CH5120-Assignment-2: State Space Model

Q.No-1 Consider the single-input system dynamics given by x' = Ax + Bu and y = Cx and choose the correct statement(s) from the following statements:

i) The system is stable in an absolute sense if all eigenvalues of A have non-negative real parts.
ii) The poles of the system is given by the eigenvalues of A.

→ The poles of the system is given by eigenvalues of 'A' is the correct statement.

Q.No-2 Identify the transfer function representation of the state space model and find the right coefficient array of the numerator, given:

x˙ = [0 1 0; 0 0 1; 1 4 3] x + [0; 0; 1] u
y = [1 0 0] x + [1] u

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \; ; \qquad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [1] u$$

By observing state equation

$$\dot{x}_3 = \dddot{x}_1 = x_1 + 4\dot{x}_1 + 3\ddot{x}_1 + u$$

$$\therefore \frac{d^3x_1}{d^3t} - 3\frac{d^2x_1}{dt^2} - 4\frac{dx_1}{dt} - x_1 = u$$

Taking laplace transform on both sides and ignoring initial conditions

$$s^3 X_1(s) - 3s^2 X_1(s) - 4sX_1(s) - X_1(s) = U(s)$$

or, $(s^3 - 3s^2 - 4s - 1) X_1(s) = U(s)$

$$\therefore \frac{X_1(s)}{U(s)} = \frac{1}{s^3 - 3s^2 - 4s - 1} \quad \text{— (i)}$$

From output equation

$$y = x_1 + u$$

Taking Laplace transform on both sides

$$Y(s) = X_1(s) + U(s) \quad \text{— (ii)}$$

$$X_1(s) = Y(s) - U(s)$$

from (i) and (ii)

$$\frac{Y(s)}{U(s)} = 1 + \frac{1}{s^3 - 3s^2 - 4s - 1}$$

$$\therefore \frac{Y(s)}{U(s)} = \frac{s^3 - 3s^2 - 4s}{s^3 - 3s^2 - 4s - 1}$$

Coefficient of numerator = [1 -3 -4 0]

Q.No-3 Obtain the transfer function form from the given state space representation and find the correct coefficients of the denominator of the transfer function.

A = [-1 -1 -1; 0 1 -1; 1 -1 1]   B = [0; 1; 0]
C = [0 0 1]   D = [0]

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-1 & -1 & -1\\ 0 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}u \; ; \quad y = \begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + [0]u$$

