Modern Control Theory

Model Predict Control - MPC – Lecture 7

Dynamic Prediction Models

Prediction using State space models (6/6)

System representation

$$x_m(k+1) = A_m x_m(k) + B_m u(k)$$
$$y(k) = C_m x_m(k)$$

System in incremental form

$$\boxed{x(k+1) = A x(k) + B \Delta u(k)}$$

$$\boxed{y(k+1) = C\, x(k)}$$

State Prediction equation

$$X(k) = \Phi\, x_0 + T\Delta U(k)$$

$$X := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{pmatrix} \quad U := \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{pmatrix}$$

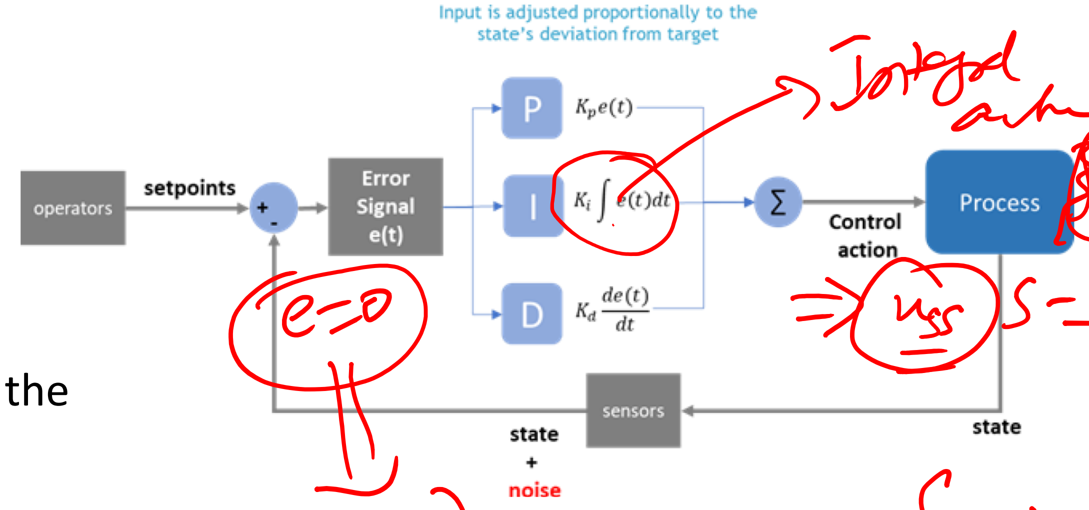
$$\Phi := \begin{pmatrix} A \\ A^2 \\ \vdots \\ A^N \end{pmatrix} \quad \Gamma := \begin{pmatrix} B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{pmatrix} \quad A = \begin{bmatrix} A_m & 0_m^T \\ C_m A_m & 1 \end{bmatrix} \quad B = \begin{bmatrix} B_m \\ C_m B_m \end{bmatrix} \quad C = \begin{bmatrix} 0_m & 1 \end{bmatrix}$$

Output Prediction equation

$$Y(k+1) = C X(k)$$

Importance of Integrators – offset free control

Classical Feedback control



Steady error of closed loop system goes to zero if the

- Controller has an integral action or
- plant already has integral action

SS

$(y_p - y_m)$ → ∫ to step response

Discrete system

$$\overbrace{\begin{bmatrix}\Delta x_m(k+1)\\ y(k+1)\end{bmatrix}}^{x(k+1)} = \overbrace{\begin{bmatrix}A_m & o_m^T\\ C_mA_m & 1\end{bmatrix}}^{A}\overbrace{\begin{bmatrix}\Delta x_m(k)\\ y(k)\end{bmatrix}}^{x(k)} + \overbrace{\begin{bmatrix}B_m\\ C_mB_m\end{bmatrix}}^{B}\Delta u(k) \qquad y(k) = \overbrace{\begin{bmatrix}o_m & 1\end{bmatrix}}^{C}\begin{bmatrix}\Delta x_m(k)\\ y(k)\end{bmatrix} + \text{free}$$

