Q.1.

a) Find 2A

$$A = \begin{bmatrix} 8 & 9 & 6 \\ 5 & 7 & 4 \\ 3 & 10 & 2 \end{bmatrix}$$

$$\Rightarrow \quad 2\begin{bmatrix} 8 & 9 & 6 \\ 5 & 7 & 4 \\ 3 & 10 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2\times 8 & 2\times 9 & 2\times 6 \\ 2\times 5 & 2\times 7 & 2\times 4 \\ 2\times 3 & 2\times 10 & 2\times 2 \end{bmatrix} = \begin{bmatrix} 16 & 18 & 12 \\ 10 & 14 & 8 \\ 6 & 20 & 4 \end{bmatrix}$$

b) find A^{-1} and B^{-1}

$$B = \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix}$$

(i) $$A^{-1} = \frac{1}{|A|} \operatorname{Adj}(A)$$

$$|A| = 8(7\times 2 - 4\times 10) - 9(5\times 2 - 4\times 3) + 6(5\times 10 - 7\times 3)$$

$$|A| = 8(14-40) - 9(10-12) + 6(50-21)$$

$$= 8(-26) - 9(-2) + 6(29)$$

$$= -208 + 18 + 174 = -16$$

$$\operatorname{Adj}(A) = \begin{bmatrix} +(14-40) & -(10-12) & +(50-21) \\ -(18-60) & +(16-18) & -(80-27) \\ +(36-42) & -(32-30) & +(56-45) \end{bmatrix}^T$$

$$\Rightarrow \begin{bmatrix} -26 & 2 & 29 \\ 42 & -2 & -53 \\ -6 & -2 & 11 \end{bmatrix}^T = \begin{bmatrix} -26 & 42 & -6 \\ 2 & -2 & -2 \\ 29 & -53 & 11 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \left(\frac{1}{-16}\right)\begin{bmatrix} -26 & 42 & -6 \\ 2 & -2 & -2 \\ 29 & -53 & 11 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 13/8 & -21/8 & 3/8 \\ -1/8 & 1/8 & 1/8 \\ -29/16 & 53/16 & -11/16 \end{bmatrix}$$

(ii)

$$B = \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} \, Adj(B)$$

$$|B| = 5\times 3 - 2\times 6 = 15 - 12 = 3$$

$$Adj(B) = \begin{bmatrix} 3 & -2 \\ -6 & 5 \end{bmatrix}^T = \begin{bmatrix} 3 & -6 \\ -2 & 5 \end{bmatrix}$$

$$B^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -6 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2/3 & 5/3 \end{bmatrix}$$

c) find det (A) and det (B)

$$|B| = det(B) = \begin{vmatrix} 5 & 6 \\ 2 & 3 \end{vmatrix} = 15 - 12 = \boxed{3}$$

$$|A| = \det(A) = \begin{vmatrix} 8 & 9 & 6 \\ 5 & 7 & 4 \\ 3 & 10 & 2 \end{vmatrix}$$

$$\Rightarrow 8(14-40) - 9(10-12) + 6(50-21)$$

$$\Rightarrow 8(-26) - 9(-2) + 6(29) \Rightarrow -208 + 18 + 174 = \boxed{-16}$$

d) find $\det(A^{-1})$ and $\det(B^{-1})$

$$|B^{-1}| = \det(B^{-1}) = \begin{vmatrix} 1 & -2 \\ -2/3 & 5/3 \end{vmatrix} = \frac{5}{3} - \frac{4}{3} = \boxed{\frac{1}{3}}$$

$$|A^{-1}| = \det(A^{-1}) = \begin{vmatrix} 13/8 & -21/8 & 3/8 \\ -1/8 & 1/8 & 1/8 \\ -29/16 & 53/16 & -11/16 \end{vmatrix}$$

$$\Rightarrow \frac{13}{8}\left(\left(\frac{1}{8}\right)\left(\frac{-11}{16}\right) - \left(\frac{1}{8}\right)\left(\frac{53}{16}\right)\right) - \left(\frac{-21}{8}\right)\left(\left(\frac{-1}{8}\right)\left(\frac{-11}{16}\right) - \left(\frac{1}{8}\right)\left(\frac{-29}{16}\right)\right)$$

