UNIT-2
Curves and Surfaces

Learning Objective

•Introduction to Splines

•Bezier Curves

•Bezier curves using Bernstein Polynomials

•Bezier bi-cubic surface patch

•Bspline Curves

•Geometrical Construction

Introduction to Splines

- Mathematical splines are motivated by the "loftsman's spline":
  - Long, narrow strip of wood or plastic
  - Used to fit curves through specified data points
  - Shaped by lead weights called "ducks"
  - Gives curves that are "smooth" or "fair"
- Such splines have been used for designing:
  - Automobiles
  - Ship hulls
  - Aircraft fuselages and wings

Requirements for Splines

- Here are some requirements we might like to have in our mathematical splines:
  - Predictable control
  - Multiple values
  - Local control
  - Versatility
  - Continuity

Useful terms about Splines

- Here are some terms we might like to use in context with splines:
  - Interpolation splines
  - Approximation splines
  - Convex hull
  - Control polygon

Parametric curves

- In general, a "parametric" curve in the plane is expressed as:

$$x = X(t) \qquad y = Y(t)$$

- Example: A circle with radius r centered at the origin is given by:

$$x = r\cos t \qquad y = r\sin t$$

- By contrast, an "implicit" representation of the circle is:

$$x^2 + y^2 = r^2$$

Parametric Plynomial curves

- A parametric “polynomial” curve is a parametric curve where each function x(t), y(t) is described by a polynomial:

$$x = \sum_{i=0}^{n} a_i t^i \qquad y = \sum_{i=0}^{n} b_i t^i$$

- Polynomial curves have certain advantages
  - Easy to compute
  - Infinitely differentiable

Piecewise Parametric Polynomial curves

- A “piecewise” parametric polynomial curve uses different polynomial functions for different parts of the curve
  - Advantage: Provides flexibility
  - Problem: How do you guarantee smoothness at the joints? (Problem known as “continuity”)

Curves and Surfaces Cont…

Conditions for smoothly joining curve segments

To ensure a smooth transition from one section of a piecewise parametric curve to the next we can impose various conditions called **continuity conditions.**

Continuities Conditions

There are two types of continuity between two adjacent curve pieces- one is the parametric continuity, denoted by **C** and other is geometric Continuity, denoted by G. Numerical superscripts are used with C and G to indicate degree of Continuity or Smoothness

Curves and Surfaces Cont…

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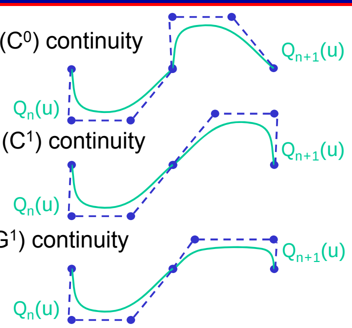
Types of Continuities:

- Zero-order Parametic / Zero-order Geometric Continuity
- First-order Parametic C^1 / First-order Geometric Continuity G^1
- Second-order Parametic C^2/ Second-order Geometric Continuity G^2

Curves and Surfaces Cont…

- Positional (C^0) continuity
- Derivative (C^1) continuity
- Tangent (G^1) continuity



Bezier Curves

Bezier Curves

- Bézier Curves were first developed in 1959 by Paul de Casteljau.
- They were popularized in 1962 by French engineer Pierre Bézier, who used them to design automobile bodies.
- A bezier curve section can be fitted to any number of control points. The number of control points to be approximated and their relative position determine the degree of Bezier Polynomial.

Bezier Curves cont..

- To construct an n^{th} degree Bezier Curve, we need n+1 control points.
- The formula for a point on the curve is:

$$P(u) = \sum_{i=0}^{n} C(n,i)\,(1-u)^{n-I}\,u^{i}\,P_i$$

Where $o \le u \le 1$

 U2. ‹#›

Bezier Curves cont..

