Projections

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Introduction to Projections

- Projection of a vector onto a subspace is finding the closest point in the subspace to the vector.
- This involves minimizing the error vector, which is orthogonal to the subspace.

$$\mathbf{p} = \mathbf{a}\hat{\mathbf{x}} = \mathbf{a}\frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}} = \mathbf{a}(\mathbf{a}^T\mathbf{a})^{-1}\mathbf{a}^T\mathbf{b}.$$

Projection onto a plane

Projection onto a plane 3

Consider the plane $x - 5y - 4z = 0$. We will compute the projection of the vector $\mathbf{b} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$ onto the plane.

Give $\hat{\mathbf{x}}$ and the projection vector $\mathbf{p}$.

$\hat{\mathbf{x}} = \begin{bmatrix} \square \\ \square \end{bmatrix}$?

$\mathbf{p} = \begin{bmatrix} \square \\ \square \\ \square \end{bmatrix}$?

Give the projection matrix P.

$P = \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix}$?

Finding a Basis for the Plane

- The plane $x - 5y - 4z = 0$ is defined by its normal vector $\mathbf{n}$.
- To find a basis, we choose vectors orthogonal to $\mathbf{n}$.
- We construct matrix $\mathbf{A}$ with these basis vectors as columns.

Computing the Projection Vector

- The projection vector $\mathbf{p}$ is found by solving $\mathbf{A}^T\mathbf{A}\hat{\mathbf{x}} = \mathbf{A}^T\mathbf{b}$.
- We calculate $\hat{\mathbf{x}}$ using the equation $\hat{\mathbf{x}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$.
- Then, $\mathbf{p} = \mathbf{A}\hat{\mathbf{x}}$.

Deriving the Projection Matrix

- The projection matrix $\mathbf{P}$ is used to project any vector onto the subspace.
- It is computed as $\mathbf{P} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$.
- This matrix represents a linear transformation mapping $\mathbb{R}^3$ onto the plane.

Projection with Invertible Matrix

Let A be an invertible matrix and let P be the projection matrix that projects vectors onto the column space of A.

Solution:

- Algebraically, $P = A(A^T A)^{-1} A^T$.
- Since A is invertible, $(A^T A)^{-1} = A^{-1}(A^T)^{-1}$, thus $P = I$.
- Explanation:
  - A is invertible, so its columns form a basis for $\mathbb{R}^n$.
  - The column space of A is $\mathbb{R}^n$.
  - Projecting any vector onto $\mathbb{R}^n$ returns the vector itself.