COMS 511 - Homework 6

Due: March 24 11:59 PM

Guidelines

- When proofs are required, you should make them both clear and rigorous. Do not hand-waive.
- Your assignment needs to be submitted via Canvas.
  - You **must** type your solutions. Please submit a PDF version.
  - Please make sure that the file you submit is not corrupted and that its size is reasonable (e.g., roughly at most 10-11 MB).

    If we cannot open your file, your homework will not be graded.
- The following are examples of activities that are prohibited:
  - Sharing solutions or fragments of solutions (e.g., via email, whiteboard, handwritten, or printed copies).
  - Post solutions or fragments of solutions in a location accessible to others.
  - Using solutions or fragments of solutions provided by other students (including students who had taken the course in the past).
  - Using solutions or solution fragments obtained on the Internet or from solution manuals for textbooks.
  - Using material from textbooks, reference books, or research articles without properly acknowledging and citing the source.
- Concerns about grading should be expressed within one week of returning the homework.
- **No late homework is accepted** with the exception of at most one late submission up to 12 hours late.

Problems

Problem 1. (50 points) In the real world, it is unrealistic to expect that each man and woman can provide a strict preference ordering for all his or her potential mates. To reflect this, let's suppose we allow each person's preference list to contain ties. Mathematically, a set W of k women forms a tie of length k in the preference list of man m if m does not prefer w_i to w_j for any $w_i, w_j \in W$ (i.e., m is indifferent between w_i and w_j), while for any other woman w not in W, either m prefers w to all women in W or m prefers all the women in W to w. A tie on a woman's list is defined analogously. This is the *Stable Marriage Problem with Ties*. Recall that in the original Stable Marriage Problem, where preference lists are strictly ordered, it is always possible to find a stable matching, where a matching is stable if there is no man x and woman y such that x and y both prefer each other over their current partners. However, for the Stable Marriage Problem with Ties, we define two different types of stability:

- Weak stability. A matching is weakly stable if there is no couple x and y, each of whom strictly prefers the other to their current partner in the matching.
- Super stability. A matching is super-stable if there is no couple x and y, each of whom either strictly prefers the other to his/her partner or is indifferent between them.

(a) Does a weakly stable matching always exist for the Stable Marriage Problem with Ties? Either prove the statement or provide a counterexample.

(b) Assume we are given an instance of the Stable Marriage Problem with Ties and a weakly stable matching M for that instance. Upon getting married to their partners assigned in M, each person's

preferences change slightly, and the married partner becomes preferred over anyone they were tied with, while other rankings remain the same. Is M now super-stable?

Problem 2. (50 points) Graph coloring is an elementary optimization problem that has been studied intensively for a long time. Here we consider the *vertex coloring problem* where colors are assigned to the vertices of a graph such that no pair of adjacent vertices have the same color and the number of colors is minimized. This minimum number is called the chromatic number of the graph and is denoted by κ. Computing the chromatic number of a graph is NP-hard.

Consequently, in practice, using an ILP solver for the vertex coloring problem might be a reasonable option to address this NP-hard problem. Give an ILP formulation for the vertex coloring problem and prove the correctness of this formulation. Also, provide a tight asymptotic bound of the size of the formulation.