$y(k) = y_{free} + y_{forced} + \ldots$

$(y_k - y_m)$

Eigen matrix of A $\quad \rho(\lambda) = \det(\lambda I - A) = \det\begin{bmatrix}\lambda I - A_m & o_m^T\\ -C_mA_m & (\lambda-1)\end{bmatrix}$

$= (\lambda\text{-}1)^*\det(\lambda I\text{-}A_m) = (\lambda\text{-}1)^*\text{Eigen}(A_m)$

$S \rightarrow 0 \leftrightarrow$ integral action

Eigenvalue

The delta model add at least one integrator to the original open loop system

For MIMO the system , the integrators would be as many states

State space model – need of observer (Estimator / State)

$$x_m(k+1) = A_m x_m(k) + B_m u(k)$$
$$y(k) = C_m x_m(k)$$

For prediction propagation, what can we say about the state measurements?

- State space model assumes the all states are measurable
- If only few measurements exist, how would be propagate the states?

The answer is Observer/ State Estimator

measure

x_0

$= Ax_0 + Bu_0$

$x_2 = Ax_1 + Bu_1$

x_3

The observer allows us to estimate the states from the measurements

(Cm, Am) are observable pair

Estimator re-cap

Let $\hat{\boldsymbol{x}}_m(k)$ denote the estimates of $x_m(k)$ *given* $\hat{\boldsymbol{x}}_m(0)$

$$\hat{\boldsymbol{x}}_m(k+1) = A_m \hat{\boldsymbol{x}}_m(k) + B_m u(k)$$ Estimated system

$$\boldsymbol{x}_m(k+1) = A_m \boldsymbol{x}_m(k) + B_m u(k)$$ Actual system

Define $\tilde{x}_m(k) = x_m(k) - \hat{x}_m(k)$

$$\tilde{x}_m(k+1) = A_m(x_m(k) - \hat{x}_m(k))$$

$$= A_m \tilde{x}_m(k)$$

$\tilde{x}_m(0) \neq 0$ $\quad \boxed{\tilde{x}_m(k) = A_m^k \tilde{x}_m(0)}$

$\boxed{x_m(k) \quad \hat{x}_m(k)}$

$\lambda < 1$

$\tilde{x}_m F$

$$\frac{1}{(s+2)(s-3)}$$

e^{3t}

e^{-2t}

This is open loop prediction with standard pitfalls:
Works only if initial condition is accurate
Convergence is not guaranteed

How do we design Estimator

Use measurement feedback to address the pitfalls

Estimator gain

$$\hat{\boldsymbol{x}}_m(k) = \underbrace{A_m\hat{\boldsymbol{x}}_m(k)A + B_m u(k)}_{\text{Original model propagation}} + \underbrace{K_{ob}\Big(y(k) - C_m\hat{\boldsymbol{x}}_m(k)\Big)}_{\text{Correction term}}$$

mea.

$\tilde{x}_m(k) = x_m(k)$

$$\tilde{x}_m(k+1) = A_m\tilde{x}_m(k) - K_{ob}C_m\tilde{x}_m(k) = (A_m - K_{ob}C_m)\tilde{x}_m(k).$$

$\Rightarrow 0$

With initial error in ***X_m(0)*** $\quad \tilde{x}_m(k) = (A_m - K_{ob}C_m)^k\tilde{x}_m(0)$

$A_m^k \, \tilde{x}_m(0)$

CLOSED LOOP POLE placement to get the error to zero with a proper design of K_{ob} = degrees of freedom

K_{ob} gives degrees of freedom

Example – SISO Estimator

Simple pendulum – linearized equation of motion $\dfrac{d^2\theta}{dt^2} + \omega^2\theta = u$

