Large Portfolio Asymptotics for Loss from Default

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Status Quo

- Dynamic reduced-form models of correlated name-by-name default timing are widely used to measure portfolio credit risk and to value securities exposed to correlated default risk
  - Default is a Poisson-type event
  - It arrives at an **intensity**, or conditional default rate
  - Intensity follows a stochastic process
- Computing the distribution of the loss from default in these models can be challenging
  - Semi-analytical transform methods have limited scope
  - Simulation methods have much wider scope, but can be slow in practice, where portfolios of several thousand names are common and relatively long time horizons may be of interest
- We are interested in credit defaults in large portfolios
  - Examples: loans, MBS, ABS, credit cards, student loans, microfinance.

Our Model

- We develop a **large portfolio approximation** (law of large numbers) for the distribution of the loss.
- Each name in the portfolio defaults at an intensity driven by
  - An idiosyncratic risk factor process
  - A **systematic risk** factor process X_t common to all names
  - A jump term modeling **contagion** within the pool. When a firm defaults, the intensities of the other firms in the pool jump.
- The limiting measure of the pool is governed by a **quasilinear stochastic PDE**
  - We develop computationally efficient numerical methods to solve the SPDE.

Prior Work

Structural models

- Static models with conditionally i.i.d. losses
  - LLN for a homogeneous pool: Vasicek (1991) and others
  - LLN for a heterogeneous pool: Gordy (2003) [Basel rules]
  - LDP for a heterogeneous pool: Dembo, Deuschel and Duffie (2004); Glasserman, Kang, and Shahabuddin (2007)
- Dynamic models with conditionally i.i.d. losses
  - LLN for a homogeneous pool: Bush, Hambly, Haworth, Jin, and Reisinger (2011)

LLNs for dynamic reduced form models

- Mean field interaction
  - Cvitanic, Ma, and Zhang (2011)
  - Dai Pra, Runggaldier, Sartori, and Tolotti (2009) and Dai Pra and Tolotti (2009)
  - This paper
- Local interaction
  - Giesecke and Weber (2006): voter model

Model Setting

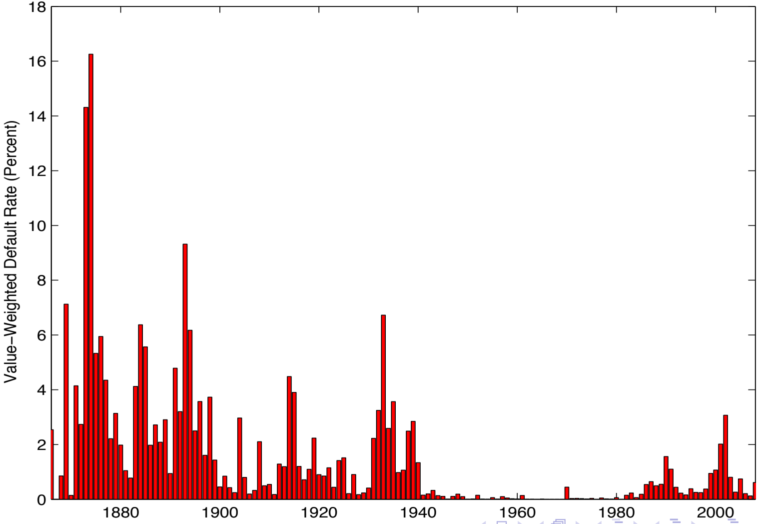
We develop a dynamic, interacting point process framework to model credit default in a pool of securities.

