Chladni Plates

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1 Abstract

1.1 Introduction

This project investigates the behavior of Chladni plates, circular metal plates on which sand is poured, at fixed frequencies. The plate, due to material properties, has "natual frequencies" where vibrations take on a specific spatial structure, seen visually by sand settling into patterns. The goal of this project is to use partial differential equations (specifically, an extension of the wave equation) to model the Chladni plate behavior and predict the spatial structure. Two models with different assumptions about plate behavior (mathematically introduced as boundary conditions) will be compared for their accuracy against experimental data.

2 Mathematical Formulation

2.1 Model

The below partial differential equation (PDE) describes small amplitude vibrations in two dimensions of a rigid plate.

$$\Delta\Delta u = -\beta \frac{\partial^2 u}{\partial t^2}, 0 < r < R, t > 0$$

We apply two sets of boundary conditions to model the Chladni Plate. The first set of boundary conditions models the plate as a circle.

$$\begin{aligned}
|u(0,t)| &< \infty, t > 0 \\
|\frac{1}{r}\frac{\partial}{\partial r}u(0,t)| &< \infty, t > 0 \\
\frac{\partial^2}{\partial r^2}u(R,t) &= 0, t > 0 \\
\frac{\partial^3}{\partial r^3}u(R,t) &= 0, t > 0
\end{aligned}$$

The second set of boundary conditions models the plate as an annulus, with an inner and outer radius to account for the bolt at the plate's center.

$$\begin{aligned}
u(R_i,t) &= 0, t > 0 \\
\frac{\partial}{\partial r}u(R_i,t) &= 0, t > 0 \\
\frac{\partial^2}{\partial r^2}u(R,t) &= 0, t > 0 \\
\frac{\partial^3}{\partial r^3}u(R,t) &= 0, t > 0
\end{aligned}$$

2.2 Let $u(r,\theta,t) = F(r)G(t)$

To solve for u we can use separation of variables. Since there is radial symmetry, there is actually no θ dependence on the boundary conditions and the final solution will only be a function of r and t.

$$\begin{aligned}
\Delta\Delta F(r)G(t) &= -\beta F(r)G^{''}(t) \\
\frac{\Delta\Delta F(r)}{F(r)} &= \frac{-\beta G^{''}(t)}{G(t)} \\
\frac{\Delta\Delta F(r)}{F(r)} &= \frac{-\beta G^{''}(t)}{G(t)} = k^4
\end{aligned}$$

2.3 Solving G equation

$$k^4 G(t) = -\beta G^{''}(t)$$

Let $G(t) = e^{rt}$

$$\begin{aligned}
k^4 e^{rt} &= -\beta r^2 e^{rt} \\
k^4 &= -\beta r^2 \\
-\frac{k^4}{\beta} &= r^2 \\
r &= \pm i \frac{k^2}{\sqrt{\beta}} \\
G(t) &= e^{\pm ik^2/\sqrt{\beta}} \\
G(t) = c_1 \cos{(k^2 t/\sqrt{\beta})} &+ c_2 \sin{(k^2 t/\sqrt{\beta})}
\end{aligned}$$

2.4 Solving F equation

$$\begin{aligned}
\Delta\Delta F(r) &= k^4 F(r) \\
\Delta\Delta F(r) - k^4 F(r) &= 0 \\
F(r)(\Delta\Delta - k^4) &= 0 \\
F(r)(\Delta^2 - (k^2)^2) &= 0 \\
F(r)(\Delta - k^2)(\Delta + k^2) &= 0
\end{aligned}$$

$$\Delta F(r,\theta) = \frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} \tag{1}$$

Equation 1 is a typical polar Laplacian (for reference).

2.5 Solving $F(r)(\Delta - k^2) = 0$

$$\begin{aligned}
F(r)(\Delta - k^2) &= 0 \\
\Delta F(r) - k^2 F(r) &= 0
\end{aligned}$$

We can substitute equation 1 in and remove the double partial of F with respect to θ because $F(r)$ doesn't depend on θ. Additionally multiply both sides by r^2 since $r \neq 0$

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} - k^2 F = 0$$
$$r^2\frac{\partial^2 F}{\partial r^2} + r\frac{\partial F}{\partial r} - r^2k^2 F = 0$$

2.6 Solving $F(r)(\Delta + k^2) = 0$

$$F(r)(\Delta + k^2) = 0$$
$$\Delta F(r) + k^2 F(r) = 0$$

We can substitute equation 1 in and remove the double partial of F with respect to θ because $F(r)$ doesn't depend on θ. Additionally multiply both sides by r^2 since $r \neq 0$

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + k^2 F = 0$$
$$r^2\frac{\partial^2 F}{\partial r^2} + r\frac{\partial F}{\partial r} + r^2k^2 F = 0$$

We now have two equations that can be combined into one. Notice that in equation 2 we have F which is a function of r (one variable) hence equation 2 is actually an ordinary differential equation.

$$r^2\frac{\partial^2 F}{\partial r^2} + r\frac{\partial F}{\partial r} \pm k^2r^2 F = 0 \tag{2}$$

2.7 Equation 2 in Bessel's Differential Equation Form

Equation 2 can be made into the form of Bessel's Differential Equation:

$$z^2\frac{\partial^2 f}{\partial z^2} + z\frac{\partial f}{\partial z} \pm (z^2 - n^2) f = 0 \tag{3}$$

The n in Equation 3 relates to the θ dependence of the plate. The radial symmetry of this plate causes $n = 0$.

