Solutions for Assignment 1

May 22, 2012

Q1

Points on a plane satisfy:

$$XN_x + YN_y + ZN_z = d \tag{1}$$

$$\Rightarrow xN_x + yN_y + N_z = \frac{d}{Z} \to 0 \text{ as } Z \to \infty \tag{2}$$

Thus the equation of the vanishing line of the plane is:

$$xN_x + yN_y + N_z = 0 \tag{3}$$

Points on a line are of the form $\mathbf{A} + \lambda\mathbf{D}$. Their projection is of the form $(\frac{A_x+\lambda D_x}{A_z+\lambda D_z}, \frac{A_y+\lambda D_y}{A_z+\lambda D_z})$. As λ approaches infinity, this becomes $(\frac{D_x}{D_z}, \frac{D_y}{D_z})$ which is the vanishing point of the line.

Since each point on the line also lies on the plane, we have $\mathbf{A}\cdot\mathbf{N}+\lambda\mathbf{D}\cdot\mathbf{N} = d$ for all λ. This implies:

$$\mathbf{D} \cdot \mathbf{N} = 0 \tag{4}$$

$$\Rightarrow \frac{D_x}{D_z}N_x + \frac{D_y}{D_z}N_y + N_z = 0 \tag{5}$$

which in turn implies that the vanishing point of the line $(\frac{D_x}{D_z}, \frac{D_y}{D_z})$ lies on the vanishing line of the plane.

Q2

As shown in Figure 1, the image plane cuts the cone of rays from the optical centre to the object, resulting in a conic section. The eccentricity of the figure formed in the image plane is thus the eccentricity of the conic section.

The eccentricity of the conic section can be estimated from the cosines of the angles α and β, shown in Figure 2. For the half angle β note that:

$$\sin\beta = \frac{r}{\sqrt{X^2+Z^2}} \Rightarrow \cos\beta = \frac{\sqrt{X^2+Z^2-r^2}}{\sqrt{X^2+Z^2}} \tag{6}$$

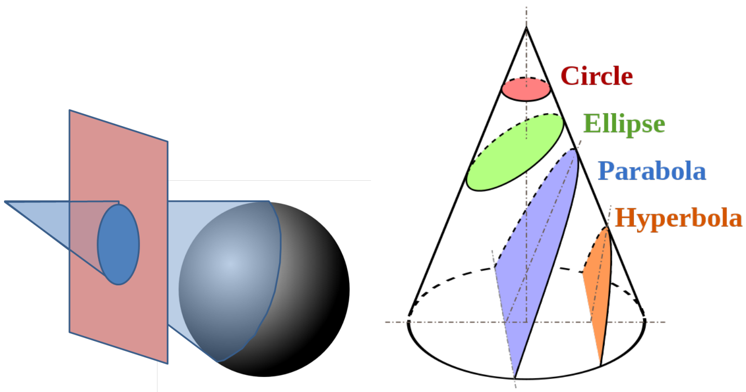

Figure 1: The setup in Q2. Note that the image plane cuts the cone of rays from the optical center to the sphere. The image formed on the image plane is therefore a “conic section”. The different kinds of conic sections are also shown (http://commons.wikimedia.org/wiki/File:Secciones_c%C3%B3nicas.svg)

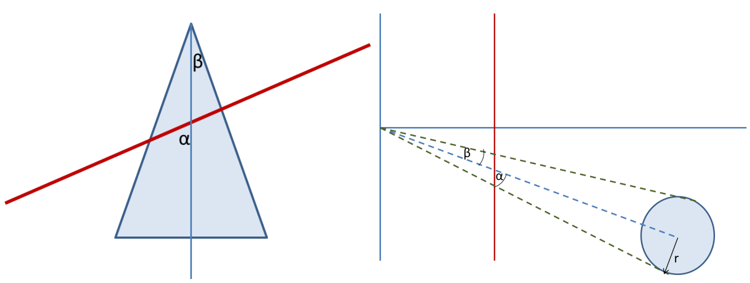

Figure 2: The eccentricity of the conic section depends on the half -angle of the cone β, and the angle the plane makes with the axis of the cone α. The eccentricity is given by $\frac{\cos\alpha}{\cos\beta}$. On the right these angles are shown in the specific context of Q2. The figure shows the top view, so that the vertical axis is the X axis and the horizontal axis is the Z axis.