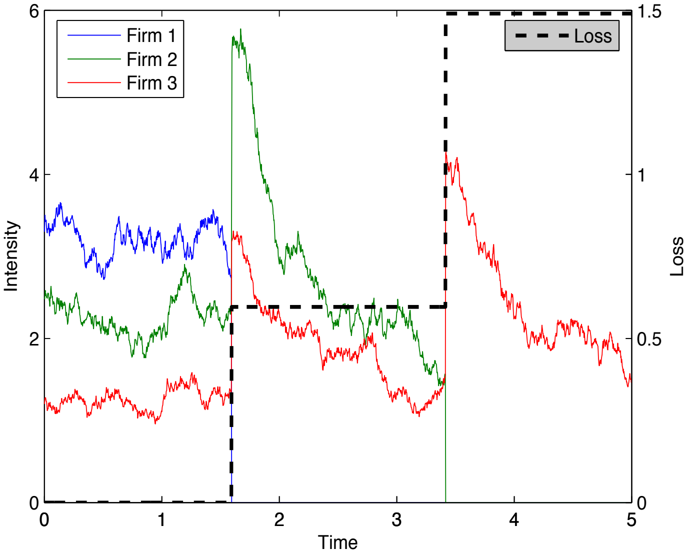
- Pool of $N \in \mathbb{N}$ names
  - **Portfolio loss rate** : $L_t^N = \frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_{\tau^n \leq t}$
  - **Systematic risk process**: $dX_t = b_0(X_t)dt + \sigma_0(X_t)dV_t$.

The default intensity of a name $n \in \{1, 2, \ldots N\}$ is

$$d\lambda_t^n = \alpha_n(\bar{\lambda}_n - \lambda_t^n)dt + \sigma_n\sqrt{\lambda_t^n}dW_t^n + \beta_n^C dL_t^N + \beta_n^S \lambda_t^n dX_t \qquad (1)$$

The model captures the three primary sources of defaults that have been observed in the empirical literature: idiosyncratic risk, contagion, and systematic risk. The latter two are responsible for default clustering.


Value–Weighted Default Rate (Percent)
0
2
4
6
8
10
12
14
16
18
1880
1900
1920
1940
1960
1980
2000


Firm 1
Firm 2
Firm 3
Loss
Intensity
Loss
Time
0
1
2
3
4
5
0
2
4
6
0
0.5
1
1.5

Homogeneous Pool

Suppose all names have the same parameters. The empirical measure for the surviving names in the pool is

$$\mu_t^N \stackrel{\text{def}}{=} \frac{1}{N}\sum_{n=1}^{N} \delta_{\lambda_t^n} \mathbf{1}_{\tau^n > t},$$

where $\mu_t^N \in D_E[0, T]$, the space of RCLL paths taking values in E, the set of sub-probability measures on $\mathbb{R}_+$. The loss is simply

$$L_t^N = 1 - \mu_t^N(\mathbb{R}_+)$$

Also, let the empirical measure of the initial conditions be

$$\Lambda_\circ^N \stackrel{\text{def}}{=} \frac{1}{N}\sum_{n=1}^{N} \delta_{\lambda_{\circ,N,n}};$$

We assume that $\Lambda_\circ^N$ weakly converges to $\Lambda_\circ \in \mathbb{R}_+$ as $N \longrightarrow \infty$.

Limiting SPDE for Homogeneous Pool

Theorem

Suppose there exists a solution υ *to the SPDE*

$$d\upsilon(t,\lambda) = \mathcal{L}^{*,X_t}\upsilon(t,\lambda)dt + \beta^S \mathcal{L}_4^{*,X_t}\upsilon(t,\lambda)dV_t,$$
$$\upsilon(t=0,\lambda) = \Lambda_\circ,$$
$$\upsilon(t,\lambda=0) = \upsilon(t,\lambda=\infty) = 0, \tag{2}$$

where $\mathcal{L}^{*,X_t} \equiv \mathcal{L}_1^* + \mathcal{L}_3^{*,X_t} + \beta^C \left(\int_0^\infty \lambda\upsilon(t,\lambda)d\lambda\right)\mathcal{L}_2^*$*. Then, the empirical measure* μ_t^N *weakly converges to* $\upsilon(t,\lambda)d\lambda$ *in* $D_E[0,T]$ *as* $N \longrightarrow \infty$*. Furthermore, the limiting loss* L_t *is simply*

