Q.1. $a_n = \frac{n^n}{n!}$ Show that $a_n \geq n$ for $n \geq 1$

Proof using mathematical induction.

a) for $n=1$

$a_n \geq n$

$\frac{n^n}{n!} \geq n \Rightarrow \frac{1^1}{1!} \geq 1$

$\Rightarrow 1 \geq 1$ which is true

for $n = K$

$\frac{K^K}{K!} \geq K$ lets assume this is true.

we have to prove for $n = K+1$

$\frac{(K+1)^{K+1}}{(K+1)!} \geq (K+1) \Rightarrow \frac{(K+1)^K \cancel{(K+1)}}{\cancel{(K+1)}\, K!} \geq K+1$

$\Rightarrow \frac{(K+1)^K}{K!} \Rightarrow \underbrace{\left[\frac{K+1}{K} \times \frac{K+1}{(K-1)} \times \frac{K+1}{(K-2)} \times \cdots \times \frac{K+1}{2}\right]} \times \left[\frac{K+1}{1}\right]$

$\Rightarrow (K+1)$ [The value here will be ≥ 1]

as numerator is greater than numerator

hence proved, as L.H.S $\geq$ R.H.S

b) $\lim_{n\to\infty} a_n = \infty \Rightarrow \lim_{n\to\infty} \frac{n^n}{n!} = \infty$

proving using definition

$\forall\, m > 0$, let the value of $N = \lceil m \rceil$

$\lceil m \rceil \geq m$

so $N \geq m$

hence, $N = \lceil m \rceil \geq m$

$\forall\, n \geq N \geq m$, $\frac{n^n}{n!} = \frac{n}{1} \times \frac{n}{2} \times \frac{n}{3} \times \cdots\cdots \times \frac{n}{n-1} \times \cancel{\frac{n}{n}}$

$\left[\frac{n}{1} \times \frac{n}{2} \times \frac{n}{3} \times \cdots\cdots \times \frac{n}{n-1}\right] \geq n$ where $n \geq m$

hence $\boxed{\lim_{n\to\infty} \frac{n^n}{n!} = \infty}$

c)

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{n^n}$

$\Rightarrow \lim_{n\to\infty} \frac{(n+1)^n \cancel{(n+1)}}{\cancel{(n+1)}\,\cancel{n!}} \times \frac{\cancel{n!}}{n^n} \Rightarrow \lim_{n\to\infty} \frac{(n+1)^n}{n^n}$

$\Rightarrow \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \Rightarrow \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \boxed{e}$

d) $\lim_{n\to\infty} \frac{n!}{n^n} \Rightarrow \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}} \times \frac{n-1}{n} \times \frac{n-2}{n} \times \cdots\cdots \times \frac{2}{n} \times \frac{1}{n}$

$\Rightarrow \lim_{n\to\infty} \frac{n-1}{n} \times \frac{n-2}{n} \times \cdots\cdots \times \frac{2}{n} \times \frac{1}{n}$

The above values is greater than 0 and minimum value $= \boxed{\frac{1}{n}}$ so using squeeze theoram,

$$\lim_{n\to\infty} 0 \le \lim_{n\to\infty} \frac{n!}{n^n} \le \lim_{n\to\infty} \frac{1}{n}$$

$$0 \le \lim_{n\to\infty} \frac{n!}{n^n} \le \frac{1}{\infty} = 0$$

hence, $\boxed{\lim_{n\to\infty} \frac{n!}{n^n} = 0}$

e) $1 \le K \le n$, To prove $a_n \ge \frac{n^K}{K!}$

$$\Rightarrow \frac{n^n}{n!} \ge \frac{n^K}{K!}$$

$$\Rightarrow \frac{n^n}{n^K} \ge \frac{n!}{K!} \Rightarrow n^{n-K} \ge \left(\frac{n}{K}\right)!$$

$n^{n-K} \geq \left(\frac{n}{K}\right)!$ proving using mathematical induction

for $n = K = 1$

$1^{1-1} \geq \left(\frac{1}{1}\right)! \Rightarrow 1^0 \geq 1$

hence True

lets assume

$$n^{n-K} \geq \left(\frac{n}{K}\right)! \quad -(1)$$

lets prove for n+1

$$(n+1)^{n+1-K} \geq \frac{(n+1)!}{K!}$$

$$\Rightarrow \cancel{(n+1)}(n+1)^{n-K} \geq \frac{\cancel{(n+1)}\, n!}{K!}$$

$$\Rightarrow (n+1)^{n-K} \geq \frac{n!}{K!} \quad -(2)$$

using (1) and (2)

$$(n+1)^{n-K} \geq (n)^{n-K}$$

$$n+1 \geq n$$ hence True

f) check if series converges.

