Continous Time Fourier Series -:

→ It is defined only for periodic signal.

→ We can convert any periodic signal into linear combination of harmonically related complex expo.
$e^{j\omega_0 t}$ $(e^{j\theta})$

(second harmonics)

eg $e^{j\underbrace{\omega_0}_{\text{funda. freq}} t}$, $e^{j\underbrace{2\omega_0}_{\text{freq}} t}$, $e^{j\underbrace{3\omega_0}_{\text{freq.}} t}$ (Third harmonics)

→ $x(t) = A_1 e^{j\omega_0 t} + A_2 e^{j2\omega_0 t} + A_3 e^{j3\omega_0 t} + \cdots\cdots$

→ Synthesis equation -:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

k represents harmonics

a_k is fourier series coeff.

→ Analysis Equation -:

$$a_k = \frac{1}{T}\int_0^T x(t) \cdot e^{-jk\omega_0 t}\, dt$$

→ $x(t) = \cos\omega_0 t = \dfrac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$

$\therefore\ a_1 = \frac{1}{2} \qquad a_{-1} = \frac{1}{2}$

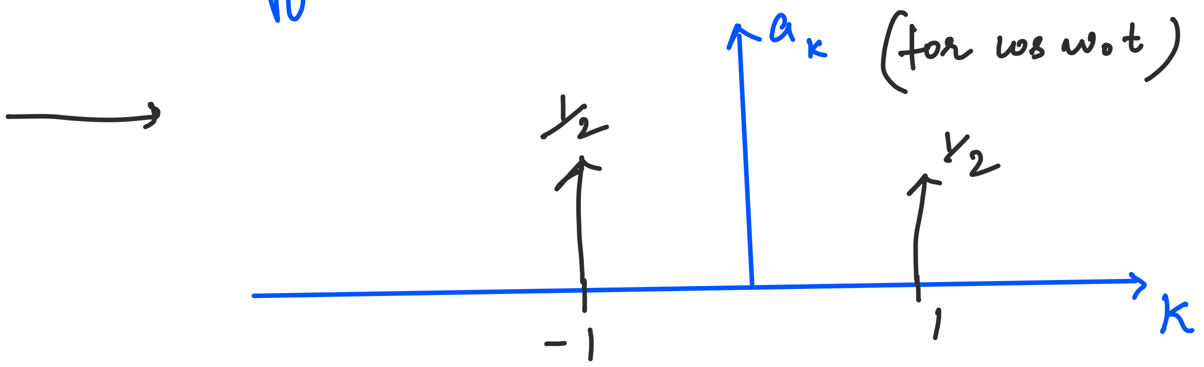
$\longrightarrow\ x(t) = \sin w_0 t = \dfrac{e^{jw_0 t} - e^{-jw_0 t}}{2j}$

$$= \underbrace{\frac{1}{2j}}\, e^{jw_0 t} - \underbrace{\frac{1}{2j}}\, e^{-jw_0 t}$$

These are fourier series coeff. a_k

$a_1 = \frac{1}{2j} \qquad a_{-1} = -\frac{1}{2j}$

$\longrightarrow$



eg find Magnitude and phase spectrum of F.S. coeff of :

$$x(t) = 4 + \underbrace{2\cos \frac{2\pi}{3} t}_{T_1} + \underbrace{4 \sin \frac{5\pi}{3} t}_{T_2}$$

$w_{0_1} = \frac{2\pi}{3} \qquad w_{0_2} = \frac{5\pi}{3}$

$T_1 = \dfrac{2\pi}{(2\pi/3)} = 3 \qquad T_2 = \dfrac{2\pi}{(5\pi/3)} = \frac{6}{5}$

Fundamental time period. $\therefore\ T = \text{LCM}\left[\frac{3}{1}, \frac{6}{5}\right] = \frac{6}{1} = 6$ sec

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{6} = \underline{\frac{\pi}{3}} \text{ rad/sec}$$

$\hookrightarrow$ fundamental frequency.

$\therefore \; 2\cos\frac{2\pi}{3}t \longrightarrow$ Represents 2^{nd} harmonics

$4\sin\frac{5\pi}{3}t \longrightarrow$ Represents 5^{th} harmonics

$$x(t) = 4e^{j(0)\frac{\pi}{3}t} + 2\left[\frac{e^{j2\left(\frac{\pi}{3}\right)t} + e^{-j2\left(\frac{\pi}{3}\right)t}}{2}\right]$$

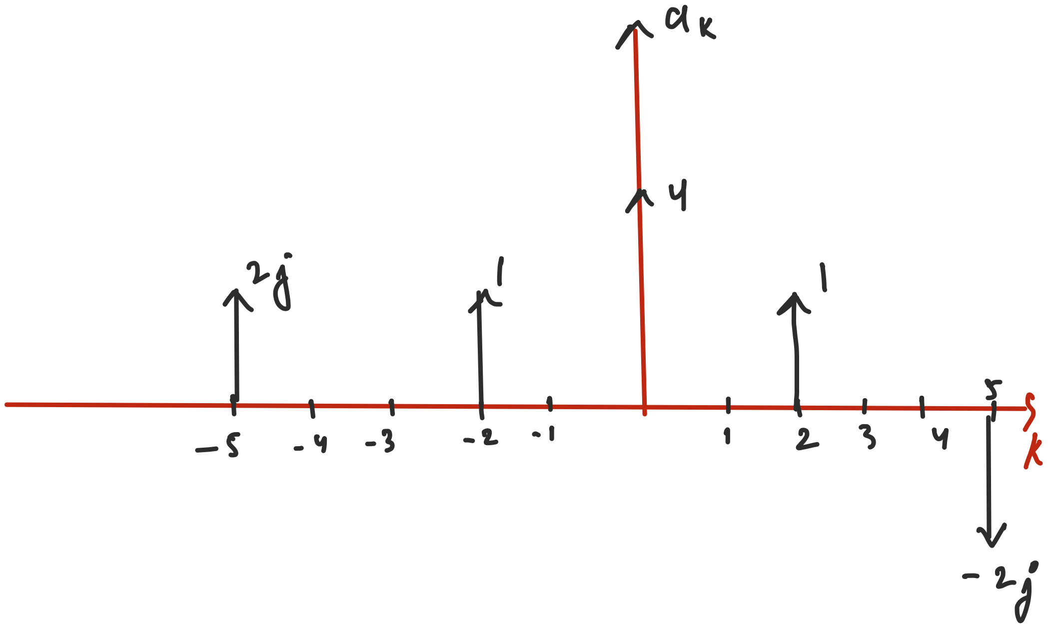
$$+ 4\left[\frac{e^{j(5)\frac{\pi}{3}t} + e^{-j(5)\frac{\pi}{3}t}}{2j}\right]$$

$$x(t) = 4e^{j0\frac{\pi}{3}t} + e^{j2\frac{\pi}{3}t} + e^{-j2\frac{\pi}{3}t} + \frac{2}{j}e^{j5\frac{\pi}{3}t} - \frac{2}{j}e^{-j5\frac{\pi}{3}t}$$