We use the following substitution to make equation 2 follow the form of equation 3 in order to apply the Bessel functions, fundamental solutions to Bessel's differential equation.

$$F(r) = f(z)$$
$$k^2r^2 = (z^2 - n^2) = z^2$$
$$kr = z$$

We apply chain rule to find $F''(r)$ in terms of $f(z)$.

$$\frac{\partial}{\partial r}F(r) = \frac{\partial z}{\partial r}\frac{\partial f}{\partial z}$$
$$F'(r) = k\frac{\partial f}{\partial z}$$
$$F''(r) = k^2\frac{\partial^2 f}{\partial z^2}$$

Plugging into eq. 2 results in the following:

$$r^2k^2\frac{\partial^2 f}{\partial z^2} + rk\frac{\partial f}{\partial z} \pm k^2r^2 f(z) = 0$$

Applying the substitution:

$$z^2\frac{\partial^2 f}{\partial z^2} + z\frac{\partial f}{\partial z} \pm z^2 f(z) = 0$$

This follows the form of Bessel's Differential Equation (see Equation 3).

2.8 Bessel Functions and Relevant Properties

Bessel functions are solutions to Bessel's Differential Equation.[3]

Bessel Function of the First Kind

$$J_n(z) = (\frac{z}{2})^n \sum_{k=0}^{\infty}(-1)^k \frac{(\frac{1}{4}z^2)^k}{k!\Gamma(n+k+1)} \tag{2.6.1}$$

Bessel Function of the Second Kind

$$Y_n(z) = \frac{J_n(z)\cos(n\pi) - J_{-n}(z)}{\sin(n\pi)} \tag{2.8.2}$$

Modified Bessel Function of the First Kind

$$I_n(z) = (\frac{z}{2})^n \sum_{k=0}^{\infty} \frac{(\frac{1}{4}z^2)^k}{k!\Gamma(n+k+1)} \tag{2.8.3}$$

Modified Bessel Function of the Second Kind

$$K_n(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \tag{2.8.4}$$

Below are some properties relevant to the mathematical formulation.[3]

Limiting Forms

$$J_0(z) \to 1, \text{if } z \to 0 \tag{2.8.5}$$
$$Y_0(z) \sim (2/\pi)\ln z, \text{if } z \to 0 \tag{2.8.6}$$
$$I_v(z) \sim (\frac{1}{2}z)^v/\Gamma(v+1), \text{when } v \text{ fixed and } z \to 0 \tag{2.8.7}$$
$$K_0(z) \sim -\ln z, z \to 0 \tag{2.8.8}$$

Recurrence Relations and Derivatives

$$J_0^{'}(z) = -J_1(z) \tag{2.8.9}$$
$$Y_0^{'}(z) = -Y_1(z) \tag{2.8.10}$$
$$I_0^{'}(z) = I_1(z) \tag{2.8.11}$$
$$K_0^{'}(z) = -K_1(z) \tag{2.8.12}$$
$$\mathscr{C}_{v-1}(z) - \mathscr{C}_{v+1} = 2\mathscr{C}_v^{'}(z) \tag{2.8.13}$$
$$\mathscr{Z}_{v-1}(z) + \mathscr{Z}_{v+1} = 2\mathscr{Z}_v^{'}(z) \tag{2.8.14}$$

$\mathscr{C}_v(z)$ represents $J_v(z)$ or $Y_v(z)$ or any nontrivial linear combination of these two functions. $\mathscr{Z}_v(z)$ represents $I_v(z)$ or $e^{v\pi i}K_v(z)$ or any nontrivial linear combination of these two functions.

2.9 Applying Bessel Functions to Solve $F(r)$

A general solution to Bessel's Differential Equation is a superposition of the Bessel functions. As such, a general solution to $f(z)$, where $n = 0$, is:

$$f(z) = AJ_0(z) + BY_0(z) + CI_0(z) + DK_0(z)$$

Therefore

$$F(r) = AJ_0(kr) + BY_0(kr) + CI_0(kr) + DK_0(kr) \tag{4}$$

2.10 First Set of Boundary Conditions

2.10.1 First Boundary Condition in First Set

$$|u(0,t)| < \infty, t > 0 \rightarrow F(0) < \infty$$

$$F(0) = AJ_0(0) + BY_0(0) + CI_0(0) + DK_0(0)$$

It may not be obvious but $Y_0(0)$ is unbounded. Looking back at limiting form (2.8.6), $Y_0(0)$ approaches $-\infty$ which results in $B = 0$. Additionally, $D = 0$ because by limiting form (2.8.8) $K_0(0)$ approaches ∞ which is unbounded hence $D = 0$.

2.10.2 Second Boundary Condition in First Set

The second boundary condition is redundant and results in the same conclusions as 2.8.1.

