Likelihood Inference for Large Financial Systems

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November 17, 2014

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Introduction

- Large interacting systems appear in many areas

- Finance: models of default and systemic risk for large credit pools and banking systems
    - Bush, Hambly, Haworth, Jin, and Reisinger (2011)
    - Cvitanic, Ma, and Zhang (2011)
    - Fouque and Ichiba (2012)
    - Garnier, Papanicolaou, and Yang (2012)
    - Giesecke, Spiliopoulos, Sowers, and Sirignano (2013)
    - Spiliopoulos, Sirignano, and Giesecke (2014)

- **Statistical estimation of model parameters is challenging** due to the size and complexity of the systems

Our Approach

- We exploit weak convergence results for large systems to develop tractable parameter estimation.
- **Large system approximation** constructed using:
  - Law of large numbers
  - Central limit theorem
- Instead of parameter estimation using the true likelihood of large but finite systems, we estimate parameters using the **approximate likelihood** from the large system approximation.
- Approximate maximum likelihood estimators (MLEs) are computationally tractable and we show they are consistent and asymptotically normal.

We consider a class of large systems:

- Interacting particle system of jump-diffusion processes
- Each particle is driven by
  - an **idiosyncratic** process
  - a **systematic** random process
  - **interaction with other particles** through a mean field term
- Default and systemic risk models mentioned earlier fall into this class of systems.

We consider a system of interacting particles $\{Y_t^n\}_{n=1}^N$ with a systematic process X and parameter vector θ:

$$dX_t = a(X_t)dt + dW_t,$$
$$dY_t^n = \nu_\theta dt + \sigma_\theta^\top dW_t^n + \gamma_\theta dJ_t^N + \zeta_\theta^\top dX_t, \quad n = 1, \ldots, N,$$

where the coefficient functions $\nu, \sigma, \gamma, \zeta$ can depend upon $\left(Y_t^n, X, \mu_t^N\right)$. The particles interact through the empirical measure

$$\mu_t^N = \frac{1}{N}\sum_{n=1}^N \delta_{Y_t^n},$$

and the jump process J_t^N, whose intensity can also depend upon $\left(Y_t^n, X, \mu_t^N\right)$. **Our goal is to estimate** θ!

Data

- Suppose we have the observation $\mathsf{D}_{M,N} = \big(\mathsf{Z}_{M,N}, \mathsf{X}_M\big)$ where

$$
\begin{aligned}
\mathsf{Z}_{M,N} &= (Z^N_{t_1}, \ldots, Z^N_{t_M}), \\
Z^N_t &= \left\langle f, \mu^N_t \right\rangle, \\
\mathsf{X}_M &= (X_t)_{t \in [0, t_M]}.
\end{aligned}
$$

- The goal is to estimate the true parameter θ_0 from the data $\mathsf{D}_{M,N}$.

Likelihood Function

- The true likelihood of the observed data is

$$\mathcal{L}_{M,N}(\theta) = p_\theta^N(Z_{t_1}^N, \ldots, Z_{t_M}^N | \mathrm{X}_M)$$

- These likelihood functions are typically computationally intractable!
  - The transition density is usually only available via Monte Carlo simulation and is computationally expensive for large N.
  - Furthermore, computing p^N typically requires filtering. Here, there are **N particles to filter!** Consequently, maximum likelihood estimation is computationally intractable.

- Our approach: use the **large system approximation** to obtain a computationally tractable **approximate likelihood function**.
- Define the empirical fluctuation process

$$\Xi_t^N = \sqrt{N}(\mu_t^N - \bar{\mu}_t).$$

- If $(\mu_t^N, \Xi_t^N, X) \xrightarrow{d} (\bar{\mu}_t, \bar{\Xi}_t, X)$, we have the second-order approximation

$$\mu_t^N \overset{d}{\approx} \bar{\mu}_t + \frac{1}{\sqrt{N}} \bar{\Xi}_t.$$

- Typically, $\bar{\mu}_t$ satisfies a nonlinear stochastic evolution equation and $\bar{\Xi}_t$ is the conditionally Gaussian solution of a linear stochastic evolution equation.

