Fluctuation Analysis for the Loss from Default

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Status Quo

- Dynamic reduced-form models of correlated name-by-name default timing are widely used to measure portfolio credit risk and to value securities exposed to correlated default risk
  - Default is a Poisson-type event
  - It arrives at an **intensity**, or conditional default rate
  - Intensity follows a stochastic process
- Computing the distribution of the loss from default in these models can be challenging in practice where portfolios of several thousands names are common and relatively long time horizons may be of interest.
  - Examples: loans, MBS, ABS, credit cards, student loans, microfinance.

Our Approach

- We develop a **large portfolio approximation** for the distribution of the loss.
  - Law of large numbers (nonlinear SPDE)
  - Fluctuation limit (linear SPDE)
- Fluctuation limit is linearized around the nonlinear dynamics of the law of large numbers (LLN)
- Efficient numerical solution using method of moments

Prior Work

Structural models

- Static models with conditionally i.i.d. losses
    - LLN for a homogeneous pool: Vasicek (1991) [Basel rules]
    - LLN for a heterogeneous pool: Gordy (2003)
    - LDP for a heterogeneous pool: Dembo, Deuschel and Duffie (2004); Glasserman, Kang, and Shahabuddin (2007)
- Dynamic models with conditionally i.i.d. losses
    - LLN for a homogeneous pool: Bush, Hambly, Haworth, Jin, and Reisinger (2011)

- Mean field interaction
    - Cvitanic, Ma, and Zhang (2011)
    - Garnier, Papanicolaou, and Yang (2012)
    - Dai Pra, Runggaldier, Sartori, and Tolotti (2009) and Dai Pra and Tolotti (2009)
    - This paper
- Local interaction
    - Giesecke and Weber (2006): voter model

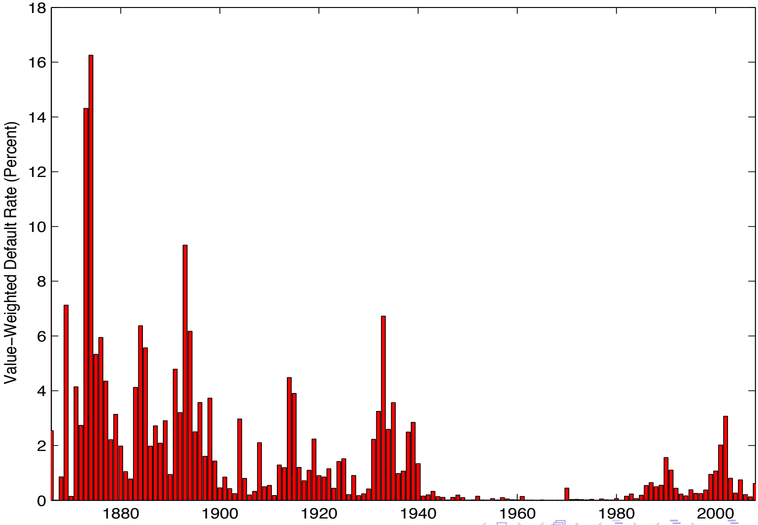
Our Model

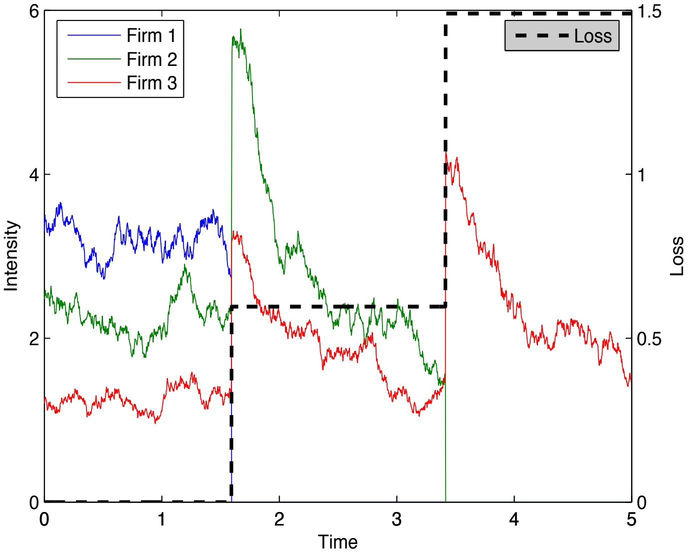
- Pool of $N \in \mathbb{N}$ names
  - **Portfolio loss rate** : $L_t^N = \frac{1}{N}\sum_{n=1}^N \mathbf{1}_{\tau^n \leq t}$
  - **Systematic risk process**: $dX_t = b_0(X_t)dt + \sigma_0(X_t)dV_t$.

