-1) + a\sum_{n=0}^{N-1} x_1(n) + b\sum_{n=0}^{N-1} x_2(n) \tag{1.57}$$

However,

$ay_1(n) + by_2(n) = ay_1(n-1) + a\sum_{n=0}^{N-1} x_1(n) + by_2(n-1) + b\sum_{n=0}^{N-1} x_2(n)$ is not equal to Eq. 1.57.

Hence (iv) is not linear, i.e. the accumulator which we proved in Ex. (i) as linear, is not linear with non-zero initial conditions ($y(n-1) \neq 0$).

Shift-invariant Systems

A discrete-time system is said to be **shift-invariant** if its input-output relationship is independent of the time of application of the input.

Thus, if $y(n)$ is the output of a discrete-time system to an input $x(n)$ and if the output of the same system to the shifted version $x(n-N)$, is equal to $y(n-N)$ and if this holds for all integers N and all input sequences $x(n)$ then the system is said to be shift invariant.

In system theoretic notation, if

$$y(n) = T[x(n)] \tag{1.58}$$

and if

$$T[x(n-N)] = y(n-N) \tag{1.59}$$

then the system is said to be shift invariant.

The L.H.S of Eq. 1.59 may be interpreted as a system operating on delayed input while the R.H.S may be interpreted as the delayed output of the same system. And if both are equal then the system is said to be shift invariant.

We shall now illustrate the above property of discrete shift-invariant systems through several examples.

Example 1.7 Determine if the following systems are shift-invariant.

(i) $y(n) = x^4(n)$

(ii) $y(n) = A + \sum_{m=0}^{2} x(n-m)$ where A is a non-zero constant

(iii) $y(n) = \sum_{m=-2}^{2} x(n-m)$

(iv) $y(n) = Ax(-n)$

(v) $y(n) = \begin{cases} x\left(\frac{n}{M}\right), & n = 0, \pm M, \pm 2M \ldots \\ 0 & \text{otherwise} \end{cases}$

Solutions

(i) Given $y(n) = x^4(n)$, thus

$$T = (\text{input})^4$$

Delayed output (obtained by replacing $(n-N)$ in the input-output relationship) is given by the equation

$$y_1(n) = y(n-N) = (x(n-N))^4 = x^4(n-N)$$

Delayed input is $x(n-N) = x_1(n)$ and the output due to system operating on delayed input is given by the equation

$$y_2(n) = T[x_1(n)] = [x(n-N)]^4 = x^4(n-N)$$

Since

$$y_1(n) = y_2(n)$$

The system is shift-invariant.

(ii) Given that

$y(n) = A + \sum_{m=0}^{2} x(n-m)$ where A is a non-zero constant.

Delayed output $y_1(n)$ (obtained by replacing $(n-N)$ in the above equation) is

$$y_1(n) = y(n-N) = A + \sum_{m=0}^{2} x(n-N-m)$$

Delayed input, $x(n-N) = x_1(n)$

Output $y_2(n)$ due to system acting on delayed input is

$$y_2(n) = A + \sum_{m=0}^{2} x_1(n-m) = A + \sum_{m=0}^{2} x_1(n-N-m)$$

Since $y_1(n) = y_2(n)$, the system is shift invariant.

(iii) $y(n) = \sum_{m=-2}^{2} x(n-m)$

Delayed output $y_1(n)$ is

$$y(n) = \sum_{m=-2}^{2} x(n-N-m)$$

Delayed input, $x(n-N) = x_1(n)$

Output, $y_2(n)$ due to system acting on delayed input is

$$y_2(n) = \sum_{m=-2}^{2} x_1(n-m) = \sum_{m=-2}^{2} x(n-N-m)$$

Since $y_1(n) = y_2(n)$, the system is shift invariant.

(iv) Given that

$$y(n) = A\,x(-n)$$

Delayed output, $y_1(n) = y(n-N) = Ax(-(n-N)) = Ax(-n+N)$

Delayed input, $x(n-N) = x_1(n)$

Output, $y_2(n)$, due to system acting on delayed input is

$$\begin{aligned} y_2(n) &= (A) \cdot (\text{Time reversed input sequence}) \\ &= A \cdot x_1(-n) = A \cdot x(-(n-N)) \\ &= A \cdot x(-n+N) \end{aligned}$$

Since $y_1(n) = y_2(n)$, the system is shift-invariant.

(v) Given that

$$y(n) = \begin{cases} x\left(\frac{n}{M}\right), & n = 0, \pm M, \pm 2M \ldots \\ 0 & \text{otherwise} \end{cases}$$

Delayed output, $y_1(n)$ is

$$y_1(n) = \begin{cases} x\left(\frac{n-N}{M}\right), & n = N, N \pm M, N \pm 2M, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Delayed input, $x(n-N) = x_1(n)$

Output, $y_2(n)$ due to system operating on delayed input is:

$$y_2(n) = \begin{cases} x_1\left(\frac{n}{M}\right), & n = 0, \pm M, \pm 2M, \ldots \\ 0 & \text{otherwise} \end{cases}$$

But $\quad x_1(n) = x(n-N)$

Hence $\quad x_1\left(\frac{n}{M}\right) = x\left(\frac{n}{M} - N\right)$

In view of the above

$$y_2(n) = \begin{cases} x_1\left(\frac{n}{M}\right) = x\left(\frac{n}{M} - N\right) = x\left(\frac{n - NM}{M}\right), & n = 0, \pm M, \pm 2M, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Obviously $y_2(n) \neq y_1(n)$.

Hence the given system is NOT shift invariant.

Linear Shift-invariant Systems

Having discussed the properties of linearity and shift-invariance we now define systems that obey these two properties simultaneously.

A system is said to be linear and shift-invariant if it satisfies both the properties of linearity and shift-invariance as defined above. Such systems are amenable to easy mathematical analysis and characterization.

For instance, the input-output relationship of a linear, shift-invariant system is given by the equation

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \tag{1.60}$$

where, $h(n)$ is the impulse response sequence (i.e. the output of the system to a unit-impulse sequence, $\delta(n)$) and $x(n)$ and $y(n)$ are the input to and output from the system respectively.

