IV Discrete –Time Fourier transform DTFT.

A. Basic Definitions

The discrete-time Fourier transform (DTFT) of x(n) is

$$X(e^{jw}) = \sum x(n)\, e^{-jwn}$$

where w is in radians. $X(e^{jw})$ is periodic with period 2π, since a function of a periodic function is periodic, and has the same period. Since the forward transform is a Fourier series, the inverse transform,

$$x(n) = 1/2\pi \int X(e^{jw})\, e^{jwn}\, dw$$

is the formula for the Fourier series coefficient. The frequency response $H(e^{jw})$ is the DTFT of the impulse response h(n). As with the continuous-time

Fourier transform, the DTFT is used because of the existence of a convolution theorem.

Amplitude and Phase of DTFT.

$$X(e^{jw}) = \mathrm{Re}\{ X(e^{jw}) \} + j\, \mathrm{Im}\{ X(e^{jw}) \}$$

$$= | X(e^{jw}) |\, e^{j\phi(w)}$$

$$| X(e^{jw}) | = \sqrt{\mathrm{Re}^2 \{X(e^{jw})\} + \mathrm{Im}^2 \{X(e^{jw})\}}$$

$$\phi(w) = \arg(X(e^{jw}))$$

$$= \tan^{-1} \frac{\mathrm{Im}(X(e^{jw}))}{\mathrm{Re}(X(e^{jw}))}$$

$$+ \pi\, u(-\mathrm{Re}(X(e^{jw})))$$

$$X(e^{jw}) = \frac{N(e^{jw})}{D(e^{jw})}$$

then,

$$| X(e^{jw}) | = \frac{| N(e^{jw}) |}{| D(e^{jw}) |}$$

If

$$X(e^{jw}) = \frac{1}{D(e^{jw})}$$

Then,

$$| X(e^{jw}) | = \frac{1}{| D(e^{jw}) |}$$

$\phi(w) = \arg(N(e^{jw})) - \arg(D(e^{jw}))$

B. Simple Examples of Forward and Inverse Transforms

Example $x(n) = \delta(n)$

$$X(e^{jw}) = \sum x(n)\, e^{-jwn} = \sum \delta(n)\, e^{-jwn}$$

$$= 1 = X(e^{jw})$$

Now Find x(n)

$$x(n) = 1/2\pi \int 1 \bullet e^{jwn}\, dw$$

$$= e^{jwn}/2\pi jn \,| \quad \text{for } n \neq 0$$

$$= 1 \quad \text{for } n = 0$$

$$= \frac{(e^{j\pi n} - e^{-j\pi n})}{2n\pi j}$$

$$= \frac{\sin(\pi n)}{n} \qquad \text{for } n \neq 0.$$

$$= 1 \qquad \text{for } n = 0$$

$$= \delta(n)$$

Example Find Transfer function or frequency response for a filter

$$h(n) = \begin{cases} 1, & 0 \leq n \leq N-1 \\ 0 & \text{elsewhere.} \end{cases}$$

$$H(e^{jw}) = \sum 1 \bullet (e^{-jw})^n = \frac{1-(e^{-jw})^N}{1- e^{-jw}}$$

for $(e^{-jw}) \neq 1$ or $w \neq 2\pi k$

$= N$ for $w = 2\pi k$

Find Amplitude and Phase

$$H(e^{jw}) = \frac{1-e^{-jwN}}{1-e^{-jw}}$$

$$= \frac{1-\cos(wN) + j\sin(wN)}{1-\cos(w) + j\sin(w)}$$

$$|H(e^{jw})| = \frac{[1-\cos(wN)]^2 + \sin^2(wN)}{[1-\cos(w)]^2 + \sin^2(w)}$$

$$= \frac{1+ \cos^2 + \ \sin^2 - 2\cos(wN)}{1+ \cos^2 + \sin^2 - 2 \cos(w)}$$

$$= \frac{2(1- \cos(wN))}{2(1- \cos(w))}$$

$$\phi(w) = \ \tan^{-1} \frac{\sin(wN)}{1- (\cos(wN)}$$

$$+ \pi\, u\, (- (1- (\cos(wN)\ ))$$

$$- \quad \tan^{-1} \frac{\sin(w)}{1-(\cos(w)}$$

$$+ \pi u(-(1-(\cos(w)))$$

Find better amplitude and phase response expressions, starting from :

