Introduction to 2D and 3D Projective Geometry

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Points and Lines in $\mathcal{P}^2$

- Points represented by: $\mathbf{x} = [x \;\; y \;\; 1]^{\mathbf{T}}$.
- Consider the line equation: $ax + by + c = 0$.
- $[a \;\; b \;\; c] \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathbf{l} \cdot \mathbf{x} = \mathbf{l}^{\mathbf{T}}\,\mathbf{x} = \mathbf{0}$, where $\mathbf{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$
- Lines are represeted by 3-vectors, just like points.
  Overall scale is unimportant.
- $$\mathbf{l}^{\mathbf{T}}\mathbf{x} = \mathbf{0}$$

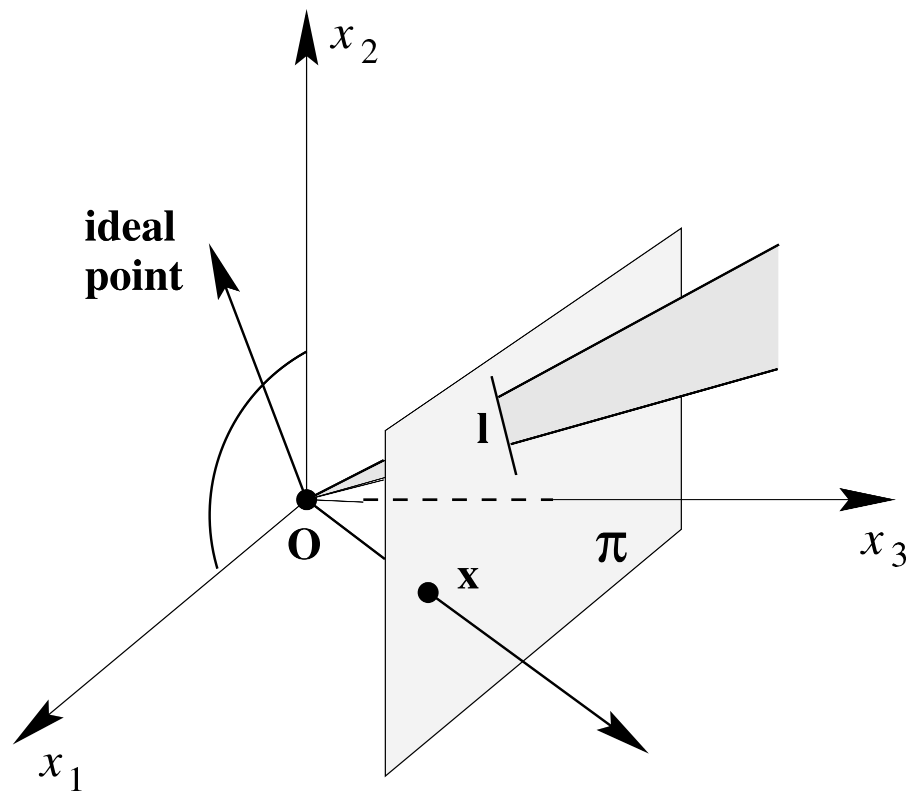
  Describes all points $\mathbf{x}$ incident on line $\mathbf{l}$
  Or all lines $\mathbf{l}$ passing through point $\mathbf{x}$.

Points at Infinity

- $\mathbf{x} = [x_1 \;\; x_2 \;\; x_3]^\mathbf{T}$ represents $(x_1/x_3, x_2/x_3)$. What happens when $x_3 \to 0$?
- Becomes **point at infinity** or **ideal point** or **vanishing point** in the direction (x_1, x_2)
- Points at infinity can be handled like any other in projective geometry.
- $[a \;\; b \;\; 0]^\mathbf{T}$ are all points at infinity on the plane. They together form a line at infinity.
- What is its representation? $\mathbf{l}_\infty = [0 \;\; 0 \;\; 1]^\mathbf{T}$.

View of Projective Representation

- $[x_1 \;\; x_2 \;\; x_3]^{\mathbf{T}}$ represent rays from origin in a 3-space.
- Any cross section perpendicular to the x_3 axis can describe the plane.



- Ideal points are rays on the $x_3 = 0$ plane.
- Lines are planes passing through the origin.
- Line at infinity l_∞ corresponds to $x_3 = 0$

Line Joining 2 Points

- Let $\mathbf{x}, \mathbf{y}$ be the points. We have: $\mathbf{l}^{\mathbf{T}}\mathbf{x} = \mathbf{l}^{\mathbf{T}}\mathbf{y} = \mathbf{0}$.
- Long route: Equation is: $y = y_1 + (x - x_1)\frac{y_2 - y_1}{x_2 - x_1}$.
- Or: $(y_2 - y_1)x - (x_2 - x_1)y + (x_2 y_1 - x_1 y_2) = 0$
- Line $\mathbf{l} = [(y_2 - y_1) \quad -(x_2 - x_1) \quad (x_2 y_1 - x_1 y_2)]^{\mathbf{T}}$.

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- Considering them as vectors in 3-space, we want to find a vector $\mathbf{L}$ orthogonal to both $\mathbf{P}$ and $\mathbf{Q}$.
- The cross-product $\mathbf{x} \times \mathbf{y}$ is a solution. Thus, $\mathbf{l} = \mathbf{x} \times \mathbf{y}$.

- $\mathbf{x} \times \mathbf{y} = [(y_2 - y_1) \quad -(x_2 - x_1) \quad (x_2 y_1 - x_1 y_2)]^{\mathbf{T}}$.

