CH5120-Assignment-3:Controllability, Observability and Pole placement

Q.No-1: Consider the state-space representation of a system described by x' = Ax + Bu and y = Cx where x is the state vector, u is the input and y is the output vector. Is the system controllable and observable?

i) A = [1 0 1; 1 1 1; -1 -1 0]; B = [0 ; 1 ; 0];C = [0 0 1];D = [0]

for controllability: $[B \quad AB \quad A^2B]$ matrix should be full rank

for observability: $\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$ matrix should be full rank

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad C = [0 \quad 0 \quad 1]$$

$$AB = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}; \quad A^2B = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix};$$

$$CA = [-1 \quad -1 \quad 0]; \quad CA^2 = [-2 \quad -1 \quad -2]$$

Controllability Matrix: $[B \quad AB \quad A^2B]$

$$= \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

By row exchange

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

↳ There are 3 pivot columns

→ Controllability matrix is full rank so, system is controllable

Observability Matrix: $\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -1 & 0 \\ -2 & -1 & -2 \end{bmatrix}$

By row exchange

$$\begin{bmatrix} -1 & -1 & 0 \\ -2 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}; \quad R_2 \to R_2 - 2R_1 \Rightarrow \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

↳ There are 3 pivot columns

Observability matrix is also full rank and hence system is observable.

```
>> B=[0;1;0];
>> C=[0 0 1];
>> D=[0];
>> sys=ss(A,B,C,D);
>> ctrb(sys)

ans =

     0     0    -1
     1     1     0
     0    -1    -1

>> rank(ctrb(sys))

ans =

     3

>> obsv(sys)

ans =

     0     0     1
    -1    -1     0
    -2    -1    -2

>> rank(obsv(sys))

ans =

     3
```

ii) A = [-1, 0, -1; 1, 1, 0 ; 0, -1, -1]; B = [0; 4; 0] ;C = [1, 1, 1]

$$A = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix}; \quad A^2B = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}; \quad CA = \begin{bmatrix} 0 & 0 & -2 \end{bmatrix}; \quad CA^2 = \begin{bmatrix} 0 & 2 & 2 \end{bmatrix}$$

Controllability Matrix: $\begin{bmatrix} B & AB & A^2B \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 4 \\ 4 & 4 & 4 \\ 0 & -4 & 0 \end{bmatrix}$$

By row exchange

$$= \begin{bmatrix} 4 & 4 & 4 \\ 0 & -4 & 0 \\ 0 & 0 & 4 \end{bmatrix}; \text{ There are 3 pivot columns}$$

Controllability matrix is full rank and hence system is controllable.

Observability Matrix: $\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 2 & 2 \end{bmatrix}$

By row exchange

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{bmatrix}; \text{ There are 3 pivot columns}$$

Observability matrix is full rank, so system is observable.

Q.No-2: Consider the following statements related to controllability and observability for a state-space system representation x' = Ax + Bu and y = Cx + Du and choose the correct option(s):

→ correct statements are:

→ Observability deals with whether or not the initial state can be observed from the output.

→ Controllability studies whether or not the state of a state-space equation can be controlled from both the input and output.

Q.No-3: Find the observability matrix for the given state space model.
Ac = [-5 -4 -1 ; 5 -9 -4 ; -4 -2 -4] Bc = [-5 ; 8 ; 7] Cc = [-2 0 1]

$$A = \begin{bmatrix} -5 & -4 & -1 \\ 5 & -9 & -4 \\ -4 & -2 & -4 \end{bmatrix}; \quad C = \begin{bmatrix} -2 & 0 & 1 \end{bmatrix}$$

Observability matrix is $\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$

$$CA = \begin{bmatrix} 16 & 6 & -2 \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} 8 & -74 & -22 \end{bmatrix}$$

$\therefore$ observability matrix is $\begin{bmatrix} -2 & 0 & 1 \\ 16 & 6 & -2 \\ 8 & -74 & -22 \end{bmatrix}$

```
A =

    -5    -4    -1
     5    -9    -4
    -4    -2    -4

>> C=[-2 0 1]

C =

    -2     0     1

>> C*A

ans =

     6     6    -2

>> C*A*A

ans =

     8   -74   -22

>> obsv(A,C)

ans =

    -2     0     1
     6     6    -2
     8   -74   -22
```