The default intensity of a name $n \in \{1, 2, \dots N\}$ is

$$d\lambda_t^n = \alpha_n(\bar{\lambda}_n - \lambda_t^n)dt + \sigma_n\sqrt{\lambda_t^n}dW_t^n + \beta_n^C dL_t^N + \beta_n^S \lambda_t^n dX_t \qquad (1)$$

The model captures the three primary sources of defaults that have been observed in the empirical literature: idiosyncratic risk, contagion, and systematic risk. The latter two are responsible for default clustering.


Value–Weighted Default Rate (Percent)
18
16
14
12
10
8
6
4
2
0
1880
1900
1920
1940
1960
1980
2000


Firm 1
Firm 2
Firm 3
Loss
Intensity
Loss
Time
0
1
2
3
4
5
0
2
4
6
0
0.5
1
1.5

Homogeneous Pool

Suppose all names have the same parameters. The empirical measure for the surviving names in the pool is

$$\mu_t^N \overset{\text{def}}{=} \frac{1}{N}\sum_{n=1}^{N} \delta_{\lambda_t^n} \mathbf{1}_{\tau^n > t},$$

where $\mu_t^N \in D_E[0,T]$, the space of RCLL paths taking values in E, the set of sub-probability measures on $\mathbb{R}_+$. The loss is simply

$$L_t^N = 1 - \mu_t^N(\mathbb{R}_+)$$

Also, let the empirical measure of the initial conditions be

$$\Lambda_\circ^N \overset{\text{def}}{=} \frac{1}{N}\sum_{n=1}^{N} \delta_{\lambda_{\circ,N,n}};$$

We assume that $\Lambda_\circ^N$ weakly converges to $\Lambda_\circ \in \mathbb{R}_+$ as $N \longrightarrow \infty$.

Law of Large Numbers for Homogeneous Pool

Theorem

Suppose there exists a solution υ to the SPDE

$$d\upsilon(t,\lambda) = \mathcal{L}^{*,X_t}\upsilon(t,\lambda)dt + \beta^S \mathcal{L}_4^{*,X_t}\upsilon(t,\lambda)dV_t,$$
$$\upsilon(t=0,\lambda) = \Lambda_\circ,$$
$$\upsilon(t,\lambda=0) = \upsilon(t,\lambda=\infty) = 0, \tag{2}$$

where $\mathcal{L}^{*,X_t} \equiv \mathcal{L}_1^* + \mathcal{L}_3^{*,X_t} + \beta^C \left(\int_0^\infty \lambda\upsilon(t,\lambda)d\lambda\right)\mathcal{L}_2^*$*. Then, the empirical measure* μ_t^N *weakly converges to* $\bar{\mu}_t(d\lambda) = \upsilon(t,\lambda)d\lambda$ *in* $D_E[0,T]$ *as* $N \longrightarrow \infty$*. Furthermore, the limiting loss* L_t *is simply*

$$L_t = 1 - \int_0^\infty \upsilon(t,\lambda)d\lambda. \tag{3}$$

The differential operators are

$$
\begin{aligned}
\mathcal{L}_1 f &= \frac{1}{2}\sigma^2\lambda\frac{\partial^2 f}{\partial\lambda^2} - \alpha(\lambda - \bar{\lambda})\frac{\partial f}{\partial\lambda} - \lambda f \\
\mathcal{L}_2 f &= \frac{\partial f}{\partial\lambda} \\
\mathcal{L}_3^x f &= \beta^S\lambda b_0(x)\frac{\partial f}{\partial\lambda} + \frac{1}{2}(\beta^S)^2\lambda^2\sigma_0^2(x)\frac{\partial^2 f}{\partial\lambda^2} \\
\mathcal{L}_4^x f &= \lambda\sigma_0(x)\frac{\partial f}{\partial\lambda}.
\end{aligned}
$$