$$H(e^{jw}) = \frac{1-e^{-jwN}}{1-e^{-jw}}$$

Example Find the DTFT of $x(n) = .5^n\, u(n)$

$$X(e^{jw}) = \sum .5^n\, (e^{-jw})^n = \sum (.5\, e^{-jw})^n$$

$$= \frac{1}{1 - .5\, e^{-jw}}$$

$$= \frac{1}{[1 - .5\cos(w)]^2 + [\, .5\sin(w)]^2}$$

$$\exp \quad -j \;\tan^{-1} \frac{.5\sin(w)}{1 - .5\cos(w)\,)}$$

Example Find the forward and inverse transforms of $x(n) = \delta(n) + \delta(n-1)$

$X(e^{jw}) = 1 + e^{-jw}$

$x(n) = 1 / 2\pi \int (1+e^{-jw})\ e^{jwn}\ dw$

$= 1 / 2\pi \int e^{jwn}\, dw\ +\ 1 /2\pi \int e^{jw(n-1)}\, dw$

$$= \frac{e^{jwn}}{2\pi jn} + \frac{e^{jw(n-1)}}{2\pi j(n-1)} \Big|$$

$$= \frac{\sin(\pi n)}{\pi n} + \frac{\sin(\pi(n-1)}{\pi(n-1)}$$

$= \mathrm{sinc}(n) + \mathrm{sinc}(n-1)\ =\ \delta(n) + \delta(n-1)$

Frequency Response From Difference Equation

Shift Theorem: $F\{x(n-n_o)\} = e^{-jwn} \cdot X(e^{jw})$
Proof: $\sum x(n-n_o)\, e^{-jwn} \,|\, n \leftarrow n + n_o = \uparrow$

Given the difference equation,

$$\sum a_k\, y(n-k) = \sum b_k\, x(n-k)$$

find the frequency response $H(e^{jw})$
Taking the DTFT of both sides,

$$F\{\sum a_k\, y(n-k)\} = F\{\sum b_k\, x(n-k)\},$$

$$\sum a_k\, F\{y(n-k)\} = \sum b_k\, F\{x(n-k)\},$$

Using the shift theorem,

$$Y(e^{jw})\sum a_k\, e^{-jwk} = X(e^{jw})\sum b_k\, e^{-jwk}$$

$$H(e^{jw}) = Y(e^{jw})/ X(e^{jw})$$

$$= \frac{\sum b_k e^{-jwk}}{\sum a_k e^{-jw}}$$

Properties of the DTFT

(1) $X(e^{jw})$ is a periodic function **of w, with period 2π**

(2) If x(n) is a real sequence, then
Re $(X(e^{jw}))$ is an even function of w
and Im $(X(e^{jw}))$ is odd

Proof: $\text{Re}\{\sum x(n)\, e^{-jwn}\} = \sum x(n) \text{Re}\{e^{-jwn}\}$

$= \sum x(n) \cos(wn) = \sum x(n) \cos(-wn)$

$= \text{Re}\,(X\,(e^{j(-w)})) = \text{Re}\,(X\,(e^{jw}))$

$\text{Im}\ \{X\ (e^{jw})\}$

$= \sum \text{Im}\ \{x(n)[\ \cos(wn)\text{-}j\ \sin(wn)]\ \}$

$= - \sum \text{Im}\ \ \{x(n)[\ \cos(wn)+ j\ \sin(wn)]\ \}$

$= - \text{Im}\ \ X(e^{-jw}\ )$

(3) if x(n) is a real sequence, then $|X\ (e^{j(w)})|$ is an even function of w and arg $\{X\ (e^{jw})\}$ is an odd function of w.

