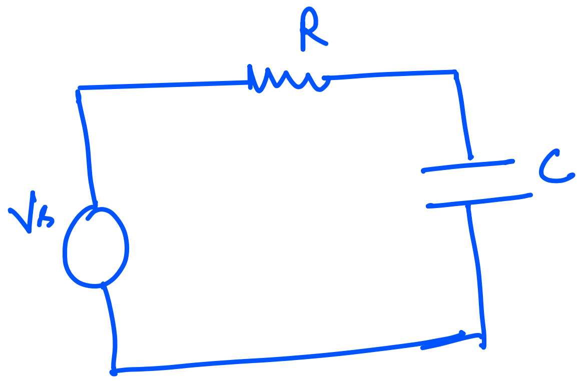
Laplace Transform -:

→ Differential Operations

$$\frac{df(t)}{dt} = s\,F(s) - f(0)$$

$f(0)$ → initial conditio

$\boxed{\text{Uni-Lateral Transform}}$



⇒ KVL

$$V_s = V_R + V_c$$

$$= I_c R + V_c$$

$$\left[I_c = C\frac{dV_c}{dt} \right]$$

$$V_s = CR\frac{dV_c}{dt} + V_c$$

$$V_s(s) = CR\left[s V_c(s) - V_c(0)\right] + V_c(s)$$

$$V_s(s) = V_c(s)\left[sCR + 1\right] - CR\,V_c(0)$$

$$V_c(s) = \frac{V_s(s)}{sCR+1} + \frac{(V_c(0))\,CR}{sCR+1}$$

$$V_s = u_s(t)$$

$$V_s(s) = \frac{1}{s}$$

$$V_c(s) = \frac{1}{s}\left[\frac{1}{sCR+1}\right] + \frac{\left[V_c(0)\right]CR}{sCR+1}$$

$$V_c(0) = 0$$

$$V_c(s) = \frac{1}{s} \cdot \frac{1}{sCR+1} + 0$$

$$= \frac{A}{s} + \frac{B}{sCR+1}$$

$$\frac{1}{s} \cdot \frac{1}{sCR+1} = \frac{A}{s} + \frac{B}{sCR+1}$$

$$A = \left. \frac{1}{sCR+1} \right|_{s=0} \Rightarrow 1$$

$$B = \left. \frac{1}{s} \right|_{s=-\frac{1}{RC}} \Rightarrow -RC$$

$$\Rightarrow \frac{1}{s} - \frac{RC}{sCR+1}$$

$$\Rightarrow \frac{1}{s} - \frac{1}{s+\frac{1}{RC}}$$

$$\Rightarrow u_s(t) - e^{-t/RC} u_s(t) = V_c(t)$$

$$\left(1 - e^{-t/RC}\right) u_s(t) = V_c(t)$$

sol^n could be . for resp. system

$$e^{-At} + e^{-Bt}$$

$$\frac{1}{s^2 + K_1 s + K_2}$$

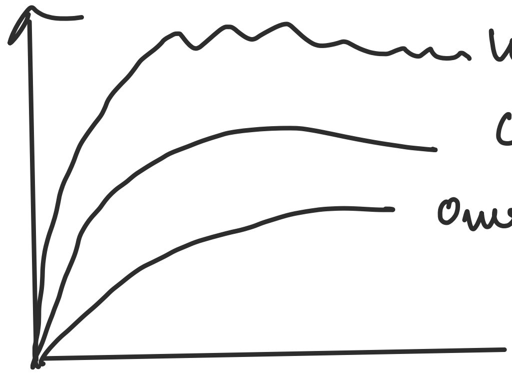
$$\frac{C_1}{s+A} + \frac{C_2}{s+B} \quad \text{(over damped)}$$

$$\frac{C_1}{s+A} + \frac{C_2}{(s+A)^2} \quad \text{(critically damped)}$$

$$\rightarrow \left(e^{-At} + t\,e^{-At} \right)$$

$$\frac{C_1}{s+(\alpha+jB)} + \frac{C_2}{s+(\alpha-jB)} \quad \text{(under-damped)}$$

$$\rightarrow e^{-\alpha t}\cos\beta t \quad \text{or} \quad -e^{-\alpha t}\sin\beta t.$$



FROM VIDEO -:

→ **Why to use Laplace Transform ?**

→ To find frequency domain of signals that are not absolutely Integrable.

→ Used to convert time domain differential equation ⟶ freq. domain algebric eqⁿ.

eg
$$4\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 3x = 0$$

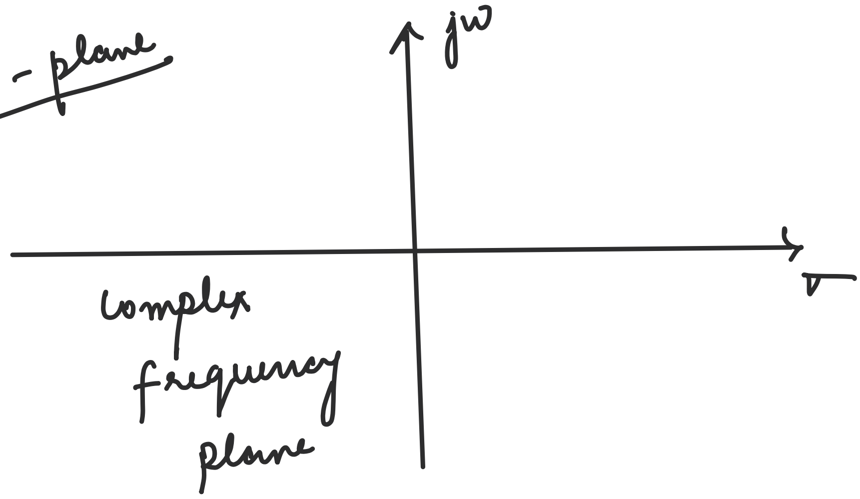
$$4x^2 + 5x + 3 = 0$$

→

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(s) = X(\sigma + j\omega)$$ where σ, ω is $-\infty$ to $+\infty$

S - plane



$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma+j\omega)t}\, dt$$

$$\boxed{X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt} \rightarrow \text{represents bilinear L.T.}$$

Note -:

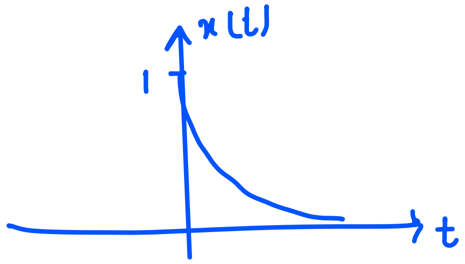
$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} \underbrace{x(t)\, e^{-\sigma t}}_{x_1(t)}\, e^{-j\omega t}\, dt \quad \text{--- (1)}$$

→ Laplace transform of $x(t)$ is nothing but F.T. of signal $x(t) \cdot e^{-\sigma t}$.

