MTL103: Optimization Methods and Applications Spring 2023

Lecture 11 — 31st January, 2023

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1 Overview

In the last lecture, we studied the general iteration of the **Simplex Method**. We understood how to select and when to remove variables from the basis vector by looking at the reduced cost $\hat{c}_j$. We realised that this method takes $O(m^3 + mn)$ operations per iteration. We proved that when it terminates, either we will obtain a BFS (basic feasible solution) or the optimal cost is ∞.

We saw that we could further optimise this by introducing the elementary row operations matrix Q, thus leading to the **Revised Simplex** method. This leads to an improvement in the time complexity to $O(m^2 + mn)$ operations per iteration.

In this lecture, we will:

- Go over the normal and revised version of the **Simplex Algorithm**.
- Develop the **Full Tableau Method** to handle our operations more conveniently.
- Solve an **example** for this method.
- Describe a method to evaluate the case of **degeneracy**.

2 Simplex Method

2.1 An iteration of the Simplex Method

1. In a typical iteration, we start with a basis consisting of the basic columns $A_{B(1)}$,...,$A_{B(m)}$, an associated basic feasible solution x, and the inverse B^{-1} of the basis matrix.
2. Compute the the reduced costs $\bar{c}_j = c_j$ - $c_B^T A_B^{-1} A_j$ for all nonbasic indices j.
   (a) If they are all non-negative, the current basic feasible solution is optimal, and the algorithm terminates;
   (b) else choose some j for which $\bar{c}_j < 0$
3. Compute $\mathbf{u} = A_B^{-1} A_j$. If no component of u is positive, the optimal cost is - ∞, and the algorithm terminates,

4. If some component of **u** is positive, let $\theta^* = min_{(i=1,...m|u_i>0)} \frac{x_{B(l)}}{u_l}$

5. Let l be such that $\theta^* = \frac{x_{B(l)}}{u_l}$. Form a new basis by replacing $A_B(l)$ with A_j. If A_j. If **y** is the new basic feasible solution, the values of the new basic variables are $y_j = \theta^*$ and $y_{B(l)} = x_{B(l)} - \theta^* u_i$,$i \neq l$

2.2 An iteration of the Revised Simplex Method

We can improve the above Simplex Method, to a revised version for a better time complexity, from $O(m^3 + mn)$ to $O(m^2 + mn)$!

1. We again start with a basis containing basic columns , i.e. $A_{B(1)}$,...,$A_{B(m)}$ and an associated basic feasible solution x, but this time we also compute A_B^{-1}

2. Now we compute the row vector $p^T = c_B^T A_B^{-1}$ and also compute the reduced costs $\bar{c}_j = c_j$ - $c_B^T A_B^{-1} A_j$ for all the non-basic indices j.
   (a) If they are all non-negative, the current basic feasible solution is optimal, and the algorithm terminates;
   (b) else choose some j for which $\bar{c}_j < 0$

3. Compute $\mathbf{u} = A_B^{-1} A_j$. If no component of u is positive, the optimal cost is - ∞, and the algorithm terminates.

4. Otherwise, if some component is positive, let $\theta^* = min_{(i=1,...m|u_i>0)} \frac{x_{B(l)}}{u_l}$

5. Let l be such that $\theta^* = \frac{x_{B(l)}}{u_l}$. Form a new basis by replacing $A_{B(l)}$ with A_j. If y is the new basic feasible solution, the new basic variables are $y_j = \theta^*$, and $y_{B(i)} = x_{B(i)} - \theta^* u_i$,

6. Finally, form the m x *(m+1)* matrix $[A_B^{-1}|$u$]$. Add to each one of its rows a multiple of the *lth* row to make the last column equal to the unit vector e_l. The first m columns of the result is the matrix A_B^{-1}

3 Full Tableau Implementation

3.1 Introduction

Let us finally describe the implementation of simplex method in terms of the so-called *full tableau.* Here, instead of maintaining and updating the matrix $\mathbf{B}^{-1}$, we maintain and update the $m \times (n+1)$ matrix $\mathbf{B}^{-1}[\mathbf{b} \mid \mathbf{A}]$.