Proof: Prove it for $|X\ (e^{j(w)})|^2$

$|\ X\ (e^{jw})|^2 =\ \ X\ (e^{jw}) \bullet X\ (e^{jw})^*$

but $X(e^{jw})^* =\ \ X\ (e^{-jw})$

so

$$| X (e^{jw})|^2 = X (e^{jw}) \bullet X (e^{-jw})$$

$$| X (e^{-jw})|^2 = X (e^{-jw}) \bullet X (e^{jw})$$

$$- \arg (X (e^{-jw}))$$

$$= - \tan^{-1} \frac{-\sum x(n)\sin(-wn)}{\sum x(n)\cos(wn)}$$

$$+ \pi u (-\sum x(n) \cos(wn))$$

$$= \tan^{-1} \frac{-\sum x(n)\sin(wn)}{\sum x(n)(\cos(wn)}$$

$$- \pi u (-\sum x(n) \cos(wn))$$

$$= \arg (X (e^{jw}))$$

$\therefore$ $\arg (X (e^{jw}))$ is an odd function.

(4) let x(n) be a real, even sequence,
$x(n) = x(-n)$. Then $X(e^{jw})$ is real and

$\mathrm{Im}\{X(e^{jw})\} = 0$.

Proof: $X(e^{jw}) = \sum x(n)\, e^{-jwn}$

$= x(0) + \sum x(n)\, e^{-jwn} + \sum x(n)\, e^{-jwn}$ |

$\sum x(-n)\, e^{jwn}$

$x(0) + \sum x(n)\, (e^{-jwn} + e^{jwn})$

$= x(0) + 2 \sum x(n) \cos(wn)$

which is real and even.

(5) let x(n) be odd and real, $x(n) = -x(-n)$

$x(0) = 0.$

Then $X(e^{jw})$ is odd and imaginary, so

$Re\{X(e^{jw})\} = 0.$

Proof: $X(e^{jw}) = \sum x(n)\, e^{-jwn}$

$$= \sum x(n)\, e^{-jwn} + \sum x(n)\, e^{-jwn}$$

$$\sum x(n)\, e^{-jwn} - \sum x(n)\, e^{jwn}$$

$$= 2j \sum x(n)\, (e^{-jwn} - e^{jwn}) / 2j$$

$$= -2j \sum x(n) \sin(wn)$$

which is odd and imaginary

(6) $F\{ x(n-m)\} = e^{-jwm} X(e^{jw})$

(7) $x(n) = e^{jwn}$ is an eigenfunction of the system, $y(n) = h(n) \quad x(n)$, the corresponding eigenvalue is $H(e^{jw})$

$$y(n) = \sum h(k)\, e^{jw(n-k)}$$

$$= e^{jwn} \sum h(k) e^{-jwk}$$

$$= e^{jwn}\, H(e^{jw})$$

Note: $\sum x(g(n))\, e^{-jf(w)g(n)} = X(e^{jf(w)})$

Convolution Theorems for the DTFT

(8) If $x(n)$, $h(n)$ and $y(n)$ have DTFT's $X(e^{jw})$, $H(e^{jw})$ and $Y(e^{jw})$, and

If $y(n) = \sum h(k)\, x(n-k)$,

then $Y(e^{jw}) = H(e^{jw})\, X(e^{jw})$

Proof: Take the Fourier Transforms of both sides as

$$Y\,(e^{jw}) = \sum \ \sum h(k)\ x(n\text{-}k)\ e^{-jwn}$$

$$e^{-jwk}\ e^{-jw(n\text{-}k)}$$

$$= \sum \ \sum h(k)\ e^{-jwk}\ x(n\text{-}k)\ e^{-jw(n\text{-}k)}$$

$$= \sum \ h(k)\ e^{-jwk} \quad \sum \ x(m)\ e^{-jwm}$$

$$= H(e^{jw}) \bullet X(e^{jw})$$

(9) $F\{x(n)\bullet h(n)\}=1/2\pi \int X(e^{j(w\text{-}u)})\ H(e^{ju})du$

First Proof :

