Q.1. Deduce the Maclaurian Series

$$\sin hx = \frac{e^{2x}-1}{2e^x} = \frac{1}{2}\left[\frac{e^{2x}}{e^x} - \frac{1}{e^x}\right] = \frac{1}{2}\left[e^x - e^{-x}\right]$$

$$\cos hx = \frac{e^{2x}+1}{2e^x} = \frac{1}{2}\left[\frac{e^{2x}}{e^x} + \frac{1}{e^x}\right] = \frac{1}{2}\left[e^x + e^{-x}\right]$$

a) for $f(x) = \sin hx$

$$f(0) = \frac{1}{2}[e^0 - e^0] = \frac{1}{2}(1-1) = \frac{0}{2} = 0$$

$$f'(x)\Big|_{x=0} = \cos hx\Big|_{x=0} = \frac{1}{2}[e^0 + e^0] = \frac{1}{2}(1+1)$$

$$= \frac{1}{2} \times 2 = 1$$

$$f''(x)\Big|_{x=0} = \sin hx\Big|_{x=0} = \frac{1}{2}(e^0 - e^0) = \frac{1}{2}(1-1) = \frac{1}{2}(0) = 0$$

$$f'''(x)\Big|_{x=0} = \cos hx\Big|_{x=0} = \frac{1}{2}(e^0 + e^0) = \frac{1}{2}(1+1) = \frac{1}{2}(2) = 1$$

$$f^4(x)\Big|_{x=0} = \sin hx\Big|_{x=0} = \frac{1}{2}(e^0 - e^0) = \frac{1}{2}(1-1) = \frac{0}{2} = 0$$

$$f^{n}(x)\Big|_{x=0} = \begin{cases} 0, & n \text{ is even} \\ 1, & n \text{ is odd} \end{cases}$$

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots + \frac{f^n(0)x^n}{n}$$

$$f(x) = 0 + \frac{1(x)}{1!} + \frac{0(x^2)}{2!} + \frac{1(x^3)}{3!} + \dots$$

$$f(x) = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

hence, $\boxed{\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}}$

radius of converge by ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2(n+1)+1}}{(2(n+1)+1)!} \times \frac{(2n+1)!}{x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{x^{2n+3}}\, x^2}{\cancel{(2n+3)!}} \times \frac{\cancel{(2n+1)!}}{\cancel{x^{2n+1}}}\right| = \lim_{n\to\infty} \frac{x^2}{(2n+3)(2n+2)}$$

$$\Rightarrow x^2 \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} = x^2(0) = 0 < 1$$

series converges for all x $\therefore$ $\boxed{\text{radius} = \infty}$

b) for $f(x) = \cosh x$

$$f(0) = \cosh x\Big|_{x=0} = \frac{1}{2}[e^0 + e^0] = \frac{1}{2}(2) = 1$$

$$f'(0) = \sinh x\Big|_{x=0} = \frac{1}{2}[e^0 - e^0] = \frac{1}{2}(1-1) = \frac{0}{2} = 0$$

$$f''(0) = \cosh x\Big|_{x=0} = \frac{1}{2}[e^0 + e^0] = \frac{1}{\cancel{2}}[\cancel{2}] = 1$$

$$f'''(0) = \sinh x\Big|_{x=0} = \frac{1}{2}[e^0 - e^0] = \frac{1}{2}(1-1) = \frac{0}{2} = 0$$

$$f''''(0) = \cosh x\Big|_{x=0} = \frac{1}{2}[e^0 + e^0] = \frac{1}{2}(2) = 1$$

hence the series is $1, 0, 1, 0, 1, \ldots$

$$f^n(x)\Big|_{x=0} = \begin{cases} 1, & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases}$$

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \ldots + \frac{f^n(0)x^n}{n}$$

$$f(x) = 1 + \frac{0x}{1!} + \frac{1(x^2)}{2!} + \frac{0(x^3)}{3!} + \frac{1(x^4)}{4!} + \ldots$$

$$f(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

hence $\boxed{\cos hx = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}}$

radius of converge by ratio test

$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{2(n+1)}}{(2(n+1))!} \times \frac{(2n)!}{x^{2n}} \right|$$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{\cancel{x^{2n+2}}\, x^2}{\cancel{(2n+2)!}} \times \frac{\cancel{(2n)!}}{\cancel{x^{2n}}} \right|$$