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

checking using comparison test

lets compare it to series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\frac{n!}{n^n} = \frac{\cancel{n}}{\cancel{n}} \times \frac{n-1}{n} \times \frac{n-2}{n} \times \cdots\cdots \times \frac{2}{n} \times \frac{1}{n}$$

$$\frac{n!}{n^n} = \frac{n-1}{n} \times \frac{n-2}{n} \times \cdots\cdots \times \frac{2}{n} \times \frac{1}{n}$$ This whole series will be < 0

$$\frac{n!}{n^n} = \frac{n\,[\underbrace{(n-1)(n-2)(n-3) \times \cdots (2) \times (1)}_{\text{greater than } 0}]}{n^n}$$

So $\frac{n!}{n^n} \geq \frac{n}{n^n}$ (2) where $\frac{n}{n^n} = \frac{1}{n^{n-1}}$ (1)

from (1) and (2)

$$\frac{n!}{n^n} \geq \frac{1}{n^{n-1}} > \frac{1}{n^2} \quad \text{for } n \geq 1$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges to a finite value

that means $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ also converges.

Q.2. Check if the series converges

$\sum_{n=5}^{\infty} \frac{6}{7^n} \Rightarrow \sum_{n=5}^{\infty} 6\left(\frac{1}{7}\right)^n$ this is in the form of geometric progression

$ar^n = \frac{a}{1-r}$

here the first element $\boxed{a = \frac{6}{7^5}}$

and $\boxed{r = \frac{1}{7}}$

$$ar^n = \frac{a}{1-r} = \frac{6/7^5}{1-\frac{1}{7}} = \frac{6/7^5}{6/7}$$

$$\Rightarrow \frac{\cancel{6}}{7^{\cancel{5}4}} \times \frac{\cancel{7}}{\cancel{6}} = \frac{1}{7^4} \neq \infty$$

as the value is finite then the series converges. and the final value is $\boxed{\frac{1}{7^4}}$.

Q.3.

a) $\lim_{x\to\infty} \frac{x^2+4x-21}{x^2-5x+6} \Rightarrow \lim_{x\to\infty} \frac{\cancel{(x-3)}(x+7)}{\cancel{(x-3)}(x-2)}$

$\Rightarrow \lim_{x\to\infty} \frac{x+7}{x-2}$ "$\frac{\infty}{\infty}$" form

divide by x raised to max power of x in denominator

$\lim_{x\to\infty} \frac{1+\frac{7}{x}}{1-\frac{2}{x}} = \boxed{1}$

b) $\lim_{x\to\infty} \frac{x^2-3x+2}{x^2+2x-3} \Rightarrow \lim_{x\to\infty} \frac{\cancel{(x-1)}(x-2)}{\cancel{(x-1)}(x+3)}$

$\Rightarrow \lim_{x\to\infty} \frac{x-2}{x+3}$ "$\frac{\infty}{\infty}$" form

divide by x raised to max power of x in denominator

$\lim_{x\to\infty} \frac{1-\frac{2}{x}}{1+\frac{3}{x}} = \boxed{1}$

c) $\lim_{x \to 1} \frac{\sin x}{x} = \frac{\sin(1)}{(1)} = \boxed{\sin(1)}$

d) $\lim_{x \to \infty} \ln\left(\frac{x^2+2x+1}{x^2-1}\right)$

$\Rightarrow \lim_{x \to \infty} \ln\left(\frac{(x+1)^{\cancel{2}}}{(x-1)\cancel{(x+1)}}\right) \Rightarrow \lim_{x \to \infty} \ln\left(\frac{x+1}{x-1}\right)$

$\Rightarrow \ln\left(\lim_{x \to \infty} \frac{x+1}{x-1}\right) \Rightarrow \ln\left(\lim_{x \to \infty} \frac{1+\frac{1}{x}}{1-\frac{1}{x}}\right)$

