Chapter 3

State Feedback - Pole Placement

Motivation

Whereas classical control theory is based on output feedback, this course mainly deals with control system design by state feedback. This model-based control strategy consists of

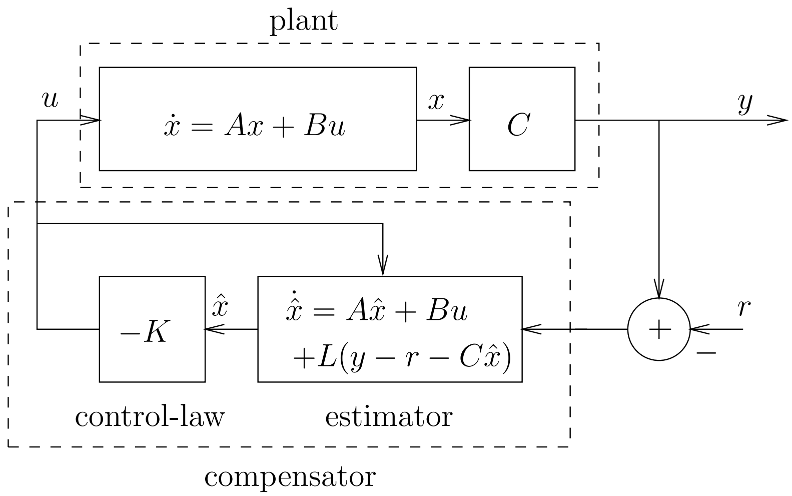
Step 1. State feedback control-law design.

Step 2. Estimator design to estimate the state vector.

Step 3. Compensation design by combining the control law and the estimator.

Step 4. Reference input design to determine the zeros.

Schematic diagram of a state-space design example



Control-law design by state feedback : a motivation

Example: Boeing 747 aircraft control

The complete lateral model of a Boeing 747 (see also page 22), including the rudder actuator (an hydraulic servo) and washout circuit (a lead compensator), is

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du. \end{aligned}$$

where

$$A = \begin{bmatrix} -10 & 0 & 0 & 0 & 0 & 0 \\ 0.0729 & -0.0558 & -0.997 & 0.0802 & 0.0415 & 0 \\ -4.75 & 0.598 & -0.115 & -0.0318 & 0 & 0 \\ 1.53 & -3.05 & 0.388 & -0.465 & 0 & 0 \\ 0 & 0 & 0.0805 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -0.3333 \end{bmatrix}$$

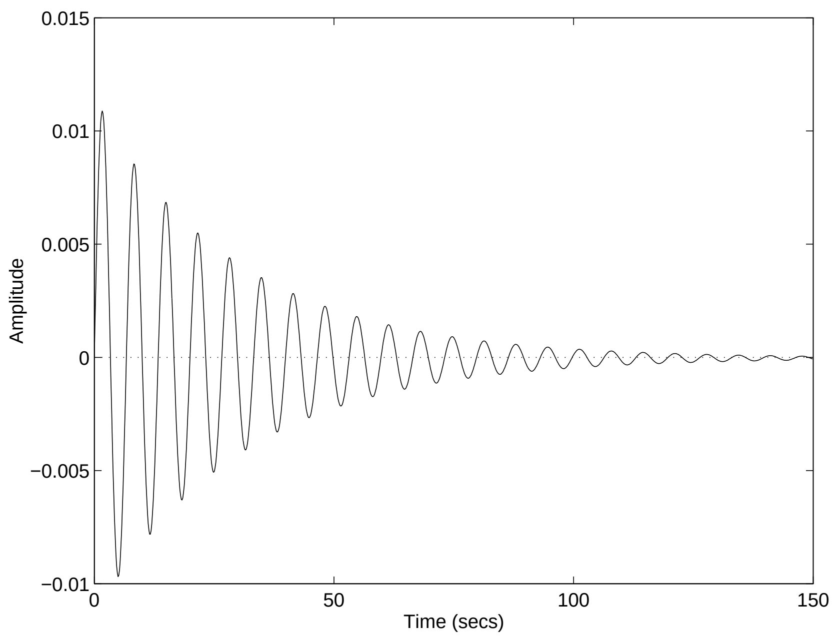
$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & -0.3333 \end{bmatrix}, D = 0.$$

The system poles are

$$-0.0329 \pm 0.9467i, \; -0.5627, \; -0.0073, \; -0.3333, \; -10.$$

The poles at $-0.0329 \pm 0.9467i$ have a damping ratio $\zeta = 0.03$ which is far from the desired value $\zeta = 0.5$. The following figure illustrates the consequences of this small damping ratio.