$a_0 = 4$ $\qquad\qquad a_{-2} = 1$

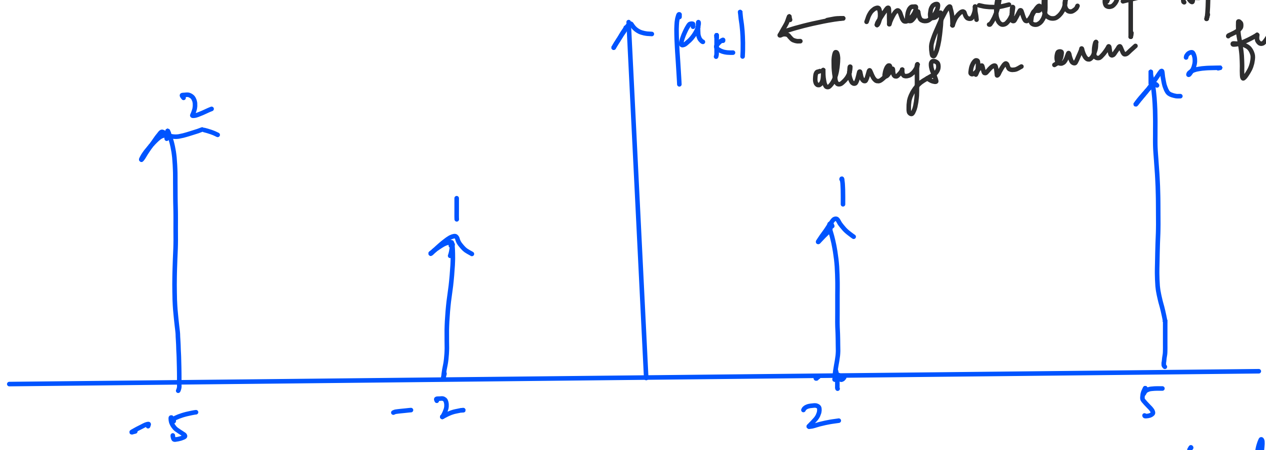
$a_2 = 1$ $\qquad\qquad a_{-5} = +2j$

$a_5 = -2j$

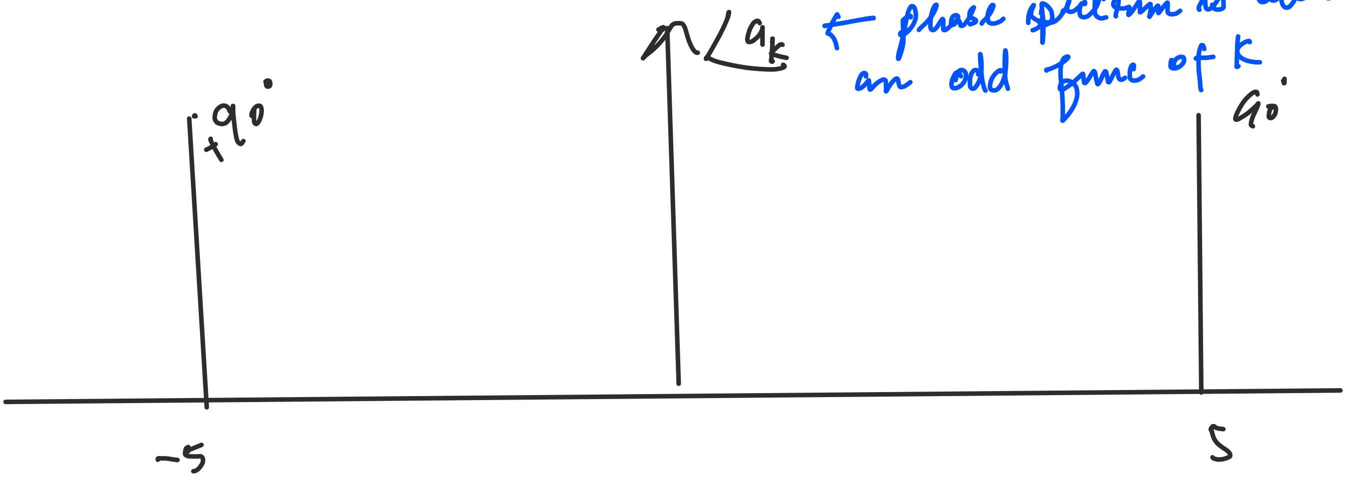


a_k	values	Magnitude	Phase angle
a_0	4	4	0°
a_2	1	1	0°
a_{-2}	1	1	0°
a_5	$-2j$	2	-90°
a_{-5}	$2j$	2	90°

← magnitude of spectrum always an even func^n of K



← phase spectrum is always an odd func of K



eg

$$x(t) = 2 + \sin w_0 t + 2\cos w_0 t + \cos\left[2w_0 t + \frac{\pi}{4}\right]$$

with w_0 = fundamental freq.

$$x(t) = 2 + \sin w_0 t + 2\cos w_0 t + \cos 2w_0 t\left(\frac{1}{\sqrt{2}}\right) - \sin 2w_0 t\left(\frac{1}{\sqrt{2}}\right)$$

$$= 2 + \frac{1}{2j}e^{jw_0 t} - \frac{1}{2j}e^{-jw_0 t} + 2\left(\frac{1}{2}\right)e^{jw_0 t} + 2\left(\frac{1}{2}\right)e^{-jw_0 t}$$

$$+\frac{1}{\sqrt{2}}\left(\frac{1}{2}\right)e^{j2\omega_0 t}+\frac{1}{\sqrt{2}}\left(\frac{1}{2}\right)e^{-j2\omega_0 t}-\frac{1}{\sqrt{2}}\left(\frac{1}{2j}\right)e^{j2\omega_0 t}$$

$$+\frac{1}{\sqrt{2}}\frac{1}{2j}e^{-j2\omega_0 t}$$

$$=\underbrace{2}_{a_0}+e^{j\omega_0 t}\overbrace{\left[\frac{1}{2j}+1\right]}^{a_1}+e^{-j\omega_0 t}\overbrace{\left[\frac{-1}{2j}+1\right]}^{a_{-1}}$$

$$+e^{j2\omega_0 t}\underbrace{\left[\frac{1}{2\sqrt{2}}-\frac{1}{j2\sqrt{2}}\right]}_{a_2}+e^{-j2\omega_0 t}\underbrace{\left[\frac{1}{2\sqrt{2}}+\frac{1}{j2\sqrt{2}}\right]}_{a_{-2}}$$

eg Consider a periodic signal with $T = 8$ sec & Fourier Series Coeff. given as

$$\left.\begin{array}{ll} a_2 = 2 & a_3 = 8j \\ a_{-2} = 2 & a_{-3} = -8j \end{array}\right\}$$

Find the signal $x(t)$

Synthesis eq^n $\quad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

Given $T = 8$ sec

$$\therefore \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4} \text{ rad/sec}$$

$$\therefore x(t) = a_2 e^{j2\omega_0 t} + a_{-2} e^{-j2\omega_0 t} + a_3 e^{j3\omega_0 t} + a_{-3} e^{-j3\omega_0 t}$$

$$2e^{j(2)\frac{\pi}{4}t} + 2e^{-j(2)\left(\frac{\pi}{4}\right)t} + 8j\, e^{j3\left(\frac{\pi}{4}\right)t} - 8j\, e^{-j3\left(\frac{\pi}{4}\right)t}$$

$$\Rightarrow 2\times 2\left[\frac{e^{j\frac{\pi}{2}t}+e^{-j\frac{\pi}{2}t}}{2}\right]+8j\left[\frac{e^{j\frac{3\pi}{4}t}-e^{-j\frac{3\pi}{4}t}}{2j}\right]$$

$$\Rightarrow \boxed{x(t) = 4\cos\frac{\pi}{2}t - 16\sin\frac{3\pi}{4}t}$$

Properties of Fourier Series -:

(1) Linearity Property -:

$$x(t) \xleftrightarrow{F.S.} a_k$$

$$y(t) \xleftrightarrow{F.S.} b_k$$

$$z(t) = \alpha\, x(t) + \beta\, y(t) \longleftrightarrow c_k$$

$$\boxed{c_k = \alpha a_k + \beta b_k}$$

eg $x(t) = \cos w_0 t$

$$a_1 = \frac{1}{2} \qquad a_{-1} = \frac{1}{2}$$

$y(t) = \sin w_0 t$

$$b_1 = \frac{1}{2j} \qquad b_{-1} = -\frac{1}{2j}$$

$$z(t) = x(t) + y(t) = \cos w_0 t + \sin w_0 t$$

$$= \frac{1}{2}e^{jw_0 t} + \frac{1}{2}e^{-jw_0 t} + \frac{1}{2j}e^{jw_0 t} - \frac{1}{2j}e^{-jw_0 t}$$

$$= \underbrace{\left[\frac{1}{2}+\frac{1}{2j}\right]}_{c_1 = a_1 + b_1} e^{jw_0 t} + \underbrace{\left[\frac{1}{2}-\frac{1}{2j}\right]}_{c_{-1} = a_{-1} + b_{-1}} e^{-jw_0 t}$$

② Time Shifting Property :-

$$x(t) \xleftrightarrow{F.S.} a_k$$

$$x(t-t_0) \xleftrightarrow{F.S.} b_k$$

$$\boxed{b_k = e^{-jk\omega_0 t_0} \, a_k}$$

$$x(t-t_0) \xleftrightarrow{F.S.} e^{jk\omega_0 t_0} \, a_k$$

eg $b_k = e^{-jk\omega_0 t_0} \, a_k$

mag :- $|b_k| = |e^{-jk\omega_0 t_0}| \, |a_k|$

$$\boxed{\therefore |b_k| = |a_k|}$$

Magnitude is not affected.

phase :- angle:

$$\angle b_k = \angle e^{-jk\omega_0 t_0} + \angle a_k$$

$$\boxed{\angle b_k = \angle a_k - k\omega_0 t_0}$$

There could be delay / advance acc. to time shift

eg $x(t) \longrightarrow a_k$

$$x_1(t) = x(t+t_0) + x(t-t_0) \longleftrightarrow b_k$$

Find b_k in terms of a_k

$$x(t) \longleftrightarrow a_k$$

$$x(t+t_0) \longleftrightarrow e^{jk\omega_0 t_0} \cdot a_k$$

$$x(t-t_0) \longleftrightarrow e^{-jk\omega_0 t_0} \cdot a_k$$

Using linearity property -:

$$b_k = e^{jk\omega_0 t_0} a_k + e^{-jk\omega_0 t_0} a_k$$

$$= a_k \left[\frac{e^{jk\omega_0 t_0} + e^{-jk\omega_0 t_0}}{2} \right] \times 2$$

$$\boxed{b_k = 2 \cos k\omega_0 t_0 \cdot a_k}$$ Ans

(3) Time Reversal Property -:

