Modern Control Theory

Model Predict Control - MPC – Lecture 9

Unconstrained Control move calculation - MIMO

Prediction using State space models (6/6)

System representation

$$x_m(k+1) = A_m x_m(k) + B_m u(k)$$
$$y(k) = C_m x_m(k)$$

System in incremental form

$$\boxed{x(k+1) = Ax(k) + B\Delta u(k)}$$

$$\boxed{y(k+1) = C\,x(k)}$$

$$x(k) = \begin{bmatrix} \Delta x_m(k) \\ y(k) \end{bmatrix}$$

$$A = \begin{bmatrix} A_m & 0_m^T \\ C_m A_m & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} B_m \\ C_m B_m \end{bmatrix}$$

$$C = \begin{bmatrix} 0_m & 1 \end{bmatrix}$$

Output Prediction equation

$$Y = \begin{bmatrix} y(k_i+1 \mid k_i) \\ y(k_i+2 \mid k_i) \\ y(k_i+3 \mid k_i) \\ \vdots \\ y(k_i+N_p \mid k_i) \end{bmatrix} \qquad \Delta U = \begin{bmatrix} \Delta u(k_i) \\ \Delta u(k_i+1) \\ \Delta u(k_i+2) \\ \vdots \\ \Delta u(k_i+N_c-1) \end{bmatrix}$$

$$\boxed{Y = F\,\hat{x}(k_i) + \Phi\,\Delta U}$$

$$F = \begin{bmatrix} CA \\ CA^2 \\ CA^3 \\ \vdots \\ CA^{N_p} \end{bmatrix}$$

Np x 1

$$\Phi = \begin{bmatrix} CB & 0 & 0 & \cdots & 0 \\ CAB & CB & 0 & \cdots & 0 \\ CA^2B & CAB & CB & \cdots & 0 \\ \vdots & & & & \\ CA^{N_p-1}B & CA^{N_p-2}B & CA^{N_p-3}B & \cdots & CA^{N_p-N_c}B \end{bmatrix}$$

Np x Nc

Unconstrained solution – SISO (2/2)

$$J = (R_s - Y)^T (R_s - Y) + \Delta U^T \bar{R} \Delta U$$

$$J = (R_s - F\hat{x}(k_i))^t (R_s - F\hat{x}(k_i)) - 2\Delta U^t \Phi^T (R_s - F\hat{x}(k_i)) + \Delta U^t (\Phi^T \Phi + R)\Delta U$$

Now $$\frac{\partial J}{\partial \Delta U} = -2\Phi^T (R_s - F\hat{x}(k_i)) + 2(\Phi^T \Phi + R)\Delta U = 0$$

$$(\Phi^T\Phi + R)\Delta U = \Phi^T (R_s - F\hat{x}(k_i))$$

Solving for ΔU we get

$$\boxed{\Delta U = (\Phi^T\Phi + R)^{-1}\Phi^T (R_s - F\hat{x}(k_i))}$$

$u_{ki} = u_{(ki-1)} + \Delta u_{ki}$ (first value in ΔU)

- ***Implementation steps ONLINE***

1. ***Obtain the current measurements of states,***
2. ***Estimate the states using Kalman filter***
3. ***Compute the optimal finite horizon control sequence (u_k, u_{k+1}, …u_M)***
4. ***Implement first move u_k***
5. ***Repeat from Step 1***

$$\Delta U(k_i) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} (\Phi^T \Phi + \bar{R})^{-1} (\Phi^T \bar{R}_s r(k_i) - \Phi^T F \hat{x}(k_i))$$

