CPSC 313 — Fall 2018
Assignment 1 — Finite Automata

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1. (a) Design a DFA with at most 5 states for the language

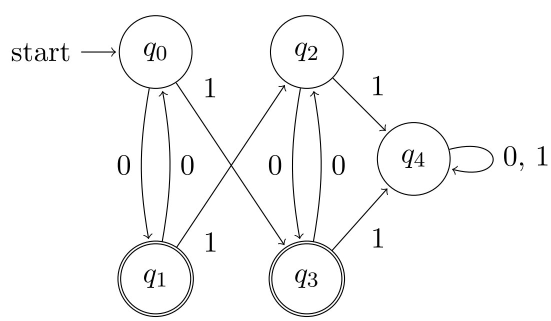
$$L_1 = \{w \in \{0,1\}^* \mid \ w \text{ contains at most one 1 and } |w| \text{ is odd}\}.$$

Give a state diagram for your DFA.

Solution.

My solution contains the following 5 states:

i. q_0: Seen zero 1s, $|w|$ is even. - start state
ii. q_1: Seen zero 1s, $|w|$ is odd. - accept state
iii. q_2: Seen one 1, $|w|$ is even.
iv. q_3: Seen one 1, $|w|$ is odd. - accept state
v. q_4: Seen two or more 1s.



(b) Prove that your DFA of part (a) accepts the language L_1, i.e. prove that $L(M) = L_1$, where M is your DFA.

Solution.

To prove that $L(M) = L_1$, we want to show that $L_1 \subseteq L(M)$, and that $L(M) \subseteq L_1$.

Claim 1: $L_1 \subseteq L(M)$.

Let $w \in L_1$. Then w contains at most one 1, and $|w|$ is odd. We want to show that $w \in L_1 \implies w \in L(M)$.

Case 0: w contains no 1s. Then $w = 0^k$, where $k \geq 1$, and k is odd, i.e. $k = 2j+1$ where $j \geq 0$. Then M starts in q_0 and transitions to q_1 after reading the first 0. After the first 0 will follow j pairs of 0s. For every pair of 0s (if there are any), M will transition to q_0 after reading the first 0 of the pair, then transition back to q_1 after reading the second 0 of the pair. Thus M will be in q_1 after reading all 0 pairs j of w. Since q_1 is an accept state, M accepts w.

Case 1: w contains exactly one 1. Then $w = 0^k10^m$, where $k, m \geq 0$, and $k+1+m$ is odd. Since $k+1+m$ must be odd, $k+m$ must be even, and therefore k, m are either both even or both odd.

Case 1a: k, m are both even. Then $k = 2j$ and $m = 2n$, where $j, n \geq 0$, and so $0^k, 0^m$ consists of some number of pairs j, n of 0s. M begins in q_0, and for each pair of 0s j (if there are any), M transitions to q_1 after reading the first 0 in the pair, then transitions back to q_0 after reading the second 0 in the pair. Thus M remains in q_0 after reading all pairs of the substring 0^k. Then M reads the 1 and transitions to q_3. Then for each pair of 0s n (if there are any), M transitions to q_2 after reading the first 0 in the pair, then transitions back to q_3 after reading the second 0 in the pair. Thus M remains in q_3 after reading all pairs of the substring 0^m. So M remains in q_3 after reading all elements of w, and since q_3 is an accept state, M accepts w.

Case 1b: k, m are both odd. Then $k = 2j+1$ and $m = 2n+1$, where $j, n \geq 0$. Then 0^k, 0^m both consist of a single 0 followed by some number of pairs j, n of 0s. M begins in q_0, and after processing the first 0 of 0^k transitions to q_1. The for each pair of 0s j in 0^k (if there are any), M transitions to q_0 after reading the first 0 of the pair, then transitions back to q_1 after reading the second 0 of the pair. Thus M remains in q_1 after reading the substring 0^k. Then M reads the 1 and transitions to q_2. Then after reading the first 0 of 0^m, M transitions to q_3. Then for each n pairs of 0s in 0^m (if there are any), M transitions to q_2 after reading the first 0 in the pair, then transitions back to q_3 after reading the second 0 in the pair. Thus M remains in q_3 after reading the substring 0^m. So, M remains in q_3 after processing all elements of w. q_3 is an accept state, so M accepts w.

In all cases, M accepts w. Therefore $w \in L_1 \implies w \in L(M)$, and $L_1 \subseteq L(M)$.

Claim 2: $L(M) \subseteq L_1$, or $w \in L(M) \implies w \in L_1$. To prove this, I will prove the contrapositive: $w \notin L_1 \implies w \notin L(M)$.

Let $w \notin L_1$. Then either w has two or more 1s, or $|w|$ is even. We want to show that $w \notin L_1 \implies w \notin L(M)$.

Case 1: w contains two or more 1s. Then w takes the form 0^k10^m1x, where $k, m \geq 0$, and $x \in \{0,1\}*$. M starts in q_0. After processing the substring 0^k, M transitions back and forth between q_1 and q_0 depending on the value of k, so M remains in either q_0 or q_1 after processing the substring 0^k. Then M reads the first 1. if M is in q_0, M will transition to q_3. If M is in q_1, M will transition to q_2. So M is in either q_2 or q_3 after reading the first 1. Reading the substring 0^m will cause M to continue to transition between q_2 and q_3 depending on the value of m, so M still remains in either q_2 or q_3 after reading the substring 0^m. After reading the second 1, M then transitions to q_4, regardless of if M was in q_2 or q_3. There is no transition exiting q_4, so M remains in q_4 after reading x, which could be any string constructed from the alphabet $\{0, 1\}$, including ϵ. Thus, M remains in q_4 after reading all elements of w. q_4 is not an accept state, so $w \notin L(M)$.