$$\dot{x} = Ax + Bu$$

$$L[\dot{x}] = L^{-1}[Ax + Bu]$$

$$\text{or, } sX(s) = AX(s) + BU(s)$$

$$(sI - A)X(s) = BU(s)$$

$$\therefore X(s) = (sI - A)^{-1}BU(s)$$

$$y = Cx + DU$$

$$\text{or, } L[y] = L[Cx] \; ; \because D = 0$$

$$\text{or, } Y(s) = CX(s)$$

$$\text{or, } Y(s) = C[(sI - A)^{-1}]BU(s)$$

$$\therefore \frac{Y(s)}{U(s)} = C[(sI - A)^{-1}]B$$

$$(sI - A) = \begin{bmatrix}s+1 & 1 & 1\\ 0 & s-1 & 1\\ -1 & 1 & s-1\end{bmatrix}$$

```
>> s=tf('s')

s =

  s

Continuous-time transfer function.

>> A=[-1 -1 -1; 0 1 -1; 1 -1 1]

A =

    -1    -1    -1
     0     1    -1
     1    -1     1

>> sI=[s 0 0; 0 s 0; 0 0 s];
>> M = inv(sI-A)


M =

  From input 1 to output...
       s^2 - 2 s - 1.943e-15
   1:  ---------------------
         s^3 - s^2 - s - 2

                -1
   2:  -----------------
       s^3 - s^2 - s - 2

              s - 1
   3:  -----------------
       s^3 - s^2 - s - 2

  From input 2 to output...
            -s + 2
   1:  -----------------
       s^3 - s^2 - s - 2

       s^2 + 1.887e-14 s - 8.535e-15
   2:  -----------------------------
             s^3 - s^2 - s - 2

            -s - 2
   3:  -----------------
       s^3 - s^2 - s - 2

  From input 3 to output...
            -s + 2
   1:  -----------------
       s^3 - s^2 - s - 2

            -s - 1
   2:  -----------------
       s^3 - s^2 - s - 2

       s^2 + 1.421e-14 s - 1
   3:  ---------------------
         s^3 - s^2 - s - 2

Continuous-time transfer function.
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>> B = [0;1;0]

B =

     0
     1
     0

>> C=[0 0 1]

C =

     0     0     1

>> TF = C*M*B

TF =

        -s - 2
  -----------------
  s^3 - s^2 - s - 2

Continuous-time transfer function.

>>
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$$\therefore \frac{Y(s)}{U(s)} = \frac{-s - 2}{s^3 - s^2 - s - 2}$$

Coefficient of denominator of transfer functions are [1 -1 -1 -2]

While writing the coefficient of denominator and numerator Constant's coefficient should be at front and further should be in increasing order of the degree of differential

The above tf num=[-2 -1 0 0] and den = [-2 -1 -1 1]

Q.No-4 Obtain the state equation in the phase variable canonical form from the 3rd order given differential equation coefficients. Given the coefficient matrix of the differential equation [a3 a2 a1 a0] is [4 -6 1 7]. The coefficient of u(t) is 2. And find the correct representation of the state matrix (A) in its simplified form.

$$a_3\frac{d^3x}{dt^3} + a_2\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0 x = K\,u(t)$$

or, $$\frac{4d^3x}{dt^3} + \frac{-6\,d^2x}{dt^2} + \frac{dx}{dt} + 7x = 2u(t)$$

$$\therefore \frac{d^3x}{dt^3} - \frac{3}{2}\frac{d^2x}{dt^2} + \frac{1}{4}\frac{dx}{dt} + \frac{7}{4}x = \frac{1}{2}u(t)$$

Let $x_1 = x$

$x_2 = \dot{x}_1 = \dot{x} \quad \Rightarrow \dot{x}_1 = x_2$

$x_3 = \dot{x}_2 = \ddot{x} \quad \Rightarrow \dot{x}_2 = x_3$

$\dot{x}_3 = \frac{1}{2}u(t) + \frac{3}{2}x_3 - \frac{1}{4}x_2 - \frac{7}{4}x_1$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7/4 & 1/4 & 3/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix} u(t)$$

Q.No-5 Discretize the given continuous-time state space system with a sample time of .1s and what is the value of A matrix in the discrete state space model given below?
Given: Ac = [-3 -6 ; 2 5] Bc = [-5 ; 8] Cc = [-4 5]

State transition matrix in continuous time domain is

$$e^{At} = L^{-1}\left[(sI - A_c)^{-1}\right]$$

$$sI - A_c = \begin{bmatrix} s+3 & 6 \\ -2 & s-5 \end{bmatrix}$$

$$(sI - A_c)^{-1} = \begin{bmatrix} \frac{(s-5)}{(s-3)(s+1)} & \frac{-6}{(s-3)(s+1)} \\ \frac{2}{(s-3)(s+1)} & \frac{s+3}{(s-3)(s+1)} \end{bmatrix}$$

$$L^{-1}(sI - A_c)^{-1} = \begin{bmatrix} \frac{1}{2}(3e^{-t} - e^{3t}) & \frac{3}{2}(e^{-t} - e^{3t}) \\ \frac{1}{2}(e^{3t} - e^{-t}) & \frac{1}{2}(3e^{3t} - e^{-t}) \end{bmatrix}$$

Sampling time $t = 0.1s$

$$A_d = e^{A_c t} = e^{A_c \times 0.1}$$

$$= \begin{bmatrix} \frac{1}{2}(3e^{-0.1} - e^{0.3}) & \frac{3}{2}(e^{-0.1} - e^{0.3}) \\ \frac{1}{2}(e^{0.3} - e^{-0.1}) & \frac{1}{2}(3e^{0.3} - e^{-0.1}) \end{bmatrix}$$

$$\therefore A_d = \begin{bmatrix} 0.682 & -0.667 \\ 0.223 & 1.572 \end{bmatrix}$$

$$B_d = \int_0^{0.1} e^{A_c\tau} B \, d\tau$$

$$e^{A_c\tau}B = \begin{bmatrix} \frac{1}{2}(3e^{-\tau} - e^{3\tau}) & \frac{3}{2}(e^{-\tau} - e^{3\tau}) \\ \frac{1}{2}(e^{3\tau} - e^{-\tau}) & \frac{1}{2}(3e^{3\tau} - e^{-\tau}) \end{bmatrix} \begin{bmatrix} -5 \\ 8 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(9e^{-\tau} - 19e^{3\tau}) \\ \frac{1}{2}(19e^{3\tau} - 3e^{-\tau}) \end{bmatrix}$$

$$B_d = \int_0^{0.1} \begin{bmatrix} \frac{1}{2}(9e^{-\tau} - 19e^{3\tau}) \\ \frac{1}{2}(19e^{3\tau} - 3e^{-\tau}) \end{bmatrix} d\tau = \begin{bmatrix} \frac{1}{2}\left(-9e^{-\tau} - \frac{19}{3}e^{3\tau}\right)\Big|_0^{0.1} \\ \frac{1}{2}\left(\frac{19}{3}e^{3\tau} + 3e^{-\tau}\right)\Big|_0^{0.1} \end{bmatrix}$$

$$\therefore B_d = \begin{bmatrix} -0.679 \\ 0.965 \end{bmatrix}$$

IN MATLAB