$$L_t = 1 - \int_0^\infty \upsilon(t,\lambda)d\lambda. \tag{3}$$

The differential operators are

$$\mathcal{L}_1 f = \frac{1}{2}\sigma^2\lambda\frac{\partial^2 f}{\partial\lambda^2} - \alpha(\lambda-\bar{\lambda})\frac{\partial f}{\partial\lambda} - \lambda f$$

$$\mathcal{L}_2 f = \frac{\partial f}{\partial\lambda}$$

$$\mathcal{L}_3^x f = \beta^S\lambda b_0(x)\frac{\partial f}{\partial\lambda} + \frac{1}{2}(\beta^S)^2\lambda^2\sigma_0^2(x)\frac{\partial^2 f}{\partial\lambda^2}$$

$$\mathcal{L}_4^x f = \lambda\sigma_0(x)\frac{\partial f}{\partial\lambda}.$$

$\mathcal{L}_1$ contains the infinitesimal generator of the idiosyncratic process and $-\lambda f$ is a kill term. The nonlinear term $\left(\int_0^\infty \lambda\upsilon(t,\lambda)d\lambda\right)\mathcal{L}_2^*\upsilon(t,\lambda)$ comes from jumps due to contagion. $\mathcal{L}_3^x f$ is related to the infinitesimal generator of the systematic risk. $\mathcal{L}_4^{*,X_t}dV_t$ is the Brownian term driving the SPDE.

Numerical Methods and Results

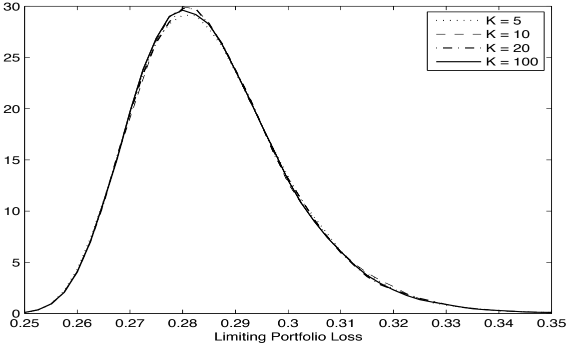
We use a method of moments to efficiently solve for the limiting loss. Consider the "moment" $u_k(t) = \int_0^\infty \lambda^k v(t,\lambda) d\lambda$ for $k \in \mathbb{N}$. Then, the limiting loss is simply $1 - u_0(t)$.

The moments $u_k(t)$ follow the SDE system

$$\begin{aligned} du_k(t) &= \Big\{ u_k(t)\big(-\alpha k + \beta^S b_0(X_t) n + \frac{1}{2}(\beta^S)^2 \sigma_0^2(X_t) k(k-1)\big) \\ &+ u_{k-1}(t)\big(\frac{1}{2}\sigma^2 k(k-1) + \alpha\bar{\lambda}k + \beta^C k u_1(t)\big) \\ &- u_{k+1}(t)\Big\} dt + \beta^S \sigma_0(X_t) k u_k(t) dV_t, \\ u_k(t=0) &= \int_0^\infty \lambda^k \Lambda_\circ(\lambda) d\lambda. \end{aligned} \tag{4}$$

We must truncate the SDE system at some level K.

The moment SDE system converges very quickly in terms of the truncation level K.



Numerical implementation details

- The moment system can be solved using an Euler scheme.
- The moments $\{u_k(t)\}_{k=1}^K$ are nonnegative, but may be become negative in a numerical scheme due to the discretization of time.
  - Creates instability
  - To avoid this, one can implement a truncated Euler scheme that immediately sets the moments to zero if they ever become negative.
- For large β^S, there is instability in the higher moments due to the exponential growth term $\frac{1}{2}(\beta^S)^2\sigma_0(X_t)^2u_k(t)dt$. This instability can be significantly reduced using the transformed moment system $w_k(t) = \exp\big(-\frac{1}{2}(\beta^S)^2\int_0^t\sigma_0(X_s)^2ds\big)u_k(t)$. Note that $w_0(t) = u_0(t)$, the quantity of interest.