$$+ \left(\frac{3}{8}\right)\left(\left(\frac{-1}{8}\right)\left(\frac{53}{16}\right) - \left(\frac{1}{8}\right)\left(\frac{-29}{16}\right)\right)$$

$$\Rightarrow \frac{13}{8}\left(\frac{-11-53}{128}\right) + \frac{21}{8}\left(\frac{11+29}{128}\right) + \frac{3}{8}\left(\frac{-53+29}{128}\right)$$

$$\Rightarrow \frac{-832+840-72}{1024} = \frac{-64}{1024} = \boxed{-16}$$

e) find BB^T and $x^T B$

$$BB^T = \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 25+36 & 10+18 \\ 10+18 & 4+9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 61 & 28 \\ 28 & 13 \end{bmatrix}$$

$$x = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad x^T = \begin{pmatrix} 2 & -1 \end{pmatrix}$$

$$x^T B = \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 10-2 & 12-3 \end{bmatrix} = \begin{bmatrix} 8 & 9 \end{bmatrix}$$

Q.2. integer $n \geq 1$

$$U = (U_{ij})_{i,j=1}^{n} \in \mathbb{R}^{n \times n} \qquad U^T U = I$$

a) $\Rightarrow$ when $n=1$, find all the possible matrices U.

when $n=1$; U becomes a 1×1 matrix:

So when $n = 1$, $U = [a]$

Since, $U^T U = I$

$\Rightarrow [a]^T [a] = [1]$

$\Rightarrow [a][a] = [1]$

$\Rightarrow a^2 = 1$

$\Rightarrow a = \pm 1$

So, the possible values of $U = [1]$ or $[-1]$

b) show that U^T is also an orthogonal matrice

So, we need to prove that $(U^T)(U^T)^T = I$ — (1)

we know $(U^T)^T = U$ — (2)

from (1) and (2)

$\Rightarrow U^T(U)$ -

in the question its given $(U^T)U = I$

hence $U^T U = I$

and U^T is also orthogonal

c) for any integer i, such that $1 \le i \le n$

find $\sum_{j=1}^{n} (U_{ij})^2$

$\Rightarrow$ As U is orthogonal $U^T U = I$

and $U^{-1} U = I$ then $\boxed{U^T = U^{-1}}$

To show we have [redacted] prove L.H.S = R.H.S

we know $U^T U = I$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & & & \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \cdots & a_{n1} \\ a_{12} & & & \\ a_{13} & & & \\ \vdots & & & \\ a_{1n} & & & \end{bmatrix} = I$$

① → from this we can say that dot product of row with itself is 1 $\Leftrightarrow$ $U_{1i} \cdot U_{1i} = 1$

② → dot product of row with other row = 0

eg:- $U_{1i} \cdot U_{2i} = 0$

~~from ①~~

so, $U_{in} \cdot U_{ni} = (U_{i1})^2 + (U_{i2})^2 + (U_{i3})^2 + \ldots\ldots + (U_{in})^2$

$$\Rightarrow (U_{i1})^2 + (U_{i2})^2 + (U_{i3})^2 + \ldots\ldots + (U_{in})^2 = 1$$

hence $\displaystyle\sum_{j=1}^{n} (U_{ij})^2 = \boxed{1}$

Q.3. $x = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ $y = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

a) find $\|x\|$

$$\|x\| = \sqrt{(x_1)^2 + (x_2)^2} = \sqrt{1^2 + 3^2}$$
$$= \sqrt{1+9} = \boxed{\sqrt{10}}$$

So, $\|x\| = \sqrt{10}$

b) $y = cx + \varepsilon \Rightarrow \begin{pmatrix} 3 \\ 1 \end{pmatrix} = c\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ 3c \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$ now multiply both sides by x

$\Rightarrow yx = (cx + \varepsilon)x = (cx)(x) + (\varepsilon x)$

$\varepsilon \cdot x = 0$ as they are perpendicular

$\Rightarrow yx = (cx)(x)$

$\Rightarrow 1 \cdot 3 + 3 \cdot 1 = c \cdot 1 + 3c \cdot 3$

$\Rightarrow 6 = c + 9c = 10c$

$c = \frac{6}{10} = 0.6$

now, $\varepsilon = y - cx$

$\varepsilon = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 0.6 \\ 1.8 \end{pmatrix} = \begin{pmatrix} 2.4 \\ -0.8 \end{pmatrix}$ or

