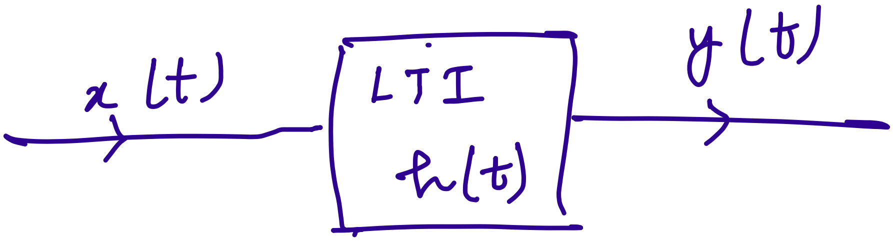
Convolution

It is used to find output of L.T.I systems for any arbitrary input provided impulse response of the system is known.



$$y(t) = x(t) * h(t)$$

LTI $\rightarrow$ Linear Time Invariant System.

Continous Time Convolution -:

$$y(t) = x(t) * h(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$\boxed{h(t-\tau) = h(-\tau+t)}$$

eg $x(t) = u(t)$, $h(t) = u(t)$

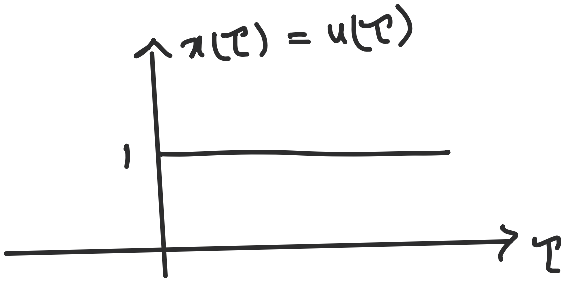
Find output $y(t) = x(t) * h(t)$

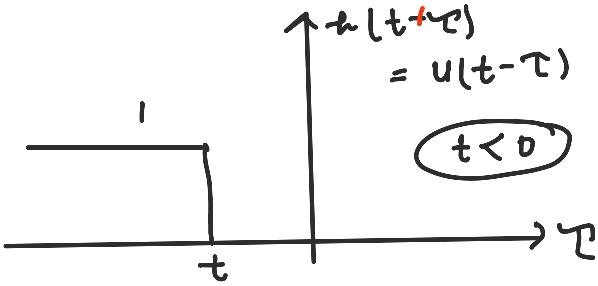
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

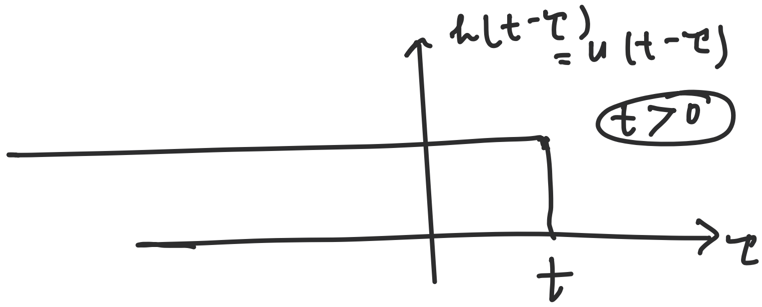
Given $x(t) = u(t) \Rightarrow x(\tau) = u(\tau)$

$h(t) = u(t) \Rightarrow h(t-\tau) = u(t-\tau)$

$= u(-\tau+t)$







$$y(t) = \int_{-\infty}^{\infty} u(\tau)\, u(t-\tau)\, d\tau$$

→ $y(t) = 0$, for $t < 0$
(As no common area exists)

→ for $t > 0$,

$$y(t) = \int_{0}^{t} \underbrace{u(\tau)}_{1}\ \underbrace{u(t-\tau)}_{1}\, d\tau$$

$$\therefore \quad = \int_{0}^{t} 1 \cdot 1\, d\tau = [\tau]_0^t = [t-0] = t$$

$$\therefore\ y(t) = \begin{cases} 0 , & t < 0 \\ t , & t \geq 0 \end{cases}$$

$$y(t) = t \cdot u(t) = r(t)$$

Result :- $\boxed{u(t) * u(t) = r(t)}$ we will use later.

eg $x(t)=u(t) \longrightarrow \boxed{h(t)=\delta(t)} \longrightarrow y(t)=\,?$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$x(t) = u(t) \qquad\qquad h(t) = \delta(t)$$

$$x(\tau) = u(\tau) \qquad\qquad h(t-\tau) = \delta(t-\tau)$$

$$\therefore\ y(t) = \int_{-\infty}^{\infty} u(\tau)\, \delta(t-\tau)\, d\tau$$

$$= u(\tau)\Big|_{\tau=t} = u(t)$$

$$\boxed{u(t) * \delta(t) = u(t)}$$

Any func^n convolved with impulse func^n resultant is same func^n.

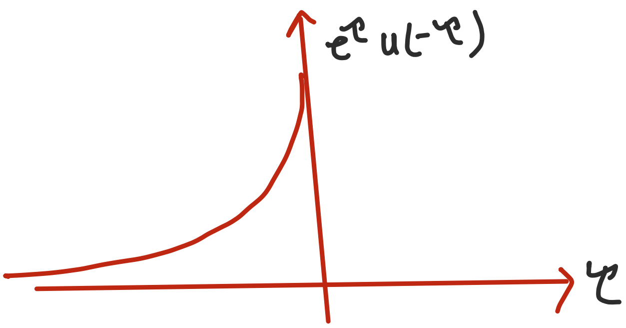
eg $x(t) = e^{t}u(-t) \qquad h(t) = u(t)$

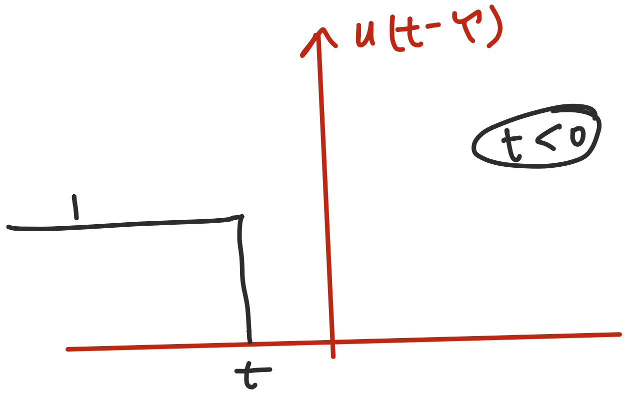
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$x(t) = e^{t}u(-t) \qquad\qquad h(t) = u(t)$$

$$x(\tau) = e^{\tau}u(-\tau) \qquad\qquad u(t-\tau) = u(t-\tau)$$

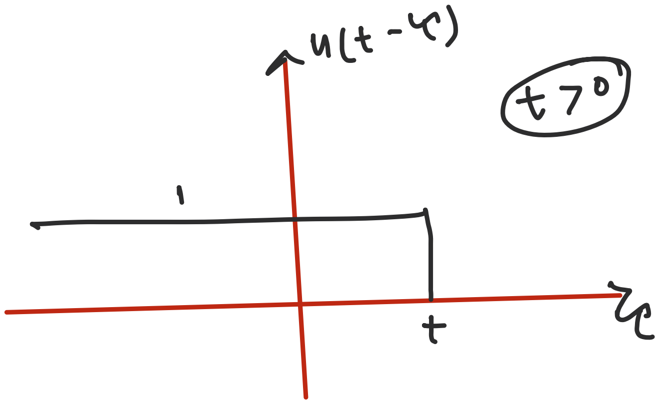
$$y(t) = \int_{-\infty}^{\infty} \underbrace{e^{\tau}u(-\tau)}_{-\infty<\tau<0} \cdot \underbrace{u(t-\tau)}\, d\tau$$





For $t < 0$

$$y(t) = \int_{-\infty}^{t} e^{\tau} \cdot d\tau$$



$$= [e^{\tau}]_{-\infty}^{t} = [e^{t} - 0]$$

$$\boxed{y(t) = e^{t}, \quad t < 0}$$

For $t > 0$

$$y(t) = \int_{-\infty}^{0} e^{\tau} d\tau = [e^{\tau}]_{-\infty}^{0} = [1 - 0] = 1$$

$$\boxed{\therefore y(t) = 1, \quad t > 0}$$

Ans. $$\therefore y(t) = \begin{cases} e^{t}, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

PROPERTIES OF CONVOLUTION -:

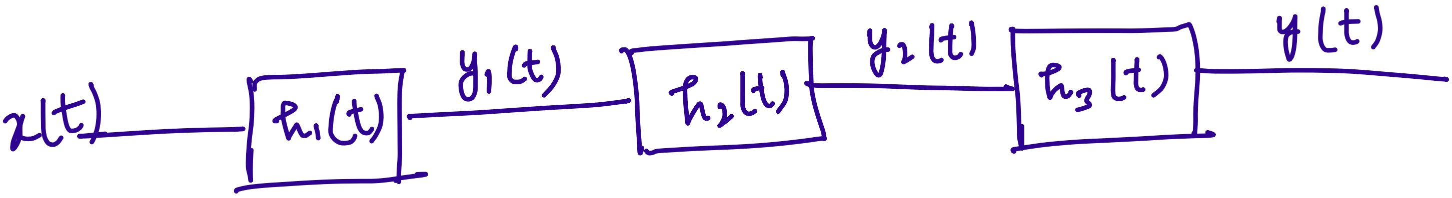
① Commutative Property -:

$$y(t) = x(t) * h(t) = h(t) * x(t)$$

$$= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$$

(2) Associative Property -:

$$y(t) = \{x_1(t) * x_2(t)\} * x_3(t)$$
$$= x_1(t) * \{x_2(t) * x_3(t)\}$$



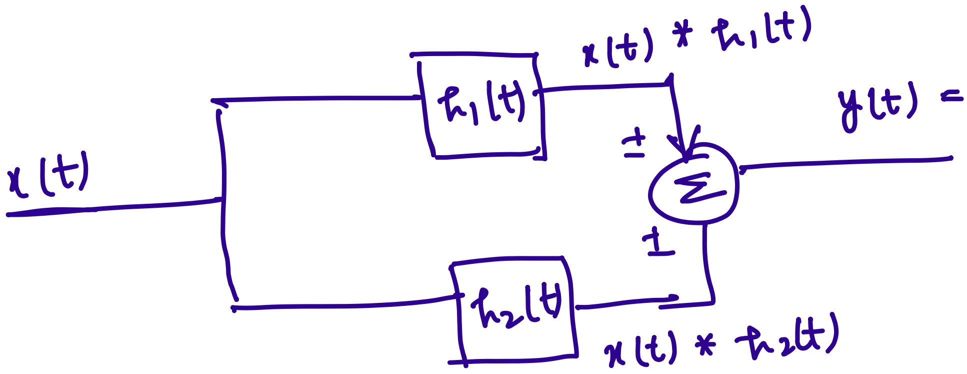
$$y_1(t) = x(t) * h_1(t)$$
$$y_2(t) = \{x(t) * h_1(t)\} * h_2(t)$$
$$y(t) = \left[\{x(t) * h_1(t)\} * h_2(t)\right] * h_3(t)$$