2.10.3 Third Boundary Condition in First Set

$$\frac{\partial^2}{\partial r^2} u(R,t) = 0, t > 0 \rightarrow F^{''}(R) = 0$$

$$F^{''}(R) = k^2 A J_0^{''}(kR) + k^2 C I_0^{''}(kR) = 0$$

Removing Derivatives. Recurrence relations and derivatives (2.8.9) can be used here

$$\begin{aligned} J_0^{'}(z) &= -J_1(z) \\ \frac{d}{dz} J_0^{'}(z) &= \frac{d}{dz} - J_1(z) \\ J_0^{''}(z) &= -J_1{'}(z) \end{aligned}$$

Using the recurrence relation (2.8.13)

$$\begin{aligned} J_{1-1}(z) - J_{1+1}(z) &= 2J_1^{'}(z) \\ -J_1^{'}(z) &= (J_2(z) - J_0(z))/2 \end{aligned}$$

Removing Derivatives. Recurrence relations and derivatives (2.8.11) can be used here

$$\begin{aligned} I_0^{'}(z) &= I_1(z) \\ \frac{d}{dz} I_0^{'}(z) &= \frac{d}{dz} I_1(z) \\ I_0^{''}(z) &= I_1{'}(z) \end{aligned}$$

Using the recurrence relation (2.8.14)

$$\begin{aligned} I_{1-1}(z) + I_{1+1}(z) &= 2I_1^{'}(z) \\ I_1^{'}(z) &= (I_2(z) + I_0(z))/2 \end{aligned}$$

Combining everything together we get

$$F^{''}(R) = \frac{k^2}{2} A(J_2(kR) - J_0(kR)) + \frac{k^2}{2} C(I_2(kR) + I_0(kR)) = 0$$

2.10.4 Fourth Boundary Condition in First Set

$$\begin{aligned} \frac{\partial^3}{\partial r^3} u(R,t) = 0, t > 0 \rightarrow F^{'''}(R) = 0 \\ F^{'''}(R) = k^3 A J_0^{'''}(kR) + k^3 C I_0^{'''}(kR) = 0 \end{aligned}$$

Removing Derivatives

$$\begin{aligned} J_0^{''}(z) &= -J_1^{'}(z) = (J_2(z) - J_0(z))/2 \\ \frac{d}{dz} J^{''}(z) &= \frac{d}{dz}(J_2(z) - J_0(z))/2 \\ J_0^{'''}(z) &= (J_2^{'}(z) - J_0^{'}(z))/2 \end{aligned}$$

Using recurrence relation (2.8.13)

$$\begin{aligned} J_0^{'}(z) &= -J_1(z) \\ J_2^{'}(z) &= (J_1(z) - J_3(z))/2 \\ J_0^{'''}(z) &= (J_1(z) - J_3(z))/4 + J_1(z)/2 \end{aligned}$$

Removing Derivatives

$$\begin{aligned} I_0^{''}(z) &= I_1^{'}(z) = (I_2(z) + I_0(z))/2 \\ \frac{d}{dz} I^{''}(z) &= \frac{d}{dz}(I_2(z) + I_0(z))/2 \\ I_0^{'''}(z) &= (I_2^{'}(z) + I_0^{'}(z))/2 \end{aligned}$$

Using recurrence relation (2.8.14)

$$\begin{aligned} I_0^{'}(z) &= I_1(z) \\ I_2^{'}(z) &= (I_1(z) + I_3(z))/2 \\ I_0^{'''}(z) &= (I_1(z) + I_3(z))/4 + I_1(z)/2 \end{aligned}$$

Combining everything together we get

$$F^{'''}(R) = \frac{k^3 A}{4}(3J_1(kR) - J_3(kR)) + \frac{k^3 C}{4}(3I_1(kR) + I_3(kR)) = 0$$

2.11 Second Set of Boundary Conditions

2.11.1 First Boundary Condition in Second Set

$$\begin{aligned} u(R_i,t) = 0, t > 0 \rightarrow F(R_i) = 0 \\ F(R_i) = AJ_0(kR_i) + BY_0(kR_i) + CI_0(kR_i) + DK_0(kR_i) = 0 \end{aligned}$$

2.11.2 Second Boundary Condition in Second Set

$$\begin{aligned} \frac{\partial}{\partial r} u(R_i,t) = 0, t > 0 \rightarrow F^{'}(R_i) = 0 \\ F^{'}(R_i) = kAJ_0^{'}(kR_i) + kBY_0^{'}(kR_i) + kCI_0^{'}(kR_i) + kDK_0^{'}(kR_i) = 0 \end{aligned}$$

Reducing primes to non primes. Recurrence relations and derivatives (2.8.9), (2.8.10), (2.8.11), (2.8.12) can be used here.

$$J_0'(z) = -J_1(z)$$
$$I_0'(z) = I_1(z)$$
$$I_0'(z) = -I_1(z)$$
$$K_0'(z) = -K_1(z)$$

Combining everything together we get

$$F'(R_i) = -k(AJ_1(kR_i) + BY_1(kR_i) - CI_1(kR_i) + DK_1(kR_i)) = 0$$

2.11.3 Third Boundary Condition in Second Set

$$\frac{\partial^2}{\partial r^2}u(R,t) = 0, t > 0 \to F''(R) = 0$$

$$F''(R) = k^2(AJ_0''(kR) + BY_0''(kR) + CI_0''(kR) + DK_0''(kR)) = 0$$

Reducing primes to non primes. Recurrence relations and derivatives (2.8.9) and (2.8.13) can be used here

$$J_0'(z) = -J_1(z)$$
$$\frac{d}{dz}J_0'(z) = \frac{d}{dz} - J_1(z)$$
$$J_0''(z) = -J_1'(z)$$
$$J_1'(z) = (J_0(z) - J_2(z))/2$$
$$J_0''(z) = (J_2(z) - J_0(z))/2$$

