Large Portfolio Asymptotics for Loss from Default

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March 23, 2012

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Introduction

- Current state of modeling for correlated credit default in a portfolio of names
  - Bottom-up models have great generality but usually must be solved using simulation
  - Reduced-form (or "top-down") models are more tractable. Semi-analytic transform methods are available in certain limited cases.
- We are interested in credit defaults in large portfolios
  - Examples: loans, MBS, ABS, credit cards, student loans, microfinance.
  - Bottom-up models become computationally intractable for such large pools. Top-down models do not consider the individual security-level dynamics and interactions in the pool.

Our Model

- We develop a **large portfolio approximation** (law of large numbers) for the distribution of the loss.
- Each name in the portfolio defaults at an intensity driven by
  - An idiosyncratic risk factor process
  - A **systematic risk** factor process X_t common to all names
  - A jump term modeling **contagion** within the pool. When a firm defaults, the intensities of the other firms in the pool jump.
- The limiting measure of the pool is governed by a **quasilinear stochastic PDE**
  - We develop computationally efficient numerical methods to solve the SPDE.

Prior Work

Structural models

- Static models with conditionally i.i.d. losses
  - LLN for a homogeneous pool: Vasicek (1991) and others
  - LLN for a heterogeneous pool: Gordy (2003) [Basel rules]
  - LDP for a heterogeneous pool: Dembo, Deuschel and Duffie (2004); Glasserman, Kang, and Shahabuddin (2007)
- Dynamic models with conditionally i.i.d. losses
  - LLN for a homogeneous pool: Bush, Hambly, Haworth, Jin, and Reisinger (2011)

LLNs for dynamic reduced form models

- Mean field interaction
  - Cvitanic, Ma, and Zhang (2011)
  - Dai Pra, Runggaldier, Sartori, and Tolotti (2009) and Dai Pra and Tolotti (2009)
  - This paper
- Local interaction
  - Giesecke and Weber (2006): voter model

Model Setting

We develop a dynamic, interacting point process framework to model credit default in a pool of securities.

- Pool of $N \in \mathbb{N}$ names
  - **Portfolio loss rate** : $L_t^N = \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{\tau^n \leq t}$
  - **Systematic risk process**: $dX_t = b_0(X_t)dt + \sigma_0(X_t)dV_t$.

The default intensity of a name $n \in \{1, 2, \ldots N\}$ is

$$d\lambda_t^n = \alpha_n(\bar{\lambda}_n - \lambda_t^n)dt + \sigma_n \sqrt{\lambda_t^n} dW_t^n + \beta_n^C dL_t^N + \beta_n^S \lambda_t^n dX_t \qquad (1)$$

The model captures the three primary sources of defaults that have been observed in the empirical literature: idiosyncratic risk, contagion, and systematic risk. The latter two are responsible for default clustering.

Homogeneous Pool

Suppose all names have the same parameters. The empirical measure for the surviving names in the pool is

$$\mu_t^N \stackrel{\text{def}}{=} \frac{1}{N}\sum_{n=1}^{N} \delta_{\lambda_t^n} \mathbf{1}_{\tau^n > t},$$

where $\mu_t^N \in D_E[0, T]$, the space of RCLL paths taking values in E, the set of sub-probability measures on $\mathbb{R}_+$. The loss is simply

$$L_t^N = 1 - \mu_t^N(\mathbb{R}_+)$$

Also, let the empirical measure of the initial conditions be

$$\Lambda_\circ^N \stackrel{\text{def}}{=} \frac{1}{N}\sum_{n=1}^{N} \delta_{\lambda_{\circ,N,n}};$$

We assume that $\Lambda_\circ^N$ weakly converges to $\Lambda_\circ \in E$ as $N \longrightarrow \infty$.

Limiting SPDE for Homogeneous Pool

Theorem. Suppose there exists a solution υ to the SPDE

$$d\upsilon(t,\lambda) = \{\mathcal{L}_1^* \upsilon(t,\lambda) + \mathcal{L}_3^{*,X_t} \upsilon(t,\lambda)$$
$$+ \beta^C \left(\int_0^\infty \lambda \upsilon(t,\lambda) d\lambda\right) \mathcal{L}_2^* \upsilon(t,\lambda)\} dt + \mathcal{L}_4^{*,X_t} \upsilon(t,\lambda) dV_t,$$
$$\upsilon(t=0,\lambda) = \Lambda_\circ,$$
$$\upsilon(t,\lambda=0) = \upsilon(t,\lambda=\infty) = 0. \tag{2}$$

Then, the empirical measure μ_t^N weakly converges to $\upsilon(t,\lambda)d\lambda$ in $D_E[0,T]$ as $N \longrightarrow \infty$. Furthermore, the limiting loss L_t is simply

$$L_t = 1 - \int_0^\infty \upsilon(t,\lambda) d\lambda. \tag{3}$$

We will state the more general convergence theorem for the heterogeneous case later.