or

$$y(t) = \{h_1(t) * h_2(t) * h_3(t)\} * x(t)$$

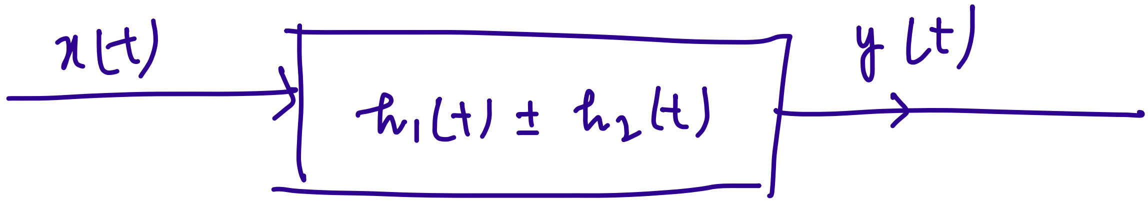
(3) Distributive Properties -:

$$y(t) = x(t) * [h_1(t) \pm h_2(t)]$$
$$= x(t) * h_1(t) \pm x(t) * h_2(t)$$



$$y(t) = \pm x(t) * h_1(t) \pm x(t) * h_2(t)$$

$\Rightarrow$



Property based on linearity :-

$\longrightarrow$ $y(t) = x(t) * h(t)$

$x'(t) * h(t) = y'(t)$

$x(t) * h'(t) = y'(t)$

$x'(t) * h'(t) = y''(t)$

Property based on time Invariancy :-

$\longrightarrow$ $x(t) * h(t) = y(t)$

$x(t-t_0) * h(t) = y(t-t_0)$

$x(t) * h(t-t_0) = y(t-t_0)$

$x(t-t_0) * h(t-t_0) = y(t-2t_0)$

eg

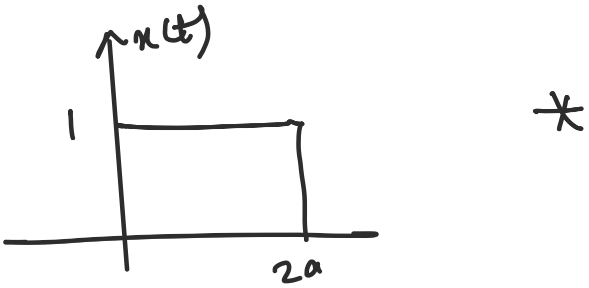
$\longrightarrow$ $u(t) * u(t-4) = r(t-4)$

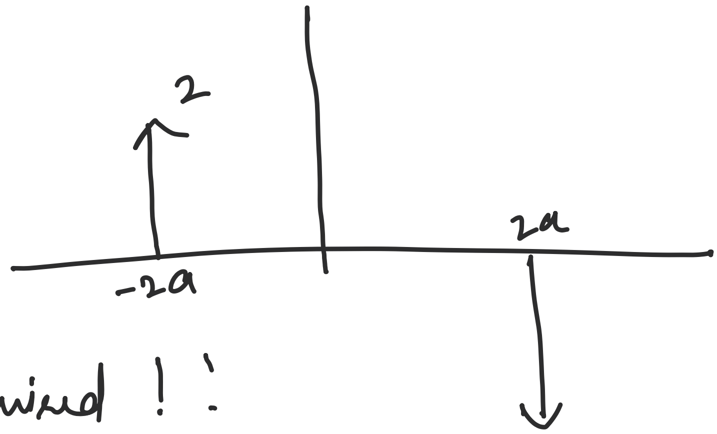
$\longrightarrow$ $u(t-3) * u(t-4) = r(t-7)$

$\longrightarrow$ $u(t-3) * u(t+4) = u(t+1)$

eg $\quad x(t) = 1 \qquad 0 \leqslant t \leqslant 2a$

$h(t) = \delta(t+2a) - \delta(t-2a)$



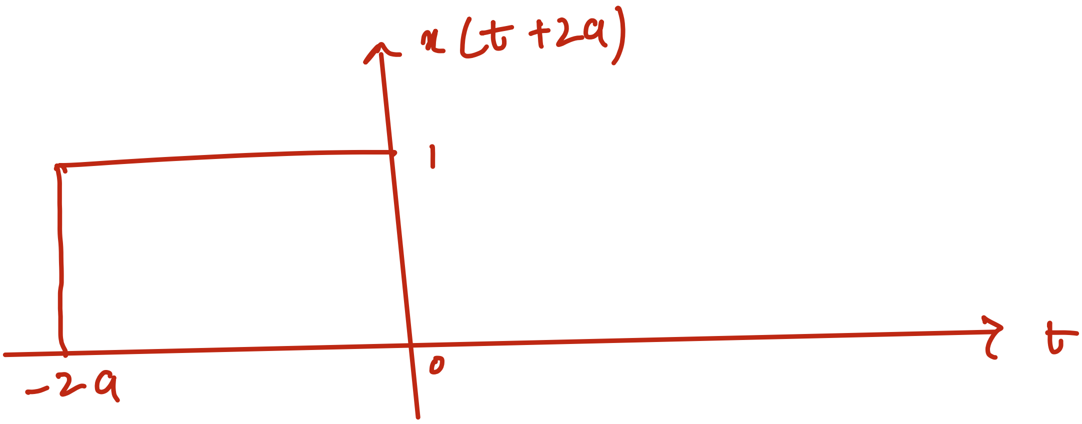


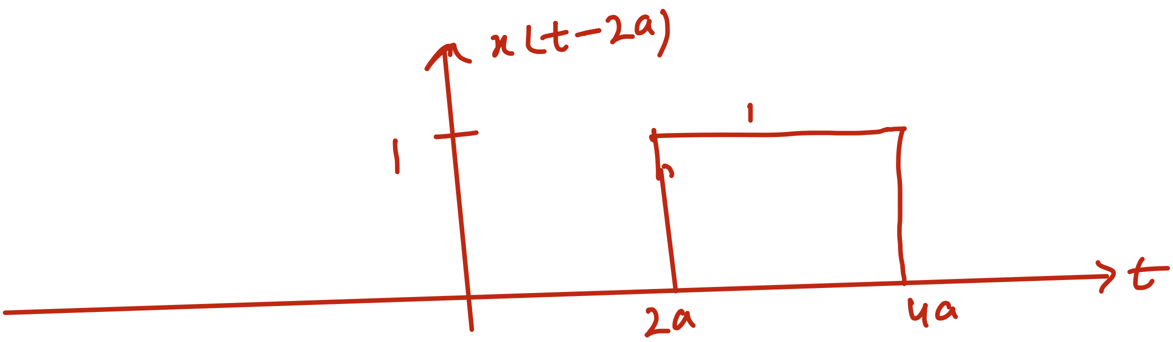
Not required !!

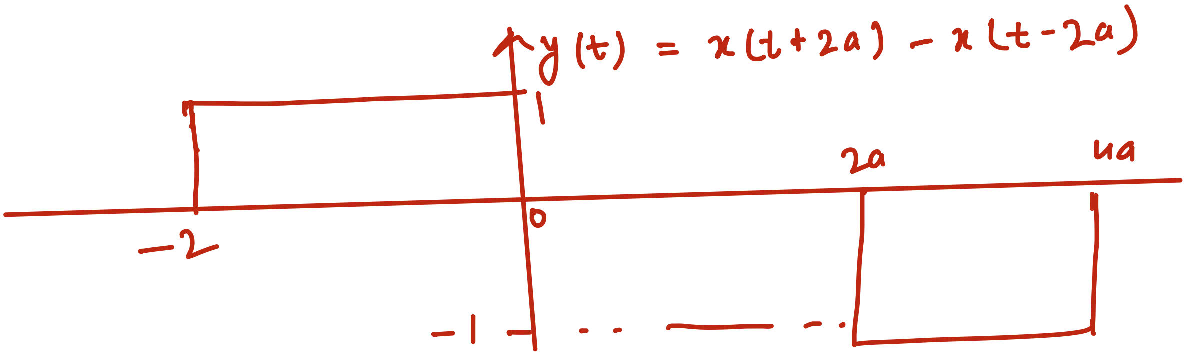
$$y(t) = x(t) * [\delta(t+2a) - \delta(t-2a)]$$

$$= [x(t) * \delta(t+2a)] - [x(t) * \delta(t-2a)]$$

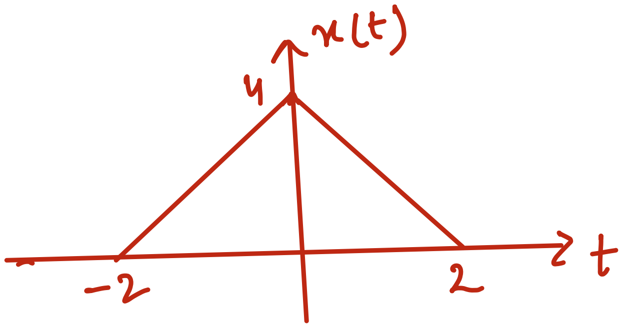
$$\boxed{y(t) = x(t+2a) - x(t-2a)}$$







eg find $x(t) * h(t)$ where



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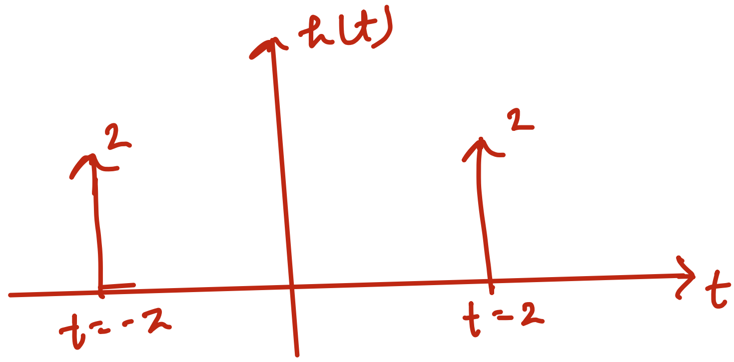