Reducing primes to non primes. Recurrence relations and derivatives (2.8.10) and (2.8.13)

$$Y_0'(z) = -Y_1(z)$$
$$\frac{d}{dz}Y_0'(z) = \frac{d}{dz} - Y_1(z)$$
$$Y_0''(z) = -Y_1'(z)$$
$$Y_1' = (Y_0(z) - Y_2(z))/2$$
$$Y_0''(z) = (Y_2(z) - Y_0(z))/2$$

Reducing primes to non primes. Recurrence relations and derivatives (2.8.11) and (2.8.14)

$$I_0'(z) = I_1(z)$$
$$\frac{d}{dz}I_0'(z) = \frac{d}{dz}I_1(z)$$
$$I_0''(z) = I_1'(z)$$
$$I_1' = (I_0(z) + I_2(z))/2$$
$$I_0''(z) = (I_2(z) + I_0(z))/2$$

Reducing primes to non primes. Recurrence relations and derivatives (2.8.12) and (2.8.14). It's important to note the $e^{v\pi i}$ in recurrence relation (2.8.14) which results in a negative being multiplied in our case below.

$$K_0'(z) = -K_1(z)$$
$$\frac{d}{dz}K_0'(z) = \frac{d}{dz} - K_1(z)$$
$$K_0''(z) = -K_1'(z)$$
$$K_1' = (K_0(z) + K_2(z))/2$$
$$K_0''(z) = (K_2(z) + K_0(z))/2$$

Combining everything together we get

$$F''(R) = \frac{k^2}{2}(A(J_2(kR) - J_0(kR)) + B(Y_2(kR) - Y_0(kR)) + C(I_0(kR) + I_2(kR)) + D(K_0(kR) + K_2(kR)) = 0$$

2.11.4 Fourth Boundary Condition in Second Set

$$\frac{\partial^3}{\partial r^3}u(R,t) = 0, t > 0 \to F'''(R) = 0$$
$$F'''(R) = k^3(AJ_0'''(kR) + BY_0'''(kR) + CI_0'''(kR) + DK_0'''(kR)) = 0$$

By the process of 2.10.4 (Fourth Boundary Condition in the First Set), we find that

$$J_0'''(z) = \frac{1}{4}(3J_1(z) - J_3(z))$$
$$I_0'''(z) = \frac{1}{4}(3I_1(z) + I_3(z))$$

To find $Y_0'''(z)$: Removing derivatives

$$Y_0''(z) = -Y_1'(z) = (Y_2(z) - Y_0(z))/2$$
$$\frac{d}{dz}Y''(z) = \frac{d}{dz}(Y_2(z) - Y_0(z))/2$$
$$Y_0'''(z) = (Y_2'(z) - Y_0'(z))/2$$

Using recurrence relations (2.8.10 and 2.8.13)

$$Y_0' = -Y_1(z)$$
$$Y_2'(z) = (Y_1(z) - Y_3(z))/2$$
$$Y_0'''(z) = (Y_1(z) - Y_3(z))/4 + Y_1(z)/2$$
$$Y_0'''(z) = \frac{1}{4}(3Y_1(z) - Y_3(z))$$

To find $K_0'''(z)$: Removing derivatives. It's important to note the $e^{v\pi i}$ in recurrence relation (2.8.14) which results in a negative being multiplied in our case below.

$$K_0^{''}(z) = -K_1^{'}(z) = (K_2(z) + K_0(z))/2$$
$$\frac{d}{dz}K^{''}(z) = \frac{d}{dz}(K_2(z) + K_0(z))/2$$
$$K_0^{'''}(z) = -(K_2^{'}(z) + K_0^{'}(z))/2$$

Using recurrence relations (2.8.10 and 2.8.12)

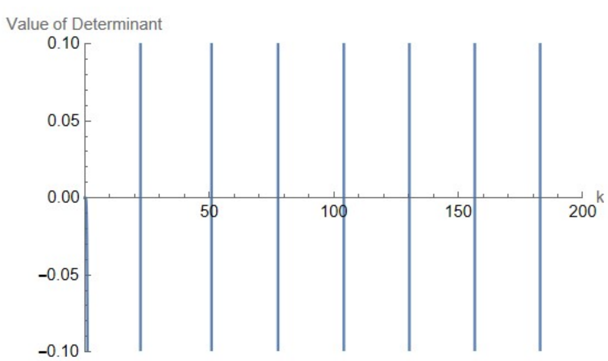
$$K_0^{'} = -K_1(z)$$
$$K_2^{'}(z) = (K_1(z) + K_3(z))/2$$
$$K_0^{'''}(z) = -((K_1(z) + K_3(z))/4 + K_1(z)/2)$$
$$K_0^{'''}(z) = -\frac{1}{4}(3K_1(z) + K_3(z))$$

Combing everything together we get

$$F^{'''}(R) = \frac{k^3A}{4}(3J_1(kR) - J_3(kR)) + \frac{k^3B}{4}(3Y_1(kR) - Y_3(kR)) + \frac{k^3C}{4}(3I_1(kR) + I_3(kR)) - \frac{k^3D}{4}(3K_1(kR) + K_3(kR)) = 0$$

2.12 Solving for the Separation Constant in the First Set

Allowable values of the separation constant, k, can be solved numerically (Appendix). The equations found by solving the boundary conditions provide a system of equations. In order to obtain a non-trivial solution for the nonzero coefficients (A, B), we set the determinant of the system of equations to 0.
The matrix for the system of equations is taken from 2.10.3 and 2.10.4. Solving for the roots of the determinant, plotted below, provides a list of allowable values of k.