Bezier curves using Bernstein Polynomials

Note:

$(a - b)^3 = a^3 - 3a^2b + 3\,ab^2 - b^3$

4 Control Point Bezier curves using Bernstein Polynomials

$$P(u) = (1-u)^3\mathbf{p}_0 + 3(1-u)^2 u\mathbf{p}_1 + 3(1-u)u^2\mathbf{p}_2 + u^3\mathbf{p}_3$$

$$P(u) = \left(-u^3 + 3u^2 - 3u + 1\right)\mathbf{p}_0 + \left(3u^3 - 6u^2 + 3u\right)\mathbf{p}_1 + \left(-3u^3 + 3u^2\right)\mathbf{p}_2 + u^3\mathbf{p}_3$$

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Bezier Curves cont..

$$P(u) = B^3_0(u)\,\mathbf{P}_0 + B^3_1(u)\,\mathbf{P}_1 + B^3_2(u)\,\mathbf{P}_2 + B^3_3(u)\,\mathbf{P}_3$$

$B^3_0(u) = -u^3 + 3u^2 - 3u + 1$

$B^3_1(u) = 3u^3 - 6u^2 + 3u$

$B^3_2(u) = -3u^3 + 3u^2$

$B^3_3(u) = u^3$

 U2. ‹#›

Bezier Curves cont..

Bezier curves can also be described in matrix form:

$$P(u) = \sum_{i=0}^{3} P_i \binom{3}{i} u^i (1-u)^{3-i}$$

$$= (1-u)^3 P_0 + 3u(1-u)^2 P_1 + 3u^2(1-u)P_2 + u^3 P_3$$

$$= \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$= \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \mathbf{M}_{Bezier} \begin{bmatrix} P_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

Curves and Surfaces Cont…

Bezier bi-cubic surface patch

Bezier Surfaces

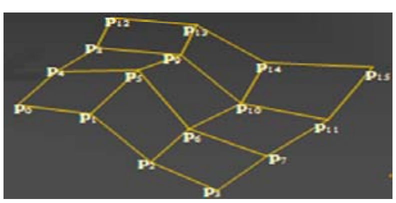
- Bezier surfaces are a straightforward extension to Bezier curves
- Instead of the curve being parameterized by a single variable u, we use two variables, u and v
- By definition, we choose to have u and v range from 0 to 1
- In computer graphics, parametric surfaces are sometimes called *patches*, *curved surfaces*, or just *surfaces*

Curves and Surfaces Cont…

Consider a *bicubic* Bezier surface

and a bicubic surface has a grid of 4x4 control points, $\mathbf{p}_0$ through $\mathbf{p}_{15}$

Total =16 Points



Curves and Surfaces Cont…

B-Spline Curves

- B-Spline Curves are more complex as compare to Bezier. Also, the Degree of B-spline can be independent of the number of Control Points(with certain limitations)

- B-spline is defined by the following formula:

$$P(u)= \sum_{i=0}^{n} P_i B_{i,d}(u)$$

where $u_{min} <= u <= u_{max}$ and $2 <= d <= n+1$

- Degree of B-spline is d-1 and n+1 are the control points.

Curves and Surfaces Cont…

Where P_i are the set of n+1 control Points and $B_{i,d}(u)$ is B-spline blending function.

$$B_{i,d}(u)= \{(u-u_i)/(u_{i+d-1}-u_i)\}B_{i,d-1}(u) + \{(u_{i+d}-u)/(u_{i+d}-u_{i+1})\}B_{i+1,d-1}(u)$$

Where $B_{i,1}(u)=1$ if $u_i <= u < u_{i+1}$
Else $B_{i,1}(u)=0$

Curves and Surfaces Cont…

- Properties of B-Spline curves:

1. The polynomial curve has degree d-1 and c^{d-2} continuity
2. For n+1 control points, the curve has n+1 blending functions
3. Each blending function Bi,d is defined over d subintervals of total range of u
4. The range of parameter u is divided into n+d subintervals by n+d+1 values specified in knot vector
5. Each section of spline curve is influenced by d control points

B-Splines are commonly described according to the selected knot-vector class :

1. Uniform
2. Open uniform
3. Nonuniform

Curves and Surfaces Cont…

- **Uniform , Periodic B-spline**

Example

{0.0,0.2,0.4,0.6,0.8,1.0}

{0,1,2,3,4,5,6,7}

Uniform B-splines have periodic blending function.

That is, for given values of n and d, all blending functions have the same shape.

$$B_{i,d}(u) = B_{i+1,d}(u+\Delta u) = B_{i+2,d}(u+2\Delta u)$$

Where Δu is the interval between adjacent knot values.

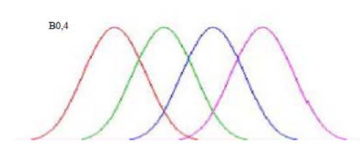
Curves and Surfaces Cont…

Bi,d (degree d-1) on each interval u , d must be at least 2 (linear) and can not be more than *n+1 (the number of control points). A knot vector (u_0 , u_1 , ... , u_{n+d}) must be* specified.

