Assignment 2 CH5120 Jasmin Karki(ge22m019)

Qno. 1 Consider the single-input system dynamics given by x' = Ax + Bu and y = Cx and choose the correct statement(s) from the following statements:

(i) The system is stable in an absolute sense if all eigenvalues of A have non-negative real parts.

False, it is stable in an absolute sense if all eigenvalues of A have **negative** real parts.

(ii) The poles of the system is given by the eigenvalues of A.

True

Qno. 2 Identify the transfer function representation of the state space model and find the right coefficient array of the numerator,

Given: x˙ = [0 1 0; 0 0 1; 1 4 3] x + [0; 0; 1] u

y = [1 0 0] x + [1] u

Qno 2

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad ; \quad y = [1\ 0\ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [1]\,U$$

From the state equation we get

$$\dot{x}_3 = x_1 + 4x_2 + 3x_3 + u$$

As $\dot{x}_3 = \dddot{x}_1$, $x_2 = \dot{x}_1$, $x_3 = \ddot{x}_1$, the eq^n becomes

$$\dddot{x}_1 = x_1 + 4\dot{x}_1 + 3\ddot{x}_1 + u$$

$$\dddot{x}_1 - 3\ddot{x}_1 - 4\dot{x}_1 - x_1 = u$$

Taking laplace transform on both sides, we get

$$\mathcal{L}\left[\frac{d^3x_1}{dt^3} - 3\frac{d^2x_1}{dt^2} - 4\frac{dx_1}{dt} - x_1\right] = \mathcal{L}[u(t)]$$

$$\Rightarrow s^3X_1(s) - 3s^2X_1(s) - 4sX_1(s) - X_1(s) = U(s)$$

$$\Rightarrow X_1(s)\,(s^3 - 3s^2 - 4s - 1) = U(s)$$

$$\Rightarrow [Y(s) - U(s)]\,[s^3 - 3s^2 - 4s - 1] = U(s)$$

$$\Rightarrow Y(s) = \frac{U(s)\,(s^3 - 3s^2 - 4s)}{s^3 - 3s^2 - 4s - 1}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{s^3 - 3s^2 - 4s}{s^3 - 3s^2 - 4s - 1}$$

$y = x_1 + U$

Taking laplace on both sides

$Y(s) = X_1(s) + U(s)$

$X_1(s) = Y(s) - U(s)$

Therefore the coefficient of numerator is [1 -3 -4] and coefficient of denominator is [1 -3 -4 -1]

```
>> A = [0 1 0; 0 0 1; 1 4 3]

A =

     0     1     0
     0     0     1
     1     4     3

>> B = [0; 0; 1]

B =

     0
     0
     1

>> C = [1 0 0]

C =

     1     0     0

>> D = [1]

D =

     1
```

```
>> [numera,denomina]=ss2tf(A,B,C,D,1);
>> tf(numera,denomina)

ans =

    s^3 - 3 s^2 - 4 s
  ---------------------
  s^3 - 3 s^2 - 4 s - 1

Continuous-time transfer function.
```

Q no 3. Obtain the transfer function form from the given state space representation and find the correct coefficients of the denominator of the transfer function.

A = [-1 -1 -1; 0 1 -1; 1 -1 1] B = [0; 1; 0]

C = [0 0 1] D = [0]

Q no 3

$$A = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C = [0 \ 0 \ 1] \quad D = [0]$$

Transfer function is $C(SI-A)^{-1}B + D$

First, $(SI-A)^{-1}$

$$|SI-A| = \begin{vmatrix} s+1 & 1 & 1 \\ 0 & s-1 & 1 \\ -1 & 1 & s-1 \end{vmatrix}$$

$$= (s+1)[(s-1)^2 - 1] - 1(+1) + 1(s-1)$$

$$= (s+1)(s^2 - 2s + \cancel{1} - \cancel{1}) - 1 + s - 1$$

$$= (s^3 - 2s^2 + s^2 - 2s + 0) - 1 + s - 1$$

$$= s^3 - \cancel{2}s^2 - s - 2$$

Adjoint of $(SI-A)$ = [Cofactor of $(SI-A)$]^T

$$\text{Cofactor of } (SI-A) = \begin{bmatrix} s^2-2s & 2-s & -s \\ -1 & s^2 & -s \\ s-1 & -s-2 & s^2-1 \end{bmatrix}$$

$$\text{Adjoint} = \begin{bmatrix} s^2-2s & -1 & s-1 \\ 2-s & s^2 & -s-2 \\ -s & -s & s^2-1 \end{bmatrix}$$