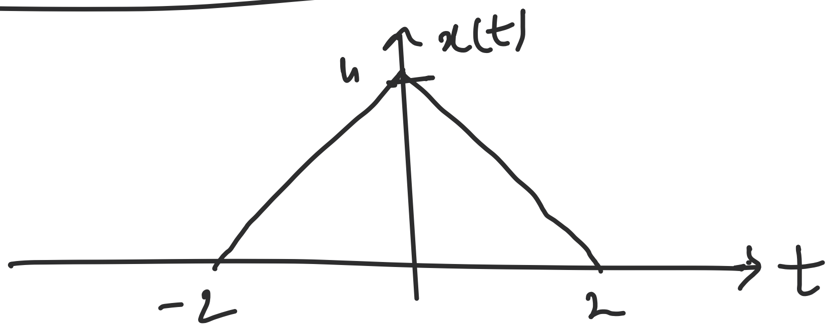
$h(t) = 2\delta(t+2) + 2\delta(t-2)$

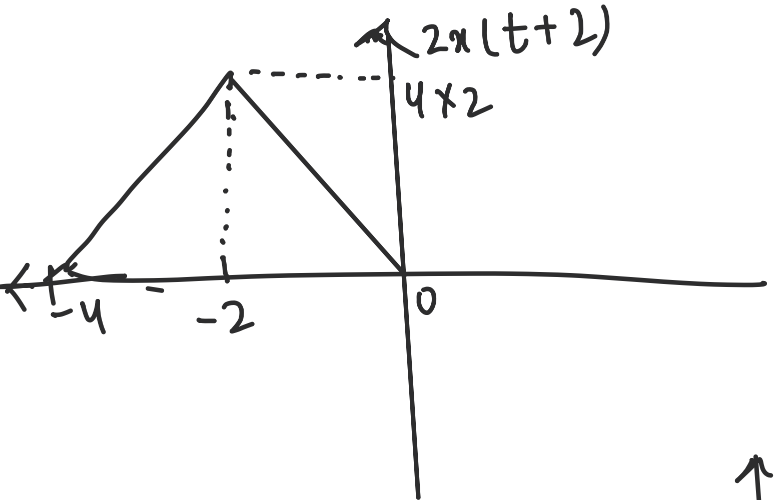
$x(t) = 4\,\text{tri}\left(\frac{t}{2}\right)$

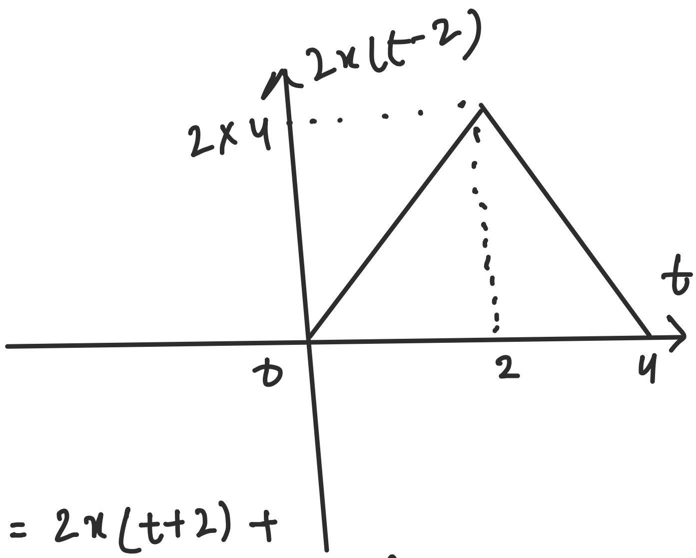
$y(t) = x(t) * h(t)$

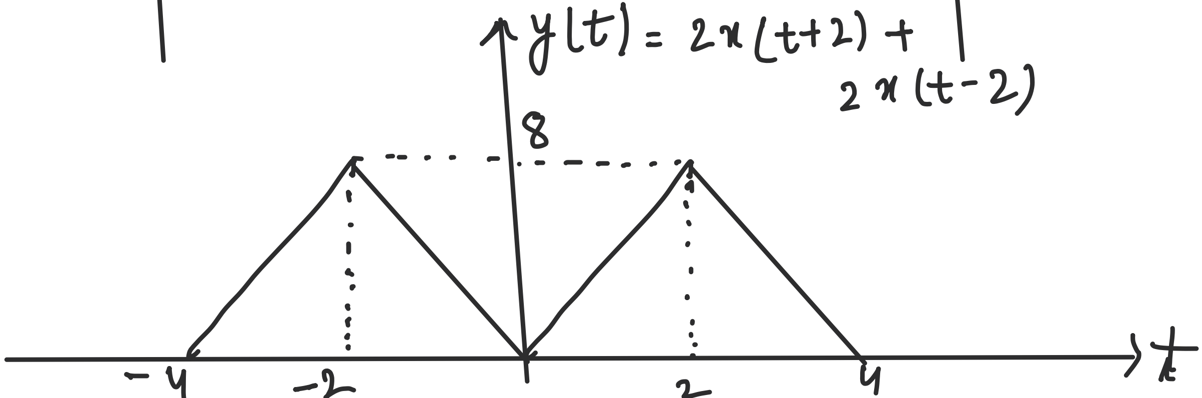
$= x(t) * [2\delta(t+2) + 2\delta(t-2)]$

$$\boxed{y(t) = 2x(t+2) + 2x(t-2)}$$

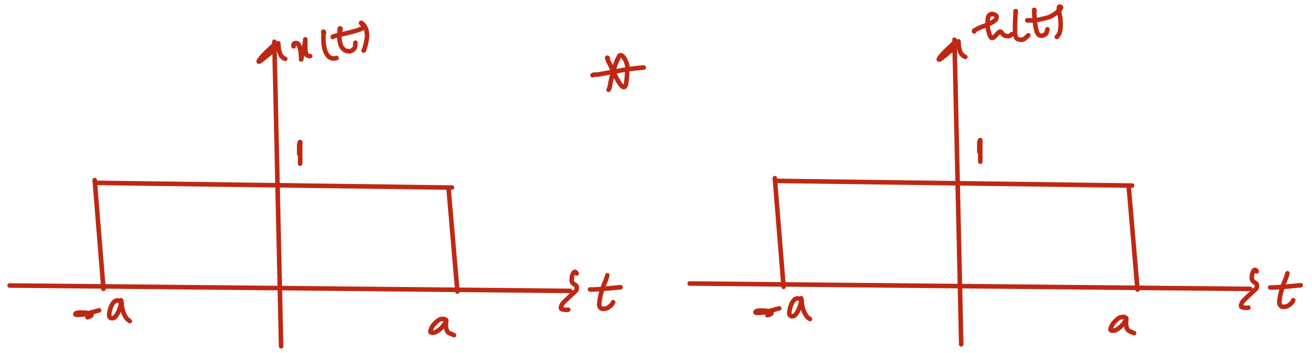








eg



$x(t) = u(t+a) - u(t-a)$

$h(t) = u(t+a) - u(t-a)$

$u(t) * u(t) = t\,u(t) = r(t)$

$y(t) = [u(t+a) - u(t-a)] * [u(t+a) - u(t-a)]$

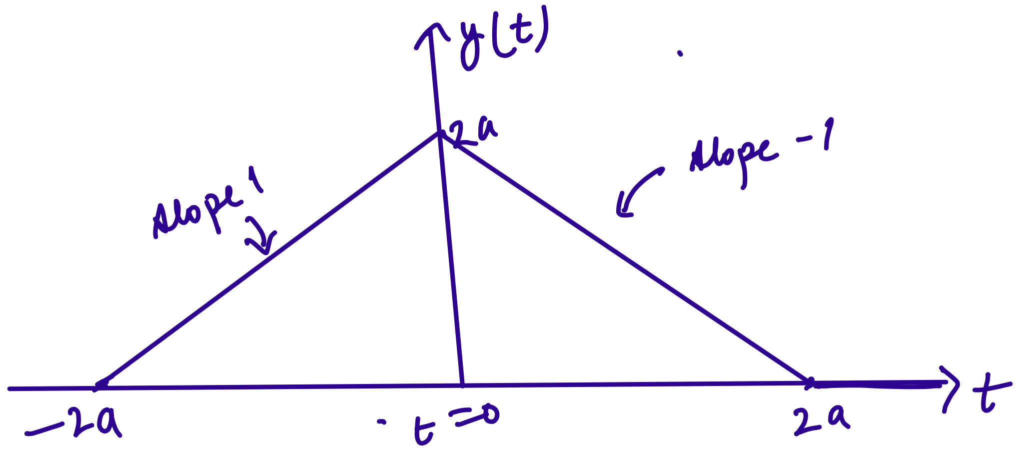
$= r[(t+2a) - r(t) - r(t) + r(t-2a)]$

$$\boxed{y(t) = r(t+2a) - 2r(t) + r(t-2a)}$$

starting	$-2a$	0	$2a$
slope	$+1$	-2	$\underbrace{+1}_{+1}$

$\underbrace{+1 \quad -2}_{-1}$

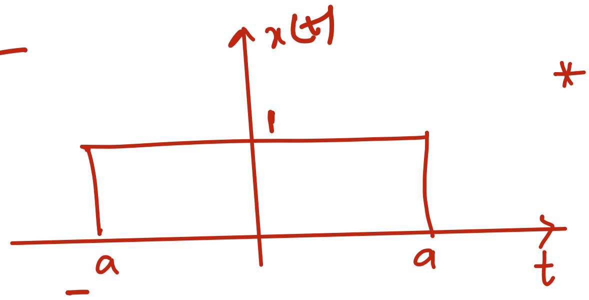
resultant slope will be 0.



left most limit of convolution = Sum of left most limits of $x(t)$ & $h(t)$

Right most limit of convolution = Sum of right limits of $x(t)$ & $h(t)$

eg



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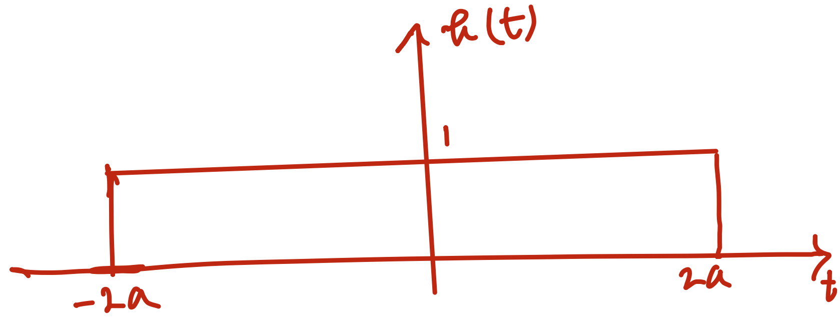