Initial condition response with $\beta = 1°$.



To improve this behavior, we want to design a control law such that the closed loop system has a pair of poles with a damping ratio close to 0.5.

General Format of State Feedback

Control law

$$u = -Kx, \quad K: \quad \text{constant matrix.}$$

For single input systems (SI):

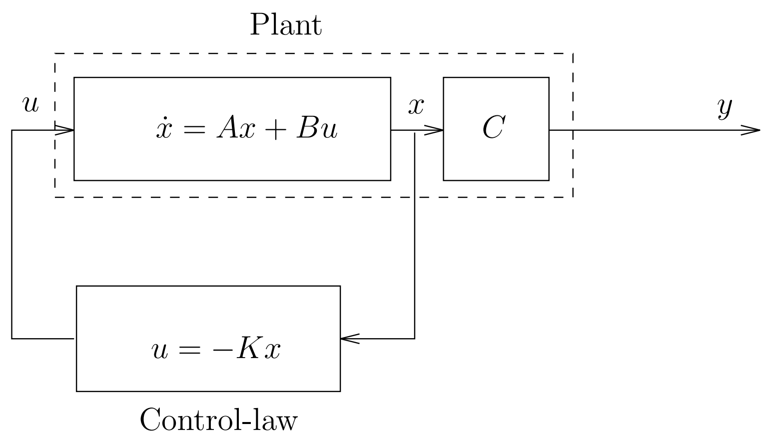
$$K = \begin{bmatrix} K_1 & K_2 & \cdots & K_n \end{bmatrix}$$

For multi input systems (MI):

$$K = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ K_{p1} & K_{p2} & \cdots & K_{pn} \end{bmatrix}$$

Note: one sensor is needed for each state $\Rightarrow$ disadvantage. We'll see later how to deal with this problem (estimator design).

Structure of state feedback control



Pole Placement

Closed-loop system:

$$\dot{x} = Ax + Bu, u = -Kx. \quad \Rightarrow \quad \dot{x} = (A - BK)x$$

poles of the closed loop system

$$\Updownarrow$$

roots of $\det\left(sI - (A - BK)\right)$

Pole-placement:

> Choose the gain K such that the poles of the closed loop systems are in specified positions.

More precisely, suppose that the desired locations are given by

$$s = s_1, s_2, \cdots, s_n$$

where $s_i, i = 1, \cdots, n$ are either real or complex conjugated pairs, choose K such that the characteristic equation

$$\alpha_c(s) \stackrel{\Delta}{=} \det\left(sI - (A - BK)\right)$$

equals

$$(s - s_1)(s - s_2)\ldots(s - s_n).$$

Pole-placement - direct method

Find K by directly solving

$$\det\left(sI - (A - BK)\right) = (s - s_1)(s - s_2)\cdots(s - s_n)$$

and matching coefficients in both sides.

Disadvantage:

- Solve nonlinear algebraic equations, difficult for n>3.

  Example Let $n = 3$, $m = 1$. Then the following 3rd order equations have to be solved to find $K = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}$:

$$\sum_{1 \le i \le 3} (a_{ii} - b_i K_i) = \sum_{1 \le i \le 3} s_i,$$

$$\sum_{1 \le i < j \le 3} \begin{vmatrix} a_{ii} - b_i K_i & a_{ij} - b_i K_j \\ a_{ji} - b_j K_i & a_{jj} - b_j K_j \end{vmatrix} = \sum_{1 \le i < j \le 3} s_i s_j,$$

$$\begin{vmatrix} a_{11} - b_1 K_1 & a_{12} - b_1 K_2 & a_{13} - b_1 K_3 \\ a_{21} - b_2 K_1 & a_{22} - b_2 K_2 & a_{23} - b_2 K_3 \\ a_{31} - b_3 K_1 & a_{32} - b_3 K_2 & a_{33} - b_3 K_3 \end{vmatrix} = s_1 s_2 s_3.$$

- You never know whether there IS a solution K. (But THERE IS one if (A, B) is controllable!)

