Quantum Entropy

Thermodynamic Entropy:

- The honour of introducing the concept entropy goes to Rudolf clausius.
- Entropy is defined as the amount of additional information needed to specify the exact physical state of a system, given its thermodynamic specification.
- According to thermodynamics the entropy change Δs of a thermodynamic system absorbing heat ΔQ is given by $\Delta S = \Delta Q/T$.
- For a natural/spontaneous process free energy (ΔG) <0 always.
- The second law of thermodynamics states that in general the total entropy of any system will not decrease other than by increasing the entropy of some other system.

Statistical Entropy:

- This considers the statistical behavior of the microscopic components of the system.
- The number of degrees of freedom of a physical system can be linked to the number of micro-states W of that system.
- The simple equation governing this is $S = K_b \log W$. K_b is Boltzman constant.

- This equation is valid only if each microstate is equally accessible (each microstate has an equal probability of occurring).
- So S is considered same as number of degrees of freedom. In fact the absolute temperature is Energy per molecular degrees of freedom.
- Boltzman distribution gives an idea of no. of particles 'i' in the state E_i.

Information Entropy:

- This is also known as Shannon classical information entropy.
- More generalized form of entropy.
- The entropy H of a random variable X is the measure of uncertainty associated with the value of X before we learn its value. Can also be viewed as information gained after knowing X.
- Suppose X takes the values {x1,x2,… } with probabilities {p1,p2,p3,…. } .Entropy does not depend on the labels.is defined as $H(X) = -\sum p_i \log p_i$ .Entropy is maximized when all the outcomes are equiprobable then $H = \log n$. $H(x) \geq 0$.
- Unless specified all the logarithms are assumed to the base 2.
- If pi =0 then it does not contribute to entropy. i.e., $0\log0 \stackrel{def}{=} 0$;

- Entropy can be viewed as the number of bits to store the information generated by the source and can be reconstructed at a later stage.
- The best definition for entropy is that it can be used to quantify the resources needed to store the information.
- Entropy is the average optimal number of bits to store the information from the source less than which lead to data being irretrievably lost.
- Example:
  Consider the basic example of coin tossing. Let us analyse the entropy behaviour when two or more probability distributions are mixed. Suppose we have 50p and 1Rs coins each coin biased with probability p1 and p2 respectively . Alice flips one of the coin and tells the result to bob. Then bob should gain atleast as much as average information on individual coins.

  $$f(px + (1 - p)y) \geq pf(x) + (1 - p)f(y).$$

  The entropy of two-outcome random variable is called binary entropy.

  $H_{bin} = -p \log p - (1-p)\log(1-p)$

Relative Entropy:

- Measure of closeness between two probability distributions.

- The probability distributions must be on the same set.
- Relative entropy of two probability distributions p(x) and q(x) is given by $H(p \| q) = \sum p \log(p/q)$;
- Most important property of relative entropy is its non-negativity i.e.,
  $H(p \| q) >= 0$ and equality holding if $p = q$
- Maximum value of entropy is a direct consequence of above inequality that $H(x) <= \log n$ where n are no. of possible outcomes. Equality holds if all outcomes are equiprobable.
- Joint entropy of two random variables (x,y) is found using combined probability distributions i.e., p(x,y).
- Conditioned entropy $H(x/y) = H(x,y) – H(y)$;
- Mutual entropy is the measure of common ness between x and y

i.e., $H(x{:}y) = H(x) – H(x/y)$;

$H(Y/X) \geq 0$ since from the above equation. So generalising

$H(x1,x2,x3,\ldots) \leq H(x1) + H(x2) + \ldots$ and equality holding if all are independent variables.

Data Processing Inequality:

This states that no clever manipulation of the data can increase the inferences that can be made from the data.

The inequality is as follows:

If X->Y->Z i.e., x ,y and z form a markov chain i.e., X and Z are conditionally independent given Y.(future and past state are independent if the present state is known.)

P(x,z/y) = p(x/y)p(z/y).

H(x) ≥ H(x:y) ≥ H(x,z)

The information capacity of a continuous channel of bandwidth B hertz, perturbed by additive white Gaussian noise of power spectral density No/2 and limited in Bandwidth to B, is given by

C = $B\log_2$ (1 + P/N0.B) bits per second

Where P is the average transmitted power.

Properties of entropy:

- H(x,y) = H(y,x)
- H(x:y) = H(y:x)
- H(x) <= H(x,y)
- H(x,y) <= H(x) + H(y)
- H(x/y,z) <= H(x/y)

Quantum Entropy:

- Also known as Von Neumann Entropy.
- Quantum Mechanics provides a mathematical and conceptual frame work for development of theories of the system.