The differential operators are

$$\mathcal{L}_1 f = \frac{1}{2}\sigma^2 \lambda \frac{\partial^2 f}{\partial \lambda^2} - \alpha(\lambda - \bar{\lambda})\frac{\partial f}{\partial \lambda} - \lambda f$$
$$\mathcal{L}_2 f = \frac{\partial f}{\partial \lambda}$$
$$\mathcal{L}_3^x f = \beta^S \lambda b_0(x)\frac{\partial f}{\partial \lambda} + \frac{1}{2}(\beta^S)^2 \lambda^2 \sigma_0^2(x)\frac{\partial^2 f}{\partial \lambda^2}$$
$$\mathcal{L}_4^x f = \beta^S \lambda \sigma_0(x)\frac{\partial f}{\partial \lambda}.$$

$\mathcal{L}_1$ contains the infinitesimal generator of the idiosyncratic process and $-\lambda f$ is a kill term. The nonlinear term $\left(\int_0^\infty \lambda \upsilon(t,\lambda)d\lambda\right)\mathcal{L}_2^* \upsilon(t,\lambda)$ comes from jumps due to contagion.
$\mathcal{L}_3^x f$ is related to the infinitesimal generator of the systematic risk.
$\mathcal{L}_4^{*,X_t} dV_t$ is the Brownian term driving the SPDE.

Numerical Methods and Results

We use a method of moments to efficiently solve for the limiting loss. Consider the "moment" $u_k(t) = \int_0^\infty \lambda^k \upsilon(t,\lambda) d\lambda$ for $k \in \mathbb{N}$. Then, the limiting loss is simply $1 - u_0(t)$.

The moments $u_k(t)$ follow the SDE system

$$\begin{aligned} du_k(t) &= \Big\{ u_k(t)\big(-\alpha k + \beta^S b_0(X_t) n + 0.5(\beta^S)^2 \sigma_0^2(X_t) k(k-1)\big) \\ &+ u_{k-1}(t)\big(0.5\sigma^2 k(k-1) + \alpha\bar{\lambda}k + \beta^C k u_1(t)\big) \\ &- u_{k+1}(t)\Big\} dt + \beta^S \sigma_0(X_t) k u_k(t) dV_t, \\ u_k(t=0) &= \int_0^\infty \lambda^k \Lambda_\circ(\lambda) d\lambda. \end{aligned} \tag{4}$$

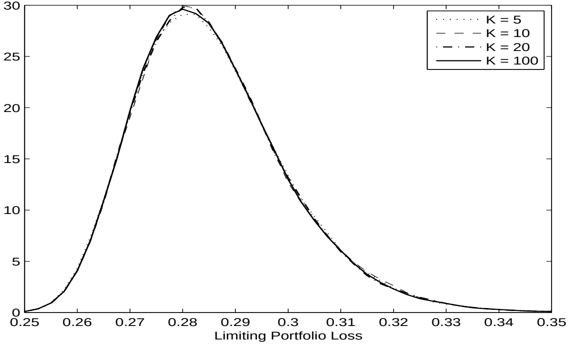
We must truncate the SDE system at some level K.

Random ODE Moment System

There exists a canonical representation of the moment system, which can be found via the variable transformation $\eta_k(t) = X_t - \frac{1}{k\beta^S}\log(u_k(t))$ for $k \geq 1$.

$$
\begin{aligned}
dX_t &= b_0(X_t)dt + \sigma_0(X_t)dV_t, \\
du_0(t) &= -e^{\beta^S(X_t - \eta_1(t))}dt, \\
d\eta_k(t) &= \Big\{ b_0(X_t) + \frac{1}{2}k\beta^S\sigma_0^2(X_t) \\
&- \frac{1}{\beta^S}\Big[-\alpha + \beta^S b_0(X_t) + \frac{1}{2}(\beta^S)^2\sigma_0^2(X_t)(k-1) \\
&+ e^{(k-1)\beta^S(X_t - \eta_{n-1}) - k\beta^S(X_t - \eta_k)}\Big(\frac{1}{2}\sigma^2(k-1) + \alpha\bar{\lambda} \\
&+ \beta^C e^{\beta^S(X_t - \eta_1)}\Big) - \frac{1}{k}e^{(k+1)\beta^S(X_t - \eta_{k+1}) - k\beta^S(X_t - \eta_k)}\Big]\Big\}dt.
\end{aligned}
$$

The moment SDE system converges very quickly in terms of the truncation level K.