→ F.T. is nothing but L.T. evaluated on $j\omega$ axis.

→ If we put $\sigma = 0$, then result is nothing but Fourier Transform.

eg $x(t) = e^{-at} u(t)$



$$X(s) = \int_{-\infty}^{\infty} \{ e^{-at} u(t) \} e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-at}\, e^{-st}\, dt = \int_{0}^{\infty} e^{-(s+a)t}\, dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty} = \frac{0-1}{-(s+a)} = \frac{1}{s+a}$$

$$\boxed{e^{-at}u(t) \xleftrightarrow{L.T.} \frac{1}{s+a}}$$ Ans

Note :- If limits are from $-\infty$ to ∞, transform is called Bilinear L.T.

But if system is causal, limits will be from 0 to ∞, transform is called uni-lateral L.T.

eg (2) $x(t) = -e^{-at}u(-t)$

$$X(s) = -\int_{-\infty}^{0} e^{-at} \cdot e^{-st}\, dt = -\int_{-\infty}^{0} e^{-(s+a)t}\, dt$$

+ve.

$$= -\left[\frac{e^{\{-(s+a)\}t}}{-(s+a)}\right]_{-\infty}^{0} = -\left[\frac{1-0}{-(s+a)}\right]$$

assuming that this term is +ve., if not the whole term will become ∞.

$$\boxed{-e^{-at}u(-t) \longleftrightarrow \frac{1}{s+a}}$$

For both ex. LT is same, therefore we can not identify the inverse. so we need more info for that. We will use ROC for it.

Note :- To find convergence of both examples :-

For $X_1(s)$

$$X_1(s) = \int_0^\infty e^{-at} e^{-(\sigma + j\omega)t} dt$$

$$= \int_0^\infty e^{-at} e^{-\sigma t} \cdot e^{-j\omega t} dt$$

$$= \int_0^\infty e^{-\underbrace{(\sigma + a)}_{+ve}t} \underbrace{e^{-j\omega t}}_{mag \to 1} dt$$

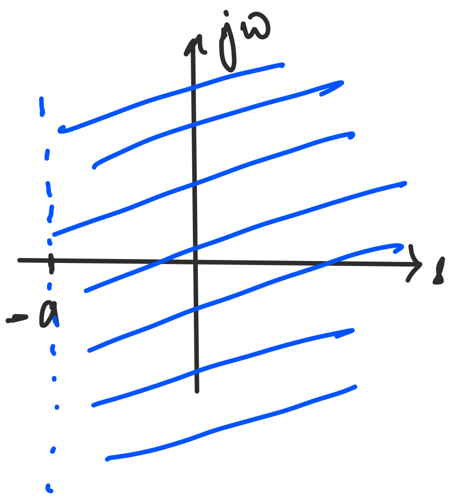
for $X_1(s)$ to converge

$\sigma + a > 0$

$\sigma > -a$

Since σ is real part of s

$$\boxed{Re\{s\} > -a}$$



For right sided signal ROC, L.T will be right sided after a point

For $X_2(s)$

$$X_2(s) = -\int_{-\infty}^0 e^{-at} e^{-(\sigma + j\omega)t} dt$$

$$= -\int_{-\infty}^0 e^{-at} e^{-\sigma t} e^{-j\omega t} dt$$

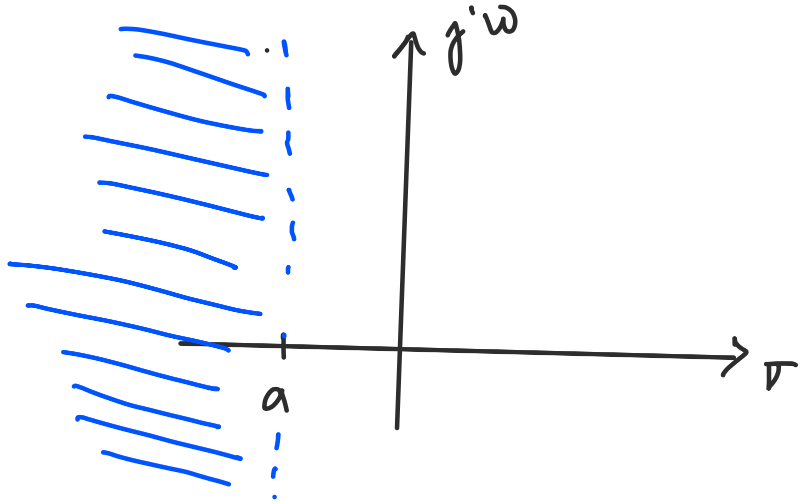
$$= -\int_{-\infty}^0 e^{-\underbrace{(\sigma + a)}_{-ve}t} \underbrace{e^{-j\omega t}}_{mag \to 1} dt$$

for $X_2(s)$ to be finite

$\sigma + a < 0$

$\sigma < -a$

$Re\{s\} < -a$



For left sided signal, ROC, L.T. will be left sided after a point.

Results :-

$$e^{-at}u(t) \longleftrightarrow \frac{1}{s+a}, \quad \text{ROC: } Re\{s\} > -a$$

$$-e^{at}u(-t) \longleftrightarrow \frac{1}{s+a}, \quad \text{ROC: } Re\{s\} < -a$$

Note :- Rational function :-

$$F(s) \Rightarrow \frac{N(s)}{D(s)}$$

Poles : $F(s) \to \infty$, $D(s) = 0$

Zeros : $F(s) \to 0$, $N(s) = 0$

eg :- $x(t) = \delta(t)$

$$X(s) = \int_{-\infty}^{\infty} \delta(t)\, e^{-st}\, dt = e^{-st}\Big|_{t=0} = 1$$

If impulse in Time domain, const in freq. domain.

eg $x(t) = \delta(t-1)$

$$X(s) = \int_{-\infty}^{\infty} \delta(t-1)\, e^{-st}\, dt = e^{-st}\Big|_{t=1} = e^{-s}$$

Region of Convergence (ROC) of X(s) :-

→ Range of σ [or $Re\{s\}$] for which L.T. of $x(t)$ or equivalently F.T. of $x(t) \cdot e^{-\sigma t}$ converges

Properties of ROC :-

→ will be a line parallel to $j\omega$ axis.