Postulates:

- Associated to any isolated physical system is a complex vector space with inner product known as state space of the system. The system is completely described by its state vector, which is a unit vector in the system's state space.
- The evolution of a closed quantum system is described by unitary transformation. The time evolution of closed quantum system is described by schrodinger equation.
- Quantum measurements are described by a collection of measurement operators . These are operators acting on the state space of the system being measured.
- The state space of a composite physical system is the tensor product of the state spaces of composite physical systems.
- Not only state vectors there is another tool called density operator to formulate quantum mechanics.

All the above postulates of state, evolution and measurement can be restated in terms of density operators

- $\rho \equiv \sum p_i |\psi_i\rangle\langle\psi_i|$. for ensemble of pure states
- $tr(\rho^2) = 1$ for pure state and < 1 for mixed state
- Reduced density operator. If we have two physical systems A and B whose state is described by density operator ρ^{AB} .Then

$\rho^A = Tr_b(\rho^{AB})$.This reduced density operator provides the correct measurement statistics for measurements on system A.

- A useful application of reduced density operator is the analysis of quantum teleportation.
- A completely mixed density operator in the D-dimensional space ,I/d has entropy log d.
  - ✧ Entropies are defined with respect to density operators.
  - ✧ $S(\rho) = -tr(\rho \log \rho)$ where log is taken to base 2.
  - ✧ If λ are eigen values of ρ then $s(\rho) = -\sum \lambda \log \lambda$.

Quantum Relative Entropy:

$S(\rho \,||\, \sigma) = tr(\rho \log \rho) - tr(\rho \log \sigma)$ where ρ and σ are two density operators.

Klein's Inequality:

Quantum Relative entropy is non-negative.

Basic Properties:

- Entropy is non-negative and is zero for a pure state since a pure state can have eigen values either zero or one.
- For a complete mixed state i.e., I/d the entropy is maximum and is given by log d.
- Suppose a composite system AB is pure state $S(A) = S(B)$.

- Suppose pi are probabilities , and the states ρi have support on orthogonal subspaces. Then

$$S(\sum p_i \rho_i) = H(p_i) + \sum p_i\, s(\rho_i)$$

- Entropy of a Tensor Product is S(ρ* σ) = S(ρ) + S(σ).
- Similar to classical entropies we can define similar terms like composite quantum entropies and conditional entropies.

  So, S(AB) = -tr(ρ_{AB} log ρ_{AB})

  S(A/B) = S(A,B) – S(B)

  Mutual Entropy: S(A:B) = S(A) + S(B) –S(AB)

  But some results of classical entropies does not hold in quantum entropy. For example H(X,Y) = H(x) + H(X/Y) => H(X,Y) >= H(X)

  But for Quantum systems conditional entropy can be negative.

  i.e., H(X,Y) can be less than H(X).

  A mixed state can be purified .Entropy of a pure state is zero.

  So, conditional entropy can be negative.

  Projective measurements Pi performed on a quantum system change density operator ρ to ρ' as follows

  $\rho' = \sum P_i\, \rho\, P_i$.

  So, entropy never decreases in such a process and increases if state changes by measurement.

  But Generalized measurements can decrease the entropy.

Schmidt Decomposition:

Let H1 and H2 be Hilbert spaces of dimensions n and m respectively. Any vector V in the tensor product H1*H2 there exists orthonormal sets {u1,u2,….} and {v1,v2,….} such that

$V = \sum a_i \ u_i * vi.$

Sub Additive:

Joint entropy of two distinct systems A and B satisfy the inequality

S(A,B) <= S(A)+S(B) which is direct consequence of Klein's inequality.

Triangle Inequality:

S(A,B) >= | S(A) – S(B)|

To prove this inequality introduce a state R which purifies A and B to state ABR. Use the above Sub Additive inequality which is trivial proof.

Entropy Concavity:

S(∑ pi ρi) >= ∑ pi s(ρi)

For proving this introduce an auxiliary system B whose orthonormal basis is |i> corresponding to density is ρi.

Use sub additive inequality and the result follows.
The equality holds if all ρi are identical.
The upper bound for $S(\sum pi\ \rho i) <= \sum pi\ s(\rho i) + H(pi)$

Strong Sub-Additivity:
For trio of quantum systems A,B and C
$S(A,B,C) + S(B) <= S(A,B) + S(B,C)$
$S(A) + S(B) <= S(A,C) + S(B,C)$ --- (2)

Proving the result is bit advanced which uses Leib'z theorem.

For classical entropies the result is obvious since $H(A) <= H(A,B)$ but may not be in case of quantum entropies.

Conditioning reduces entropy $S(A/B,C) <= S(A/B)$

Sub-additivity of conditional entropy:
$S(A,B / C,D) <= S(A/B) + S(C/D)$

Remarks:
The applications discussed here show the power and versatility of relative entropy in attacking problems of quantum information theory.
The relative entropy techniques are central to further work in quantum information theory. They are very promising in solving the perplexing additivity problems.