Comparison with Explicit Finite Difference Scheme

An alternative to the method of moments is direct finite difference of the original SPDE. However, due to the requirement that the stochastic diffusion term be nonanticipative, second-order accurate schemes in time such as Crank-Nicholson cannot be used.

Explicit finite difference suffers from

- Discretization of space. If one chooses a space grid with J points, the FD scheme has order of complexity of at least J coupled ODEs.
- Due to the scheme being explicit, there is instability. This instability increases with the parameter β^S (i.e., as the "randomness" in the SPDE increases).

Given a desired accuracy in time of $\mathcal{O}(\bar{\Delta})$, we estimate the ratio of the computational cost of the explicit finite difference to that of the method of moments for the case of $\sigma_0 = 1$ to be

$$\frac{\text{Cost of Explicit FD}}{\text{Cost of Method of Moments}} \approx \frac{J}{K} \min\left(\frac{(\beta^S \lambda_{max})^2}{\delta^2}, \frac{1}{\bar{\Delta}}\right). \tag{5}$$

Time Step	Method of Moments	Explicit Finite Difference
10^{-2}	2.4890 seconds	Unstable
10^{-3}	25.7241 seconds	Unstable
10^{-4}	254.6494 seconds	Unstable
10^{-5}	2561.9614 seconds	8512.3172 seconds

Table: Actual computational times using $K = 100$ moments and 1000 Monte Carlo trials.

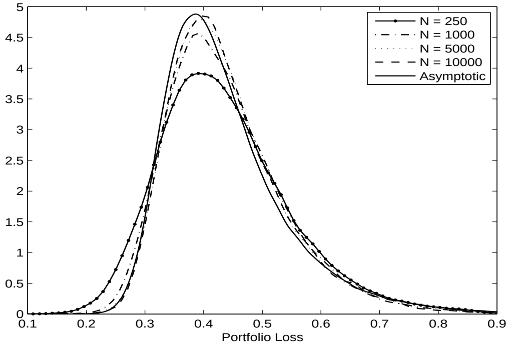
Comparison of simulation of asymptotic SPDE and simulation of the finite system.

Method of Moments with $K = 200$	6.24984 seconds
Finite System with $N = 500$	13.57564 seconds
Finite System with $N = 1000$	22.12235 seconds
Finite System with $N = 5000$	236.23656 seconds
Finite System with $N = 10000$	441.80289 seconds
Finite System with $N = 25000$	1048.80624 seconds

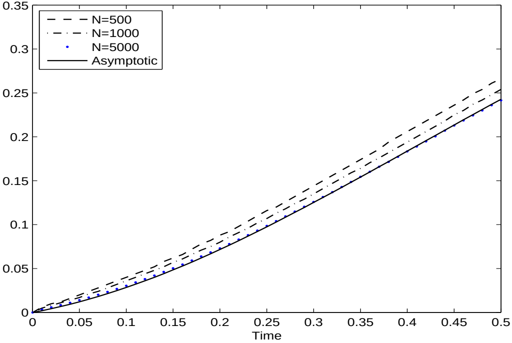
Table: 1000 Monte Carlo samples and a time horizon of $T = 1$

Numerical Convergence

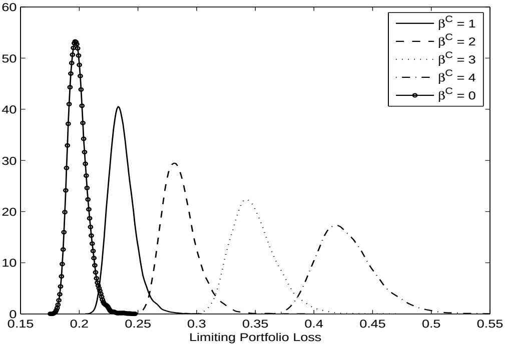
Numerical demonstration of convergence of distribution of the loss for the finite system to limiting loss distribution as N becomes large.



