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Note Title 4/30/2012

RIGID BODY MOTIONS

$\rightarrow \quad \underset{\sim}{a}, \underset{\sim}{b} \qquad \|\underset{\sim}{a} - \underset{\sim}{b}\| = \|\psi(\underset{\sim}{a}) - \psi(\underset{\sim}{b})\|$ definition

<u>example</u> Translation $\psi(\underset{\sim}{a}) = \underset{\sim}{a} + \underset{\sim}{t}$

<u>ROTATIONS</u>

- LINEAR TRANSFORMATIONS

$$\underset{\sim}{a} \longmapsto \psi(\underset{\sim}{a}) \qquad \underset{\sim}{b} \longmapsto \psi(\underset{\sim}{b})$$

$$\alpha \underset{\sim}{a} + \beta \underset{\sim}{b} \longmapsto \alpha \psi(\underset{\sim}{a}) + \beta \psi(\underset{\sim}{b})$$ <u>DEFINITION</u>

<u>Another way</u> $\psi(\underset{\sim}{a}) = A\, a$

matrix

vector

Suppose we have points in $\mathbb{R}^2$

then $\underset{\sim}{a}$ is a vector e.g. $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

any linear transformation corresponds to a matrix multiplication

$$\chi(\underset{\sim}{a}) = \underbrace{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}_{\underset{\sim}{a}}$$

Points are going to be column vectors

Linear Transformations from $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are represented by 2×2 matrices as

If we are in $\mathbb{R}^3$

then a point is of type $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ a column vector.

Any linear transformation from $\mathbb{R}^3 \to \mathbb{R}^3$ corresponds to multiplication by a 3×3 matrix

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{21} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\underset{\sim}{c} = A \underset{\sim}{b}$$

is the matrix corresponding to the linear transformation ψ

We want to understand rigid body motions that are <u>also</u> linear transformations.

<u>Defn</u> An orthogonal transformation is a linear transformation that preserves inner products.

$$\underset{\sim}{a} \longrightarrow \psi(\underset{\sim}{a})$$

$$\underset{\sim}{b} \longrightarrow \psi(\underset{\sim}{b})$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \psi(\underset{\sim}{a}) \cdot \psi(\underset{\sim}{b})$$

(Inner product $\underset{\sim}{a} \cdot \underset{\sim}{b}$ is also called scaler product

$\underset{\sim}{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ $\underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1 b_1 + a_2 b_2$

Properties of orthogonal transformations

- Orthogonal transformations preserve norms

$$\underset{\sim}{a} \longrightarrow \psi(a)$$

$$\underset{\sim}{a} \longrightarrow \psi(a)$$

$$\underset{\sim}{a} \cdot \underset{\sim}{a} = \psi(a) \cdot \psi(a)$$

before   after

$$\| \underset{\sim}{a} \|^2 = \| \psi(a) \|^2$$

LENGTH IS PRESERVED

More generally

$$\underset{\sim}{a} \longrightarrow \psi(a)$$
$$\underset{\sim}{b} \longrightarrow \psi(b)$$

TRUE

proved previously $\rightarrow$ $\rightarrow$ $\left\{ \begin{array}{l} \|\psi(a)\|^2 = \|a\|^2 \\ \|\psi(b)\|^2 = \|b\|^2 \\ \psi(a)\cdot\psi(b) = \underset{\sim}{a}\cdot\underset{\sim}{b} \end{array} \right.$ $\leftarrow$ dfn

need to show that

ψ is an rigid body motion $\rightarrow$ $\boxed{\|\underset{\sim}{a} - \underset{\sim}{b}\| = \|\psi(\underset{\sim}{a}) - \psi(\underset{\sim}{b})\|}$ Proved

if ψ is an orthogonal transformation

Proof

$$\big(\psi(a) - \psi(b)\big)\cdot\big(\psi(a) - \psi(b)\big) \overset{?}{=} (\underset{\sim}{a} - \underset{\sim}{b})\cdot(\underset{\sim}{a} - \underset{\sim}{b})$$

$$\underbrace{\cancel{\|\psi(a)\|^2} - 2\,\cancel{\psi(a)\cdot\psi(b)} + \cancel{\|\psi(b)\|^2}}_{} \overset{?}{=} \underbrace{\cancel{\|a\|^2} + \cancel{\|b\|^2} - 2\,\cancel{\underset{\sim}{a}\cdot\underset{\sim}{b}}}_{}$$

These two are equal!! ✓✓ ☺

An orthogonal transformation is a rigid body transformation

<u>defn</u> of orthogonal transformation
- linear
- preserves inner products $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{\psi(a)} \cdot \underset{\sim}{\psi(b)}$

<u>defn</u> of rigid body transformation

$$\| \underset{\sim}{a} - \underset{\sim}{b} \| = \| \psi(a) - \psi(b) \|$$

- We proved that orthogonal transformations are rigid body transformations
- Earlier we proved that translations are rigid body transformations