Pole-placement for SI: Ackermann's method

Let A_c, B_c be in a control canonical form, then

$$A_c - B_c K_c = \begin{bmatrix} -a_1 - K_1 & -a_2 - K_2 & \cdots & \cdots & -a_n - K_n \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

and

$$\det(sI - (A_c - B_c K_c)) =$$

$$s^n + (a_1 + K_1)s^{n-1} + (a_2 + K_2)s^{n-2} + \cdots + (a_n + K_n)$$

If

$$(s - s_1)(s - s_2)\cdots(s - s_n) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n$$

then

$$K_1 = -a_1 + \alpha_1, K_2 = -a_2 + \alpha_2, \cdots, K_n = -a_n + \alpha_n.$$

Design procedure (when A,B are not in control canonical form):

- Transform (A, B) to a control canonical form (A_c, B_c) with a similarity transformation T.
- Find control law K_c with the procedure on the previous page.
- Transform K_c: $K = K_c T^{-1}$.

Note that the system (A, B) must be controllable.

Property:
For SI systems, control law K is unique!

The design procedure can be expressed in a more compact form :

$$K = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \mathcal{C}^{-1} \alpha_c(A),$$

where $\mathcal{C}$ is the controllability matrix:

$$\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

and

$$\alpha_c(A) = A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \cdots + \alpha_n I.$$

Pole-placement for MI - SI generalization

Fact: If (A, B) is controllable, then for almost any $K_r \in \mathbb{R}^{m\times n}$ and almost any $v \in \mathbb{R}^m$, $(A - BK_r, Bv)$ is controllable.

$$\Downarrow$$

From the pole placement results for SI, there is a $K_s \in \mathbb{R}^{1\times n}$ so that the eigenvalues of $A - BK_r - (Bv)K_s$ can be assigned to desired values.

$$\Downarrow$$

Also the eigenvalues of $A - BK$ can be assigned to desired values by choosing a state feedback in the form of

$$u = -Kx = -(K_r + vK_s)x.$$

Design procedure:

- Arbitrarily choose K_r and v such that $(A - BK_r, Bv)$ is controllable.
- Use Ackermann's formula to find K_s for $(A - BK_r, Bv)$.
- Find state feedback gain $K = K_r + vK_s$.

Pole-placement for MIMO - Sylvester equation

Let Λ be a real matrix such that the desired closed-loop system poles are the eigenvalues of Λ. A typical choice of such a matrix is:

$$\Lambda = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ -\beta_1 & \alpha_1 & & & \\ & & \ddots & & \\ & & & \lambda_1 & \\ & & & & \ddots \end{bmatrix},$$

which has eigenvalues: $\alpha_1 \pm j\beta_1, \ldots, \lambda_1, \ldots$. which are the desired poles of the closed-loop system. For controllable systems (A, B) with static state feedback,

$$A - BK \sim \Lambda.$$

$\Rightarrow$ There exists a similarity transformation X such that:

$$X^{-1}(A - BK)X = \Lambda,$$

or

$$AX - X\Lambda = BKX.$$

The trick to solve this equation: split up the equation by introducing an arbitrary auxiliary matrix G:

$$\begin{aligned} AX - X\Lambda &= BG, \quad \text{(Sylvester equation in } X\text{)} \\ KX &= G. \end{aligned}$$

The Sylvester equation is a matrix equation that is linear in X. If X is solved for a known G, then

$$K = GX^{-1}.$$

Design procedure:

- Pick an arbitrary matrix G.
- Solve the Sylvester equation for X.
- Obtain the static feedback gain $K = GX^{-1}$.

Properties:

- There is always a solution for X if A and Λ have no common eigenvalues.
- For SI, K is unique, hence independent of the choice of G.
- For certain special choices of G this method may fail ($e.g. X$ not invertible or ill conditioned). Then just try another G.

Examples for pole placement

Example Boeing 747 aircraft control - control law design with pole placement (Ackermann's method)

Desired poles:

$$-0.0051,\ -0.468,\ -1.106,\ -9.89,\ -0.279 \pm 0.628i$$

which have a maximum damping ratio 0.4.