Examples

- Line through $(5,2),(3,2)$: $\begin{bmatrix} i & j & k \\ 5 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

  That is the line $y = 2$.
- Ideal point of line $[0\ \ 1\ \ -2]^{\mathbf{T}}$ is $[1\ \ 0\ \ 0]^{\mathbf{T}}$, same as that of $[0\ \ 1\ \ k]^{\mathbf{T}}$ for any k.
- Line joining $[3\ \ 4\ \ 0]^{\mathbf{T}}$ and $[2\ \ 3\ \ 0]^{\mathbf{T}}$ is: $[0\ \ 0\ \ 1]^{\mathbf{T}}$, or $\mathbf{l}_\infty$.

Point of Intersection of 2 Lines

- Two lines $\mathbf{l}, \mathbf{m}$ intersect in a point with $\mathbf{l}^{\mathbf{T}}\mathbf{x} = \mathbf{m}^{\mathbf{T}}\mathbf{x} = \mathbf{0}$.
- $\mathbf{x} = \mathbf{l} \times \mathbf{m}$.
- $y = -(a_1 x + c_1)/b_1$. And, $a_2 x - b_2(a_1 x + c_1)/b_1 + c_2 = 0$.
- $x = (b_2 c_1 - b_1 c_2)/(a_2 b_1 - a_1 b_2)$.
  $y = (a_1 c_2 - a_2 c_1)/(a_2 b_1 - a_1 b_2)$.
- $\mathbf{x} = [(b_2 c_1 - b_1 c_2) \quad (a_1 c_2 - a_2 c_1) \quad (a_2 b_1 - a_1 b_2)]^{\mathbf{T}} = \mathbf{l} \times \mathbf{m}$
- Duality at work: points and lines are interchangeable

Examples

- Intersection of $x = 1$ and $y = 2$: $\begin{bmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

  Same as $(1, 2)$.

- Intersection of $x = 1$ and $x = 2$: $\begin{bmatrix} i & j & k \\ 1 & 0 & -1 \\ 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

- Ideal point of line $\mathbf{l} = [a \;\; b \;\; c]^{\mathbf{T}}$ is $[b \;\; -a \;\; 0]^{\mathbf{T}} = \mathbf{l} \times \mathbf{l}_\infty$, the intersection of $\mathbf{l}$ with line at infinity!

Conics: Second Order Entities

- General quadratic entity:
  $ax^2 + bxy + cy^2 + dx + ey + f = 0$.
- Rewrite using homogeneous coordinates as:
  $ax^2 + bxy + cy^2 + dxw + eyw + fw^2 = 0$.
- Rewrite as: $[x \;\; y \;\; w] \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$
- A symmetric $\mathbf{C}$ represents a conic: $\mathbf{x}^{\mathbf{T}}\mathbf{C}\mathbf{x} = \mathbf{0}$.
  Covers circle, ellipse, parabola, hyperbola, etc.
- Degenerate conics include line ($a = b = c = 0$) and two lines when $\mathbf{C} = \mathbf{l}\mathbf{m}^{\mathbf{T}} + \mathbf{m}\mathbf{l}^{\mathbf{T}}$.

Properties of Conics

- $\mathbf{C}\mathbf{x}$ gives the tangent line to the conic at $\mathbf{x}$.

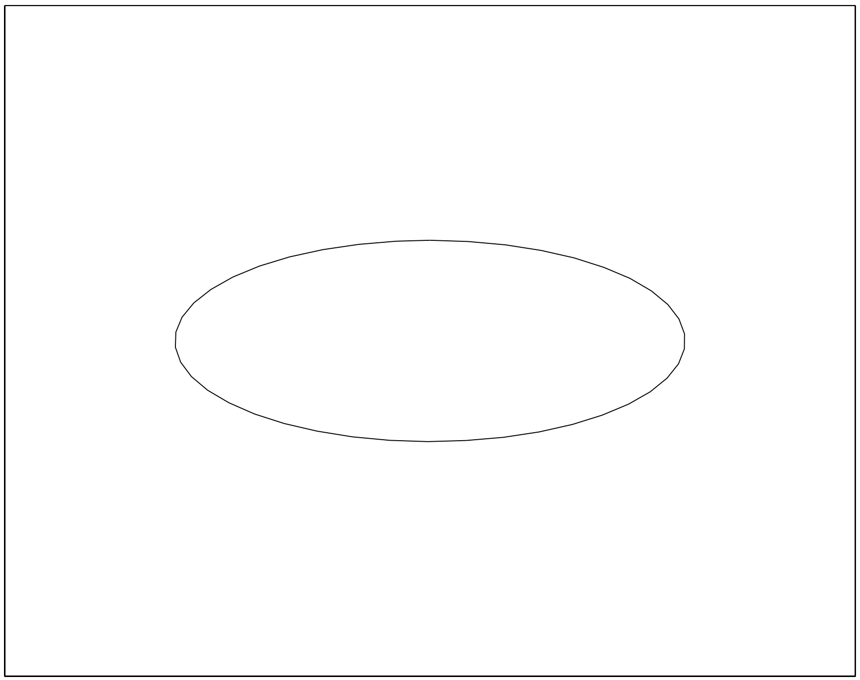
  A point $\mathbf{x}$ on the conic is on line $\mathbf{l} = \mathbf{C}\mathbf{x}$ as $\mathbf{x}^{\mathbf{T}}\ (\mathbf{C}\mathbf{x}) = \mathbf{0}$. If $\mathbf{l}$ intersects the conic in another point $\mathbf{y}$, $\mathbf{y}^{\mathbf{T}}\mathbf{C}\mathbf{y} = \mathbf{0}$ due to the conic and $(\mathbf{C}\mathbf{x})^{\mathbf{T}}\mathbf{y} = \mathbf{x}^{\mathbf{T}}\mathbf{C}\mathbf{y} = \mathbf{0}$ due to line. Thus, $\mathbf{C}\mathbf{y}$ is a line joining $\mathbf{x}$ and $\mathbf{y}$. That is $\mathbf{C}\mathbf{y} = \mathbf{C}\mathbf{x}$ or $\mathbf{x} = \mathbf{y}$

