1 Where do state space equations come from?

State space prototype form for this course:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad x(0) = x_0, \\ y(t) &= C(t)x(t) + D(t)u(t). \end{aligned}$$

Most of the time we will deal with LTI.
We will also play with discrete-time systems.

1.1 Modeling is art!

- Basic principles (physical, chemical, monetary, etc).
- System identification (black box, parametric, grey box, etc).

1.2 Example 1: Vector second order systems

Mechanical systems (y = position, $\dot{y}$ = velocity, $\ddot{y}$ = acceleration)

$$M\ddot{y} + D\dot{y} + Ky = u. \tag{1}$$

Define

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \dot{y} \\ y \end{pmatrix}, \quad \Rightarrow \quad \dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \ddot{y} \\ \dot{y} \end{pmatrix},$$

and add the dummy equation

$$\dot{x}_2 = x_1.$$

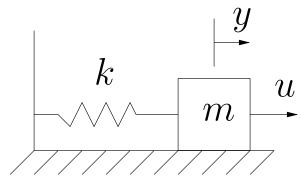
Assume M is invertible to rewrite (1) as

$$\dot{x}_1 = -M^{-1}Dx_1 - M^{-1}Kx_2 + M^{-1}u,$$

so that

$$\begin{aligned} \underbrace{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}}_{\dot{x}} &= \underbrace{\begin{bmatrix} -M^{-1}D & -M^{-1}K \\ I & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{Ax} + \underbrace{\begin{bmatrix} M^{-1} \\ 0 \end{bmatrix} u.}_{Bu} \\ y &= \underbrace{\begin{bmatrix} 0 & I \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{Cx} \end{aligned}$$

1.3 Example 2: Mass-spring system



Balance of forces

$$m\ddot{y} = u - ky - d\dot{y},$$

where m : mass
k : spring coefficient (stiffness)
d : viscous friction coefficient

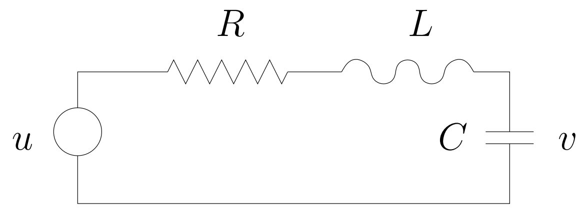
Rearranging terms

$$m\ddot{y} + d\dot{y} + ky = u,$$

and using formula for vector second order system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -d/m & -k/m \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1/m \\ 0 \end{bmatrix} u, \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x. \end{aligned}$$

1.4 Example 3: RLC circuit



Balance of voltages and currents

$$Ri + L\frac{di}{dt} + v = u,$$
$$i = C\frac{dv}{dt}.$$

where R : resistance
C : capacitance
L : inductance

Rearranging terms

$$\frac{di}{dt} = \frac{1}{L}(-Ri - v + u),$$
$$\frac{dv}{dt} = \frac{1}{C}i.$$

Define

$$x = \begin{pmatrix} i \\ v \end{pmatrix},$$

to write

$$\dot{x} = \begin{bmatrix} -R/L & -1/L \\ 1/C & 0 \end{bmatrix} x + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

1.5 More examples

Check out Skelton, DSC, Chapter 3.

2 From differential equations to transfer functions

Starting with the linear differential equation

$$\ddot{y} + a_1 \dot{y} + a_2 y = b_0 \ddot{u} + b_1 \dot{u} + b_2 u$$

take Laplace transforms

$$(s^2 + a_1 s + a_2) Y(s) = (b_0 s^2 + b_1 s + b_2) U(s),$$

so that

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2}.$$

Prototype LTI is

$$H(s) = \frac{N(s)}{D(s)},$$

where $N(s)$ and $D(s)$ are polynomials, $H(s)$ is rational.

WARNING: When $H(s)$ is not rational, system is still LTI but will not fit our prototype state space form. Example: linear system with delay

$$\dot{y}(t) = u(t - \tau) \quad \Leftrightarrow \quad H(s) = e^{-s\tau}/s \quad \Leftrightarrow \quad \begin{cases} \dot{x}(t) = u(t) \\ y(t) = x(t - \tau) \end{cases}$$

By the way, what is the state in this case?

2.1 Discrete-time systems

Starting with the linear difference equation

$$y(k+2) + a_1 y(k+1) + a_2 y(k) = b_0 u(k+2) + b_1 u(k+1) + b_2 u(k)$$

take Z transforms

$$(z^2 + a_1 z + a_2) Y(z) = (b_0 z^2 + b_1 z + b_2) U(z),$$

so that

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2}.$$

2.2 More terminology

A rational transfer function H is
Proper when degree of $D \geq$ degree of N,
Strictly proper when degree of $D >$ degree of N.

2.3 Properness and causality

If H is not proper, then H is not causal.
In discrete time:

$$H(z) = z \quad \Leftrightarrow \quad y(k) = u(k+1)$$

In continuous time:

$$H(s) = s \quad \Leftrightarrow \quad y(t) = \dot{u}(t)$$

Remember that

$$\dot{u}(t) = \frac{du}{dt} = \lim_{\epsilon \to 0} \frac{u(t+\epsilon) - u(t)}{\epsilon}$$

This requires continuity at all t, i.e.,

$$\frac{du}{dt} = \lim_{\epsilon \to 0^-} \frac{u(t+\epsilon) - u(t)}{\epsilon} = \lim_{\epsilon \to 0^+} \frac{u(t+\epsilon) - u(t)}{\epsilon}$$