Case 2: $|w|$ is even. Then w consists of some $2k$ number of characters from the alphabet, where $k \geq 0$. The flow of M will proceed as follows:

i. M begins in q_0.
ii. After processing the first character of w, M will transition to either q_1 or q_3.
iii. After processing the second character of w, if M was in q_1 it will transition to q_0 or q_2. If M was in q_3, it will transition to q_0, q_2, or q_4.
iv. After processing the next (odd) character of w, if M was in q_0, it will transition to q_1 or q_3. If M was in q_2, it will transition to q_1, q_3, or q_4. If M was in q_4, it will remain in q_4.
v. After processing the next (even) character of w, if M was in q_1, it will transition to q_0 or q_2. If M was in q_3, it will transition to q_0, q_2, or q_4. If M was in q_4, it will remain in q_4.
vi. Repeat from step iv. for the remaining characters in w.

Once M has finished reading w, since w must have an even number of characters, M will always finish after step i., iii., or v. from above. Therefore, M will always finish in either q_0, q_2, or q_4 after processing all characters of w. None of these states are accept states, so $w \notin L(M)$.

In both cases, M does not accept w. Therefore $w \notin L_1 \implies w \notin L(M)$. So $w \in L(M) \implies w \in L_1$ and thus $L(M) \subseteq L_1$.

Both Claim 1 and Claim 2 are true, therefore $L(M) = L_1$.

2. (a) Design an NFA with at most 5 states for the language

$$L_2 = \{w \in \{0,1\}^* \mid \ w \text{ contains the substring } 0101\,\}.$$

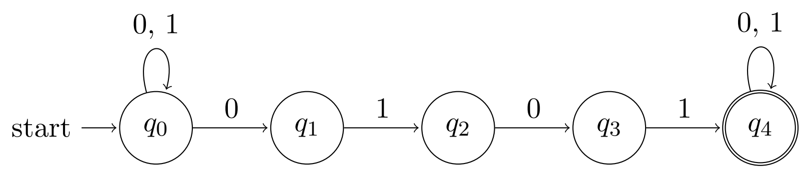
Give a state diagram for your NFA.

Solution.

My solution contains the following 5 states:

i. q_0: Seen no elements of the substring 0101. - start state
ii. q_1: Seen the first 0 of the substring.
iii. q_2: Seen the first 1 of the substring.

iv. q_3: Seen the second 0 of the substring.
v. q_4: Seen the second 1 of the substring. - accept state



(b) Prove that your NFA of part (a) accepts the language L_2, i.e. prove that $L(N) = L_2$, where N is your NFA.

Solution.

To prove that $L(N) = L_2$, we want to show that $L_2 \subseteq L(N)$, and that $L(N) \subseteq L_2$.

Claim 1: $L_2 \subseteq L(N)$. We want to show that $w \in L_2 \implies w \in L(N)$.

Let $w \in L_2$. Then w contains the substring 0101 and takes the form $x0101y$, where $x, y \in \{0,1\}*$. It is sufficient to show that there exists a sequence of states that N moves through on input w such that N ends up in an accepting state after processing w. Thus N moves through the following sequence of states:

i. N remains in q_0 after processing x (including the case where $x = \epsilon$).
ii. N transitions to q_1 after reading the first 0 of the substring 0101.
iii. N transitions to q_2 after reading the first 1 of the substring 0101.
iv. N transitions to q_3 after reading the second 0 of the substring 0101.
v. N transitions to q_4 after reading the second 1 of the substring 0101.
vi. N remains in q_4 after processing y (including the case where $y = \epsilon$).

So N remains in q_4 after processing all elements of w. q_4 is an accept state, so N accepts w. Therefore $w \in L_2 \implies w \in L(N)$, and $L_2 \subseteq L(N)$.

Claim 2: $L(N) \subseteq L_2$, or $w \in L(N) \implies w \in L_2$. To prove this, I will prove the contrapositive: $w \notin L_2 \implies w \notin L(N)$.

Let $w \notin L_2$. Then w does not contain the substring 0101. To prove by way of contradiction, suppose N accepts w. Then N is in q_4 after processing w, and w takes the form $w_1 w_2 ... w_n$, where $w_i \in \{0,1\}*$ for $1 \leq i \leq n$. All arrows transitioning from q_4 lead back into q_4, so N can never leave q_4 once it reaches this state.

Let w_{p-1} be the first symbol effecting a transition from a state different from q_4 into q_4. Then N moves to, and remains in, state q_4 on input symbols $w_p, w_{p+1}, ..., w_n$. The only transition into q_4 from a state q different from q_4 is the transition $q_3 \xrightarrow{1} q_4$. So $w_{p-1} = 1$ and $q = q_3$. Furthermore, the only way to reach q_3 is from q_2 on input symbol 0, so $w_{p-2} = 0$. The only way to reach q_2 is from q_1 on input symbol 1, so $w_{p-3} = 1$. The only way to reach q_1 is from q_0 on input symbol 0, so $w_{p-4} = 0$. Thus w contains the substring $w_{p-4}w_{p-3}w_{p-2}w_{p-1} = 0101$, and thus $w \in L_2$. This contradicts our assumption that $w \notin L_2$, thus our assumption that N accepts w is false and N does not accept w. Therefore $w \notin L(N)$, and so $w \notin L_2 \implies w \notin L(N)$. Then $w \in L(N) \implies w \in L_2$, and $L(N) \subseteq L_2$.

Both Claim 1 and Claim 2 are true, therefore $L(N) = L_2$.