$$x(t) \xleftrightarrow{F.S.} a_k$$

$$x(-t) \xleftrightarrow{F.S.} b_k$$

$$\boxed{b_k = a_{-k}}$$

Case I : If $x(t)$ is an even signal

$$x(-t) = x(t)$$

$$\boxed{b_k = a_{-k} = a_k}$$

Case II -: If $x(t)$ is an odd signal

$$x(-t) = -x(t)$$

$$x(t) \longleftrightarrow a_k$$

$$x(-t) = -x(t) \longleftrightarrow -a_k$$

$$\boxed{\therefore \quad b_k = a_{-k} = -a_k}$$

eg Given $x(t) \longleftrightarrow a_k$

$$x_1(t) = x(1-t) + x(t-1) \longleftrightarrow b_k$$

Find b_k in terms of a_k

$\longrightarrow$ $x(t) \longleftrightarrow a_k$

$$x(t+1) \longleftrightarrow e^{jk\omega_0 1} a_k$$

$$x(-t+1) \longleftrightarrow e^{-jk\omega_0} a_{-k} \quad \text{— (1)}$$

$\longrightarrow$ $x(t) \longleftrightarrow a_k$

$$x(t-1) \longleftrightarrow e^{-jk\omega_0} a_k \quad \text{— (2)}$$

$$\therefore \quad b_k = e^{-jk\omega_0} a_{-k} + e^{-jk\omega_0} a_k$$

$$\boxed{b_k = e^{-jk\omega_0}\left[a_{-k} + a_k\right]} \quad \text{— (3)}$$

Case I : $x(t)$ is even

$$a_{-k} = a_k$$

$$\boxed{b_k = 2e^{-jk\omega_0} a_k}$$

Case II : $x(t)$ is odd

$$a_{-k} = -a_k$$

$$\therefore \boxed{b_k = 0}$$

(4) Time Scaling Property :-

$$x(t) \xleftrightarrow{F.S} a_k$$

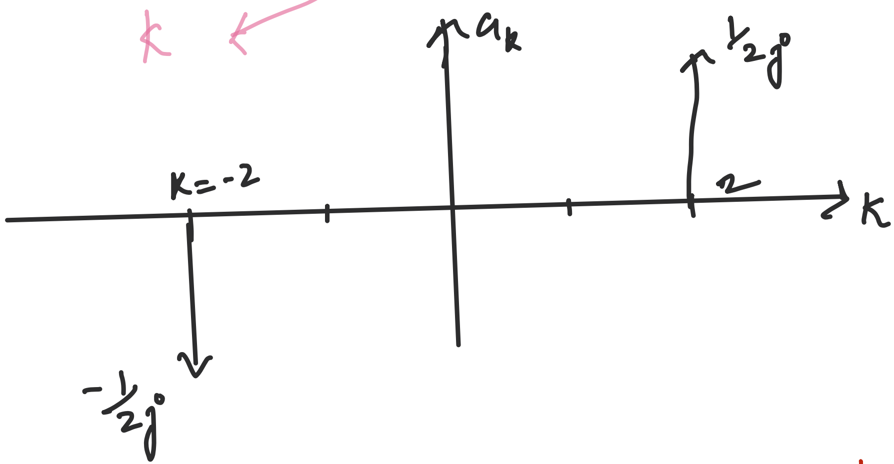
$$x(\alpha t) \xleftrightarrow{F.S.} b_k$$

$$\boxed{b_k = a_k}$$

→ coeff will be same but k is diff.

eg $x(t) = \sin 2\omega_0 t$, ω_0 = fundamental freq.

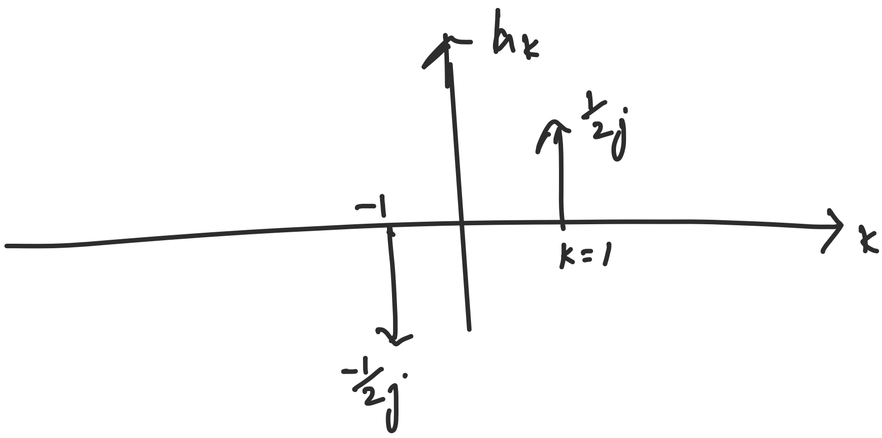
k



$$x_1(t) = x\left(\frac{t}{2}\right) = \sin\left(\cancel{2}\omega_0\left(\frac{t}{\cancel{2}}\right)\right) = \sin \omega_0 t$$

$b_1 = \frac{1}{2j}$

$b_{-1} = -\frac{1}{2j}$



(5) Differentiation Property :-

$$x(t) \xleftrightarrow{F.S} a_k$$

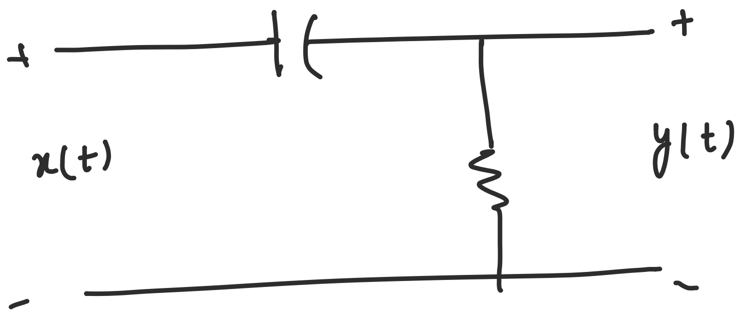
$$\frac{d}{dt} x(t) \xleftrightarrow{F.S.} b_k$$

$$\boxed{b_k = jk\omega_0 . a_k}$$

$$|b_k| = |jk\omega_0|\,|a_k| = k\omega_0 |a_k|$$

$$\angle b_k = \angle jk\omega_0 + \angle a_k = \angle a_k + 90$$

A differentiator acts as an ideal High Pass Filter



It provides a phase shift of $+\pi/2$

It adds a zero at origin.

⑥ Integration Property :-

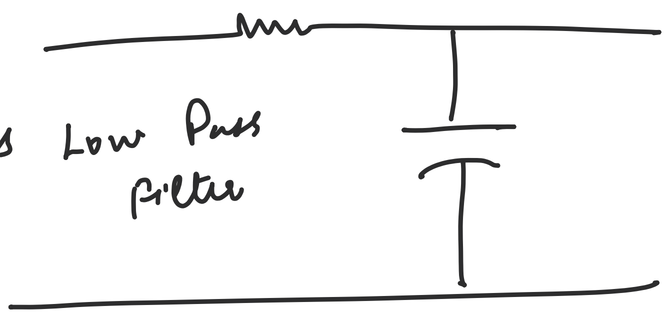
$$x(t) \xleftrightarrow{F.S.} a_k$$

$$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{F.S.} b_k$$

$$\boxed{b_k = \frac{1}{jk\omega_0}\, a_k}$$

$$|b_k| = \frac{|a_k|}{k\omega_0}$$

$$\angle b_k = \angle a_k - \pi/2$$



Acts as Low Pass filter

(7) Frequency shifting Property -:

$$x(t) \xleftrightarrow{F.S.} a_k$$

Integer

$$e^{jm\omega_0 t} \cdot x(t) \longleftrightarrow b_k$$

Freq

then $\boxed{b_k = a_{k-m}}$

(8) Multiplication Property -:

$$x(t) \longleftrightarrow a_k$$

$$y(t) \longleftrightarrow b_k$$

$$z(t) = x(t)\, y(t) \longleftrightarrow c_k$$

$$\boxed{c_k = a_k * b_k}$$

discrete convolution.

eg $x(t) = 5 + 2\sin 4\pi t$

$$y(t) = 6\cos 6\pi t + 4\sin 2\pi t$$

$$z(t) = x(t)\, y(t) \longleftrightarrow c_k = ?$$

Sol

$$\omega_0 = HCF\,[4\pi, 6\pi, 2\pi]$$

$$\omega_0 = 2\pi \text{ rad/sec}$$

(1) $x(t) = 5 + \frac{2}{2j} e^{j4\pi t} - \frac{2}{2j} e^{-j4\pi t}$

$a_0 = 5$, $a_2 = -j$ $a_{-2} = j$

$a_k = \{0, -j, 0, \underset{\uparrow}{5}, 0, -j, 0\}$

(2) $y(t) = \frac{6}{2} e^{j6\pi t} + \frac{6}{2} e^{-j6\pi t} + \frac{4}{2j} e^{j2\pi t} - \frac{4}{2j} e^{-j2\pi t}$

$b_3 = 3$ $b_{-3} = 3$ $b_1 = -2j$ $b_{-1} = 2j$

$b_k = \{3, 0, 2j, \underset{\uparrow}{0}, -2j, 0, 3\}$

(9) Parseval's Theorem -:

Total average power in any periodic signal is always equal to the sum of average power contained in all of its harmonic components.

$$P_x = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

eg $x(t) = \cos w_0 t$

We know, $\boxed{\text{Power} = \frac{A^2}{2} = \frac{1}{2} W}$

Using P. theorem,

$a_1 = \frac{1}{2}$ $a_{-1} = \frac{1}{2}$

$$\sum_{k=-\infty}^{\infty} |a_k|^2 = |a_1|^2 + |a_{-1}|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} W.$$

