Row Reduced Echelon Form and Subspaces

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Row-Reduced Echelon Form (RREF)

The RREF of a matrix is characterized by the following conditions:

1. Each non-zero row starts with a leading 1 (pivot).
2. Each leading 1 is the only non-zero element in its column.
3. Leading 1s shift to the right as you move down the rows.
4. Rows with all zeros are at the bottom.

Example of a matrix in RREF:

$$\begin{bmatrix} \color{red}{1} & \color{blue}{*} & 0 & \color{blue}{*} & \color{blue}{*} \\ 0 & 0 & \color{red}{1} & \color{blue}{*} & \color{blue}{*} \\ 0 & 0 & 0 & \color{red}{1} & \color{blue}{*} \\ 0 & 0 & 0 & 0 & \color{red}{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Red indicates leading 1s, Blue represents any value (*).

Question 1

Solving Ax = b with some solutions

Let $A = \begin{bmatrix} 1 & 2 & 7 & 3 \\ 1 & 3 & 10 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -7 \\ -10 \end{bmatrix}$. Consider the linear equation $A\mathbf{x} = \mathbf{b}$.

Solve for $\mathbf{x}$ and give two different vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ that satisfy the equation.

$\mathbf{x}_1 = \begin{bmatrix} \square \\ \square \\ \square \\ \square \end{bmatrix}$?

$\mathbf{x}_2 = \begin{bmatrix} \square \\ \square \\ \square \\ \square \end{bmatrix}$?

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Step 1: Augmented Matrix

Set up the augmented matrix for the system $Ax = b$.

$$\left[\begin{array}{cccc|c} 1 & 2 & 7 & 3 & -7 \\ 1 & 3 & 10 & 4 & -10 \end{array}\right]$$

Step 2: Row Reduction

Subtract row 1 from row 2 to eliminate the first element of row 2.

$$\begin{array}{c}
R2 \leftarrow R2 - R1 \\
\hline\hline
\left[\begin{array}{cccc|c} 1 & 2 & 7 & 3 & -7 \\ 0 & 1 & 3 & 1 & -3 \end{array}\right]
\end{array}$$

Step 3: Pivot to Leading 1

The pivot in row 2 is already a leading 1.

$$\left[\begin{array}{cccc|c} 1 & 2 & 7 & 3 & -7 \\ 0 & 1 & 3 & 1 & -3 \end{array}\right]$$

Step 4: Eliminate Second Element in Row 1

Subtract 2 times row 2 from row 1.

$$\begin{array}{c} R1 \quad \leftarrow \quad R1 \quad - \quad 2 \cdot R2 \\ \hline \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 3 & 1 & -3 \end{array}\right] \end{array}$$

Step 5: Reduced Row Echelon Form

The matrix is now in reduced row echelon form (RREF). The solutions can be read directly from the RREF.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 3 & 1 & -3 \end{array}\right]$$

Step 6: Particular Solutions

Let t_1 and t_2 be free variables corresponding to the 3rd and 4th columns.

$$\begin{aligned} x_1 &= -1 - t_1 - t_2 \\ x_2 &= -3 - 3t_1 - t_2 \\ x_3 &= t_1 \\ x_4 &= t_2 \end{aligned}$$

Thus, the solution set is a line in $\mathbb{R}^4$ parametrized by t_1 and t_2.

Step 7: A Particular Solution

To find a particular solution, we can set the free variables t_1 and t_2 to zero.

$$
\begin{aligned}
x_1 &= -1 - (0) - (0) = -1 \\
x_2 &= -3 - 3(0) - (0) = -3 \\
x_3 &= 0 \\
x_4 &= 0
\end{aligned}
$$

The particular solution is:

$$
\mathbf{x_1} = \begin{bmatrix} -1 \\ -3 \\ 0 \\ 0 \end{bmatrix}
$$

This solution satisfies the equation $Ax = b$.

Step 7: A Particular Solution

To find another solution, we can set the free variables t_1 and t_2 to 1 and 0.

$$\begin{aligned} x_1 &= -1 - (1) - (0) = -2 \\ x_2 &= -3 - 3(1) - (0) = -6 \\ x_3 &= 1 \\ x_4 &= 0 \end{aligned}$$

The particular solution is:

$$\mathbf{x_2} = \begin{bmatrix} -2 \\ -6 \\ 1 \\ 0 \end{bmatrix}$$

This solution satisfies the equation $Ax = b$.

