CH5120-Assignment-1: Linear Algebra

Q.No-1 What is the correct vector space and corresponding dimensions of the four fundamental subspaces of the matrix M (C(M), R(M), N(M), L(M))? (Note: C(M) is the column space of M, N(M) is the null space of M, R(M) is the row space of M, and L(M) is the left null space of M. Dimension of a subspace is the number of basis vectors spanning the subspace)

$$M = \begin{bmatrix} 0 & 2 & 3 & 1 \\ -6 & -1 & -1 & 1 \\ 8 & 8 & 6 & 3 \end{bmatrix}$$

Reducing matrix M to row echelon form

$R_1 \rightarrow R_3$

$$M = \begin{bmatrix} 8 & 8 & 6 & 3 \\ -6 & -1 & -1 & 1 \\ 0 & 2 & 3 & 1 \end{bmatrix}$$

$R_2 \rightarrow 4R_2 + 3R_1$

$$M = \begin{bmatrix} 8 & 8 & 6 & 3 \\ 0 & 20 & 14 & 13 \\ 0 & 2 & 3 & 1 \end{bmatrix}$$

$R_1 \rightarrow 5R_1 - 2R_2$; $R_3 \rightarrow 10R_3 - R_2$

$$M = \begin{bmatrix} 40 & 0 & 2 & -11 \\ 0 & 20 & 14 & 13 \\ 0 & 0 & 16 & -3 \end{bmatrix}$$

$R_1 \rightarrow 8R_1 - R_3$; $R_2 \rightarrow 8R_2 - 7R_3$

$$M = \begin{bmatrix} 320 & 0 & 0 & -85 \\ 0 & 160 & 0 & 125 \\ 0 & 0 & 16 & -3 \end{bmatrix}$$

column space, C(M) = linear combination of vectors corresponding to pivot columns

$$\therefore C(M) = \text{span}\left(\begin{bmatrix} 0 \\ -6 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix} \right)$$

$$\text{Dim}\left(C(M)\right) = 3$$

To find Null space:

$$A\vec{x} = \vec{0}$$

$$\begin{bmatrix} 320 & 0 & 0 & -85 \\ 0 & 160 & 0 & 125 \\ 0 & 0 & 16 & -3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$320u - 85x = 0$; $160v + 125x = 0$; $16w - 3x = 0$

$\Rightarrow u = \frac{85}{320}x = \frac{17}{64}x$ $\quad \Rightarrow v = -\frac{125}{160}x = \frac{-25}{32}x$ $\quad \therefore w = \frac{3}{16}x$

$$\therefore \text{Null space, } N(M) = \text{span}\left(\begin{bmatrix} 17/64 \\ -25/32 \\ 3/16 \\ 1 \end{bmatrix} \right) ; \quad \text{Dim}(N(M)) = 1$$

$$\text{Row space, } R(M) = \text{span}\left(\begin{bmatrix} 0 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ 8 \\ 6 \\ 3 \end{bmatrix} \right) ; \quad \text{Dim}(R(M)) = 3$$

To find left Null space:

$$M^T = \begin{bmatrix} 0 & -6 & 8 \\ 2 & -1 & 8 \\ 3 & -1 & 6 \\ 1 & 1 & 3 \end{bmatrix}$$

$R_1 \rightarrow R_4$

$$M^T = \begin{bmatrix} 1 & 1 & 3 \\ 2 & -1 & 8 \\ 3 & -1 & 6 \\ 0 & -6 & 8 \end{bmatrix} ; \quad \begin{aligned} R_2 &\rightarrow R_2 - 2R_1 \\ R_3 &\rightarrow R_3 - 3R_1 \end{aligned}$$