$\mathcal{L}_1$ contains the infinitesimal generator of the idiosyncratic process and $-\lambda f$ is a kill term. The nonlinear term $\left(\int_0^\infty \lambda v(t,\lambda)d\lambda\right)\mathcal{L}_2^* v(t,\lambda)$ comes from jumps due to contagion. $\mathcal{L}_3^x f$ is related to the infinitesimal generator of the systematic risk. $\mathcal{L}_4^{*,X_t} dV_t$ is the Brownian term driving the SPDE.

Second-Order Approximation

- We are interested in the empirical fluctuation process $\Xi_t^N = \sqrt{N}(\mu_t^N - \bar{\mu}_t)$ which has RCLL paths taking values in the set of signed measures.
- Can view a limit $\bar{\Xi}_t$ of the fluctuation process Ξ_t^N as a **dynamic central limit theorem**.
- Such a limit yields the **second-order approximation**

$$\mu_t^N(d\lambda) \approx \bar{\mu}(t, d\lambda) + \frac{1}{\sqrt{N}} \bar{\Xi}_t(d\lambda).$$

- This suggests the second-order approximation to the loss

$$L_t^N \approx L_t - \frac{1}{\sqrt{N}} \int_{\mathbb{R}^+} \bar{\Xi}_t(d\lambda).$$

Fluctuation Limit

Theorem

The signed measures Ξ_t^N weakly converge as $N \longrightarrow \infty$ to $\bar{\Xi}_t$. With probability 1, for any $f \in W_0^4(w, \rho)$, $\bar{\Xi}_t$ satisfies the stochastic evolution equation

$$d \left\langle f, \bar{\Xi}_t \right\rangle = \left\langle \mathcal{G}_{X_t, \bar{\mu}} f, \bar{\Xi}_s \right\rangle dt + \left\langle \mathcal{L}_4^{X_t} f, \bar{\Xi}_t \right\rangle dV_t + d \left\langle f, \bar{\mathcal{M}}_t \right\rangle, \ \ a.s.$$

The limiting stochastic evolution equation has a unique solution in $W_0^{-4}(w, \rho)$.

Theorem

Conditional on the σ-algebra $\mathcal{V}_t = \sigma(V_s, s \leq t)$, $\bar{\mathcal{M}}_t$ is a centered Gaussian with covariance function, for $f, g \in W_0^4(w, \rho)$, given by

$$\operatorname{Cov}\left[\left\langle f, \bar{\mathcal{M}}_{t_1}\right\rangle, \left\langle g, \bar{\mathcal{M}}_{t_2}\right\rangle \,\middle|\, \mathcal{V}_{t_1 \vee t_2}\right] = \mathbb{E}\left[\int_0^{t_1 \wedge t_2} \left[\langle \mathcal{L}_5(f,g), \bar{\mu}_s\rangle + \langle \mathcal{L}_6(f,g), \bar{\mu}_s\rangle + (\beta^C)^2 \langle \mathcal{L}_2 f, \bar{\mu}_s\rangle \langle \mathcal{L}_2 g, \bar{\mu}_s\rangle \langle \lambda, \bar{\mu}_s\rangle - \langle \mathcal{L}_7 g, \bar{\mu}_s\rangle \langle \mathcal{L}_2 f, \bar{\mu}_s\rangle - \langle \mathcal{L}_7 f, \bar{\mu}_s\rangle \langle \mathcal{L}_2 g, \bar{\mu}_s\rangle\right] ds \,\middle|\, \mathcal{V}_{t_1 \vee t_2}\right]$$