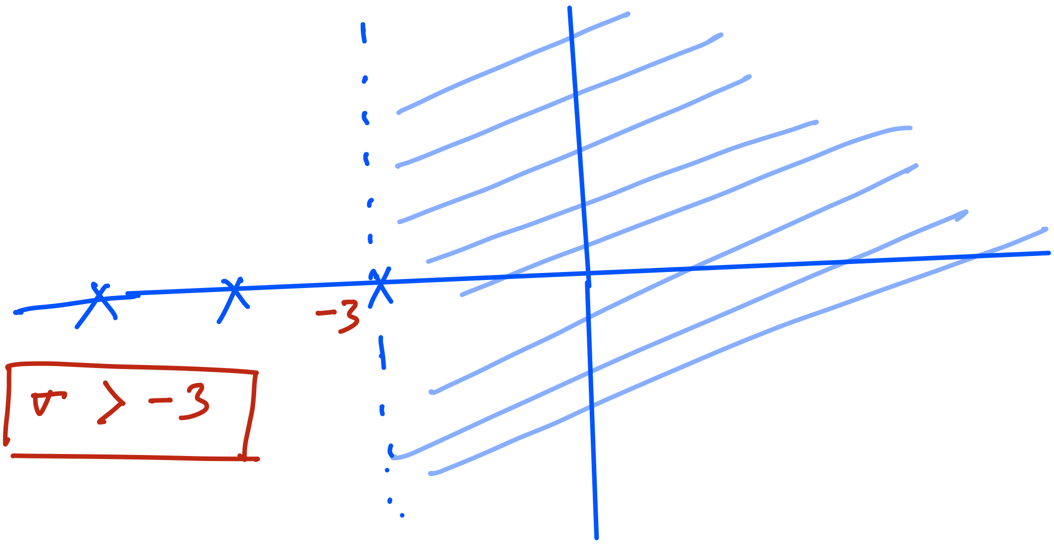
→ ROC only depends on Real values of s i.e. 'σ'

→ for rational L.T. $X(s)$, ROC does not contains any poles

→ because at Poles, L.T. will be ∞.

→ If $x(t)$ is r.s.s. If rational L.T. of $x(t)$ is $X(s)$,
↳ right sided signal.

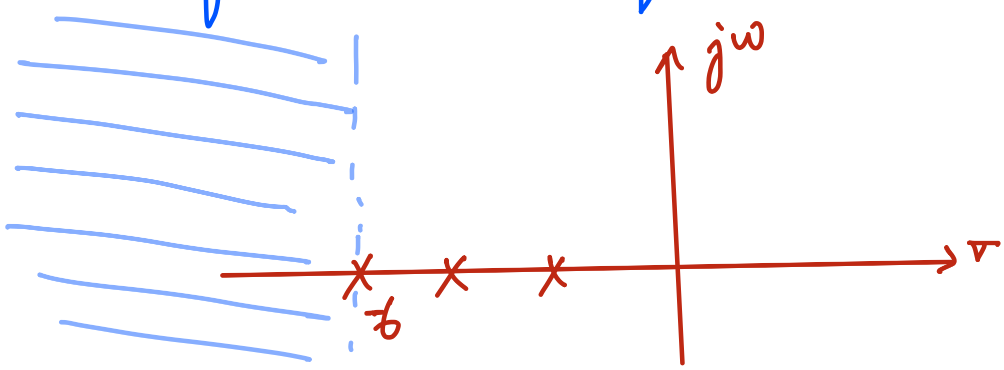
then ROC of $X(s)$ will be the region in s-plane, right to the right most pole.



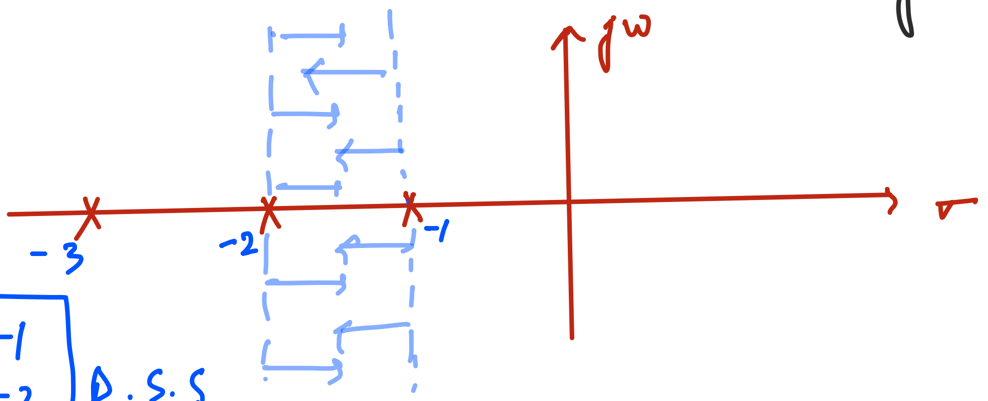
$\boxed{\sigma > -3}$

→ If $x(t)$ is a l.s.s (left sided signal) and its rational L.T. is $X(s)$, then ROC of $X(s)$ will be left to the leftmost pole

$\boxed{\sigma < -6}$



→ If $x(t)$ is a double sided signal and it's L.T is $X(s)$, then ROC of $X(s)$ will be a strip b/w two poles, but will not contain any pole.



$$\boxed{\begin{array}{c} -2 < \sigma < -1 \\ -3 < \sigma < -2 \end{array}}$$ D.S.S.

→ If $x(t)$ is a R.S.S. and ROC of it's L.T $X(s)$ contains a line $\sigma = \sigma_0$, then all possible values of s, for which $\sigma > \sigma_0$ will also be in the ROC.

→ If $x(t)$ is a L.S.S. & ROC of it's L.T. includes a line $\sigma = \sigma_0$, then all possible values of s, for which $\sigma < \sigma_0$, will also be in the ROC

→ If $x(t)$ is a finite duration signal, then ROC of $X(s)$ will be entire s-plane, except possible removal of $\sigma = 0$ and/or $\pm\infty$.

eg $x(t) = \delta(t) \longleftrightarrow 1$, ROC : Entire s plane

eg $x(t) = \delta(t-1) \longleftrightarrow e^{-s}$, ROC : ?

eg $x(t) = \begin{cases} e^{-at} & , \; 0 < t \leq T \\ 0 & , \; \text{otherwise} \end{cases}$

$$X(s) = \int_0^T e^{-at} e^{-st} \, dt = \int_0^T e^{-(s+a)t} dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^T$$

$$X(s) = \frac{e^{-(s+a)T} - 1}{-(s+a)}$$

$$X(s) = \frac{1 - e^{-(s+a)T}}{s+a} = \frac{(1 - e^{-aT} e^{-sT})}{s+a}$$

$$X(s)\Big|_{s=-a} = \left. \frac{0 - e^{-aT}(-T)\, e^{-sT}}{1} \right|_{s=-a}$$

$$= T \, e^{-aT} e^{aT} = T$$

entire plane !!