Q.No-4: Find the controllability matrix for the given state space model.
Ac = [8 1 -6 ; -8 5 -3 ; -9 -4 -6] Bc = [0 ; 8 ; 2] Cc = [-8 -2 -8]

$$A = \begin{bmatrix} 8 & 1 & -6 \\ -8 & 5 & -3 \\ -9 & -4 & -6 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 8 \\ 2 \end{bmatrix}$$

controllability matrix is $\begin{bmatrix} B & AB & A^2B \end{bmatrix}$

$$= \begin{bmatrix} 0 & -4 & 266 \\ 8 & 34 & 334 \\ 2 & -44 & 164 \end{bmatrix}$$

```
A =

     8     1    -6
    -8     5    -3
    -9    -4    -6

>> B=[0;8;2]

B =

     0
     8
     2

>> ctM=[B A*B (A^2)*B]

ctM =

     0    -4   266
     8    34   334
     2   -44   164

>> rank(ctM)

ans =

     3
```

Q.No-5: Is the system represented by the transfer function H(s) = (5*s + 5)/(s^2 + 2*s + 1), observable and controllable?

$$H(s) = \frac{5s+5}{s^2+2s+1}$$

Let $H(s) = \frac{W(s)}{U(s)} \times \frac{Y(s)}{W(s)}$; where $\frac{W(s)}{U(s)} = \frac{1}{s^2+2s+1}$; $\frac{Y(s)}{W(s)} = 5s+5$

$$\frac{W(s)}{U(s)} = \frac{1}{s^2+2s+1}$$

or, $s^2 W(s) + 2s W(s) + W(s) = U(s)$

Taking inverse laplace transform {initial condition zero}

$$\frac{d^2 w(t)}{dt^2} + 2\frac{dw(t)}{dt} + w(t) = u(t)$$

The state equation in statespace becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t)$$

{Assumption
$x_1(t) = w(t)$
$x_2(t) = \frac{dw(t)}{dt}$}

$$\frac{Y(s)}{W(s)} = 5s+5$$

or, $Y(s) = 5sW(s) + 5W(s)$

Taking inverse laplace transform

$$y(t) = 5\frac{dw(t)}{dt} + 5w(t)$$

output equation/Measurement equation in State-space form is

$$[y] = \underbrace{[5 \quad 5]}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

controllability matrix is $[B \quad AB]$

$$= \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$; controllability matrix is full rank, So system is Controllable

observability matrix is $\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ -5 & -5 \end{bmatrix}$

→ observability matrix is rank dificient, so system is not observable.

```
>> den=[1 2 1];
>> [A,B,C,D]=tf2ss(num,den)

A =

    -2    -1
     1     0

B =

     1
     0

C =

     5     5

D =

     0

>> ctrb(A,B)

ans =

     1    -2
     0     1

>> obsv(A,C)

ans =

     5     5
    -5    -5
```

Q.No-6: Obtain the gain matrix for the given system such that the closed loop poles are placed at -5 and -4
A = [1 2; 1 1]; B = [1; 0]; C = [1 1]; D = [0]

Desired location of poles: -5, -4;

So desired characteristics equation of the system is

$$(s+5)(s+4)=0$$

or, $s^2+9s+20=0$ —— (i)

The characteristics equation for closed loop control of above system will be

$\det(sI-A+BK)=0$ 'K' is gain matrix

Let $K=[K_1, K_2]$ then

$$BK=\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}K_1 & K_2\end{bmatrix}=\begin{bmatrix}K_1 & K_2\\0 & 0\end{bmatrix}$$

$$\det(sI-A+BK)=\begin{vmatrix}s-1+K_1 & -2+K_2\\-1 & s-1\end{vmatrix}$$

$$=(s-1+K_1)(s-1)+(K_2-2)$$

$$=s^2+(K_1-2)s-K_1+1+K_2-2$$

$$=s^2+(K_1-2)s+(K_2-K_1-1)$$

∴ Characteristics equation is

$s^2+(K_1-2)s+(K_2-K_1-1)=0$ —— (ii)

Comparin coefficient of equation (i) and (ii)

$K_1-2=9$ $\quad$ $K_2-K_1-1=20$

$\therefore K_1=11$ $\quad$ $\therefore K_2=20+11+1=32$

Hence gain matrix $K=\begin{bmatrix}11 & 32\end{bmatrix}$

```
>> B = [1;0];
>> C=[1 1];
>> D=[0];
>> K=[11 32];
>> Acl=A-B*K;
>> sys=ss(Acl,B,C,D)

sys =

  A =
         x1   x2
   x1   -10  -30
   x2     1    1

  B =
        u1
   x1    1
   x2    0

  C =
        x1  x2
   y1    1   1

  D =
        u1
   y1    0

Continuous-time state-space model.

>> pole(sys)

ans =

   -5.0000
   -4.0000
```