Equation 1.60 is called the **convolution summation.**

Since these systems are easy to characterize their design becomes relatively simple. And a number of highly useful digital signal processing algorithms have been developed using this class of systems.

An example of such a system is the familiar linear, constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k) \tag{1.61}$$

where $x(n)$ and $y(n)$ are, respectively, the input and output of the system and a_k's b_k's are constants.

The order of such a system is given by **maximum** (N, M) and is the order of the difference equation characterizing the system. As can be observed from Eq. 1.61, it is easy to implement such a system using the basic operations on sequences defined earlier.

Also the output $y(n)$ of the system defined by Eq. 1.61 can be computed recursively from the equation

$$y(n) = -\sum_{k=1}^{N} \frac{a_k}{a_0}\, y(n-k) + \sum_{k=0}^{M} \frac{b_k}{a_0}\, x(n-k) \tag{1.62}$$

if $a_0 \neq 0$ and the system is causal[*] with the given initial conditions

$y(0), y(-1), y(-2), \ldots, y(-N)$.

[*] Causal systems are defined in the section following the next.

Insofar as implementation of the difference Eq. 1.62 is concerned, it is pretty much straightforward, as shown in Fig. 1.16, without loss of generality, for the case $N = 2$ that is

$$y(n) = a_1 y(n-1) - a_2 y(n-2) + b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) \tag{1.63}$$

This is referred to as the Direct Form Structure[†] and requires $(2N + 1)$ multiplications and $2N$ additions for the computation of each sample, $y(n)$. The number of delays is N which is the order of the filter.

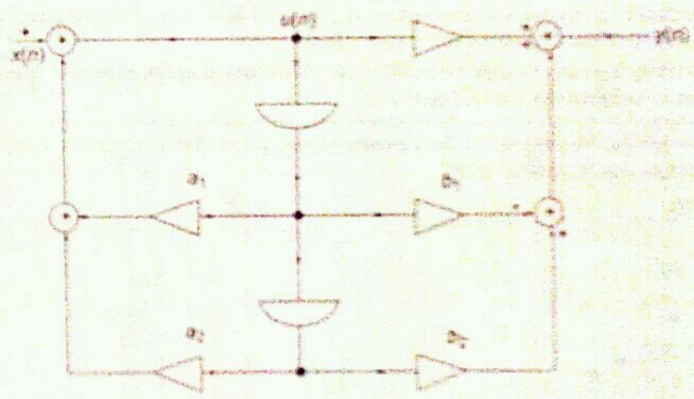


Fig. 1.16

A linear shift invariant system (direct form structure)

For the special case where all a_k's are identically zero except a_0, the implementation scheme becomes even simpler as shown in Fig. 1.17 for $a_0 = 1$. Such a system is known as a Finite Impulse Response (FIR) filter.

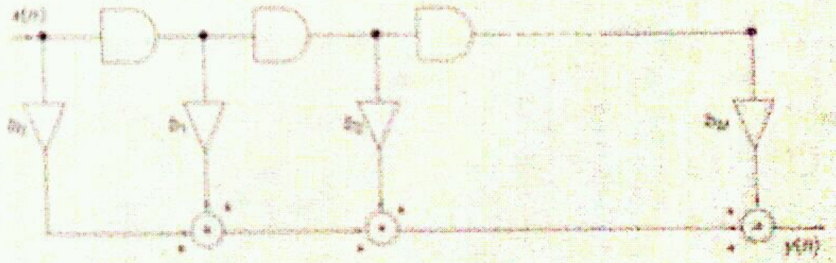


Fig. 1.17

An FIR filter (direct form structure)

[†] That this structure represents Eq. 1.63 may be verified using the transfer function concept where $\frac{Y(z)}{X(z)} = \frac{Y(z)}{U(z)} \cdot \frac{U(z)}{X(z)}$

And the structure is referred to as a Direct Form Structure as before. As can be seen from the structure, to realize it one requires $(N + 1)$ multipliers, N adders and N delay elements.

Note that FIR structures do not have any feedback paths unlike Fig. 1.16 and the difference equation describing it is referred to as being *non-recursive*.

The Convolution Summation

It was mentioned that the input-output relationship of a linear shift-invariant system can be described by a convolution summation and that a wide variety of digital signal processing algorithms have been designed using it as a basis.

This being the case it might be worthwhile illustrating through examples, the mechanics involved in carrying out the said operation.

Example 1.8 Determine the outputs of the systems below, given that the impulse responses and input sequences are as shown in (Fig. 1.18).