Hence proved last property.

Note :- $X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$

Taking conjugate on both sides,

$$X^*(s) = \int_{-\infty}^{\infty} x^*(t)\, e^{-s^* t}\, dt$$

Replacing s by s^*

$$X^*(s^*) = \int_{-\infty}^{\infty} x^*(t)\, e^{-(s^*)^* t}\, dt$$

$$\boxed{X^*(s^*) = \int_{-\infty}^{\infty} x^*(t)\, e^{-st}\, dt}$$

Result

$$x(t) \xleftrightarrow{L.T} X(s)$$

$$\boxed{x^*(t) \xleftrightarrow{L.T.} X^*(s^*)}$$

Review :-

$$x(t) \xleftrightarrow{LT} X(s)$$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} \underbrace{x(t)\, e^{-st}}_{x_1(t)}\, e^{-j\omega t}\, dt$$

$$L[x(t)] = F[\underbrace{x(t)\, e^{-st}}_{\text{Absolutely Integrable.}}]$$

ROC $\longrightarrow$ depends only on 'σ'

$\sigma \longrightarrow$ real part of s

$$F[x(t)] = L[x(t)]_{\sigma=0}$$

$$\underbrace{e^{-at}u(t)}_{R.S.S.} \longleftrightarrow \frac{1}{s+a}, \quad Re\{s\} > -a$$

$$\underbrace{e^{-at}u(-t)}_{L.S.S.} \longleftrightarrow \frac{1}{s+a}, \quad Re\{s\} < -a$$

eg $x(t) = u(t)$

$$X(s) = \int_0^\infty 1.\, e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \left[\frac{0-1}{-s}\right] = \frac{1}{s}$$

$$X(s) = \int_0^\infty e^{-\sigma t} e^{-j\omega t}\, dt \Rightarrow \sigma > 0 \quad \underline{\underline{ROC}}$$

$$\Rightarrow L[u(t)] = \frac{1}{\sigma + j\omega} = \frac{1}{s}, \quad \sigma > 0.$$

eg $X(s) = \dfrac{1}{s+4}$, find causal inverse $x(t)$.

$\sigma > -4$

$$\boxed{\begin{array}{ll} e^{-at}u(t) \longleftrightarrow \dfrac{1}{s+a}, & \sigma > -a \\ -e^{-at}u(t) \longleftrightarrow \dfrac{1}{s+a}, & \sigma < -a \end{array}}$$

$$x(t) = e^{-4t}u(t)$$

eg $X(s) = \frac{1}{s+4}$, $\sigma < -4$

Find inverse L.T. if $x(t)$ is L.S.S.

$$x(t) = -e^{-4t} u(-t)$$

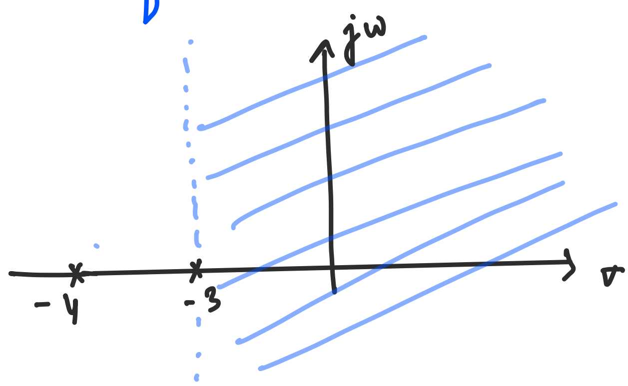
eg $X(s) = \frac{1}{(s+3)(s+4)}$, Find all possible inverse $x(t)$. Write all possible ROC's of $X(s)$

$$\frac{1}{(s+3)(s+4)} = \frac{A}{s+3} + \frac{B}{s+4}$$

$$A = 1 \quad , \quad B = -1$$

$$X(s) = \frac{1}{s+3} - \frac{1}{s+4}$$

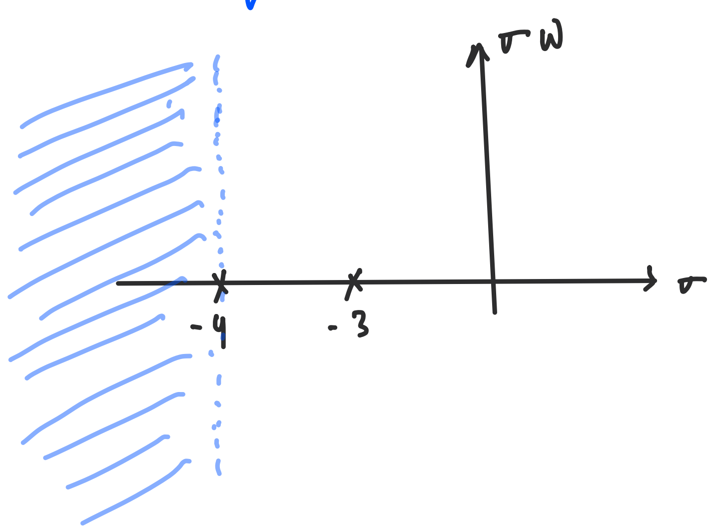
Case 1 :- If $x(t)$ is a R.S.S.



ROC : $Re\{s\} > -3$

$$x(t) = e^{-3t} u(t) - e^{-4t} u(t)$$

Case II : If $x(t)$ is a L.S.S.

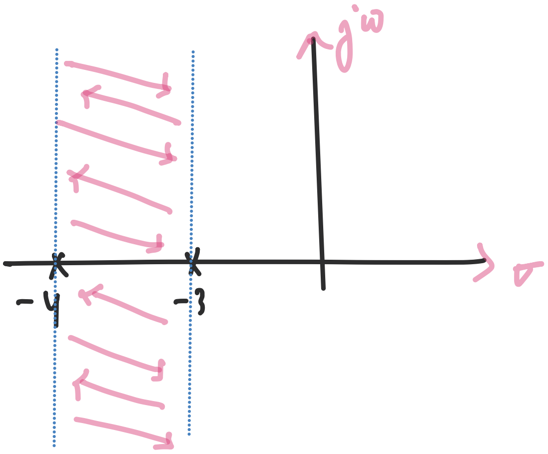


ROC : $Re\{s\} < -4$

$$x(t) = -e^{-3t} u(-t) - (-) e^{-4t} u(-t)$$

$$x(t) = -e^{-3t} u(-t) + e^{-4t} u(-t)$$

Case III : If $x(t)$ is a double signal -:



ROC : $-4 < \sigma < -3$

$$x(t) = -e^{-3t}u(-t) - e^{-4t}u(t)$$

Note :: To determine is given system is stable or not -:

If jw-axis is not included in ROC.

