Classical Control Theory	Modern Control Engineering
Deals with only SISO systems	Can deal with both the SISO and MIMO systems
The system must be linear time invariant (LTI)	The system can be both LTI or non-linear, time variant.
Mainly complex frequency domain approach	Mainly time domain approach

State:

The minimum set of variables such that the knowledge of these variables at time $t = t_o$, along with the knowledge of the input for $t \geq t_o$, completely describes the behavior of the system for any time $t \geq t_o$.

Concept of state is not just limited to physical systems. It is applicable to biological systems, economic systems, social systems and others.

State Variables:

The state variables are the smallest set of variables that are enough to describe the dynamics of a system. The variables that does not represent any physical parameters and the variables that are neither measurable nor observable can also be chosen as state variables.

State Vector:

If 'n' state variables are required to describe the dynamics of a given system, then 'n' state variables can be considered as 'n' components of a state vector. Such vector is called as state vector.

State Space:

The n dimensional space whose coordinate axes consist of the x_1 axis, x_2 axis, … x_n axis is called state space. Here $x_1, x_2, \ldots x_n$ are state variables.

State Space Equations:

The state space representation for a given system is not unique, except that the number of state variables is same for any of the different state space representations of the same system.

State equation is the set of `n` simultaneous first order differential equation with 'n' state variables.

Output Equation:

Output equation is the linear combinations of the state variables and the inputs.

State Space Representation (Mathematical Analysis):

In state space representation the nth order differential equation is represented by 'n' number of first order differential equation. Consider a multiple input multiple outputs system. Let $x_1, x_2, \ldots x_n$ be the state variables. Assume that there are 'r' inputs $u_1, u_2, \ldots u_r$ and 'm' outputs $y_1, y_2, \ldots y_m$. Then the system may be described by:

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= f_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t)\\
\frac{dx_2(t)}{dt} &= f_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t)\\
&\;\;. \\
&\;\;. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{------(i)}\\
&\;\;. \\
\frac{dx_n(t)}{dt} &= f_3(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t)
\end{aligned}$$

The outputs of the system may be given by:

$$\begin{aligned}
y_1(t) &= g_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t)\\
y_2(t) &= g_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t)\\
&\;\;. \\
&\;\;. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{------(ii)}\\
&\;\;. \\
y_m(t) &= g_m(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t)
\end{aligned}$$

If we define

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix} \; ; \quad \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{t}) = \begin{bmatrix} f_1(x_1, x_2, \ldots x_n; u_1, u_2, \ldots u_r; t) \\ f_2(x_1, x_2, \ldots x_n; u_1, u_2, \ldots u_r; t) \\ \cdot \\ \cdot \\ \cdot \\ f_n(x_1, x_2, \ldots x_n; u_1, u_2, \ldots u_r; t) \end{bmatrix}$$

$$\boldsymbol{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \cdot \\ \cdot \\ \cdot \\ y_m(t) \end{bmatrix} \; ; \quad \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{t}) = \begin{bmatrix} g_1(x_1, x_2, \ldots x_n; u_1, u_2, \ldots u_r; t) \\ g_2(x_1, x_2, \ldots x_n; u_1, u_2, \ldots u_r; t) \\ \cdot \\ \cdot \\ \cdot \\ g_m(x_1, x_2, \ldots x_n; u_1, u_2, \ldots u_r; t) \end{bmatrix} \; ; \quad \boldsymbol{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ \cdot \\ \cdot \\ u_r(t) \end{bmatrix}$$

Then above state equation (i) and output equation (ii) may be expressed as:

$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t) ----(iii)$ is state equation
$\boldsymbol{y}(t) = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, t) ----(iv)$ is the output equation of the system.

When the vector functions ‘f’ and ‘g’ involves time t explicitly, then the system is called a time varying system. If the equation (iii) and (iv) are linearized about the operating state, then we have the following linearized state equation and output equation.

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t) ----(v)$$
$$\boldsymbol{y}(t) = \boldsymbol{C}(t)\boldsymbol{x}(t) + \boldsymbol{D}(t)\boldsymbol{u}(t) ----(vi)$$

Where $\boldsymbol{A}(t)$ is called the system matrix or state matrix, $\boldsymbol{B}(t)$ is the input matrix, $\boldsymbol{C}(t)$ is the output matrix and $\boldsymbol{D}(t)$ is the direct transmission matrix.

If the vector functions ‘f’ and ‘g’ do not involve time ‘t’ explicitly then the system is called a time invariant system. In this case, equation (v) and (vi) can be simplified to

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{Ax}(t) + \boldsymbol{Bu}(t) ----(vii)$$
$$\boldsymbol{y}(t) = \boldsymbol{Cx}(t) + \boldsymbol{Du}(t) ----(viii)$$

Above equation (vii) and (viii) are the state equation and output equation of linear time-invariant system.

Selection of State Variables

Number of state variables should be minimum

The number of state variables must be linearly independent

The number of state variables is equal to the number of energy storage element in the system.

The number of state variables is equal to the order of differential equation

The number of state variables are equal to the number of integrators.

General guideline for the selection of state is that select the parameter of energy storing element as a state.

For example

$$\frac{d^n y(t)}{dt^n} + a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} + a_2 \frac{d^{n-2} y(t)}{dt^{n-2}} + ... + a_{n-2} \frac{d^2 y(t)}{dt^2} + a_{n-1} \frac{dy(t)}{dt} + a_o y(t) = \frac{d^n u(t)}{dt^n} + \frac{d^n u(t)}{dt^n} ... + \frac{d^n u(t)}{dt^n} + \frac{d^n u(t)}{dt^n}$$

is a nth order differential equation in y(t) involving derivative terms of the input