Measurement is angular velocity – $d\theta/dt$ Estimate pendulum angle θ

$\dot{x}_2 = \dfrac{d^2\theta}{dt^2} = -\omega^2 x_1 + u$

$\dot{x}_1 = \dfrac{d\theta}{dt} = x_2$

Let ω = 2 rad/s, sampling interval = 0.1 sec

States: $x_1(t) = \theta$ and $x_2(t) = \dot{\theta}$ *($d\theta/dt$)*

Model $$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Discrete model: $$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.9801 & 0.0993 \\ -.3973 & 0.9801 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.0050 \\ 0.0993 \end{bmatrix} u(k) \qquad y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Example – SISO Estimator

$\det(\lambda I - A) = (\lambda - 0.1)(\lambda - 0.2)$

$\lambda = 0.1,\ \lambda = 0.2$

System initial condition $\theta(0) = x_1(0) = 1,\ x_2(0) = 0$

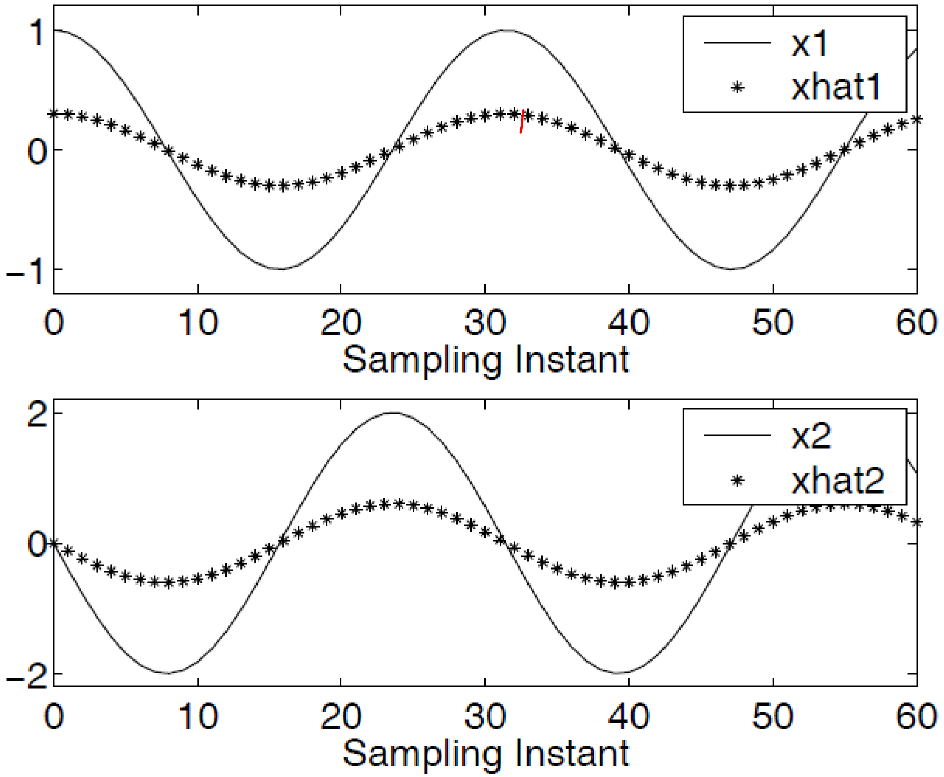
$$\tilde{x}_m(k) = (A_m - K_{ob} C_m)^k \tilde{x}_m(0)$$