Curves and Surfaces Cont…

Cubic B-Spline curves using uniform knot vectors

n=3 and d=4



Curves and Surfaces Cont…

- **Open, Uniform B-splines**

Example

{0,0,1,2,3,3} d=2,n=3

{0,0,0,0,1,2,2,2,2} d=4,n=4

We can generate open, uniform knot vector by using following calculations:

$$U_j = \begin{cases} 0, & \text{for } 0 \le j < d \\ j-d+1, & \text{for } d \le j \le n \\ n-d+2, & \text{for } j > n \end{cases}$$

Curves and Surfaces Cont…

- **NonUniform B-splines**

Example

{0,1,2,3,3,4}

{0,2,2,3,3,6}

- Nonuniform B-splines provide increased flexibility in controlling a curve shape
- Multiple knot values reduce the continuity by 1 for each repeat of a particular value
- We can obtain blending functions using methods similar to those discussed for uniform and open uniform B-splines

Curves and Surfaces Cont…

Effect of multiple control points at same location

- Adding multiple control points at a same position/location in space will add more weight to that point "pulling" the Bezier curve towards it.

Curves and Surfaces Cont…

Controlling the shape of the curves

- Can control the shape through
  - Control points
    - ✓Overlapping the control points to make it pass through a specific point
  - Knots
    - ✓Changing the continuity by increasing the multiplicity at some knot (non-uniform B-splines)

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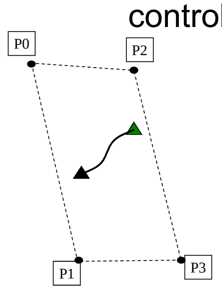
Curves and Surfaces Cont…

Controlling the shape through control points

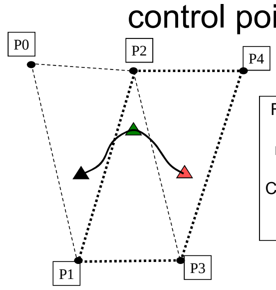


First knot shown with 4 control points, and their convex hull.

 U2. ‹#›

Curves and Surfaces Cont…

Controlling the shape through control points



First two curve segments shown with their respective convex hulls.

Centre Knot must lie in the intersection of the 2 convex hulls.

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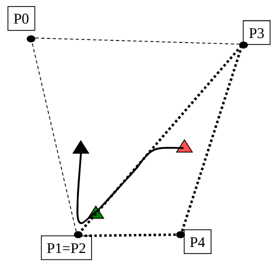
Curves and Surfaces Cont…

Repeated control point.

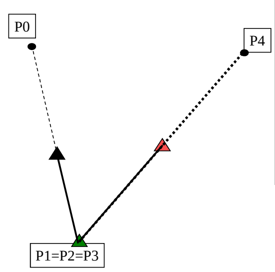


First two curve segments shown with their respective convex hulls.

The curve is forced to lie on the line that joins the 2 convex hulls.

Curves and Surfaces Cont…

Triple control point.



First two curve segments shown with their respective convex hulls.

Both convex hulls collapse to straight lines – all the curve must lie on these lines.

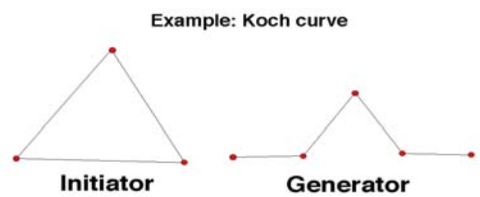
Curves and Surfaces Cont…

Fractal-Geometric Methods

- Fractal is a geometrical shape that is repeated at every scale and can not be represented by classic geometry
- Natural objects can be realistically described with fractal-geometric methods
- A fractal object has two basic characteristics :
  - Infinite details at every point
  - Self-similarity between object parts and overall features of the object
- Example : Natural shapes like mountains and clouds

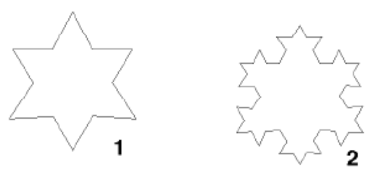
Curves and Surfaces Cont…

Geometrical Construction

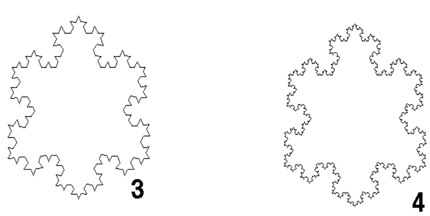


Curves and Surfaces Cont…

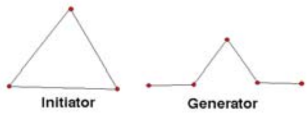
Resulting Koch curves



Curves and Surfaces Cont…



Curves and Surfaces Cont…



Recursive function

Pass all parts to next level

Replace part with the generator, scaled to same length

Stop at desired recursion depth or when sections are small enough (e.g. 1 pixel long)

