Assignment 3: Controllability, Observability and Pole Placement

Q.1 i) Is the system controllable and observable?

⇒ CONTROLLABLE:- If $[B \;\; AB \;\; A^2B \; \dots]$ matrix is non singular.

OBSERVABLE:- If $\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix}$ matrix isn't singular.

Both of these matrix should have FULL RANK.

Given,

$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \quad C = [0 \;\; 0 \;\; 1] \quad D = [0]$$

$$AB = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad A^2B = A\,AB = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

$$[B \;\; AB \;\; A^2B] = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} +1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$ is full rank. The system is controllable.

$$C = [0 \;\; 0 \;\; 1] \qquad CA = [0 \;\; 0 \;\; 1]\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}^T \qquad CA^2 = [-1 \;\; -1 \;\; 0]\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$= [-2 \;\; -1 \;\; -2]$$

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -1 & 0 \\ -2 & -1 & -2 \end{bmatrix} = \begin{bmatrix} -2 & -1 & -2 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ -2 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Swapping rows $= \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ is full rank. The system is observable

```
>> A = [1 0 1; 1 1 1; -1 -1 0];
>> B=[0; 1 ; 0];
>> C = [0 0 1];
>> D = [0];
>> ctrb(A,B)

ans =

     0     0    -1
     1     1     0
     0    -1    -1

>> rank(ctrb(A,B))

ans =

     3
```

```
>> obsv(A,C)

ans =

     0     0     1
    -1    -1     0
    -2    -1    -2

>> rank(obsv(A,C))

ans =

     3
```

Both [B AB] and [C; CA] matrices are full rank.

2. Consider the following statements related to controllability and observability for a state-space system representation x' = Ax + Bu and y = Cx + Du and choose the correct option(s):

The correct options are:

iii) Observability deals with whether or not the initial state can be observed from the output.

(iv) Controllability studies whether or not the state of a state-space equation can be controlled from both the input and output.

(Q1 ii) $\Rightarrow$ $A = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$, $C = [1 \ 1 \ 1]$

$$AB = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix} \qquad A^2B = A \cdot AB = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}$$

$$[B \quad AB \quad A^2B] = \begin{bmatrix} 0 & 0 & 4 \\ 4 & 4 & 4 \\ 0 & -4 & 0 \end{bmatrix} \cong \begin{bmatrix} 4 & 4 & 4 \\ 0 & -4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$ is full rank. The system is controllable

$$CA = [1 \ 1 \ 1]\begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} = [0 \ 0 \ -2] \qquad CA^2 = [0 \ 0 \ -2]\begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} = [0 \ 2 \ 2]$$

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 2 & 2 \end{bmatrix} \underset{\text{Swapping rows}}{\cong} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$ is full rank. The system is observable.

Therefore, the system is both controllable and observable.

```
>> A = [-1, 0, -1; 1, 1, 0 ; 0, -1, -1];
>> B = [0; 4; 0] ;
>> C = [1, 1, 1];
>> ctrb(A,B)

ans =

     0     0     4
     4     4     4
     0    -4     0

>> rank(ctrb(A,B))

ans =

     3
```

```
>> obsv(A,C)

ans =

     1     1     1
     0     0    -2
     0     2     2

>> rank(obsv(A,C))

ans =

     3
```

3. Find the observability matrix for given state space model

$A_c = [-5\ -4\ -1;\ 5\ -9\ -4;\ -4\ -2\ -4]$

$B_c = [-5;8;7]$ $C_c = [-2\ 0\ 1]$

$$\Rightarrow A_c = \begin{bmatrix} -5 & -4 & -1 \\ 5 & -9 & -4 \\ -4 & -2 & -4 \end{bmatrix} \quad B = \begin{bmatrix} -5 \\ 8 \\ 7 \end{bmatrix}, \quad C_c = [-2\ 0\ 1]$$