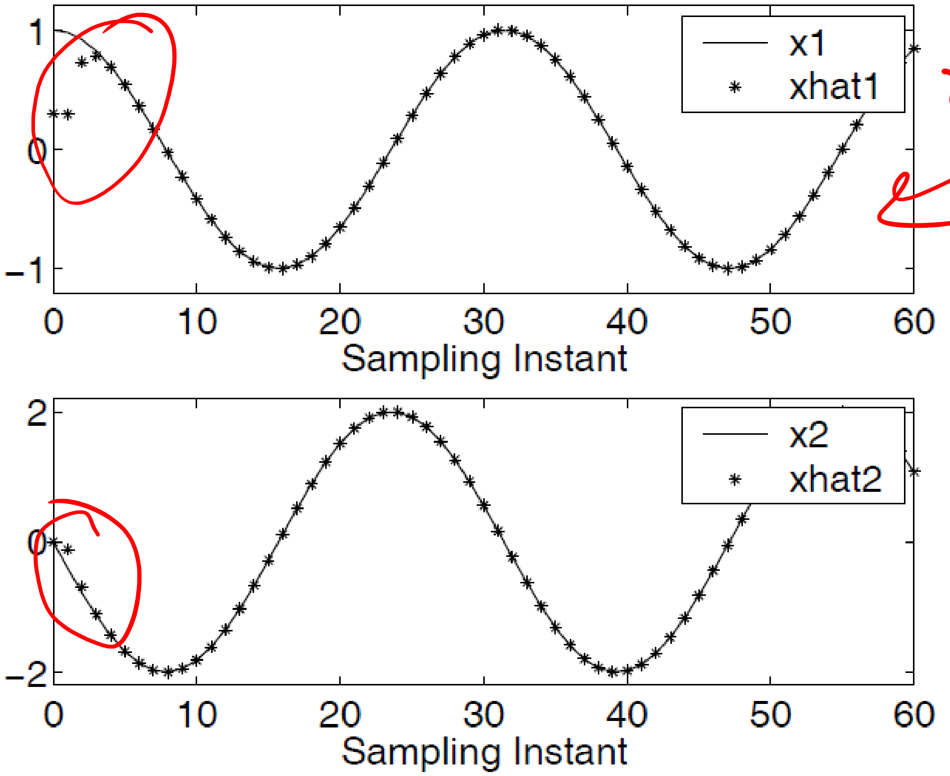
Estimator initial condition $\hat{x}_1(0) = 0.3$ and $\hat{x}_2(0) = 0$

Open loop observer with $K_{ob} = 0$

Estimator with closed loop poles 0.1 and 0.2

$$\begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix} = \begin{bmatrix} -1.6284 \\ 1.6601 \end{bmatrix}$$





noise

MIMO system – Kalman filter Estimator

- Real life, state and measurements have noise
- Propagation of noise must be handled properly

$$x_m(k+1) = A_m x_m(k) + B_m u(k) + d(k)$$
$$y(k) = C_m x_m(k) + \xi(k),$$

d(k) = state noise

ξ(k) = measurement noise

estimate

Prediction will be using the estimated state

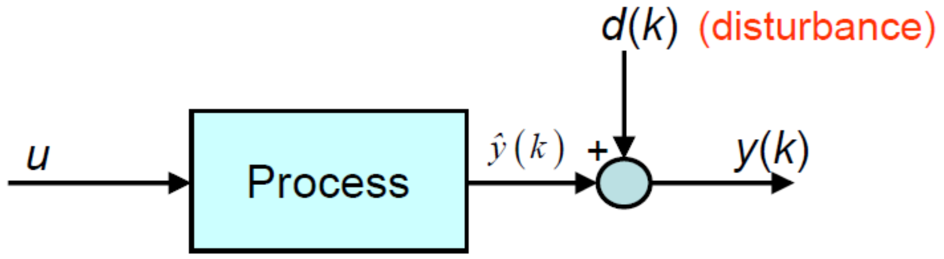
$$\hat{X}(k) = \Phi \hat{x}_0 + T\Delta U(k)$$

Np

Kalman filter
Q – State Noise Covariance,
R – Measurement Noise covariance

Nu

Difference with step response prediction – output feedback



When can predictions drift away actual?

- Inaccurate model
- Unmeasured disturbances

x

$y = Cx$

How do we correct the model predictions?