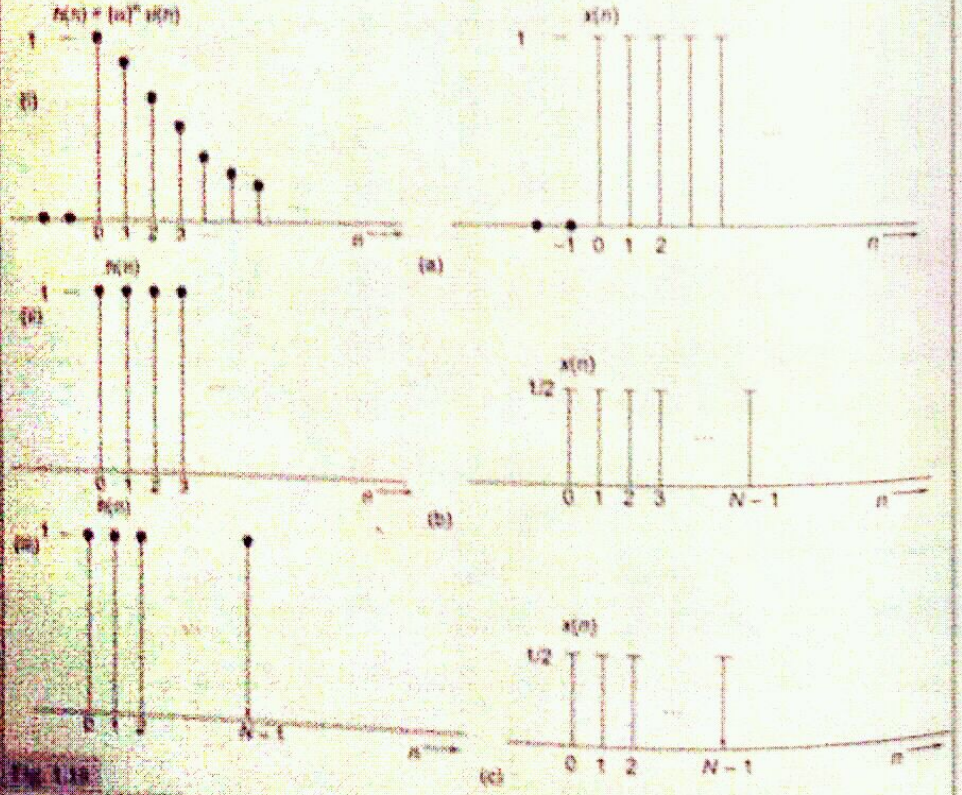


Fig. 1.18 *Three different kinds of systems with differing inputs*

Solution (i) Recall that the output of a linear shift-invariant system is given by the convolution summation

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

To gain an insight into its operation we shall perform the convolution graphically. The various steps are as shown in Fig. 1.19.

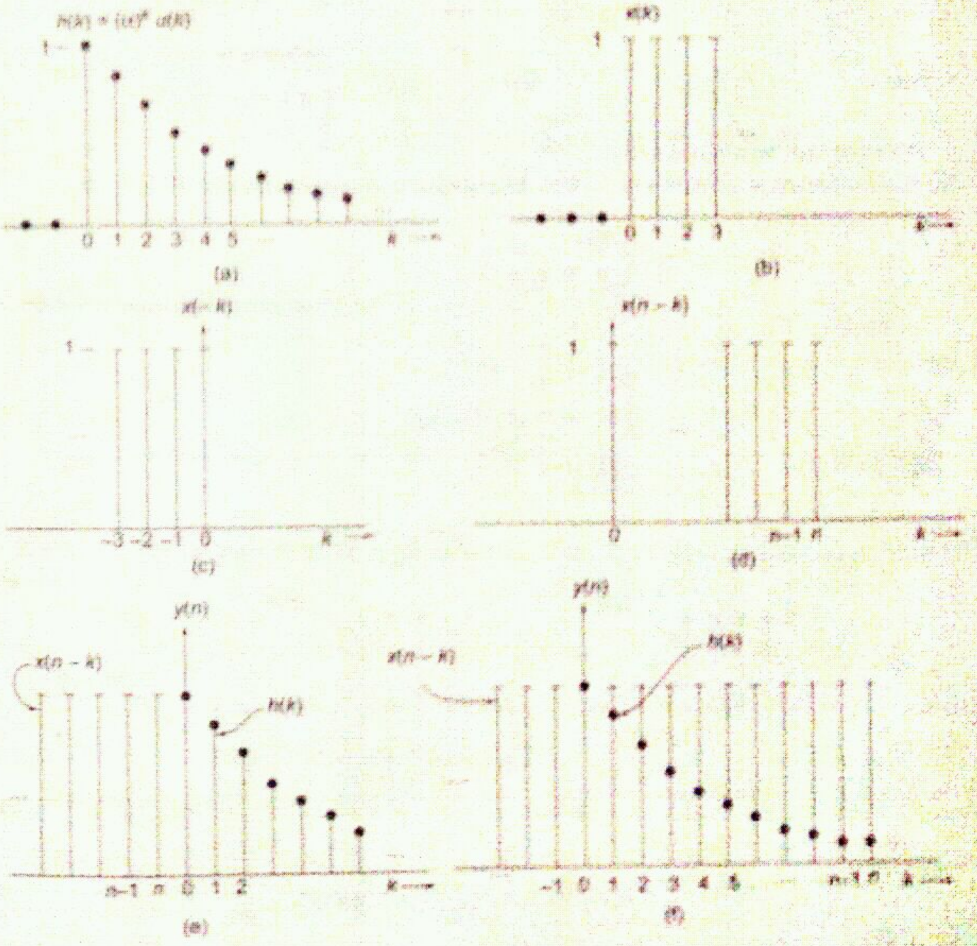


Fig. 1.19 *Graphical convolution case (i)*

From Fig. 1.19(e) it is evident that

$$y(n) = 0 \qquad \text{for } -\infty < n < 0$$

From Fig. 1.19(f) it follows that

$$y(n) = \sum_{k=0}^{\infty} (\alpha^k)(1) \qquad \text{for } 0 \le n < \infty$$

$$= \sum_{k=0}^{n} \alpha^k = \frac{1-(\alpha)^n}{1-\alpha} = \left(\frac{1-\alpha^n}{1-\alpha}\right)$$

Hence

$$y(n) = \begin{cases} 0, & -\infty < n < 0 \\ \dfrac{1-(\alpha)^n}{1-\alpha}, & 0 \le n < \infty \end{cases}$$

This completes the solution to (i).

(ii) Consider example given in Fig. 1.18(b). Its graphical convolution is shown in Fig. 1.20.