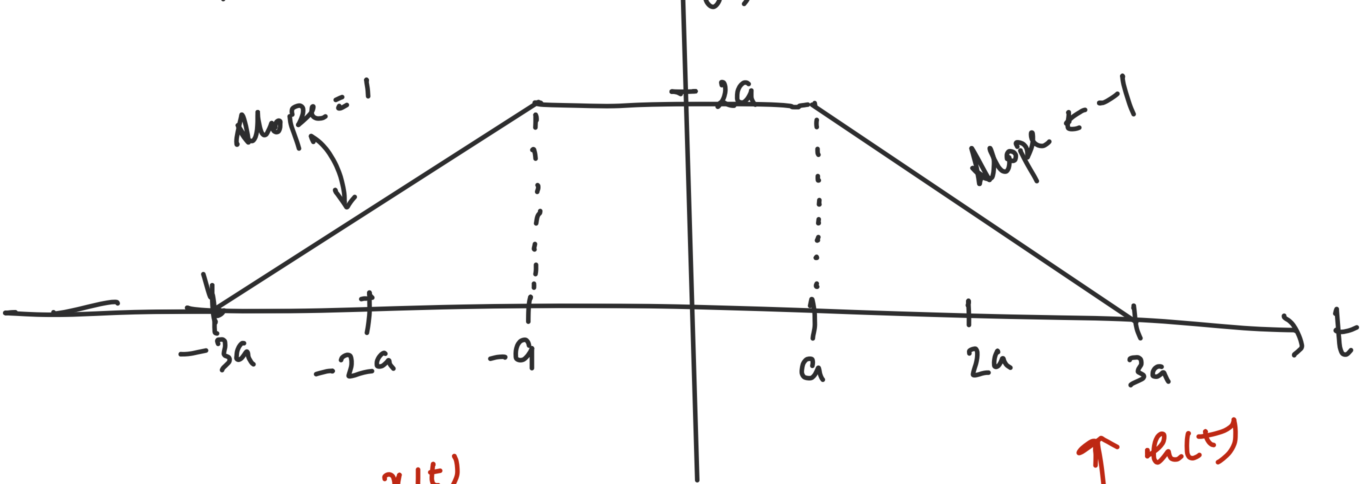
$x(t) = u(t+a) - u(t-a)$ $\qquad$ $h(t) = u(t+2a) - u(t-2a)$

$y(t) = x(t) * h(t)$

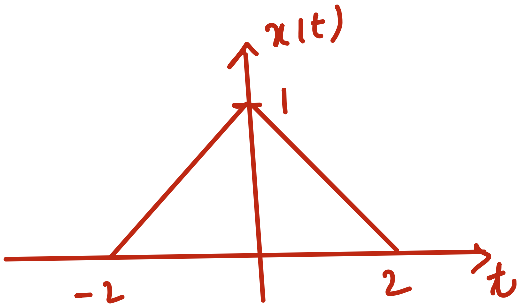
$= [u(t+a)\,u(t+2a) - u(t+a)\,u(t-2a) - u(t-a)\,u(t+2a) + u(t-a)\,u(t-2a)]$

$y(t) = r(t+3a) - r(t-a) - r(t+a) + r(t-3a)$

$-3a$	a	$-a$	$3a$
1	-1	-1	1



eg



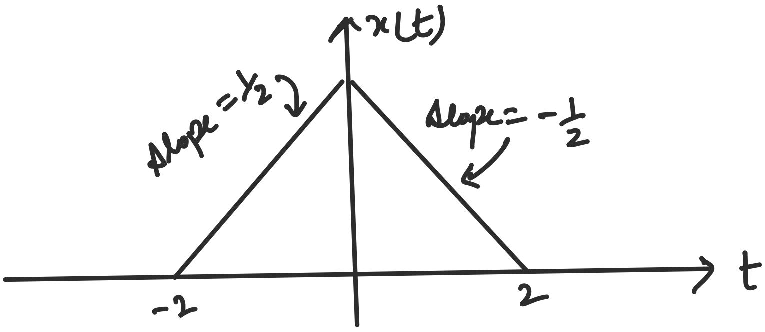
$*$

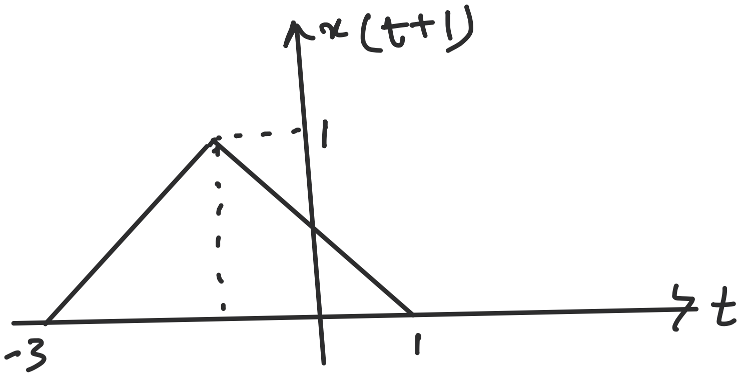
$y(t) = x(t) * h(t)$

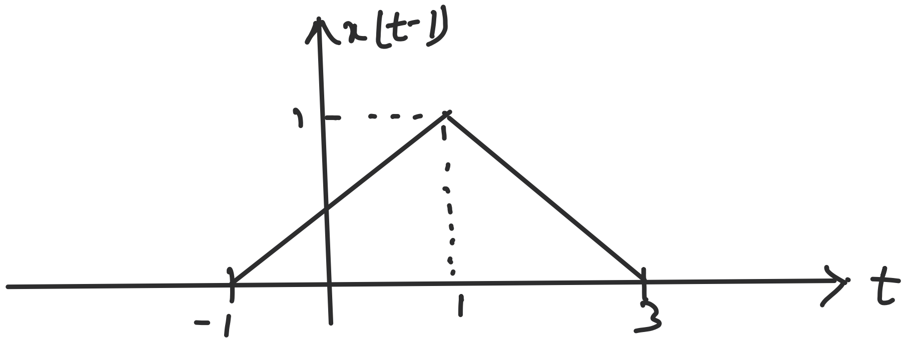
$h(t) = \delta(t+1) * \delta(t-1)$

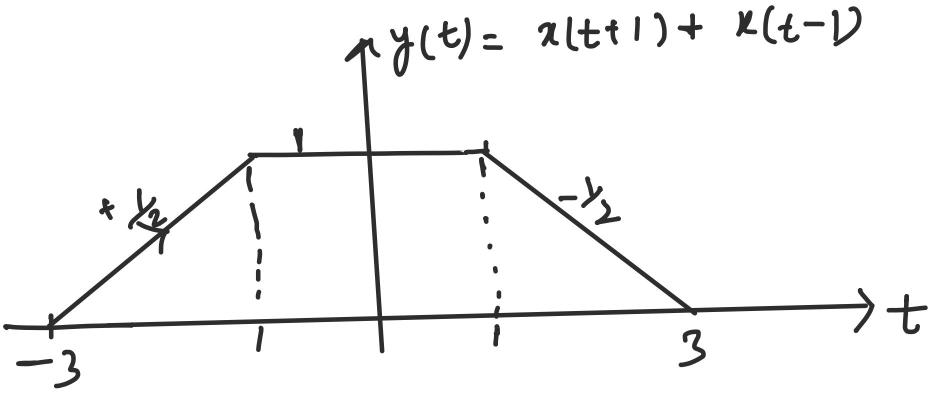
$\therefore\; y(t) = x(t) * [\delta(t+1) + \delta(t-1)]$

$$\boxed{y(t) = x(t+1) + x(t-1)}$$









Note: $x(t) * h(t) = y(t)$

$$\boxed{x'(t) * h(t) = y'(t)}$$

eg $\quad x(t) = u(t+a) - u(t-a) \qquad h(t) = \text{sgn}(t)$

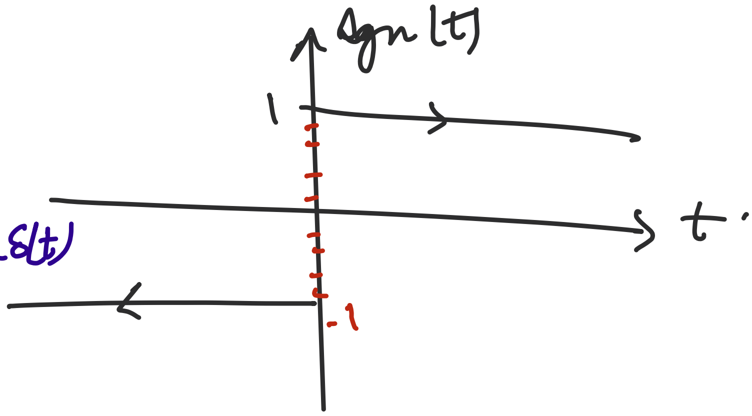
$x(t) * h(t) = y(t)$

$x(t) * h'(t) = y'(t)$

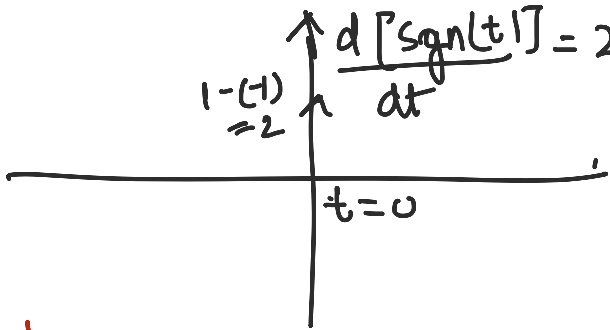
$\therefore y'(t) = [u(t+a) - u(t-a)] * 2\delta(t)$

$y'(t) = 2u(t+a) - 2u(t-a)$

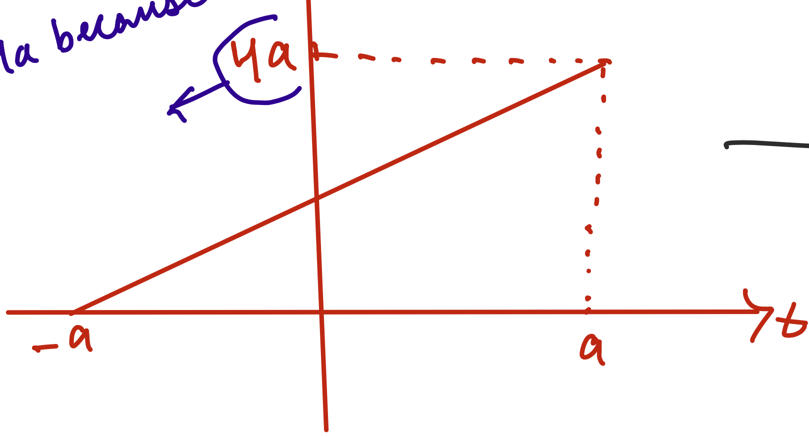
$y(t) = 2r(t+a) - 2r(t-a)$



⇓ differentiate because there is a discontinuity



4a because Amp is 2 time dist covered in x axis.



