Modern Control Theory

Model Predict Control - MPC – Lecture 10

Constrained Control move calculation - MIMO

Prediction using State space models - MIMO

m inputs, q outputs and n1 states

For good control q <=m

$$x_m(k+1) = A_m x_m(k) + B_m u(k) + B_d \omega(k)$$
$$y(k) = C_m x_m(k),$$

w(k) is integrated white noise

$$\omega(k) - \omega(k-1) = \epsilon(k)$$

Am is n1 x n1 Bm is n1 x m Bd is n1 x m Cm is q x n1

u is m x 1 y is q x 1

$$\Delta x_m(k) = x_m(k) - x_m(k-1)$$
$$\Delta u(k) = u(k) - u(k-1)$$
$$\Delta y(k+1) = y(k+1) - y(k)$$

$$\begin{bmatrix} \Delta x_m(k+1) \\ y(k+1) \end{bmatrix} = \overset{A_e}{\begin{bmatrix} A_m & o_m^T \\ C_m A_m & I_{q \times q} \end{bmatrix}} \begin{bmatrix} \Delta x_m(k) \\ y(k) \end{bmatrix} + \overset{B_e}{\begin{bmatrix} B_m \\ C_m B_m \end{bmatrix}} \Delta u(k) + \begin{bmatrix} B_d \\ C_m B_d \end{bmatrix} \epsilon(k)$$

$$y(k) = \underset{C_e}{\begin{bmatrix} o_m & I_{q \times q} \end{bmatrix}} \begin{bmatrix} \Delta x_m(k) \\ y(k) \end{bmatrix}$$

Incremental form

$$x(k+1) = Ax(k) + B\Delta u(k) + B_\epsilon \epsilon(k) \qquad y(k) = Cx(k)$$

Augmented
'q' Integrators

$$\rho(\lambda) = \det \begin{bmatrix} \lambda I - A_m & o_m^T \\ -C_m A_m & (\lambda - 1) I_{q \times q} \end{bmatrix} = (\lambda - 1)^q \det(\lambda I - A_m) = 0$$

Prediction using State space models - MIMO

System representation

$$x_m(k+1) = A_m x_m(k) + B_m u(k)$$
$$y(k) = C_m x_m(k)$$

System in incremental form

$$\boxed{x(k+1) = Ax(k) + B\Delta u(k)}$$

$$\boxed{y(k+1) = C\,x(k)}$$

Output Prediction equation

q*NpX 1

$$Y = \begin{bmatrix} y(k_i+1) \mid k_i \\ y(k_i+2) \mid k_i \\ y(k_i+3) \mid k_i \\ \vdots \\ y(k_i+N_p) \mid k_i \end{bmatrix}$$

Output Prediction equation

m*Nc X 1

$$\Delta U = \begin{bmatrix} \Delta u(k_i) \\ \Delta u(k_i+1) \\ \Delta u(k_i+2) \\ \vdots \\ \Delta u(k_i+N_c-1) \end{bmatrix}$$

$$\boxed{Y = F\,\hat{x}(k_i) + \Phi\,\Delta U}$$

(q*Np x (n1+q)

$$F = \begin{bmatrix} CA \\ CA^2 \\ CA^3 \\ \vdots \\ CA^{N_p} \end{bmatrix}$$

q*Np x m*Nc

$$\Phi = \begin{bmatrix} CB & 0 & 0 & \cdots & 0 \\ CAB & CB & 0 & \cdots & 0 \\ CA^2B & CAB & CB & \cdots & 0 \\ \vdots \\ CA^{N_p-1}B & CA^{N_p-2}B & CA^{N_p-3}B & \cdots & CA^{N_p-N_c}B \end{bmatrix}$$

Unconstrained solution – MIMO

$$J = (R_s - F\hat{x}(k_i))^t (R_s - F\hat{x}(k_i)) - 2\Delta U^t \Phi^T (R_s - F\hat{x}(k_i)) + \Delta U^t (\Phi^T \Phi + R)\Delta U$$