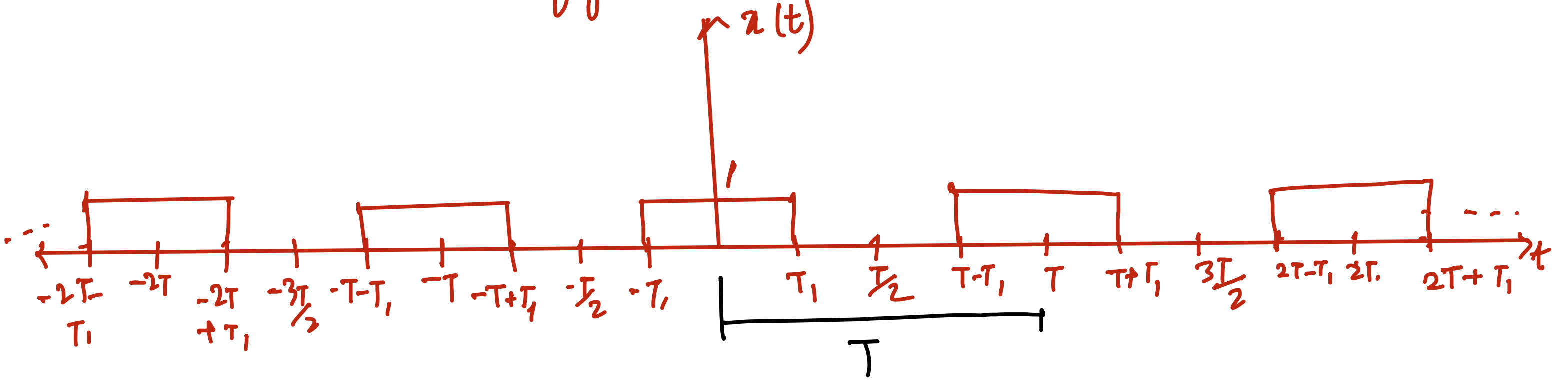
(10) Convolution Property :-

Time period = T $\begin{cases} x_1(t) \longleftrightarrow a_k \\ x_2(t) \longleftrightarrow b_k \end{cases}$

$$x(t) = x_1(t) * x_2(t) \longleftrightarrow c_k$$

$$\boxed{c_k = T\left[a_k \cdot b_k\right]}$$

eg Find F.S. coeff. for the periodic gate pulses shown in figure -:



$$a_K = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T_1}^{T_1} 1 . e^{-jk\omega_0 t}\, dt = \frac{1}{T}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{-T_1}^{T_1}$$

$$= \frac{1}{T}\left[\frac{e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}}{-jk\omega_0}\right]$$

$$= \frac{1 \times 2}{T k\omega_0}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{j \times 2}\right] = \frac{2 \sin k\omega_0 T_1}{T k\omega_0}$$

$$[T\omega_0 = 2\pi]$$

$$a_K = \frac{\cancel{2} \sin k\omega_0 T_1}{\cancel{2} k\pi} = \frac{\sin k\omega_0 T_1}{k\pi}$$

All coeff could be found using this formula $\frac{\sin k\omega_0 T_1}{k\pi}$ except a_K when $k=0$.

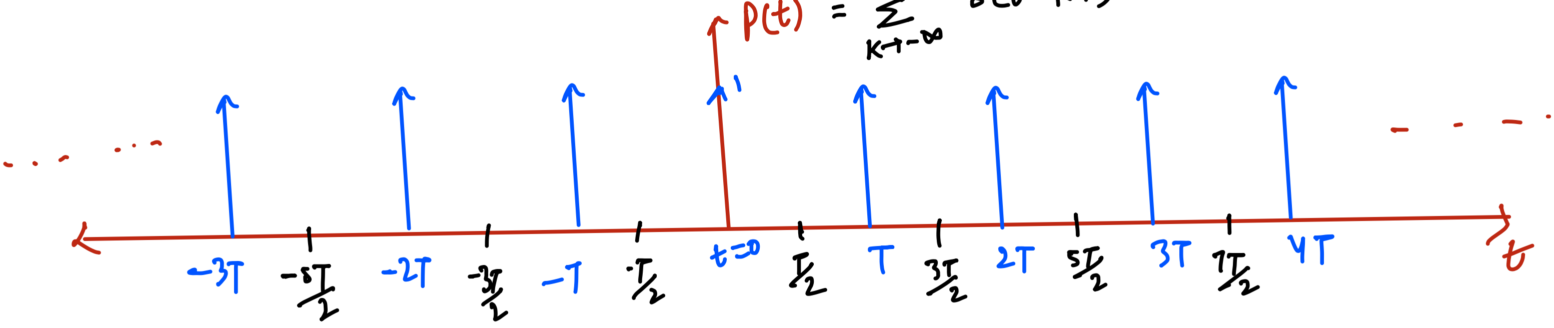
<u>For a_0</u> Using L-Hospital's Rule -:

$$a_0 = \lim_{k\to 0} \frac{(\cos k\omega_0 T_1)(\omega_0 T_1)}{\pi} = \frac{\omega_0 T_1}{\pi} = \frac{2\pi}{T} \cdot \frac{T_1}{\pi}$$

$$\boxed{\therefore \quad a_0 = \frac{2T_1}{T}}$$

eg. Find F.S. coeff. for periodic impulse train (dirac delta comb) shown in figure :-

$$p(t) = \sum_{k \to -\infty}^{\infty} \delta(t - kT)$$



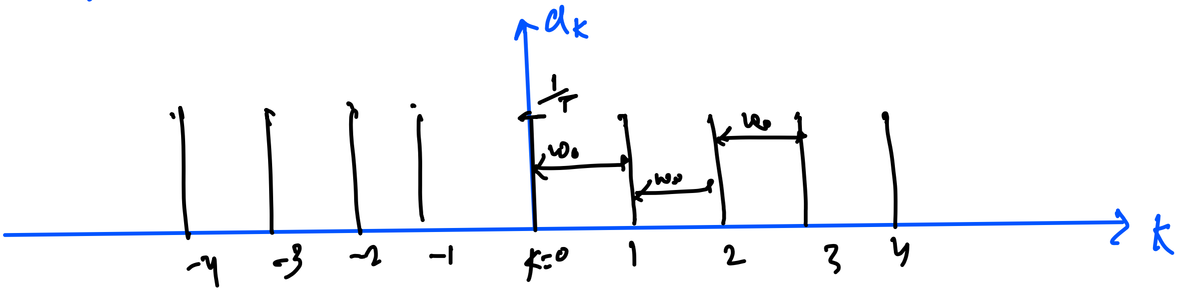
Analysis equation:

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

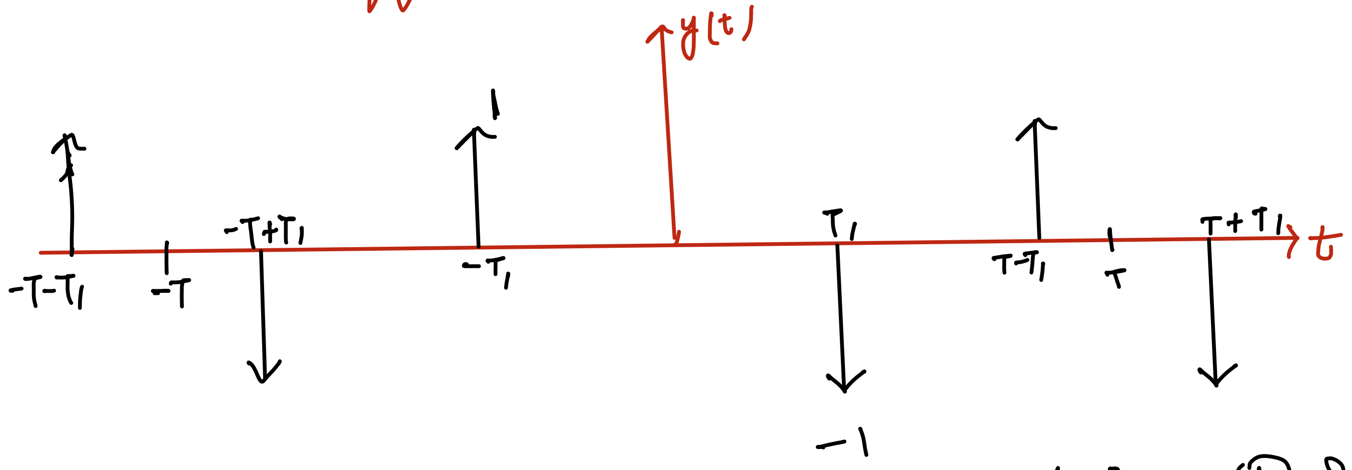
$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T} e^{-jk\omega_0 t} \Big|_{t=0} = \frac{1}{T}$$

Result → F.S. coeff (a_k) for impulse train are independent of 'k'.

$$\sum_{k \to -\infty}^{\infty} \delta(t - kT) \longleftrightarrow \frac{1}{T}, \quad \forall k.$$