eg ② $\quad x(t) = u(t+a) - u(t-a)$

$h(t) = u(t)$

$\frac{dy}{dt} = y'(t) = h'(t) * x(t) = [u(t+a) - u(t-a)] * \delta(t)$

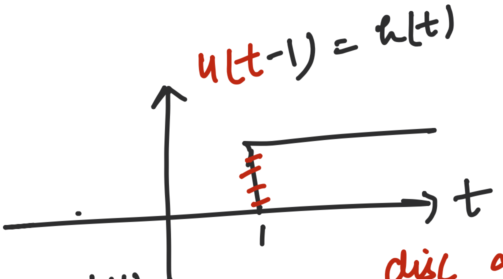
$= u(t+a) - u(t-a)$

$y(t) = r(t+a) - r(t-a)$

eg $\quad x(t) = t\,u(t) = r(t)$

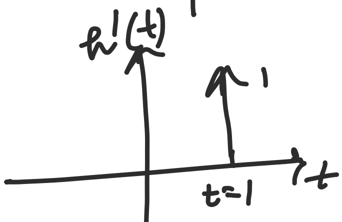
$h(t) = u(t-1)$

$y'(t) = x(t) * h'(t)$



disc at $t = 1$

$\therefore$ diff of this func is $\delta(t-1)$



$$y'(t) = t\,u(t) * \delta(t-1)$$
$$= r(t) * \delta(t-1) = r(t-1)$$
$$y(t) = p(t-1) = \frac{(t-1)^2}{2}\, u(t-1) \qquad \left[p(t) = \frac{t^2}{2} u(t) \right]$$

parabola)

Note → $p(t) = \frac{t^2}{2} u(t)$ $\qquad \frac{d\,[\text{Parbolic}]}{dt} = \text{Ramp}$

→ $r(t) = t\,u(t)$ $\qquad \frac{d\,[\text{ramp}]}{dt} = \text{step}$

→ $u(t) = 1 \cdot u(t)$ $\qquad \frac{d\,[\text{unit}]}{dt} = \text{Impulse.}$

→ $\delta(-t) = \delta(t)$

$\therefore$ <u>$\delta(t)$ is even funcn</u>.

eg Find: $y(t) = u(t-2) * \delta(4-t)$

We know, $\delta(4-t) = \delta(-t+4)$
$$= \delta\{-(t-4)\} = \delta(t-4)$$

$$\therefore\ y(t) = u(t-2) * \delta(t-4)$$
$$y(t) = u(t-6)$$

eg Find $y(t) = u(t) * \delta(3-4t)$

We know $\delta\{-(4t-3)\} = \delta(4t-3)$
$$\delta\{4(t-\tfrac{3}{4})\} = \frac{1}{4}\,\delta(t-\tfrac{3}{4})$$

$\therefore$ Simplified eq^n is

$$y(t) = u(t) * \frac{\delta(t - \frac{3}{4})}{4}$$

$$\boxed{y(t) = \frac{1}{4} u\left(t - \frac{3}{4}\right)}$$

~~eg~~ $y(t) = e^{-2t} u(t) * \delta(5-2t)$

$$\delta(5-2t) = \delta(2t-5) = \frac{\delta(t - 5/2)}{2}$$

$$y(t) = e^{-2t} u(t) * \frac{\delta(t - 5/2)}{2}$$

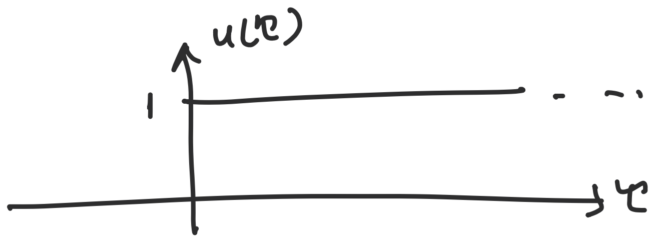
$$y(t) = \frac{e^{-2(t-5/2)} u(t - 5/2)}{2}$$ Ans

Graphical convolution example :-

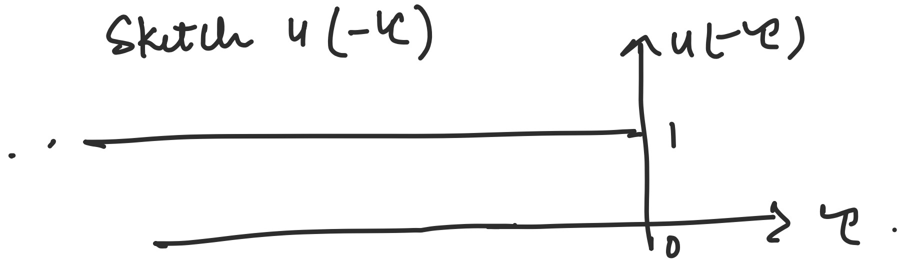
eg ① $y(t) = u(t) * u(t)$

$$\Rightarrow y(t) = \underbrace{\int_{-\infty}^{\infty} u(\tau)\, u(t-\tau)\, d\tau}_{\text{defination of } u(t) * u(t)}$$

Sketch $u(\tau)$

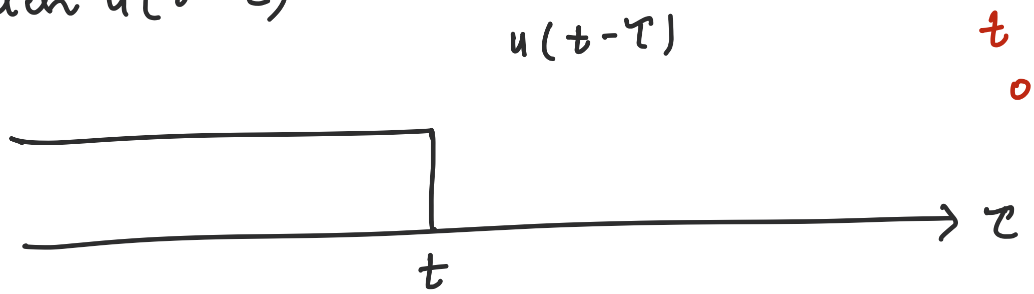


Sketch $u(-\tau)$

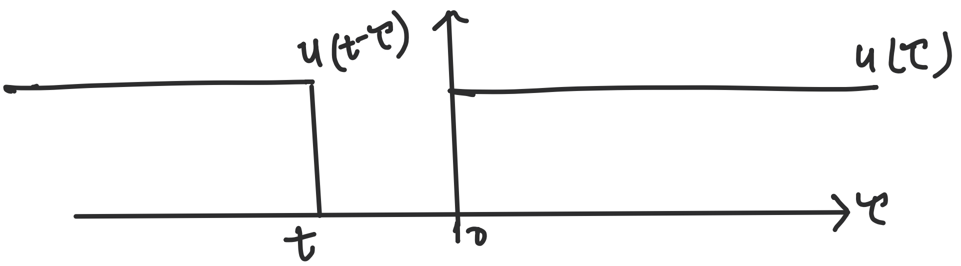


Sketch $u(t-\tau)$

t could be < 0 or t > 0



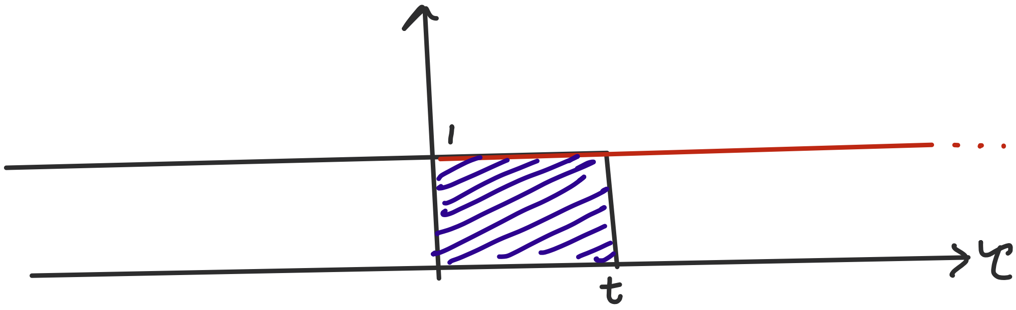
Case 1 : $t < 0$



$u(\tau)\, u(t-\tau) = 0$

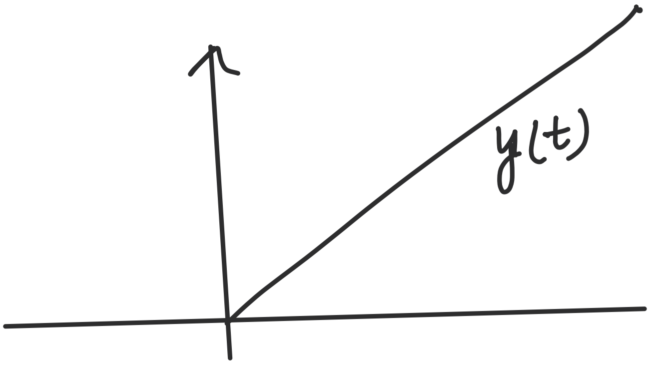
$$\Rightarrow \int_{-\infty}^{\infty} u(\tau)\, u(t-\tau)\, d\tau = 0.$$

Case 2: $t \geq 0$



$$\int_{-\infty}^{\infty} u(\tau)\, u(t-\tau)\, d\tau = \int_{0}^{t} 1\, d\tau = t$$

Combine Cases -: $y(t) = u(t) * u(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}$



$\Rightarrow$ $\boxed{u(t) * u(t) = t\, u(t)}$

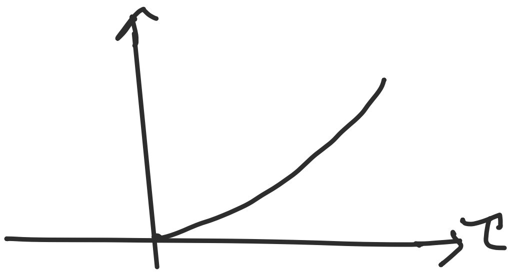
example 2 -:

$x(t) = t^3 u(t)$ & $y(t) = t^2 u(t)$

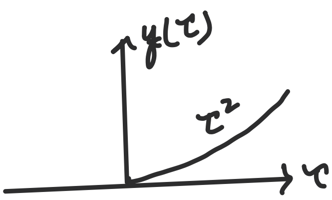
compute $z(t) = x(t) * y(t)$

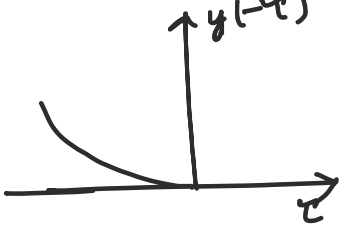
$$\Rightarrow z(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$$