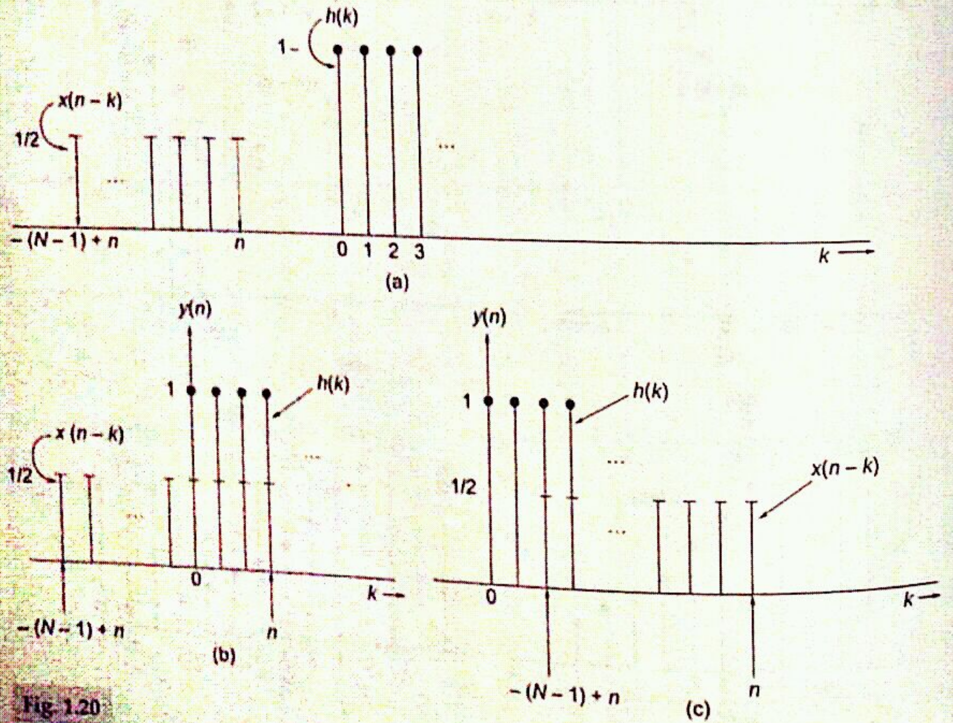


Fig. 1.20 Graphical convolution case (ii)

From Fig. 1.20(a), it is seen that

$$y(n) = 0 \qquad -\infty < n < 0$$

From Fig. 1.20(b), it is observed that

$$y(n) = \sum_{k=0}^{n} (1)\left(\frac{1}{2}\right) = \frac{1}{2}\sum_{k=0}^{n}(1) = \frac{1}{2}(n+1) \qquad \text{for } 0 \le n \le N-1$$

From Fig. 1.20(c), it is seen that

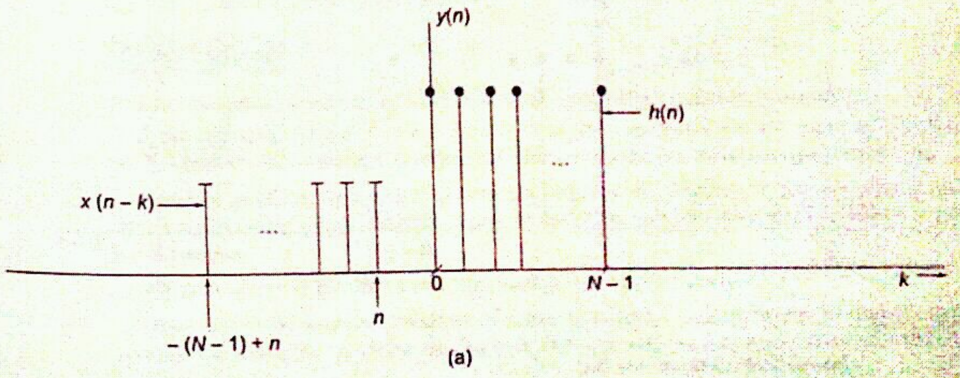
$$y(n) = \sum_{k=-(N-1)+n}^{n} (1)\left(\frac{1}{2}\right) = \frac{1}{2}\sum_{k=n-(N-1)}^{n}(1) \qquad \text{for } N-1 < n < \infty$$

$$= \frac{1}{2}[N]$$

Thus, the complete solution to (ii) is

$$y(n) = \begin{cases} 0 & -\infty < n < 0 \\ \dfrac{1}{2}(n+1) & 0 \le n \le N-1 \\ \dfrac{N}{2} & N-1 < n < \infty \end{cases}$$

Let us consider the final example as given in Fig. 1.18c. Its graphical solution is shown in Fig. 1.21.



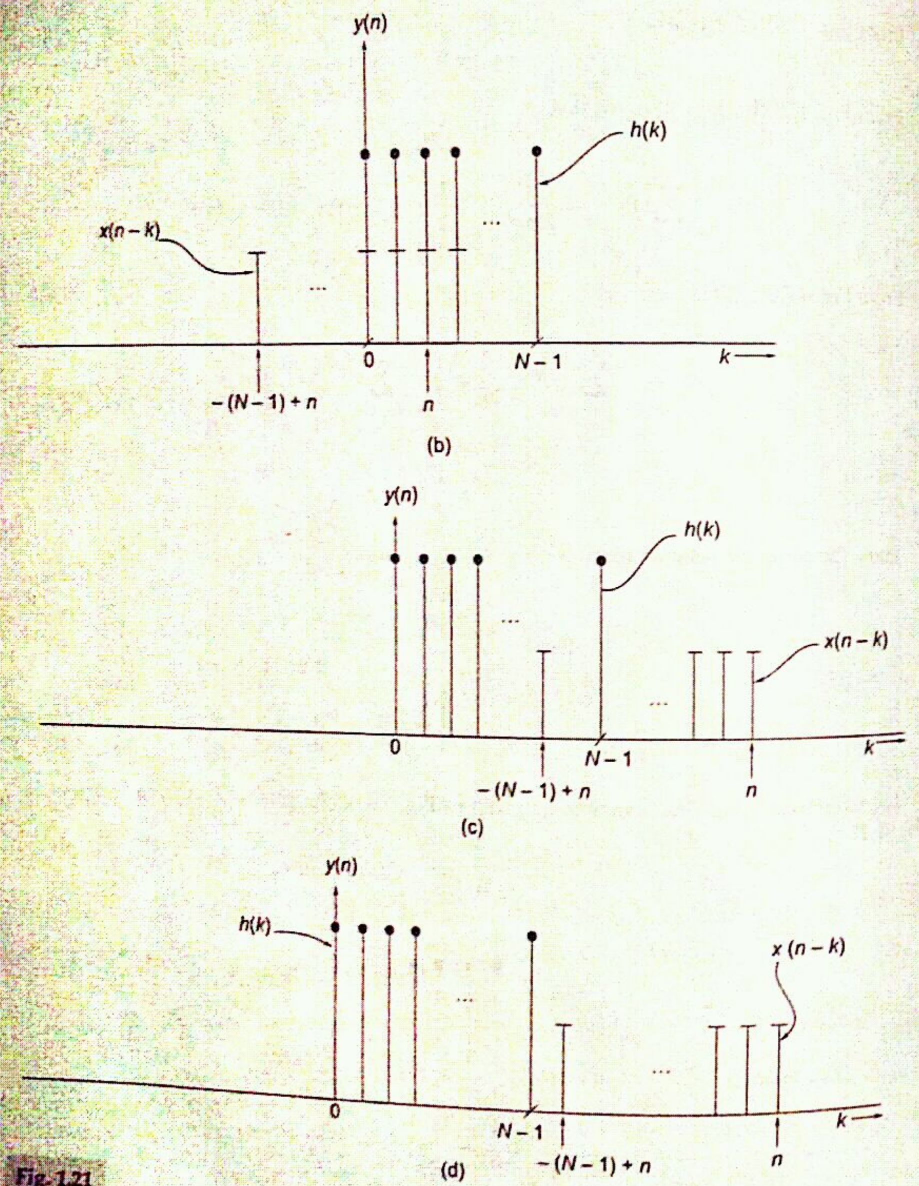


Fig. 1.21 Graphical convolution case (iii)