$= \begin{pmatrix} 12/5 \\ -4/5 \end{pmatrix}$

c) find angle between x and y

$$x \cdot y = \|x\| \cdot \|y\| \cos\theta$$

$$\Rightarrow \cos\theta = \frac{x \cdot y}{\|x\|\|y\|} \quad \Rightarrow \cos\theta = \frac{(1)(3)+(3)(1)}{(\sqrt{1^2+3^2})(\sqrt{3^2+1^2})}$$

$$\cos\theta = \frac{3+3}{(\sqrt{10})(\sqrt{10})} = \frac{\cancel{6}\,3}{\cancel{10}\,5}$$

$$\cos\theta = \frac{3}{5}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{3}{5}\right)$$

hence $\theta \simeq 53.13$

Q.4.

a) let $A = \{5m : m \in Z\}$, $B = \{5^n \times m : m, n \in Z$ and $n \geq 1\}$

show $A = B$

we need to prove $A \subseteq B$ and $B \subseteq A$

$\forall m \in A$, $5m = 5^1 \times m \Rightarrow 5^1 \times m \in B$ — ①

as $5^1 \times m$ is in form of $5^n \times m$

$\forall n \in B$, $5^n \times m \Rightarrow (5^{n-1})(5)(m)$

$\Rightarrow 5^{n-1} \times (5m)$ where $5m$ can be any integer $\in Z$

so $5^{n-1} \times p$ where $p \in \mathbb{Z}$

so $5^{n-1} \times p \in A$ — ②

from ① and ②
we can say $A \subseteq B$ & $B \subseteq A$
hence $A = B$

b) let $C = \{10t : t \in \mathbb{Z}\}$. Prove or disprove $A = C$

To prove $A = C \iff A \subseteq C$ & $C \subseteq A$

let $p = 5m$

$\forall p \in A \quad p = 5m = 2\left(\frac{5m}{2}\right) = 10\left(\frac{m}{2}\right)$

here m can be odd or even

So $10\left(\frac{2z}{2}\right) = 10z \in C$ — ①

or

$10\left(\frac{2z}{2} + \frac{1}{2}\right) = 10\left(z + \frac{1}{2}\right) \in C$

now, let $q = 10t$

$\forall q \in C$, $10t = 5 \times 2 \times t$

$= 5 \times (2 \times t)$

② — $= 5 \times m$ as $2 \times t \in \mathbb{Z}$

from ① and ②

$A \subseteq C$ & $C \subseteq A$

hence $A = C$

c) let $C = \{10t : t \in Z\}$. suppose $2n = 5m$ for some integers m and n. show that $2n \in C$.

Ans from $2m = 5m$ we can say m = even

as (even) = odd × even

$\underset{\uparrow\ 2n}{(\text{even})} = \underset{\uparrow\ 5}{\text{odd}} \times \underset{\uparrow\ m}{\text{even}}$ ①

$2n = 5m$

$n = \frac{5m}{2}$

for n to be integer m needs to be even — ②

so let m be $= 2L$ from ① and ②

$2n = 5m \iff 2n = 5(2L)$

$\Rightarrow \boxed{2n = 10L}$

10L is in the form of 10t where L and t are both integers

hence $2n \in C$

Q.5. Construct truth table for $(P \vee q) \wedge (\sim P)$

P	q	$\sim P$	$P \vee q$	$(P \vee q) \wedge (\sim P)$
T	T	F	T	F
T	F	F	T	F
F	T	T	T	T
F	F	T	F	F

Q.6 $$(1+x)^n = \sum_{K=0}^{n} \binom{n}{K} x^K$$

a) $$\binom{5}{3} = \frac{5!}{3!\,(5-3)!} = \frac{\cancel{5!}^{\,5\times \cancel{4}\,2}}{\cancel{3!}\,\cancel{2}!} = 10$$

b) Show that for any $x \geq 0$, $(1+x)^{100} \geq 1+100x$

Proof using mathematical induction

for $x=0$

$$(1+0)^{100} \geq 1+100(0)$$

$$\Rightarrow 1^{100} \geq 1+0 \Rightarrow 1 \geq 1$$

hence it is True

lets assume for $x=K$ the above equation is True

$$(1+K)^{100} \geq 1+100K$$

need to prove for $n=K+1$ it is true.