Random ODE Moment System

There exists a canonical representation of the moment system, which can be found via the variable transformation $\eta_k(t) = X_t - \frac{1}{k\beta^S}\log(u_k(t))$ for $k \geq 1$.

$$
\begin{aligned}
dX_t &= b_0(X_t)dt + \sigma_0(X_t)dV_t, \\
du_0(t) &= -e^{\beta^S(X_t - \eta_1(t))}dt, \\
d\eta_k(t) &= \Big\{ b_0(X_t) + \frac{1}{2}k\beta^S\sigma_0^2(X_t) \\
&- \frac{1}{\beta^S}\Big[-\alpha + \beta^S b_0(X_t) + \frac{1}{2}(\beta^S)^2\sigma_0^2(X_t)(k-1) \\
&+ e^{(k-1)\beta^S(X_t - \eta_{n-1}) - k\beta^S(X_t - \eta_k)}\Big(\frac{1}{2}\sigma^2(k-1) + \alpha\bar{\lambda} \\
&+ \beta^C e^{\beta^S(X_t - \eta_1)}\Big) - \frac{1}{k}e^{(k+1)\beta^S(X_t - \eta_{k+1}) - k\beta^S(X_t - \eta_k)}\Big]\Big\}dt.
\end{aligned}
$$

Comparison with Explicit Finite Difference Scheme

An alternative to the method of moments is direct finite difference of the original SPDE. However, due to the requirement that the stochastic diffusion term be nonanticipative, second-order accurate schemes in time such as Crank-Nicholson cannot be used.

Explicit finite difference suffers from

- Discretization of space. If one chooses a space grid with J points, the FD scheme has order of complexity of at least J coupled ODEs.
- Due to the scheme being explicit, there is instability. This instability increases with the parameter β^S (i.e., as the "randomness" in the SPDE increases).

Given a desired accuracy in time of $\mathcal{O}(\bar{\Delta})$, we estimate the ratio of the computational cost of the explicit finite difference to that of the method of moments for the case of $\sigma_0 = 1$ to be

$$\frac{\text{Cost of Explicit FD}}{\text{Cost of Method of Moments}} \approx \frac{J}{K} \min\Big(\frac{(\beta^S \lambda_{max})^2}{\delta^2}, \frac{1}{\bar{\Delta}}\Big). \tag{5}$$

Time Step	Method of Moments	Explicit Finite Difference
10^{-2}	2.4890 seconds	Unstable
10^{-3}	25.7241 seconds	Unstable
10^{-4}	254.6494 seconds	Unstable
10^{-5}	2561.9614 seconds	8512.3172 seconds

Table: Actual computational times using $K = 100$ moments and 1000 Monte Carlo trials.

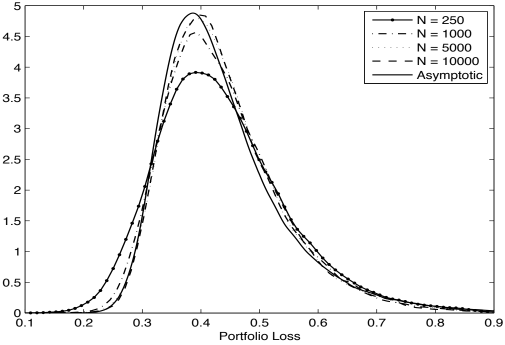
Comparison of simulation of asymptotic SPDE and simulation of the finite system.

Method of Moments with $K = 200$	6.24984 seconds
Finite System with $N = 500$	13.57564 seconds
Finite System with $N = 1000$	22.12235 seconds
Finite System with $N = 5000$	236.23656 seconds
Finite System with $N = 10000$	441.80289 seconds
Finite System with $N = 25000$	1048.80624 seconds

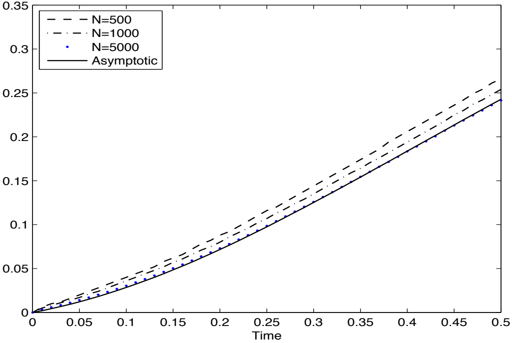
Table: 1000 Monte Carlo samples and a time horizon of $T = 1$

