MATH7501 Mathematics for Data Science 1, Assignment 1

Due time: 16:00 April 04, 2023, Tuesday.

Full marks: 115.

1. Let

$$A = \begin{pmatrix} 8 & 9 & 6 \\ 5 & 7 & 4 \\ 3 & 10 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix}, \quad \text{and } x = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

   (a) (5 pts) Find $2A$.

   (b) (5 pts) Find A^{-1} and B^{-1}.

   (c) (5 pts) Find $\det(A)$ and $\det(B)$.

   (d) (5 pts) Find $\det(A^{-1})$ and $\det(B^{-1})$.

   (e) (5 pts) Find BB^T and $x^T B$.

2. Let $n \geq 1$ be an integer. Let $U = (U_{ij})_{i,j=1}^n \in \mathbb{R}^{n \times n}$ be an n-by-n orthogonal matrix. That is, $U^T U = I$, where I is the n-by-n identity matrix.

   (a) (5 pts) When $n = 1$, find all the possible matrices U.

   (b) (5 pts) Show that U^T is also an orthogonal matrix.

   (c) (5 pts) For any integer i such that $1 \leq i \leq n$, find

$$\sum_{j=1}^{n} \cancel{U_{ij}^2}.$$

$(U_{ij})^2$

3. Let

$$x = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \text{and } y = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

   (a) (5 pts) Find $\|x\|$.

   (b) (5 pts) Write $y = cx + \xi$, where $c \in \mathbb{R}$ and $\xi \perp x$ (i.e., the inner product $\langle \xi, x \rangle = 0$, or in another way of saying, ξ is perpendicular to x.) Find ξ.

   (c) (5 pts) Let θ be the angle between x and y. Find $\cos\theta$.

4. Recall the fact that for any prime numbers p, q with $p \neq q$, if $p|m$ and $q|m$ for some integer m, then $(pq)|m$.

   (a) (5 pts) Let $A = \{5m : m \in \mathbb{Z}\}$ and $B = \{5^n \times m : m, n \in \mathbb{Z} \text{ and } n \geq 1\}$. Show that $A = B$.

   (b) (5 pts) Let $C = \{10t : t \in \mathbb{Z}\}$. Prove or disprove the statement that $A = C$.

(c) (5 pts) Let $C = \{10t : t \in \mathbb{Z}\}$. Suppose $2n = 5m$ for some integers m and n. Show that $2n \in C$.

5. (5 pts) Construct a truth table for $(p \vee q) \wedge (\sim p)$.

6. Recall the binomial expansion

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \quad \text{for any } n \in \mathbb{Z} \text{ with } n \geq 0, \text{ and any } x \in \mathbb{R}.$$

Note that in the above equation, we use $0^0 := 1$.

(a) (5 pts) Evaluate $\binom{5}{3}$.

(b) (5 pts) Show that for any $x \geq 0$, $(1+x)^{100} \geq 1 + 100x$.

(c) (5 pts) Find the limit

$$\lim_{n\to\infty} n\left(\left(1+\frac{1}{n}\right)^2 - 1\right).$$

(d) (5 pts) Show that for any integers n, k with $1 \leq k \leq n\cancel{+1 \text{ and } n \geq 1}$, one has

$$\binom{n}{k}\frac{1}{n^k} = \frac{1}{k!}\prod_{i=0}^{k-1}\left(1-\frac{i}{n}\right), \quad \text{and} \quad \binom{n+1}{k}\frac{1}{(n+1)^k} = \frac{1}{k!}\prod_{i=0}^{k-1}\left(1-\frac{i}{n+1}\right).$$

(e) (5 pts) Use (d) to show that for any integers n, k with $0 \leq k \leq n\cancel{+1}$ and $n \geq 1$, one has

$$\binom{n}{k}\frac{1}{n^k} \leq \binom{n+1}{k}\frac{1}{(n+1)^k} \leq \frac{1}{k!}.$$

(f) (5 pts) Use (e) to show that the sequence $(a_n := (1+\frac{1}{n})^n)_{n=1}^{\infty}$ is increasing, i.e., $a_n \leq a_{n+1}$ for any integer $n \geq 1$.

(g) (5 pts) Show that for any integer $k \geq 2$, $\frac{1}{k!} \leq \frac{1}{k-1} - \frac{1}{k}$, and use (e) to show that for any integer $n \geq 2$

$$\left(1+\frac{1}{n}\right)^n \leq 2 + 1 - \frac{1}{n} < 3.$$

(h) (5 pts) Using the above arguments, show that the following limit exists,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Note that you should not use the theorem we studied in the lecture that this limit equals e.

*** End ***