$$(1+(K+1))^{100} \geq 1+100(K+1)$$

L.H.S Expanding by binomial expansion

$$\Rightarrow \binom{100}{0}(1+K)^{100} + \binom{100}{1}(1+K)^{99} + \dots + \binom{100}{100}(1+K)^{0}$$

$$\Rightarrow \frac{\cancel{100}}{\cancel{100}}(1+K)^{100} + 100(1+K)^{99} + \dots + \underbrace{100(K+1)+1}_{\text{same as R.H.S}}$$

R.H.S $\Rightarrow 1+100(K+1)$

in L.H.S $\left[(1+K)^{100} + 100(1+K)^{99} + \dots + \binom{100}{98}(1+K)^{2} + \text{R.H.S}\right]$

hence L.H.S > R.H.S and ∴ Proved

c) find the limit $\lim_{n\to\infty} n\left(\left(1+\frac{1}{n}\right)^2 - 1\right)$

$$\Rightarrow \lim_{n\to\infty} n\left(\cancel{1} + \frac{1}{n^2} + \frac{2}{n} - \cancel{1}\right)$$

$$\Rightarrow \lim_{n\to\infty} \cancel{n}\left(\frac{1+2n}{n^{\cancel{2}}}\right) \Rightarrow \lim_{n\to\infty} \frac{1}{n} + \frac{2\cancel{n}}{\cancel{n}}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n} + 2 = \frac{1}{\infty} + 2 = 0 + 2 = \boxed{2}$$

$\forall \varepsilon > 0$, let $N = \lceil 1/\varepsilon \rceil$

Then $\forall n \geq N \geq 1/\varepsilon$ we get $\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \varepsilon$

Therefore $\lim_{n\to\infty} \frac{1}{n} = 0$

hence, the final answer $= 0+2 = \boxed{2}$

d) Show that for n, K with $1 \leq K \leq n$

(i) $\Rightarrow \binom{n}{K}\frac{1}{n^K} = \frac{1}{K!}\prod_{i=0}^{K-1}\left(1 - \frac{i}{n}\right)$

To prove, we have to show L.H.S = R.H.S

L.H.S $\binom{n}{K}\left(\frac{1}{n^K}\right) = \frac{1}{n^K}\left(\frac{n!}{K!\,(n-K)!}\right) = \frac{1}{K!\,n^K}\left(\frac{n!}{(n-K)!}\right)$

R.H.S $\frac{1}{K!}\left[\prod_{i=0}^{K-1}\left(1-\frac{i}{n}\right)\right] = \frac{1}{K!}\left[\prod_{i=0}^{K-1}\left(\frac{n-i}{n}\right)\right]$

$\Rightarrow \frac{1}{K!}\left[\left(\frac{n-0}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\cdots\left(\frac{n-(K-1)}{n}\right)\right]$

$\Rightarrow \frac{1}{K!}\left[\frac{(1)(n-1)(n-2)\cdots\cdots(n-(K-1))}{n^{K-1}} \times \frac{n}{n}\right]$

$\Rightarrow \frac{1}{K!\,n^K}\left[(1)(n-1)(n-2)\cdots(n-(K-1))(n)\right]$

$\Rightarrow \frac{1}{K!\,n^K}\left[(n)(n-1)(n-2)\cdots(n-(K-1)) \times \frac{(n-K)!}{(n-K)!}\right]$

$\Rightarrow \frac{1}{K!\,n^K}\left[\frac{(n)(n-1)(n-2)\cdots(n-K+1)(n-K)!}{(n-K)!}\right]$

$\Rightarrow \frac{1}{K!\,n^K}\cdot\left[\frac{n!}{(n-K)!}\right]$

hence, L.H.S = R.H.S

d)

(ii) show $\binom{n+1}{K}\dfrac{1}{(n+1)^K} = \dfrac{1}{K!}\prod_{i=0}^{K-1}\left(1-\dfrac{i}{n+1}\right)$

To prove, we have to show L.H.S = R.H.S

L.H.S $\binom{n+1}{K}\left(\dfrac{1}{(n+1)^K}\right) = \dfrac{(n+1)!}{(K!)(n+1-K)!}\left(\dfrac{1}{(n+1)^K}\right)$