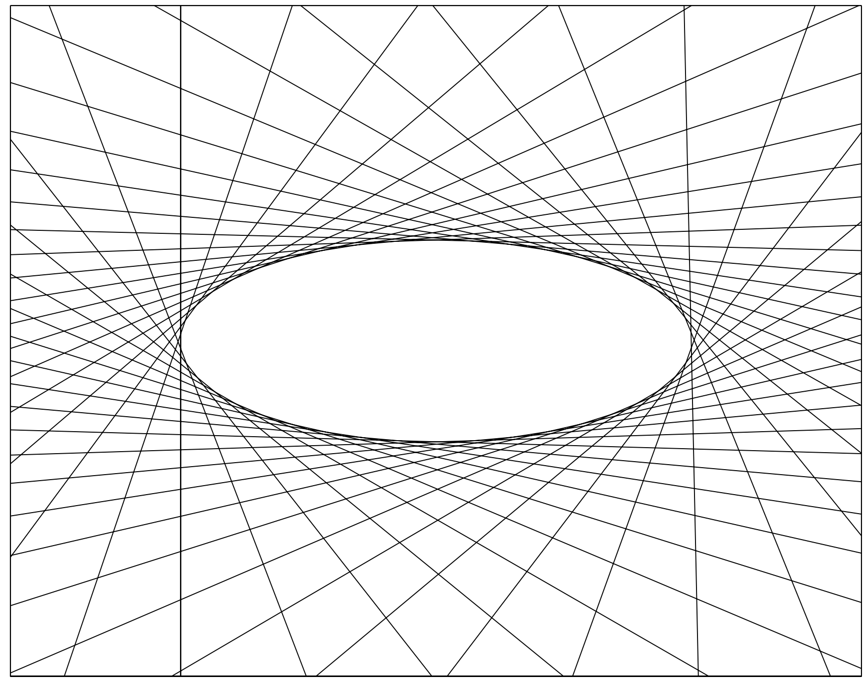
- Dual conic: conic defined by its tangent lines. Given by $\mathbf{l}^{\mathbf{T}}\ \mathbf{C}^*\ \mathbf{l} = \mathbf{0}$. Generally, the adjoint matrix $\mathbf{C}^*$ or inverse $\mathbf{C}^{\text{-}1}$ will work.

- Point of tangency of $\mathbf{l}$ and $\mathbf{C}$ is given $\mathbf{C}^{-\mathbf{T}}\ \mathbf{l} = \mathbf{C}^{-1}\ \mathbf{l}$ due to symmetry.

  Since $\mathbf{l}^{\mathbf{T}}\ \mathbf{C}^{-1}\ \mathbf{l} = \mathbf{0}$, the point $\mathbf{x} = \mathbf{C}^{-1}\mathbf{l}$ is on line $\mathbf{l}$. It is also on the conic as:
  $\mathbf{x}^{\mathbf{T}}\mathbf{C}\mathbf{x} = (\mathbf{C}^{-1}\mathbf{l})^{\mathbf{T}}\ \ \mathbf{C}\ \ (\mathbf{C}^{-1}\mathbf{l}) = \mathbf{l}^{\mathbf{T}}\mathbf{C}^{-\mathbf{T}}\ \ (\mathbf{C}\mathbf{C}^{-1})\ \ \mathbf{l} = \mathbf{l}^{\mathbf{T}}\ \ \mathbf{C}^{-1}\ \ \mathbf{l} = \mathbf{0}$

Point and Line Conics





Projective Transformations

- A general non-singular 3×3 matrix $\mathbf{H}$ transforms points to other points. Overall scale of $\mathbf{H}$ is unimportant.
- $\mathbf{x}' = \mathbf{H}\,\mathbf{x}$ gives the transformed point.
- $\mathbf{l}' = \mathbf{H}^{-\mathbf{T}}\,\mathbf{l}$ gives the transformed line.
- $\mathbf{C}' = \mathbf{H}^{-\mathbf{T}}\,\mathbf{C}\,\mathbf{H}^{-\mathbf{1}}$ is the transformed conic.
- $\mathbf{C}^{*\prime} = \mathbf{H}\,\mathbf{C}^{*}\,\mathbf{H}^{\mathbf{T}}$ is the transformed dual conic.
- Linearity is preserved. $\mathbf{p}', \mathbf{q}', \mathbf{r}'$ collinear if $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are. In fact, that is the basic definition of the basic **projectivity** tranformation.
- Such a transformation is called:
  collineation, homography, projective transformation.

Isometric Transformation

- Transformations of the form, with $\delta = \pm 1$:

$$\begin{bmatrix} \delta \cos\theta & -\sin\theta & a \\ \delta \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{bmatrix}$$

- Includes rotations, translations, reflections.
- Called *Euclidean* and *rigid* transformations.
- Preserves distance measurements, angles, parallelism, etc.

Similarity Transformations

- Transformations of the form for nonzero s:

$$\begin{bmatrix} s\cos\theta & -s\sin\theta & a \\ s\sin\theta & s\cos\theta & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$$

- Includes rotations, translations, uniform scaling
- Preserves angles, parallelism, ratio of distances, ratio of areas, circular points $\mathbf{I}$, $\mathbf{J}$
- 4 degrees of freedom; needs 2 point matches to estimate
- Geometric structure that is defined upto an unknown similarity transformation is called **metric structure**

Affine Transformations

- Transformations of the form: $\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$
- Includes rotations, translations, nonuniform scaling, shearing, etc.
- Preserves parallelism, ratio of lengths of parallel lines, ratio of areas, centroid, $\mathbf{l}_\infty$
- 6 degrees of freedom; needs 3 point matches
- Points at infinity map to other points at infinity

$$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix}$$

Projective Transformation

- Any general matrix $\mathbf{H}$, a general transformation.