Now $$\frac{\partial J}{\partial \Delta U} = -2\Phi^T (R_s - F\hat{x}(k_i)) + 2(\Phi^T \Phi + R)\Delta U = 0$$

$$(\Phi^T\Phi + R)\Delta U = \Phi^T (R_s - F\hat{x}(k_i))$$

Solving for ΔU we get

$$\boxed{\Delta U = (\Phi^T\Phi + R)^{-1}\Phi^T (R_s - F\hat{x}(k_i))}$$

$u_{ki} = u_{(ki-1)} + \Delta u_{ki}$ (first m value in ΔU)

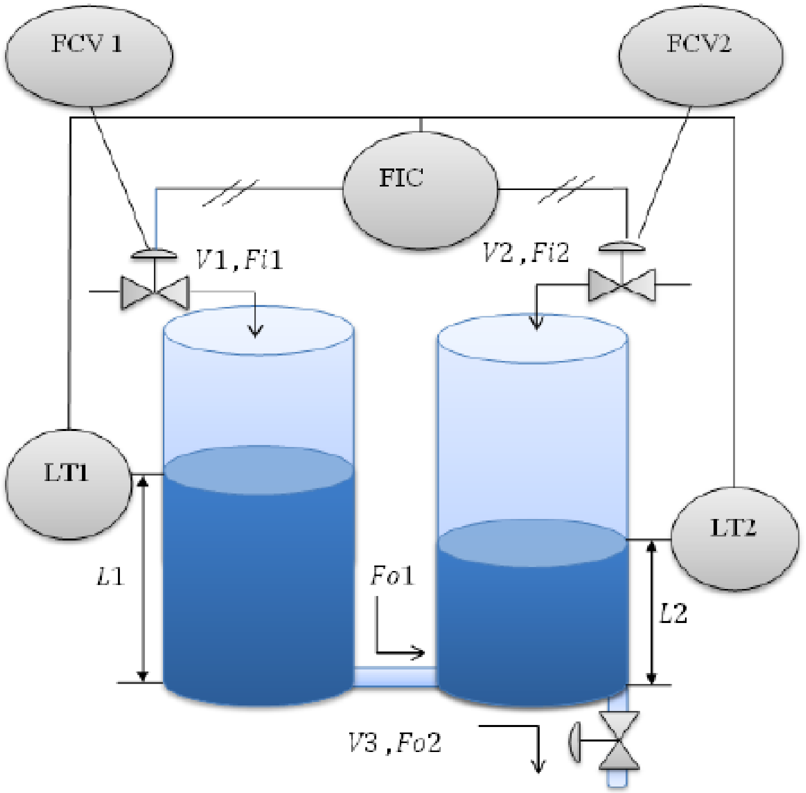
$$\Delta u(k_i) = \overbrace{\begin{bmatrix} I_m & o_m & \ldots & o_m \end{bmatrix}}^{N_c} (\Phi^T\Phi + \bar{R})^{-1} (\Phi^T \bar{R}_s r(k_i) - \Phi^T F x(k_i))$$

- ***Implementation steps ONLINE***

1. ***Obtain the current measurements of states,***
2. ***Estimate the states using Kalman filter***
3. ***Compute the optimal finite horizon control sequence (u_k, u_{k+1}, …u_M)***
4. ***Implement first move u_k for each of the inputs***
5. ***Repeat from Step 1***

Example problem

Level control of coupled tanks



$$A1.\frac{dL1}{dt} = Fi1 - Fo1 \qquad A2.\frac{dL2}{dt} = Fi2 - Fo2 + Fo1$$

$$Fo1 = \alpha1.\sqrt{L1 - L2} \qquad Fo2 = \alpha2.\sqrt{L2}$$

Linearize around some operating point (h1.h2), we get

$$x = \begin{pmatrix} h1 \\ h2 \end{pmatrix} \quad u = \begin{pmatrix} fi1 \\ fi2 \end{pmatrix}$$