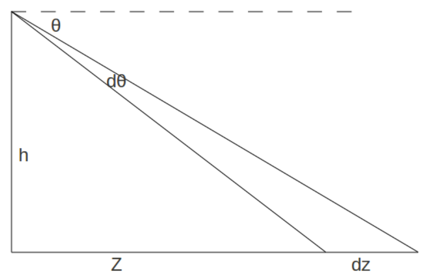

Figure 3: Figure for Q 3

The angle α is the same as the angle made by the axis of the cone with the XY plane. The axis of the cone is the line joining the optical center with the center of the sphere $(X, 0, Z)$. This angle can be easily calculated to be given by:

$$\cos\alpha = \frac{X}{\sqrt{X^2 + Z^2}} \tag{7}$$

The eccentricity is therefore $\frac{\cos\alpha}{\cos\beta} = \frac{X}{\sqrt{X^2+Z^2-r^2}}$.

The eccentricity of the parabola is equal to 1, while the eccentricity of a hyperbola is greater than 1. For the eccentricity to be equal to 1, we will want that $Z^2 - r^2 = 0 \Rightarrow Z = r$. For the eccentricity to be greater than 1, we will similarly want that $Z < r$.

In spherical projection, note that the intersection of a sphere with a cone whose vertex is at the center of the sphere is always a circle.

Q3

Figure 3 shows the setup. When the object moves back a distance dZ, the angle subtended at the eye changes by $d\theta$. We need to find the value of dZ for which $d\theta$ falls below 1 minute (= 1/60 degrees).

The crucial observation we make is the relationship between θ and Z: $\tan\theta = \frac{h}{Z}$. This allows us to find the dZ for which θ changes by 1 minute. Fortunately, this change is so small that we can use a differential approximation:

$$\tan\theta = \frac{h}{Z} \tag{8}$$

$$\Rightarrow \sec^2\theta d\theta = -\frac{h}{Z^2} dZ \tag{9}$$

$$\Rightarrow dZ = -\frac{h^2 + Z^2}{h} d\theta \tag{10}$$

Q4

An important thing to note here is that the matrix describing reflection about the $\theta = \beta$ line is given by $\begin{bmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{bmatrix}$. In other words, the matrix contains 2β and not β. A simple way to verify this is to look at the image of the point $(1, 0)$. This point lies at an angle β from the $\theta = \beta$ line, and so its image should also lie at an angle β from the $\theta = \beta$ line.

Once we have the reflection matrix, simple matrix multiplication in conjunction with trigonometric identities ($\sin(A-B) = \sin A \cos B - \cos A \sin B$, $\cos(A-B) = \cos A \cos B + \sin A \sin B$) gives us what we need:

$$\begin{bmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{bmatrix} \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix} = \begin{bmatrix} \cos 2(\beta-\alpha) & -\sin 2(\beta-\alpha) \\ \sin 2(\beta-\alpha) & \cos 2(\beta-\alpha) \end{bmatrix} \tag{11}$$

Q5

We need to show that $e^{\phi\hat{s}} = \mathbf{I} + \sin\phi\hat{s} + (1-\cos\phi)\hat{s}^2$. This is easy once we know the Taylor expansions of exponential, sin and cos:

$$e^{\mathbf{X}} = \mathbf{I} + \mathbf{X} + \frac{\mathbf{X}^2}{2!} + \frac{\mathbf{X}^3}{3!} + \ldots \tag{12}$$

$$\sin\phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + \ldots \tag{13}$$

$$\cos\phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} + \ldots \tag{14}$$