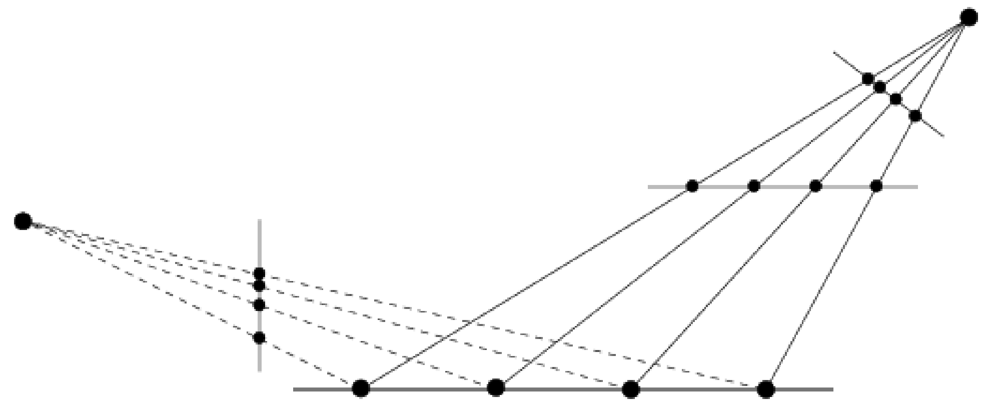
$$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^{\mathbf{T}} & 1 \end{bmatrix} = \mathbf{H_P H_A H_S} = \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{v}^{\mathbf{T}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & 0 \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$$

where $\mathbf{K}$ is upper triangular with determinant 1

- Finite points can map to ideal points and vice versa.
- The impact of even slight projectivity is serious and non-intuitive. Yet, it models a pin-hole camera.
- Doesn't preserve parallelism, lengths, angles, or ratios of lengths. But, preserves cross-ratios.
- 8 degrees of freedom; needs 4 point matches

Cross-Ratios on a Line

Consider 4 points $\mathbf{X}_i, i = 1 \cdots 4$ on a line and its different projections $\mathbf{x}_i$



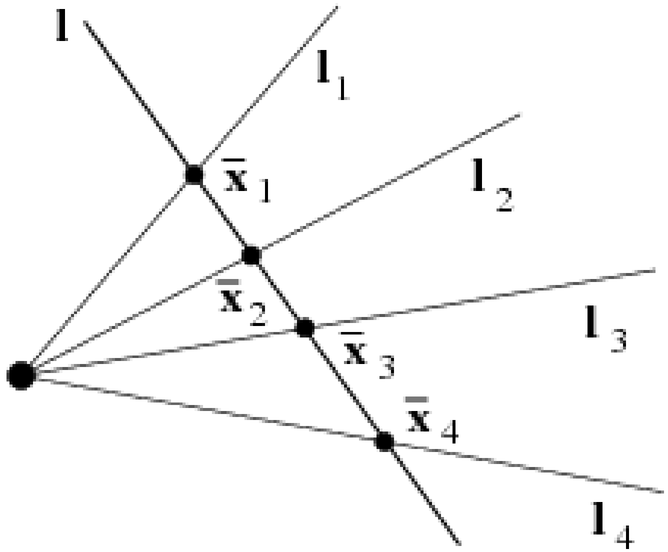
Cross ratio of 4 points defined as:

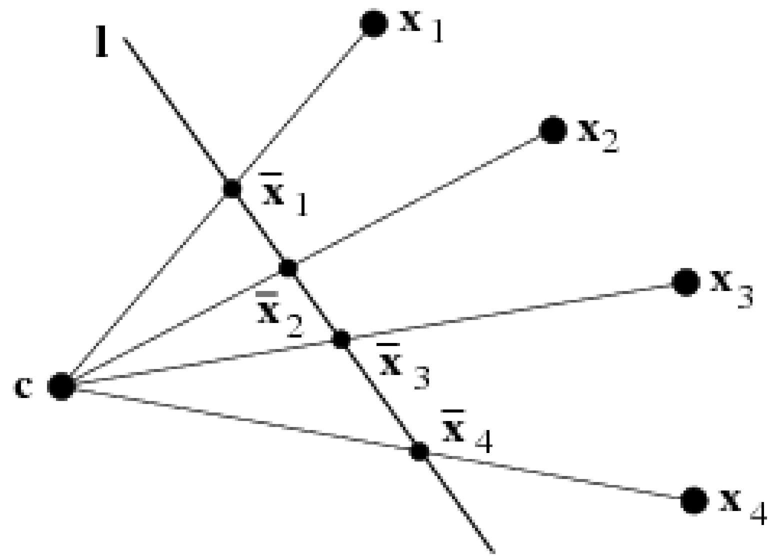
$$\text{Cross}(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x_4}) = \frac{|\mathbf{x_1 x_2}||\mathbf{x_3 x_4}|}{|\mathbf{x_1 x_3}||\mathbf{x_2 x_4}|}, \quad \text{with } |\mathbf{xy}| = \det \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

$|\mathbf{xy}|$ is the 1D signed distance along the line.

Homogeneous scale factors cancel each other. Hence cross-ratio is preserved under **any** projective transformation and can be measured in any projection.

Concurrent Lines





- For 4 concurrent co-planar lines, cross ratio of 4 points $\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4$ measured on any line is constant.
- For 4 coplanar points and given a projection centre in the plane, Cross$(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4)$ on any line is constant.

