General Rules for constructing Root Locus

Obtain the characteristics equation of the system $1 + G(s)H(s) = 0$. Rearrange the characteristics equation in the form

$$1 + K\frac{(s+z_1)(s+z_2)\ldots(s+z_n)}{(s+p_1)(s+p_2)\ldots(s+p_n)} = 0$$

Or, $1 + K\dfrac{Q(s)}{P(s)} = 0$

The gain parameter K is increased from zero to infinity and the roots of the characteristics equation (or poles of system) are plotted in s-plane for all values of gain K.

Where does root loci start and how does it move?

$$1 + K\frac{Q(s)}{P(s)} = 0$$

Or, $P(s) + KQ(s) = 0$

When K = 0;

$P(s) = 0$

When K→∞; The KQ(s) term becomes dominant and P(s) can be neglected, so

$Q(s) = 0$

So we can say that when ‘K’ is increased from 0 to ∞ , the closed loop poles of a system (roots of polynomial P(s) + KQ(s)) moves from open loop poles (roots of polynomial P(s)) to open loop zeros (roots of polynomial Q(s)).

Rule-1: ***As K is increased from 0 to ∞ , the roots of the characteristics equation or closed loop poles move from the open loop poles of the system to the open loop zeros of the system.***

Nature of Root Locus about Real Axis:

The poles and zeros of any physical system are either real or complex. The complex poles and zeros always exist in conjugate pair. So the root locus is symmetrical about the real axis.

Rule-2: ***The Root Locus is symmetrical about the real axis.***

Roots and Parameter K Relationship:

No value of ‘s’ corresponds to more than one value of ‘K’. But for single value of ‘K’ there may exist multiple roots.

Rule 3: ***Root locus branch will never cross it’s own path. However the two paths originating from different root may cross each other.***

Existance of Root Locus on Real axis:

Consider a point ‘p’ on the real axis. Then the open loop poles or zeros of the system which are at right of the test point ‘p’ will contribute the angle of 180^o to phase of G(s)H(s). Similarly any poles or zeros of the system which are at left of the test point ‘p’ will contribute angle of 0^o to phase of G(s)H(s). The net angle of contribution of a complex conjugate pole or zero is always zero.

Any point on s-plane is part of the root locus if the phase angle criterion is satisfied at that point. To satisfy this condition the total number of open loop poles and zeros to the right of any point should be odd.
Rule-4: ***If the total number of real poles and real zeros to the right of the any segment on the real axis is odd, then that segment of the real axis is part of the root locus.***

Number of Branches and Nature of Branches:

We know that the root locus travels from open loop poles to open loop zeros when K is varied from 0 to ∞. What if the number of open loop zeros are less than that of number of poles? What if the number of open loop zeros are more than that of the number of open loop poles?

Rule-5-a: ***There are ‘n’ lines or loci where ‘n’ is the degree of Q(s) {open loop zeros } or P(s) {open loop poles }, whichever is the greater.***

Rule-5-b: ***If there are same number of open loop poles and open loop zeros then each pole and zero forms a pair. The root travels from a pole to a corresponding zero.***

Rule-5-c: ***If there are more open loop poles than the open loop zeros then the line from extra poles go off to infinity.***

Rule-5-d: ***If there are more number of open loop zeros than the number of open loop poles then the extra line come from infinity.***

Root Locus entering or leaving the real axis

When the root locus leave the real axis in search of open loop zeros or when the root locus enters the real axis in search of open loop zeros they do so at 90^o with the real axis.

Rule-6: ***Root locus leave or enter into the real axis always at*** $\pm 90^o$ ***with the positive real axis.***

Root Locus Going to Infinity or Coming From Infinity

We know that the root locus may go to infinity in search of open loop pole or come from infinity to a finite open loop zeros. But in which direction the root locus goes or come from infinity? At what angle with the real axis the root locus goes or come from infinity?

Let us consider a open loop zero at ∞ . The open loop zero is far away from the real axis. If we see the finite poles and zeros in the comlex s-plane from this open loop zero at infinity then these finite poles and zeros seems like concentrated at a point. So we can consider that all the vectors running from these open loop poles and zeros to open loop zero at the infinity make same angle θ with the real axis. This single line going to open loop pole at infinity is a asymptote. The root locus will go to infinity along this asympote.