R.H.S $\dfrac{1}{K!}\prod_{i=0}^{K-1}\left(\dfrac{n+1-i}{n+1}\right) = \dfrac{1}{K!}\left[\prod_{i=0}^{K-1}\left(\dfrac{n+1-i}{n+1}\right)\right]$

$$\Rightarrow \frac{1}{K!}\left[\frac{(n+1-0)}{(n+1)}\cdot\left(\frac{n+1-1}{n+1}\right)\cdot\left(\frac{n+1-2}{n+1}\right)\cdots\cdots\left(\frac{n+1-(K-1)}{n+1}\right)\right]$$

$$\Rightarrow \left[\frac{1}{K!}\right]\left[\frac{1\cdot(n)\cdot(n-1)\cdots\cdots(n-K+2)}{(n+1)^{K-1}}\times\frac{n+1}{n+1}\right]$$

$$\Rightarrow \frac{1}{K!}\times\frac{1}{(n+1)^K}\left[(n+1)(n)(n-1)\cdots\cdots(n-k+2)\times\frac{(n-K+1)!}{(n-K+1)!}\right]$$

$$\Rightarrow \frac{1}{K!\,(n+1)^K\,(n-K+1)!}\left[(n+1)!\right]$$

$$\Rightarrow \frac{(n+1)!}{K!\,(n+1)^K(n-K+1)!}$$

Hence, L.H.S = R.H.S

$\therefore \binom{n+1}{K}\left(\dfrac{1}{(n+1)^K}\right) = \dfrac{1}{K!}\prod_{i=0}^{K-1}\left(1-\dfrac{i}{n+1}\right)$

e) Show that for n, K with $0 \le K \le n$ and $n \ge 1$ show:-

$$\binom{n}{K}\frac{1}{n^K} \le \binom{n+1}{K}\frac{1}{(n+1)^K} \le \frac{1}{K!}$$

first we shall see $\binom{n}{K}\frac{1}{n^K} \le \binom{n+1}{K}\frac{1}{(n+1)^K}$

$$\Rightarrow \frac{n!}{K!\,(n-K)!}\left(\frac{1}{n^K}\right) \le \frac{(n+1)!}{K!\,(n+1-K)!}\left(\frac{1}{n+1}\right)^K$$

$$\Rightarrow \cancel{\frac{n!}{K!\,(n-K)!}}\left(\frac{1}{n^K}\right) \le \frac{(n+1)\,\cancel{n!}}{\cancel{K!}\,(n+1-K)\,\cancel{(n-K)!}} \times \frac{1}{(n+1)^K}$$

$$\Rightarrow \left(\frac{n+1}{n}\right)^K \le \frac{n+1}{(n+1-K)} \Rightarrow \left(1+\frac{1}{n}\right)^K \le \frac{n+1}{n+1-K}$$

for $0 \le K \le n$ and $n \ge 1$
we can clearly say,

$$\binom{n}{K}\frac{1}{n^K} \le \binom{n+1}{K}\frac{1}{(n+1)^K} \qquad \text{— (1)}$$

now, to prove

$$\binom{n+1}{K}\frac{1}{(n+1)^K} \le \frac{1}{K!}$$

$$\Rightarrow \underline{\underline{L.H.S}} \qquad \binom{n+1}{k} \frac{1}{(n+1)^k}$$

$$\Rightarrow \frac{(n+1)!}{k!\,(n+1-k)!} \times \frac{1}{(n+1)^k} = \frac{(n+1)(n)(n-1)\dots \cancel{(n+1-(k-1))}!}{k!\,(n+1)^k \quad \cancel{(n+1-k)}!}$$

$$\Rightarrow \frac{(n+1)(n)\dots\dots(n-k+2)}{k! \quad (n+1)^k}$$

$$\Rightarrow \frac{1}{k!}\left[\frac{\cancel{n+1}}{\cancel{n+1}} \times \frac{n}{n+1} \times \frac{n-1}{n+1} \times \dots\dots \times \frac{n-k+2}{n+1}\right]$$

The value in the braket is a fraction which is less than 1

hence $L.H.S \leq R.H.S$

$$\frac{1}{k!}\left[\frac{n}{n+1} \times \frac{n-1}{n+1} \times \dots\dots \times \frac{n-k+2}{n+1}\right] \leq \frac{1}{k!} \qquad -(2)$$

from (1) and (2)

$$\binom{n}{k}\frac{1}{n^k} \leq \binom{n+1}{k}\frac{1}{(n+1)^k} \leq \frac{1}{k!}$$

f) show that $\left(a_n := \left(1+\frac{1}{n}\right)^n\right)_{n=1}^{\infty}$ is increasing i.e. $a_n \leq a_{n+1}$ for $n \geq 1$