From Fig. 1.21(a), it is evident that

$$y(n) = 0 \qquad \text{for } -\infty < n < 0$$

From Fig. 1.2(b), it is seen that

$$y(n) = \sum_{k=0}^{n} (1)\left(\frac{1}{2}\right) = \frac{1}{2}(n+1) \qquad \text{for } 0 \le n \le (N-1)$$

From Fig. 1.21(c), it is observed that

$$y(n) = \sum_{k=-(N-1)+n}^{N-1} (1)\left(\frac{1}{2}\right) \qquad \text{for } (N-1) < n \le 2(N-1)$$

From Fig. 1.21(d), it is evident that

$$y(n) = 0\,; \qquad \text{for } 2(N-1) < n < \infty$$

Thus the complete solution to (iii) is

$$y(n) = \begin{cases} 0 & ; \quad -\infty < n < 0 \\ \frac{1}{2}(n+1) & ; \quad 0 \le n \le N-1 \\ \frac{1}{2}(2N-n-2) & ; \quad N-1 < n \le 2(N-1) \\ 0 & ; \quad 2(N-1) < n < \infty \end{cases}$$

It should be clear from the above examples that convolution summation is a rather difficult operation and the student has to be careful in carrying out the operations, especially with regard to fixing the limits on summation.

Causal Systems

Another property of discrete-time systems that is of practical interest is causality.

A discrete-time system is said to be causal, if the response of the system at n_0^{th} instant, $y(n_0)$, depends only on input samples, $x(n)$, for $n \le n_0$ and not on input values for $n > n_0$.

Roughly speaking, for a causal system it means that the response should not precede the stimulus or in other words changes, if any, in output samples do not precede changes in the input samples.

We shall illustrate the above concept through a series of examples.

Observe that the way the definition of causality is given, it is applicable to discrete-time systems for which the sampling rate for both the input and the output are the same [1].

Example 1.9 Determine if the following systems are causal.

(i) $y(n) = a_1 y(n-1) + a_2 y(n-2) + b_0 x(n) + b_1 x(n-1)$

(ii) $y(n) = n^2 x(n)$

(iii) $y(n) = x(-n)$

(iv) $y(n) = x(n) + \frac{1}{3}[x(n-1) + x(n+2)] + \frac{2}{3}[x(n-2) + x(n+1)]$

Solution

(i) We are given that

$$y(n) = a_1 y(n-1) + a_2 y(n-2) + b_0 x(n) + b_1 x(n-1)$$

Now at $n = 0$,

$$y(0) = a_1 y(-1) + a_2 y(-2) + b_0 x(0) + b_1 x(-1)$$

Thus, the output depends on not only the present value of the input but also on the past value of the input, i.e. $x(-1)$ and not on future values of the input.

Hence, the system is Causal.

Note in this example $y(-1)$ and $y(-2)$ are called the initial conditions.

(ii) $y(n) = n^2 x(n)$

Now at $n = 0$,

$y(0) = 0 \cdot x(0) = 0$

Further the output at any instant $n > 0$ depends on the present value of the input. Hence the system is Causal.

(iii) $y(n) = x(-n)$

At $n = 0$, $y(0) = x(0)$

and $y(1) = x(-1)$ and so on

Hence the output at any instant depends on the past values of the input but not on the future values of the input. Hence the system is Causal.

(iv) We are given that

$$y(n) = x(n) + \frac{1}{3}[x(n-1) + x(n+2)] + \frac{2}{3}[x(n-2) + x(n+1)]$$

At $n = 0$,

$$y(0) = x(0) + \frac{1}{3}[x(-1) + x(+2)] + \frac{2}{3}[x(-2) + x(1)]$$

Thus, the output at $n = 0$ depends not only the present and past values of the input but also future values of the input. Hence, the system is non-causal.

Stable Systems

Yet another classification of discrete-time systems is based on the concept of stability of a system. We define a discrete-time system to be stable if and only if, for every bounded input, the output is also bounded. The implication here is that if the response $y(n)$ to the input sequence $x(n)$ is such that for all $x(n)$

$$|x(n)| < K_1 \tag{1.64}$$

and for all values of n, then

$$|y(n)| < K_2 \tag{1.65}$$

and for all values of n, where K_1 and K_2 are finite constants.

This type of stability is called bounded-input bounded-output (BIBO) stability.

For the case of linear time-invariant systems it can be shown [5] that BIBO stability is equivalent to satisfying the condition

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \tag{1.66}$$

where

$h(n)$ is the impulse response of the discrete-time system.

Note that Eq. 1.66 implies that the impulse response must be absolutely summable.

We shall now illustrate this concept of stability through a few examples.

Example 1.10 Determine if the following discrete systems are stable.

(i) $y(n) = x^4(n)$

(ii) $y(n) = \sum_{m=0}^{N-1} x(n-m)$

(iii) $y(n) = A\,x(-n)$ where A is a non-zero constant

(iv) $y(n) = n^2 x(n)$

(v) $y(n) = \sum_{n=0}^{\infty} (a)^n \cdot x(n)$ where $x(n)$ is the unit sequence

Solution

(i) We are given that the input-output relationship:

$$y(n) = x^4(n)$$

Thus if $x(n)$ is bounded say, $|x(n)| < M < \infty$

then $$|y(n)| = |x^4(n)| = M^4 < \infty$$

Hence $|y(n)|$ is bounded and the system is said to be BIBO stable.