Curves and Surfaces Cont…

Example: Generation of plants

Curves and Surfaces Cont…

Hermite Interpolation

(Named after the French mathematician Charles Hermite)

Given end slopes for a specified curve segment

- (a) Coordinates at Point 0,
- (b) Slope at Point 0,
- (c) Coordinates at Point 1, and
- (d) Slope at Point 1.

- Let the parametric curve be

  $P(u) = au^3 + bu^2 + cu + d$.

  where u is the parameter that ranges from 0 to 1

Curves and Surfaces Cont…

- A defined curve has a defined set of coefficients a, b, c, d.
- Substituting a value u into the equation gives a point on the curve.
- Substituting many values of u from 0 to 1 will trace out the curve.
- The two points and two slopes, p_0, p_1, p_0' and p_1',

our objective is to find the coefficients a, b, c, d.

Curves and Surfaces Cont…

$$P(u) = au^3 + bu^2 + cu + d$$

$$P(u) = [\,u^3 \quad u^2 \quad u \quad 1\,] \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \qquad \text{Derivative } P'(u) = [\,3u^2 \quad 2u \quad 1 \quad 0\,] \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\text{But, } P(0) = p_0 = a \times 0^3 + b \times 0^2 + c \times 0 + d \times 1$$

$$P(1) = p_1 = a \times 1^3 + b \times 1^2 + c \times 1 + d \times 1$$

$$P'(0) = p_0' = a \times 3 \times 0^2 + b \times 2 \times 0 + c \times 1 + d \times 0$$

$$P'(1) = p_1' = a \times 3 \times 1^2 + b \times 2 \times 1 + c \times 1 + d \times 0$$

Curves and Surfaces Cont…

We get: $a = 2p_0 - 2p_1 + p_0' + p_1'$ $\quad b = -3p_0 + 3p_1 - 2p_0' - p_1'$ $\quad c = p_0'$ $\quad d = p_0$

Next, substitute back to equation $P(u) = au^3 + bu^2 + cu + d$

We get: $P(u) = (2p_0 - 2p_1 + p_0' + p_1')u^3 + (-3p_0 + 3p_1 - 2p_0' - p_1')u^2 + p_0'u + p_0$

Rearranging, $\boxed{P(u) = p_0(2u^3 - 3u^2 + 1) + p_1(-2u^3 + 3u^2) + p_0'(u^3 - 2u^2 + u) + p_1'(u^3 - u^2)}$

Conclusion

In this unit Curves and Surfaces we learn Bezier Curves, Bezier curves using Bernstein Polynomials, Bezier bi-cubic surface patch, Bspline Curves, Geometrical Construction

Summary

Bézier Curves were first developed in 1959 by Paul de Casteljau. They were popularized in 1962 by French engineer Pierre Bézier, who used them to design automobile bodies.

Bezier surfaces are a straightforward extension to Bezier curves. Instead of the curve being parameterized by a single variable *t*, we use two variables, *s* and *t.*

Review Questions cont..

Short answer type Questions

1. What do you mean by Knot vector?
2. What is the difference between Uniform and Non-Uniform B-Spline?
3. Explain C and G with reference to Continuity.
4. What is the Effect of multiple control points at same location?
5. Give the Conditions for smoothly joining curve segments.
6. What do you mean by Bezier bi-cubic surface patch?

Review Questions cont..

Long answer type Questions

1. Write the pseudo code for generating Bezier Curve
2. What do you mean by Cubic B-Spline curves using uniform knot vectors?
3. Explain Koch Curve? With the help of an example.
4. What do you mean by Second order parametric and geometric continuity.
5. Explain 4 point and 5 point Bezier curves using Bernstein Polynomials
6. Explain Computing control points given end slopes for a specified curve segment.

Suggested Reading/References

[1]. Donnald Hearn and M. Pauline Baker, “Computer Graphics”, PHI.

[2]. Foley James D, “Computer Graphics”, AW 2nd Ed.

[3]. Rogers, “Procedural Element of Computer Graphics”, McGraw Hill.

[4]. Newman and Sproul, “Principal of to Interactive Computer Graphics”, McGraw Hill.