Q.No-7: Consider the state space model of the single input system given below. Derive the gain matrix of the state feedback system such that the system is supposed to have the following eigenvalues. $\lambda 1 = 0$; $\lambda 2 = -0.5 - j\,0.5$; $\lambda 3 = -0.5 + j\,0.5$

$$\begin{bmatrix} x1(k+1) \\ x2(k+1) \\ x3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -2 & -1 \end{bmatrix} \begin{bmatrix} x1(k) \\ x2(k) \\ x3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

Desired characteristics equation is

$$S(S+0.5+j0.5)(S+0.5-j0.5)=0$$

or, $S\left((S+0.5)^2-(j0.5)^2\right)=0$

or, $S\left[S^2+S+\frac{1}{4}+\frac{1}{4}\right]=0$

$\therefore S^3+S^2+0.5S=0$ —— (i)

Let $K=[K_1 \quad K_2 \quad K_3]$ be the gain matrix.

$$BK=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}[K_1 \quad K_2 \quad K_3] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ K_1 & K_2 & K_3 \end{bmatrix}$$

Characteristics equation for the closed loop system will be

$$\det(SI-A+BK)=0$$

or, $$\begin{vmatrix} S & -1 & 0 \\ 0 & S & -1 \\ K_1+4 & K_2+2 & S+K_3+1 \end{vmatrix}=0$$

$$S^3-(-K_3-1)S^2+S(K_2+2)+K_1+4=0 \quad \text{—— (ii)}$$

Comparing coefficients of equation (i) and (ii)

$-(-K_3-1)=1$ $\qquad K_2+2=0.5$ $\qquad K_1+4=0$

or, $K_3+1=1$ $\qquad \therefore K_2=-\frac{3}{2}$ $\qquad \therefore K_1=-4$

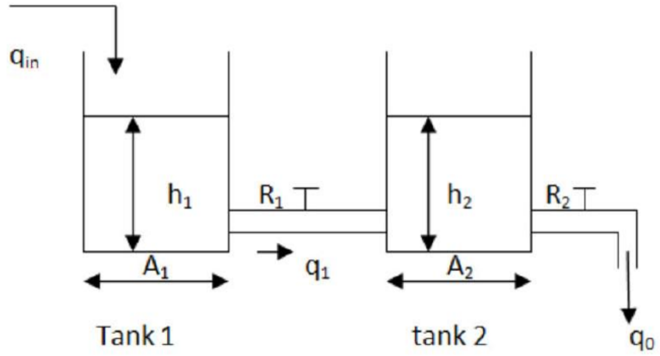
$\therefore K_3=0$

Hence gain matrix is $K=[-4 \quad -1.5 \quad 0]$

Verification in MATLAB

```
>> s*(s+0.5+0.5j)*(s+0.5-0.5j)

ans =

  s^3 + s^2 + 0.5 s

Continuous-time transfer function.

>> A=[0 1 0; 0 0 1; -4 -2 -1];
>> B=[0;0;1];
>> C=[0 0 0];
>> D=[0];
>> syms k1 k2 k3 l;
>> K=[k1 k2 k3];
>> M=l*eye(3) - (A - B*K)

M =

[      l,      -1,          0]
[      0,       l,         -1]
[ k1 + 4, k2 + 2, k3 + l + 1]

>> det(M)

ans =

k1 + 2*l + k2*l + k3*l^2 + l^2 + l^3 + 4

>> K = [ -4 -1.5 0]

K =

   -4.0000   -1.5000         0

>> Acl= A - B*K

Acl =

         0    1.0000         0
         0         0    1.0000
         0   -0.5000   -1.0000

>> system=ss(Acl,B,C,D);
>> pole(system)

ans =

   0.0000 + 0.0000i
  -0.5000 + 0.5000i
  -0.5000 - 0.5000i
```

Q.No-8: The figure shows the interaction of two tanks. Assuming the rate of flow from each tank, q(t) = driving force (h(t))/resistance (R), find the coefficient array of the denominator of the transfer function h2(s)/qin(s) along with the controllability and observability matrix.
Given: A1 = 1; A2 = 1; R1 = 2; R2 = 2



$$q_1 = \frac{h_1(t) - h_2(t)}{R_1} \quad \text{—— (i)}$$

$$q_o = \frac{h_2(t)}{R_2} \quad \text{—— (ii)}$$

$$A_1 \frac{dh_1(t)}{dt} = q_{in}(t) - q_1(t)$$

$$= q_{in}(t) - \frac{1}{R_1}\left(h_1(t) - h_2(t)\right)$$

$$\therefore q_{in}(t) = A_1 \frac{dh_1(t)}{dt} + \frac{1}{R_1}\left[h_1(t) - h_2(t)\right] \quad \text{—— (iii)}$$