→ we cannot calculate F.T

→ And system is not BIBO or unstable

→ Case I → stable

Case II → unstable

Case III → unstable.

eg Find LT of $x(t) = e^{-a|t|}$?

$$x(t) = \underbrace{e^{-at}u(t)}_{x_1(t)} + \underbrace{e^{at}u(-t)}_{x_2(t)}$$

$$x_2(t) = e^{at}u(-t)$$

Transform

$$X_2(s) = \int_{-\infty}^{0} e^{at}e^{-st}\,dt = \int_{-\infty}^{0} e^{(a-s)t}\,dt = \left[\frac{e^{(a-s)t}}{(a-s)}\right]_{-\infty}^{0}$$

$$= \frac{1}{-s+a} = \frac{-1}{s-a}$$

ROC

$$X_2(s) = \int_{-\infty}^{0} e^{at}e^{-\sigma t}e^{-j\omega t}\,dt = \int_{-\infty}^{0} e^{(a-\sigma)t}e^{-j\omega t}\,dt$$

For $X_2(s)$ to be finite $a - \sigma > 0$

$a > \sigma$ $\Rightarrow Re\{s\} < a$

$$X_1(s) = \frac{1}{s+a}, \quad Re\{s\} > -a$$

$$X_2(s) = \frac{-1}{s+a}, \quad Re\{s\} < a$$

$$X(s) = X_1(s) + X_2(s), \quad ROC :- \; -a < Re\{s\} < a$$

$$= \frac{1}{s+a} - \frac{1}{s-a} = \frac{s-a-s-a}{(s+a)(s-a)} = \frac{-2a}{s^2-a^2}, \quad -a < Re\{s\} < a$$

~~eg~~ $x(t) \longleftrightarrow X(j\omega)$

$$IFT[X(j\omega)] = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$X(\sigma + j\omega) = X(s) = L.T[x(t)] = FT[x(t) \cdot e^{-\sigma t}]$$

$$x(t) \cdot e^{-\sigma t} \longleftrightarrow X(\sigma + j\omega)$$

$$IFT[X(\sigma + j\omega)] = x(t)\, e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)\, e^{j\omega t}\, d\omega$$

$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)\, e^{(\sigma + j\omega)t}\, d\omega$$

Put $s = \sigma + j\omega$

$$\frac{ds}{d\omega} = j \Rightarrow d\omega = \frac{ds}{j}$$

when $\omega \longrightarrow -\infty$, $s = \sigma - j\omega$

when $\omega \longrightarrow +\infty$, $s = \sigma + j\omega$

$$x(t) = \frac{1}{2\pi}\int_{\sigma - j\omega}^{\sigma + j\omega} X(s)\, e^{st}\, \frac{ds}{j}$$

$$\Rightarrow \boxed{\therefore \quad x(t) = \frac{1}{2\pi j} \int_{\sigma - j\omega}^{\sigma + j\omega} X(s)\, e^{st}\, ds}$$

I.L.T. i.e. $x(t)$

Avoid this formula. Use only when transform is not rational.

Properties of L.T. :-

① Linearity Property :-

$x_1(t) \longleftrightarrow X_1(s)$, ROC : R_1

$x_2(t) \longleftrightarrow X_2(s)$, ROC : R_2

$$\boxed{\alpha\, x_1(t) + \beta\, x_2(t) \longleftrightarrow \alpha X_1(s) + \beta X_2(s) \qquad \text{ROC}: R_1 \cap R_2 \text{ (common area)}}$$

② Time Shifting Property :-

$x(t) \longleftrightarrow X(s)$, ROC : R

$$\boxed{x(t-t_0) \longleftrightarrow e^{-st_0} X(s) \, , \quad \text{ROC : containing } R}$$

③ Time Reversal Property :-

$x(t) \longleftrightarrow X(s)$, ROC : R

$x(-t) \longleftrightarrow X(-s)$, ROC : $\frac{1}{R}$

if ROC : $R \Rightarrow \sigma > -4$ $\qquad$ $R : \sigma < 4$

$\frac{1}{R} \Rightarrow \sigma < +4$ $\qquad$ $\frac{1}{R} : \sigma > -4$

eg $x(t) = e^{-at}u(t) \longleftrightarrow \frac{1}{s+a}$, $Re\{s\} > -a$

$x(-t) = e^{at}u(-t) \longleftrightarrow \frac{1}{-s+a}$, $Re\{s\} < +a$

(4) Time Scaling Property -:

$x(t) \longleftrightarrow X(s)$, ROC : R

$x(\alpha t) \longleftrightarrow \frac{1}{|\alpha|} X\left(\frac{s}{\alpha}\right)$, ROC : αR

eg $x(t) = e^{-2t}u(t) \longleftrightarrow \frac{1}{s+2}$

ROC : $Re\{s\} > -2$

$x_1(t) = x(3t) = e^{-6t}u(t) \longleftrightarrow \frac{1}{s+6}$

ROC : $Re\{s\} > -6$

(5) Frequency Shifting Property -:

$x(t) \longleftrightarrow X(s)$, ROC : R

$e^{s_0 t}x(t) \longleftrightarrow X(s - s_0)$, ROC : $R + Re\{s_0\}$

eg $x(t) = e^{-3t}u(t)$

$X(s) = \frac{1}{s+3}$, ROC : $Re\{s\} > -3$

$e^{-2t}x(t) \longleftrightarrow \frac{1}{(s+2)+3}$, $Re\{s\} > -3+(-2)$

$= \frac{1}{s+5}$, $Re\{s\} > -5$

(6) Convolution in time property -:

$$x_1(t) \longleftrightarrow X_1(s) \ , \quad ROC: R_1$$

$$x_2(t) \longleftrightarrow X_2(s) \ , \quad ROC: R_2$$

$$\boxed{x_1(t) * x_2(t) \longleftrightarrow X_1(s)\, X_2(s) \ , \quad ROC: R_1 \cap R_2}$$

(7) Differentiation in time -:

$$x(t) \longleftrightarrow X(s) \ , \quad ROC: R$$

$$\frac{d\, x(t)}{dt} \longleftrightarrow s\, X(s) \ , \quad ROC: \text{containing } R$$

For uni-lateral , $\frac{d\, x(t)}{dt} \longleftrightarrow s\, X(s) - x(0^-)$

$$\frac{d^2 x(t)}{dt} \longleftrightarrow s^2 X(s) - s\, x(0^-) - x'(0^-)$$

$$\boxed{\frac{d^n}{dt^n} x(t) \longleftrightarrow s^n X(s)}$$

general case !!