$$y = \begin{pmatrix} h1 \\ h2 \end{pmatrix}$$

$$A_m = \begin{pmatrix} -\frac{\alpha1}{2.A1}.\frac{1}{\sqrt{(L1-L2)}} & \frac{\alpha1}{2.A1}.\frac{1}{\sqrt{(L1-L2)}} \\ \frac{\alpha1}{2.A2}.\frac{1}{\sqrt{(L1-L2)}} & \frac{-1}{2.A2}.(\frac{\alpha1}{\sqrt{(L1-L2)}} + \frac{\alpha2}{\sqrt{L2}}) \end{pmatrix}$$

$$B_m = \begin{pmatrix} \frac{1}{A1} & 0 \\ 0 & \frac{1}{A2} \end{pmatrix} \quad C_m = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad D_m = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Matrices to compute:

Φ, F, $\Phi^T\Phi$, $\Phi^T R_s$, $\Phi^T F$

$$\Delta U = (\Phi^T\Phi + R)^{-1}\Phi^T(R_s - F\hat{x}(k_i))$$

Demo

Linearize around
L1 = 4m, L2 = 3.5m $A_m = \begin{pmatrix} -7.923 & 7.923 \\ 9.781 & -12.97 \end{pmatrix}$ $B_m = \begin{pmatrix} 5.093 & 0 \\ 0 & 6.288 \end{pmatrix}$ $C_m = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- Sampling time = 0.05 secs, discretize
- Np = 10, Nc = 3

Handling constraints

Inputs

- Rate of change constraints on the input – (Δu)
- Absolute constraint (u)

Output

- Output limits - y

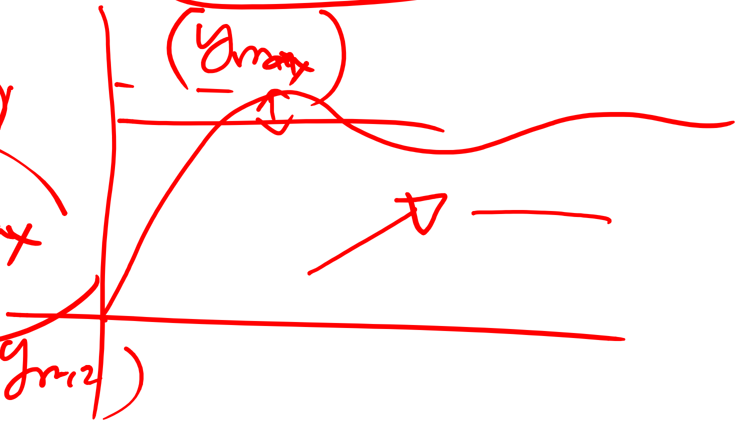
Inputs – Hard constraints – cannot be violated

Output – Soft constraints – violation allowed with some penalty

Safety limit

(y_{max})

(u_{min}) y_{max}



(y_{min})

u_{max} 100% → ~~100~~.1%

$y > y_{max}$

$y + \varepsilon \le y_{max}$

Infeasible

Handling input constraints - incremental change in inputs

$$J = (R_s - Y)^T (R_s - Y) + \Delta U^T \bar{R} \Delta U$$

Decision variable is Δu

Rate of input change: Most easy constraint to include

$$\begin{bmatrix} \Delta u_1^{max} & \Delta u_2^{max} & \ldots & \Delta u_m^{max} \end{bmatrix}$$

$$\begin{bmatrix} \Delta u_1^{min} & \Delta u_2^{min} & \ldots & \Delta u_m^{min} \end{bmatrix}$$

$$\begin{aligned} \Delta u_1^{min} &\leq \Delta u_1(k) \leq \Delta u_1^{max} \\ \Delta u_2^{min} &\leq \Delta u_2(k) \leq \Delta u_2^{max} \\ &\vdots \\ \Delta u_m^{min} &\leq \Delta u_m(k) \leq \Delta u_m^{max} \end{aligned}$$