Since this open loop zero at infinity will be part of root locus, we can apply the angle criterion at this zero.

$$\angle G(s)H(s) = \angle \pm 180(2k+1)$$
$$number\ of\ zeros\ x\ \theta - number\ of\ poles\ x\ \theta = \angle \pm 180(2k+1)$$
$$(n_z - n_p)\ x\ \theta = \angle \pm 180(2k+1)$$
$$\text{Angle of Asymptote, } \theta = \frac{\angle \pm 180(2k+1)}{n_z - n_p}$$

where 'n_p' is number of finite open loop poles
'n_z' is number of finite open loop zeros

Rule-6-a: ***The root loci goes to infinity along the asymptotes.***

Rule-6-b: ***There are as many asympotes as there are unmatched pole-zero pair.***

Rule-6-c: ***The angle of asymptotes is given by***

$$Angle\ of\ Asymptotes, \theta_A = \frac{\angle \pm 180^o(2k+1)}{n_p - n_z}$$

Rule-6-d: ***The asympotes starts at centroid of all finite open loop poles and zeros. The centroid of asymptote is given as*** $Centroid\ of\ Asymptotes, C = \frac{\sum finite\ poles - \sum finite\ zeros}{n_p - n_z}$

Some Example of Asymptotes angle calculation:

For 1 extra pole; $\Phi_{A1} = \frac{2x0+1}{1} \times 180^o = 180^o$

For 2 extra pole; $\Phi_{A1} = \frac{2x0+1}{2} \times 180^o = 90^o$; $\Phi_{A2} = \frac{2x1+1}{2} \times 180^o = 270^o = -90^o$

For 3 extra pole; $\Phi_{A1} = \frac{2x0+1}{3} \times 180^o = 60^o$; $\Phi_{A2} = \frac{2x1+1}{3} \times 180^o = 180^o$; $\Phi_{A3} = \frac{2x2+1}{3} \times 180^o = 300^o = -60^o$

For 4 extra pole; $\Phi_{A1}=\frac{2x0+1}{4}\times 180^o = 45^o;\quad \Phi_{A2}=\frac{2x1+1}{4}\times 180^o = 135^o$

$\Phi_{A3}=\frac{2x2+1}{4}\times 180^o = 225^o = -135^o;\quad \Phi_{A4}=\frac{2x3+1}{4}\times 180^o = 315^o = -45^o$

Crashing of Root Loci

Sometime the two root loci come towards each other and crash each other at some point either in the real axis or at some complex point. After crashing the individual root locus will move in different directions searching for open loop zero. The point where two root locus meet is known as breakaway point or break in point. But at what point in the s-plane two root locus carash?

We have characteristics polynomial of a closed loop system as $f(s) = 1 + G(s)H(s)$

Case-I: When system has simple non repeated closed loop poles

$$f(s) = 1 + G(s)H(s) = (s+p_1)(s+p_2)\ldots(s+p_n)$$

Differentiating above equation with respect to 's' we have

$$\frac{df(s)}{ds} = \frac{d[(s+p_1)(s+p_2)\ldots(s+p_n)]}{ds}$$

$$\frac{df(s)}{ds} = (s+p_2)\ldots(s+p_n)\frac{d(s+p_1)}{ds} + (s+p_1)\frac{d[(s+p_2)\ldots(s+p_n)]}{ds}$$

$$\frac{df(s)}{ds} = (s+p_2)..(s+p_n)+(s+p_1)(s+p_3)..(s+p_n)\frac{d(s+p_2)}{ds}+(s+p_1)(s+p_2)\frac{d[(s+p_3)..(s+p_n)]}{ds}$$

$$\frac{df(s)}{ds} = \sum_{i=1}^{i=n}\left(\prod_{j=1}^{j=n}\frac{s+p_j}{s+p_i}\right)$$

For unique poles the above first derivative of characteristics polynomial will never equal to zero.