Let $X(e^{j(w\text{-}u)}) = \sum \ x(n)\ e^{-jn(w\text{-}u)}$ and

$H(e^{ju}) = \sum \ h(m)\ e^{-jum}$ on the right hand side above. This gives

$$1/\,2\pi \int \sum \sum x(n)\, h(m)\, e^{-jnw}\, e^{ju(n-m)}\, du$$

$$= 1/\,2\pi \sum \sum x(n)\, h(m)\, e^{-jnw} \int e^{ju(n-m)}\, du$$

$$= \sum x(n)\, h(n)\; e^{-jnw}$$

$$= F\,\{\, x(n) \bullet h(n) \,\}$$

Second Proof for Property (9) Let

$x(n) = 1/\,2\pi \int X(e^{jv})\, e^{jnv}\, dv$ and

$h(n) = 1/\,2\pi \int H(e^{ju})\, e^{jnu}\, du$ to get

$$F\{x(n) \bullet h(n)\} = \sum x(n)\, h(n)\, e^{-jnw}$$

$$= 1/4\pi^2 \sum \int \int X(e^{jv})\, H(e^{ju})\, e^{jn(u+v-w)}\, du\, dv$$

$$=1/4\pi^2 \int \int X(e^{jv})\, H(e^{ju})[\sum e^{jn(u+v-w)}]\, du\, dv$$

$$2\pi \sum \delta(u+v-w-2\pi n) \text{ since}$$

$$\sum e^{-jnTw} = 2\pi/T \sum \delta(w-2\pi n / T)$$

$$= 1/2\pi \int \int X(e^{jv})\; H(e^{ju})\, [\, \delta(u + v - w) +$$

$$\delta(u+v-w-2\pi)\, du\, dv$$

$$u = w-v,\; u = w + 2\pi - v$$

$= 1/2\pi \int X(e^{jv}) \; H(e^{j(w-v)})\, dv$

Third Proof for property (9)

Let $y(n) = h(n) \bullet x(n)$.

Find $Y(e^{jw})$ as a function of $H(e^{jw})$ and $X(e^{jw})$

$h(n) = 1/2\pi \int H(e^{j\theta})\, e^{jn\theta}\, d\theta$

$Y(e^{jw}) = \sum x(n)\, h(n)\; e^{-jnw}$

$= \sum 1/\, 2\pi \int H(e^{j\theta})\, e^{jn\theta}\, d\theta\; x(n)\; e^{-jnw}$

$= 1/\, 2\pi \int H(e^{j\theta})\, [\, \sum x(n)\, e^{-jn(w-\theta)}]\, d\theta$

$= 1/\, 2\pi \int X(e^{j(w-\theta)})\, H(e^{j\theta})\; d\theta$

(10) Parseval's Equation

$$\sum_{n=-\infty}^{\infty} h(n)x^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{jw})X^*(e^{jw})dw$$

$$x^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{jw})e^{-jwn}dw \quad \text{so LHS} =$$

$$\sum_{n=-\infty}^{\infty} h(n)\frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{jw})e^{-jwn}dw = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{jw})\sum_{n=-\infty}^{\infty} h(n)e^{-jwn}dw$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{jw})H(e^{jw})dw = \text{RHS}$$

Ex. Let H(e^{jw}) be a causal, stable allpass filter, i.e. |H(e^{jw}) | = 1 for all w. Prove that h(n) is shift-orthogonal, i.e.

$$\sum_{n=-\infty}^{\infty} h(n)h(n+m) = \delta(m)$$ From (10),

$$\sum_{n=-\infty}^{\infty} h(n)h^*(n+m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{jw})|^2 e^{-jwm}dw$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-jwm}dw = \delta(m)$$

Example IIR and FIR filters

Ex. Zero Phase averaging filter, FIR, non-recursive

$$y(n) = \frac{1}{(1+2M)} \sum x(n-k)$$

$$h(k) = \frac{1}{(1+2M)} \qquad \text{if } |k| \le M$$

$$H(e^{jw}) = \frac{1}{(1+2M)} \sum e^{-jnw}$$

$$= \frac{(e^{jMw} - e^{-jw(M+1)})}{(1-e^{-jw})} \frac{1}{(1+2M)}$$

$$= \frac{(e^{j(M+1/2)w} - e^{-jw(M+1/2)})}{(e^{jw/2} - e^{-jw/2})} \frac{1}{(1+2M)}$$

$$= \frac{1}{(1+2M)} \frac{\sin((M+1/2)w)}{\sin(w/2)}$$

Example Find $G(e^{jw})$ if $g(n) = x(2n)$

$g(n) = 1/\,2\pi \int X(e^{j\theta})\, e^{j(2n)\theta}\, d\theta$

$$G(e^{iw}) = \sum \ g(n) \ e^{-jnw}$$

$$= \sum 1/ 2\pi \int \ X(e^{j\theta}) \ e^{j2n\theta} \, d\theta \ e^{-jnw}$$

$$= 1/ 2\pi \ \int \ X(e^{j\theta}) \ [\ \sum e^{-jn(w-2\theta)}] \ d\theta$$

Change of variable to simplify the exponent.