$$\text{Now, } C(SI-A)^{-1}B + D = [0 \ 0 \ 1] \frac{\begin{bmatrix} s^2-2s & -1 & s-1 \\ 2-s & s^2 & -s-2 \\ -s & -s & s^2-1 \end{bmatrix}^T}{s^3 - s^2 - s - 2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + [0]$$

$$= \frac{-s-2}{s^3 - s^2 - s - 2}$$

$$\therefore G(s) = \frac{-s-2}{s^3 - s^2 - s - 2}$$

```
>> A = [ -1 -1 -1; 0 1 -1; 1 -1 1]

A =

    -1    -1    -1
     0     1    -1
     1    -1     1

>> B = [0; 1; 0]

B =

     0
     1
     0

>> C = [ 0 0 1 ]

C =

     0     0     1

>> D = [0]

D =

     0
```

```
>> [nume,deno] = ss2tf(A,B,C,D,1);
>> tf(nume,deno)

ans =

       -s - 2
  -----------------
  s^3 - s^2 - s - 2

Continuous-time transfer function.
```

Q no. 4

Qno 4

$$a3\frac{d^3x}{dt^3} + a2\frac{d^2x}{dt^2} + a1\frac{dx}{dt} + a0x = u(t)$$

Given, $[a3\ a2\ a1\ a0] = [4\ -6\ 1\ 7]$

and $u(t) = 2$

$$4\frac{d^3x}{dt^3} - 6\frac{d^2x}{dt^2} + \frac{dx}{dt} + 7x = 2$$

$$\frac{d^3x}{dt^3} - \frac{6}{4}\frac{d^2x}{dt^2} + \frac{dx}{dt}\times\frac{1}{4} + \frac{7}{4}x = \frac{2}{4}$$

$$\frac{d^3x}{dt^3} = \frac{3}{2}\frac{d^2x}{dt^2} - \frac{1}{4}\frac{dx}{dt} - \frac{7}{4}x + \frac{2}{4}$$

$$\dddot{x} = 1.5\ddot{x} - 0.25\dot{x} - 1.75x + 0.5$$

$x_1 = x$, $x_2 = \dot{x}$, $x_3 = \ddot{x}$

$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

$\dot{x}_3 = 1.5x_3 - 0.25x_2 - 1.75x_1 + 0.5$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1.75 & -0.25 & 1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix} u(t)$$

$$\therefore A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1.75 & -0.25 & 1.5 \end{bmatrix}$$

Q no 5

Q no 5

Sample time = 0.1 s

$$A_c = \begin{bmatrix} -3 & -6 \\ +2 & 5 \end{bmatrix}, \quad B_c = \begin{bmatrix} -5 \\ 8 \end{bmatrix} \quad C_c = \begin{bmatrix} 4 & 5 \end{bmatrix}$$

State transition matrix in continuous time domain is

$$e^{A_c t} = \mathcal{L}^{-1}[(sI - A)^{-1}]$$

$$\boxed{\begin{array}{ll} \multicolumn{2}{l}{\text{Inverse laplace formula}} \\ \dfrac{1}{(s+a)(s+b)} & \dfrac{1}{(b-a)}\left(e^{-at} - e^{-bt}\right) \\ \dfrac{(s+d)}{(s+a)(s+b)} & \dfrac{1}{(b-a)}\left[(d-a)e^{-at} - (d-b)e^{-bt}\right] \end{array}}$$

$$sI - A_c = \begin{bmatrix} s+3 & 6 \\ -2 & s-5 \end{bmatrix}$$

$$|sI - A_c| = (s+3)(s-5) + (2)\times 6$$
$$= s^2 - 2s - 3$$
$$= (s+1)(s-3)$$

$$\cancel{(sI - A_c)^{-1} = \begin{bmatrix} \frac{s+3}{(s+1)(s+3)} & \frac{6}{(s+1)(s+3)} \\ \frac{-2}{(s+1)(s+3)} & \frac{s-5}{(s+1)(s+3)} \end{bmatrix} \xrightarrow{\mathcal{L}^{-1}} \begin{bmatrix} \frac{1}{(-3-1)}(3-1)e^{-t} - 6e^{3t} & \frac{1}{(-3-1)} \\ \frac{1}{(-3-1)} & \frac{1}{(-3-1)} \end{bmatrix}}$$

$$(sI - A_c)^{-1} = \begin{bmatrix} \dfrac{s-5}{(s-3)(s+1)} & \dfrac{-6}{(s-3)(s+1)} \\ \dfrac{2}{(s-3)(s+1)} & \dfrac{s+3}{(s-3)(s+1)} \end{bmatrix}$$

$$\mathcal{L}^{-1}(sI - A_c)^{-1} = \begin{bmatrix} \frac{1}{1+3}\left((-5+3)e^{+3t} - (-5-1)e^{-t}\right) & \frac{-6}{1+3}\left(e^{+3t} - e^{-t}\right) \\ \frac{2}{1+3}\left(e^{+3t} - e^{-t}\right) & \frac{1}{1+3}\left((3+3)e^{+3t} - (3-1)e^{-t}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\left(-e^{3t} + 3e^{-t}\right) & -\frac{3}{2}\left(e^{3t} - e^{-t}\right) \\ \frac{1}{2}\left(e^{3t} - e^{-t}\right) & \frac{1}{2}\left(3e^{3t} - e^{-t}\right) \end{bmatrix}$$