For example: Let $f(s) = 1 + G(s)H(s) = (s+2)(s+3)(s-4)$ then;

$$\frac{df(s)}{ds} = \frac{d[(s+2)(s+3)(s-4)]}{ds}$$

Or, $\frac{df(s)}{ds} = (s+3)(s-4)\frac{d(s+2)}{ds} + (s+2)\frac{d[(s+3)(s-4)]}{ds}$

Or, $\frac{df(s)}{ds} = (s+3)(s-4)+(s+2)(s-4)\frac{d(s+3)}{ds}+(s+2)(s+3)\frac{d(s-4)}{ds}$

Or, $\frac{df(s)}{ds} = (s+3)(s-4)+(s+2)(s-4)+(s+2)(s+3)$

No values of 's' can make the above first derivative equal to zero.

Case-II: When system has repeated closed loop poles:

$$f(s) = 1 + G(s)H(s) = (s + p_1)^r (s + p_2) \ldots (s + p_n)$$

Differentiating above equation with respect to 's' we have

$$\frac{df(s)}{ds} = \frac{d[(s + p_1)^r (s + p_2) \ldots (s + p_n)]}{ds}$$

Or, $$\frac{df(s)}{ds} = (s + p_2) \ldots (s + p_n) \frac{d(s + p_1)^r}{ds} + (s + p_1)^r \frac{d[(s + p_2) \ldots (s + p_n)]}{ds}$$

Or, $$\frac{df(s)}{ds} = r(s+p_2)..(s+p_n)(s+p_1)^{r-1} + (s+p_1)(s+p_3)..(s+p_n)\frac{d(s+p_2)}{ds} + (s+p_1)(s+p_2)\frac{d[(s+p_3)..(s+p_n)]}{ds}$$

For s = -p_1 the above first derivative of characteristics polynomial will equal to zero.

i.e $$\left.\frac{df(s)}{ds}\right|_{s=-p_1} = 0$$

For example: Let $f(s) = 1 + G(s)H(s) = (s + 2)^2 (s + 3)(s - 4)$ then;

$$\frac{df(s)}{ds} = \frac{d[(s + 2)^2 (s + 3)(s - 4)]}{ds}$$

Or, $$\frac{df(s)}{ds} = (s + 3)(s - 4)\frac{d(s + 2)^2}{ds} + (s + 2)\frac{d[(s + 3)(s - 4)]}{ds}$$

Or, $$\frac{df(s)}{ds} = 2(s+3)(s-4)(s+2) + (s+2)(s-4)\frac{d(s+3)}{ds} + (s+2)(s+3)\frac{d(s-4)}{ds}$$

Or, $$\frac{df(s)}{ds} = 2(s+3)(s-4)(s+2) + (s+2)(s-4) + (s+2)(s+3)$$

If we evaluate above derivative at s = -2 then

$$\left.\frac{df(s)}{ds}\right|_{s=-2} = 0 + 0 + 0 = 0$$

This is the unique property of the characteristics polynomial. If characteristics polynomial has multiple roots (at least two roots) at some point 's' in s-plane then at that point the first derivative of the characteristics polynomial will be zero.

At the breakaway or breakin point of two root locus the system has two poles. If 'n' root locus crash at a point in s-plane then the system will have 'n' number of miltiple poles at that point. So using the above property of characteristics polynomial $\frac{df(s)}{ds} = 0$, we can solve for the value of 's' at which the root locus crash.

We have characteristics polynomial of a closed loop system as

$$f(s) = 1 + G(s)H(s)$$

Or, $f(s) = 1 + K\dfrac{Q(s)}{P(s)}$

The characteristics equation is f(s) = 0

Or, $1 + K\dfrac{Q(s)}{P(s)} = 0$

$$\therefore P(s) + KQ(s) = 0 ------(i)$$

From equation (i)

$$K = -\frac{P(s)}{Q(s)}$$

$$\frac{dK}{ds} = -\frac{Q(s)\dfrac{d\ P(s)}{ds} - P(s)\dfrac{d\ Q(s)}{ds}}{Q^2(s)}$$

$$\frac{dK}{ds} = \frac{P(s)Q'(s) - Q(s)P'(s)}{Q^2(s)} ------(ii)$$

At root locus crashing point 's'

$$\frac{df(s)}{ds} = 0$$

Or, $0 + K\dfrac{d\left(\dfrac{Q(s)}{P(s)}\right)}{ds} = 0$

Or, $K\dfrac{P(s)\ Q'(s) - Q(s)\ P'(s)}{P^2(s)} = 0$

$$\therefore P(s)\ Q'(s) - Q(s)\ P'(s) = 0------(iii)$$

From equation ii and iii

$$\frac{dK}{ds} = \frac{P(s)Q'(s) - Q(s)P'(s)}{Q^2(s)} = 0------(iv)$$

Using Equation (iv) we can solve for the value of 's'. The values of 's' that lies in the root locus will be the actual breakaway or breakin point of the crashing root locus.