Invariants for Different Types

Property	Euclidean	Similarity	Affine	Projective
Length				
Angle				
Length ratio				
Area ratio				
Parallelism				
Centroid				
Ratio of len ratio				
Collinearity				

Invariants for Different Types

Property	Euclidean	Similarity	Affine	Projective
Length	**Yes**	**No**	**No**	**No**
Angle	**Yes**	**Yes**	**No**	**No**
Length ratio	**Yes**	**Yes**	**No**	**No**
Area ratio	**Yes**	**Yes**	**Yes**	**No**
Parallelism	**Yes**	**Yes**	**Yes**	**No**
Centroid	**Yes**	**Yes**	**Yes**	**No**
Ratio of len ratio	**Yes**	**Yes**	**Yes**	**Yes**
Collinearity	**Yes**	**Yes**	**Yes**	**Yes**

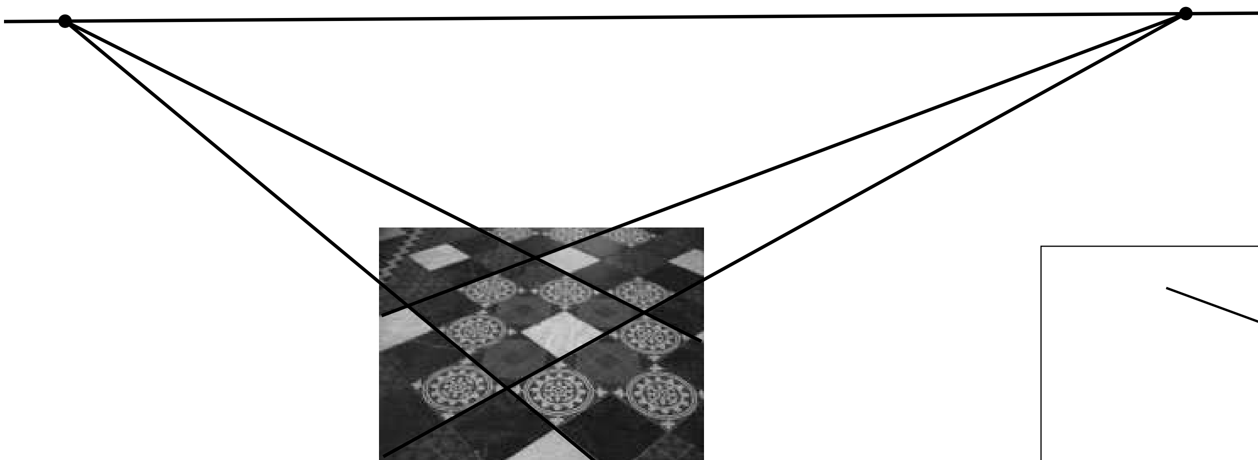
Line at Infinity

- Affinity maps $\mathbf{l}_\infty$ to itself. Conversely, any transformation that does it is an affine one
- General projectivity can map $\mathbf{l}_\infty$ to a finite line and vice versa
- A circle intersects $\mathbf{l}_\infty$ at **circular points**. Canonical (Euclidean) circle is: $x^2 + y^2 + dxw + eyw + fw^2 = 0$.
- Points on $\mathbf{l}_\infty$ have $w = 0$. Thus, $x^2 + y^2 = 0$.
- Circular points are given canonically by:

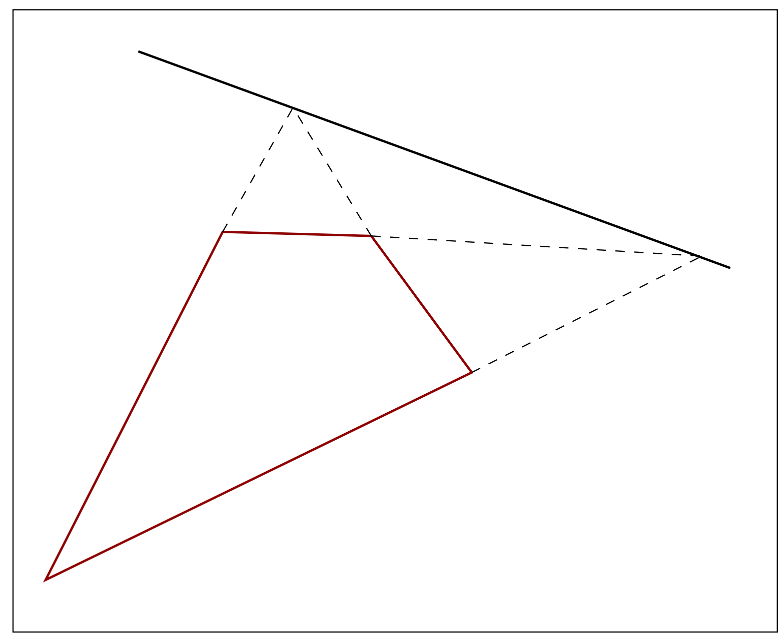
$$I = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

Finding Line at Infinity

Line at infinity can be found in the image from 2 sets of parallel lines.