$u= w- 2\theta$, $du = -2d\theta \quad d\theta = -1/2 \ du$.

$u_1 = w +2\pi$ = lower limit

$u_2 = w - 2\pi$ = upper limit
Switch limits and change signs.

$$G(e^{iw}) = 1/ 4\pi \int \ X(e^{j(w-u)/2}) \ [\ \sum e^{-jnu}] \ du$$

But

$$\sum e^{-jnTw} = \frac{2\pi}{T} \sum \delta(w-2\pi n / T)$$

so

$$\sum e^{-jnu} = 2\pi \sum \delta(u - 2\pi n)$$

$$G(e^{jw}) = ½ \sum \int X(e^{j(w-u)/2}) \, \delta(u-2\pi n) \, du$$

$$= 1/2 \; [\;\; X(e^{jw/2}) + X \, (e^{j(w/2-\pi)}) \;]$$

$$= \; 1/2 \; [\;\; X(e^{jw/2}) + X \, (- e^{j(w/2)}) \;]$$

$$= G(e^{jw})$$

E. More Examples

Example $y(n) - a\, y(n-1) = x(n) - b\, x(n-1)$

$y(n) = x(n) - b\, x(n-1) + a\, y(n-1)$

Given a, find b such that the system is all-pass. Frequency response is

$$H(e^{jw}) = \frac{(1-be^{-jw})}{(1-ae^{-jw})}$$

$$|H(e^{jw})|^2 = \frac{|(1-be^{-jw})|^2}{|(1-ae^{-jw})|^2}$$

$$= \frac{(1+b^2 - 2b\cos(w))}{(1+a^2 - 2a\cos(w))}$$

$$= b^2 \frac{(1+ (1/b^2) - 2 (1/b) \cos(w))}{(1+ a^2 - 2a \cos(w))}$$

try b = 1/a

$$= \frac{1}{a^2} \frac{(1+ a^2 - 2 a \cos(w))}{(1+ a^2 - 2a \cos(w))}$$

Example

Let $X(e^{jw}) = F\{ x(n)\}$ Find the sequence y(n) in terms of x(n) if

$Y(e^{jw}) = X^2(e^{jw})$

$X^2(e^{jw}) = (\sum\ x(n)\ e^{-jnw})\ (\sum\ x(m)\ e^{-jmw})$

$= \sum\ y(k)\ e^{-jkw}$ note difference indexes

$\therefore\ \sum \sum\ x(n)\ x(m)\ e^{-jw(n+m)}$

$= \sum\ y(k)\ e^{-jkw}$

$e^{-jw(n+m)} = e^{-jkw}$ solve for n as

$n+m = k$

$m = k-n$ m is fixed now, and sum over m disappears.

$\therefore\ y(k) = \sum\ x(n)\ x(k-n)$

$y(n) = \sum x(k)\, x(n-k)$

Alternately; use convolution theorem.

Example

$g(n) = \begin{cases} x(n / 2), & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}$

Find $G(e^{jw}) = \sum g(n)\, e^{-jnw} = \sum x(n/2)\, e^{-jnw}$

$= \sum x(n)\, e^{-jn(2w)}$ so $X(e^{j2w}) = G(e^{jw})$

Hard Method

g(n) = Same Definition.

$g(n) = x(n/2) = 1/ 2\pi \int X(e^{j\theta})\, e^{j(n/2)\theta}\, d\theta$

$$G(e^{iw}) = \sum 1/\,2\pi \int X(e^{j\theta})\, e^{j(n/2)\theta}\, d\theta\, e^{-jnw}$$

$$= 1/\,2\pi \int X(e^{j\theta})\ [\sum e^{-j(n/2)(\theta-2w)}]\, d\theta$$

$$\sum e^{-jn(\theta-2w)} = 2\pi \sum \delta(\theta - 2w - 2\pi n)$$

use

$$\sum e^{-jnTw} = \frac{2\pi}{T} \sum \delta(w - 2\pi n / T)$$

$$= \sum \int X(e^{j\theta})\, \delta(\theta - 2w - 2\pi n)\, d\theta$$

Question; how many values of n will generate a non-zero $\delta(\theta - 2w - 2\pi n)$,given constant w.