The observability matrix is $[C;\ CA;\ CA^2]$

$$CA = [-2\ 0\ 1] \begin{bmatrix} -5 & -4 & -1 \\ 5 & -9 & -4 \\ -4 & -2 & -4 \end{bmatrix}$$

$$= [6\ \ 6\ \ -2]$$

$$CA^2 = CA.A = [6\ \ 6\ \ -2] \begin{bmatrix} -5 & -4 & -1 \\ 5 & -9 & -4 \\ -4 & -2 & -4 \end{bmatrix}$$

$$= [8\ \ -74\ \ -22]$$

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 6 & 6 & -2 \\ 8 & -74 & -22 \end{bmatrix}$$

$$\text{Determinant} = -2(6\times-22-74\times2)+1(6\times-74-6\times8)$$
$$= 68$$
$$\neq 0$$

Therefore, the matrix is observable.

Therefore, the matrix is observable.

4. Find the controllability matrix for the given state space model.

$A_c = [8\ 1\ -6;\ -8\ 5\ -3;\ -9\ -4\ -6]$

$B_c = [0;8;2]$ $C_c = [-8\ -2\ -8]$

$$\Rightarrow A_c = \begin{bmatrix} 8 & 1 & -6 \\ -8 & 5 & -3 \\ -9 & -4 & -6 \end{bmatrix} \quad B_c = \begin{bmatrix} 0 \\ 8 \\ 2 \end{bmatrix} \quad C_c = [-8\ -2\ -8]$$

The controllability matrix is $[B\ AB\ A^2B]$

$$AB = \begin{bmatrix} 8 & 1 & -6 \\ -8 & 5 & -3 \\ -9 & -4 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 2 \end{bmatrix} = \begin{bmatrix} 8-12 \\ 40-6 \\ -32-12 \end{bmatrix} = \begin{bmatrix} -4 \\ 34 \\ -44 \end{bmatrix}$$

$$A^2B = A.AB = \begin{bmatrix} 8 & 1 & -6 \\ -8 & 5 & -3 \\ -9 & -4 & -6 \end{bmatrix} \begin{bmatrix} -4 \\ 34 \\ -44 \end{bmatrix}$$

$$= \begin{bmatrix} -32+34+264 \\ 32+34*5+44*3 \\ 36-34*4+6*44 \end{bmatrix} = \begin{bmatrix} 266 \\ 334 \\ 164 \end{bmatrix}$$

$$[B\ AB\ A^2B] = \begin{bmatrix} 0 & -4 & 266 \\ 8 & 34 & 334 \\ 2 & -44 & 164 \end{bmatrix}$$

Determinant = $-119640 \neq 0$

Therefore, the matrix is controllable

5) Is the system represented by transfer function observable & controllable?

$$H(s) = \frac{5s+5}{s^2+2s+1}$$

$\Rightarrow$ Soln:-

The state space matrix of the given transfer function is:-

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [5 \quad 5] \quad D = [0]$$

For controllability, $[B \quad AB]$ matrix should be full rank / non singular

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad AB = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$[B \quad AB] = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$ which is full rank.

Therefore the system is controllable.

For the system to be observable $\begin{bmatrix} C \\ CA \end{bmatrix}$ matrix should be full rank.

$$C = [5 \quad 5] \quad CA = [5 \quad 5]\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = [-5 \quad -5]$$

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ -5 & -5 \end{bmatrix}$$ is not full rank.

Therefore, the system is not observable.

6) Obtain the gain matrix for the given system such that the closed loop poles are placed at -5 and -4.