Numerical Convergence

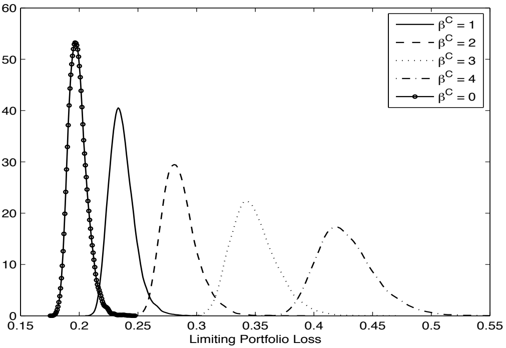
Numerical demonstration of convergence of distribution of the loss for the finite system to limiting loss distribution as N becomes large.



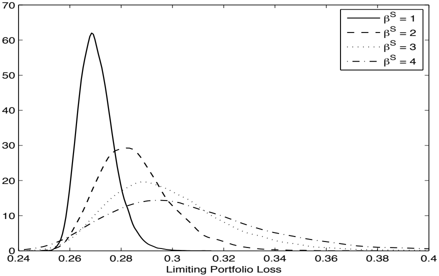
Numerical demonstration of convergence of value at risk (VaR) for the finite system to the VaR of the limiting loss.



- Large β^C (level of contagion) shifts the loss distribution to the right and causes heavy right tails.
- Volatility of the loss increases as the connectivity between the firms increases.

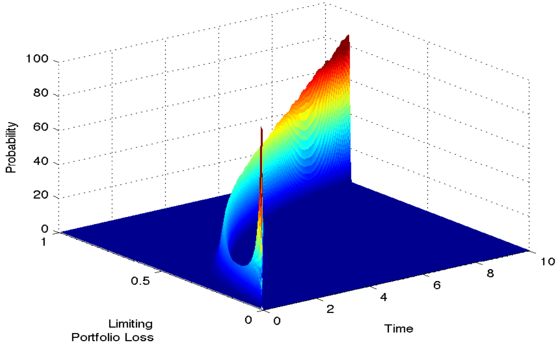


- Large β^S (level of systematic risk) shifts the loss distribution to the right and causes heavy right tails.

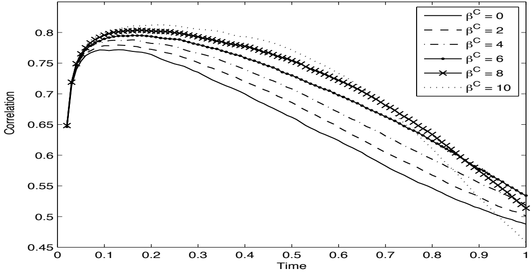


Loss Surface

- The model gives the loss distribution for many time horizons simultaneously.
- Model simulates dynamic evolution of loss



- The correlation between the systematic risk and the loss increases as β^C (which indicates the level of contagion) increases.
- There is a **complex interaction between contagion and systematic risk** in the model. The system becomes increasingly vulnerable to shocks from the systematic risk as the contagion channel in the system becomes stronger.



Heterogeneous Pool

The model allows for a significant amount of bottom-up heterogeneity; the intensity dynamics of each name can be different. Define the "types"

$$\mathrm{p}^{N,n} = (\alpha_{N,n}, \bar{\lambda}_{N,n}, \sigma_{N,n}, \beta^{C}_{N,n}, \beta^{S}_{N,n}) \in \mathcal{P} \stackrel{\text{def}}{=} \mathbb{R}^4_+ \times \mathbb{R};$$