$x(\tau) = \tau^3 u(\tau)$

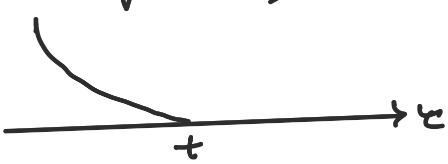


$y(t-\tau) = (t-\tau)^2 u(t-\tau)$

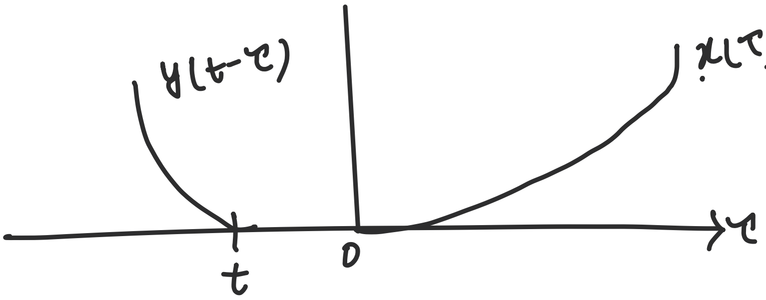




$y(t-\tau)$



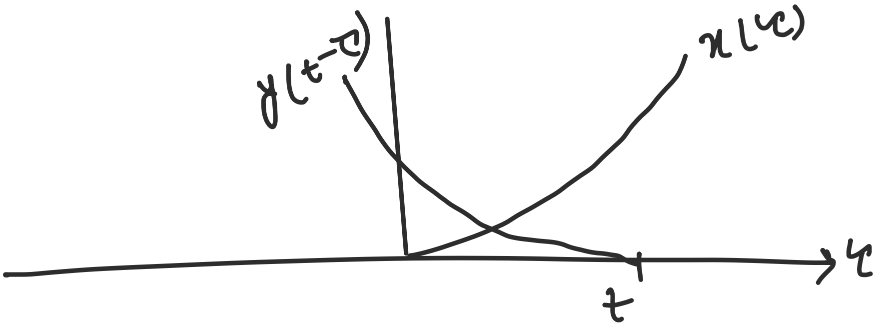
Case 1 : $t \leq 0$



$$\Rightarrow x(\tau)\, y(t-\tau) = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} x(\tau) y(t-\tau)\, d\tau = 0$$

Case 2 : $t > 0$



$$\Rightarrow z(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \tau^3 u(\tau)\, (t-\tau)^2 u(t-\tau)\, d\tau$$

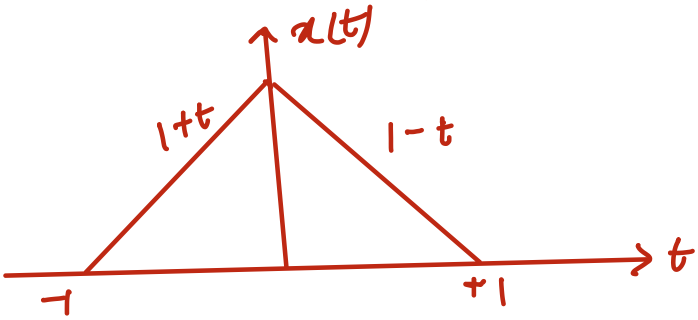
$$= \int_0^t \tau^3 (t-\tau)^2\, d\tau = \int_0^t \tau^3 \left[t^2 - 2t\tau + \tau^2\right] d\tau$$

$$= \int_0^t \tau^3 t^2 - 2\tau^4 t + \tau^5\, d\tau = \left. \frac{\tau^4 t^2}{4} - \frac{2\tau^5 t}{5} + \frac{\tau^6}{6} \right|_0^t$$

$$= \frac{t^6}{4} - \frac{2t^6}{5} + \frac{t^6}{6} = \frac{t^6}{60}$$

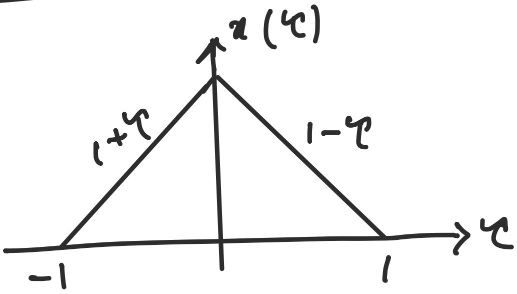
---

eg 3 Consider the CT signal

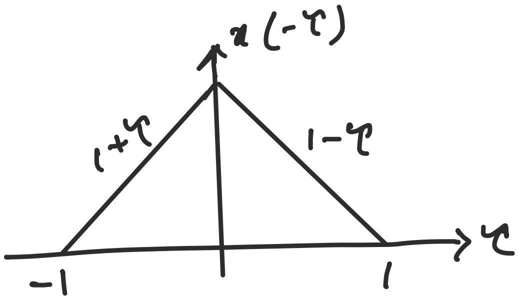


Compute $y(t) = x(t) * x(t)$

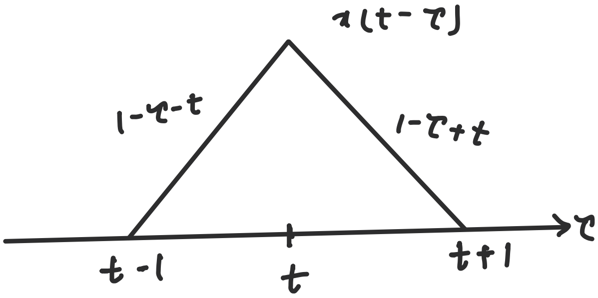
Sketch $x(\tau)$



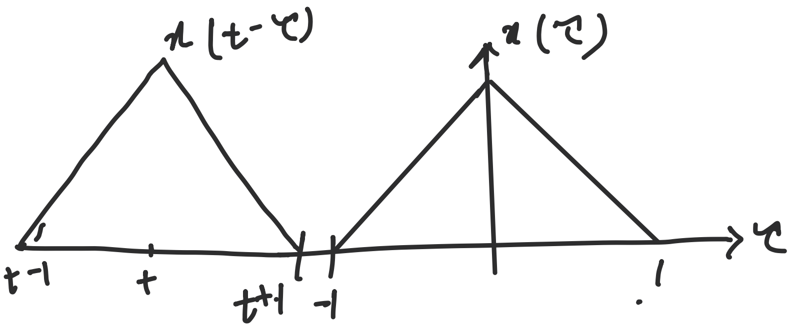
Sketch $x(-\tau)$



Sketch $x(t-\tau)$



Case 1 :



$\Rightarrow$ No overlap

$\Rightarrow x(\tau)\, x(t-\tau) = 0$

$\Rightarrow y(t) = 0$

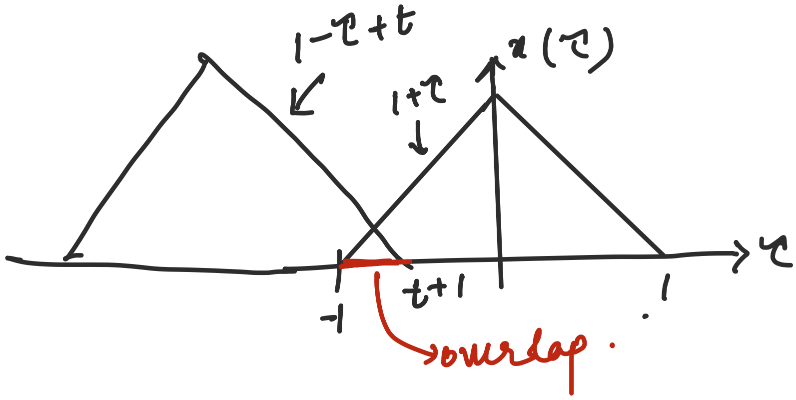
Case 2 :

$t+1 \geq -1 \Rightarrow t \geq -2$

$t+1 < 0$

$t < -1$

$\therefore -2 < t < -1$



$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, x(t-\tau)\, d\tau$$

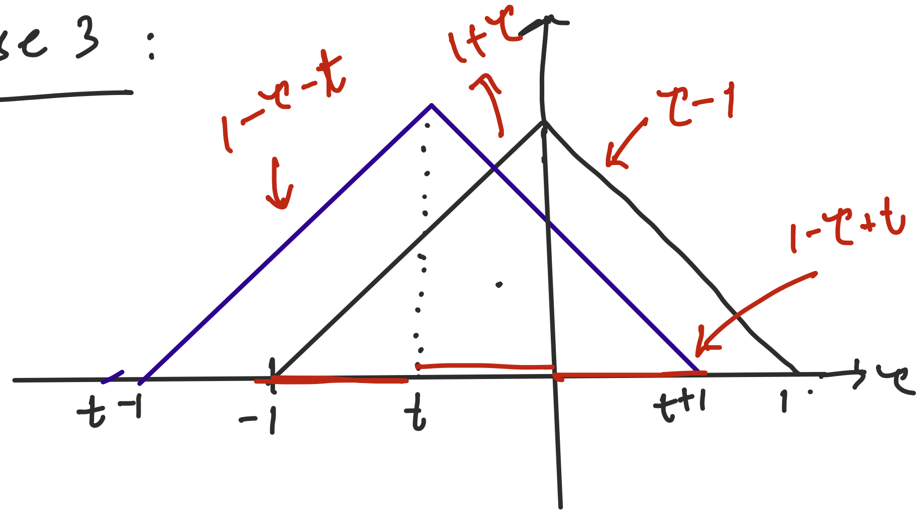
$$= \int_{-1}^{t+1} (1+\tau)(1-\tau+t)\, d\tau$$

$$= \frac{1}{6}(t+2)^3$$

Case 3 :