(8) Differentiation in frequency domain -:

$$x(t) \longleftrightarrow X(s) \ , \quad ROC: R$$

$$-t\, x(t) \longleftrightarrow \frac{d\, X(s)}{ds} \ , \quad ROC: R$$

Remember -:

① $t^n \cdot e^{-at} u(t) \longleftrightarrow \dfrac{n!}{(s+a)^{n+1}}$, $\sigma > -a$

$t^n u(t) \longleftrightarrow \dfrac{n!}{s^{n+1}}$

⑨ Integration in time domain -:

$x(t) \longleftrightarrow X(s)$, ROC : R

$\int_{-\infty}^{t} x(\tau)\, d\tau \longleftrightarrow \dfrac{X(s)}{s}$, ROC : $R \cap [Re\{s\} > 0]$

eg $x(t) = \delta(t)$ $X(s) = 1$, ROC: entire s-plane.

$x_1(t) = \int_{-\infty}^{t} x(\tau)\, d\tau = \int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t) \longleftrightarrow \dfrac{1}{s}$, $Re\{s\} > 0$

⑩ Integration in frequency domain -: (division by 't' property)

$x(t) \longleftrightarrow X(s)$, ROC : R

$\dfrac{x(t)}{t} \longleftrightarrow \int_{s}^{\infty} X(s)\, ds$, ROC : R

(11) Initial and Final Value theorems -:

→ Initial Value theorem -:

Pre-conditions -:

→ $x(t) = 0 \quad \forall \quad t < 0$, $x(t) = x_1(t)\, u(t)$

→ It must not contain impulses or its higher order derivatives.

$$\boxed{\lim_{t \to 0^+} x(t) = \lim_{s \to \infty} sX(s)}$$

definition of theorem.

=

$$\boxed{x(0^+) = \lim_{s \to \infty} sX(s)}$$

Also remember,

$$\boxed{\int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = sX(s) - x(0^+)}$$

Now, we know

$$\lim_{t \to 0^+} x(t) = x(0^+) = \lim_{s \to \infty} sX(s)$$

If $X(s)$ is a rational function

$$X(s) = \frac{N(s)}{D(s)}$$

Case I :- Degree of $N(s)$ > Degree of $D(s)$

$$X(s) = \frac{s^2 + 8s + 16}{s+4} = \underline{(s+4)}$$

$\delta(t) \longleftrightarrow 1$

$\frac{d}{dt}\delta(t) \longleftrightarrow s$

$$\times \quad x(t) = \delta'(t) + 4\delta(t)$$

If degree of $N(s) > D(s)$ then, we cannot calculate initial value.

If signal includes impulse, you cannot calculate initial value.

Case II :- degree of $N(s)$ = degree of $D(s)$

$$X(s) = \frac{s+4}{s+4} = 1$$

$\therefore$ cannot calculate initial value of system for same reason in case I.

Case III To find Initial value,

Degree of $D(s)$ > Degree of $N(s)$

eg $$X(s) = \frac{s+2}{s^3 + 2s^2 + 3s + 4}$$

$$x(0^+) = \lim_{s \to 0} \frac{s \cdot s\left(1 + \frac{2}{s}\right)}{s^3\left(1 + \frac{2}{s} + \frac{3}{s^2} + \frac{4}{s^3}\right)} = \frac{1}{\infty} = 0$$

If [degree of $D(s)$ − degree of $N(s)$] ≥ 1

Then $x(0^+) = 0$

Final Value Theorem :-

→ $x(t) = 0, \quad t < 0$

→ $x(t)$ → shouldn't include impulse or higher order derivatives.

→ Poles of $sX(s)$ must lie in LHS of s-plane

→ For unbounded and periodic function are not applicable.

→ $\boxed{\lim_{t \to \infty} x(t) = x(\infty) = \lim_{s \to 0} sX(s)}$ ← Final value theorem

eg $X(s) = \int_{0^+}^{\infty} x(t)\, e^{-st}\, dt$

$$\int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st}\, dt = sX(s) - x(0^+)$$

$$\int_{0^+}^{\infty} \frac{dx(t)}{dt} \underbrace{e^{-st}}_{1}\, dt = \lim_{s \to 0} [sX(s) - x(0^+)]$$

$$\int_{0^+}^{\infty} \frac{dx(t)}{dt}\, dt = \lim_{s \to 0} sX(s) - x(0^+)$$

$$[x(t)]_{0^+}^{\infty} = x(\infty) - \cancel{x(0^+)} = \lim_{s \to 0} sX(s) - \cancel{x(0^+)}$$

$$\boxed{\therefore \; x(\infty) = \lim_{s \to 0} sX(s)}$$

This is final value theorem.

Summary of all Properties :-

1. Linearity : $\alpha x_1(t) + \beta x_2(t) \longleftrightarrow \alpha X_1(s) + \beta X_2(s)$, ROC: $R_1 \cap R_2$

2. Time shifting: $x(t-t_0) \longleftrightarrow e^{-st_0} X(s)$, ROC: containing R.

3. Time Reversal : $x(-t) \longleftrightarrow X(-s)$, ROC: $\frac{1}{R}$

4. Scaling : $x(\alpha t) \longleftrightarrow \frac{1}{|\alpha|} X\left(\frac{s}{\alpha}\right)$, ROC : αR

5. Convolution : $x_1(t) * x_2(t) \longleftrightarrow X_1(s)\, X_2(s)$, ROC: $R_1 \cap R_2$

6. Multiplication : $x_1(t)\, x_2(t) \longleftrightarrow \frac{1}{2\pi j}\left[X_1(s) * X_2(s)\right]$, ROC: $R_1 \cap R_2$