$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $C = [1 \quad 1]$ $D = [0]$

$\dot{x} = Ax + Bu$

$= Ax + B(r - kx)$ Feedback of $-kx$ is sent to input. So, $u = r - kx$

$= (A - BK)x + Br$

Assuming $r = 0$, $u = -kx = -[k_1 \quad k_2]\,x$

So, $(A - Bk) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} [k_1 \quad k_2]$

$= \begin{pmatrix} 1-k_1 & 2-k_2 \\ 1 & 1 \end{pmatrix}$

And, $|sI - (A - BK)| = \begin{vmatrix} s-1+k_1 & -2+k_2 \\ -1 & s-1 \end{vmatrix}$

$= s^2 - s + k_1 s - s + 1 - k_1 - 2 + k_2$

The values of s can be obtained by equating determinant to zero.
eigenvalues $s^2 + (k_1 - 2)s + (k_2 - k_1 - 1)$ is the characteristic equation

Since the closed loop poles are -5 and -4. They are the eigenvalues and the characteristic equation corresponding to it is

$(s+5)(s+4) = 0$

$s^2 + 9s + 20 = 0$

Comparing with the above characteristic eqn, we get, $K = [k_1 \quad k_2] = [11 \quad 32]$

7. Consider the state space model of the single input system given below.

$$\begin{bmatrix} x1(k+1) \\ x2(k+1) \\ x3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -2 & -1 \end{bmatrix} \begin{bmatrix} x1(k) \\ x2(k) \\ x3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

Derive the gain matrix of the state feedback system such that the system is supposed to have following eigenvalues.

$\lambda_1 = 0 \quad \lambda_2 = -0.5 - j0.5, \quad \lambda_3 = -0.5 + j0.5$

⇒ The characteristic equation with desired eigenvalue is

$(s-\lambda_1)(s-\lambda_2)(s-\lambda_3) = 0$

$s(s+0.5+j0.5)(s+0.5-j0.5) = 0$

$s[(s+0.5)+j0.5][(s+0.5)+j0.5] = 0$

$s(s^2+s+0.25-j^2 0.25) = 0$

$s^3+s^2+0.5s = 0$

Now we find

$|sI-(A-BK)|$ for which we first find,

$$A_{cl} = (A-BK) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -2 & -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [k_1 \ k_2 \ k_3] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4-k_1 & -2-k_2 & -1-k_3 \end{bmatrix}$$

$$\text{Then } |sI-A_{cl}| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 4+k_1 & 2+k_2 & s+k_3+1 \end{vmatrix}$$

$= s(s(s+1+k_3)+2+k_2)+1(4+k_1)$

$= s(s^2+s+k_3 s+2+k_2)+4+k_1$

$= s^3+s^2+k_3 s^2+2s+k_2 s+4+k_1$

$= s^3+(1+k_3)s^2+(2+k_2)s+4+k_1$

Comparing the determinant with the characteristic eqn above, we get

$(1+k_3) = 1 \qquad 2+k_2 = 0.5 \qquad 4+k_1 = 0$

$k_3 = 0 \qquad k_2 = -1.5 \qquad k_1 = -4$

$\therefore K = [k_1 \ k_2 \ k_3] = [-4 \ -1.5 \ 0]$

8.

$$q_{in} - q_o = A_1 \frac{dh_1}{dt}$$

$$q_1 - q_o = A_2 \frac{dh_2}{dt}$$

$$q_1 = \frac{h_1 - h_2}{R_1}, \quad q_o = \frac{h_2}{R_2}$$

$$q_{in} = A_1 \frac{dh_1}{dt} + \frac{h_1 - h_2}{R_1}$$

Likewise, $A_2 \frac{dh_2}{dt} + q_o = \frac{h_1 - h_2}{R_1}$

$$\boxed{h_1 = R_1 A_2 \frac{dh_2}{dt} + \frac{h_2}{R_2} \times R_1 + h_2}$$

Substituting h_1,

$$q_{in} = A_1 \frac{d}{dt}\left[R_1 A_2 \frac{dh_2}{dt} + \frac{h_2}{R_2} \times R_1 + h_2\right] + \frac{1}{R_1}\left(R_1 A_2 \frac{dh_2}{dt} + \frac{h_2}{R_2} R_1 + h_2\right) - \frac{h_2}{R_1}$$

$$= A_1 R_1 A_2 \frac{d^2 h_2}{dt^2} + \frac{A_1 R_1}{R_2} \frac{dh_2}{dt} + A_1 \frac{dh_2}{dt} + A_2 \frac{dh_2}{dt} + \frac{\cancel{h_2} R_1 h_2}{R_1 R_2} + \cancel{\frac{h_2}{R_1}} - \cancel{\frac{h_2}{R_1}}$$