From the desired poles

$$\alpha_c(s) = s^6 + 12.03s^5 + 23.01s^4 + 19.62s^3 + 10.55s^2 + 2.471s + 0.0123,$$

$$\alpha_c(A) = \begin{bmatrix} 8844.70 & 0 & 0 & 0 & 0 & 0 \\ 368.20 & 7.88 & 2.43 & -0.61 & -0.16 & 0 \\ 4220.20 & -2.47 & 5.93 & -0.46 & -0.23 & 0 \\ -1459.94 & -2.75 & -25.73 & 2.78 & 1.09 & 0 \\ 115.27 & 25.80 & -6.71 & -0.82 & 0.08 & 0 \\ -436.60 & -5.95 & 0.06 & 0.28 & 0.06 & 0.13 \end{bmatrix}$$

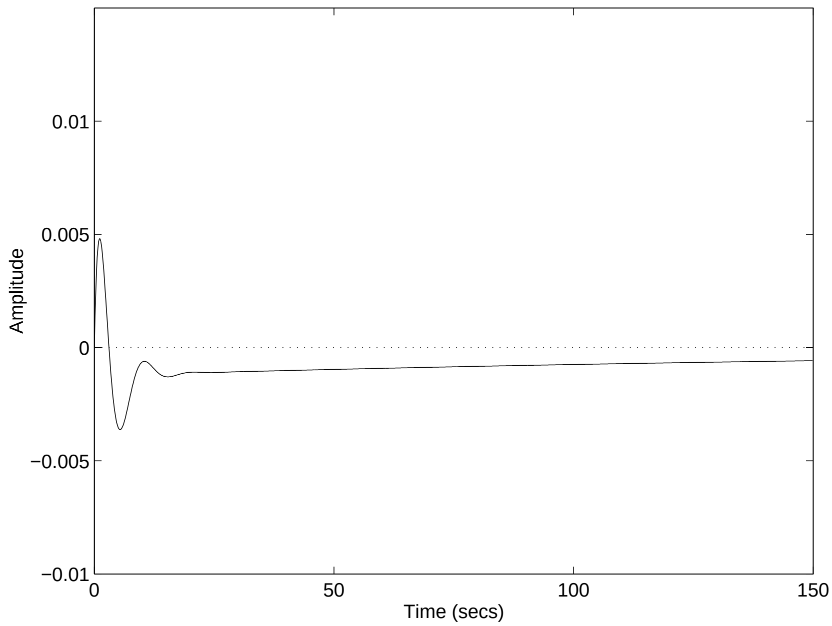
is obtained and hence the controllability matrix is given by

$$\mathcal{C} = \begin{bmatrix} 1.00 & -10.00 & 100.00 & -1000.00 & 10000.00 & -100000 \\ 0 & 0.07 & 4.12 & -42.22 & 418.15 & -4180.52 \\ 0 & -4.75 & 48.04 & -477.48 & 4774.34 & -47746.07 \\ 0 & 1.53 & -18.08 & 167.47 & -1664.37 & 16651.01 \\ 0 & 0 & 1.15 & -14.21 & 129.03 & -1280.03 \\ 0 & 0 & -4.75 & 49.62 & -494.03 & 4939.01 \end{bmatrix}.$$

Then the control law is

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathcal{C}^{-1} \alpha_c(A)$$
$$= \begin{bmatrix} 1.06 & -0.19 & -2.32 & 0.10 & 0.04 & 0.49 \end{bmatrix}.$$

Plot of the initial condition response with $\beta_0 = 1^\circ$:



Much better!

Example Tape drive control - control law design with pole placement (Sylvester equations)
Desired poles:

$$-0.451 \pm 0.937i,\ -0.947 \pm 0.581i,\ -1.16,\ -1.16.$$

Take an arbitrary matrix G:

$$G = \begin{bmatrix} 1.17 & 0.08 & -0.70 & 0.06 & 0.26 & -1.45 \\ 0.63 & 0.35 & 1.70 & 1.80 & 0.87 & -0.70 \end{bmatrix}.$$