```
>> A=[-3 -6; 2 5];
>> B = [-5;8];
>> C=[-4 5];
>> D=[0];
>> X = ss(A,B,C,D)

X =

  A =
        x1   x2
   x1   -3   -6
   x2    2    5

  B =
        u1
   x1   -5
   x2    8

  C =
        x1   x2
   y1   -4    5

  D =
        u1
   y1    0

Continuous-time state-space model.
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>> U = c2d(X,0.1,'zoh')

U =

  A =
             x1       x2
   x1    0.6823  -0.6675
   x2    0.2225    1.572

  B =
             u1
   x1   -0.6797
   x2    0.9651

  C =
         x1   x2
   y1    -4    5

  D =
         u1
   y1     0

Sample time: 0.1 seconds
Discrete-time state-space model.

>>
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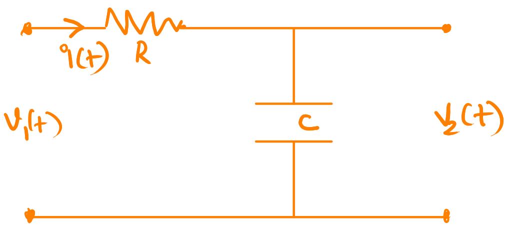
Q.No-6 Identify the state space representation (in the form x ̇(t) = A x(t) + Bu(t) and y(t) = C x(t) + D u(t)) for the transient response of the circuit shown in the figure and choose the right option for the A matrix. Such that the input to the system is the voltage u1(t) and the output is the voltage u2(t). Given: R = 130Ω, C = 2.000000e-06F



$$\frac{V_1(t) - V_2(t)}{R} = \frac{C\,dV_2(t)}{dt}$$

$$\text{or,}\quad \frac{C\,dV_2(t)}{dt} + \frac{1 V_2(t)}{R} = \frac{1}{R} V_1(t)$$

$$\frac{dV_2(t)}{dt} + \frac{1}{RC} V_2(t) = \frac{1}{RC} V_1(t)$$

Let $y = V_2(t)$

$x_1 = y = V_2(t)$

$\dot{x}_1 = \frac{dV_2(t)}{dt} = \frac{1}{RC} V_1(t) - \frac{1}{RC} x_1$

$$\therefore \begin{bmatrix} \dot{x}_1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} x_1 \end{bmatrix} + \begin{bmatrix} \frac{1}{RC} \end{bmatrix} V_1(t)$$

$$y = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} x_1 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} V_1(t)$$

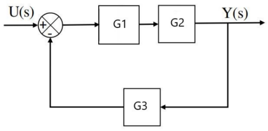
$$\therefore A = \begin{bmatrix} -\frac{1}{RC} \end{bmatrix} = \begin{bmatrix} -3846.15 \end{bmatrix}$$

$B = \begin{bmatrix} 3846.15 \end{bmatrix}$

$C = \begin{bmatrix} 1 \end{bmatrix}$

$D = \begin{bmatrix} 0 \end{bmatrix}$

Q.No-7 If the state-space model is represented as x' = Ax + Bu and y = Cx + Du where x, u, and y are the state variable vector, input vector, and measurement vector respectively. What is the A matrix for the following closed loop system?
Given: G1 = 7/(s+9), G2 = 9/(s+1), G3 = 3/(s+1)



$$\frac{Y(s)}{U(s)} = \frac{G_1G_2}{1+G_1G_2G_3}$$

$$= \frac{7\times 9\times(s+1)}{(s+9)(s+1)(s+1)+7\times 9\times 3}$$

$$\frac{Y(s)}{U(s)} = \frac{63s+63}{s^3+11s^2+19s+198}$$

Let $\frac{Y(s)}{U(s)} = \frac{W(s)}{U(s)} \times \frac{Y(s)}{W(s)}$

Let $\frac{W(s)}{U(s)} = \frac{1}{s^3+11s^2+19s+198}$

$$\frac{Y(s)}{W(s)} = 63s+63$$

$$\frac{W(s)}{U(s)} = \frac{1}{s^3+11s^2+19s+198}$$

or, $W(s)(s^3+11s^2+19s+198) = U(s)$

Taking inverse laplace transform

$$\frac{d^3w(t)}{dt^3} + 11\frac{d^2w(t)}{dt^2} + 19\frac{dw}{dt} + 198\,w(t) = U(t)$$