$$\Delta u_1^{min} \leq \Delta u_1(k_i + 1) \leq \Delta u^{max}$$

$$\Delta u_1^{min} \leq \Delta u_1(k_i + 2) \leq \Delta u^{max}$$

$\Delta u_{2min} \leq \Delta u_2(k_i+1)$

$\vdots$

$$\Delta U^{min} \leq \Delta U \leq \Delta U^{max}$$

$$\begin{aligned} -\Delta U &\leq -\Delta U^{min} \\ \Delta U &\leq \Delta U^{max}. \end{aligned}$$

$$\begin{bmatrix} -I \\ I \end{bmatrix} \Delta U \leq \begin{bmatrix} -\Delta U^{min} \\ \Delta U^{max} \end{bmatrix}$$

Handling input constrains – Amplitude of control signal u

$$J = (R_s - Y)^T(R_s - Y) + \Delta U^T \bar{R} \Delta U$$

Decision variable is Δu

$$
\begin{aligned}
u_1^{min} &\le u_1(k) \le u_1^{max} \\
u_2^{min} &\le u_2(k) \le u_2^{max} \\
&\vdots \\
u_m^{min} &\le u_m(k) \le u_m^{max}.
\end{aligned}
$$

Example: SISO Nc = 4

$$
\begin{aligned}
u(k_i) &= u(k_i - 1) + \Delta u(k_i) \\
&= u(k_i - 1) + [1\ 0\ 0\ 0]\,\Delta U
\end{aligned}
$$

$$
\begin{aligned}
u(k_i + 1) &= u(k_i) + \Delta u(k_i + 1) \\
&= u(k_i - 1) + \Delta u(k_i) + \Delta u(k_i + 1) \\
&= u(k_i - 1) + [1\ 1\ 0\ 0]\,\Delta U
\end{aligned}
$$

$$
\begin{bmatrix} u(k_i) \\ u(k_i+1) \\ u(k_i+2) \\ \vdots \\ u(k_i+N_c-1) \end{bmatrix}
=
\begin{bmatrix} I \\ I \\ I \\ \vdots \\ I \end{bmatrix} u(k_i - 1) +
\begin{bmatrix} I & 0 & 0 & \dots & 0 \\ I & I & 0 & \dots & 0 \\ I & I & I & \dots & 0 \\ \vdots \\ I & I & \dots & I & I \end{bmatrix}
\begin{bmatrix} \Delta u(k_i) \\ \Delta u(k_i+1) \\ \Delta u(k_i+2) \\ \vdots \\ \Delta u(k_i+N_c-1) \end{bmatrix}
$$

C1 C2

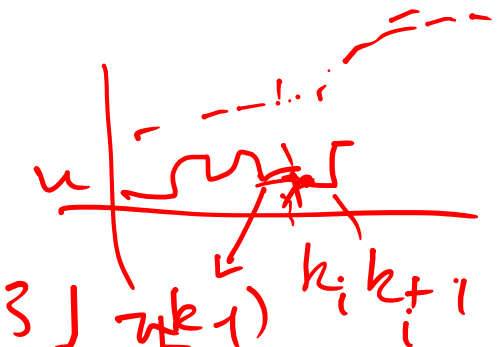
$Ax \le b$

$$-C_2 \Delta U \le -U^{min} + C_1 u(k_i - 1)$$

$$C_2 \Delta U \le U^{max} - C_1 u(k_i - 1)$$

$\Delta u(k_i)$ $\Delta u(k_i+1)$

$\Delta u(k_i+2)$ $\Delta u(k_i+3)$



Handling output constraints – control limits

$y1_{min} < y1_{pred}(k) < y1_{max}$

$y2_{min} < y2_{pred}(k) < y2_{max}$

....

$yq_{min} < yq_{pred}(k) < yq_{max}$

$Y_{min} < Y_{pred}(k) < Y_{max}$

$Y_{min} < Y_{pred}(k+1) < Y_{max}$

...