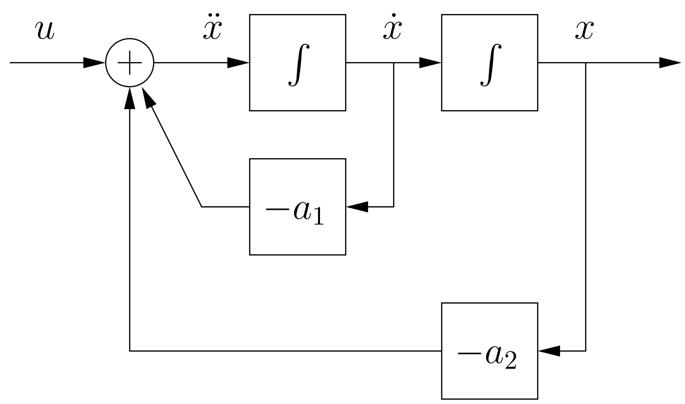
3 Realizations of differential equations (analog simulation)

3.1 Kelvin's scheme

How can we simulate the linear differential equation

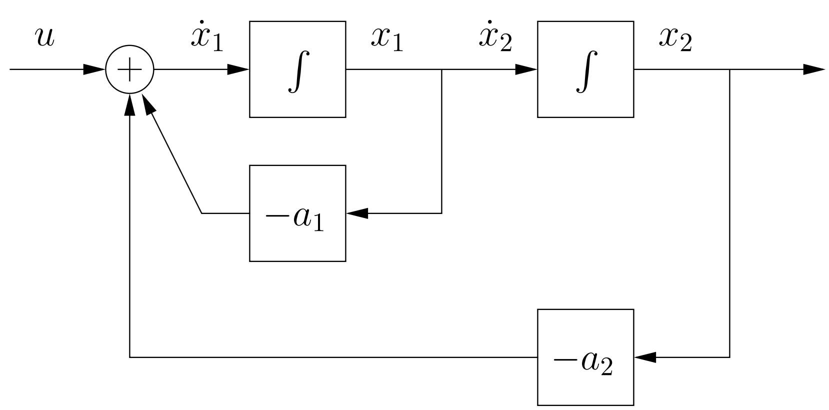
$$\ddot{x} + a_1\dot{x} + a_2 x = u.$$

We cannot use differentiator (non causal, noisy, ...). Solution: INTEGRATORS!



From the diagram to state space

1. Each integrator output is a state
2. Each integrator input is an equation



State: (x_1, x_2)
Equations:

$$\begin{cases} \dot{x}_1 = -a_1 x_1 - a_2 x_2 + u, \\ \dot{x}_2 = x_1 \end{cases} \quad \Longrightarrow \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

3.2 A more general problem

How can we simulate the linear differential equation

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_1\ddot{u} + b_2\dot{u} + b_3 u.$$

3.2.1 First solution: controller realization

Consider

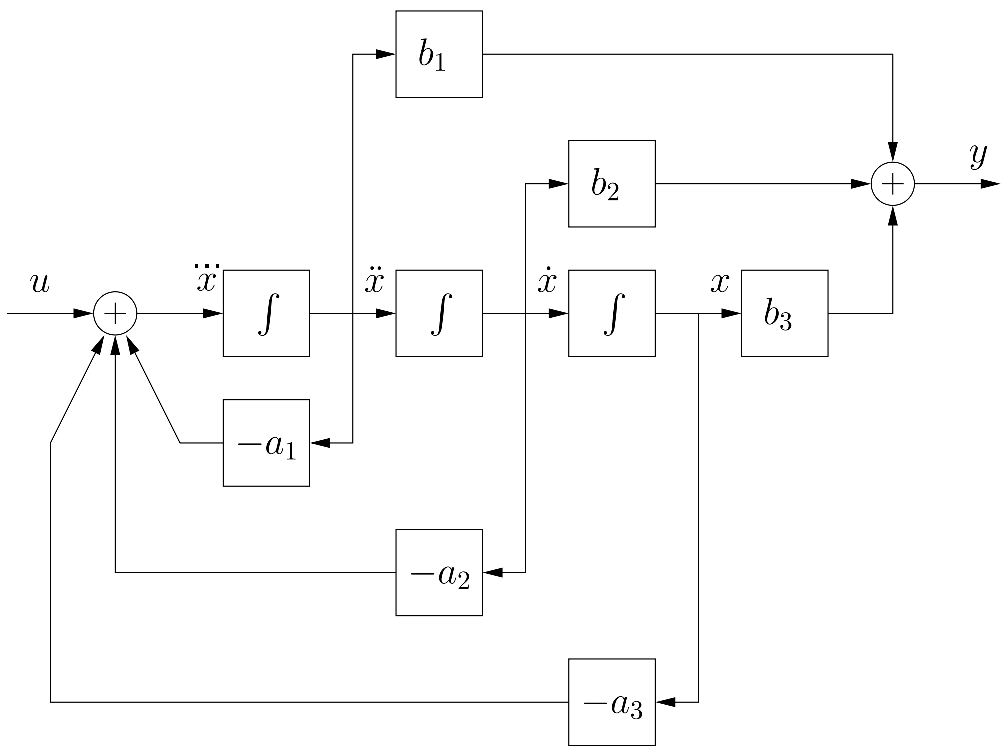
$$\dddot{x} + a_1\ddot{x} + a_2\dot{x} + a_3 x = u.$$

Then use linearity! Recall that

$$u \to x, \qquad \dot{u} \to \dot{x}, \qquad \ddot{u} \to \ddot{x}$$

so that

$$y = b_1\ddot{x} + b_2\dot{x} + b_3 x.$$



State space equations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$