Solve the Sylvester equation for X:

$$AX - X\Lambda_c = BG,$$

where

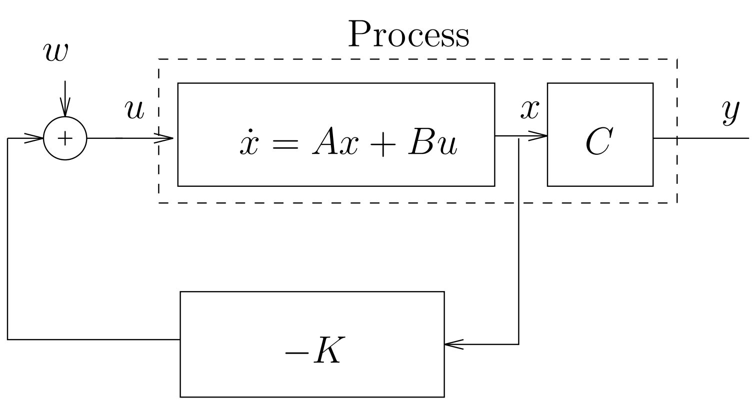
$$\Lambda_c = \begin{bmatrix} -0.451 & 0.937 & & & & \\ -0.937 & -0.451 & & & & \\ & & -0.947 & 0.581 & & \\ & & -0.581 & -0.947 & & \\ & & & & -1.16 & \\ & & & & & -1.16 \end{bmatrix}.$$

$$X = \begin{bmatrix} -0.27 & -1.83 & 1.00 & 0.86 & 2.31 & -10.78 \\ 0.92 & 0.29 & -0.23 & -0.70 & -1.34 & 6.25 \\ 0.56 & -0.76 & -2.26 & -6.25 & 6.42 & -5.74 \\ 0.23 & 0.43 & -0.75 & 3.62 & -3.72 & 3.33 \\ -0.62 & 0.91 & 0.17 & 1.19 & 1.40 & -7.87 \\ -0.58 & 0.33 & 2.99 & -3.15 & 4.75 & -3.76 \end{bmatrix}.$$

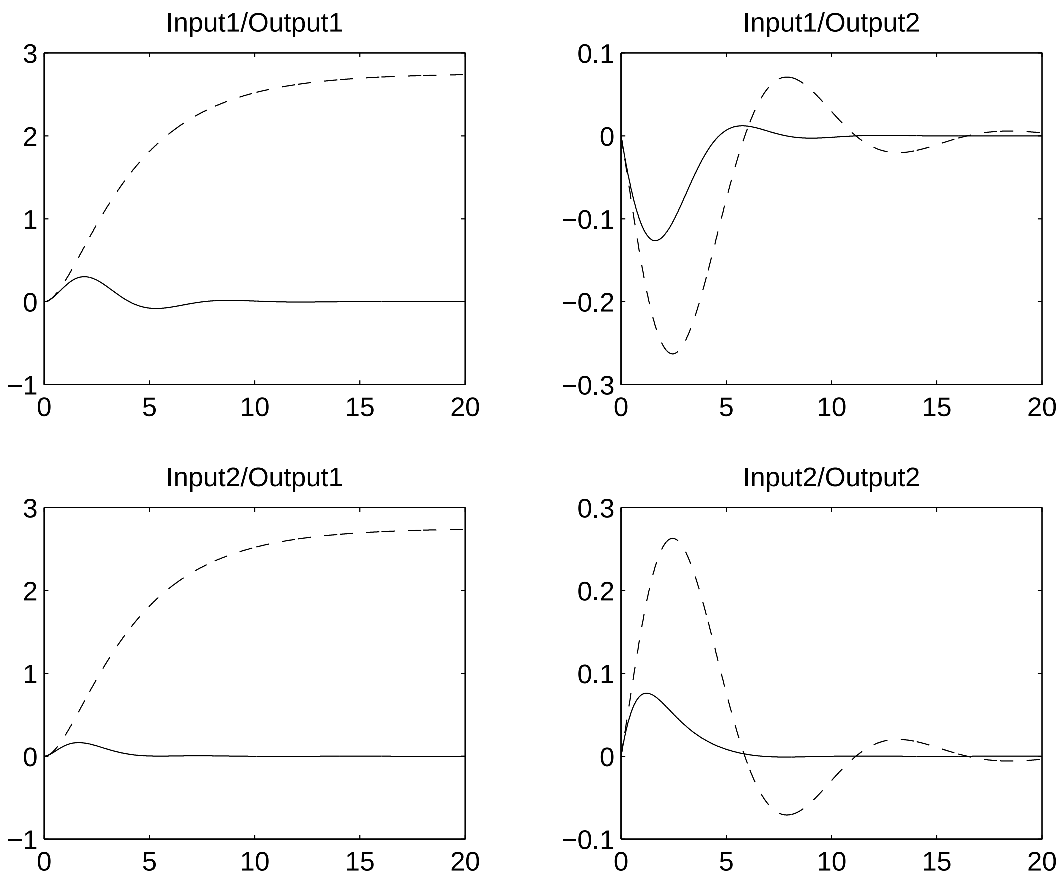
Obtain the static feedback gain $K = GX^{-1}$:

