Fullness of the Mandelfungus Product

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Let $S = \{(c,z) : c \in M, z \in K_c\}$. We show that S is full in $\mathbb{C}^2$.

First note that both M and K_c for every $c \in M$ are both contained in the disk $D = \{z \in \mathbb{C} : |z| \leq 2\}$.

To see that this is true, first suppose that $|c| > 2$, and let $f(z) = z^2 + c$. We claim that $|f^{\circ n}(0)| \geq |c|(|c|-1)^n$. This would then show that the orbit of 0 diverges, showing in turn that $c \notin M$. To prove the claim, note first that $|f(0)| = |c^2 + c| \geq |c|^2 - |c| = |c|(|c|-1)$. Now inductively,

$$\begin{aligned}
|f^{\circ n+1}(0)| &= |f^{\circ n}(0)^2 + c| \\
&\geq |f^{\circ n}(0)|^2 - |c| \\
&= |f^{\circ n}(0)|(|f^{\circ n}(0)| - \frac{|c|}{|f^{\circ n}(0)|} \\
&\geq |c|(|c|-1)^n(|c|-1),
\end{aligned}$$

proving the claim. Therefore, $M \subseteq D$.

Now if $c \in M$, we also want to show that $K_c \subseteq D$. This follows in much the same way as above. For if $|z| \geq 2 \geq |c|$, we have that

$$\begin{aligned}
|z^2 + c| &\geq |z|^2 - |c| \\
&\geq |z|^2 - |z| \\
&\geq |z||c| - |z| \\
&\geq |z|(|c|-1).
\end{aligned}$$

and so

$$|f^{\circ n}(z)| \geq (|c|-1)^n |z|.$$

Therefore, the orbit of such z is unbounded, and therefore $K_c \subseteq D$.

This shows that the set S is clearly bounded. We just need to show that S^c is connected. But we have that

$$S^c = \{(c,z) : c \in M, z \notin K_c\} \cup \{(c,z) : c \notin M, z \in \mathbb{C}\} = A \cup B.$$

Note that clearly B, as a product of two connected spaces, is connected.

Now consider the space A. For each $c \in M$, we know that since the single critical point of the map $f(z) = z^2 + c$ lies in K_c, the entire basin of ∞ in $\mathbb{C}$, namely $\hat{\mathbb{C}} \setminus K_c$, is homeomorphic to a punctured disk $D \setminus \{0\}$ via a homeomorphism φ_c (see, for example, Milnor Theorem 9.3). We show that A is the total space of a fiber bundle $(E = A, B, \pi, F)$ with base space $B = M$, projection $\pi : E \to B$ given by $\pi(c,z) = c$, and fiber $F \cong D \setminus \{0\}$.

To see this, let $(z_0, c_0) \in E$ and consider $c_0 = \pi(c_0, z_0)$. Let $U \subseteq M$ be an open neighborhood of c_0. Then we have the following diagram:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "proj_1"] \\
U &
\end{tikzcd}$$

with $\varphi(c,z) = (c, \varphi_c(z))$, and it is easy to see that this diagram commutes. Hence we see that E is in fact the total space of a fiber bundle.

Now we claim that, given a fiber bundle with connected base space B and connected fibers F, the total space E must be connected as well. To see this, suppose by means of contradiction that $E = A_1 \sqcup A_2$.Let $x \in A_1$, with U a (connected) neighborhood of $\pi(x)$. Then $\pi^{-1}(U) \cap A_1 \neq \varnothing$ since $x \in \pi^{-1}(U) \cap A_1$, and $\pi^{-1}(U) \cong U \times F$ which, as a product of connected spaces, must be connected. Therefore, we must have that $\pi^{-1}(U)$ must be entirely contained in A_1. Similarly, if $y \in A_2$ with neighborhood V of $\pi(y)$, Then $\pi^{-1}(V)$ must be entirely contained in A_2.

But then, since A_1 and A_2 disconnect E and hence are open, and since π is an open map, $\pi(A_1)$ and $\pi(A_2)$ are open sets with empty intersection whose union is B, and hence $\pi(A_1)$ and $\pi(A_2)$ disconnected B, a contradiction.

Therefore, we see that our original set $A \subseteq S^c$, as the total space of a fiber bundle with connected fibers and connected base space, must also be connected.

Then only other option is that A and B themselves give a separation of E. However, note if we let $c \in \partial M$ and $z \in K_c$, then $(z, c) \in A \cap \overline{B}$, so this is not the case. Therefore, E^c is connected and therefore E is full.