$t+1 \geq 0 \Rightarrow t \geq -1$

$t+1 < 1 \Rightarrow t < 0$



$$\int_{-1}^{t} (1+\tau)(1+\tau-t)\,d\tau + \int_{t}^{0} (1+\tau)(1-\tau+t)\,d\tau$$

$$+ \int_{0}^{t+1} (1-\tau)(1-\tau+t)\,d\tau$$

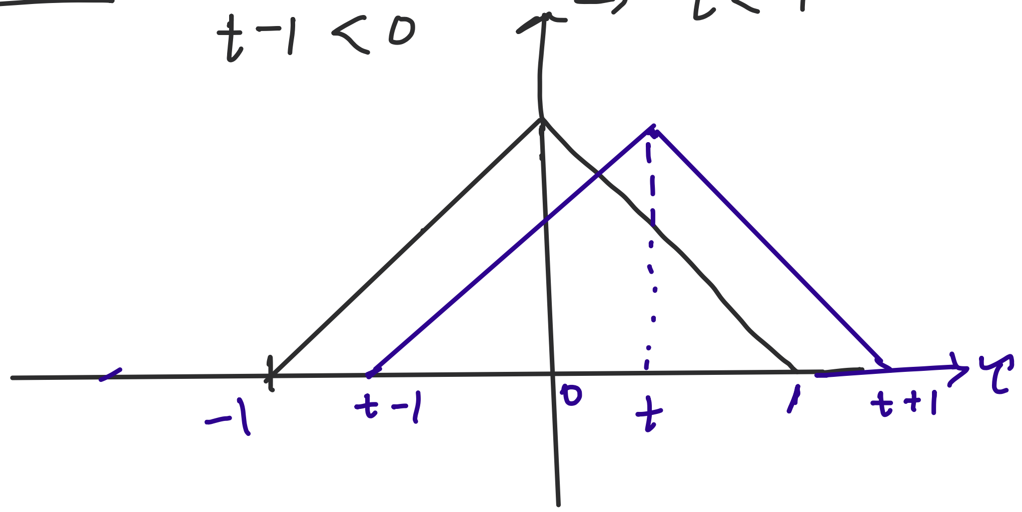
$$= \frac{1}{6}[-t^3+3t+2] - \frac{1}{6}t[t^2+6t+6] + \frac{1}{6}[-t^3+3t+2]$$

$$= \frac{1}{3}[-t^3+3t+2] - \frac{1}{6}t[t^2+6t+6]$$

Case 4 :

$t-1 \geq -1 \Rightarrow t \geq 0$

$t-1 < 0 \Rightarrow t < 1$



$$\int_{t-1}^{0} (\tau+1)(1-\tau-t) + \int_{0}^{t} (\tau-1)(1-\tau-t) + \int_{t}^{1} (\tau-1)(1-\tau+t)$$

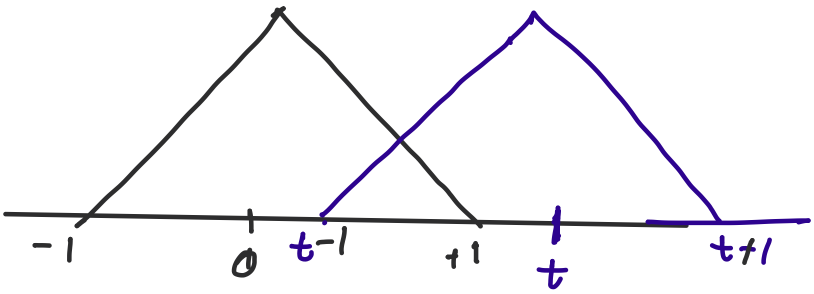
$$= \frac{1}{6}(t^3-3t+2) + \frac{1}{6}t(t^2-6t+6) + \frac{1}{6}(t^3-3t+2)$$

$$= \frac{1}{3}(t^3-3t+2) + \frac{1}{6}t(t^2-6t+6)$$

Case 5 :

$t-1 \geq 0 \Rightarrow t \geq 1$

$t-1 < 1 \Rightarrow t < 2$

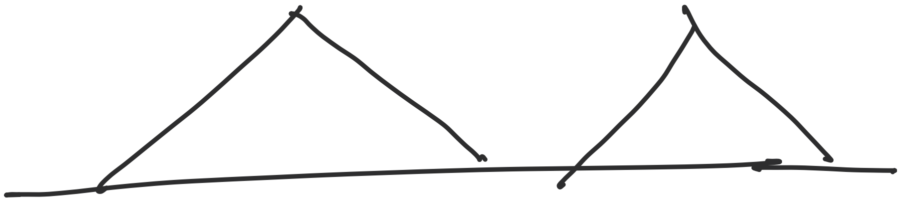


$$y(t) = \int_{t-1}^{1} (1-\tau)(1+\tau-t)\, d\tau = -\frac{1}{6}(t-2)^3$$

Case 6 :

$t-1 \geq 1$

$t \geq 2$

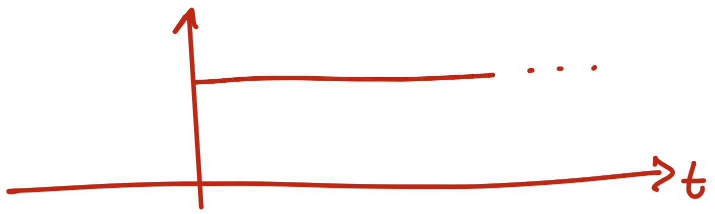


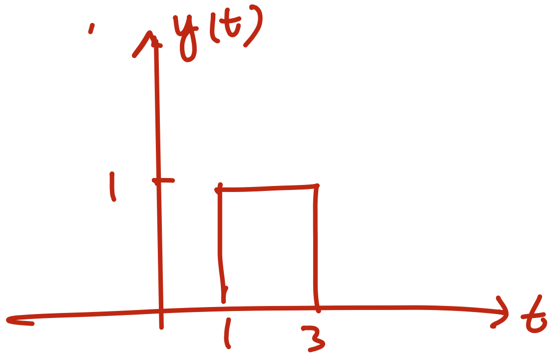
$\Rightarrow$ No overlap $\Rightarrow y(t) = 0$

$$\therefore\ y(t) = \begin{cases} \frac{1}{6}(t+2)^3 & -2 \leq t < -1 \\ \frac{1}{3}(-t^3+3t+2) - \frac{1}{6}t(t^2-6t+6) & -1 \leq t < 0 \\ \frac{1}{3}(t^3-3t+2) + \frac{1}{6}t(t^2-6t+6) & 0 \leq t < 1 \\ -\frac{1}{6}(t-2)^3 & 1 \leq t < 2 \\ 0 & \text{elsewhere.} \end{cases}$$

Example 4 :- Consider the CT signal

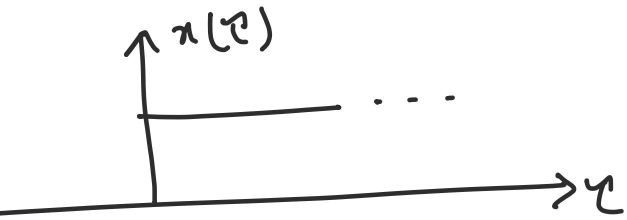
$x(t) = u(t)$



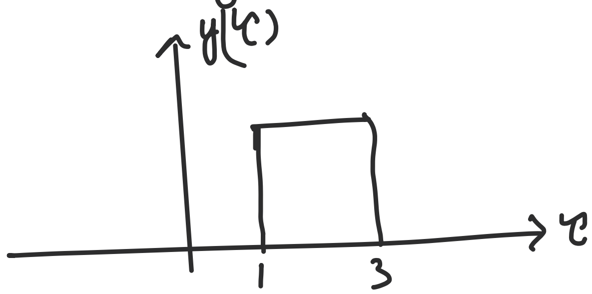


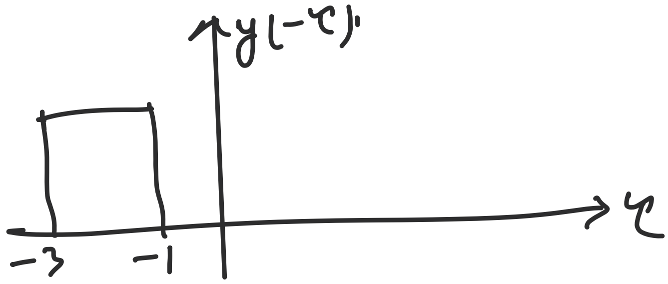
Compute $z(t) = x(t) * y(t)$

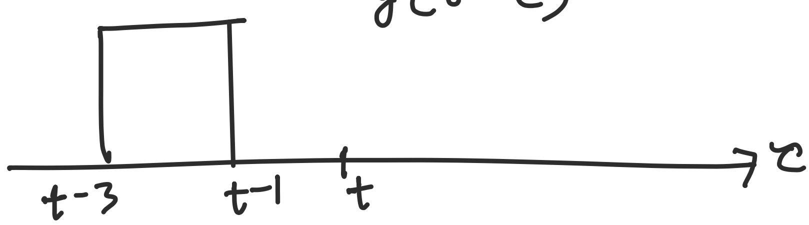
Sketch $x(\tau)$



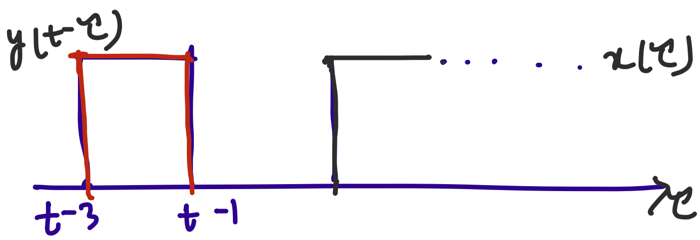
Sketch $y(t-\tau)$







Case 1 :-



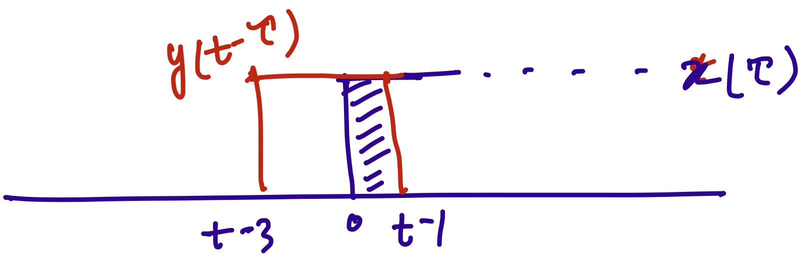
$t-1 \leq 0$

$t \leq -1$

$\Rightarrow$ No overlap

$\Rightarrow z(t) = 0$

Case 2



$t-1 > 0$

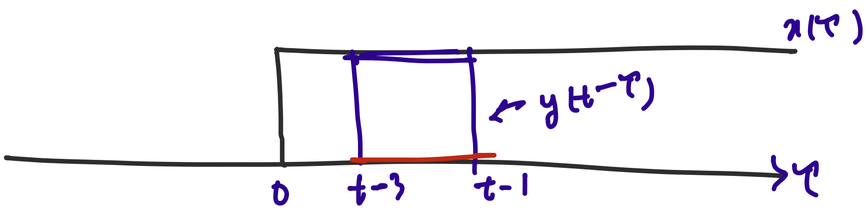
$t > 1$

$t-3 \leq 0$

$t \leq 3$

$$z(t) = \int_0^{t-1} 1 \cdot 1 \, d(\tau) = \int_0^{t-1} d\tau = t-1$$

Case 3 -!