$$\mathcal{G}_{x,\mu} f = \mathcal{L}_1 f + \mathcal{L}_3^x f + \beta^C \langle \lambda, \mu\rangle + \mathcal{L}_2 f + \beta^C \langle \mathcal{L}_2 f, \mu\rangle \lambda$$

$$\mathcal{L}_5(f,g) = \sigma^2 \frac{\partial f}{\partial \lambda} \frac{\partial g}{\partial \lambda} \lambda$$

$$\mathcal{L}_6(f,g) = f(\lambda) g(\lambda) \lambda$$

$$\mathcal{L}_7 f = \beta^C f(\lambda) \lambda$$

Method of Moments

- Consider the moments $u_k(t) = \int_0^\infty \lambda^k \bar{\mu}_t(d\lambda)$ and $v_k(t) = \int_0^\infty \lambda^k \bar{\Xi}_t(d\lambda)$.
- The zeroth moments directly yield the second-order approximation of the loss

$$L_t^N \approx L_t - \frac{1}{\sqrt{N}} \int_0^\infty \bar{\Xi}_t(d\lambda) = 1 - u_0(t) - \frac{1}{\sqrt{N}} v_0(t).$$

- The moments $\{u_k(t)\}_{k=0}^\infty$ and $\{v_k(t)\}_{k=0}^\infty$ satisfy two respective systems of SDEs which can be solved efficiently.
  - The systems are not closed, so one must truncate at some level K.

LLN Moment System

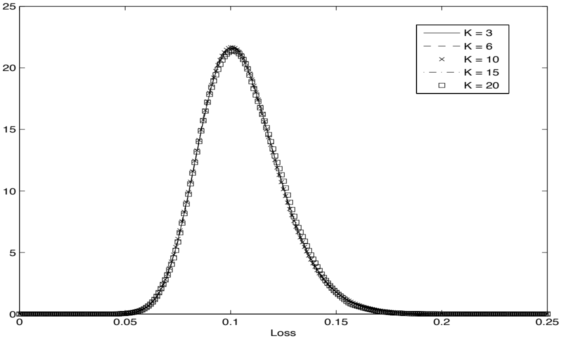
$$
\begin{aligned}
du_k(t) &= \Big\{u_k(t)\big(-\alpha k + \beta^S b_0(X_t)k + \frac{1}{2}(\beta^S)^2\sigma_0^2(X_t)k(k-1)\big) \\
&+ u_{k-1}(t)\big(\frac{1}{2}\sigma^2 k(k-1) + \alpha\bar{\lambda}k + \beta^C k u_1(t)\big) \\
&- u_{k+1}(t)\Big\}dt + \beta^S \sigma_0(X_t) k u_k(t) dV_t, \\
u_k(t=0) &= \int_0^\infty \lambda^k \bar{\mu}_{t=0}(d\lambda).
\end{aligned}
$$

Fluctuation Moment System

$$
\begin{aligned}
dv_k(t) &= \Big\{\beta^C k u_{k-1}(t) v_1(t) + \Big[\frac{\sigma^2}{2} k(k-1) + \alpha\bar{\lambda} k \\
&+ k\beta^C u_1(t)\Big] v_{k-1}(t) - v_{k+1}(t) + \Big[k\beta^S b_0(X_t) - k\alpha \\
&+ 0.5k(k-1)\left(\beta^S \sigma_0(X_t)\right)^2\Big] v_k(t)\Big\} dt \\
&+ k\beta^S \sigma_0(X_t) v_k(t) dV_t + d\mathbf{M}_k(t), \\
v_k(0) &= \int_0^\infty \lambda^k \Xi_{t=0}(d\lambda),
\end{aligned}
$$

where $\mathbf{M}_k(t) = \langle \lambda^k, \bar{\mathcal{M}}_t \rangle$ and the quadratic covariation $[\mathbf{M}_k(t), \mathbf{M}_j(t)] = \left(\Sigma_{\mathcal{M}}(t)\right)_{kj} dt$. Note that the quadratic covariation depends upon the path of X.