Output feedback based on the latest measurement

$$\tilde{y}(k+j) \triangleq \hat{y}(k+j) + \underbrace{\left[y(k) - \hat{y}(k)\right]}_{\text{Bias term or correction term or Estimated disturbance}}$$

N_P

N_u

MIMO Model prediction with bias correction

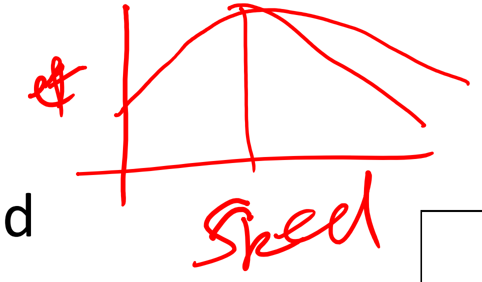
'r' inputs $\mathbf{u} = [u_1 \quad u_2 \quad \cdots \quad u_r]^T$

'm' outputs $\mathbf{y} = [y_1 \quad y_2 \quad \cdots \quad y_m]^T$

$$\tilde{Y}(k+1) = S\Delta U(k) + \hat{Y}^o(k+1) + \Phi\left[\mathbf{y}(k) - \hat{\mathbf{y}}(k)\right]$$

Φ is matrix of '1' with dimension mP xm

Cost function



- Minimize the error between set-point and predicted response at all steps
  - This leads faster response

more

- Minimize the control that minimized the error between set-point and predicted response at all steps
  - This leads trade off between faster response and use of input energy

Δu Δu Δu^2

$$J = (R_s - Y)^T (R_s - Y) + \Delta U^T \bar{R} \Delta U$$

bike

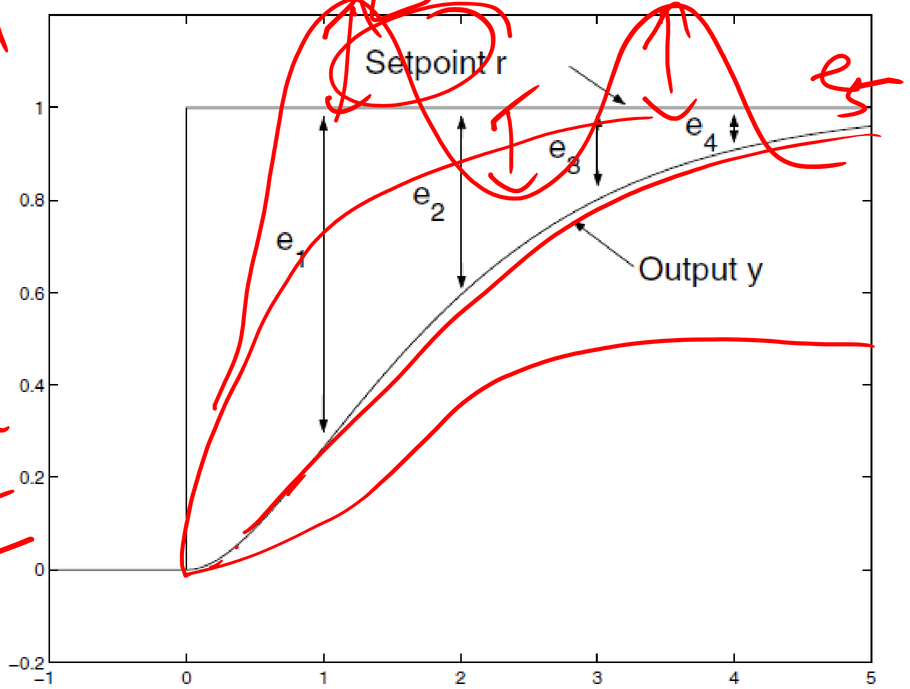
$$\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0.01 \end{pmatrix}$$

0 to 100 kmhs

0 to 5 km → fuel con

0 → 20 s → fuel con



e_1, e_2, e_3, e_4

$\boxed{e_5 = 0}$