$(2n+1)(2n+2)$

$$\Rightarrow \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+1)} = x^2(0) = 0$$

$$0 < 1$$

Series converges for all x

$\therefore$ $\boxed{\text{radius} = \infty}$

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Career Development

Q.2 Show using Riemann integral

$$\lim_{n\to\infty}\left(\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\dots+\frac{n^2}{(2n-1)^3}+\frac{1}{8n}\right)=\frac{3}{8}$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{n^2}{n^3}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\dots+\frac{n^2}{8n^3}\right)$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{n^2}{(n+0)^3}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\dots+\frac{n^2}{(2n)^3}\right)$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{n^2}{(n+0)^3}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\dots+\frac{n^2}{(n+n)^3}\right)$$

$$\Rightarrow \lim_{n\to\infty}\sum_{i=0}^{n}\frac{n^2}{(n+i)^3} \Rightarrow \lim_{n\to\infty}\sum_{i=0}^{n}\frac{1}{n}\left(\frac{n\times n^2}{(n+i)^3}\right)$$

$$\Rightarrow \lim_{n\to\infty}\sum_{i=0}^{n}\frac{1}{n}\left(\frac{n^3}{(n+i)^3}\right)$$

$$\Rightarrow \lim_{n\to\infty}\sum_{i=0}^{n}\left(\frac{1}{n}\right)\left(\frac{1}{\left(\frac{n}{n}+\frac{i}{n}\right)^3}\right)$$

$$\Rightarrow \lim_{n\to\infty} \sum_{i=0}^{n} \left(\frac{1}{n}\right)\underbrace{\left(\frac{1}{\left(1+\frac{i}{n}\right)^3}\right)}_{f(x)}$$

$$\Rightarrow \int_0^1 \frac{1}{(1+x)^3}\,dx = \int_0^1 (1+x)^{-3}\,dx$$

$$= \left.\frac{(1+x)^{-2}}{-2}\right|_0^1 = \frac{-1}{2}\left((1+1)^{-2} - (1+0)^{-2}\right)$$

$$= \frac{-1}{2}\left(2^{-2} - 1^{-2}\right) = \frac{-1}{2}\left(\frac{1}{4} - 1\right) = \frac{-1}{2}\left(\frac{1-4}{4}\right)$$

$$= \frac{(-1)(-3)}{8} = \frac{3}{8} = R.H.S$$

hence proved.

Q.3. for $a > 0$,

compute, $\int_{-a}^{a} |x(x-a)(x-2a)(x-3a)| \, dx$

finding critical points $f(x) = 0$

$(x)(x-a)(x-2a)(x-3a) = 0$

so, $x = 0, a, 2a, 3a$

we have to find between a and $-a$

So 2 regions $-a < x < 0$ & $0 \leq x < a$

hence, for $-a < x < 0$

$x < 0$

$x - a < 0$

$x - 2a < 0$

$x - 3a < 0$

hence equation is positive

for $0 < x < a$

$x > 0$

$x - a < 0$

$x - 2a < 0$

$x - 3a < 0$

hence equation is negative.

$$(x)(x-a)(x-2a)(x-3a) = x^4 - 6ax^3 + 11a^2x^2 - 6a^3x$$

$$\Rightarrow \int_{-a}^{0} \left(x^4 - 6ax^3 + 11a^2x^2 - 6a^3x\right) dx$$

$$\Rightarrow \left[\frac{x^5}{5} - \frac{6ax^4}{4} + \frac{11a^2x^3}{3} - \frac{6a^3x^2}{2}\right]_{-a}^{0}$$

$$\Rightarrow \left[\frac{x^5}{5} - \frac{3a}{2}x^4 + \frac{11a^2}{3}x^3 - 3a^3x^2\right]_{-a}^{0}$$

$$\Rightarrow (0-0+0-0) - \left[\frac{(-a)^5}{5} - \frac{3a}{2}(-a)^4 + \frac{11a^2}{3}(-a)^3 - 3a^3(-a)^2\right]$$

$$\Rightarrow -\left[\frac{-a^5}{5} - \frac{3a^5}{2} - \frac{11}{3}a^5 - 3a^5\right]$$

$$\Rightarrow \frac{a^5}{5} + \frac{3a^5}{2} + \frac{11a^5}{3} + 3a^5 = \frac{6a^5 + 45a^5 + 110a^5 + 90a^5}{30}$$

$$= \frac{251}{30}a^5 \text{ — } (1)$$

$$\int_0^a -\left(x^4 - 6ax^3 + 11a^2x^2 - 6a^3x\right)\,dx$$

$$= -\left[\frac{x^5}{5} - \frac{3ax^4}{2} + \frac{11a^2x^3}{3} - 3a^3x^2\right]_0^a$$

$$= -\left(\left[\frac{a^5}{5} - \frac{3a^5}{2} + \frac{11a^5}{3} - 3a^5\right] - [0]\right)$$

$$= -\left(\frac{6a^5 - 45a^5 + 110a^5 - 90a^5}{30}\right)$$

$$= -\left(\frac{-19a^5}{30}\right) = \frac{19}{30}a^5 \quad - (2)$$

Adding (1) and (2)

$$\frac{251}{30}a^5 + \frac{19}{30}a^5 = \frac{\overset{9}{\cancel{270}}\,a^5}{\cancel{30}}$$

$$= \boxed{9a^5}$$ final answer

Q.4 Compute $\int_0^2 \lfloor x^2 \rfloor \, dx$

using piecewise.