The LLN and fluctuation moment systems converge very quickly in terms of the truncation level K.



The fundamental solution $\Psi : [0, T] \times \Omega \longrightarrow \mathbb{R}^{K+1,K+1}$ satisfies

$$d\Psi(t) = A(t)\Psi(t)dt + \beta^S B\Psi(t)dX_t,$$
$$\Psi(t = 0) = I.$$

If $\beta^S = 0$, $\mathbf{v}(t)$ is a Gaussian process with mean zero and covariance

$$\Sigma(t) = \Psi(t)\Big[\int_0^t \Psi^{-1}(s)\Sigma_{\mathcal{M}}(s)\big(\Psi^{-1}(s)\big)^\top ds\Big]\Psi(t)^\top \tag{4}$$

Therefore, we can compute the solution completely **semi-analytically**. Furthermore, if $\beta^S = 0$ and $\beta^C = 0$, there is a **closed-form solution**.

$$\Sigma(t) = \int_0^t e^{A(t-s)}\Sigma_{\mathcal{M}}(s)e^{A^\top(t-s)}ds$$

- Simulate paths $X^1, \dots, X^M$ of the systematic risk process on $[0, T]$
- Conditional upon each path X^m, the law of large numbers moments $\mathbf{u}^m(t)$ are deterministic and the fluctuation moments $\mathbf{v}^m(t)$ have a Gaussian solution. Furthermore, we can **semi-analytically** compute the distribution of $\mathbf{v}^m(t)$.
- Approximate the unconditional loss distribution by $\frac{1}{M}\sum_{m=1}^{M} \mathbb{P}^m(\cdot)$, where $\mathbb{P}^m$ is a Gaussian measure with mean $L^m = 1 - u_0^m$ and variance $\mathrm{Var}[v_0^m(t)]/N$.

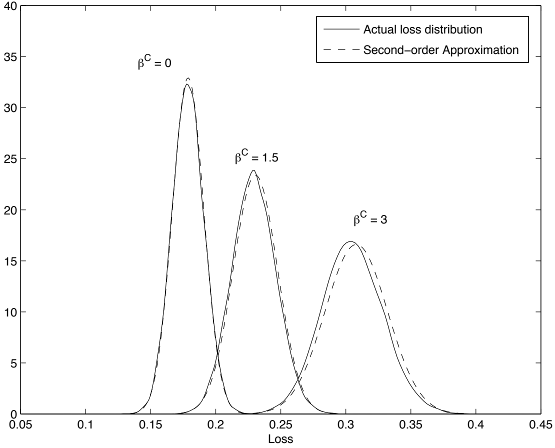


Figure: Comparison of approximate loss distribution and actual loss distribution in the finite system at $T = 1$ for $N = 1,000$. The parameter case is $\sigma = .9$, $\alpha = 4$, $\lambda_0 = .2$, $\bar{\lambda} = .2$, and $\beta^S = 0$.

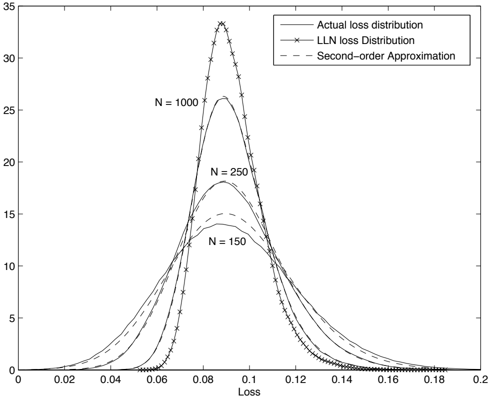


Figure: Comparison of approximate loss distribution and actual loss distribution in the finite system at $T = .5$. Parameter case is $\sigma = .9, \alpha = 4, \lambda_0 = .2, \bar{\lambda} = .2, \beta^C = 0$, and $\beta^S = 1$. X is an OU process with mean 1, reversion speed 2, and volatility 1.