2.13 Solving for Separation Constant in the Second Set

The same process as above is applied to the second set, but with all coefficients A, B, C, and D. The system of equations from 2.11 produces a matrix. The determinant is plotted below.

The roots of the determinant produce the k values.

2.14 Separation Constant Results

Number of Nodes	Circular Model k-value	Annulus Model k-value
1	22.434	13.458
2	50.965	37.560
3	77.671	65.128
4	104.088	92.061
5	130.403	118.854

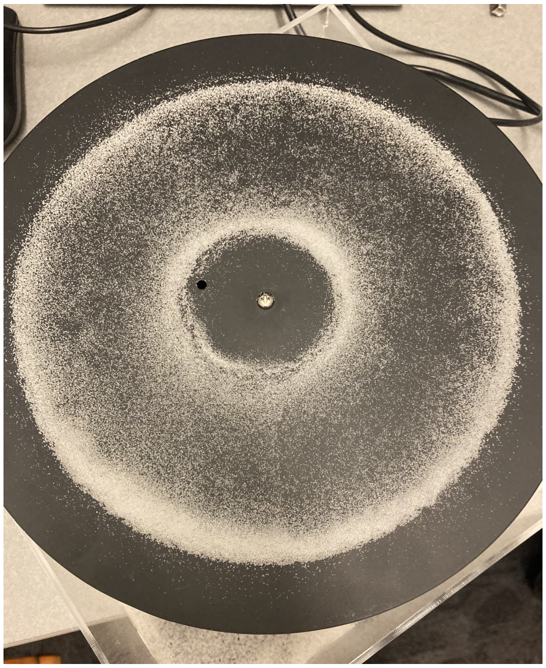
3 Experimental Results

3.1 Physical Data

One Node

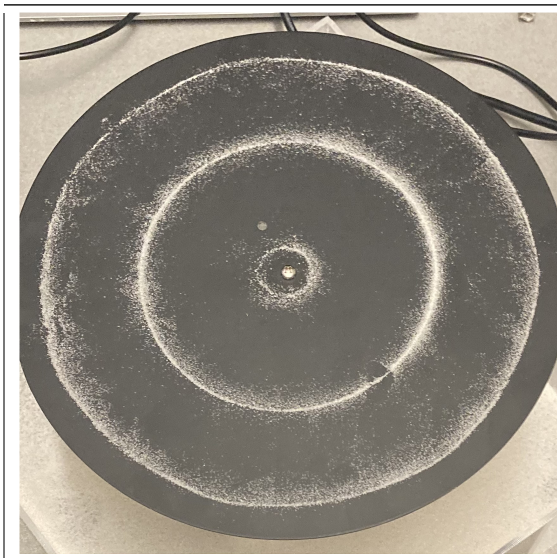
Stagnation Frequency (Hz)	Radius (cm)
98	$r_1 = 7.5$

Two Nodes	



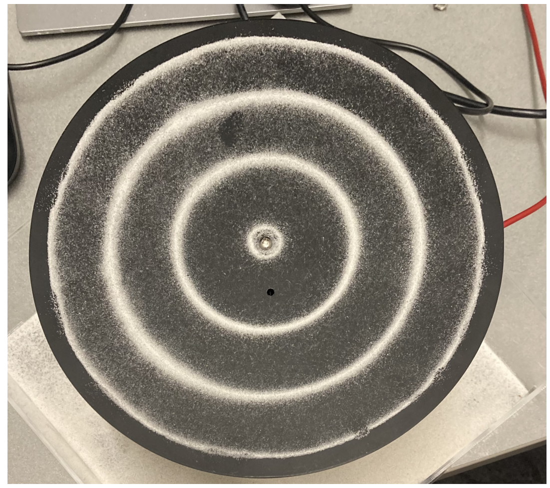
Stagnation Frequency (Hz)	Radii (cm)
380	$r_1 = 2.5$ $r_2 = 9.5$

Three Nodes	



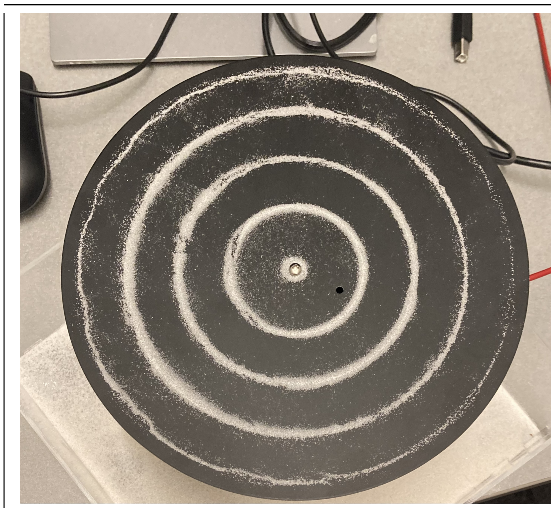
Stagnation Frequency (Hz)	Radii (cm)
925	$r_1 = 1$ $r_2 = 6.4$ $r_3 = 10.5$

Four Nodes	



Stagnation Frequency (Hz)	Radii (cm)
1653	$r_1 = 0.8$ $r_2 = 4.5$ $r_3 = 8$ $r_4 = 10.75$

Five Nodes	



Stagnation Frequency (Hz)	Radii (cm)
2800	$r_1 = 0.75$ cm $r_2 = 3.5$ cm $r_3 = 6$ cm $r_4 = 8.75$ cm $r_5 = 11$ cm

4 Model Comparison to Experimental Results

4.1 Spatial Zeros of F(r)

The k-values show the spatial zeros of the $F(r)$ equation, where the sand physically comes to rest. We plot $F(r)$ for each k, and the roots of $F(r)$ give the distance between the plate center and each ring of sand.