$$\lfloor x^2 \rfloor = \begin{cases} 0 & , 0 \le x < 1 \\ 1 & , 1 \le x < \sqrt{2} \\ 2 & , \sqrt{2} \le x < \sqrt{3} \\ 3 & , \sqrt{3} \le x < 2 \end{cases}$$

hence, $\int_0^2 \lfloor x^2 \rfloor \, dx = \int_0^1 0 \, dx + \int_1^{\sqrt{2}} 1 \, dx + \int_{\sqrt{2}}^{\sqrt{3}} 2 \, dx$

$+ \int_{\sqrt{3}}^{2} 3 \, dx$

$\Rightarrow$

$0 + x\Big|_1^{\sqrt{2}} + 2x\Big|_{\sqrt{2}}^{\sqrt{3}} + 3x\Big|_{\sqrt{3}}^{2}$

$\Rightarrow (\sqrt{2} - 1) + 2(\sqrt{3} - \sqrt{2}) + 3(2 - \sqrt{3})$

$\Rightarrow \sqrt{2} - \cancel{1} + 2\sqrt{3} - 2\sqrt{2} + \cancel{6} - 3\sqrt{3}$

$= \boxed{5 - \sqrt{2} - \sqrt{3}} =$

Q.5. evaluate

$$\lim_{h\to 0}\left[\frac{1}{h}\int_{1}^{1+h}\sqrt{x^4+1}\,dx\right] \quad -(1)$$

let $F(x)$ be anti-derivative of $\sqrt{x^4+1}$

so $F'(x) = \sqrt{x^4+1}$

so from (1), $\lim_{h\to 0}\left[\frac{1}{h}\, F(x)\Big|_{1}^{1+h}\right]$

$$\Rightarrow \lim_{h\to 0}\left[\frac{1}{h}\left(F(1+h) - F(1)\right)\right]$$

$$\Rightarrow \lim_{h\to 0}\frac{F(1+h) - F(1)}{h} \quad -(2)$$

from (2) and definition of derivative

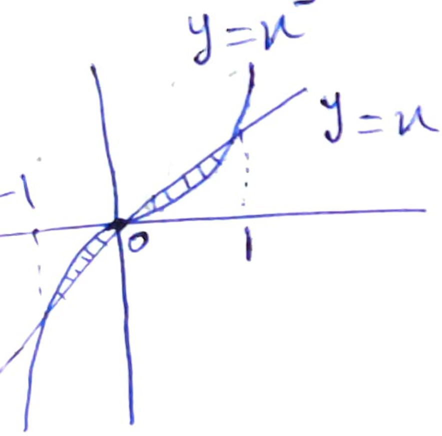
$$\lim_{h\to 0}\frac{F(1+h)-F(1)}{h} = F'(1)$$

$f'(x) = \sqrt{x^4+1}$

So, $f'(1) = \sqrt{1^4+1} = \sqrt{1+1} = \boxed{\sqrt{2}}$

final answer

Q.6. area between graph $y = x$ & $y = x^3$



point of intersection

$\Rightarrow x^3 = x \Rightarrow x^3 - x = 0$

$\Rightarrow x(x^2-1) = 0 \Rightarrow x(x-1)(x+1) = 0$

$\boxed{x = 0, 1, -1}$

The area bounded above the x-axis is same as area below the x-axis

So, $\int_{-1}^{1} x - x^3 \, dx = \int_{-1}^{0} -x + x^3 \, dx + \int_{0}^{1} x - x^3 \, dx$

$= 2 \int_{0}^{1} x - x^3 \, dx = 2\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$

$\Rightarrow 2\left[\frac{1}{2} - \frac{1}{4} - 0 + 0\right] = 2\left[\frac{1}{2} - \frac{1}{4}\right] = 2\left[\frac{1}{4}\right] = \boxed{\frac{1}{2}}$

final answer

Q.7 we are given

$$f(\lambda x, \lambda y) = \lambda^n f(x,y) \text{ for } \lambda \neq 0 \quad -① $$

assume, $F(t) = f(tx, ty)$

Then $F'(t) = \frac{d}{dt} f(tx, ty) = \frac{d}{dt} t^n f(x,y)$

using ①

So $F'(t) = n t^{n-1} f(x,y)$

$F'(1) = n f(x,y) \quad - ②$

let $(tx, ty) = L(t)$ where $L(1) = (x,y)$

By appling chain rule

So, $F'(t) = f(L(t))$

$$F'(t) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\left(\frac{d}{dt} L(t)\right)$$

$$F'(t) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)(x, y)$$

so, $F'(1) = \frac{\partial f}{\partial x} x + \frac{\partial f}{\partial y} y = 1\left(\frac{\partial f}{\partial x} x + \frac{\partial f}{\partial y} y\right)$