$$A_2 \frac{dh_2(t)}{dt} = q_1(t) - q_o(t)$$

$$A_2 \frac{dh_2(t)}{dt} = \frac{1}{R_1}\left[h_1(t) - h_2(t)\right] - \frac{1}{R_2} h_2(t)$$

$$\text{or, } A_2 \frac{dh_2(t)}{dt} = \frac{1}{R_1} h_1(t) - \frac{1}{R_1} h_2(t) - \frac{1}{R_2} h_2(t)$$

$$= \frac{1}{R_1} h_1(t) - \frac{R_1 + R_2}{R_1 R_2} h_2(t)$$

$$\therefore h_1(t) = A_2 R_1 \frac{dh_2(t)}{dt} + \frac{R_1 + R_2}{R_2} h_2(t) \quad \text{—— (iv)}$$

using equation (iv) in equation (iii)

$$\therefore q_{in}(t) = A_1 \frac{d}{dt}\left(A_2 R_1 \frac{dh_2(t)}{dt} + \frac{R_1+R_2}{R_2} h_2(t)\right) + \frac{1}{R_1}\left[A_2 R_1 \frac{dh_2(t)}{dt} + \frac{R_1+R_2}{R_1} h_2(t) - h_2(t)\right]$$

$$= A_1 A_2 R_1 \frac{d^2 h_2(t)}{dt^2} + A_1 \frac{(R_1+R_2)}{R_2} \frac{dh_2(t)}{dt} + A_2 \frac{dh_2(t)}{dt} + \frac{R_1+R_2}{R_1^2} h_2(t) - \frac{1}{R_1} h_2(t)$$

$$\therefore q_{in}(t) = A_1 A_2 R_2 \frac{d^2 h_2(t)}{dt^2} + \left(A_1 \frac{(R_1+R_2)}{R_2} + A_2\right) \frac{dh_2}{dt} + \frac{R_2}{R_1^2} h_2(t)$$

$$Q_{in}(s) = A_1 A_2 R_2 s^2 H_2(s) + \left(A_1 \left(\frac{R_1+R_2}{R_2}\right) + A_2\right) s H_2(s) + \frac{R_2}{R_1^2} H_2(s)$$

$$\therefore \frac{H_2(s)}{Q_{in}(s)} = \frac{1}{A_1 A_2 R_2 s^2 + \left(A_1 \frac{(R_1+R_2)}{R_2} + A_2\right)s + \frac{R_2}{R_1^2}}$$

or, $\frac{H_2(s)}{Q_{in}(s)} = \frac{1}{2s^2 + 3s + 1/2}$

coefficients of denominator of transfer functions $\frac{H_2(s)}{Q_{in}(s)}$ are $Den = [2 \quad 3 \quad 1/2]$

Let $y = h_2(t)$

$x_1(t) = y = h_2(t)$

$x_2(t) = \dot{x}_1(t) = \frac{dh_2(t)}{dt}$

$$\frac{d^2h_2(t)}{dt^2} = \frac{1}{2} q_{in}(t) - \frac{3}{2} x_2(t) - \frac{1}{2} x_1(t)$$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1/2 & -3/2 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}}_{B} q_{in}(t)$$

$$y(t) = \underbrace{[1 \quad 0]}_{C} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Controllability Matrix: $[B \quad AB] = \begin{bmatrix} 0 & 1/2 \\ 1/2 & -3/4 \end{bmatrix}$

Observability Matrix: $\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Q.No-9: Is the system (MIMO) represented by the state space matrices controllable and observable?
A = [0.9984, 0, 0.0042, 0; 0, 0.9989, 0, 0.0033; 0, 0, 0.9958, 0; 0, 0, 0, 0.9967]
B = [0.0083, 0; 0, 0.0063; 0, 0.0048; 0.0031, 0]   C = [0.5, 0, 0, 0; 0, 0.5, 0, 0]

```
>> A=[0.9984 0 0.0042 0; 0 0.9989 0 0.0033; 0 0 0.9958 0; 0 0 0 0.9967];
>> B=[0.0083 0; 0 0.0063; 0 0.0048; 0.0031 0];
>> C=[0.5 0 0 0; 0 0.5 0 0];
>> D=[0];
>> system=ss(A,B,C,D);
>> contMat=ctrb(system);
>> obsvMat=obsv(system);
>> nulContMat = size(null(contMat),2)

nulContMat =

     4
```
{Nullity = 4 ⇒ Rank deficient}

```
>> nulObsvMat= size(null(obsvMat),2)

nulObsvMat =

     0
```
{Nullity = 0 ⇒ Full rank}

```
>>
```

→ Controllability matrix is rank deficient, hence system is not controllable.

→ Observability matrix is full rank, hence system is observable.