(ii) We are given that

$$y(n) = \sum_{m=0}^{N-1} x(n-m)$$

As before, let $|x(n)| < M < \infty$
then

$$y(n) = \left|\sum_{m=0}^{N-1} x(n-m)\right| \le \left|\sum_{m=0}^{N-1} x(n-m)\right| < \sum_{m=0}^{N-1} (M) \cdot N \cdot < \infty \quad \text{where } N \text{ is finite}$$

Hence the given system is BIBO stable.

(iii) We are given that
$y(n) = A\,x(-n)$
As before, let $|x(n)| < M < \infty$.

This implies that $|x(-N)| < M$ since it is a mirror image of $x(n)$ around $n = 0$.

Now, $$|y(n)| = |Ax(-n)| \le |A||x(-n)| < A.\,M < \infty$$

Hence the given system is BIBO stable.

(iv) We are given that

$$y(n) = n^2 x(n)$$

As before, let $|x(n)| < M < \infty$.
Hence it follows from the above that
$y(n) = n^2 x\,(n) < |n^2|\,M < \infty$ as long as n is finite
Hence the given system is BIBO stable.

(v) We are given that

$$y(n) = \sum_{n=0}^{\infty} (a)^n \cdot x(n)$$

$$y(n) = \sum_{n=0}^{\infty} (a)^n \cdot (1) \qquad \text{since } x(n) = u(n) = 1$$

Now,

$$\sum_{n=0}^{\infty} (a)^n = \frac{1}{1-a} \quad \text{if } |a| < 1$$

Consequently,

$y(n) = \dfrac{1}{1-a}$ for all n as long as $|a| < 1$

and

$|y(n)| = \dfrac{1}{1-a} < \infty$ for all n for $|a| < 1$

Hence the given system is BIBO stable for $|a| < 1$
On the other hand, if $|a| \ge 1$, then the system is not BIBO stable.
With this, we come to the end of the topic on discrete-time sequences and systems that play an important role in digital signal processing applications.

Summary

The chapter started with the concept of a signal as a quantity which in some manner conveys information. The problem of mathematical representation of signals, in particular discrete signals, was then posed. The characterisation, classification and time-domain representation of discrete-time sequences that are equally spaced was then taken up. Typical sequences that play an important role in digital signal processing such as, the unit sample sequence, the unit step sequence, the sinusoidal sequence, etc. were introduced. Important differences that exist between continuous and discrete sinusoids were highlighted and examples given to drive home the point. Basic operations on sequences that lead to the development of discrete-time systems were defined. The problem of classification of discrete-time systems was dealt in detail and several examples given to illustrate how they can be classified.

The important class of linear time-invariant (shift-invariant) systems was then introduced in detail and how they can be realized by constant coefficient difference equations. The operation of convolution summation was discussed and how it can be used to characterise the input-output relationship of a linear shift-invariant system shown. A number of examples were given to illustrate the mechanics of this operation graphically. The concepts of causality and stability, with respect to discrete-time systems, were also introduced and examples given to illustrate the underlying ideas.

Key Terms

Signal	Deterministic signal	Bounded signal
Continuous-time signal	Random signal	Duration-limited signal
Discrete-time-signal	Time-series	Delay operator
Biomedical signal	Unit sample sequence	Modulator
Basis Functions	Step sequence	Multiplier
Waveform	Exponential sequence	Discrete-time systems
Sampled signal	Single-frequency sequence	Linear system
Digital signal	Discrete-time sinusoid	Non-linear system
Discrete signal	Symmetric sequence	Shift-invariant
Sampled data signal	Odd and even sequences	Convolution

Direct form structure
FIR filter
Graphical convolution
Periodic
Aperiodic
Causal
Non-causal
Sampling rate
Stable
BIBO
Difference equation
Constant coefficient
Uniform interval
Characterization
Rational number
Integer
Power
Energy
Conjugate
Anti-symmetric
Absolutely summable
Band-limited
Square summable

FOR PRACTICE

Objective-type Questions

1. Fill in the blanks with appropriate words from those given in parenthesis.
   - (i) In case of a continuous-time signal it is the ________ variable that needs to be continuous.
     (dependent, continuous, independent, amplitude)
   - (ii) The independent variable of discrete signals is ________
     (continuous, discrete, deterministic, distributed)
   - (iii) A discrete signal with amplitudes that are continuous is called a ________ signal.
     (digital, continuous, sampled-data, discrete)
   - (iv) A discrete signal with amplitudes that are discrete-valued is called a ________ signal.
     (discrete, digital, sampled-data, causal)
   - (v) A finite energy signal with ________ average power is called an energy signal.
     (finite, zero, bounded, infinite, periodic, aperiodic)
   - (vi) An infinite energy signal with ________ average power is called a power signal.
     (zero, finite, an, periodic)
   - (vii) For a system to be linear the superposition property must hold for any ________ constants, α and β, and for all ________ inputs, $x_1(n)$ and $x_2(n)$, where $y_1(n)$ is the response to $x_1(n)$ and $y_2(n)$ is the response to $x_2(n)$ of the system.
     (permissible, plausible, possible, real, complex, arbitrary)
   - (viii) For a shift-invariant system the relation between the input and output must hold for any ________ input sequence and its ________ output.
     (related, corresponding, real complex, arbitrary, unique)
2. Indicate the most appropriate answer by a √ mark for the statements given above them.
   - (i) If $y_1(n)$ and $y_2(n)$ are the responses of a causal discrete-time system to the inputs $u_1(n)$ and $u_2(n)$ respectively, then