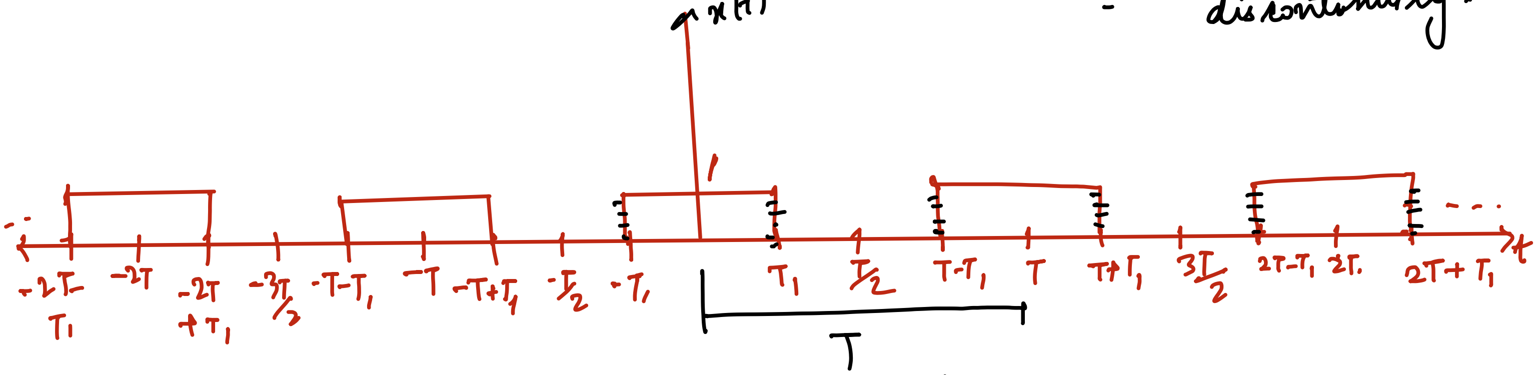
eg Find F.S. coeff for the signal shown below -:

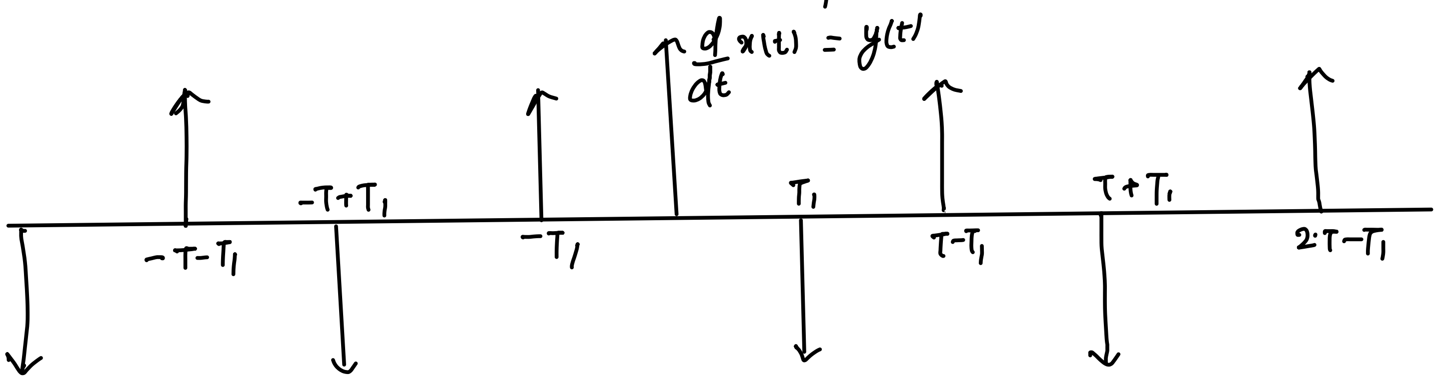


We can solve using results found in eg ① & eg ② -:

Method ① -:

≡ ← represents discontinuity.





We know $x(t) \longleftrightarrow \dfrac{\sin k\omega_0 T_1}{k\pi}$

$$y(t) = \frac{d}{dt}x(t) \longleftrightarrow jk\omega_0 \frac{\sin k\omega_0 T_1}{k\pi}$$

$$= jk\frac{2\pi}{T}\frac{\sin k\omega_0 T_1}{\pi k} = \frac{2j}{T}\sin k\omega_0 T_1$$

Note

Expansion Exp F.S. : $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

Expansion Trig. F.S: $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$

avg value of a signal.

For a_0

$$\int_0^T x(t)\,dt = \int_0^T a_0\,dt + 0 + \cancel{0} = a_0 T$$

$$\boxed{a_0 = \frac{1}{T}\int_0^T x(t)\,dt}$$ Avg value

or Avg value = $\frac{\text{Area (1 Time period)}}{T}$

For a_n

$$\boxed{a_n = \frac{2}{T}\int_0^T x(t) \cos n\omega_0 t \; dt}$$

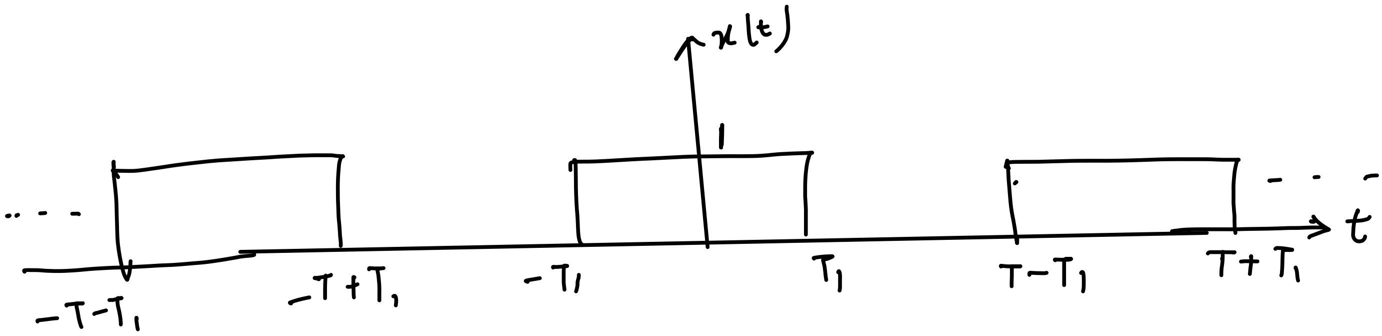
For b_n

$$\boxed{b_n = \frac{2}{T}\int_0^T x(t) . \sin n\omega_0 t \; dt}$$

Symmetry In F.S. -:

① Even Symmetric Signal -:

$x(-t) = x(t)$



→ $a_0 \neq 0$

→ $\angle a_k = 0^\circ$ or $\pm 180^\circ$

Polar $a_k \to$ complex	$a_k = \lvert a_k \rvert \angle a_k = \lvert a_k \rvert e^{j\theta}$

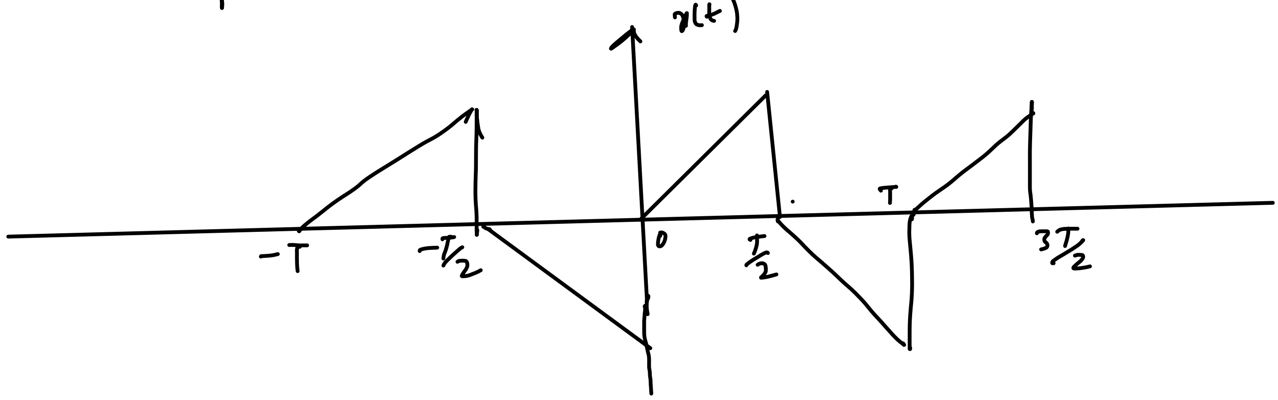
→ $a_n = \frac{2}{T} \int_0^T x(t) \cos n\omega_0 t \, dt$

→ $b_n = \frac{2}{T} \int_0^T x(t) \sin n\omega_0 t \, dt = 0$ ✓
$\quad (E) \times (O) = (O)$

$\therefore$ This coeff. can not exist.