Prediction using State space models - MIMO

m inputs, q outputs and n1 states

For good control q <=m

$$x_m(k+1) = A_m x_m(k) + B_m u(k) + B_d \omega(k)$$
$$y(k) = C_m x_m(k),$$

w(k) is integrated white noise

$$\omega(k) - \omega(k-1) = \epsilon(k)$$

Am is n1 x n1 Bm is n1 x m Bd is n1 x m Cm is q x n1

u is m x 1 y is q x 1

$$\Delta x_m(k) = x_m(k) - x_m(k-1)$$

$$\Delta u(k) = u(k) - u(k-1)$$

$$\Delta y(k+1) = y(k+1) - y(k)$$

$$\begin{bmatrix} \Delta x_m(k+1) \\ y(k+1) \end{bmatrix} = \begin{bmatrix} A_m & o_m^T \\ C_m A_m & I_{q \times q} \end{bmatrix} \begin{bmatrix} \Delta x_m(k) \\ y(k) \end{bmatrix} + \begin{bmatrix} B_m \\ C_m B_m \end{bmatrix} \Delta u(k) + \begin{bmatrix} B_d \\ C_m B_d \end{bmatrix} \epsilon(k)$$

$$y(k) = \begin{bmatrix} o_m & I_{q \times q} \end{bmatrix} \begin{bmatrix} \Delta x_m(k) \\ y(k) \end{bmatrix}$$

Incremental form

$$x(k+1) = Ax(k) + B\Delta u(k) + B_\epsilon \epsilon(k) \qquad y(k) = Cx(k)$$

Augmented
'q' Integrators

$$\rho(\lambda) = \det \begin{bmatrix} \lambda I - A_m & o_m^T \\ -C_m A_m & (\lambda - 1) I_{q \times q} \end{bmatrix} = (\lambda - 1)^q \det(\lambda I - A_m) = 0$$

Prediction using State space models - MIMO

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Output Prediction equation

q*NpX 1

$$Y = \begin{bmatrix} y(k_i+1) \mid k_i \\ y(k_i+2) \mid k_i \\ y(k_i+3) \mid k_i \\ \vdots \\ y(k_i+N_p) \mid k_i \end{bmatrix}$$

Output Prediction equation

m*Nc X 1

$$\Delta U = \begin{bmatrix} \Delta u(k_i) \\ \Delta u(k_i+1) \\ \Delta u(k_i+2) \\ \vdots \\ \Delta u(k_i+N_c-1) \end{bmatrix}$$

$$\boxed{Y = F\,\hat{x}(k_i) + \Phi\,\Delta U}$$

$$F = \begin{bmatrix} CA \\ CA^2 \\ CA^3 \\ \vdots \\ CA^{N_p} \end{bmatrix}$$

(q*Np x (n1+q)

$$\Phi = \begin{bmatrix} CB & 0 & 0 & \cdots & 0 \\ CAB & CB & 0 & \cdots & 0 \\ CA^2B & CAB & CB & \cdots & 0 \\ \vdots & & & & \\ CA^{N_p-1}B & CA^{N_p-2}B & CA^{N_p-3}B & \cdots & CA^{N_p-N_c}B \end{bmatrix}$$