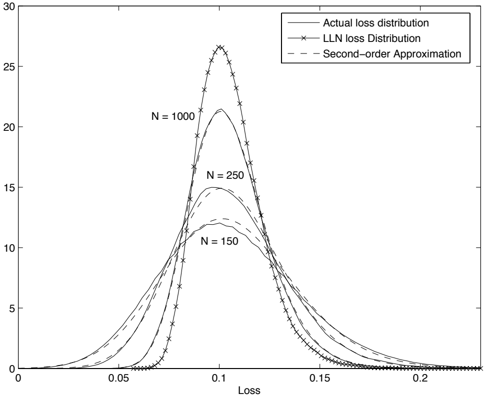


Figure: Comparison of approximate loss distribution and actual loss distribution in the finite system at $T = .5$. Parameter case is $\sigma = .9, \alpha = 4, \lambda_0 = .2, \bar{\lambda} = .2, \beta^C = 1$, and $\beta^S = 1$. X is an OU process with mean 1, reversion speed 2, and volatility 1.

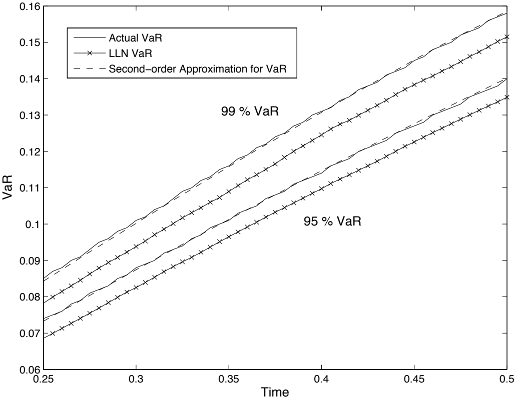


Figure: Comparison of first- and second-order approximations with actual Value at Risks for $N = 1,000$. Parameter case is $\sigma = .9, \alpha = 4, \lambda_0 = .2, \bar{\lambda} = .2, \beta^C = 1$, and $\beta^S = 1$. X is an OU process with mean 1, reversion speed 2, and volatility 1.

Conclusion

- We have developed a **second-order approximation** of the loss in a large credit pool which is accurate even for moderately-sized pools.
- Computationally efficient numerical scheme using a method of moments.
- Tractable parameter estimation for large stochastic systems is possible using the second-order approximation.
- Next step: empirical studies for large credit pools (for example, mortgage-backed securities).

Heterogeneous Case

The model allows for a significant amount of bottom-up heterogeneity; the intensity dynamics of each name can be different. Define the "types"

$$\mathrm{p}^{N,n} = (\alpha_{N,n}, \bar{\lambda}_{N,n}, \sigma_{N,n}, \beta^C_{N,n}, \beta^S_{N,n}) \in \mathcal{P} \stackrel{\text{def}}{=} \mathbb{R}^4_+ \times \mathbb{R};$$

For each $N \in \mathbb{N}$, let $\hat{\mathrm{p}}^{N,n}_t \stackrel{\text{def}}{=} (\mathrm{p}^{N,n}, \lambda^{N,n}_t) \in \hat{\mathcal{P}} \stackrel{\text{def}}{=} \mathcal{P} \times \mathbb{R}_+$ for all $n \in \{1, 2, \dots, N\}$. The empirical measure for the surviving names in the pool is

$$\mu^N_t \stackrel{\text{def}}{=} \frac{1}{N} \sum_{n=1}^{N} \delta_{\hat{\mathrm{p}}^{N,n}_t} \mathbf{1}_{\tau^n > t},$$

Also, let the empirical measure of the intensity processes' parameters be $\pi^N \stackrel{\text{def}}{=} \frac{1}{N} \sum_{n=1}^{N} \delta_{\mathrm{p}^{N,n}}$ and assume that π^N weakly converges to $\pi \in \mathcal{P}$ as $N \longrightarrow \infty$.