This matrix is usually augmented by a top row, whose leftmost entry is $-\mathbf{c}_B^T \mathbf{x}_B$, which is also equal to $-\mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{b}$. The rest of the row is the row vector of reduced costs $\bar{\mathbf{c}}^T = \mathbf{c}^T - \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{A}$. Thus, the structure of the tableau is:

$-\mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{b}$	$\mathbf{c}^T - \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{A}$
$\mathbf{B}^{-1}\mathbf{b}$	$\mathbf{B}^{-1}\mathbf{A}$

or, in more detail:

$-\mathbf{c}_B^T\mathbf{x}_B$	$\bar{c}_1^T$	...	$\bar{c}_n^T$
$x_{B(1)}$	$\vert$	...	$\vert$
$\vdots$	$\mathbf{B}^{-1}\mathbf{A}_1$	...	$\mathbf{B}^{-1}\mathbf{A}_n$
$x_{B(m)}$	$\vert$	...	$\vert$

3.2 Some Terminology

Simplex Tableau: The matrix $\mathbf{B}^{-1}[\mathbf{b} \mid \mathbf{A}]$

Zeroth Row: The top row of the tableau, containing reduced costs

Zeroth Column: The column $\mathbf{B}^{-1}\mathbf{b}$

i^{th} *Column:* The column $\mathbf{B}^{-1}\mathbf{A}_i$

Pivot Column: The column $\mathbf{B}^{-1}\mathbf{A}_j$ corresponding to the variable that enters the basis

Pivot Row: If the l^{th} basic variable exits the basis, the l^{th} row of the tableau is called the pivot row

Pivot Element: The element belonging to both the pivot row and the pivot column

3.3 The Mechanics

An iteration of the full tableau implementation

1. We start with the tableau associated with a basis matrix $\mathbf{B}$ and the corresponding BFS x.
2. Examine the reduced costs in the zeroth row of the tableau. If they are all non-negative, the current basic feasible solution is optimal, and the algorithm terminates; Else choose some j for which $\bar{\mathbf{c}}_{\mathbf{j}} < 0$
3. Consider the vector $\mathbf{u}$=$\mathbf{B}^{-1}\mathbf{A}_j$, which is the j^{th} column (the pivot column) of the tableau. If no component of u is positive, the optimal cost is $-\infty$, and the algorithm terminates.
4. For each i for which u_i is positive, compute the ratio $x_{B(i)}/u_i$. Let l be the index of a row that corresponds to the smallest ratio. The column $\mathbf{A}_{B(l)}$ exits the basis and the column $\mathbf{A}_j$ enters the basis.
5. Add to each row of the tableau a constant multiple of the l^{th} row (the pivot row) so that u_l (the pivot element) becomes one and all other entries of the pivot column become zero

3.4 An Example

Consider the problem:

$$
\begin{array}{rrcrcrl}
\text{minimize} & -10x_1 & - & 12x_2 & - & 12x_3 & \\
\text{subject to} & x_1 & + & 2x_2 & + & 2x_3 & \leq 20 \\
 & 2x_1 & + & x_2 & + & 2x_3 & \leq 20 \\
 & 2x_1 & + & 2x_2 & + & x_3 & \leq 20 \\
x_1, x_2, x_3 \geq 0 & & & & & &
\end{array}
$$

After introducing slack variables, we obtain the following standard form problem:

$$
\begin{array}{rrcrcrcccl}
\text{minimize} & -10x_1 & - & 12x_2 & - & 12x_3 & & & & \\
\text{subject to} & x_1 & + & 2x_2 & + & 2x_3 & + x_4 & & & = 20 \\
 & 2x_1 & + & x_2 & + & 2x_3 & & + x_5 & & = 20 \\
 & 2x_1 & + & 2x_2 & + & x_3 & & & + x_6 & = 20 \\
x_1, ..., x_6 \geq 0 & & & & & & & & &
\end{array}
$$

Note that (0, 0, 0, 20, 20, 20) is a BFS and can be used to start the algorithm. So we have B(1)=4, B(2)=5, B(3)=6 and the basis matrix happens to be 3×3 identity matrix $\mathbf{I}$. To obtain the zeroth row of the initial tableau, we note that $\mathbf{c}_B$=$\mathbf{0}$, and therefore $\mathbf{c}_B^T\mathbf{x}_B$=0 and $\bar{\mathbf{c}}$=$\mathbf{c}$. Hence we have the initial tableau:

		x_1	x_2	x_3	x_4	x_5	x_6
	0	-10	-12	-12	0	0	0
$x_4 =$	20	1	2	2	1	0	0
$x_5 =$	20	2*	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