② **Odd Symmetry :-**

if $x(-t) = -x(t)$



For odd symmetric signal

→ $a_0 = 0$

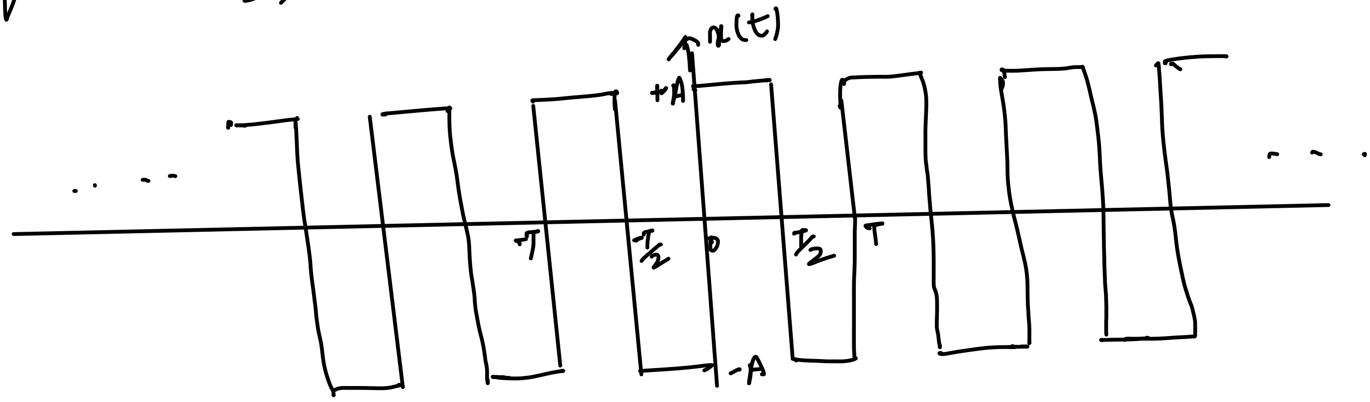
→ $\angle a_k = \pm 90^\circ$

→ $a_n = \frac{2}{T} \int_0^T x(t) \cos n\omega_0 t \, dt = 0.$

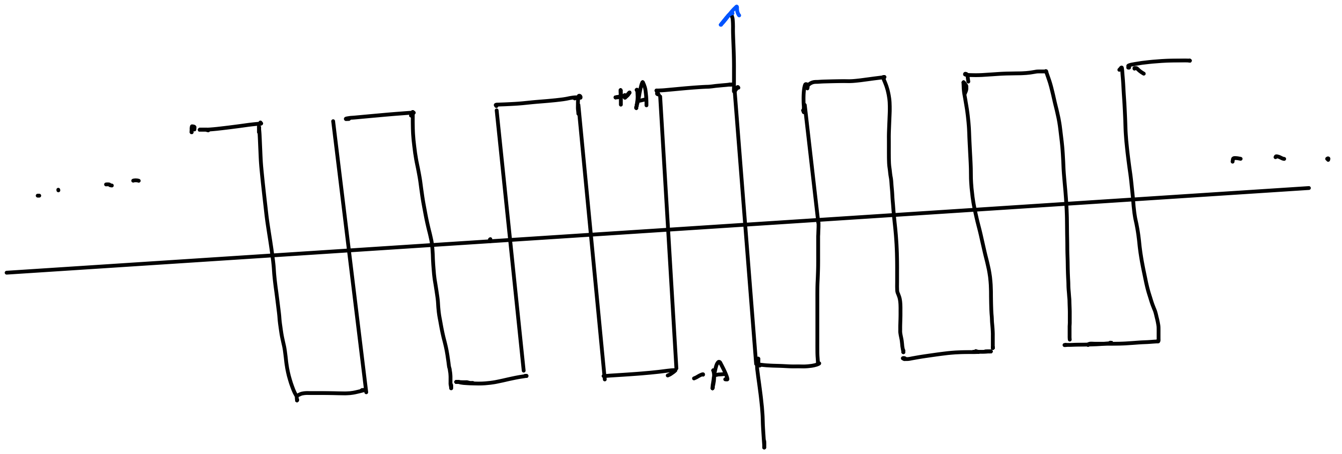
→ b_n will exist.

(3) Half wave Symmetric :-

If $x(t \pm \frac{T}{2}) = -x(t)$



$x(t+\frac{T}{2}) = x(t-\frac{T}{2}) = -x(t)$



→ $a_0 = 0$

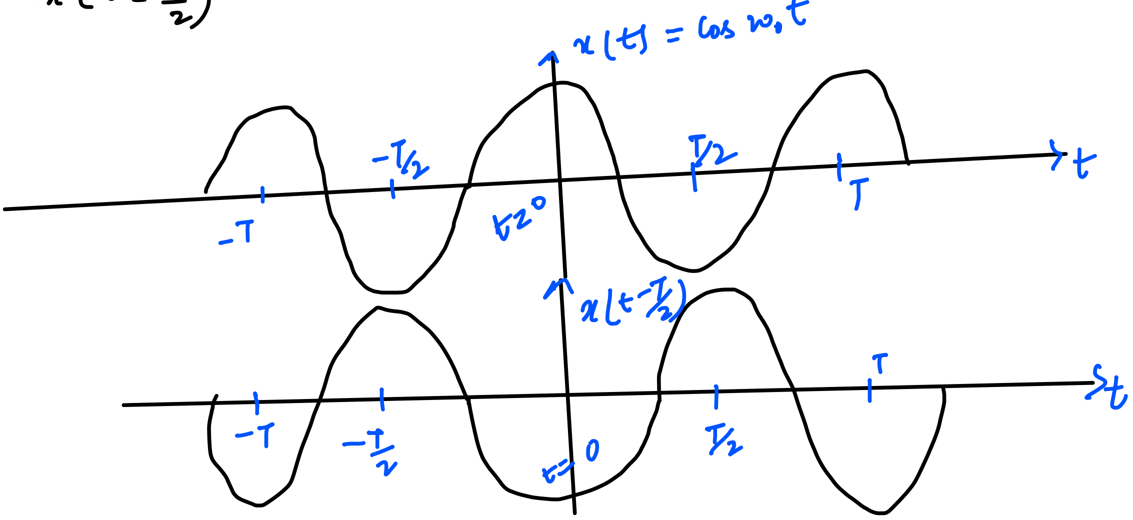
→ $a_k = 0$ for $k =$ even

→ (No even harmonics will be present)

(4) Even + half wave symmetry :-

$x(-t) = x(t)$

& $x(t \pm \frac{T}{2}) = -x(t)$



→ $a_0 = 0$

→ $a_k = 0$ for $k =$ even

→ $\angle a_k = \pm 180^\circ$

(5) Odd + Half wave symmetry :-

→ $x(-t) = -x(t)$

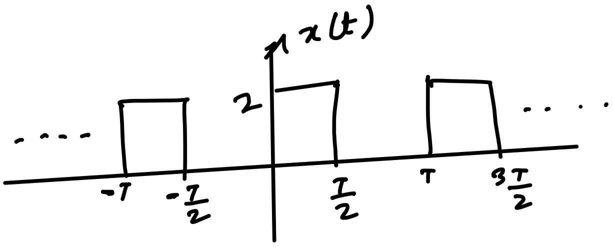
→ $x\left(t \pm \frac{T}{2}\right) = -x(t)$