$$M^T = \begin{bmatrix} 1 & 1 & 3 \\ 0 & -3 & 2 \\ 0 & -4 & -3 \\ 0 & -6 & 8 \end{bmatrix} ; \quad \begin{aligned} R_1 &\rightarrow R_1 + \tfrac{1}{3}R_2 \\ R_3 &\rightarrow R_3 - \tfrac{4}{3}R_2 \\ R_4 &\rightarrow R_4 - 2R_2 \end{aligned}$$

$$M^T = \begin{bmatrix} 1 & 0 & 11/3 \\ 0 & -3 & 2 \\ 0 & 0 & -17/3 \\ 0 & 0 & 4 \end{bmatrix} ; \quad \begin{aligned} R_1 &\rightarrow R_1 + \tfrac{3}{17} \times \tfrac{11}{3} R_3 \\ R_2 &\rightarrow R_2 + \tfrac{3}{17} \times 2 R_3 \\ R_4 &\rightarrow R_4 + \tfrac{3}{17} \times 4 R_3 \end{aligned}$$

$$M^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -17/3 \\ 0 & 0 & 0 \end{bmatrix}$$

So no non trivial left null space exist.

$\therefore$ Left null space, $L(M) = \{\vec{0}\} =$

$$\text{Dim}\big(L(M)\big) = 0$$

Q.No-4 Find the sum of the eigen values of the 3-dimensional matrix

$$A = \begin{bmatrix} -6 & -9 & 6 \\ -2 & 3 & -4 \\ 7 & -4 & -2 \end{bmatrix}$$

$\rightarrow$ sum of eigen values of A = trace of $A = -6 + 3 - 2 = -5$

Q.No-5 Find the product of the eigen values of the 3-dimensional matrix

$$A = \begin{bmatrix} -8 & -4 & -9 \\ -5 & 5 & -4 \\ 2 & -4 & 1 \end{bmatrix}$$

$\rightarrow$ Product of eigen values = determinant of matrix

$= 10$

```
>> A=[-8 -4 -9; -5 5 -4; 2 -4 1]

A =

    -8    -4    -9
    -5     5    -4
     2    -4     1

>> det(A)

ans =

   10.0000 ←

>> L=eig(A)

L =

   -8.8015
    6.9647
   -0.1631

>> L(1)*L(2)*L(3)

ans =

   10.0000
```

Q.No-6 Find the left eigen vector of the given matrix

$$M = \begin{bmatrix} -7 & -1.5 & 7 \\ -1.5 & 2 & -5 \\ 7 & -5 & 0 \end{bmatrix}$$

```
>> M = [-7 -1.5 7; -1.5 2 -5; 7 -5 0]

M =

   -7.0000   -1.5000    7.0000
   -1.5000    2.0000   -5.0000
    7.0000   -5.0000         0

>> A = transpose(M)

A =

   -7.0000   -1.5000    7.0000
   -1.5000    2.0000   -5.0000
    7.0000   -5.0000         0

>> [E,D]=eig(A)

E =

   -0.8257   -0.4232   -0.3731
    0.1131   -0.7720    0.6254
    0.5527   -0.4742   -0.6853
      ↑          ↑          ↑
D =

  -11.4804         0         0
         0   -1.8932         0
         0         0    8.3736
```

→ left eigen vector is a row vector x such that

$$\vec{x}M = \lambda\vec{x}$$

$$(\vec{x}M)^T = (\lambda\vec{x})^T$$

$$M^T\vec{x}^T = \lambda\vec{x}^T$$

→ left eigen vector is a transpose of right eigen vector of M^T

left eigen vectors are

$$e_1 = (-7 \quad -1.5 \quad 7)$$

$$e_2 = (-1.5 \quad 2 \quad 5)$$

$$e_3 = (7 \quad -5 \quad 0)$$

Q.No-7 Compute the singular values of the given matrix

$$M = \begin{bmatrix} -4 & 2 & 4 & -8 \\ 1 & -5 & 5 & -5 \\ -6 & 3 & -1 & 8 \end{bmatrix}_{3\times 4}$$

```
>> M = [-4 2 4 -8; 1 -5 5 -5; -6 3 -1 8]

M =

    -4     2     4    -8
     1    -5     5    -5
    -6     3    -1     8

>> MT = transpose(M)

MT =

    -4     1    -6
     2    -5     3
     4     5    -1
    -8    -5     8

>> L=eig(M*MT)

L =

   21.6793
   67.7193
  196.6014

>> S = [sqrt(L(1)) sqrt(L(2)) sqrt(L(3))]

S =

    4.6561    8.2292   14.0215
```