Replacing value $A_1 = A_2 = 1$ and $R_1 = R_2 = 2$ we get,

$$q_{in} = 2 \frac{d^2 h_2}{dt^2} + 3 \frac{dh_2}{dt} + \frac{1}{2} h_2$$

Taking laplace on both sides, we get

$$Q(s) = 2s^2 H_2(s) + 3sH_2(s) + 0.5 H_2(s)$$

$$\frac{H_2(s)}{Q(s)} = \frac{1}{2s^2 + 3s + 0.5}$$

Coefficient array of denominator = [2 3 0.5]

9. Is the system (MIMO) represented by state space matrices controllable and observable?

$$A = \begin{bmatrix} 0.9984 & 0 & 0.0042 & 0 \\ 0 & 0.9989 & 0 & 0.0033 \\ 0 & 0 & 0.9958 & 0 \\ 0 & 0 & 0 & 0.9967 \end{bmatrix} \quad B = \begin{bmatrix} 0.0083 & 0 \\ 0 & 0.0063 \\ 0 & 0.0048 \\ 0.0031 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \end{bmatrix}$$

→ Soln:

$$AB = \begin{bmatrix} 0.00828672 & 0.0000201 \\ 0.00001023 & 0.006293 \\ 0 & 0.004779 \\ 0.0030897 & 0 \end{bmatrix} \quad A*AB = \begin{bmatrix} 0.0083 & 0 \\ 0 & 0.0063 \\ 0 & 0.0048 \\ 0.0031 & 0 \end{bmatrix}$$

$$A*A^2B = \begin{bmatrix} 0.0083 & 0.0001 \\ 0 & 0.0063 \\ 0 & 0.0047 \\ 0.0031 & 0 \end{bmatrix}$$

$$[B \; AB \; A^2B \; A^3B]$$

$$\begin{bmatrix} 0.0083 & 0 & 0.0083 & \cdots & 0.0083 & 0.0001 \\ 0 & 0.0063 & 0 & \cdots & 0 & 0.0063 \\ 0 & 0.0048 & 0 & \cdots & 0 & 0.0047 \\ 0.0031 & 0 & 0.0031 & \cdots & 0.0031 & 0 \end{bmatrix}$$

```
>> ctrb(A,B)

ans =

  Columns 1 through 6

    0.0083         0    0.0083    0.0000    0.0083    0.0000
         0    0.0063    0.0000    0.0063    0.0000    0.0063
         0    0.0048         0    0.0048         0    0.0048
    0.0031         0    0.0031         0    0.0031         0

  Columns 7 through 8

    0.0083    0.0001
    0.0000    0.0063
         0    0.0047
    0.0031         0

>> rank(ctrb(A,B))

ans =

     4
```

```
>> ctrb(A,B)

ans =

  Columns 1 through 6

    0.0083         0    0.0083    0.0000    0.0083    0.0000
         0    0.0063    0.0000    0.0063    0.0000    0.0063
         0    0.0048         0    0.0048         0    0.0048
    0.0031         0    0.0031         0    0.0031         0

  Columns 7 through 8

    0.0083    0.0001
    0.0000    0.0063
         0    0.0047
    0.0031         0

>> rank(ctrb(A,B))

ans =

     4
```

Both [B AB A^2B A^3B] and [C; CA; CA^2; CA^3] matrices are full rank. So it is both controllable and observable.