Question 2

Solving Ax = 0

Let $A = \begin{bmatrix} 1 & 0 & 0 & -5 \\ 1 & 0 & -2 & 11 \\ 4 & -2 & -8 & 32 \end{bmatrix}$. Compute its row-reduced echelon form R.

$R = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix}$?

Find the special solutions for $R\mathbf{x} = \mathbf{0}$ by setting each free variable to 1 and the other free variables to zero. Submit the special solutions as columns of a single matrix N.

$N =$ matrix (rtol=0.01, atol=1e-08) ?

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Solving $Ax = 0$

Given the matrix A:

$$A = \begin{bmatrix} 1 & 0 & 0 & -5 \\ 1 & 0 & -2 & 11 \\ 4 & -2 & -8 & 32 \end{bmatrix}$$

we want to compute its row-reduced echelon form R and find the special solutions for $Rx = 0$.

The RREF of A

The row-reduced echelon form R is:

$$R = \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -8 \end{bmatrix}$$

The RREF of A

The row-reduced echelon form R is:

$$R = \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -8 \end{bmatrix}$$

This indicates that the fourth variable is free.

Special Solution

The special solution for $Rx = 0$ corresponding to the free variable in the fourth position set to 1 is:

$$\begin{bmatrix} 5 \\ -6 \\ 8 \\ 1 \end{bmatrix}$$

This vector satisfies the equation $Ax = 0$ and represents the special solution where the free variable is 1 and the other variables are adjusted accordingly.

Question 3: Column spaces

Column spaces

Consider the matrix $A = \begin{bmatrix} 8 & -12 & 1 & -12 \\ 6 & -9 & 0 & -9 \\ 12 & -18 & 1 & -18 \\ 12 & -18 & 2 & -18 \\ 10 & -15 & 0 & -15 \end{bmatrix}$.

The column space of A can be described geometrically as:

- ○ (a) a 3D-space
- ○ (b) a 2D-space (plane)
- ○ (c) a 0D-space (point)
- ○ (d) a 5D-space

Calculating the RREF

To find the linearly independent columns, we first compute the row-reduced echelon form (RREF) of A.

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 1 \\ 0 & 1 & \frac{1}{4} & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns in the RREF correspond to the linearly independent columns in the original matrix.

Identifying Linearly Independent Columns

The columns of A that correspond to the pivot columns in the RREF are linearly independent.
The rank of A, which is the number of pivot columns in the RREF, tells us the dimension of the column space of A.
Since the rank of A is 2, The correct answer is (b) a 2D-space (plane).

Question 4

Row-reduced echelon forms 4

Consider the matrix $R = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -7 \end{bmatrix}$, which is in row-reduced echelon form. Show that two different matrices other than R can arrive at the same row-reduced echelon form matrix by giving two different matrices that result in R after performing elimination.

$A = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix}$?

$B = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix}$?

Constructing Matrix A

To construct a matrix A that row-reduces to R, we can just multiply the first row by any scalar, say 5

$$A = \begin{bmatrix} 5 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -7 \end{bmatrix}$$

This matrix will reduce back to R through row reduction.

Constructing Matrix B

Similarly, for matrix B, we can choose a different set of scalars and rows. Scale second row by 8

$$B = \begin{bmatrix} 1 & 0 & 0 & 16 \\ 0 & 8 & 0 & -16 \\ 0 & 0 & 1 & -7 \end{bmatrix}$$

Matrix B also row-reduces to R.

Question 5

Null spaces 2

Construct two different matrices A and B so that the null spaces of both matrices consist only of all scalar multiples of $\begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix}$.

$A =$	matrix (rtol=0.01, atol=1e-08)	?
$B =$	matrix (rtol=0.01, atol=1e-08)	?

Understanding Null Space and Rank

- Null Space: Set of vectors that, when multiplied by a matrix, yield the zero vector.
- Rank: Number of linearly independent columns (dimension of column space).
- Dimension of null space (columns - rank).
- In our case, Dimension of null space = 1, since we want scalar multiples of v, which has rank 1.
- A and B have 4 columns. Hence, rank of A and B = 4 - 1 = 3

Constructing Matrix A

To construct A, we need to ensure that:

1. $Av = 0$ for the given vector v.
2. Rank of A is maintained.

Constructing Matrix A (Continued)

A maintains the rank and $Av = 0$ for v.
Matrix A:

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Now, A satisfies the desired properties.

Constructing Matrix B

B maintains the rank and $Bv = 0$ for v. Matrix B:

$$B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$