→ singular values of M are square root of eigen values of MM^T or M^TM.

singular values are 4.66, 8.23, 14.02

Q.No-8 Compute the matrix M when M is rotated clockwise by 90 degree

$$M = \begin{bmatrix} -9 & -7 \\ 9 & 8 \end{bmatrix}$$

90° clockwise rotation matrix is $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$RM = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -9 & -7 \\ 9 & 8 \end{bmatrix} = \begin{bmatrix} 9 & 8 \\ -9 & -7 \end{bmatrix}$$

Q.No-10 M is a square matrix of dimension 3. Perform the eigen value decomposition of M and calculate the trace of the inverse of the eigen vectors of matrix M.

$$M = \begin{bmatrix} 101 & -2 & 14 \\ -2 & 25 & 26 \\ 14 & 26 & 76 \end{bmatrix}$$

$D = \text{invE} \times M \times E$

E is a matrix of eigen vectors

invE is inverse of E

```
>> M = [101 -2 14; -2 25 26; 14 26 76]

M =

   101    -2    14
    -2    25    26
    14    26    76

>> [E L] = eig(M)

E =

   -0.0846   -0.4878    0.8689
   -0.9132    0.3868    0.1282
    0.3986    0.7826    0.4782


L =

   13.4653         0         0
         0   80.1249         0
         0         0  108.4098

>> invE = inv(E)

invE =

   -0.0846   -0.9132    0.3986
   -0.4878    0.3868    0.7826
    0.8689    0.1282    0.4782

>> D = invE*M*E

D =

   13.4653    0.0000    0.0000
   -0.0000   80.1249    0.0000
    0.0000    0.0000  108.4098
```

Q.No-2 State true/false for the following statements with reasoning.

If all entries of A are positive, then A is positive-definite matrix

→ False; suppose $A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$; $|A| = -14$ which is negative

If A is positive-definite matrix, then Inverse(A) is also a positive-definite matrix.

→ True

for positive definite matrix, eigen values (λ_i) are positive

we know eigen values of A^{-1} are reciporcal of eigen values of A

i.e. $\lambda_i(A^{-1}) = \dfrac{1}{\lambda_i(A)}$

$\Rightarrow \lambda_i(A^{-1})$ are positive

$\Rightarrow A^{-1}$ is positive-definite matrix

Q.No 9 For the given matrix (A), calculate exp(At) such that f(x)=exp(xt) is a characteristic polynomial of A. What is the expression for b1 if exp(At) = b0+b1A+b2A2 ?

$$A = \begin{bmatrix} 3.5 & 0 & 0.5 \\ 1 & 2 & 1 \\ 0.5 & 0 & 3.5 \end{bmatrix}$$