$\Rightarrow \ln(1) = \boxed{0}$

e) $\lim_{x \to \infty} \frac{x^2(\sin x + (\cos x)^3)}{(x^2+1)\quad(x-3)}$

$\Rightarrow \lim_{x \to \infty} \frac{x^2 \sin x}{(x^2+1)(x-3)} + \lim_{x \to \infty} \frac{x^2 (\cos x)^3}{(x^2+1)(x-3)}$

e) $\lim_{x\to\infty} \dfrac{x^2(\sin x + (\cos x)^3)}{(x^2+1)\ (x-3)}$

$\Rightarrow \lim_{x\to\infty} \dfrac{x^2 \sin x}{(x^2+1)(x-3)} + \lim_{x\to\infty} \dfrac{x^2 (\cos x)^3}{(x^2+1)(x-3)}$

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e) using squeeze theoram,

$-1 \leq \sin x \leq 1$ / $-1 \leq (\cos x)^3 \leq 1$

$\dfrac{-x^2}{(x^2+1)(x-3)} \leq \dfrac{x^2 \sin x}{(x^2+1)(x-3)} \leq \dfrac{x^2}{(x^2+1)(x-3)}$ /

$\dfrac{-x^2}{(x^2+1)\ (x-3)} \leq \dfrac{x^2 (\cos x)^3}{(x^2+1)(x-3)} \leq \dfrac{x^2}{(x^2+1)(x-3)}$

$\Rightarrow \lim_{x\to\infty} \dfrac{-x^2}{(x^2+1)(x-3)}$ $\quad \lim_{x\to\infty} \dfrac{x^2}{(x^2+1)\ (x-3)}$

Divide by x raised to max power of x in the denominator

$\Rightarrow \lim_{x\to\infty} \dfrac{\frac{-1}{x}}{1-\frac{3}{x}+\frac{1}{x^2}-\frac{3}{x^3}} = \dfrac{0}{1} = 0$

$\Rightarrow \lim_{x\to\infty} \dfrac{\frac{1}{x}}{1-\frac{3}{x}+\frac{1}{x^2}-\frac{3}{x^3}} = \dfrac{0}{1} = 0$

So, $\lim_{x \to \infty} \frac{x^2 \sin x}{(x^2+1)(x-3)} = 0$

$$\lim_{x \to \infty} \frac{x^2 (\cos x)^3}{(x^2+1)(x-3)} = 0$$

So the final value is $0 + 0 = \boxed{0}$

Q.4. let f be continous one-to-one function
To show:- f is strictly increasing or strictly decreasing
Proof using intermediate Value theorem (IVT) and contradition.

lets assume function $f(x)$ is continous for interval $[a, b]$

then there exists a value K where

$\rightarrow f(a) \leq K \leq f(b)$

① —— $\rightarrow f(a) \leq f(c) \leq f(b)$ where $f(c) = K$

Q.4.

lets assume there exists points x, y, z

x, y, z in $[a, b]$ and it ~~is~~

either $f(x) < f(y) < f(z)$ or $f(x) > f(y) > f(z)$

Since function f is one-to-one

$f(x) \neq f(y) \neq f(z)$

$[a, b]$ is continous

~~[illegible]~~ if $[a < x < y < z < b]$

then by IVT there exists a point k

between $[x\ z]$ such that

$f(x) \leq k \leq f(z)$

$f(x) \leq f(M) \leq f(z)$

This $f(M) = f(y)$

Since, f is one to one $M \neq y$ and let $M < y$

so $x < M < y < z$ where

$f(x) < f(M) = f(y) < f(z)$

but the statement ~~[illegible]~~ contradicts that $f(x) < f(M)$ and $f(M) < f(y)$

Therefore, we can conclude by contradiction that f is either strictly increasing or strictly decreasing on $[a\ b]$