     $u_1(n) = u_2(n)$ for $n < N$
     implies that
     (a) $y_1(n) = y_2(n)$ for $n > N$
     (b) $y_1(n) = y_2(n)$ for $n = N$
     (c) $y_1(n) = y_2(n)$ for $n < N$
     (d) $y_1(n) = y_2(n)$ for $N < n < N$
   - (ii) For a causal system, changes in output samples
     (a) do follow changes in the input
     (b) do not follow changes in the input
     (c) do not predict changes in the input
     (d) do predict changes in the input.
   - (iii) For a linear discrete time-invariant system, BIBO stability is
     (a) a necessary condition
     (b) a sufficient condition
     (c) a necessary and sufficient condition
   - (iv) The impulse response completely characterizes a
     (a) Linear system
     (b) Linear discrete system
     (c) Linear time-varying system
     (d) Linear shift-invariant system
   - (v) The unit of the normalized digital angular frequency is
     (a) Hertz
     (b) cycles per second
     (c) radians per second
     (d) radians per sample
   - (vi) It is possible to represent an arbitrary sequence as a
     (a) Combination of unit step functions
     (b) Combination of delayed unit step functions
     (c) Combination of weighted unit step functions
     (d) Combination of delayed weighted unit step functions

Short Questions

1. Distinguish between
   - (i) a digital signal and a discrete signal.
   - (ii) a continuous signal and an analog signal.
   - (iii) a sampled-data signal and a digital signal.
   - (iv) a stochastic signal and a random signal.
   - (v) the symbols $X(n)$ and $x(n)$ describing a random process.
   - (vi) a band-limited signal and a full band signal.

2. Elucidate the two important tasks of digital signal processing.
3. Explain why the sequence $x(n) = A\cos(\omega_0 n + \phi)$ is not referred to as a single frequency signal.
4. How does one perform the operation of summation and modulation with unequal length sequences, if so desired? Explain.
5. If $y(n) = x(n) * h(n)$ and the length of the sequence $x(n)$ is N, and that of the sequence $h(n)$ is M, what is the length of the sequence $y(n)$? Specify.
6. If $y(n) = x(n) * h(n)$ and that $x(n)$ is periodic with period N what is the nature of $y(n)$? Explain.
7. What do you understand by the term system operation? Explain.
8. How does one obtain a causal discrete system from a non-causal discrete system? Explain.

Problems

1. Determine the periods of the following periodic sequences
   (i) $x_1(n) = 2e^{-j0.4\pi n}$
   (ii) $x_2(n) = e^{-j\left(0.4\pi n + \frac{\pi}{4}\right)}$
   (iii) $x_3(n) = \sin(0.5\,\pi\, n + .5\,\pi)$
   (iv) $x_4(n) = \cos(0.6\,\pi\, n + 2\pi)$
   (v) $x_5(n) = \tan(0.4\,\pi\, n + \pi/4)$
2. If $x_1(n)$ and $x_2(n)$ are two periodic sequences with periods N_1 and N_2 respectively, is a linear continuation of the two sequences a periodic sequence? If so, what is the fundamental period. Repeat for three periodic sequences.
3. Determine the fundamental period of the following sequences.
   (i) $x_1(n) = \sin(0.7\pi n) + 3\cos(1.1\pi n - \pi/2)$
   (ii) $x_2(n) = 4\sin(1.3\pi n) - 3\cos(0.3n\pi + .3\pi)$
   (iii) $x_3(n) = 2\sin(1.2\,\pi n + 0.5\pi) + 4\sin(0.8n\pi + \pi/4) + \cos(0.8n\pi)$
4. Identify which of the following sequences are bounded sequences.
   (i) $x_1(n) = \{2a^n u(n)\}$ where $|a| < 1$
   (ii) $x_2(n) = \{3\sin\omega_0 n\}$
   (iii) $x_3(n) = 4\cos^2(\omega_0 n)$
   (iv) $x_4(n) = \cos(\omega_0 n^2)$
   (v) $x_5(n) = A^n u(n)$ where A is a complex number
   (vi) $x_6(n) = \sum_{n=1}^{\infty} n\alpha^n$; $|\alpha| < 1$
   (vii) $x_7(n) = \sum_{n=1}^{\infty} \frac{\alpha^n}{n!}$

5. Determine the energy of the following N – length sequences
   (i) $x_1(n) = A\sin\left(\frac{2k\pi n}{N}\right)$ $\quad 0 \le n \le N-1$
   (ii) $x_2(n) = e^{j\left(\frac{2\pi k}{N}\right)}$ $\quad 0 \le n \le N-1$
   (iii) $x_3(n) = n^2$ $\quad 0 \le n \le N-1$
6. Compute the average power and energy of the following sequences, if they exist.
   (i) $x_1(n) = \delta(n)$
   (ii) $x_2(n) = u(n)$
   (iii) $x_3(n) = n\,u(n)$
   (iv) $x_4(n) = e^{j\omega_0 n}$
7. Determine if the following systems are
   (a) Linear (b) Causal (c) Stable (d) Shift-invariant
   (i) $y_1(n) = 3x(n)$
   (ii) $y_2(n) = 3x(n) + 5$
   (iii) $y_3(n) = n\,x(n)$
   (iv) $y_4(n) = n^3 x(n)$
   (v) $y_5(n) = \sum_{k=0}^{n} x(k)$
   (vi) $y_6(n) = x(-n) + x(n)$
   (vii) $y_7(n) = \sum_{k=n-5}^{n+5} x(k)$
   (viii) $y_8(n) = x(n-8)$
   (ix) $y_9(n) = x(n+1) - 2x(n) + x(n-1)$
   (x) $y_{10}(n) = x(n) - y^2(n-1) + y(n-1)$
8. Determine $y(n)$ where $y(n) = x(n) * h(n)$ and
   (i) $x(n) = \{1, 2, 3, 4\}$; $h(n) = \{6, 4, 2\}$
   (ii) $x(n) = \{1, 1, 1, 1\}$; $h(n) = \{a^n ; n = 0, 1, 2\}$
   (iii) $x(n) = \{1, 1, \ldots 1\}$; $h(n) = a^n u(n)$