For each $N \in \mathbb{N}$, let $\hat{\mathrm{p}}^{N,n}_t \stackrel{\text{def}}{=} (\mathrm{p}^{N,n}, \lambda^{N,n}_t) \in \hat{\mathcal{P}} \stackrel{\text{def}}{=} \mathcal{P} \times \mathbb{R}_+$ for all $n \in \{1, 2, \ldots, N\}$. The empirical measure for the surviving names in the pool is

$$\mu^N_t \stackrel{\text{def}}{=} \frac{1}{N} \sum_{n=1}^{N} \delta_{\hat{\mathrm{p}}^{N,n}_t} \mathbf{1}_{\tau^n > t},$$

Also, let the empirical measure of the intensity processes' parameters be $\pi^N \stackrel{\text{def}}{=} \frac{1}{N} \sum_{n=1}^{N} \delta_{\mathrm{p}^{N,n}}$ and assume that π^N weakly converges to $\pi \in \mathcal{P}$ as $N \longrightarrow \infty$.

Limiting SPDE for Heterogeneous Pool

Theorem

If there exists a solution to the SPDE

$$d\upsilon(t,\hat{\mathsf{p}}) = \Big\{\mathcal{L}_1^{*}\upsilon(t,\hat{\mathsf{p}}) + \mathcal{L}_3^{*,X_t}\upsilon(t,\hat{\mathsf{p}}) + \left(\int_{\hat{\mathsf{p}}'\in\hat{\mathcal{P}}} \mathcal{Q}(\hat{\mathsf{p}}')\upsilon(t,\hat{\mathsf{p}}')d\hat{\mathsf{p}}'\right)\mathcal{L}_2^{*}\upsilon(t,\hat{\mathsf{p}})\Big\}dt + \mathcal{L}_4^{*,X_t}\upsilon(t,\hat{\mathsf{p}})dV_t,\ \hat{\mathsf{p}}\in\hat{\mathcal{P}},$$

where $\mathcal{L}_i^{*}$ denote adjoint operators, with initial condition

$$\lim_{t\searrow 0}\upsilon(t,\hat{\mathsf{p}})d\hat{\mathsf{p}} = \pi\times\Lambda_{\circ},$$

and boundary conditions

$$\upsilon(t,\lambda=0,\mathsf{p}) = \upsilon(t,\lambda=\infty,\mathsf{p}) = 0.$$

Then, μ_t^N weakly converges to $\upsilon(t,\hat{\mathsf{p}})d\hat{\mathsf{p}}$ in $D_E[0,T]$ as $N\longrightarrow\infty$.

Gaussian Correction

We have developed a second-order accurate approximation using a type of "**dynamic central limit theorem**".

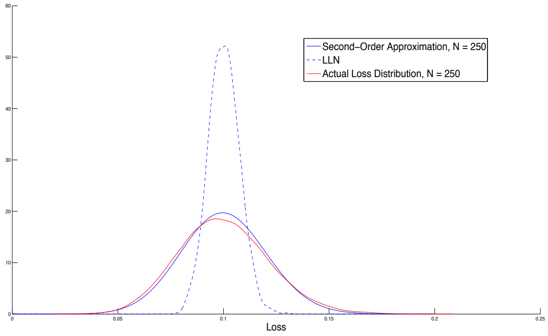


Figure: Comparison of second-order approximate, LLN, and actual loss distributions in the finite system at $T = .5$.

Maximum Likelihood

- Problem: statistical inference for a **large stochastic system**.
  - Example: Observe losses $L_{t_1}^N, L_{t_2}^N, \ldots, L_{t_M}^N$ in a credit portfolio of size N and wish to fit model parameters using maximum likelihood.
- Approach: Instead of working with the original stochastic system, perform filtering and maximum likelihood estimation using the **second-order approximation** which makes use of the LLN result here and the CLT result.
- If X_t's path is completely observed, semi-analytic computation of likelihood is available (**no Monte Carlo simulation**).

Conclusion

- We prove a law of large numbers for the loss in a credit portfolio.
- The limiting measure solves an SPDE, which can be efficiently solved using the method of moments.
- We account for the main economic driving factors behind default clustering. The complex interaction between contagion and systematic risk is a central and unique feature of the model.
- Many applications, including: MBS, large ABS or corporate credit portfolios, microfinance, student loans.