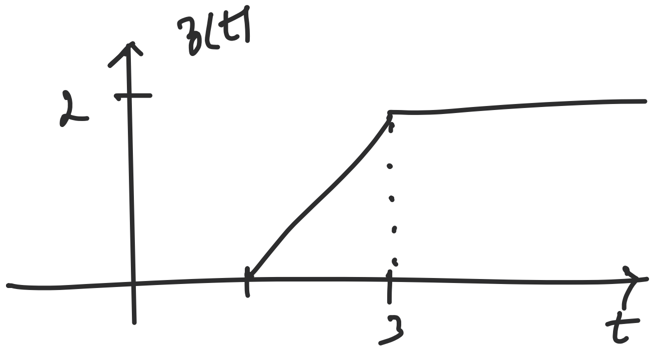


$t-3 > 0$

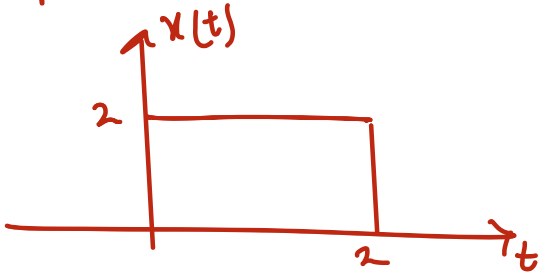
$t > 3$

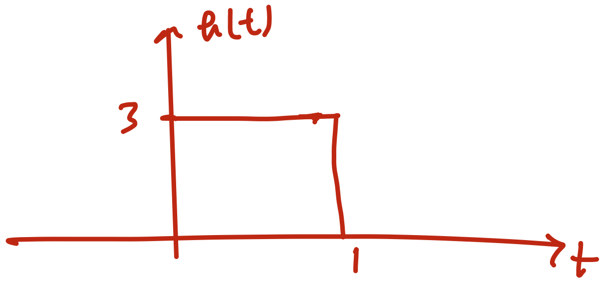
$$z(t) = \int_{t-3}^{t-1} 1 \cdot 1 \cdot d\tau = \int_{t-3}^{t-1} d\tau = \tau \Big|_{t-3}^{t-1} = (t-1)-(t-3) = 2$$

$$\Rightarrow z(t) = \begin{cases} 0 & t \leq 1 \\ t-1 & 1 < t \leq 3 \\ 2 & t > 3 \end{cases}$$



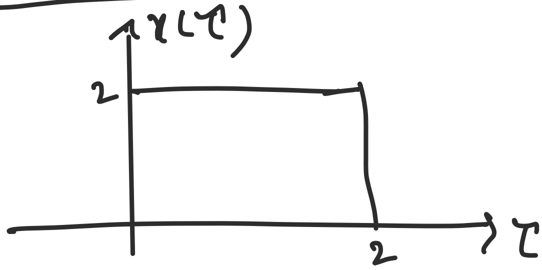
Example 5 :- Convolve these two signals :-



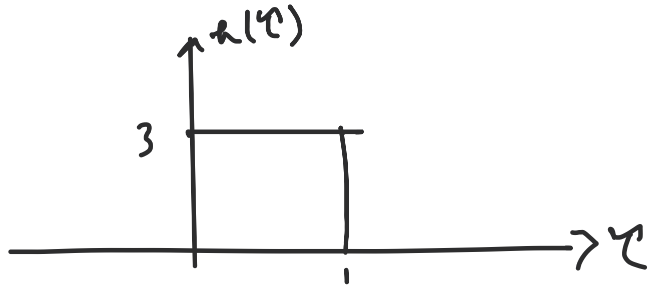


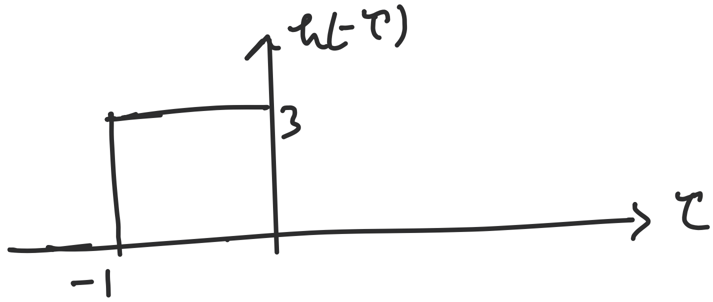
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau) = ?$$

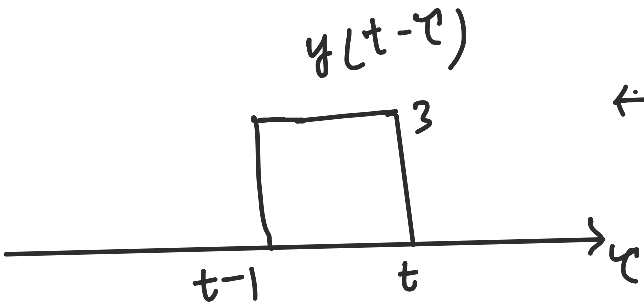
Sketch $x(\tau)$



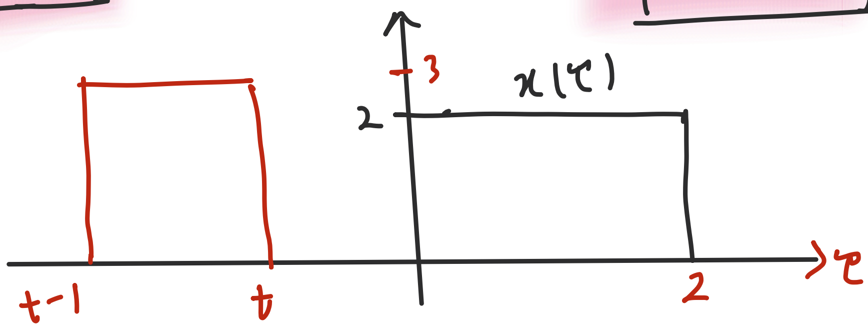
Sketch $h(t-\tau)$





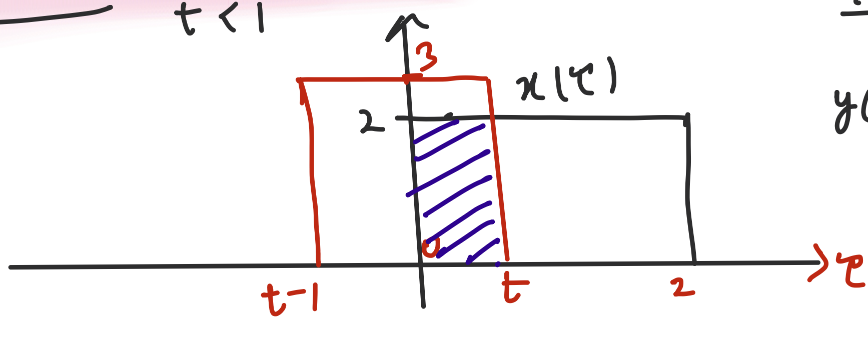


Case 1 $\boxed{t < 0}$



No overlapping $\therefore y(\tau) = 0$

Case 2 :- $t-1 < 0$, $t < 1$ $\quad t > 0$ $\quad \boxed{0 \le t \le 1}$



$$y(t) = \int_0^t 2(3)\,d\tau$$

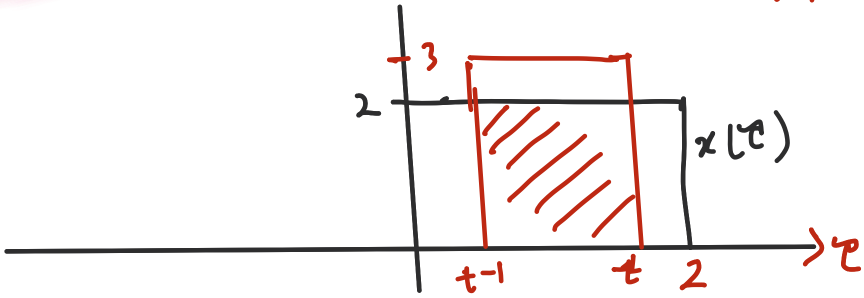
$$= 6\tau\Big|_0^t = 6t$$

Case 3 :-

$t-1>0$ $\quad t\leq 2$

$t>1$

$1\leq t\leq 2$



$$y(t)=\int_{t-1}^{t} 6\,d\tau = 6\tau\Big|_{t-1}^{t}$$

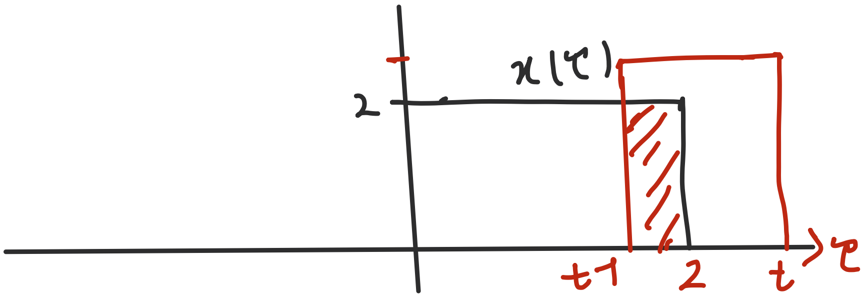
$$= 6t-6t+6$$

$$y(t)=6$$

Case 4 :-

$t-1<2$ $\quad t<3$ $\quad t>2$

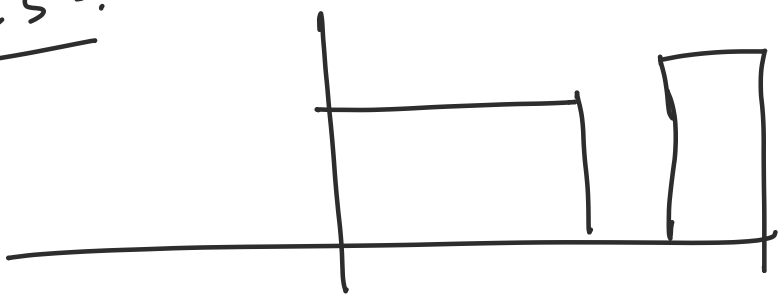
$2<t<3$



$$y(t)=\int_{t-1}^{2} 6\,d\tau = 6\tau\Big|_{t-1}^{2}$$

$$= -6t+18$$

case 5 :-



$t>3$

no overlapping.

$$\therefore\quad y(t)=\begin{cases} 6t & 0<t\leq 1 \\ 6 & 1<t\leq 2 \\ -6t+18 & 2<t\leq 3 \\ 0 & \text{otherwise} \end{cases}$$