As sampling time, $t = 0.1s$ then

$$A_d = e^{A_c t} = e^{0.1A}$$

$$= \begin{bmatrix} \frac{1}{2}\left(3e^{-0.1} - e^{0.3}\right) & -\frac{3}{2}\left(e^{0.3} - e^{-0.1}\right) \\ \frac{1}{2}\left(e^{0.3} - e^{-0.1}\right) & \frac{1}{2}\left(3e^{0.3} - e^{-0.1}\right) \end{bmatrix} = \begin{bmatrix} 0.682 & -0.667 \\ 0.223 & 1.572 \end{bmatrix}$$

As sampling time, $t = 0.1s$ then

$$A_d = e^{A_c t} = e^{0.1A}$$

$$= \begin{bmatrix} \frac{1}{2}(3e^{-0.1} - e^{0.3}) & -\frac{3}{2}(e^{0.3} - e^{-0.1}) \\ \frac{1}{2}(e^{0.3} - e^{-0.1}) & \frac{1}{2}(3e^{0.3} - e^{-0.1}) \end{bmatrix} = \begin{bmatrix} 0.682 & -0.667 \\ 0.223 & 1.572 \end{bmatrix}$$

$$B_d = \int_0^{0.1} e^{A_c\tau} B \, d\tau$$

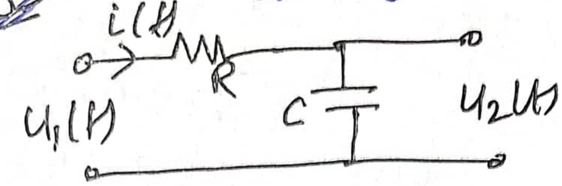
$$= \int_0^{0.1} \begin{bmatrix} \frac{5}{2}(-e^{3\tau} + 3e^{-\tau}) - \frac{3}{2} \times 8 (e^{3\tau} - e^{-\tau}) \\ -\frac{5}{2}(e^{3\tau} - e^{-\tau}) + \frac{1}{2} \times 8 (3e^{3\tau} - e^{-\tau}) \end{bmatrix} d\tau$$

$$= \int_0^{0.1} \begin{bmatrix} -\frac{19}{2} e^{3\tau} - \frac{9}{2} e^{-\tau} \\ -\frac{19}{2} e^{3\tau} - \frac{3}{2} e^{-\tau} \end{bmatrix} d\tau$$

$$B_d = \begin{bmatrix} -0.679 \\ 0.965 \end{bmatrix}$$

Q no 6.

Q6 One energy storage system → 1 degree equation



$R = 130\,\Omega$

$C = 2e^{-6}$ F

Using KVL, we get

$$u_1(t) = i(t)R + u_2(t) \quad \text{—(i)}$$

Since, $i(t) = C\dfrac{du_2(t)}{dt}$

1

Equation (i) becomes,

$$u_1(t) = \frac{C\,du_2(t)}{dt} * R + u_2(t)$$

$$\dot{u}_2 = \frac{-u_2 + u_1}{CR}$$

$$\dot{u}_2 = \left(-\frac{1}{CR}\right)u_2 + \left(\frac{1}{CR}\right)u_1$$

Comparing with $\dot{x} = Ax + Bu$

we get, $A = \left(\dfrac{-1}{CR}\right)$

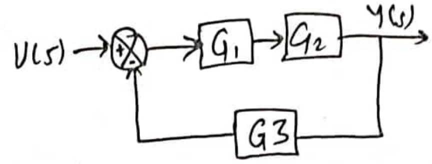
Putting value for C and R,

$$A = \frac{-1}{130 \times 2 \times 10^{-6}}$$

$$\boxed{A = -3846.15\,\Omega^{-1}F^{-1}}$$

Q no 7

Q.7.