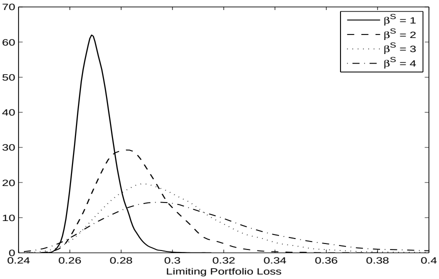
Numerical demonstration of convergence of value at risk (VaR) for the finite system to the VaR of the limiting losses.



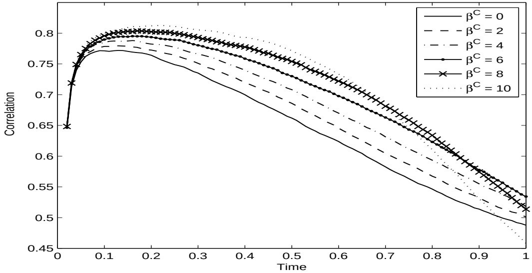
- Large β^C (level of contagion) shifts the loss distribution to the right and causes heavy right tails.
- Volatility of the loss increases as the connectivity between the firms increases.



- Large β^S (level of systematic risk) shifts the loss distribution to the right and causes heavy right tails.
- This is an example of the **complex interaction between contagion and systematic risk** in the model. Shocks to the system can spread through the contagion channel. The joint presence of systematic and contagion risk greatly magnifies the likelihood of extreme default events.

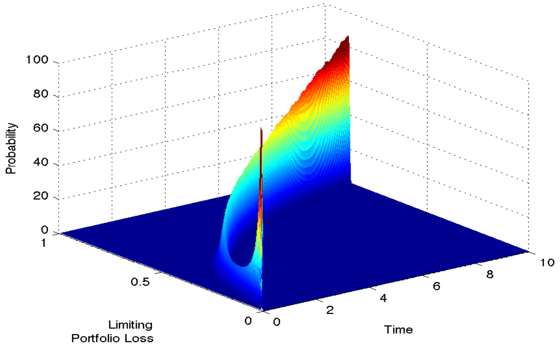


- The correlation between the systematic risk and the loss increases as β^C (which indicates the level of contagion) increases.
- The system becomes increasingly vulnerable to stresses from the systematic risk as the contagion channel in the system becomes stronger.



Loss Surface

- The model gives the loss distribution for many time horizons simulataneously.
- When the pool is almost exhausted, the dynamics of the system become more certain.



Heterogeneous Pool

The model allows for a significant amount of bottom-up heterogeneity; the intensity dynamics of each name can be different. Define the "types"

$$\mathrm{p}^{N,n} = (\alpha_{N,n}, \bar{\lambda}_{N,n}, \sigma_{N,n}, \beta^{C}_{N,n}, \beta^{S}_{N,n}) \in \mathcal{P} \overset{\text{def}}{=} \mathbb{R}^4_+ \times \mathbb{R};$$

For each $N \in \mathbb{N}$, let $\hat{\mathrm{p}}^{N,n}_t \overset{\text{def}}{=} (\mathrm{p}^{N,n}, \lambda^{N,n}_t) \in \hat{\mathcal{P}} \overset{\text{def}}{=} \mathcal{P} \times \mathbb{R}_+$ for all $n \in \{1, 2, \ldots, N\}$. The empirical measure for the surviving names in the pool is

$$\mu^N_t \overset{\text{def}}{=} \frac{1}{N} \sum_{n=1}^{N} \delta_{\hat{\mathrm{p}}^{N,n}_t} \mathbf{1}_{\tau^n > t},$$

Also, let the empirical measure of the intensity processes' parameters be $\pi^N \overset{\text{def}}{=} \frac{1}{N} \sum_{n=1}^{N} \delta_{\mathrm{p}^{N,n}}$ and assume that π^N weakly converges to $\pi \in \mathcal{P}$ as $N \longrightarrow \infty$.

