Gaussian Approximations for Portfolio Credit Risk

Justin A. Sirignano

Management Science and Engineering
Stanford University
jasirign@stanford.edu

July 11, 2012

Joint work with **Kay Giesecke** (Stanford) and **Konstantinos Spiliopoulos** (Boston University)

Status Quo

- Dynamic reduced-form models of correlated name-by-name default timing are widely used to measure portfolio credit risk and to value securities exposed to correlated default risk
  - Default is a Poisson-type event
  - It arrives at an **intensity**, or conditional default rate
  - Intensity follows a stochastic process
- Computing the distribution of the loss from default in these models can be challenging
  - Semi-analytical transform methods have limited scope
  - Simulation methods have much wider scope, but can be slow in practice, where portfolios of several thousand names are common and relatively long time horizons may be of interest

- We are interested in credit defaults in **large portfolios**
  - Examples: loans, MBS, ABS, credit cards, student loans, microfinance
- "Law of large numbers": the limiting measure of the portfolio satisfies a quasilinear stochastic partial differential equation (SPDE)
- Second-order accurate approximation using a type of "**dynamic central limit theorem**"
  - Linear SPDE with conditionally Gaussian solution
  - We develop a numerical scheme to solve the SPDE
  - Very accurate even for relatively "small" portfolios

Prior work on CLTs

CLTs for dynamic reduced form models

- Mean field interaction
  - Dai Pra, Runggaldier, Sartori, and Tolotti (2009)
  - Dai Pra and Tolotti (2009)
  - This paper
- Local interaction
  - Giesecke and Weber (2006): voter model

Model Setting

We develop a dynamic, interacting point process framework to model credit default in a pool of securities.

- Pool of $N \in \mathbb{N}$ names
  - **Portfolio loss rate** : $L_t^N = \frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_{\tau^n \leq t}$
  - **Systematic risk process**: $dX_t = b_0(X_t)dt + \sigma_0(X_t)dV_t$.

The default intensity of a name $n \in \{1, 2, \ldots N\}$ is

$$d\lambda_t^n = \alpha_n(\bar{\lambda}_n - \lambda_t^n)dt + \sigma_n\sqrt{\lambda_t^n}dW_t^n + \beta_n^C dL_t^N + \beta_n^S \lambda_t^n dX_t \qquad (1)$$

The model captures the three primary sources of defaults that have been observed in the empirical literature: idiosyncratic risk, contagion, and systematic risk. The latter two are responsible for default clustering.

Homogeneous Pool

Suppose all names have the same parameters. The empirical measure for the surviving names in the pool is

$$\mu_t^N \stackrel{\text{def}}{=} \frac{1}{N} \sum_{n=1}^{N} \delta_{\lambda_t^n} \mathbf{1}_{\tau^n > t},$$

where $\mu_t^N \in D_E[0, T]$, the space of RCLL paths taking values in E, the set of sub-probability measures on $\mathbb{R}_+$. The loss is simply

$$L_t^N = 1 - \mu_t^N(\mathbb{R}_+)$$

Limiting SPDE for Homogeneous Pool

Theorem

Suppose there exists a solution υ to the SPDE

$$
\begin{gathered}
d\upsilon(t,\lambda) = \mathcal{L}_1^{*,X_t}\upsilon(t,\lambda)dt + \mathcal{L}_2^{*,X_t}\upsilon(t,\lambda)dV_t, \\
\upsilon(t=0,\lambda) = \Lambda_\circ, \\
\upsilon(t,\lambda=0) = \upsilon(t,\lambda=\infty) = 0.
\end{gathered}
$$

Then, the empirical measure μ_t^N weakly converges to $\upsilon(t,\lambda)d\lambda$ in $D_E[0,T]$ as $N \longrightarrow \infty$. Furthermore, the limiting loss L_t is simply

$$
L_t = 1 - \int_0^\infty \upsilon(t,\lambda)d\lambda.
$$

Gaussian Correction

Consider the empirical fluctuation process $\eta_t^N = \sqrt{N}(\mu_t^N - \upsilon)$ which has RCLL paths taking values in the set of signed measures.

Theorem

Suppose that there exists a solution η to the SPDE

$$d\eta = \mathcal{G}^{*,X_t,\upsilon}\eta dt + \mathcal{L}_2^{*,X_t}\eta dV_t + d\bar{\mathcal{M}}_t.$$

Then, η_t^N weakly converges to $\eta(t,\lambda)d\lambda$ in the space $D_{W_0^{-4}(w,\rho)}[0,T]$ as $N \longrightarrow \infty$ where $W_0^{-4}(w,\rho)$ is a weighted Sobolev space. Furthermore, η is unique.