Given, $G_1 = \frac{7}{s+9}$

$G_2 = \frac{9}{s+1}$

$G_3 = \frac{3}{s+1}$

$G_1 G_2 = \frac{7}{s+9} \times \frac{9}{s+1} = \frac{63}{s^2+10s+9}$

$G_1 G_2 G_3 = \frac{63}{(s^2+10s+9)} \times \frac{3}{(s+1)} = \frac{189}{(s^2+10s+9)(s+1)}$

$= \frac{189}{s^3+11s^2+19s+9}$

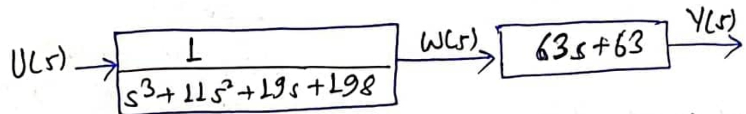
$\frac{Y(s)}{U(s)} = \frac{G_1G_2}{1+G_1G_2G_3}$ $\left[\because \frac{Y(s)}{U(s)} = \frac{G}{1+GH} \right.$ is the representation for closed loop system$\left. \right]$

$\frac{Y(s)}{U(s)} = \frac{G_1G_2}{1+G_1G_2G_3}$

$= \frac{\cancel{\frac{63}{s^2+10s+9}}}{\frac{189+(s^2+10s+9)(s+1)}{\cancel{(s+1)(s^2+10s+9)}}}$

$= \frac{63(s+1)}{189+(s+1)(s^2+10s+9)}$

$\frac{Y(s)}{U(s)} = \frac{63s+63}{s^3+11s^2+19s+198}$



$W(s) = \frac{U(s)}{s^3+11s^2+19s+198}$

$Y(s) = (63s+63)W(s)$

$= 63s\,W(s) + 63W(s)$

$U(s) = s^3 W(s) + 11s^2 W(s) + 19s W(s) + 198 W(s)$

In time domain, we get

$u(t) = \dddot{w} + 11\ddot{w} + 19\dot{w} + 198w$

$y(t) = 63\dot{w} + 63w$

$-11\ddot{w} - 19\dot{w} - 198w - u(t) = \dddot{w}$

$\dot{w}_1 = w_2$

$\dot{w}_2 = w_3$

$\dot{w}_3 = -u(t) - 198w_1 - 19w_2 - 11w_3$

Now, Representing in matrix form, we get

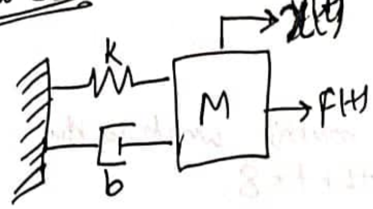
$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -198 & -19 & -11 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} u(t)$$

Comparing with $\dot{x} = Ax + Bu$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -198 & -19 & -11 \end{bmatrix}$$

Q no 8.

Qno 8



The mass damper equation is

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f(t)$$

Here, displacement $= x(t)$

$$m\ddot{x} + b\dot{x} + kx = f(t)$$

$$x_1 = x, \quad x_2 = \dot{x}, \quad x_3 = \ddot{x}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{b}{m}x_2 - \frac{k}{m}x_1 - f(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -b/m \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f(t)$$

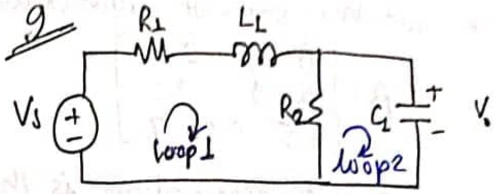
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Comparing with $\dot{x} = Ax + Bu$

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{3}{4} & \frac{-2e-1}{4} \end{bmatrix}$$

Therefore, $A = \begin{bmatrix} 0 & 1 \\ -0.75 & -0.05 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

9



In loop 1,

$$V_s - i(t)R_1 - L_1\frac{di(t)}{dt} - V_0 = 0$$

$$\frac{di(t)}{dt} = \frac{V_0 + i(t)R_1 - V_s}{L_1}$$

$$= \frac{V_0}{L_1} + \frac{i(t)}{L_1}R_1 - \frac{V_s}{L_1}$$

Likewise, current divider rule in loop2

$$i(t) = \frac{V_0}{R_2} + C_1\frac{dV_0}{dt}$$

$$\frac{dV_0}{dt} = \frac{i(t)}{C_1} - \frac{V_0}{R_2C_1}$$

$$x = \begin{pmatrix} i \\ V_0 \end{pmatrix}$$

$$\dot{x} = \begin{bmatrix} R_1/L_1 & 1/L_1 \\ 1/C_1 & -\frac{1}{R_2C_1} \end{bmatrix} x(t) + \begin{bmatrix} -1/L_1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Comparing with $\dot{x} = Ax + Bu$, we get

$$A = \begin{bmatrix} 2 & 1 \\ \frac{10}{3} & -\frac{1}{2.1} \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$