$$e^{At} = L^{-1}\left[(sI-A)^{-1}\right]$$

$$(sI-A)^{-1} = \begin{bmatrix} \dfrac{s-3.5}{s^2-7s+12} & 0 & \dfrac{0.5}{s^2-7s+12} \\ \dfrac{s-3}{s^3-9s^2+26s-24} & \dfrac{1}{s-2} & \dfrac{s-3}{s^3-9s^2+26s-24} \\ \dfrac{0.5}{s^2-7s+12} & 0 & \dfrac{s-3.5}{s^2-7s+12} \end{bmatrix}$$

```
>> s = tf('s')

s =

  s

Continuous-time transfer function.

>> A=[3.5 0 0.5; 1 2 1; 0.5 0 3.5]

A =

    3.5000         0    0.5000
    1.0000    2.0000    1.0000
    0.5000         0    3.5000

>> sI = [s 0 0; 0 s 0; 0 0 s];
>> M=inv(sI-A)

M =

  From input 1 to output...
           s - 3.5
   1:  --------------
       s^2 - 7 s + 12

                   s - 3
   2:  ----------------------
       s^3 - 9 s^2 + 26 s - 24

             0.5
   3:  --------------
       s^2 - 7 s + 12

  From input 2 to output...
   1:  0

        1
   2:  -----
       s - 2

   3:  0

  From input 3 to output...
             0.5
   1:  --------------
       s^2 - 7 s + 12

                   s - 3
   2:  ----------------------
       s^3 - 9 s^2 + 26 s - 24

           s - 3.5
   3:  --------------
       s^2 - 7 s + 12

Continuous-time transfer function.
```

```
>> syms s
>> M = [ (s-3.5)/(s^2 -7*s + 12) (s-3)/(s^3 - 9*s^2 +26*s -24) 0.5/(s^2 -7*s+12); 0 1/↙
(s-2) 0; 0.5/(s^2 -7*s +12) (s-3)/(s^3-9*s^2 +26*s-24) (s-3.5)/(s^2 -7*s +12)]

M =

[ (s - 7/2)/(s^2 - 7*s + 12), (s - 3)/(s^3 - 9*s^2 + 26*s - 24),     1/(2*(s^2 - 7*s +↙
12))]
[                          0,                        1/(s - 2),↙
0]
[      1/(2*(s^2 - 7*s + 12)), (s - 3)/(s^3 - 9*s^2 + 26*s - 24), (s - 7/2)/(s^2 - 7*s +↙
12)]

>> for i=1:3
for j=1:3
stM(i,j)=ilaplace(M(i,j));
end
end
>> stM

stM =

[ exp(3*t)/2 + exp(4*t)/2, exp(4*t)/2 - exp(2*t)/2, exp(4*t)/2 - exp(3*t)/2]
[                       0,                exp(2*t),                        0]
[ exp(4*t)/2 - exp(3*t)/2, exp(4*t)/2 - exp(2*t)/2, exp(3*t)/2 + exp(4*t)/2]
```

$$\therefore e^{At} = \begin{bmatrix} \frac{1}{2}(e^{3t}+e^{4t}) & 0 & \frac{1}{2}(e^{4t}-e^{3t}) \\ \frac{1}{2}(e^{4t}-e^{2t}) & e^{2t} & \frac{1}{2}(e^{4t}-e^{2t}) \\ \frac{1}{2}(e^{4t}-e^{3t}) & 0 & \frac{1}{2}(e^{3t}+e^{4t}) \end{bmatrix}$$

Q.No 3 For the given matrix (A), calculate exp(At) such that f(x)=exp(xt) is a characteristic polynomial of A. What is the expression for b3 if exp(At) is expressed as [a1 b1 c1; a2 b2 c2; a3 b3 c3] ?