9. Evaluate $y(n)$ where $x(n)$ and $h(n)$ are as given in Fig. 1.22.

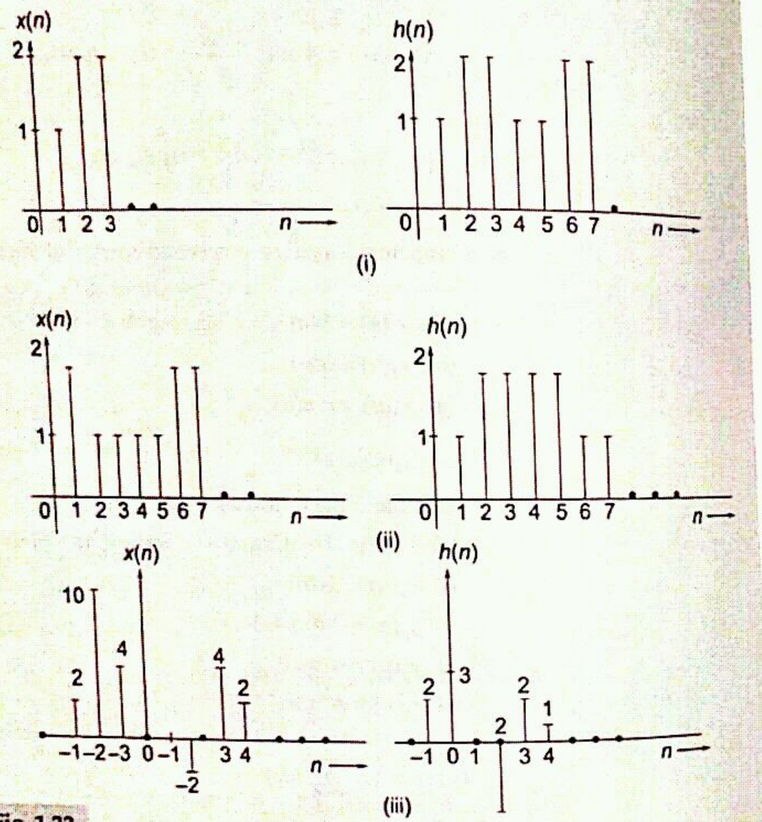


Fig. 1.22

10. Let $y(n) = h(n) * x(n)$, where both $x(n)$ and $h(n)$ are causal. For each pair $y(n)$ and $h(n)$ given below, determine $x(n)$. The first sample in each sequence is its value at $n = 0$.

    (i) $\{y(n)\} = \{-2, -2, 22, -6, 60, 56, 96\}$;
    $\{h(n)\} = \{-1, 2, 3, 4\}$

    (ii) $\{y(n)\} = \{1, 3, 6, 10, 15, 12, 12, 9, 5\}$
    $\{h(n)\} = \{2, 4, 6, 8, 5\}$

11. Consider the causal sequence defined by

$$x(n) = \begin{cases} 2(-1)^n & n \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

What is its energy? Determine its average power.

Study Questions

1. Prove that any arbitrary complex valued sequence $x(n)$, can be decomposed into

$$x(n) = x_e(n) + x_0(n)$$

where

$$x_e(n) = x_e^*(-n) \qquad \text{(conjugate symmetric)}$$

$$x_0 = (n) = -x_0^*(-n) \qquad \text{(conjugate anti-symmetric)}$$

2. The operation of signal dilation is defined by

$$y(n) = x(n\,M)$$

where M is an integer.

   (i) Explain why the operation is called dilation.
   (ii) Is there a loss of information in such a process? Explain.

3. Prove the following properties of linear convolution

   (i) $f_1(n) * f_2(n) = f_2(n) * f_1(n)$ (Commutative)
   (ii) $[f_1(n) * f_2(n)] * f_3(n) = f_1(n) * [f_2(n) * f_3(n)]$ (Associative)
   (iii) $f_1(n) * [f_2(n) + f_3(n)] = f_1(n) * f_2(n) + f_1(n) * f_3(n)$ (Distributive)
   (iv) $f(n) * \delta(n - n_0) = f(n - n_0)$

4. Determine if the

   (i) Even part of a real sequence is even.
   (ii) Odd part of a real sequence is odd.

5. Prove that the energy of the infinite sequence

$$x_1(n) = \begin{cases} \frac{1}{n}; & n \geq 1 \\ 0 & \text{elsewhere} \end{cases}$$

has an energy given by the equation; $\xi_{x_1} = \sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right)$

that converges to $\frac{\pi^2}{6}$.

6. Consider the sequence

$$x_2(n) = \begin{cases} \frac{1}{\sqrt{n}} & ; \quad n \geq 1 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

that has the energy given by the equation

$$\xi_{x_2} = \sum_{n=1}^{\infty} \frac{1}{n}$$

What can you say about its energy? Explain.

Computer Simulation

1. Generate and plot each of the following sequences over the indicated interval
   (i) $x_1(n) = 3\delta(n+3) - \delta(n-3)$; $-5 \le n \le 5$
   (ii) $x_2 n = 2n[u(n) - u(n-10] + 8e^{-0.2u(n-10)}[u(n-10) - u(n-20)]$; $0 \le n \le 20$
   (iii) $x_3(n) = 2e^{-\left(0.5+\frac{j\pi}{5}\right)n}$; $0 \le n \le 100$
2. Generate a sinusoidal sequence

   $x(n) = A\cos(\omega_0 n + \theta)$

   and plot the sequence for different values of length of the sequence, amplitude A, angular frequency ω_0 and phase θ, where $0 < \omega_0 < \pi$ and $0 \le \theta \le 2\pi$
3. Let $f(n) = u(n) - u(n-10)$

   Decompose $f(n)$ into even and odd components where

   $$f_e(n) = \frac{1}{2}[f(n) + f(-n)] \quad \text{and} \quad f_0(n) = \frac{1}{2}[f(n) + f(-n)].$$
4. Determine the output of the system that is characterized by the impulse response $h(n) = (0.8)^n u(n)$

   and input

   $x(n) = u(n) - u(n-10)$.
5. Given the following two sequences

   $$x(n) = \{3, 9, \underset{\substack{\uparrow \\ \text{origin}}}{-8}, 7, -1, 4, -2\}; \quad -3 \le n \le 3$$

   $$h(n) = \{2, \underset{\substack{\uparrow \\ \text{origin}}}{4}, 4, -5, -3, 1\}; \quad -1 \le n \le 4.$$
6. Determine and plot the impulse response of the system of below for $0 \le n \le 100$.

   $$y(n) - 0.25y(n-1) + 0.5y(n-2) = x(n) + x(n-1) + \frac{1}{3}x(n-3)$$

   Repeat for

   $x(n) = [2 + 3\sin(0.5\pi n)]u(n)$.

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