Number of Nodes	k value	Circular Model Roots of F(r) (cm)	Experimental Roots of F(r) (cm)
1	22.4338	$r_1 = 8.72$	$r_1 = 7.5$
2	50.9653	$r_1 = 4.76$ $r_2 = 10.19$	$r_1 = 2.5$ $r_2 = 9.5$
3	77.6713	$r_1 = 3.10$ $r_2 = 7.13$ $r_3 = 10.75$	$r_1 = 1$ $r_2 = 6.4$ $r_3 = 10.5$
4	104.088	$r_1 = 2.31$ $r_2 = 5.30$ $r_3 = 8.33$ $r_4 = 11.05$	$r_1 = 0.8$ $r_2 = 4.5$ $r_3 = 8$ $r_4 = 10.75$
5	130.403	$r_1 = 1.84$ $r_2 = 4.23$ $r_3 = 6.64$ $r_4 = 9.05$ $r_5 = 11.23$	$r_1 = 0.75$ cm $r_2 = 3.5$ cm $r_3 = 6$ cm $r_4 = 8.75$ cm $r_5 = 11$

Number of Nodes	k value	Annulus Model Roots of F(r) (cm)	Experimental Roots of F(r) (cm)
1	13.4579	$r_1 = 0.21$	$r_1 = 7.5$
2	37.2673	$r_1 = 0.21$ $r_2 = 9.74$	$r_1 = 2.5$ $r_2 = 9.5$
3	65.1375	$r_1 = 0.21$ $r_2 = 6.26$ $r_3 = 10.53$	$r_1 = 1$ $r_2 = 6.4$ $r_3 = 10.5$
4	92.0606	$r_1 = 0.21$ $r_2 = 4.46$ $r_3 = 7.86$ $r_4 = 10.93$	$r_1 = 0.8$ $r_2 = 4.5$ $r_3 = 8$ $r_4 = 10.75$
5	118.854	$r_1 = 0.21$ $r_2 = 3.50$ $r_3 = 6.12$ $r_4 = 8.78$ $r_5 = 11.16$	$r_1 = 0.75$ cm $r_2 = 3.5$ cm $r_3 = 6$ cm $r_4 = 8.75$ cm $r_5 = 11$

We see that, for both models, the spatial zeros of F(r) differ most from the experimental spatial zeros when the r is small (and therefore close to the center of the plate). This is attributed to the presence of the bolt connecting the wave driver to the plate, which is the source of major variation from the model. Theoretically, modelling as an annulus accounts for the bolt better than in the circular case. We see that the model r values are closer to the from the experimental values using the annulus BCs, supporting this theoretical conclusion. This indicates that the circular model does not accurately depict the behavior at the bolt, and lends preference to the annulus case.

Between the two models, we similarly see the largest discrepancies at small r. This difference is attributed to the change in wave behavior near a hole (in the annulus model) vs. in a solid plate.

Overall, the annulus case ends up with more precise spatial zeros, most notably at the center of the plate. Model adjustments to more accurately account, such as friction, for the bolt would be necessary.

4.2 Stagnation Frequencies

The theoretical stagnation frequency is the frequency of the $G(t)$ equation at the allowable values of k (found in section 2.14). The frequency of $G(t)$ is given by $\frac{k^2}{\sqrt{\beta}*2\pi}$. Here, β represents the physical constants of the plate, such that $\beta = \rho h/d$.

Quantity	Symbol	Value	Units
Density	ρ	2700	kg/m^3
Plate height	h	0.00095	m
Young's Modulus	E	6.89E+10	kg/ms^2
Poisson Ratio	ν	0.33	
	D	5.524365	
	β	46.4307	

Solving $\frac{k^2}{\sqrt{\beta}*2\pi}$ produces the theoretical stagnation frequencies. The below table shows compares the model frequencies to the experimental frequencies for the first five k values. Adjusting the β value can produce the same theoretical and experimental frequency, as shown below.

k values	Circular Model Theoretical Frequency ($k^2/(2\pi\sqrt{\beta})$) (Hz)	Experimental Frequency (Hz)	Adjusted β
22.434	117.55	98	66.80
50.965	606.68	380	118.35
77.671	1409.07	925	107.74
104.088	2530.57	1653	108.81
130.403	3971.84	2800	93.42

The same process was completed for the annulus model (second set of boundary conditions).

k values	Annulus Model Theoretical Frequency ($k^2/(2\pi\sqrt{\beta})$) (Hz)	Experimental Frequency (Hz)	Adjusted β
13.4579	42.30	98	8.65
37.2673	329.51	380	34.91
65.1375	990.78	925	53.26
92.0606	1979.52	1653	66.58
118.854	3299.26	2800	64.46

We see that, for the annulus model, the expected frequencies are close to the experimental frequencies. However, slight adjustments in β are needed to resolve variations between the model and the experiment. Notably, for the smallest k-value where there is one ring of sand, the adjustment in β is the greatest.