$$K = \begin{bmatrix} 0.55 & 1.58 & 0.32 & 0.56 & 0.67 & 0.05 \\ 0.60 & 0.60 & 0.68 & 3.24 & -0.21 & 1.74 \end{bmatrix}.$$

The closed loop system looks like :



Impulse response (to w):



Dashed line: without feedback, sensitive to process noise.

Solid line: with state feedback, much better!

Property:

Static state feedback does not change the transmission zeros of a system:

$$\text{zeros}(A, B, C, D) = \text{zeros}(A - BK, B, C - DK, D)$$

Proof :

If ζ is a zero of (A, B, C, D), then (when ζ is not a pole), there are u and v such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \zeta$$

Now let $\bar{u} = u$, $\bar{v} = Ku + v$, then

$$\begin{bmatrix} A - BK & B \\ C - DK & D \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix} \zeta$$

$\Rightarrow \zeta$ is a zero of $(A - BK, B, C - DK, D)$.

Pole location selection

Dominant second-order poles selection

Use the relations between the time specifications (rise time, overshoot and settling time) and the second-order transfer function with complex poles at radius ω_n and damping ratio ζ.

1. Choose the closed-loop poles for a high-order system as a desired pair of dominant second-order poles.
2. Select the rest of the poles to have real parts corresponding to sufficiently damped modes, so that the system will mimic a second-order response with reasonable control effort.
3. Make sure that the zeros are far enough into the left half-plane to avoid having any appreciable effect on the second-order behavior.

Prototype design

An alternative for higher-order systems is to select prototype responses with desirable dynamics.

- ITAE transfer function poles : a prototype set of transient responses obtained by minimizing a certain criterion of the form

$$J = \int_0^\infty t|e|dt.$$

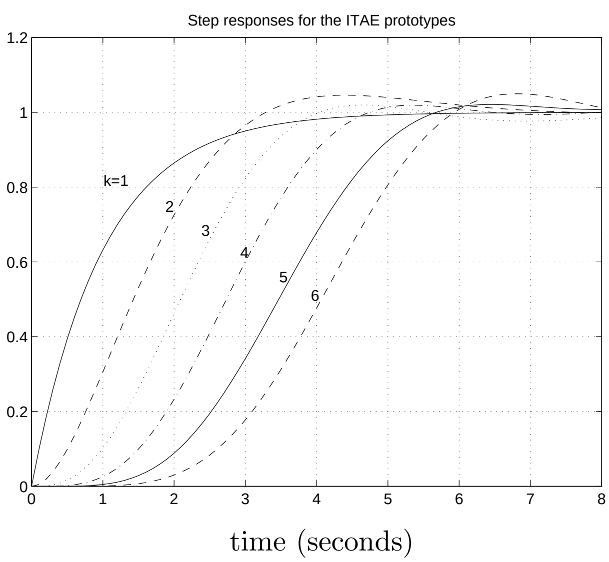
  Property: fast but with overshoot.