7. Diff in time : $\frac{d}{dt} x(t) \longleftrightarrow s X(s)$, ROC: R

8. Diff in freq : $-t\, x(t) \longleftrightarrow \frac{d}{ds} X(s)$, ROC : R

9. Int in time : $\int_{-\infty}^{t} x(\tau)\, d\tau \longleftrightarrow \frac{X(s)}{s}$, ROC : $R \cap [Re\{s\} > 0]$

10. Int in freq: $\frac{x(t)}{t} \longleftrightarrow \int_{s}^{\infty} X(s)\, ds$, ROC : R

11. Initial Value Theorem: $\lim_{t \to 0^+} x(t) = x(0^+) = \lim_{s \to \infty} s X(s)$

12. Final Value Theorem: $\lim_{t \to \infty} x(t) = x(\infty) = \lim_{s \to 0} s X(s)$

eg Find L.T of $x(t) = 4e^{-5t}\,u(-t-3)$

We know that

$$u(t) \longleftrightarrow \frac{1}{s}, \quad \text{ROC}: Re\{s\} > 0$$

using time shifting prop.

$$u(t-3) \longleftrightarrow \frac{e^{-3s}}{s}, \quad Re\{s\} > 0$$

using time reversal prop.

$$u(-t-3) \longleftrightarrow \frac{e^{3s}}{(-s)}, \quad Re\{s\} < 0$$

using frequency shifting property.

$$e^{-5t}\,u(-t-3) \longleftrightarrow \frac{e^{3(s+5)}}{-(s+5)}, \quad Re\{s\} < 0 + (-5)$$

$$Re\{s\} < -5$$

eg $x(t) = \cos w_0 t \; u(t)$.

$$x(t) = \frac{1}{2}\left[e^{jw_0 t} + e^{-jw_0 t}\right] u(t)$$

$$= \frac{1}{2} e^{jw_0 t} u(t) + \frac{1}{2} e^{-jw_0 t} u(t)$$

We know,

$$e^{-at} u(t) \longleftrightarrow \frac{1}{s+a}, \quad Re\{s\} > -a$$

$$x(t) = \frac{1}{2} e^{-(-jw_0)t} u(t) + \frac{1}{2} e^{-(jw_0)t} u(t)$$

$$X(s) = \frac{1}{2}\left[\frac{1}{s+(-jw_0)} + \frac{1}{s+jw_0}\right]$$

$$= \frac{1}{2}\left[\frac{s+jw_0+s-jw_0}{s^2-(jw_0)^2}\right] = \frac{s}{s^2+w_0^2}$$

$$\text{ROC}: Re\{s\} > 0$$

Remember :-

$x(t)$		$X(s)$	ROC: R
$\delta(t)$	$\longleftrightarrow$	1	, Entire s-plane
$u(t)$	$\longleftrightarrow$	$\frac{1}{s}$	, $Re\{s\} > 0$
$-u(-t)$	$\longleftrightarrow$	$\frac{1}{s}$	, $Re\{s\} < 0$
$e^{-at}u(t)$	$\longleftrightarrow$	$\frac{1}{s+a}$	, $Re\{s\} > -a$
$-e^{-at}u(-t)$	$\longleftrightarrow$	$\frac{1}{s+a}$	, $Re\{s\} < -a$
$t^n u(t)$	$\longleftrightarrow$	$\frac{n!}{(s)^{n+1}}$	, $Re\{s\} > 0$
$-t^n u(-t)$	$\longleftrightarrow$	$\frac{n!}{(s)^{n+1}}$	, $Re\{s\} < 0$
$t^n e^{-at}u(t)$	$\longleftrightarrow$	$\frac{n!}{(s+a)^{n+1}}$	, $Re\{s\} > -a$
$-t^n e^{-at}u(-t)$	$\longleftrightarrow$	$\frac{n!}{(s+a)^{n+1}}$	, $Re\{s\} < -a$
$\cos w_0 t \, u(t)$	$\longleftrightarrow$	$\frac{s}{s^2+w_0^2}$	, $Re\{s\} > 0$
$\sin w_0 t \, u(t)$	$\longleftrightarrow$	$\frac{w_0}{s^2+w_0^2}$	, $Re\{s\} > 0$

Causality And Stability :-

$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) = x(t) * h(t)$

→ Causal : If $h(t) = 0 \quad \forall \; t < 0$

→ Non - Causal : $h(t) \neq 0 \quad \forall \; t = 0$

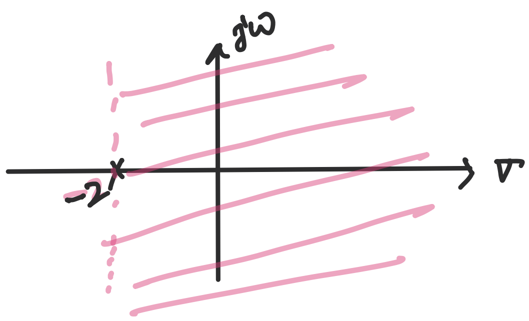
→ Anti - Causal : $h(t) = 0 \;, \; \forall \; t > 0$

Property :- ① If a system is causal, then ROC of its system func^n must be right to the right most pole.

ROC : Right half plane.

② If system func^c is rational, ROC is right to the rightmost pole, then system will be causal.

eg $H(s) = \dfrac{1}{s+2}$, $Re\{s\} > -2$
Is it causal?



$\therefore$ Causal.

eg $h(t) = e^{-a|t|}$

$h(t) = \underbrace{e^{at} u(-t)}_{\downarrow} + e^{-at} u(t)$

$\therefore$ Non causal.

eg $H(s) = \dfrac{-2a}{s^2 - a^2}$, $-a < Re\{s\} < a$

Is it causal.

→ system is rational

→ but ROC is not right to rightmost pole

→ ROC is in between 2 poles

→ ∴ system is non-causal.

eg $H(s) = \underbrace{\dfrac{e^{s}}{s+1}}_{\text{system is not rational}}$, $Re\{s\} > -1$

Is it causal?

but ROC is right to rightmost pole.

$e^{-at} u(t) \longleftrightarrow \dfrac{1}{s+a}$

$x(t) \longleftrightarrow X(s)$

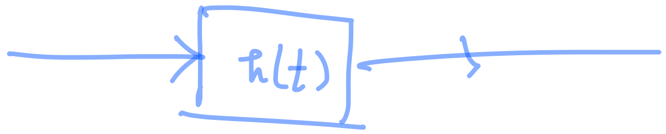
$x(t+t_0) \longleftrightarrow e^{st_0} X(s)$

$h(t) = e^{-(t+1)} u(t+1)$

$h(t) \neq 0 , \quad t < 0$

∴ Non causal.