In the circular model, the theoretical frequencies appear to be consistently greater than the experimental frequency by an increasing amount of Hz for each number of nodes. This is a larger variation than the difference in the annulus case, again supporting the conclusion from 4.1 that the annulus model is more accurate.

In order to remedy the large difference in the circular case, a dramatic adjustment in β is needed. Unlike in the annulus case, where the adjusted β values are clustered around a similar value, the circular case shows a decreasing trend in adjusted β with increasing k-value. This could support model inadequacy for lower numbers of nodes.

5 Discussion and Conclusions

5.1 Experimental Design

We note that sources of error in experimental component of this project could impact the conclusions drawn about model accuracy. Human error in radius and frequency measurement could be present. Different orientations of the plate (such as slight tilting) could impact wave behavior. Multiple trials would account for this variation. Limitations on precision of numeric solvers could impact model data for theoretical k, r, and frequency values.

5.2 Model Accuracy and Comparison

The results of the spatial zero and stagnation frequency analysis in Section 4 show that, for both data points, the annulus model results in theoretical values closer to the experimental data. We note that the annulus model is still imperfect, specifically in modelling the behavior near the wave driver bolt. The largest model variation is at small r-values or low numbers of nodes. In order to reconcile the circular model to the experimental data, β requires considerable and inconsistent adjustment. This implies a poor model, if the variable containing physical information about the system needs to change from node to node. For the annulus case, the adjusted β doesn't vary as much with the number of nodes, and is close to the theoretical β.

6 Appendix

```
Boundary Conditions For the Circle Model
(*Defining the Different Boundary Conditions↩
    *)
J11 = (BesselJ[2, k*x] - BesselJ[0, k*x])*(k↩
    ^2/2);
J12 = (BesselI[0, k*x] + BesselI[2, k*x])*(k↩
    ^2/2);
J21 = (3*BesselJ[1, k*x] - BesselJ[3, k*x])↩
    *(k^3/4);
J22 = (3*BesselI[1, k*x] + BesselI[3, k*x])↩
    *(k^3/4);
(*Fixing x to R=0.12 meters*)
x = 0.12;
(*Putting BCs into 2x2 matrix with J Bessel ↩
    Functions in First
Column, I in the Second*)
det = Det[
{"J11", "J12"}  ,
{"J21", "J22"}],
det = FullSimplify[det] (*Simplifying ↩
    Expression for Easier Use*)

(*Plotting Determinant Expression Set to ↩
    Zero to Find k Solution
Values*)
Plot[det == 0, {k, 0, 200}, PlotRange -> ↩
    {{0, 200}, {-.1, .1}},
 AxesLabel -> {"k", "Value of Determinant"}]
```

```
(*Using FindRoot Function to Get More Exact ↩
    k Values (Repeat For
First Five)*)
FindRoot[
 1/8 k^5 (-((3 BesselI[1, 0.12` k] +
         BesselI[3, 0.12` k]) (BesselJ[0, ↩
             0.12` k] -
         BesselJ[2, 0.12` k])) - (BesselI[0,↩
             0.12` k] +
       BesselI[2, 0.12` k]) (3 BesselJ[1, ↩
```

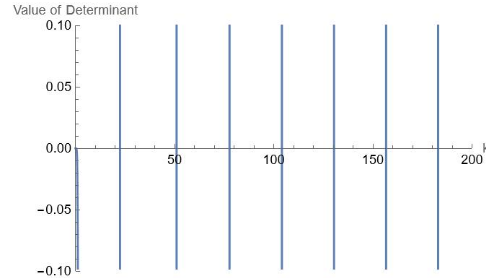


Figure 1: *Determinant of Circle BCs Set to Zero*

```
           0.12` k] -
       BesselJ[3, 0.12` k])), {k, 160}]
(*Defining k Values Found*)
k1 = 22.4338;
k2 = 50.9653;
k3 = 77.6713;
k4 = 104.088;
k5 = 130.403;
k6 = 156.671;
(*Plotting F Equation with Solved ↩
    Coefficients to Visualize r Value \
Solutions, FindRoot to Get More Exact r ↩
    Values*)
(*Repeat for Each k Value and Root*)

FindRoot[(-(BesselI[2, k5*0.12] + BesselI[0,↩
     k5*0.12])*
      BesselJ[0, k5*l])/(BesselJ[2, k5*0.12]↩
         - BesselJ[0, k5*0.12]) +
   BesselI[0, k5*l] == 0, {l, 0.01, .12}]
Plot[(-(BesselI[2, k5*0.12] + BesselI[0, k5↩
    *0.12])*
      BesselJ[0, k5*l])/(BesselJ[2, k5*0.12]↩
         - BesselJ[0, k5*0.12]) +
   BesselI[0, k5*l] == 0, {l, 0, 0.12},
 AxesLabel -> {"r values", "F Equation Set ↩
     to Zero"}]
```