q*Np x m*Nc

Cost function

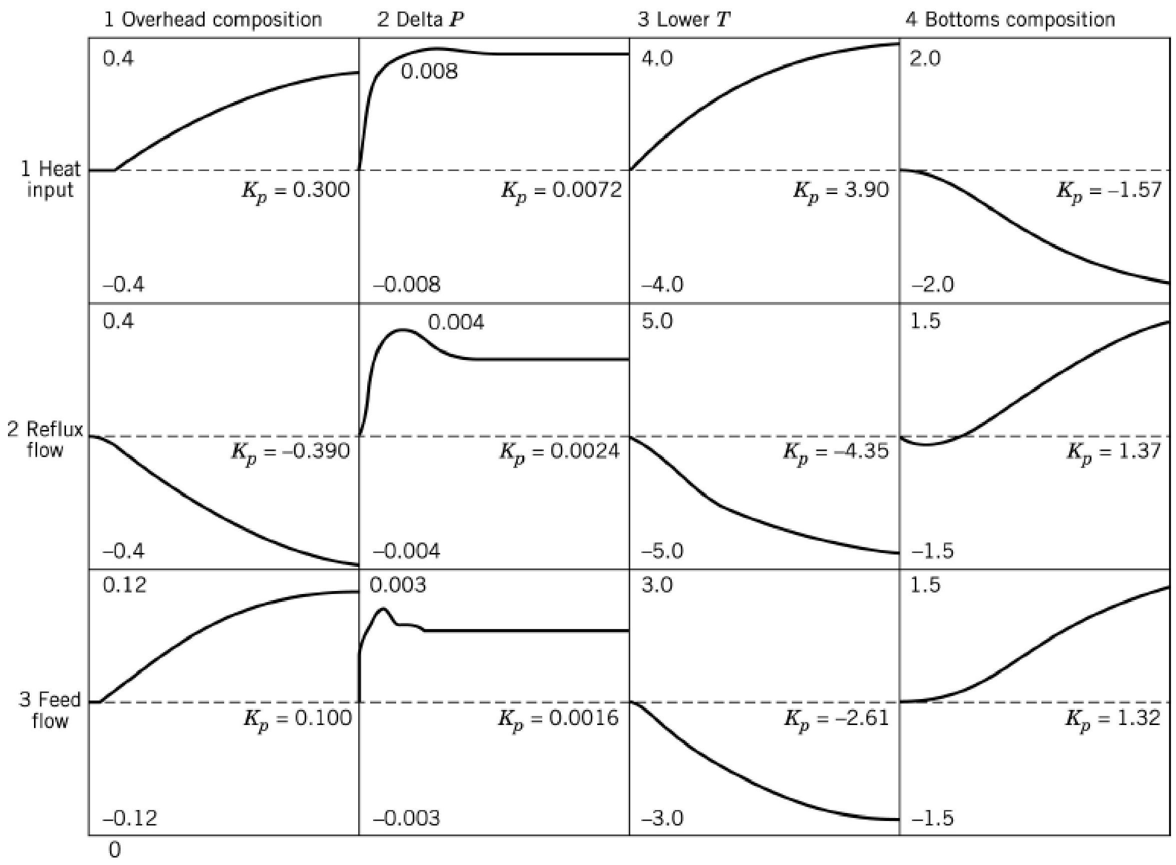
- Minimize the error between set-point and predicted response at all steps
  - This leads faster response

- Minimize the control that minimized the error between set-point and predicted response at all steps

  This leads trade off between faster response and use of input energy/

  constraints on rate of change of input

$$J = (R_s - Y)^T (R_s - Y) + \Delta U^T \bar{R} \Delta U$$



For MIMO system:

Y = step prediction for output (q*Np x 1)

Rs = Set-point trajectory into the future (q*Np x 1)

∆u = Nc future control moves (m*Nc x 1)

R – penalty on control move (m*Nc x m*Nc)

Unconstrained solution – MIMO

$$J = (R_s - F\hat{x}(k_i))^t (R_s - F\hat{x}(k_i)) - 2\Delta U^t \Phi^T (R_s - F\hat{x}(k_i)) + \Delta U^t (\Phi^T \Phi + R)\Delta U$$

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$u_{ki} = u_{(ki-1)} + \Delta u_{ki}$ (first m value in ΔU)