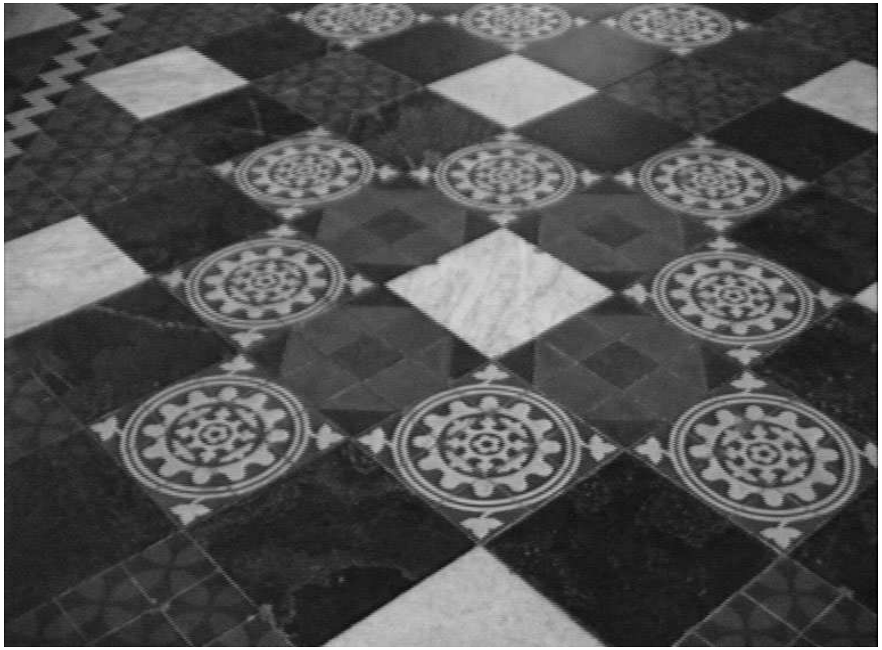
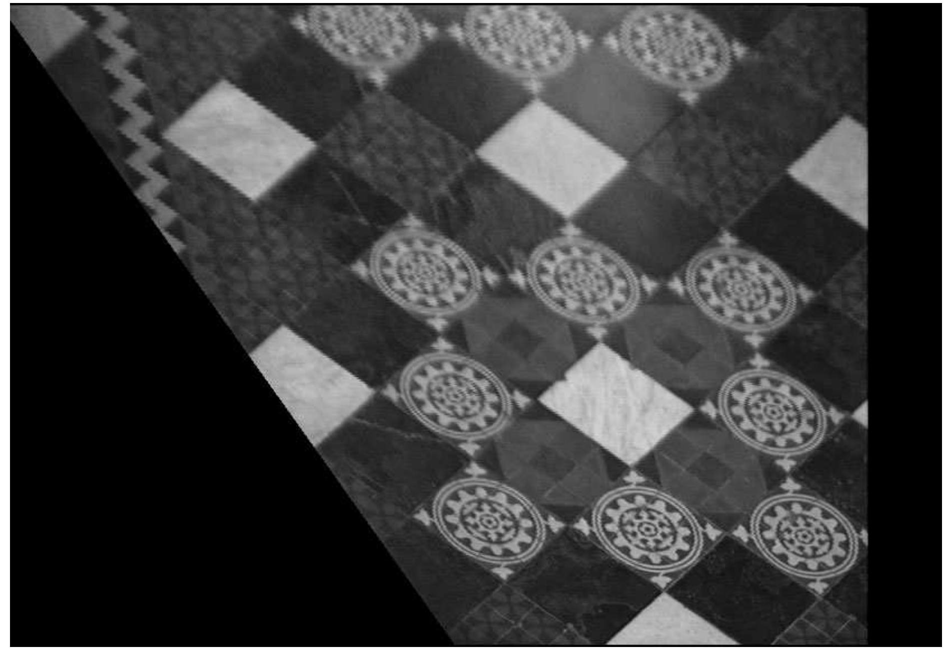
Rectangle imaged as:



Affine Structure from Images

- Affine structure gives parallelism, ratio of areas, centroid, etc., and can be the basis of many decisions.
- Find $\mathbf{l}_\infty$ in image using parallel lines.
- Apply a transformation $\mathbf{H}$ that maps the line to $[0\ 0\ 1]^\mathbf{T}$
- Any $\mathbf{H} = \mathbf{H_A} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$ sends $\begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$ to $[0\ 0\ 1]^\mathbf{T}$ as $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}^{-\mathbf{T}} \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = \begin{bmatrix} l_3 & 0 & -l_1 \\ 0 & l_3 & -l_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Affine Rectification

Parallel lines are parallel, but right angles are not right angles.

Circular Points & Similarity

- Circular points are fixed under similarity

$$\begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} = s(\cos\theta + i\sin\theta) \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

- Conversely, any transformation that fixes circular points is a similarity.
- Thus, a transformation $\mathbf{H}$ that sends the circular points to their canonical form $\mathbf{I}$ and $\mathbf{J}$ leaves only a similarity transformation.

Dual Conic to Circular Points

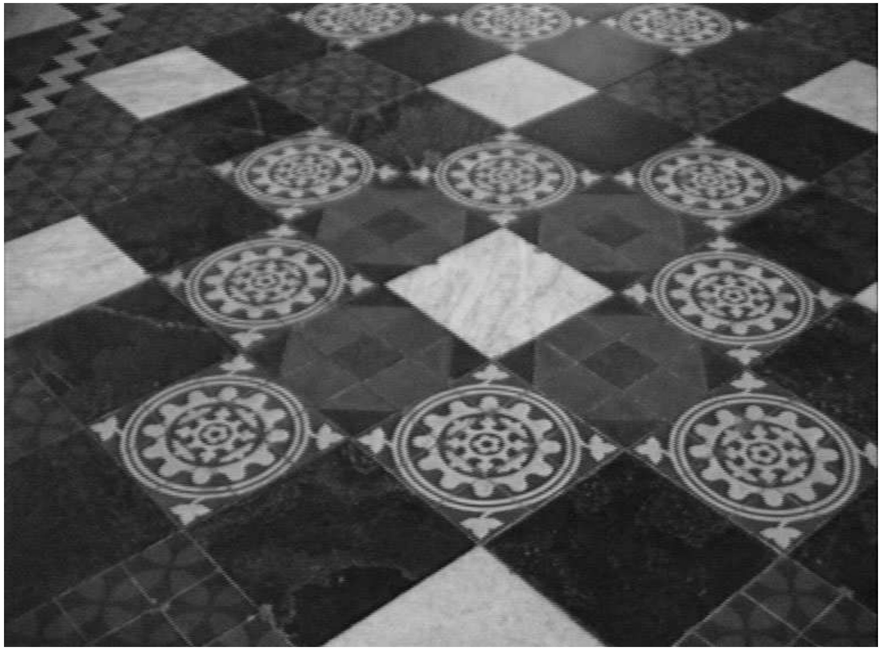
- $\mathbf{C}^*_\infty = \mathbf{I}\mathbf{J}^\mathbf{T} + \mathbf{J}\mathbf{I}^\mathbf{T}$ is a dual conic defined by the circular points. It is fixed under similarity also.
- In canonical or Euclidean frame, $\mathbf{C}^*_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
- We can see