Next, verify that $\hat{s}^2 = ss^T - \mathbf{I}$, so that $\hat{s}^3 = -\hat{s}, \hat{s}^4 = -\hat{s}^2$ and so on. Thus the even powers give $\hat{s}^2$ with alternating signs and the odd powers give $\hat{s}$ with alternating signs. Collecting these two groups separately, we get:

$$e^{\phi\hat{s}} = \mathbf{I} + \hat{s}(\phi - \frac{\phi^3}{3!} + \ldots) + \hat{s}^2(\frac{\phi^2}{2!} - \frac{\phi^4}{4!} + \ldots) \tag{15}$$

$$= \mathbf{I} + \hat{s}\sin\phi + \hat{s}^2(1-\cos\phi) \tag{16}$$

Q6

Rodriguez' formula gives us:

$$\mathbf{R} = \mathbf{I} + \hat{s}\sin\phi + \hat{s}^2(1-\cos\phi) \tag{17}$$

Use the fact (proved above) that $\hat{s}^2 = ss^T - \mathbf{I}$, so:

$$\mathbf{R} = \hat{s}\sin\phi + ss^T(1-\cos\phi) + \mathbf{I}\cos\phi \tag{18}$$

Take trace on both sides, and use the fact that the trace of a skew symmetric matrix is 0, while the trace of ss^T is 1, to get:

$$\text{trace}(\mathbf{R}) = (1 - \cos\phi) + 3\cos\phi \tag{19}$$

$$\Rightarrow \cos\phi = \frac{1}{2}(\text{trace}(\mathbf{R}) - 1) \tag{20}$$

Points on the axis are unaffected by the rotation. Thus, the axis is an eigenvector with eigenvalue 1. The other two eigenvalues give $\cos\phi + i\sin\phi$ and its conjugate.

Q7

$$\mathbf{R} - \mathbf{R}^T = \sin\phi(\hat{s} - \hat{s}^T) + (1 - \cos\phi)(\hat{s}^2 - \hat{s}^{2T}) \tag{21}$$

$\hat{s}$ is skew-symmetric and so $\hat{s} - \hat{s}^T = 2\hat{s}$. We know from a previous question that $\hat{s}^2 = ss^T - I$ and is therefore symmetric. Thus

$$\mathbf{R} - \mathbf{R}^T = 2\sin\phi\hat{s} \tag{22}$$

Let $\mathbf{A} = \mathbf{R} - \mathbf{R}^T$. The vector $[A_{32}, -A_{31}, A_{21}]$ equals $2\sin\phi s$. Normalize this vector to get s. Take its magnitude to get $\sin\phi$. You can get $\cos\phi$ from the trace of $\mathbf{R}$. Use the sin and cos to figure out ϕ. Note that rotation by ϕ about s is the same as rotation by $-\phi$ about $-s$.

Q8

Consider first the case when there is no translation. In this case we want to find the rotation matrix $\mathbf{R}$ which solves the following optimization problem:

$$\min_{\mathbf{R} \text{ is a rotation}} \sum_i \|\mathbf{R}u_i - v_i\|^2 \tag{23}$$

Note that this problem is slightly more complicated than naive least squares because $\mathbf{R}$ is restricted to be an orthogonal matrix. Let us look at one of the terms in the objective:

$$\begin{aligned} \|\mathbf{R}u_i - v_i\|^2 &= \|\mathbf{R}u_i\|^2 + \|v_i\|^2 - 2v_i^T\mathbf{R}u_i & (24) \\ &= \|u_i\|^2 + \|v_i\|^2 - 2v_i^T\mathbf{R}u_i & (25) \end{aligned}$$

where the last step follows from the fact that for orthogonal matrices, $\|\mathbf{R}u\| = \|u\|$.