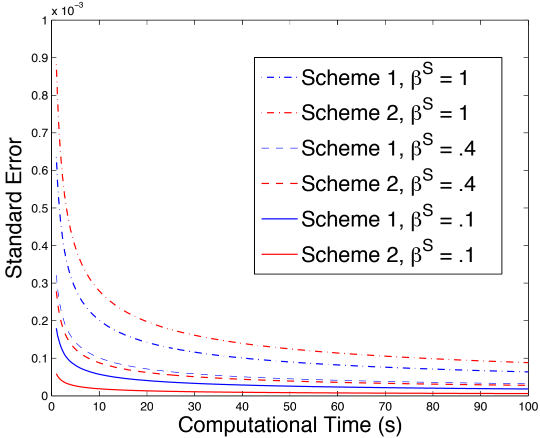


Figure: Comparison of standard error for direct simulation and conditionally semi-analytic simulation. Parameter case is $T = .5, N = 500, \sigma = .9, \alpha = 4, \lambda_0 = .2, \bar{\lambda} = .2$, and $\beta^C = 1$. X_t is an OU process with mean 1, reversion speed 2, and volatility 1.

A collection of measures $\{P_n\}_{n=1}^{\infty}$ on $(\Omega, \mathcal{F})$ is tight if, for any $\epsilon > 0$, there exists a compact subset K_ϵ of Ω such that for every $n \in \mathbb{N}$

$$P_n(K_\epsilon^c) < \epsilon \tag{5}$$

In addition, Prokhorov's theorem tells us that a collection of measures $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ is tight if and only if the closure of $\mathcal{P}$ is sequentially compact. We say that $\mathcal{P}$ is relatively compact if any sequence has a weakly convergent subsequence.

- Tightness implies relative compactness
- If $(\Omega, \mathcal{F})$ is a Polish space, relative compactness implies tightness.

Tightness in the Skorokhod space $D_E([0, T])$ can be proven using the dual criterion in Ethier and Kurtz (compact containment and a regularity condition).

The weighted Sobolev space $W_0^J(w,\rho)$ is the closure of C_0^∞ in the norm

$$||f||_{W_0^J(w,\rho)} = \Big(\sum_{k\leq J}\int w^2(\lambda)|\rho^k(\lambda)D^k f(\lambda)|^2 d\lambda\Big)^{\frac{1}{2}} \tag{6}$$

$W_0^J(w,\rho)$ is a Hilbert space for every $J \geq 2$. We also require that for every $k \leq J$, the functions $\rho^{k-1}D^k\rho$ and $w^{-1}\rho^k D^k w$ are bounded. An example for $J = 2$ that satisfies this requirement is

$\rho = \sqrt{1+\lambda^2}$ and $w = (1+\lambda^2)^\beta$ for $\beta < -3$.

Proving tightness for the fluctuation case is much more difficult than for the LLN case.

- Space of signed measures endowed with the weak topology is not metrizable
- "Standard approach": consider a weighted Sobolev space
  - Identifying the correct weights is difficult due to the growth and degeneracy of the coefficient functions
- Consider an orthonormal basis $\{f_a\}_{a=1}^{\infty}$ of $W_0^4(w,\rho)$ to prove (using Parseval's Identity)

$$\sup_{N\in\mathbb{N}} \sup_{0\leq t\leq T} \mathbb{E}[||\Xi_t^N||^2_{W_2^{-4}(w,\rho)}] = \sup_{N\in\mathbb{N}} \sup_{0\leq t\leq T} \sum_{a\geq 1} \mathbb{E}\langle \Xi_t^N, f_a \rangle^2 \leq C \tag{7}$$

- Convergence of martingale term essentially follows from a martingale central limit theorem.