Q.5. To determine if a function f is continous, we use that a function is continous at point if and only if $\lim\limits_{x \to a} f(x) = f(a)$

let
$$f(x) = \begin{cases} x^2, & \text{if } x > 0 \\ x, & \text{if } x \leq 0 \end{cases}$$

then at $x =$ — (1)

$$\lim_{h \to 0} [f(x+h) - f(x-h)] =$$

$$\lim_{h \to 0} [\cancel{x^2} + \cancel{h^2} + 2xh - \cancel{x^2} - \cancel{h^2} + 2xh]$$

$$= \lim_{h \to 0} 2xh = 2(1)(0) = 0$$

this $0 \neq 1$ from (1)

hence the function is discontinous. as it will not be true for any arbitary value of x.

$$\lim_{x \to 0^-} \neq \lim_{x \to 0} \neq \lim_{x \to 0^+}$$

Q.6. $f(x) = x^2 \cos x$

a) $f'(x) = \cos x \frac{d(x^2)}{dx} + x^2 \frac{d}{dx} \cos x$

$f'(x) = (2x)\cos x - x^2 \sin x$

$\boxed{f'(x) = 2x\cos x - x^2 \sin x}$

b) $f(x) = \frac{2x+1}{3x-2}$

$f'(x) = \frac{(3x-2)\frac{d}{dx}(2x+1) - (2x+1)\frac{d}{dx}(3x-2)}{(3x-2)^2}$

$f'(x) = \frac{(3x-2)(2) - (2x+1)(3)}{(3x-2)^2}$

$f'(x) = \frac{\cancel{6x}-4-\cancel{6x}-3}{(3x-2)^2} = \boxed{\frac{-7}{(3x-2)^2}}$

c) $f(x) = \cos(\sin x) = \frac{d \cos(\sin x)}{dx} \frac{d}{dx}(\sin x)$

$\boxed{f'(x) = -\sin(\sin x)\cos x}$

Q.7. $\lim_{x \to \frac{\pi}{4}} \dfrac{\sqrt{2}\cos x - 1}{1 - (\tan x)^2}$ "$\frac{0}{0}$" form

$$\Rightarrow \lim_{x \to \pi/4} \frac{\sqrt{2}\cos x - 1}{1 - \frac{(\sin x)^2}{(\cos x)^2}} \times \frac{\sqrt{2}\cos x + 1}{\sqrt{2}\cos x + 1}$$

$$\Rightarrow \lim_{x \to \pi/4} \frac{2\cos^2 x - 1}{\cos^2 x - \sin^2 x} \times \frac{\cos^2 x}{\sqrt{2}\cos x + 1}$$

$$\Rightarrow \lim_{x \to \pi/4} \frac{\cos^2 x + (\cos^2 x - 1)}{\cos^2 x - \sin^2 x} \times \frac{\cos^2 x}{\sqrt{2}\cos x + 1}$$

$$\Rightarrow \lim_{x \to \pi/4} \frac{\cancel{\cos^2 x - \sin^2 x}}{\cancel{\cos^2 x - \sin^2 x}} \times \frac{\cos^2 x}{\sqrt{2}\cos x + 1}$$

$$\Rightarrow \lim_{x \to \pi/4} \frac{\cos^2 x}{\sqrt{2}\cos x + 1} = \frac{\left(\frac{1}{\sqrt{2}}\right)^2}{\cancel{\sqrt{2}} \times \frac{1}{\cancel{\sqrt{2}}} + 1}$$

$$= \frac{\frac{1}{2}}{2} = \boxed{\frac{1}{4}}$$

Q.8 $f(x) = \frac{1}{1+|x|} + \frac{1}{1+|x-2|}$

lets use piece wise to calculate different cases

$|x| = 0 \Rightarrow x = 0$

$|x-2| = 0 \Rightarrow x = \pm 2$

case 1 when $x < -2$, $|x| = -x$, $|x-2| = -(x-2)$

$$f(x) = \frac{1}{1-x} + \frac{1}{1-(x-2)} = \frac{1}{1-x} + \frac{1}{1-x+2} = \frac{1}{1-x} + \frac{1}{3-x}$$

$$f'(x) = \frac{d}{dx}(1-x)^{-1} + \frac{d}{dx}(3-x)^{-1} = -1(1-x)^{-2}(-1) - 1(3-x)^{-2}(-1)$$

$$= \frac{1}{(1-x)^2} + \frac{1}{(3-x)^2}$$

Case 2 when $-2 \leq x < 0$, $|x| = -x$, $|x-2| = -(x-2)$

$$f(x) = \frac{1}{1-x} + \frac{1}{1-(x-2)} = \frac{1}{1-x} + \frac{1}{1-x+2} = \frac{1}{1-x} + \frac{1}{3-x}$$