Answer; only one, use n = 0.

$\therefore$ use $\delta(\theta - 2w)$ and $\theta = 2w$

so = $\int X(e^{j\theta})\, \delta(\theta - 2w)\, d\theta = X(e^{j2w})$

Ex. Ideal LP Filter

$$h(n) = (1 / 2\pi) \int e^{jwn}\, dw = \left. \frac{e^{jwn}}{2\pi jn} \right|$$

$$= \frac{e^{j(wc)n} - e^{-j(wc)n}}{(2j)\, \pi n}$$

$$= \frac{\sin(w_c n)}{\pi n}$$

Ex. Ideal BP Filter

$$h(n) = \frac{\sin(w_{c2} n) - \sin(w_{c1} n)}{\pi n}$$

How do we implement $y(n) = h(n) * x(n)$ in pseudocode if

$$h(n) = \frac{\sin(w_{c2} n) - \sin(w_{c1} n)}{\pi n}$$

F. Advanced Topic Number 1

Problems:

Some applications, such as communication systems, have a continuous stream of samples coming in, and spectral information is needed. Using past samples up to time n, we can calculate the DFT of the data in several ways.

Solution 1

With samples starting at time 0, and continuing up to time n, we get

$$X_n(e^{jw}) = \sum_{m=0}^{n} x(m)e^{-jwm}$$

If x(n) is real, the number of real multiplies is $N_M = 2(n+1)$. The problems here are that:

(1) N_M quickly becomes too large to update in real time,
(2) The time variable n causes overflow.

Solution 2

We can solve the first problem by defining a spectrum over a fixed window of N samples, starting at time n-(N-1), as

$$X_n(e^{jw}) = \sum_{m=n-(N-1)}^{n} x(m)e^{-jwm}$$

N_M is 2N with N fixed, and n increases as new data comes in. Although the limits on the sum increase, this could be fixed by using a shift register that keeps only the most recent N samples. However, the exponent of e still grows without bound, leading to overflow.

Solution 3

We can solve the exponent problem by re-defining the spectrum as

$$X_n(e^{jw}) = \sum_{m=n-(N-1)}^{n} x(m)e^{-jw(m-n)}$$

which can be re-written as

$$X_n(e^{jw}) = e^{jwn} \sum_{m=n-(N-1)}^{n} x(m)e^{-jwm}$$

Now, since

$$X_{n-1}(e^{jw}) = e^{jw(n-1)} \sum_{m=n-(N-1)-1}^{n-1} x(m)e^{-jwm}$$

we can write

$$X_n(e^{jw}) = e^{jw} X_{n-1}(e^{jw}) + x(n) - x(n-N)e^{jwN}$$

Now the exponents are well-behaved, and we have N_M = 6 real multiplies.

Ex: Suppose that a signal x(n) is being monitored, where

$$x(n) = \cos(w_o(n)\cdot n + \phi(n)) + n(n)$$

and where n(n) represents noise. Here $w_o(n)$ denotes a frequency that is slowly changing with time.
(a) Indicate a method for calculating and updating a relevant feature vector, over a moving window of N time samples
(b) Give a method for estimating $w_o(n)$.

Solution: Given the number of features N_F, define evenly spaced frequencies as $w(k) = (\pi/(N_F-1))(k-1)$.

(a) For k between 1 and N_F , the kth complex feature X(k), in the feature vector **X**, is calculated and updated as

$X(k) = X_n(e^{jw(k)})$ on the previous page.

(b) At each time n, estimate w_o as:

Xmax = $|X(1)|$, $w_o = w(1)$

For $2 \le k \le N_F$
If($|X(k)| >$ Xmax)Then
Xmax = $|X(k)|$
$w_o = w(k)$
Endif
End