$\Rightarrow$ we will show $a_n \leq a_{n+1}$ ~~by~~ or
$a_{n+1} - a_n \geq 0$

expanding by binomial theoram.

$$a_{n+1} = \left(1 + \frac{1}{n+1}\right)^{n+1} \Big/ a_n = \left(1+\frac{1}{n}\right)^n$$

$$a_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1} = 1 + \binom{n+1}{1}\left(\frac{1}{n+1}\right)^1 + \binom{n+1}{2}\left(\frac{1}{n+1}\right)^2 + \ldots\ldots + \binom{n+1}{n+1}\left(\frac{1}{n+1}\right)^{n+1} \quad -①$$

$$a_n = \left(1+\frac{1}{n}\right)^n = 1 + \binom{n}{1}\left(\frac{1}{n}\right)^1 + \binom{n}{2}\left(\frac{1}{n}\right)^2 + \ldots\ldots + \binom{n}{n}\left(\frac{1}{n}\right)^n \quad -②$$

Subtracting ① and ②

$$\left[1 + \binom{n+1}{1}\left(\frac{1}{n+1}\right)^1 + \binom{n+1}{2}\left(\frac{1}{n+1}\right)^2 + \ldots + \binom{n+1}{n+1}\left(\frac{1}{n+1}\right)^{n+1}\right] -$$

$$\left[1 + \binom{n}{1}\left(\frac{1}{n}\right)^1 + \binom{n}{2}\left(\frac{1}{n}\right)^2 + \ldots\ldots + \binom{n}{n}\left(\frac{1}{n}\right)^n\right]$$

comparing every value when we subtract

1^{st} term

$$1-1=0$$

2^{nd} term $$\binom{n+1}{1}\left(\frac{1}{n+1}\right)^1 - \binom{n}{1}\left(\frac{1}{n}\right)^1 = 0$$

3^{rd} term $$\binom{n+1}{2}\left(\frac{1}{n+1}\right)^2 - \binom{n}{2}\left(\frac{1}{n}\right)^2$$

$$\Rightarrow \frac{(n+1)(n)!}{2!\,(n-1)(n-2)!}\left(\frac{1}{(n+1)^2}\right) - \frac{n!}{2!\,(n-2)!}\left(\frac{1}{n^2}\right)$$

$$\Rightarrow \frac{n!}{2!\,(n-2)!}\left[\frac{\cancel{n+1}}{(n+1)^{\cancel{2}}(n-1)} - \frac{1}{n^2}\right] \Rightarrow \frac{n!}{2!\,(n-2)!}\left[\frac{1}{n^2-1} - \frac{1}{n^2}\right]$$

$$\Rightarrow \frac{n!}{2!\,(n-2)!}\left[\frac{1}{n^2(n^2-1)}\right]$$

hence the value is positive

which means $a_{n+1} - a_n \geq 0$

hence $a_{n+1} \geq a_n$

9) show that for integer $k \geq 2$,

(i) $\rightarrow \frac{1}{k!} \leq \frac{1}{k-1} - \frac{1}{k}$, show for $n \geq 2$

(ii) $\left(1+\frac{1}{n}\right)^n \leq 2+1-\frac{1}{n} < 3$

(ii)
for first

Proving by mathematical induction

for $K=2$

L.H.S $\Rightarrow \frac{1}{2!}$ R.H.S $\Rightarrow \frac{1}{2-1} - \frac{1}{2}$

$\Rightarrow \frac{1}{2}$ $\Rightarrow \frac{1}{1} - \frac{1}{2} \Rightarrow \frac{1}{2}$

hence L.H.S = R.H.S

for $K=n$ let it be true

$\Rightarrow \frac{1}{n!} \leq \frac{1}{n-1} - \frac{1}{n}$ — (1)

for $K=n+1$ we have to prove

$\Rightarrow \frac{1}{(n+1)!} \leq \frac{1}{n} - \frac{1}{n+1}$

$\Rightarrow \frac{1}{(n+1)!} = \left(\frac{1}{n+1}\right)\left(\frac{1}{n!}\right) \Rightarrow \frac{1}{(n+1)!} \leq \frac{1}{n(n+1)}$

multiplying by $\frac{1}{n+1}$ on (1)

$$\Rightarrow \left(\frac{1}{n+1}\right)\left(\frac{1}{n!}\right) \le \frac{1}{n+1}\left[\frac{1}{n-1} - \frac{1}{n}\right]$$

$$\Rightarrow \frac{1}{(n+1)!} \le \left(\frac{1}{n+1}\right)\left(\frac{\cancel{n} - \cancel{n}+1}{(n)(n-1)}\right)$$

$$\Rightarrow \frac{1}{(n+1)!} \le \left(\frac{1}{n+1}\right)\left(\frac{1}{(n)(n-1)}\right)$$