$$f'(x) = \frac{1}{(1-x)^2} + \frac{1}{(3-x)^2}$$

Case 3 when $0 \leq x < 2$, $|x| = x$, $|x-2| = -(x-2)$

$$f(x) = \frac{1}{1+x} + \frac{1}{1-(x-2)} = \frac{1}{1+x} + \frac{1}{1-x+2}$$

$$f(x) = \frac{1}{(1+x)} + \frac{1}{3-x}$$

$$f'(x) = \frac{d(1+x)^{-1}}{dx} + \frac{d}{dx}(3-x)^{-1} = -1(1+x)^{-2} \cdot -1(3-x)^{-2}(-1)$$

$$f'(x) = \frac{1}{(3-x)^2} - \frac{1}{(1+x)^2}$$

Case 4 when $2 \le x$, $|x| = x$, $|x-2| = x-2$

$$f(x) = \frac{1}{1+x} + \frac{1}{1+x-2} = \frac{1}{1+x} + \frac{1}{x-1}$$

$$f'(x) = \frac{d(x+1)^{-1}}{dx} + \frac{d(x-1)^{-1}}{dx} = -1(x+1)^{-2} - (x-1)^{-2}$$

now, $f'(x) = 0$

case 1

$$\frac{1}{(1-x)^2} = \frac{-1}{(3-x)^2} \Rightarrow (3-x)^2 = -(1+x)^2 \quad \text{Not defined}$$

Case 2

$$\frac{1}{(1-x)^2} + \frac{1}{(3-x)^2} = 0 \Rightarrow (3-x)^2 = -(1-x)^2. \quad \text{Not defined}$$

Case 3

$$\frac{1}{(3-x)^2} - \frac{1}{(1+x)^2} = 0 \Rightarrow x^2+9-6x = x^2+1+2x$$

$$8 = 8x$$

$$\boxed{1 = x}$$

case 4.

$$\frac{-1}{(x+1)^2} - \frac{1}{(x-1)^2} = 0 \Rightarrow -(x-1)^2 = (x+1)^2 \quad \text{Not defined}$$

hence, $\boxed{\text{critical point} = 1}$

and global minima is

$$\Rightarrow \frac{1}{1+|1|} + \frac{1}{1+|1-2|} \Rightarrow \frac{1}{1+1} + \frac{1}{1+|-1|}$$

$$\Rightarrow \frac{1}{2} + \frac{1}{2} = \boxed{1}$$ ← global maximum

Q.9. for $a>0$, $b>0$

determine global maximum of

$$f(x) = x^a (1-x)^b$$

To find the global maxima we need to first find the critical points of $f(x)$

$$f'(x) = x^a \frac{d}{dx}(1-x)^b + (1-x)^b \frac{d}{dx} x^a$$

$$f'(x) = (x^a)(b(1-x)^{b-1})(-1) + (1-x)^b a x^{a-1}$$

now $f'(x) = 0$

$$0 = x^a (b(1-x)^{b-1})(-1) + (1-x)^b a x^{a-1}$$

$$\Rightarrow x^a (b(1-x)^{b-1}) = (1-x)^b (a)(x^{a-1})$$

$$\Rightarrow \frac{b\cancel{(x^a)}\cancel{(1-x)^b}}{(1-x)} = \frac{\cancel{(1-x)^b}(a)\cancel{(x^a)}}{x}$$

$$\Rightarrow \frac{b}{1-x} = \frac{a}{x} \qquad \Rightarrow bx = a - ax$$

$$\Rightarrow (a+b)x = a$$

$$\Rightarrow \boxed{x = \frac{a}{a+b}}$$

This value lies in the defined domain $[0,1]$

global maxima at $x = \frac{a}{a+b}$

$$f\left(\frac{a}{a+b}\right) = \left(\frac{a}{a+b}\right)^a \left(1 - \frac{a}{a+b}\right)^b$$

$$= \frac{a^a}{(a+b)^a} \times \left(\frac{b^b}{(a+b)^b}\right)$$

$$= \boxed{\frac{a^a b^b}{(a+b)^{a+b}}}$$

← global maxima