The space-time random process $\bar{\mathcal{M}}_t$ is **conditionally Gaussian** given the σ-algebra $\mathcal{V} = \sigma(\{V_t\}_{t=0}^T)$ with zero mean and covariance

$$\begin{aligned}\mathrm{Cov}[\langle \phi, \bar{\mathcal{M}}_t \rangle, \langle \psi, \bar{\mathcal{M}}_t \rangle] = \sigma^2 \int_0^t \langle \phi' \psi' \lambda, \upsilon \rangle ds + \int_0^t \langle \lambda \phi \psi, \upsilon \rangle ds \\ -\beta^C \int_0^t \langle \phi', \upsilon \rangle \langle \psi \lambda, \upsilon \rangle ds - \beta^C \int_0^t \langle \psi', \upsilon \rangle \langle \phi \lambda, \upsilon \rangle ds \\ +(\beta^C)^2 \int_0^t \langle \phi', \upsilon \rangle \langle \psi', \upsilon \rangle \langle \lambda, \upsilon \rangle ds.\end{aligned}$$

The fluctuation limit $\eta(t, \lambda)$ can be used to develop a **second-order approximation** for a pool of size N.

$$\mu_t^N(\lambda) \approx \upsilon(t, \lambda) + \frac{1}{\sqrt{N}} \eta(t, \lambda)$$

The loss in the finite pool can be approximated as

$$L_t^N \approx L_t - \frac{1}{\sqrt{N}} \xi_t,$$

where $L_t = 1 - \int_0^\infty \upsilon(t, \lambda) d\lambda$ and $\xi_t = \int_0^\infty \eta(t, \lambda) d\lambda$.

We reduce the SPDE to a system of SDEs using a method of moments where $v_k(t) = \int_0^\infty \lambda^k \eta(t,\lambda) d\lambda$. This system is non-closed, so we truncate at level K.

$$\begin{aligned} d\mathbf{v}(t) &= A(t)\mathbf{v}(t)dt + \beta^S B\mathbf{v}(t)dX_t + d\mathbf{M}(t) \\ \mathbf{v}(t=0) &= 0 \end{aligned} \tag{2}$$

where $\mathbf{M}_k(t) = \left\langle \lambda^k, \bar{\mathcal{M}}(t) \right\rangle$ and $[\mathbf{M}_k(t), \mathbf{M}_j(t)] = \left(\Sigma_{\mathcal{M}}(t)\right)_{kj} dt$.

The fundamental solution $\Psi : [0,T] \times \Omega \longrightarrow \mathbb{R}^{K+1,K+1}$ satisfies

$$\begin{aligned} d\Psi(t) &= A(t)\Psi(t)dt + \beta^S B\Psi(t)dX_t \\ \Psi(t=0) &= I \end{aligned} \tag{3}$$

If $\beta^S = 0$, $\mathbf{v}(t)$ is a Gaussian process with mean zero and covariance

$$\Sigma(t) = \Psi(t)\Big[\int_0^t \Psi^{-1}(s)\Sigma_{\mathcal{M}}(s)\big(\Psi^{-1}(s)\big)^\top ds\Big]\Psi(t)^\top$$

Therefore, we can compute the solution completely **semi-analytically**.

If $\beta^S = 0$ and $\beta^C = 0$, there is a **closed-form solution**.

$$\Sigma(t) = \int_0^t e^{A(t-s)}\Sigma_{\mathcal{M}}(s)e^{A^\top(t-s)}ds$$

General Case: $\beta^C, \beta^S > 0$

- Simulate paths $X^1, \ldots, X^M$ of the systematic risk process on $[0, T]$
- Conditional upon each path X^m, there is a Gaussian solution $\mathbf{v}^m(t)$ to the moment system whose distribution can be semi-analytically calculated.
  - Solve for the LLN loss L^m and the fundamental solution $\Psi^m(t)$
  - Compute the conditional covariance matrix using the closed-form formula given earlier. This yields $\mathrm{Var}[v_0^m(t)]$.
- Approximate the unconditional distribution as $\mathbb{P}[L_t^N \in \cdot] \approx \frac{1}{M}\sum_{m=1}^M p^m(\cdot)$ where p^m is a normal probability density function with mean L^m and variance $\frac{\mathrm{Var}[v_0^m(t)]}{N}$.