Since $\|u_i\|^2$ and $\|v_i\|^2$ are constant and independent of $\mathbf{R}$, we can ignore them and end up with the following optimization problem:

$$\max_{\mathbf{R} \text{ is a rotation}} \sum_i v_i^T\mathbf{R}u_i \tag{26}$$

Now, since we are dealing with 2D points, we know that $\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ for some θ. Putting this in (25) and simplifying, we get the following optimization problem:

$$\max_{\theta}(\sum_i u_{ix}v_{ix} + u_{iy}v_{iy})\cos\theta + (\sum_i u_{ix}v_{iy} - v_{ix}u_{iy})\sin\theta \tag{27}$$

Write $x = [\cos\theta \ \ \sin\theta]^T$ so that $\|x\| = 1$. As θ varies, x spans all 2D unit vectors. Write $p = [\sum_i(u_{ix}v_{ix} + u_{iy}v_{iy}) \ \ \sum_i(u_{ix}v_{iy} - v_{ix}u_{iy})]^T$. Then the above optimization problem can be written as:

$$\max_{x:\|x\|=1} p^T x \tag{28}$$

It is easy to see that the solution is $x = \frac{p}{\|p\|}$. From this we can find out θ and hence $\mathbf{R}$.

Now suppose we also have a translation. Thus, we want to solve:

$$\min_{\mathbf{R} \text{ is a rotation}, t} \sum_i \|\mathbf{R}u_i + t - v_i\|^2 \tag{29}$$

Differentiating w.r.t t and setting it to 0, we get:

$$t = \frac{1}{n}\sum_i(v_i - \mathbf{R}u_i) = \bar{v} - \mathbf{R}\bar{u} \tag{30}$$

Here n is the number of points and $\bar{u} = \frac{\sum_i u_i}{n}$ is the mean of the $u_i's$ and $\bar{v} = \frac{\sum_i v_i}{n}$. Substituting this value of t in (28), we get the following optimization problem:

$$\min_{\mathbf{R} \text{ is a rotation}} \sum_i \|\mathbf{R}(u_i - \bar{u}) - (v_i - \bar{v})\|^2 \tag{31}$$

This is similar to (22) (i.e the case of no translation) and can be solved similarly to get $\mathbf{R}$. We can then substitute this value of $\mathbf{R}$ in (29) to get t.

For the given set of points this gives $\mathbf{R} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $t = [0 \ 0.75]^T$.

Q9

Let the three non-collinear points on the plane be $\vec{p}_1, \vec{p}_2$ and $\vec{p}_3$. Without loss of generality let us assume $\vec{p}_1$ to be the origin. The two "axes" of the coordinate system are then $\vec{q}_1 = \vec{p}_2 - \vec{p}_1$ and $\vec{q}_2 = \vec{p}_3 - \vec{p}_1$. Any other point on the plane can then be represented as $\vec{p}_1 + \alpha\vec{q}_1 + \beta\vec{q}_2$. The coordinates are thus α and β.

Now an affine transformation takes the point $\vec{x}$ to the point $\mathbf{A}\vec{x} + \vec{t}$ for some $\mathbf{A}, \vec{t}$. This means that the new origin and axes are, respectively:

$$\vec{p}_1' = \mathbf{A}\vec{p}_1' + \vec{t} \tag{32}$$
$$\vec{q}_1' = \mathbf{A}\vec{q}_1 \tag{33}$$
$$\vec{q}_2' = \mathbf{A}\vec{q}_2 \tag{34}$$

Now if a point $\vec{x}$ has coordinates α, β, and is mapped to $\vec{x}'$, then:

$$\begin{aligned}
\vec{x}' &= \mathbf{A}\vec{x} + \vec{t} && (35)\\
&= \mathbf{A}(\vec{p}_1 + \alpha\vec{q}_1 + \beta\vec{q}_2) + \vec{t} && (36)\\
&= (\mathbf{A}\vec{p}_1 + \vec{t}) + \alpha(\mathbf{A}\vec{q}_1) + \beta(\mathbf{A}\vec{q}_2) && (37)\\
&= \vec{p}_1' + \alpha\vec{q}_1' + \beta\vec{q}_2' && (38)
\end{aligned}$$

Thus the coordinates remain unchanged.