State equation in state-space representation is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -198 & -19 & -11\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}}_{B}U(t)$$

$$\frac{Y(s)}{W(s)} = 63s+63$$

$$Y(s) = 63s\,W(s) + 63\,W(s)$$

Taking inverse laplace transform

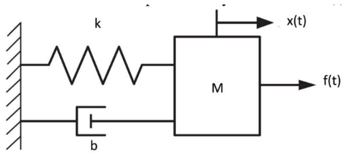
$$y(t) = 63\frac{dw(t)}{dt} + 63\,w(t)$$

or, $y(t) = 63\,x_2(t) + 63\,x_1(t)$

output equation in state space form is

$$[y] = \underbrace{[63 \quad 63 \quad 0]}_{C}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \underbrace{[0]}_{D}U(t)$$

Q.No-8 Identify the state space representation (in the form $\dot{x} = Ax + Bu$ and $y = Cx + Du$) of the Mass-Spring-Damper model shown in the figure and find the matrix A. Note that the input to the system is force f(t) and output is displacement x(t).
Given: M = 4; k = 3; b = 2.000000e-01



$$M\ddot{x}(t) = f(t) - Kx(t) - b\dot{x}(t)$$

or, $$\ddot{x}(t) = \frac{1}{M}f(t) - \frac{b}{M}\dot{x}(t) - \frac{K}{M}x(t)$$

Assumptions:
$y = x_1 = x(t)$
$x_2 = \dot{x}_1$

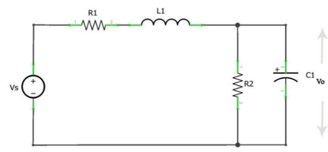
In state-space representation

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -K/M & -b/M\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{M}\end{bmatrix}f(t)$$

$$[y] = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]\,f(t)$$

$$\therefore A = \begin{bmatrix}0 & 1\\ -3/4 & -\frac{0.2}{4}\end{bmatrix}; \quad B = \begin{bmatrix}0\\ 1/4\end{bmatrix}$$

Q.No-9 Identify the state space representation (in the form $\dot{x} = Ax + Bu$ and $y = Cx + Du$) of the circuit shown in the figure and find the matrix A. Given: R1 = 2; R2 = 7; L1 = 1; C1 = 3.000000e-01. Follow SI units for R, L, and C. Assume Zero initial conditions. Also, consider the circuit to be at time t=0 just after it is switched on.



$$V_s - R_1 i(t) - L\frac{di(t)}{dt} - V_0 = 0$$

$$i(t) = \frac{V_0}{R_2} + C\frac{dV_0}{dt}$$

$$\therefore V_s - R_1\left(\frac{V_0}{R_2} + C\frac{dV_0}{dt}\right) - L\frac{d}{dt}\left(\frac{V_0}{R_2} + C\frac{dV_0}{dt}\right) - V_0 = 0$$

or, $$V_s - \frac{R_1 V_0}{R_2} - R_1 C\frac{dV_0}{dt} - \frac{L}{R_2}\frac{dV_0}{dt} - LC\frac{d^2V_0}{dt^2} - V_0 = 0$$

or, $$LC\frac{d^2V_0}{dt^2} = V_s - \left(R_1 C + \frac{L}{R_2}\right)\frac{dV_0}{dt} - \left(\frac{R_1}{R_2} + 1\right)V_0$$

$$\therefore \frac{d^2V_0}{dt^2} = \frac{1}{LC}V_s - \frac{1}{LC}\left(R_1 C + \frac{L}{R_2}\right)\frac{dV_0}{dt} - \frac{1}{LC}\left(\frac{R_1}{R_2} + 1\right)V_0$$

In state space representation:

Let $y = x_1 = V_0$

$$x_2 = \dot{x}_1 = \frac{dV_0}{dt}$$

$$\dot{x}_2 = \ddot{x}_1 = \frac{d^2 V_0}{dt^2}$$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC}\left(\frac{R_1}{R_2}+1\right) & -\frac{1}{LC}\left(RC + \frac{L}{R_2}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} V_S$$

$$[y] = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\, U$$

$$\therefore A = \begin{bmatrix} 0 & 1 \\ -30/7 & -52/21 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 \\ 10/3 \end{bmatrix}$$