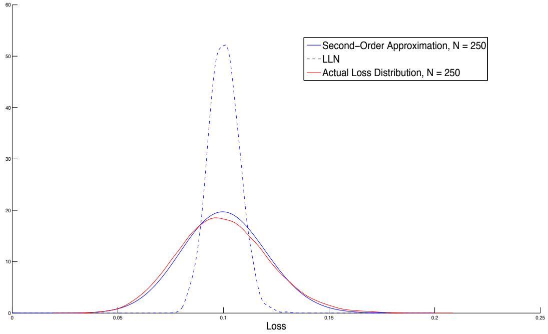


Figure: Comparison of second-order approximate loss distribution, LLN loss distribution, and actual loss distribution in the finite system at $T = .5$. Parameter case is $\sigma = .9, \alpha = 4, \lambda_0 = .2, \bar{\lambda} = .2, \beta^C = .5$, and $\beta^S = .5$. X_t is an OU process with mean 1, reversion speed 2, and volatility 1.

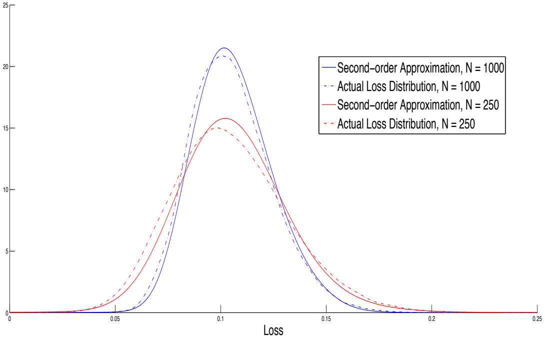
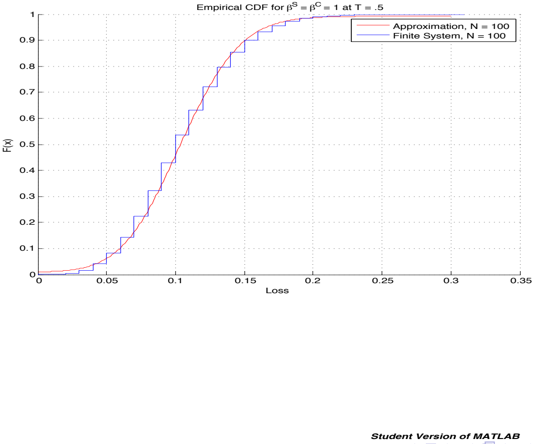


Figure: Comparison of approximate loss distribution and actual loss distribution in the finite system at $T = .5$. Parameter case is $\sigma = .9, \alpha = 4, \lambda_0 = .2, \bar{\lambda} = .2, \beta^C = 1$, and $\beta^S = 1$. X_t is an OU process with mean 1, reversion speed 2, and volatility 1.


Empirical CDF for $\beta^S = \beta^C = 1$ at T = .5
Approximation, N = 100
Finite System, N = 100
F(x)
Loss
Student Version of MATLAB

Conclusion

- We extend previous results on a law of large numbers by developing a central limit theorem.
- The limiting fluctuation measure solves a linear SPDE.
- Semi-analytic and analytic solutions available in some parameter cases.
- We propose a general numerical method to solve the CLT and LLN SPDEs.
- Many applications, including: MBS, large ABS or corporate credit portfolios, microfinance, student loans.

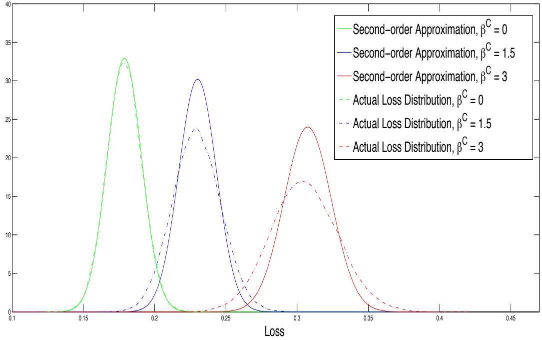


Figure: Comparison of approximate loss distribution and actual loss distribution in the finite system at $T = 1$ for $N = 1000$. Parameter case is $\sigma = .9, \alpha = 4, \lambda_0 = .2, \bar{\lambda} = .2$, and $\beta^S = 0$.