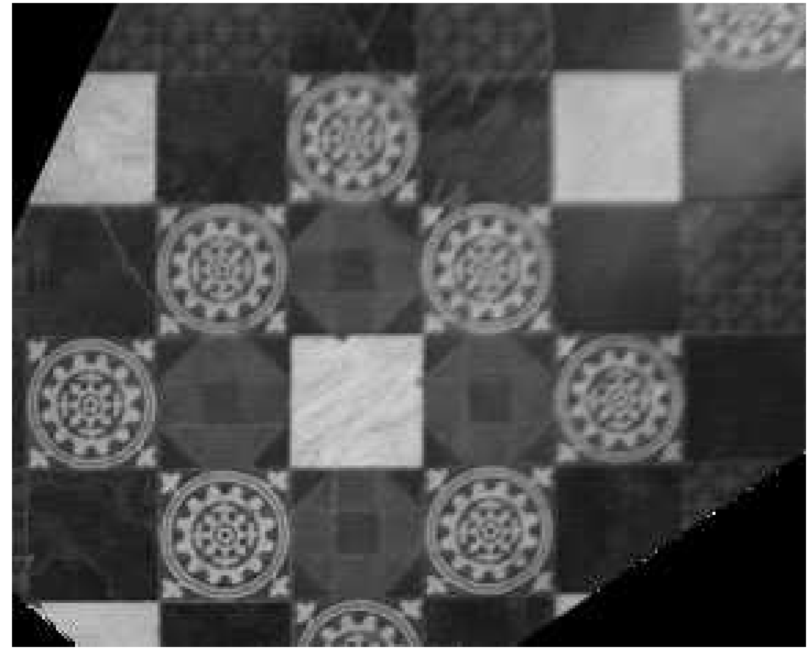
$$\begin{bmatrix} s\mathbf{R}^\mathbf{T} & \mathbf{0} \\ \mathbf{t}^\mathbf{T} & 1 \end{bmatrix} \mathbf{C}^*_\infty \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} s^2 & 0 & 0 \\ 0 & s^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \equiv \mathbf{C}^*_\infty$$

Metric Structure from Images

- Identify circular points in the image.
- This can be done by finding a world circle in the image as a conic, finding the $\mathbf{l}_\infty$ in the image and finding their intersection
- Map one circular point to $\mathbf{I}$ and the other to $\mathbf{J}$. The transformation $\mathbf{H}$ that does it metric rectifies the image.
- $\mathbf{l}_\infty$ gives affine structure, the circle gives metric structure.
- Can be done using 2 non-parallel orthogonal line pairs instead of a circle or 5 orthogonal line pairs from projective!

Metric Rectification





Angles in Projective Space

- Angle between Euclidean lines $\mathbf{l}$ and $\mathbf{m}$ is given by $\cos\theta = \mathbf{l} \cdot \mathbf{m}$
- Analog in projective space involves $\mathbf{C}^*_\infty$.

$$\cos\theta = \frac{\mathbf{l}^\mathbf{T}\mathbf{C}^*_\infty\mathbf{m}}{\sqrt{\mathbf{l}^\mathbf{T}\mathbf{C}^*_\infty\mathbf{l}}\sqrt{\mathbf{m}^\mathbf{T}\mathbf{C}^*_\infty\mathbf{m}}}$$

is invariant in projective transformations!

- $\cos\theta = 0$ if $\mathbf{l}$ and $\mathbf{m}$ are othogonal.

3D Projective Geometry

- Points represented using 4-vectors $\mathbf{X} = [x_1 \; x_2 \; x_3 \; x_4]^{\mathbf{T}}$
- Planes also represented by 4-vectors, such that $\pi^{\mathbf{T}}\mathbf{X} = \mathbf{0}$ is the plane equation.
- Lines represented using two 4-vectors: line joining two points or intersection of two plans.
- Quadrics are general order-2 surfaces: $\mathbf{X}^{\mathbf{T}}\mathbf{Q}\mathbf{X} = \mathbf{0}$
- Lines have points at infinity. Points at infinity on a plane lie on the line at infinity for it. All points/lines at infinity lie on the **plane at infinity** denoted by π_∞

Hierarchy of Transformations

- Plane at infinity π_∞ is **fixed** for, and only for, **affine** transformations.
- The **absolute conic** Ω_∞ on π_∞ is the intersection of all spheres with π_∞. In canonical coordinates, $x_1^2 + x_2^2 + x_3^2 = 0 = x_4$. Thus, $\mathbf{\Omega}_\infty = \mathbf{I}$ on π_∞.
- **Absolute dual quadric** $\mathbf{Q}_\infty^*$ is defined using planes. Planes enveloping it meet π_∞ in lines that are tangent to Ω_∞. It is given in Euclidean/canonical coords as $\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}^\mathbf{T} & 0 \end{bmatrix}$
- $\mathbf{Q}_\infty^*$ and $\mathbf{\Omega}_\infty$ are **fixed** under **similarity** transformations.