$$A = \begin{bmatrix} 5 & 1 & 1 \\ -2 & 8 & 1 \\ 2 & -2 & 5 \end{bmatrix}$$

$$e^{At} = L^{-1}\left[(sI-A)^{-1}\right]$$

$$(sI-A)^{-1} = \begin{bmatrix} \frac{s^2-13s+42}{s^3-18s^2+07s-210} & \frac{s-7}{s^3-18s^2+107s-210} & \frac{s-7}{s^3-18s^2+107s-210} \\ \frac{-2s+12}{s^3-18s^2+107s-210} & \frac{s^2-10s+23}{s^3-18s^2+107s-210} & \frac{s-7}{s^3-18s^2+107s-210} \\ \frac{2s-12}{s^3-18s^2+107s-210} & \frac{-2s+12}{s^3-18s^2+107s-210} & \frac{s^2-13s+42}{s^3-18s^2+107s-210} \end{bmatrix}$$

```
>> s = tf('s')

s =

  s

Continuous-time transfer function.

>> A=[5 1 1; -2 8 1; 2 -2 5]

A =

     5     1     1
    -2     8     1
     2    -2     5

>> sI = [s 0 0; 0 s 0; 0 0 s];
>> M=inv(sI-A)

M =

  From input 1 to output...
             s^2 - 13 s + 42
   1:  ---------------------------
       s^3 - 18 s^2 + 107 s - 210

                -2 s + 12
   2:  ---------------------------
       s^3 - 18 s^2 + 107 s - 210

                 2 s - 12
   3:  ---------------------------
       s^3 - 18 s^2 + 107 s - 210

  From input 2 to output...
                  s - 7
   1:  ---------------------------
       s^3 - 18 s^2 + 107 s - 210

             s^2 - 10 s + 23
   2:  ---------------------------
       s^3 - 18 s^2 + 107 s - 210

                -2 s + 12
   3:  ---------------------------
       s^3 - 18 s^2 + 107 s - 210

  From input 3 to output...
                  s - 7
   1:  ---------------------------
       s^3 - 18 s^2 + 107 s - 210

                  s - 7
   2:  ---------------------------
       s^3 - 18 s^2 + 107 s - 210

             s^2 - 13 s + 42
   3:  ---------------------------
       s^3 - 18 s^2 + 107 s - 210
```

```
>> syms s
>> M = transpose([ (s^2-13*s+42)/(s^3-18*s^2+107*s-210) (-2*s+12)/(s^3-18*s^2+107*s-
210) (2*s-12)/(s^3-18*s^2+107*s-210); (s-7)/(s^3-18*s^2+107*s-210) (s^2 -10*s+23)/
(s^3-18*s^2+107*s-210) (-2*s+12)/(s^3-18*s^2+107*s-210); (s-7)/(s^3-18*s^2+107*s-210)
(s-7)/(s^3-18*s^2+107*s-210) (s^2-13*s+42)/(s^3-18*s^2+107*s-210)])

M =

[ (s^2 - 13*s + 42)/(s^3 - 18*s^2 + 107*s - 210),           (s - 7)/(s^3 - 18*s^2 +
107*s - 210),            (s - 7)/(s^3 - 18*s^2 + 107*s - 210)]
[       -(2*s - 12)/(s^3 - 18*s^2 + 107*s - 210), (s^2 - 10*s + 23)/(s^3 - 18*s^2 +
107*s - 210),            (s - 7)/(s^3 - 18*s^2 + 107*s - 210)]
[        (2*s - 12)/(s^3 - 18*s^2 + 107*s - 210),       -(2*s - 12)/(s^3 - 18*s^2 +
107*s - 210), (s^2 - 13*s + 42)/(s^3 - 18*s^2 + 107*s - 210)]

>> for i=1:3
for j=1:3
stM(i,j)=ilaplace(M(i,j));
end
end
>> stM

stM =

[             exp(5*t),            exp(6*t) - exp(5*t), exp(6*t) - exp(5*t)]
[ exp(5*t) - exp(7*t), exp(6*t) - exp(5*t) + exp(7*t), exp(6*t) - exp(5*t)]
[ exp(7*t) - exp(5*t),             exp(5*t) - exp(7*t),            exp(5*t)]
```

$$\therefore e^{At} = \begin{bmatrix} e^{5t} & e^{6t}-e^{5t} & e^{6t}-e^{5t} \\ e^{5t}-e^{7t} & e^{6t}-e^{5t}+e^{7t} & e^{6t}-e^{5t} \\ e^{7t}-e^{5t} & e^{5t}-e^{7t} & e^{5t} \end{bmatrix}$$