③

using ② and ③
The above is in the form

$$n f(x,y)$$ hence proved

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f(x,y)$$

Q.8. linear approximation at $(a,b)=(1,1)$

$$f(x,y) = \frac{x^2 y}{x^4+y^2+1}$$

$$f(x,y) \simeq f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$f(a,b) = f(1,1) = \frac{1^2(1)}{1^4+1^2+1} = \boxed{\frac{1}{3}} \quad -①$$

$$f_x'(a,b) = \frac{(x^4+y^2+1)(2xy) - (x^2y)(4x^3)}{(x^4+y^2+1)^2}$$

$$f_x'(1,1) = \frac{(1^4+1^2+1)(2(1)(1)) - (1^2(1))(4(1)^3)}{(1^4+1^2+1)^2}$$

$$f_x'(1,1) = \frac{3(2) - (1)(4)}{3^2} = \frac{6-4}{9} = \boxed{\frac{2}{9}} \quad -②$$

$$f_y'(a,b) = \frac{(x^4+y^2+1)(x^2) - (x^2y)(2y)}{(x^4+y^2+1)^2}$$

$$f'y(1,1) = \frac{(1^4+1^2+1)(1^2) - (1^2(1))(2(1))}{(1^4+1^2+1)^2}$$

$$f'(y)(1,1) = \frac{(3)-2}{3^2} = \boxed{\frac{1}{9}} \quad -③$$

Substituting (1), (2) and (3) in main formula

$$\frac{1}{3} + \frac{2}{9}(x-1) + \frac{1}{9}(y-1)$$

$$= \frac{1}{3} + \frac{2x-2+y-1}{9} \Rightarrow \boxed{\frac{2x+y-3}{9} + \frac{1}{3}}$$

final answer

<u>Q.9.</u> $f(p,q,r) = \frac{p^2 - r}{q^4}$

where $p(x) = x^3+7$, $q(x) = \cos 2x$, $r(x) = 4x$

$$\frac{df}{dx} = \frac{\partial f}{\partial p} \times \frac{dp}{dx} + \frac{\partial f}{\partial r} \times \frac{dr}{dx} + \frac{\partial f}{\partial q} \times \frac{dq}{dx}$$

$$\frac{\partial f}{\partial p} = \frac{\partial}{\partial p}\left(\frac{p^2}{q^4} - \frac{r}{q^4}\right) = \boxed{\frac{2p}{q^4}}$$

$$\frac{\partial f}{\partial r} = \frac{\partial}{\partial r}\left(\frac{p^2}{q^4} - \frac{r}{q^4}\right) = \boxed{\frac{-1}{q^4}}$$

$$\frac{\partial f}{\partial q} = \frac{\partial}{\partial q}\left((p^2-r)(q^{-4})\right) = \underline{(p^2-r)(-4)(q^{-5})}$$

$$\boxed{\frac{\partial f}{\partial q} = \frac{-4(p^2-r)}{q^5}}$$

$$\frac{dp}{dx} = \frac{d}{dx}(x^3+7) = \boxed{3x^2}$$

$$\frac{dq}{dx} = \frac{d}{dx}\cos 2x = \boxed{(-\sin 2x)(2)}$$

$$\frac{dr}{dx} = \frac{d}{dx}(4x) = \boxed{4}$$

Substituting in main function

$$\Rightarrow \frac{2p}{q^4} \times (3x^2) + (4)\left(-\frac{1}{q^4}\right) + (-2)(\sin 2x) \times \frac{(-4)(p^2-r)}{q^5}$$

substituting value of p, q, r.

$$\Rightarrow \frac{2(x^3+7)(3x^2)}{(\cos 2x)^4} - \frac{4}{(\cos 2x)^4} + \frac{8\sin 2x\left((x^3+7)^2-4x\right)}{(\cos 2x)^5}$$

substituting value of p, q, r

$$\Rightarrow \frac{2(x^3+7)(3x^2)}{(\cos 2x)^4} - \frac{4}{(\cos 2x)^4} + \frac{8\sin 2x\,((x^3+7)^2-4x)}{(\cos 2x)^5}$$

$$\Rightarrow \frac{6(x^5+7x^2)}{(\cos 2x)^4} - \frac{4}{(\cos 2x)^4} + \frac{8(\sin 2x)(x^6+49+14x^3-4x)}{(\cos 2x)^5}$$

= final answer.