Assumptions

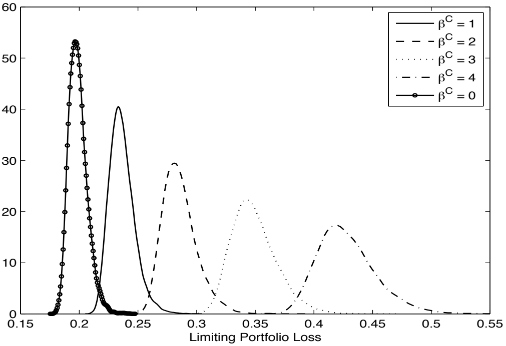
- Compact containment of parameters
- Weak convergence of parameters and initial conditions
- X has a unique, strong solution. Moreover, there is a function $u(x)$ such that $\sigma_0(x)u(x) = -b_0(x)$ and for every $T > 0$ we have

$$\mathbb{E}[e^{\frac{1}{2}\int_0^T |u(X_s)|^2 ds}] < \infty. \tag{8}$$

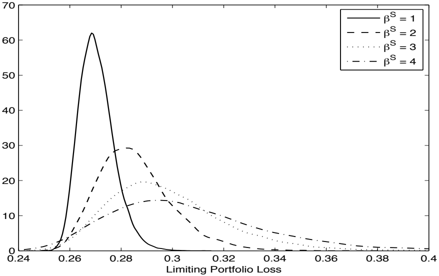
- For the fluctuation equation, we also need

$$\mathbb{E}\Big[\int_0^T \big[|b_0(X_s)|^2 + |\sigma_0(X_s)|^4\big] ds < \infty. \tag{9}$$

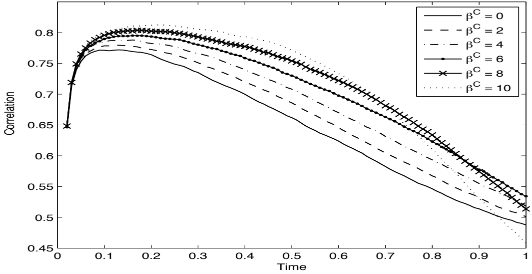
- Large β^C (level of contagion) shifts the loss distribution to the right and causes heavy right tails.
- Volatility of the loss increases as the connectivity between the firms increases.



- Large β^S (level of systematic risk) shifts the loss distribution to the right and causes heavy right tails.



- The correlation between the systematic risk and the loss increases as β^C (which indicates the level of contagion) increases.
- There is a **complex interaction between contagion and systematic risk** in the model. The system becomes increasingly vulnerable to shocks from the systematic risk as the contagion channel in the system becomes stronger.



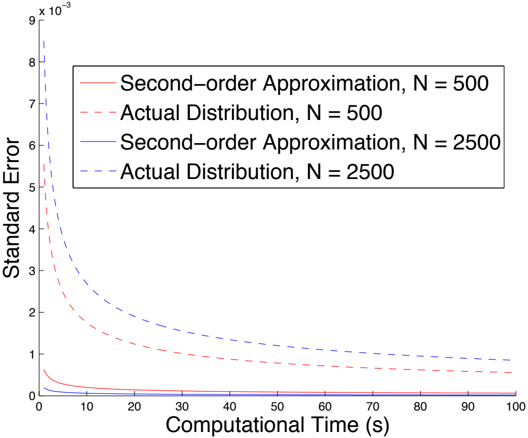


Figure: Comparison of standard error for direct simulation of the finite system and the second-order approximation . Parameter case is $T = .5, \sigma = .9, \alpha = 4, \lambda_0 = .2, \bar{\lambda} = .2, \beta^C = 1$, and $\beta^S = 1$. X_t is an OU process with mean 1, reversion speed 2, and volatility 1.

- Truncation level of $K = 6$ and a time step of .005 for both asymptotic and finite systems.
- Total computational time of 50 seconds.
- For the standard errors to be equal, we must invest more computational resources into the finite system simulation:
  - 2 orders of magnitude more for $N = 500$
  - 4 orders of magnitude more for $N = 2,500$

	Finite System	Second-order Approximation
Standard Error ($N = 500$)	7.8436×10^{-4}	8.9991×10^{-5}
Standard Error ($N = 2,500$)	1.2022×10^{-3}	2.7449×10^{-5}