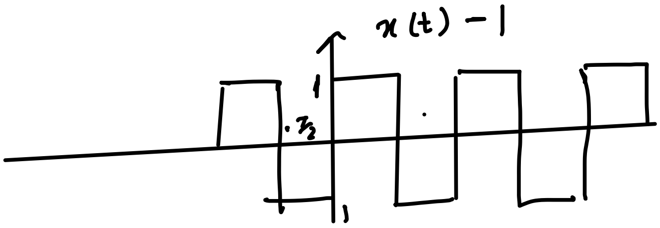
→ $a_0 = 0$

→ $a_k = 0$ for $k =$ even

→ $\angle a_k = \pm 90^\circ$

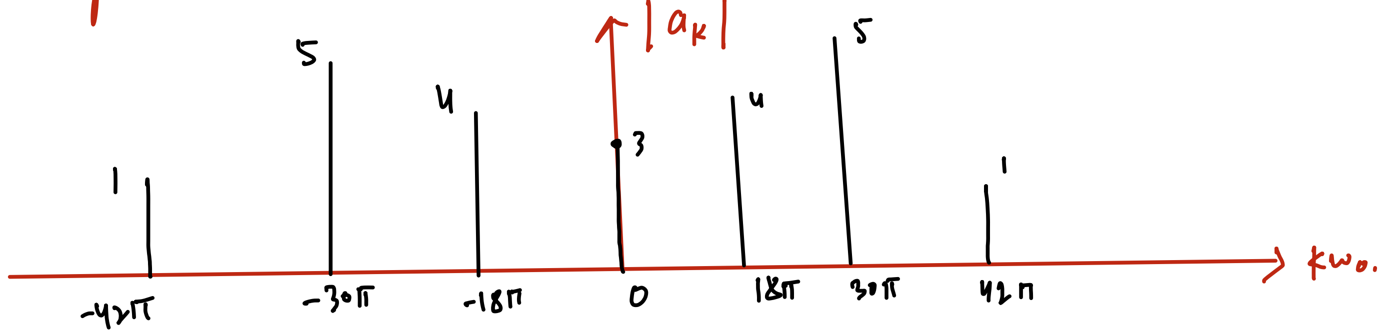
(6) Hidden Symmetry :-

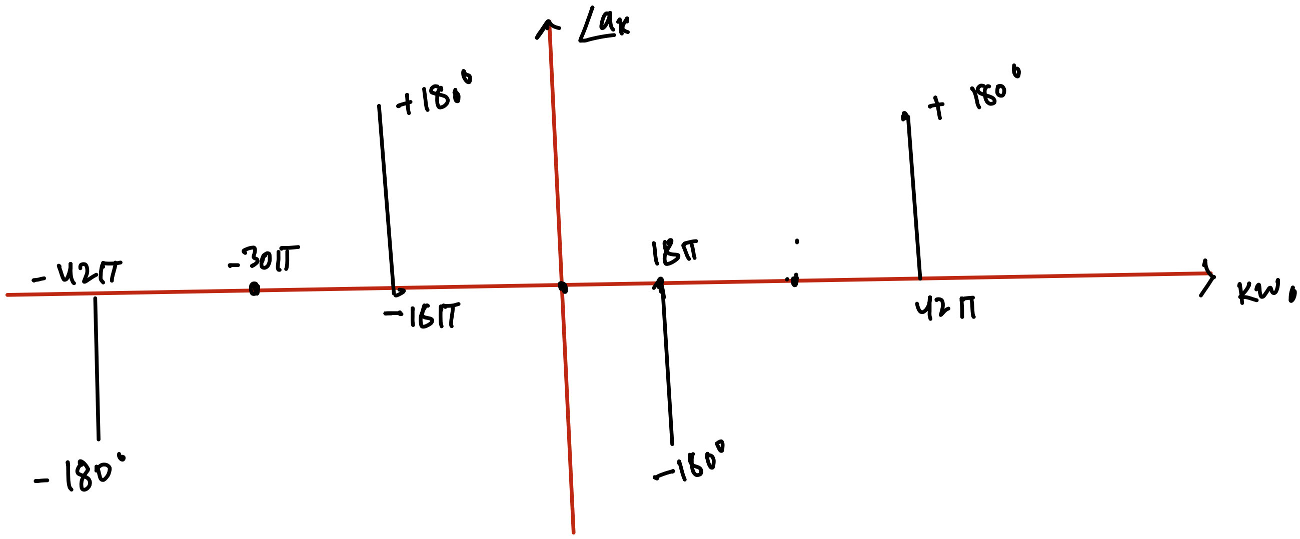




$a_0 \neq 0$ ✓ always remember!!

eg Consider the periodic signal $x(t)$ with magnitude & phase spectrum shown below -:





Find :
1. Fundamental frequency
2. Existing symmetry
3. Total power
4. RMS value
5. Signal $x(t)$

(1) Fundamental frequency -: $\omega_0 = GCD\,[18\pi, 30\pi, 42\pi]$

$$\boxed{\omega_0 = 6\pi}$$

(2) To find symmetry -:

$\longrightarrow$ $a_0 \neq 0$ [It is 3] $\longrightarrow$ hidden

$\longrightarrow$ $\angle a_k = \pm 180°$

Present components of 3^{rd}, 5^{th} & 7^{th} harmonic.

$a_k = 0$, $k =$ even. $\longrightarrow$ Half wave

$\longrightarrow$ ∴ Signal posses hidden + half wave symmetry

③ Total power -:

Parseval's theorm -:

$$P = \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$P = (1)^2 + (5)^2 + (4)^2 + (3)^2 + (4)^2 + (5)^2 + (1)^2$$
$$= 1 + 25 + 16 + 9 + 16 + 25 + 1 = 93\ W.$$

④

RMS value $= \sqrt{\text{Power}} = \sqrt{93} =$ ____

⑤

$$a_k = |a_k|\, e^{j\theta}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = a_{-7} e^{-j7\omega_0 t} + a_7 e^{j7\omega_0 t}$$
$$+ a_{-5} e^{-j5\omega_0 t} + a_5 e^{j5\omega_0 t} + a_5 e^{-j3\omega_0 t} + a_5 e^{j3\omega_0 t} + a_0$$

$a_{-7} = 1\, e^{-j180^\circ}$, $a_7 = 1\, e^{j180^\circ}$, $a_5 = 5 e^{j0^\circ}$, $a_{-5} = 5 e^{j0^\circ}$

$a_3 = 4\, e^{-j180^\circ}$, $a_{-3} = 4\, e^{j180^\circ}$, $a_0 = 3 e^{j0}$

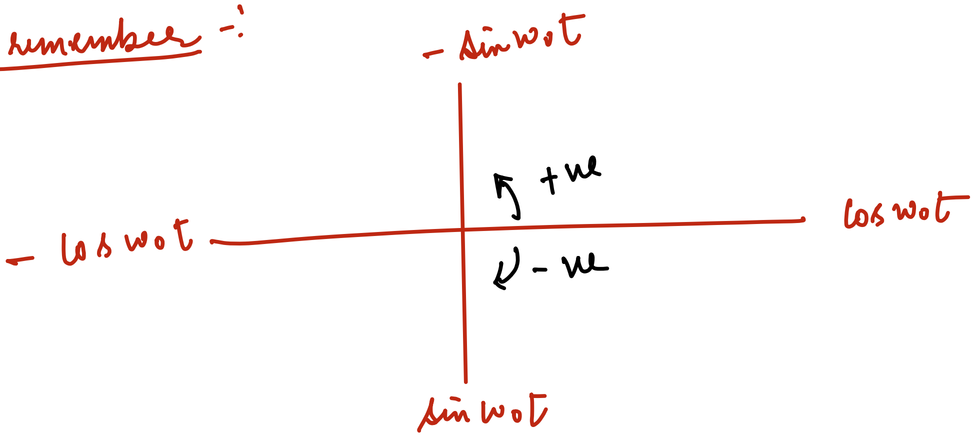
$$x(t) = 1 e^{j180^\circ} e^{j42\pi t} + 1 e^{-j180^\circ} e^{-j42\pi t} + 5 e^{j30\pi t} + 5 e^{-j30 t}$$
$$+ 4 e^{-j180^\circ} e^{j18\pi t} + 4 e^{j180^\circ} e^{-j18\pi t} + 3$$

$$= 3 + \left[e^{j(42\pi t + 180^\circ)} + e^{-j(42\pi t + 180)} \right] + 5\left[e^{j30\pi t} + e^{-j30\pi t} \right] + 4\left[e^{j(18\pi t - 180^\circ)} + e^{-j(18\pi t - 180)} \right]$$

$$= 3 + 2\cos 42\pi t + 180^\circ + 10\cos 30\pi t + 8\cos(18\pi t - 180^\circ)$$

$$\boxed{= 3 - 2\cos 42\pi t + 10\cos 30\pi t - 8\cos 18\pi t}$$ Ans

To remember -:



Relation between F. S. coeff -:

Analysis eqn -:

$x(t) \xleftrightarrow{F.S.} c_k$

$$c_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \text{—— (1)}$$

Taking conjugate both sides.

this should be -ve.

$$c_k^* = \frac{1}{T}\int_0^T x^*(t)\, .\, e^{+jk\omega_0 t}\, dt$$

Replacing k by -k

$x^*(t) \longleftrightarrow c_{-k}^*$

$$c_{-k}^* = \frac{1}{T}\int_0^T x^*(t)\, .\, e^{-jk\omega_0 t}\, dt \quad \text{—— (2)}$$

Results :

$$x(t) \longleftrightarrow c_k$$

$$x^*(t) \longleftrightarrow c_{-k}^*$$

Case I : If $x(t)$ is real

$$x^*(t) = x(t)$$

then $\boxed{c_{-k}^* = c_k}$

$\boxed{c_k^* = c_{-k}}$

Nature of signal $x(t)$	Nature of F.S. coeff. (c_k)
→ Even Conjugate Symmetric	Conjugate Symmetric [$c_{-k}^* = c_k$] Even
→ Odd Conjugate Symmetric	Conjugate Antisymmetric [$c_{-k}^* = -c_k$] Odd
→ Real + Even	Real + Even
→ Img + Even	Imag + Even
→ Real + odd	Imag + Odd
→ Imag + odd	Real + odd

$$y(t) = \text{Re}\{x(t)\} = \frac{1}{2}\left(x(t) + x^*(t)\right)$$

from tutorials!

→ Relation between F.S. coeff :-

Complex Expo. F.S. : $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$

Trignometric. F.S. : $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$

$$\Rightarrow x(t) = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2} \left\{ e^{j\omega_0 n t} + e^{-jn\omega_0 t} \right\}$$

$$+ \sum_{n=1}^{\infty} \frac{b_n}{2j} \left\{ e^{jn\omega_0 t} - e^{-jn\omega_0 t} \right\}$$

$$\Rightarrow x(t) = a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} + \frac{b_n}{2j} \right) e^{jn\omega_0 t} + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} - \frac{b_n}{2j} \right) e^{-jn\omega_0 t}$$

$$\Rightarrow x(t) = a_0 + \sum_{n=1}^{\infty} \underbrace{\frac{1}{2}(a_n - jb_n)}_{C_n} e^{jn\omega_0 t} + \sum_{n=1}^{\infty} \underbrace{\frac{1}{2}(a_n + jb_n)}_{C_{-n}} e^{-jn\omega_0 t}$$

$$\Rightarrow x(t) = a_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} c_{-n} e^{-jn\omega_0 t}$$

where
$$\boxed{\begin{aligned} c_n &= \frac{1}{2}(a_n - jb_n) \\ c_0 &= a_0 \\ c_{-n} &= \frac{1}{2}(a_0 + jb_n) \end{aligned}}$$