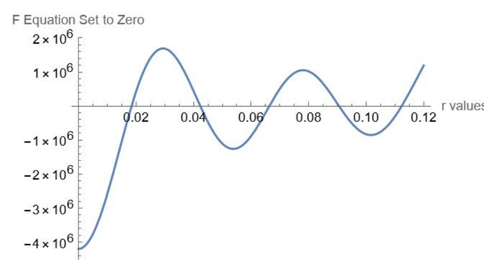


Figure 2: *F Equation with k=130.403*

```
(*Defining Boundary Conditions with ↩
    Respective Bessel Functions*)
D11 = BesselJ[0, w*Ri];
D12 = BesselY[0, w*Ri];
D13 = BesselI[0, w*Ri];
D14 = BesselK[0, w*Ri];
D21 = -w*BesselJ[1, w*Ri];
D22 = -w*BesselY[1, w*Ri];
D23 = w*BesselI[1, w*Ri];
D24 = -w*BesselK[1, w*Ri];
D31 = (w^2/2)*(BesselJ[2, w*R] - BesselJ[0, ↩
```

```
    w*R]);
D32 = (w^2/2)*(BesselY[2, w*R] - BesselY[0, w*R]);
D33 = (w^2/2)*(BesselI[0, w*R] + BesselI[2, w*R]);
D34 = (w^2/2)*(BesselK[0, w*R] + BesselK[2, w*R]);
D41 = (w^3/4)*(3*BesselJ[1, w*R] - BesselJ[3, w*R]);
D42 = (w^3/4)*(3*BesselY[1, w*R] - BesselY[3, w*R]);
D43 = (w^3/4)*(3*BesselI[1, w*R] + BesselI[3, w*R]);
D44 = (w^3/4)*(-3*BesselK[1, w*R] - BesselK[3, w*R]);
(*Setting Inner and Outer Radius Values*)
R = 0.12;
Ri = 0.0021;
(*Setting Matrix Form and Determinant Set to Zero*)
mat = {{D11, D12, D13, D14}, {D21, D22, D23, D24}, {D31, D32, D33,
    D34}, {D41, D42, D43, D44}};
deter = Det[mat];
deter = FullSimplify[deter];
(*Plotting Determinant Set to Zero To Find k Values*)
Plot[deter == 0, {w, -1, 500}, PlotRange -> {{0, 150}, {-.01, .01}},
 AxesLabel -> {"k Values", "Determinant of BCs Set to Zero"}]
(*Finding Exact k Values*)
FindRoot[deter == 0, {w, 110, 120}]
```

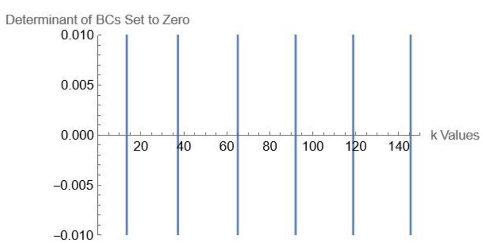


Figure 3: *Determinant of Annulus BCs Set to Zero*

```
k = {13.4579, 37.2673, 65.1375, 92.0606, 118.854}
Rinner = 0.0021;
Router = 0.12;

Equation1 =
 A*BesselJ[0, k*Rinner] + B*BesselY[0, k*Rinner] +
  CC*BesselI[0, k*Rinner] + D*BesselK[0, k*Rinner]
Equation2 = -k (A*BesselJ[1, k*Rinner] + B*BesselY[1, k*Rinner] -
    CC*BesselI[1, k*Rinner] + D*BesselK[1, k*Rinner])
Equation3 = (k^2/2) (A*(BesselJ[2, k*Router] - BesselJ[0, k*Router]) +
     B*(BesselY[2, k*Router] - BesselY[0, k*Router]) +
    CC*(BesselI[2, k*Router] + BesselI[0, k*Router]) +
    D*(BesselK[2, k*Router] + BesselK[0, k*Router]))
Equation4 =
 A*(k^3/4)*(3 BesselJ[1, k*Router] - BesselJ[3, k*Router]) +
  B*(k^3/4)*(3 BesselY[1, k*Router] - BesselY[3, k*Router]) +
  CC*(k^3/4)*(3 BesselI[1, k*Router] + BesselI[3, k*Router]) +
  D*(k^3/4)*(-3 BesselK[1, k*Router] - BesselK[3, k*Router])
(*Solve 6d BC for A with Arbitrary Variable as the Bessel Functions*)
Solve[Equation4 == 0, A]
(*Solve 6c for B*)
Solve[Equation3 == 0, B]
(*Solve 6b for C*)
Solve[Equation2 == 0, CC]

(* Creating our F equation to plot *)
Fequation[
  r_] := (A*BesselJ[0, k*r] + B*BesselY[0, k*r] + CC*BesselI[0, k*r] +
     D*BesselK[0, k*r])/D

(* Plotting out F equation *)
Plot[Fequation[r], {r, 0, 0.12}]

(* Using a reasonable guess to find our roots/(r values) *)
FindRoot[Fequation[r], {r, 0.12}]
```

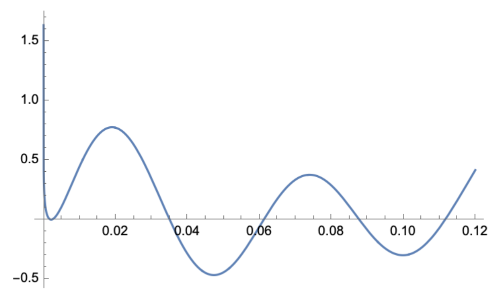


Figure 4: *F Equation with k=118.854*

7 References

1. Haberman, R. Applied Partial Differential Equations with Fourier Series and Boundary Value Problems, 4th ed., Pearson Prentice Hall; 2004.

2. Hildebrand, F.B. Advanced Calculus for Applications, 2nd ed., Prentice Hall; 1976, Chaps 4.8 - 4.10.

3. The Digital Library of Mathematical Functions, NIST. dlmf.nist.gov/10