Now, since reduced cost corresponding to x_1 is negative, let x_1 enter the basis. The pivot column is $\mathbf{u}$=(1, 2, 2). We compute the ratios $x_{B(i)}/u_i$ for i=1,2,3; the smallest ratio corresponds to i=2 and i=3. We break the tie by choosing l=2. This determines the pivot element, which has been indicated by an asterisk. $x_{B(2)}$ exits the basis, and the new basis becomes $\bar{B}(1)$=4, $\bar{B}(2)$=1, $\bar{B}(3)$=6. We multiply the pivot row by 5 and add it to the zeroth row. We multiply the pivot row by 1/2 and subtract it from the first row. We subtract the pivot row from the third row. Finally, we divide the pivot row by 2. This leads us to the new tableau:

		x_1	x_2	x_3	x_4	x_5	x_6
	100	0	-7	-2	0	5	0
$x_4 =$	10	0	1.5	1*	1	-0.5	0
$x_1 =$	10	1	0.5	1	0	0.5	0
$x_6 =$	0	0	1	-1	0	-1	1

The corresponding basic feasible solution is $\mathbf{x} = (10, 0, 0, 10, 0, 0)$. Note that this is a degenerate basic feasible solution, because the basic variable x_6 is equal to zero. The variables x_2 and x_3 have negative reduced costs. Let us choose x_3 to be the one that enters the basis. The pivot column is $\mathbf{u} = (1, 1, -1)$. Since $u_3 < 0$, we only form the ratios $x_{B(i)}/u_i$, for i=1,2. There is again a tie, which we break by letting l=1, and the first basic variable x_4 exits the basis. The pivot element is again indicated by an asterisk. After carrying out the necessary elementary row operations, we obtain the following new tableau:

		x_1	x_2	x_3	x_4	x_5	x_6
	120	0	-4	0	2	4	0
$x_3 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	0	1	-1	0	-1	1	0
$x_6 =$	10	0	2.5*	0	1	-1.5	1

At this point, x_2 is the only variable with negative reduced cost. We bring x_2 into the basis, x_6 exits, and the resulting tableau is:

		x_1	x_2	x_3	x_4	x_5	x_6
	136	0	0	0	3.6	1.6	1.6
$x_3 =$	4	0	0	1	0.4	0.4	-0.6
$x_1 =$	4	1	0	0	-0.6	0.4	0.4
$x_2 =$	4	0	1	0	0.4	-0.6	0.4

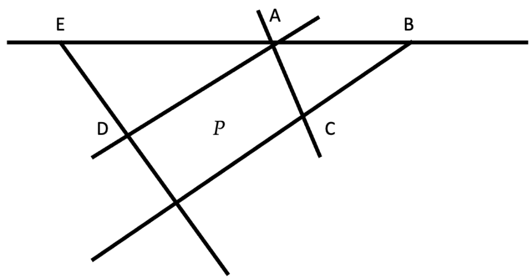
We have now reached the point (x_1, x_2, x_3)=(4, 4, 4). Its optimality is confirmed by observing that all reduced costs are non-negative.

4 Degenerate Cases

Recall, degeneracy occurs when one or more of the basic variables in a basic solution x is 0. In the Simplex method it thus occurs when the leaving index l of basis B gives us $\theta^* = \frac{x_{B(l)}}{u_l} = 0$.

Degeneracy can cause substantial problems, including the possibility of nonterminating behaviour (cycling). This is because in the presence of degeneracy, a change of basis may keep us at the same the basic feasible solution, with no cost improvement resulting.

For example, consider the figure shown below (where P is a polyhedron).



Tracing the path $BCDE$ we get back to the same point A and end up in a cycle.

Thus, cycling needs to be avoided. For this, we use the Bland's Rule. This is a suitable rule to apply for choosing the entering and exiting variables (pivoting rule).

Smallest Index Pivoting Rule or Bland's Rule

1. Find the smallest j for which the reduced cost vector $\hat{c_j}$ is negative and have the column A_j enter the basis.

2. Out of all the variables, x_i that are tied in the test for choosing an exiting variable, select the one with the smallest value of i.

References

[1] *Introduction to Linear Optimizationby Dimitris Bertsimas and John Tsitsiklis*

[2] *Lecture slides for the course, MTL103: Optimization Methods and Applications by Prof. Minati De*