Calibrated Camera

- Internal matrix $\mathbf{K}$ is known. In camera's Euclidean frame, a point $\mathbf{x}$ maps to a line $\lambda\mathbf{d}$ where $\mathbf{d}$ is the direction.
- Then, $\mathbf{x} = \mathbf{K}\mathbf{d}$ upto scale. Or $\mathbf{d} = \mathbf{K}^{-1}\mathbf{x}$.
- Given $\mathbf{d_1}, \mathbf{d_2}$ corresponding to points $\mathbf{x_1}, \mathbf{x_2}$, the angle between the rays is:

$$\cos\theta = \frac{\mathbf{x_1^T K^{-T} K^{-1} x_2}}{\sqrt{\mathbf{x_1^T (K K^T)^{-1} x_1}}\sqrt{\mathbf{x_1^T (K K^T)^{-1} x_2}}}$$

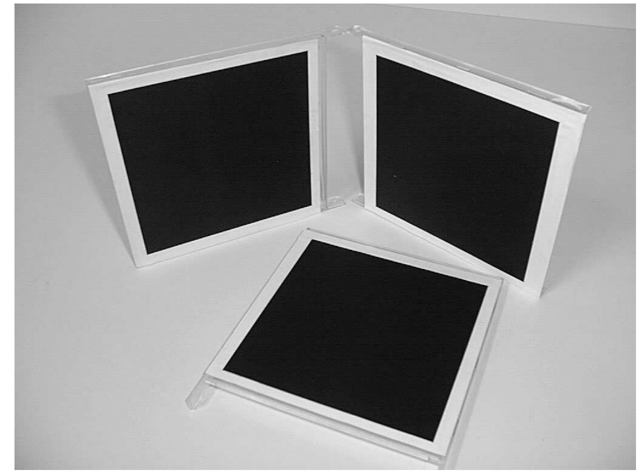
- An image line $\mathbf{l}$ defines a world plane in camera's Euclidean frame given by $\mathbf{n} = [(\mathbf{K^T l})^\mathbf{T} \;\; \mathbf{0}]^\mathbf{T}$.

(Image points $\mathbf{x}$ on $\mathbf{l}$ project to directions $\mathbf{d} = \mathbf{K}^{-1}\mathbf{x}$. Points on the plane are $[\mathbf{d}^\mathbf{T} \;\; w]^\mathbf{T}$. The plane through origin $[\mathbf{n}^\mathbf{T} \;\; \mathbf{0}]^\mathbf{T}$ contains these points if $\mathbf{n^T d} = \mathbf{0}$ or $\mathbf{n^T K^{-1} x} = \mathbf{0}$. Since $\mathbf{l^T x} = \mathbf{0}$, this is true if $\mathbf{n} = \mathbf{K^T l}$)

Image of the Absolute Conic

- A camera projects points $[\mathbf{d} \;\; \mathbf{0}]^{\mathbf{T}}$ on π_∞ to $\mathbf{x} = \mathbf{KRd}$. Thus, $\mathbf{KR}$ is the homograhy from π_∞ to the image.
- Image of the Absolute Conic (IAC) is given by: $\omega = (\mathbf{KK}^{\mathbf{T}})^{-\mathbf{1}} = \mathbf{K}^{-\mathbf{T}}\mathbf{K}^{-\mathbf{1}}$. Dual-IAC is: $\omega^* = KK^{\mathbf{T}}$
  $\omega = \mathbf{H}^{-\mathbf{T}}\mathbf{\Omega}_\infty\mathbf{H}^{-\mathbf{1}} = (\mathbf{KR})^{-\mathbf{T}}\mathbf{I}(\mathbf{KR})^{-\mathbf{1}} = \mathbf{K}^{-\mathbf{T}}\mathbf{RR}^{-\mathbf{1}}\mathbf{K}^{-\mathbf{1}}$ as $\mathbf{\Omega}_\infty = \mathbf{I}$ on π_∞.
- Properties of IAC ω
  - ω depends only on $\mathbf{K}$ and not on $\mathbf{R}, \mathbf{t}$.
  - Angle between two rays: $\cos\theta = \mathbf{x}_\mathbf{1}^{\mathbf{T}}\omega\mathbf{x}_\mathbf{2}/(\cdots)$
  - Two rays are othogonal iff $\mathbf{x}_\mathbf{1}^{\mathbf{T}}\omega\mathbf{x}_\mathbf{2} = \mathbf{0}$.
  - We can calibrate the camera given ω, using Cholesky factorization.
  - Circular points of all planes lie on Ω_∞ and are imaged onto intersection of plane's $\mathbf{l}_\infty$ and ω

Camera Calibration Using IAC



- Consider 3 non-parallel planes with squares. Assume the corners are $(0,0),(0,1),(1,1),(1,0)$.
- Compute homography $\mathbf{H}$ for each plane to the image plane.
- Imaged circular points $\mathbf{h}_1 \pm i\mathbf{h}_2$ are on ω; hence $\mathbf{x}^\mathbf{T}\omega\mathbf{x} = \mathbf{0}$
- Two equations from each plane (real/imaginary parts): $\mathbf{h_1^T}\omega\mathbf{h_2} = \mathbf{0}$ and $h_1^\mathbf{T}\omega h_1 = h_2^\mathbf{T}\omega h_2$. The conic ω has 5 dof. We need 5 equations and hence 3 planes.
- Factorize $\omega = (\mathbf{K}\mathbf{K}^\mathbf{T})^{-\mathbf{1}}$ to get $\mathbf{K}$.
- Image line $\omega\mathbf{x}$ corresponds to a world plane orthogonal to ray of $\mathbf{x}$.

Line $\omega\mathbf{x}$ is tangent to ω at $\mathbf{x}$. If $\mathbf{y}$ is a point on the line, $\mathbf{y}^\mathbf{T}\omega\mathbf{x} = \mathbf{0}$, or rays of $\mathbf{x}, \mathbf{y}$ are orthogonal. Since this is true for any $\mathbf{y}$, the plane of $\omega\mathbf{x}$ is orthognal to the ray of $\mathbf{x}$.

Thank You!

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Many figures are from the book

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