Stochastic Evolution Equations

- The limiting measures satisfy stochastic evolution equations:

$$
\begin{aligned}
d\bar{\mu}_s &= \mathcal{A}^1_{X_s,\bar{\mu}_s}\bar{\mu}_s ds + \mathcal{A}^2_{X_s,\bar{\mu}_s}\bar{\mu}_s dX_s, \\
d\bar{\Xi}_s &= \mathcal{G}^1_{X_s,\bar{\mu}_s}\bar{\Xi}_s ds + \mathcal{G}^2_{X_s,\bar{\mu}_s}\bar{\Xi}_s dX_s + d\bar{\mathcal{M}}_s,
\end{aligned}
$$

- Reduce $\bar{\Xi}_s$ to a system of SDEs:

$$
d\bar{\mathbf{v}}_t = A_1(X_t)\bar{\mathbf{v}}_t dt + A_2(X_t)\bar{\mathbf{v}}_t dX_t + Bd\mathbf{B}_t,
$$

- Use the fundamental solution $\Psi(s,t)$!

$$
\bar{\mathbf{v}}_t = \Psi(s,t)\bar{\mathbf{v}}_s + \Psi(s,t)\int_s^t \Psi^{-1}(s,u)Bd\mathbf{B}(u).
$$

Approximate Likelihood Function

- The approximate likelihood is

$$\mathcal{L}^A_{M,N}(\theta) = \bar{p}^N_\theta(Z^N_{t_1}, \dots, Z^N_{t_M} | \mathrm{X}_M),$$

  where $\bar{p}^N_\theta$ is the density under the second-order approximation $\mu^N_t \overset{d}{\approx} \bar{\mu}_t + \frac{1}{\sqrt{N}} \bar{\Xi}_t$.

- The approximate estimator is

$$\theta^A_{M,N} = \arg\max_{\theta \in \Theta} \mathcal{L}^A_{M,N}(\theta).$$

- $\bar{p}^N_\theta(\mathrm{Z}_{M,N} | \mathrm{X}_M)$ can be calculated **semi-analytically** since the large system approximation is **conditionally Gaussian**!

Theorem (Consistency of Approximate Estimator)

Suppose the data $\mathrm{D}_{M,N}$ *is produced by the true parameter* θ_0*. Provided certain technical assumptions hold (see Appendix), then the approximate estimator is consistent:* $\theta^A_{M,N} \xrightarrow{p} \theta_0$ *as* $N \to \infty$*.*

Theorem (Asymptotic Normality of Approximate Estimator)

In addition, provided certain technical assumptions, the approximate estimator is asymptotically normal: $\sqrt{N}(\theta^A_{M,N} - \theta_0) \xrightarrow{d} Q$*, where* Q *is conditionally Gaussian given* X_M*.*

Relationship with Least Squares

- The approximate estimator converges to a weighted least squares estimator as $N \to \infty$:

$$\theta^A_{M,\infty} \quad \in \quad \arg\max_{\theta \in \Theta} -\frac{1}{2}(Z_{M,\infty} - m^M_\theta)^\top (\Sigma^M_\theta)^{-1}(Z_{M,\infty} - m^M_\theta).$$

- Σ^M_θ is the covariance from the CLT with parameter θ and m^M_θ is the LLN with parameter θ.
- In fact, the (unweighted) least squares estimator using just the LLN is a consistent estimator of θ_0!

Consider the mean-field system:

$$
\begin{aligned}
dY_t^{N,n} &= \beta^C \left\langle y, \mu_t^N \right\rangle dt + \sqrt{Y_t^{N,n}}\, dW_t^{N,n} + \beta^S Y_t^{N,n} dX_t, \\
dX_t &= dW_t. \qquad (1)
\end{aligned}
$$

We wish to estimate the parameters $\theta = (\beta^C, \beta^S)$.

N	Approx. Estimator for β^C	Approx. Estimator for β^S
100	.2397	.008325
500	.2635	.008660
1,000	.2429	.01122
10,000	.2492	.01035
100,000	.2495	.009931

Table : Approximate estimators for the system (1) with true parameter $\theta_0 = (.25, .01)$. N is the size of the system.

Extensions

- Discrete observations of X
- Continuous observation of Z_t^N
- Simultaneous parameter estimation for the systematic process X
- Discrete instead of continuous-time dynamics

Conclusion

- Parameter estimation for a class of interacting systems is computationally intractable
- Approximate likelihood estimators are obtained via a large system approximation
- Approximate estimators are consistent and asymptotically normal