Time Domain Representation	Frequency Domain
Time domain graph shows how signal changes over time	Frequency domain graph shows how much of signal lies within each frequency
In a time domain graph, the magnitude x(t) of the signal at each time instant is represented	Signal is represented by a sum of sinusoids of different frequencies (w/f), each with certain magnitude X(w). Phase shift of each of sinusoids are plotted.

Consider the mass-spring-damper system as:

$$M\ddot{x} + B\dot{x} + Kx = 0$$

Solution to this equation is of the form: $x(t) = e^{-\sigma t}(A\sin(\omega t + \phi))$

The solution of above form of differential equation can have both the exponential and sinusoidal terms. Exponential term is due to damper while sinusoidal term is due to interconnections between mass and spring.

It is observed that the solutions of differential equations of LTI systems are either exponential or sinusoids or combination of both.

In frequency domain transformation, signals are decomposed into sinusoids described by an amplitude and phase at each frequency. To account for exponential response as well, we extend the idea of frequency domain representation. A new transformation is defined such that signals are decomposed into both sinusoids and exponential.

Laplace Transform

Laplace transform decomposes signals in time domain into a domain of both sine and exponential functions. This domain is called as s-domain. s is a complex number i.e. s-plane is two dimensional: one dimension to describe the frequency of sine wave (ω) and another to describe the exponential term (σ).

Given a signal x(t), its Laplace transform is given by:

$$L(x(t)) = X(s) = \int_0^\infty x(t)e^{-st}dt$$

Where ‘s’ is complex variable defined as $s = \sigma + j\omega$

Existence of Laplace transform depends on the convergence of above integral which depends on the value of σ .

---xxx---	Time domain signal	Laplace transform
Impulse function	$\delta(t)$	1
Exponential Function	e^{-at}	$\frac{1}{s+a}$
	t	$\frac{1}{s^2}$
	t^n	$\frac{n!}{s^{n+1}}$
	$\sin(at)$	$\frac{a}{s^2 + a^2}$
	$\cos(at)$	$\frac{s}{s^2 + a^2}$
Linearity	if $L[x_1(t)] = X_1(s)$ and $L[x_2(t)] = X_2(s)$ then, $L[ax_1(t) + bx_2(t)] = aX_1(s) + bX_2(s)$	
Time Shifting	$L[x(t)] = X(s)$ then, $L[x(t - t_o)] = e^{-st_o}X(s)$	
Time shifting Example	t	$\frac{1}{s^2}$
	$(t - a)$	$e^{-sa}\frac{1}{s^2}$
Time Scaling	$L[x(t)] = X(s)$ then, $L[x(at)] = \frac{1}{\vert a\vert}X(\frac{s}{a})$	
Time Scaling Example	$\sin(t)$	$\frac{1}{s^2 + 1}$
	$\sin(at)$	
Time Scaling Example	e^{-t}	$\frac{1}{s + 1}$
	e^{-at}	$\frac{1}{\vert a\vert}\times\frac{1}{\frac{s}{a} + 1} = \frac{1}{s + a}$

Time scaling Example:

$$L[\sin(t)] = \frac{1}{s^2 + 1}; \quad L[\sin(at)] = \frac{1}{|a|}\times\frac{1}{\left(\frac{s}{a}\right)^2 + 1} = \frac{a}{s^2 + a^2}$$

$$L[e^{-t}] = \frac{1}{s + 1}; \quad L[e^{-at}] = \frac{1}{|a|}\times\frac{1}{\frac{s}{a} + 1} = \frac{1}{s + a}$$

$$L[e^{t}] = \frac{1}{s - 1}; \quad L[e^{at}] = \frac{1}{|a|}\times\frac{1}{\frac{s}{a} - 1} = \frac{1}{s - a}$$

Time scaling and Time Shifting Combined

$$L[e^{-t}] = \frac{1}{s + 1} \;; \quad L[e^{-(t-t_o)}] = e^{-st_o}\frac{1}{s + 1}$$

$$L[e^{-(at-t_o)}] = L[e^{-a\left(t - \frac{t_o}{a}\right)}] = \frac{1}{|a|}e^{\frac{-st_o}{a^2}}\frac{1}{\frac{s}{a} + 1} = e^{\frac{-st_o}{a^2}}\frac{1}{s + a}$$

$$L[\sin(at)] = \frac{1}{|a|}\times\frac{1}{\left(\frac{s}{a}\right)^2 + 1} = \frac{a}{s^2 + a^2} \;; \quad L[\sin(at-\phi)] = e^{-s\phi}\frac{a}{s^2 + a^2}$$

Time Reversal Property:

$$L[x(t)] = X(s) \;;\; L[x(-t)] = X(-s)$$

Time Differentiation:

if $L[x(t)] = X(s)$, then

$$L\left[\frac{dx(t)}{dt}\right] = sX(s) - x(0)$$

$$L\left[\frac{d^2x(t)}{dt^2}\right] = s^2X(s) - sx(0) - x'(0)$$

$$L\left[\frac{d^nx(t)}{dt^n}\right] = s^nX(s) - s^{n-1}x(0) - s^{n-2}x'(0) \;\ldots\; -x^{n-1}(0)$$

Time Integration

$$\text{if } L[x(t)] = X(s), \text{ then}$$
$$L\left[\int_0^t x(\tau)d\tau\right] = \frac{1}{s}X(s)$$

Frequency Differentiation

$$\text{if } L[x(t)] = X(s), \text{ then}$$
$$L[t\ x(t)] = -\frac{d\ X(s)}{ds}$$
$$L[t^n\ x(t)] = (-1)^n \frac{d^n\ X(s)}{ds^n}$$

Frequency Differentiation Example

$$L[\sin(at)] = \frac{a}{s^2 + a^2}$$
$$L[t\ \sin(at)] = -\frac{d}{ds}\left(\frac{a}{s^2 + a^2}\right) = -\frac{(0 - a(2s+0))}{(s^2 + a^2)^2} = -\frac{2as}{(s^2 + a^2)^2}$$

Frequency Integration

$$\text{if } L[x(t)] = X(s), \text{ then}$$
$$L[\frac{1}{t}\ x(t)] = \int_s^\infty X(u)du$$

Frequency Shifting:

$$\text{if } L[x(t)] = X(s), \text{ then}$$
$$L[e^{s_o t}\ x(t)] = X(s - s_o)$$

Example of frequency shifting:

$$L[\sin(at)] = \frac{a}{s^2 + a^2}$$
$$L[e^{bt}\ \sin(at)] = \frac{a}{(s-b)^2 + a^2}$$
$$L[e^{at}\ \sin(at)] = \frac{a}{(s-a)^2 + a^2} = \frac{a}{s^2 - 2sa + 2a^2}$$

Frequency Scaling

Periodic Function

Laplace transform of a piecewise periodic function f(t) with period p is given by:

$$X(s) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} x(t)\,dt$$

Initial Value Theorem

Relates the s-domain expression to the time domain behavior as time approaches zero.

$\lim_{t \to 0} x(t) = \lim_{s \to \infty} sX(s)$ provided that the Laplace transform exist and its limit exist as s approaches infinity.

Final Value Theorem

Relates the s-domain expression to the time domain behavior as time approaches infinity.

$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$ provided that the Laplace transform X(s) exist and its limit exist as s approaches 0, and also final value should exist.