Limiting SPDE for Heterogeneous Pool

Theorem. If there exists a solution to the SPDE

$$d\upsilon(t,\hat{\mathsf{p}}) = \Big\{\mathcal{L}_1^*\upsilon(t,\hat{\mathsf{p}}) + \mathcal{L}_3^{*,X_t}\upsilon(t,\hat{\mathsf{p}}) + \left(\int_{\hat{\mathsf{p}}'\in\hat{\mathcal{P}}} \mathcal{Q}(\hat{\mathsf{p}}')\upsilon(t,\hat{\mathsf{p}}')d\hat{\mathsf{p}}'\right)\mathcal{L}_2^*\upsilon(t,\hat{\mathsf{p}})\Big\}dt + \mathcal{L}_4^{*,X_t}\upsilon(t,\hat{\mathsf{p}})dV_t,\ \hat{\mathsf{p}}\in\hat{\mathcal{P}},$$

where $\mathcal{L}_i^*$ denote adjoint operators, with initial condition

$$\lim_{t\searrow 0}\upsilon(t,\hat{\mathsf{p}})d\hat{\mathsf{p}} = \pi\times\Lambda_\circ,$$

and boundary conditions

$$\upsilon(t,\lambda=0,\mathsf{p}) = \upsilon(t,\lambda=\infty,\mathsf{p}) = 0.$$

Then, μ_t^N weakly converges to $\upsilon(t,\hat{\mathsf{p}})d\hat{\mathsf{p}}$ in $D_E[0,T]$ as $N\longrightarrow\infty$.

Central Limit Theorem

Consider the process $\eta_t^N = \sqrt{N}(\mu_t^N - \upsilon)$. Then, in the case of $\beta^S = 0$, η_t^N weakly converges to $\eta(t, \lambda)$ as $N \longrightarrow \infty$ where $\eta(t, x)$ satisfies the SPDE

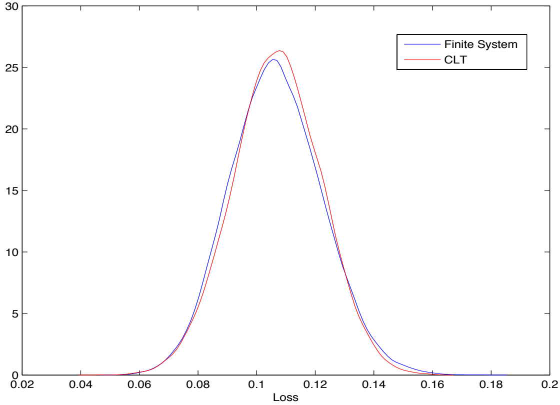
$$d\eta = \{\mathcal{L}_1^* \eta - \beta^C \big(\int_0^\infty \lambda \upsilon d\lambda \big) \frac{\partial \eta}{\partial \lambda} - \beta^C \big(\int_0^\infty \lambda \eta d\lambda \big) \frac{\partial \upsilon}{\partial \lambda} \} + dZ(t, \lambda), \tag{6}$$

where $Z(t, \lambda)$ is a **Gaussian process in both time and space** defined by the covariance relationship

$$\begin{aligned} Cov[\langle \phi, Z \rangle, \langle \psi, Z \rangle] = \sigma^2 \int_0^t \langle \phi' \psi' \lambda, \upsilon \rangle ds + \int_0^t \langle \lambda \phi \psi, \upsilon \rangle ds \\ - \beta^C \int_0^t \langle \phi', \upsilon \rangle \langle \psi \lambda, \upsilon \rangle ds - \beta^C \int_0^t \langle \psi', \upsilon \rangle \langle \phi \lambda, \upsilon \rangle ds \\ + (\beta^C)^2 \int_0^t \langle \phi', \upsilon \rangle \langle \psi', \upsilon \rangle \langle \lambda, \upsilon \rangle ds. \end{aligned} \tag{7}$$

$$\mu_t^N \approx \upsilon + N^{-\frac{1}{2}}\eta.$$

For $N = 500$ and $\beta^C = 1$, we compare the finite portfolio's loss distribution with the Gaussian approximation's loss distribution.



Conclusion

- We prove a law of large numbers for the loss in a credit portfolio
- The limiting measure solves an SPDE, which can be efficiently solved using the method of moments.
- We account for the main economic driving factors behind default clustering. The complex interaction between contagion and systematic risk is a central and unique feature of the model.
- Many applications, including: MBS, large ABS or corporate credit portfolios, microfinance, student loans.

Ongoing work includes:

- Developing maximum likelihood methods for estimation of model parameters. Standard particle filtering methods for hidden Markov models do not apply.
- Use of the limiting mean field approximation to facilitate simulation of the finite system
- Developing a central limit theorem (CLT)
- A similar mean field model can be developed for mortgage pools. The model captures prepayment, burnout, and correlation between default and prepayment. The limiting SPDE depends on a systematic risk X_t and the interest rate R_t.