Rule-7: ***The breakaway or breakin point of the root loci can be determined by solving for the roots of the equation*** $\dfrac{dK}{ds} = \dfrac{P(s)Q'(s) - Q(s)P'(s)}{Q^2(s)} = 0$ ***. Those roots that lies in the root locus are the breakaway or breakin point.***

Root Locus at Complex Pole or Zero

On increasing the parameter under consideration the root locus depeart away from the complex pole. When the gain is infinite the root locus will arrive at the open loop zero. At what angle the root locus arrive at complex open loop zero? At what angle the root locus depart away from the complex open loop pole?

Let us consider a system with ‘n’ number of open loop poles and ‘m’ number of open loop zeros. Consider a test point S' near a complex pole S_{pj}. Let the phasor from the complex pole S_{pj} to the test point S' makes an angle ϕ_d with the real axis. Since the test point is very near to the complex pole S_{pj}, we cosider the angle contribution of other finite open loop poles and zeros to the test point S' to be same as the angle contribution to the complex pole S_{pj}. Let α_{pi-pj} represent the angle of a vector from the ith open loop pole to the complex pole under consideration S_{pj}, β_{zi-pj} represent the angle of a vector from the ith open loop zero to the complex pole under consideration S_{pj}.

For the test point S' to be part of root locus it must satisfy the angle criterion. So applying the angle criterion at the test point.

$$\sum_{i=1}^{i=m} \beta_{zi-pj} - \sum_{i=1}^{i=n} \alpha_{pi-pj} - \phi_d = \pm 180(2k+1)$$

$$\text{Angle of departure at complex pole } S_{pj},\ \phi_d = \pm 180(2k+1) + \sum_{i=1}^{i=m} \beta_{zi-pj} - \sum_{i=1}^{i=n} \alpha_{pi-pj}$$

Similarly, the angle of arrival ϕ_a at any complex zero S_{zj} can be written as

$$\sum_{i=1}^{i=m} \gamma_{zi-zj} + \phi_a - \sum_{i=1}^{i=n} \theta_{pi-zj} = \pm 180(2k+1)$$

$$\text{Angle of arrival at complex zero } S_{zj},\ \phi_a = \pm 180(2k+1) + \sum_{i=1}^{i=n} \theta_{pi-zj} - \sum_{i=1}^{i=m} \gamma_{zi-zj}$$

θ_{pi-zj} represents the angle from ith open loop pole to complex zero under consideration S_{zj} where as, γ_{zi-zj} represents the angle from ith open loop zero to complex zero under consideration S_{zj}.

Rule-8-a: ***The angle of departure of a root locus from a complex pole is given by***

$$\phi_d = \pm 180(2k+1) + \sum_{i=1}^{i=m} \beta_{zi-pj} - \sum_{i=1}^{i=n} \alpha_{pi-pj}$$

Rule-8-b: ***The angle of arrival of a root locus at complex zero is given by***

$$\phi_a = \pm 180(2k+1) + \sum_{i=1}^{i=n} \theta_{pi-zj} - \sum_{i=1}^{i=m} \gamma_{zi-zj}$$

Root Locus and Imaginary Axis

The root locus might cross the imaginary axis. If it does the point of intersection of the root locus with the imaginary axis can be found either by using the Routh Stability Criterion or by substituting s=jw in the characteristics equation. We can solve for the value of w and gain K. The value of w is the frequency at which the root locus cross the imaginary axis.

Parameter K at any point on Root Locus

The value of parameter 'K' can be determined at any point in the root locus using the magnitude criterion. The value of 'K' at any test point 'S' on the root locus is

$$K = \frac{\text{Product of Lengths of the line joining the open loop poles and test point S}}{\text{Product of Lenghts of the line joining the open loop zeros and test point S}}$$