$Y_{min} < Y_{pred}(k+Np) < Y_{max}$

$Y_{pred} = Fx(k_i) + \Phi\Delta U$

$$Y^{min} \leq Fx(k_i) + \Phi\Delta U \leq Y^{max}$$

$$: \begin{bmatrix} -\Phi \\ \Phi \end{bmatrix} \Delta U \leq \begin{bmatrix} -Y^{min} + Fx(k_i) \\ Y^{max} - Fx(k_i) \end{bmatrix}$$

$Ax \leq b$

SISO Example – input constraints

$-1.5 \leq \Delta u(k) \leq 3;$

$-3 \leq u(k) \leq 6.$

Nc = 3

$$\begin{bmatrix} -I \\ I \end{bmatrix} \Delta U \leq \begin{bmatrix} -\Delta U^{min} \\ \Delta U^{max} \end{bmatrix}$$

$$C_2 \Delta U \quad -U^{min} + C_1 u(k_i - 1)$$

$$C_2 \Delta U \leq U^{max} - C_1 u(k_i - 1)$$

$$\begin{array}{r} \\ I \\ \\ \\ -I \\ \\ \\ U_{max} \\ \\ \\ U_{min} \\ \\ \end{array}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & -1 & 0 \\ -1 & -1 & -1 \end{bmatrix}
\underbrace{\begin{bmatrix} \Delta u(k_i) \\ \Delta u(k_i + 1) \\ \Delta u(k_i + 2) \end{bmatrix}}_{\text{3 Steps of } \Delta u \text{ computation}}
\leq
\begin{bmatrix} 3 \\ 3 \\ 3 \\ 1.5 \\ 1.5 \\ 1.5 \\ 6 - u(k_i - 1) \\ 6 - u(k_i - 1) \\ 6 - u(k_i - 1) \\ 3 + u(k_i - 1) \\ 3 + u(k_i - 1) \\ 3 + u(k_i - 1) \end{bmatrix}$$

Handling constrains

$Y = F\hat{x}(k_i) + \Phi \Delta U$

$J = (x_1 - 2)^2 + 5 + 10$

$J|_2$

$x_1 = 2$

Min
Δu $J = (R_s - Y)^T (R_s - Y) + \Delta U^T \bar{R} \Delta U$

Min
Δu $J = \cancel{(R_s - F\hat{x}(k_i))^t (R_s - F\hat{x}(k_i))} - 2\Delta U^t \Phi^T (R_s - F\hat{x}(k_i)) + \Delta U^t (\Phi^T \Phi + R) \Delta U$

Min
Δu $J/2 = \frac{1}{2} \Delta U^T (\Phi^T \Phi + R) \Delta U - \Delta U^T \Phi^T (R_s - F\hat{x}(k_i))$

$\boxed{E = \Phi^T \Phi + R}$ $\boxed{F = \Phi^T (R_s - F\hat{x}(k_i))}$

$|y|$ E F $(N_c = 1)$ $m = 1$

Min
Δu $J' = \frac{1}{2} \Delta U^T E \Delta U + \Delta U^T F$

This is standard Quadratic programming (QP) with E called the Hessian – symmetric positive definite

Subject to $\begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} \Delta U \le \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}$

$M_1 = \begin{bmatrix} -C_2 \\ C_2 \end{bmatrix}$ $N_1 = \begin{bmatrix} -U^{min} + C_1 u(k_i - 1) \\ U^{max} - C_1 u(k_i - 1) \end{bmatrix}$ Input amplitude (abs)

$M_2 = \begin{bmatrix} -I \\ I \end{bmatrix}$ $N_2 = \begin{bmatrix} -\Delta U^{min} \\ \Delta U^{max} \end{bmatrix}$ Input rate of change

Linear in constraints $\boxed{M \Delta U \le \gamma;}$ x

$M_3 = \begin{bmatrix} -\Phi \\ \Phi \end{bmatrix}$ $N_3 = \begin{bmatrix} -Y^{min} + Fx(k_i) \\ Y^{max} - Fx(k_i) \end{bmatrix}$ Output