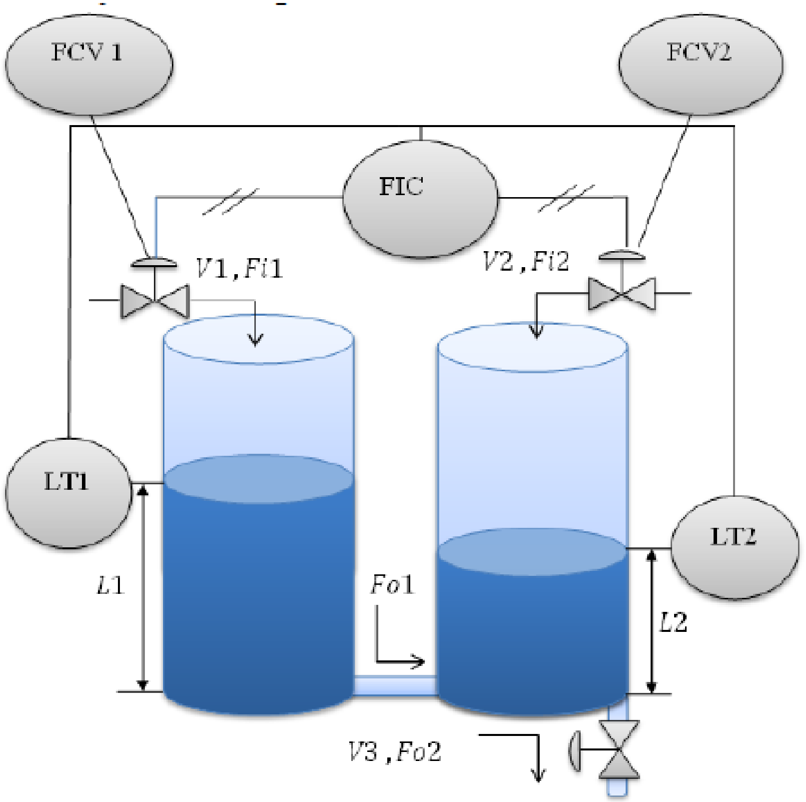
- ***Implementation steps ONLINE***

1. ***Obtain the current measurements of states,***
2. ***Estimate the states using Kalman filter***
3. ***Compute the optimal finite horizon control sequence (u_k, u_{k+1}, …u_M)***
4. ***Implement first move u_k for each of the inputs***
5. ***Repeat from Step 1***

$$\Delta u(k_i) = \overbrace{\begin{bmatrix} I_m & o_m & \ldots & o_m \end{bmatrix}}^{N_c} (\Phi^T \Phi + \bar{R})^{-1} (\Phi^T \bar{R}_s r(k_i) - \Phi^T F x(k_i))$$

Example problem

Level control of coupled tanks



$$A1.\frac{dL1}{dt} = Fi1 - Fo1 \qquad A2.\frac{dL2}{dt} = Fi2 - Fo2 + Fo1$$

$$Fo1 = \alpha 1.\sqrt{L1 - L2} \qquad Fo2 = \alpha 2.\sqrt{L2}$$

Linearize around some operating point (h1.h2), we get

$$x = \begin{pmatrix} h1 \\ h2 \end{pmatrix} \quad u = \begin{pmatrix} fi1 \\ fi2 \end{pmatrix} \quad y = \begin{pmatrix} h1 \\ h2 \end{pmatrix}$$

$$A_m = \begin{pmatrix} -\frac{\alpha 1}{2.A1}.\frac{1}{\sqrt{(L1-L2)}} & \frac{\alpha 1}{2.A1}.\frac{1}{\sqrt{(L1-L2)}} \\ \frac{\alpha 1}{2.A2}.\frac{1}{\sqrt{(L1-L2)}} & \frac{-1}{2.A2}.\left(\frac{\alpha 1}{\sqrt{(L1-L2)}} + \frac{\alpha 2}{\sqrt{L2}}\right) \end{pmatrix}$$

$$B_m = \begin{pmatrix} \frac{1}{A1} & 0 \\ 0 & \frac{1}{A2} \end{pmatrix} \quad C_m = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad D_m = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Matrices to compute:

Φ, F, $\Phi^T\Phi$, $\Phi^T R_s$, $\Phi^T F$

$$\Delta U = (\Phi^T\Phi + R)^{-1}\Phi^T (R_s - F\hat{x}(k_i))$$

Demo

Linearize around
L1 = 4m, L2 = 3.5m

$$A_m = \begin{pmatrix} -7.923 & 7.923 \\ 9.781 & -12.97 \end{pmatrix} \quad B_m = \begin{pmatrix} 5.093 & 0 \\ 0 & 6.288 \end{pmatrix} \quad C_m = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Sampling time = 0.05 secs, discretize
- Np = 10, Nc = 3