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}}$$

$C_n \longrightarrow$ Complex $\longrightarrow Re\{C_n\} + Img\{C_n\}$

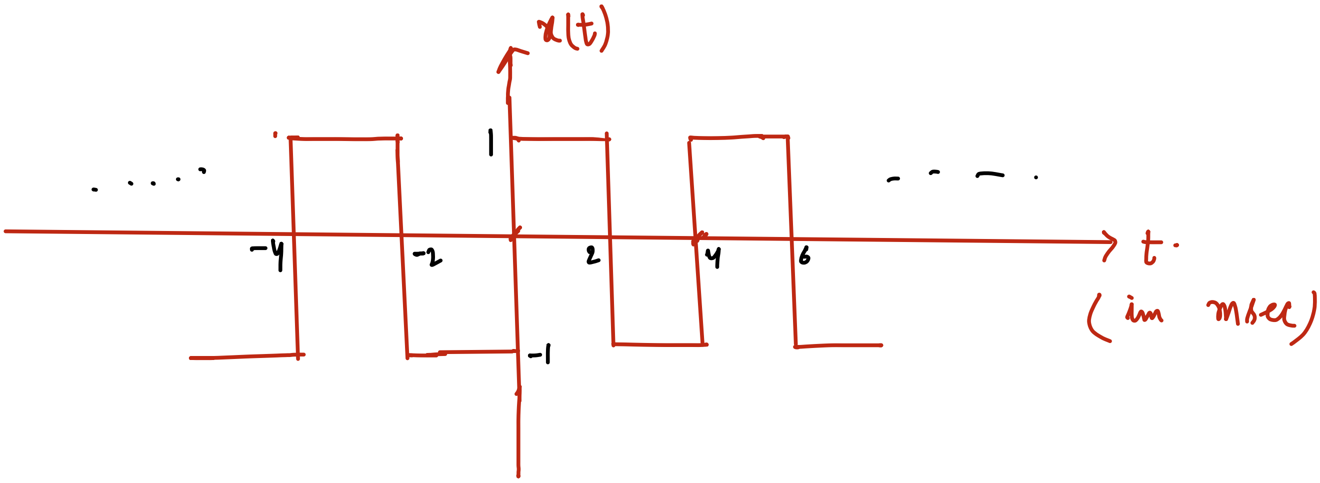
$$\frac{1}{2}a_n - \frac{1}{2}jb_n = Re\{C_n\} + Img\{C_n\}$$

$$\frac{1}{2}a_n = Re\{C_n\}$$

$$-\frac{1}{2}b_n = Img\{C_n\}$$

$$\Rightarrow \boxed{\begin{aligned} a_n &= 2Re\{C_n\} \\ b_n &= -2Img\{C_n\} \\ a_0 &= C_0 \end{aligned}}$$

eg ①



Freq of 5th harmonic will be

(A) 100 Hz

(B) 200 Hz

(C) 250 Hz

(D) 1250 Hz.

$f_0, \omega_0 \longrightarrow$

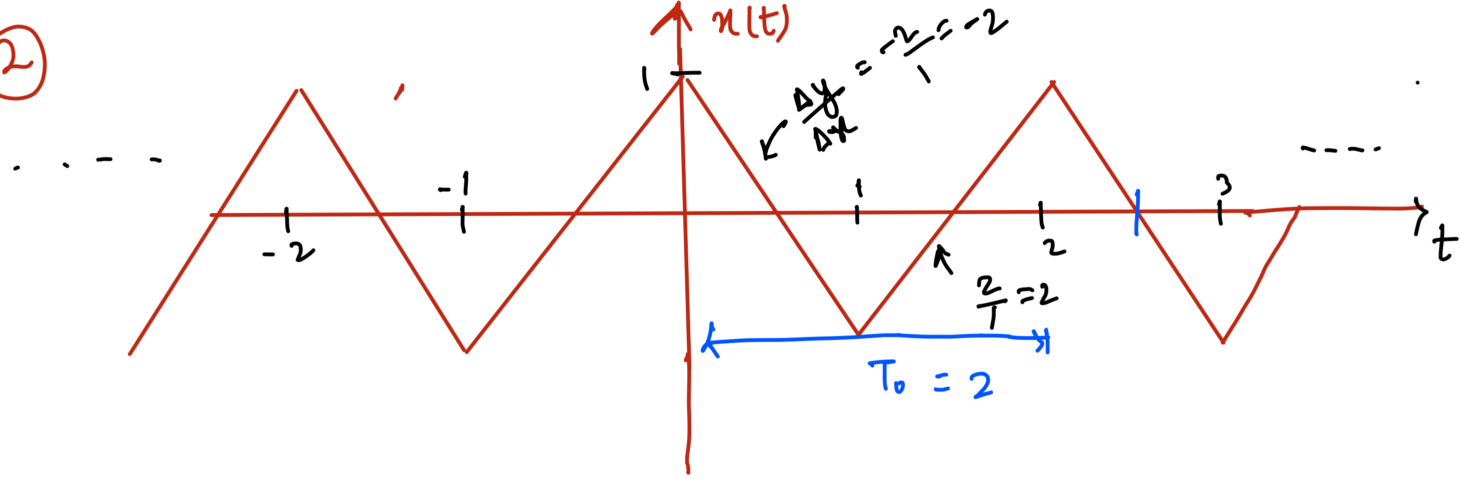
$5f_0, 5\omega_0 \longrightarrow$ Fifth harmonics

$$f_0 = \frac{1}{T} = \frac{1}{4ms} = \frac{1}{4}KHz$$

$$5f_0 = \frac{5}{4}KHz = 1.25\,KHz$$

Ans

eg ②



What is F. S. representation for $x(t)$?

$x(-t) = x(t) \longrightarrow$ Even

$x(t+\frac{T}{2}) = -x(t) \longrightarrow$ Half wave

$\therefore$ signal is Even + Half wave symmetric.

$b_n = 0$, $a_0 = 0$, $a_K = 0$
K is even

$$a_n = \frac{2}{T}\int_0^T x(t) \cdot \cos n\omega_0 t \, dt$$

$$x(t) = \begin{cases} -2t+1 & , \quad 0 < t < 1 \;\; \left(\frac{T}{2}\right) \\ 2t-3 & , \quad \frac{T}{2} < t < T \end{cases}$$

$$a_n = \frac{2}{T}\int_0^{T/2} -2t+1 \;\cos n\omega_0 t \, dt + \frac{2}{T}\int_{T/2}^{T} 2t-3 \;\cos n\omega_0 t \, dt$$

$= \underline{\hspace{4cm}}$ Ans

~~eg~~ 3 The fourier series expansion for $f(x) = \sin^2 x$ is

(A) $\sin x + \sin 2x$

(B) $1 - \cos 2x$

(C) $\cos x + \sin 3x$

✓(D) $0.5 - 0.5\cos 2x$

$$f(x) = \frac{1 - \cos 2x}{2} = 0.5 - 0.5\cos 2x$$

Neso Academy -:

Fourier Series Expansion is used for periodic signals to expand them in terms of their harmonics which are sinusoidal and orthogonal to one another.

Fourier Series (Periodic signal)
Fourier Transform (Non-periodic signal)] Used for analysis purpose

Laplace Transform] Used for design purpose and check for stability of a system.

Dirichlet conditions -:

→ Signal should have finite number of maxima & minima over the range of time period.

→ Signal should have finite number of discontinuity over the range of time period.

Useful Integrals in Fourier series :-

① $\int_0^T \sin n\omega t \, dt = 0$

② $\int_0^T \cos n\omega t \, dt = 0$

③ $\int_0^T \sin n\omega t \cos m\omega t \, dt = 0 \qquad n \neq m$

④ $\int_0^T \sin n\omega t \sin m\omega t \, dt = 0 \qquad n \neq m$

⑤ If $n = m$

i.e. $\int_0^T \sin^2 n\omega t \, dt = \frac{T}{2}$

$\int_0^T \cos^2 n\omega t \, dt = \frac{T}{2}$

$\sin(n\pi) = 0$

$\left.\begin{array}{ll} \cos(2n\pi) = 1 & (\text{even}) \\ \cos((2n+1)\pi) = -1 & (\text{odd}) \end{array}\right\} \cos(n\pi) = (-1)^n$

Integration by parts -:

$$\int u\,dv = uv - \int v\,du$$

$u = x$

$du = 1\,dx$

$V = -\cos x$

$dv = \sin x\,dx$

ILATE

$$\int x \sin x \, dx = -x\cos x - \int -\cos x\,dx$$
$$= -x\cos x + \sin x + C$$

$$\int \underset{dv}{x^2}\ \underset{u}{\ln x}\ \underset{dv}{dx}$$

$u = \ln x$

$du = \frac{1}{x}dx$

$V = x^3/2$

$dv = x^2\,dx$

$$= uv - \int v\,du$$
$$= \ln x \; 2x - \int \frac{x^3}{3}\frac{1}{x}\,dx$$
$$= 2x\ln x - \frac{1}{3}\frac{x^3}{3} = 2x\ln x - \frac{x^3}{9} + C$$

$$\int \underset{dv}{e^x}\ \underset{u}{\sin x}\ \underset{dv}{dx}$$

ILATE

$u = \sin x$

$du = \cos x\,dx$

$v = e^x$

$dv = e^x\,dx$