← we know $n \geq 2$ so, minimum value 2 passed

$$\Rightarrow \frac{1}{(n+1)!} \le \left(\frac{1}{n+1}\right)\left(\frac{1}{n}\right)\left(\frac{1}{2-1}\right)$$

$$\Rightarrow \frac{1}{(n+1)!} \le \frac{1}{(n)(n+1)}$$

hence proved.

(ii) now to prove

$$\left(1+\frac{1}{n}\right)^n \le 2+1-\frac{1}{n} < 3$$

for $n \geq 2$

By binomial expansion,

$$\Rightarrow \left(1+\frac{1}{n}\right)^{n} \Rightarrow \sum_{k=0}^{\infty} \binom{n}{k}\left(\frac{1}{n}\right)^{k}$$

$$= \frac{\cancel{n!}}{0!\,\cancel{(n)!}}\left(\frac{1}{n}\right)^{0} + \frac{\cancel{n!}}{1!\,\cancel{(n-1)!}}\left(\frac{1}{n}\right)^{1} + \frac{n!}{2!\,(n-2)!}\left(\frac{1}{n}\right)^{2}$$

$$+ \dots + \frac{n!}{n!\,0!}\left(\frac{1}{n}\right)^{n}$$

$$\Rightarrow \left(1+\frac{1}{n}\right)^{n} = 1 + 1 + \frac{n!}{2!\,(n-2)!}\left(\frac{1}{n}\right)^{2} + \dots + \left(\frac{1}{n}\right)^{n}$$

$$\Rightarrow \left(1+\frac{1}{n}\right)^{n} = 2 + \frac{n!}{2!\,(n-2)!}\left(\frac{1}{n^2}\right) + \frac{n!}{3!\,(n-3)!}\left(\frac{1}{n}\right)^{3}$$

$$+ \dots + \left(\frac{1}{n}\right)^{n}$$

here the third term is :-

$$\frac{n(n-1)\cancel{(n-2)!}}{2\,\cancel{(n-2)!}\;n^{2}} = \frac{n^{2}-n}{2n^{2}} = \frac{n-1}{2n}$$

$$\Rightarrow \frac{\cancel{n}}{2\cancel{n}} - \frac{1}{2n} = \boxed{\frac{1}{2} - \frac{1}{2n}}$$

for $n \geq 2$ the above term and every following term is positive and can be dropped.

$\Rightarrow \left(1+\frac{1}{n}\right)^n \leq 1+1+\frac{n-1}{2n}$ [redacted] $- (1)$

here $\frac{n-1}{2n} \Rightarrow \frac{1}{2}\left[\frac{n-1}{n}\right] < \frac{n-1}{n}$ $- (2)$

from (1) and (2)

$\Rightarrow \left(1+\frac{1}{n}\right)^n \leq 1+1+\frac{n-1}{n}$

$\Rightarrow \left(1+\frac{1}{n}\right)^n \leq 2+1-\frac{1}{n}$

for $2+1-\frac{1}{n} < 3 \Rightarrow 3-\frac{1}{n} < 3$

for any $n \geq 2$ it is less than 3

h) prove $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists

from previous question, we have

$$\left(1+\frac{1}{n}\right)^n \leq 2+\left(1-\frac{1}{n}\right) < 3 \quad -①$$

$\forall\, n \geq 2$

so, $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \Rightarrow \lim_{n\to\infty}\left(2+1-\frac{1}{n}\right)$

$\Rightarrow \lim_{n\to\infty}\left(3-\frac{1}{\infty}\right) = 3$

so using ① and above argument

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists and tends to 3

since $\lim_{n\to\infty}\left(2+1-\frac{1}{n}\right)$ exists.

hence proved.