Stability :-



→ $\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$, BIBO

→ $h(t) \xleftrightarrow{L.T.} H(s)$, ROC : R

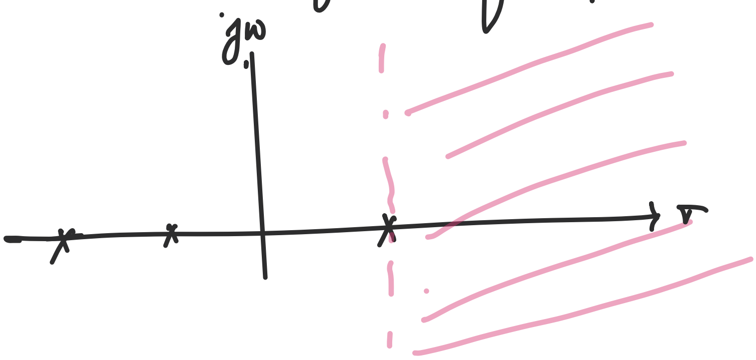
$h(t)$ — impulse response; $H(s)$ — system func^n

→ $j\omega$ axis must be included in ROC of it's system func^n ($H(s)$).

Case I : Right Sided : (causal)

→ ROC : Right to the right most pole

→ For a causal system to be stable, all poles must lie on left half of that plane.



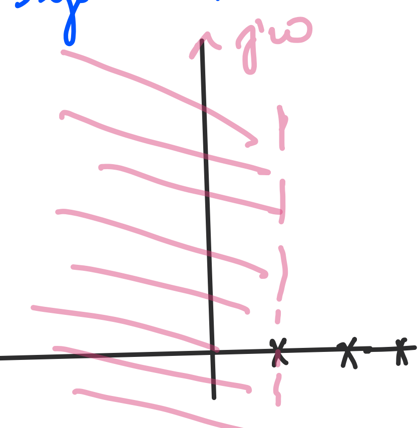
Not stable!

$j\omega$ axis not included

Case II : Left sided (Non-causal) System :-

→ ROC : left to the left most plane

→ poles should be on right half of plane for stability.

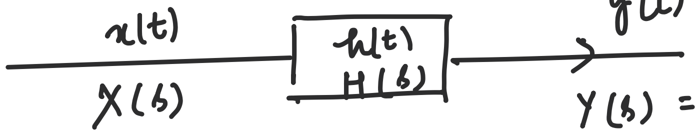


system is stable ∴ includes $j\omega$ axis.

eg If Input - Output relation of an LTI system is given by a differential equation as:

$$\frac{d^2 y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$$

Find all possible impulse responses & comment on stability of the system in each case.



$y(t) = x(t) * h(t)$

$Y(s) = X(s) H(s)$

$\therefore H(s) = \dfrac{Y(s)}{X(s)}$

Taking L.T. both sides :-

$$s^2 Y(s) - s\,Y(s) - 2\,Y(s) = X(s)$$

$$Y(s)\left[s^2 - s - 2\right] = X(s)$$

$$\therefore \quad H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 - s - 2} = \frac{1}{s^2 + s - 2s - 2}$$

$$H(s) = \frac{1}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}$$

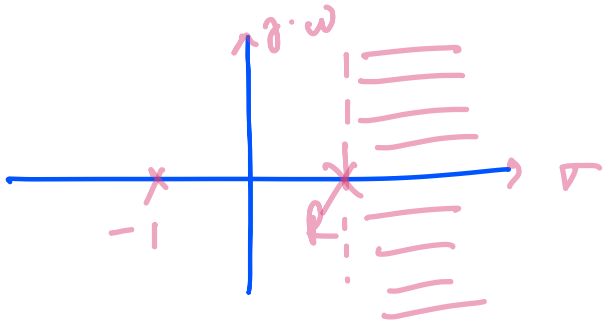
$$A = \left.\frac{1}{s-2}\right|_{s=-1} = -\frac{1}{3}$$

$$B = \left.\frac{1}{s+1}\right|_{s=2} = \frac{1}{3}$$

$$H(s) = \frac{\left(-\frac{1}{3}\right)}{s+1} + \frac{\frac{1}{3}}{s-2}$$

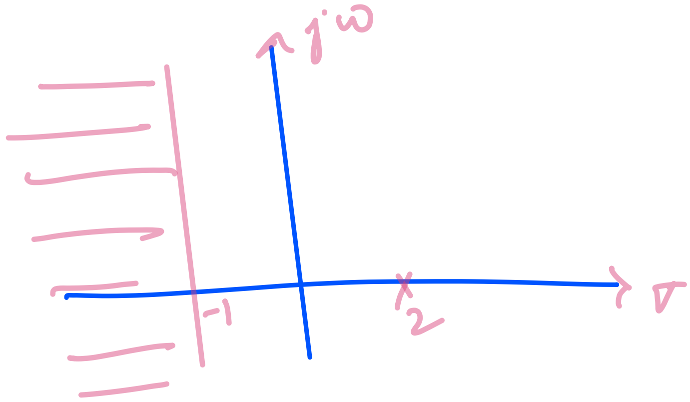
Case I : h(t) is right sided

$$h(t) = -\frac{1}{3} e^{-t} u(t) + \frac{1}{3} e^{2t} u(t)$$



$\therefore$ unstable

Case II : h(t) is left sided



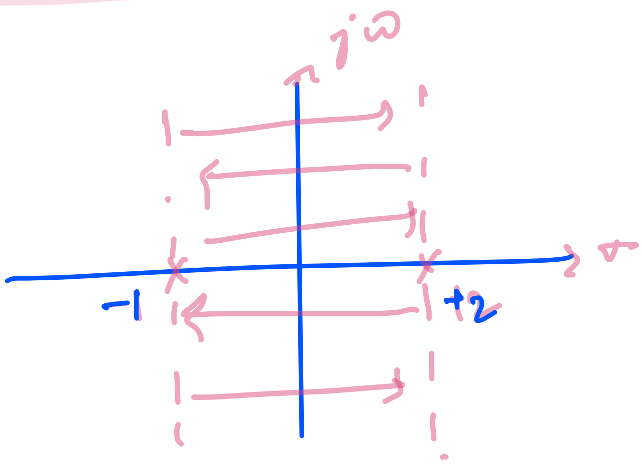
$Re\{s\} < -1 \longrightarrow$ unstable

$$h(t) = -\frac{1}{3}(-1) e^{-t} u(-t) + \frac{1}{3}(-1) e^{2t} u(-t)$$

Case III : h(t) is double sided

$$\boxed{-1 < Re\{s\} < 2}$$

stable



$$h(t) = -\frac{1}{3} e^{-t} u(t) + \frac{1}{3}(-1) e^{2t} u(-t)$$