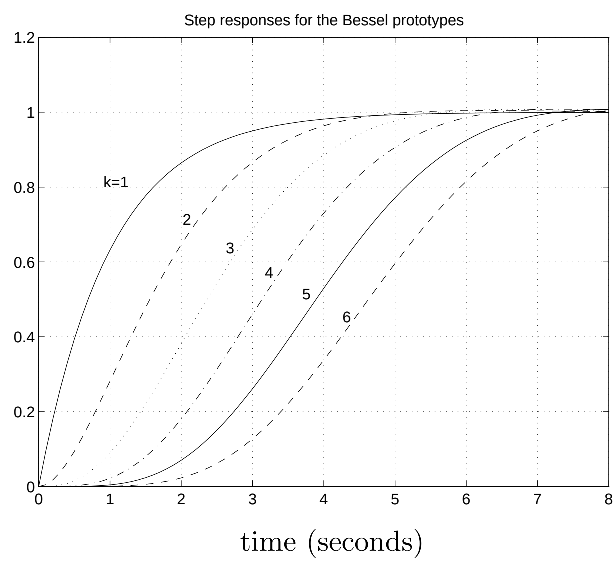
- Bessel transfer function poles : a prototype set of transfer functions of $1/B_n(s)$ where $B_n(s)$ is the nth-degree Bessel polynomial.

  Property: slow without overshoot.

Prototype Response Poles:

	k	Pole Location
(a) ITAE	1	-1
T.F. poles	2	$-0.7071 \pm 0.7071j$
	3	$-0.7081,\ -0.5210 \pm 1.068j$
	4	$-0.4240 \pm 1.2630j,\ 0.6260 \pm 0.4141j$
(b) Bessel	1	-1
T.F. poles	2	$-0.8660 \pm 0.5000j$
	3	$-0.9420, -0.7455 \pm 0.7112j$
	4	$-0.6573 \pm 0.8302j,\ -0.9047 \pm 0.2711j$





Pole locations should be adjusted for faster/slower response. A time scaling with factor α can be applied by replacing the Laplace variable s in the transfer function by s/α.

Examples

Example Tape drive control - Selection of poles. The poles selection methods above are basically for SI systems. Thus consider only the one-motor (the left one) of the tape drive system and set the inertia J of the right wheel 3 times larger than the left one. Then

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{aligned}$$

where

$$x = \begin{bmatrix} p_1 \\ \omega_1 \\ p_2 \\ \omega_2 \\ i_1 \end{bmatrix}, \; A = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ -0.1 & -0.35 & 0.1 & 0.1 & 0.75 \\ 0 & 0 & 0 & 2 & 0 \\ 0.1 & 0.1 & -0.1 & -0.35 & 0 \\ 0 & -0.03 & 0 & 0 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T, \; C = \begin{bmatrix} 0.5 & 0 & 0.5 & 0 & 0 \\ -0.2 & -0.2 & 0.2 & 0.2 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; y = \begin{bmatrix} p_3 \\ T \end{bmatrix}, \; u = e_1.$$

Specification: the position p_3 has no more than 5% overshoot and a rise time of no more than 4 sec. Keep the peak tension as low as possible.

Pole placement as a dominant second-order system:

Using the formulas on page 79, we find

$$\text{Overshoot } M_p < 5\% \Rightarrow \text{damping ratio } \zeta = 0.6901.$$
$$\text{Rise time } t_r < 4 \text{ sec.} \Rightarrow \text{natural frequency } \omega_n = 0.45$$

The formulas on page 79 are only approximative, therefore we take some safety margin and choose for instance $\zeta = 0.7$ and $\omega_n = 1/1.5$.

$$\Downarrow$$

$$\text{Poles}: \frac{-0.707 \pm 0.707j}{1.5}$$

Other poles far to the left: $-4/1.5$, $-4/1.5$, $-4/1.5$. $\Rightarrow$

$$K = \begin{bmatrix} 8.5123 & 20.3457 & -1.4911 & -7.8821 & 6.1927 \end{bmatrix}.$$

Control law:

$$u = -Kx + 7.0212r,$$

where r is the reference input such that y follows r (in steady state $y = r$).

Pole placement using an ITAE prototype:

Check the step responses of the ITAE prototypes and observe that the rise time for the 5th order system is about 5 sec. So let $\alpha = 5/4 = 1.25$. From the ITAE poles table, the following poles are selected:

$(-0.8955, -0.3764 \pm 1.2920j, -0.5758 \pm 0.5339j) \times 1.25.$

$\Rightarrow K = \left[\begin{array}{ccccc} 1.9563 & 4.3700 & 0.5866 & 0.8336 & 0.7499 \end{array}\right].$

Control law:

$$u = -Kx + 2.5430r.$$

Pole placement using a Bessel prototype:

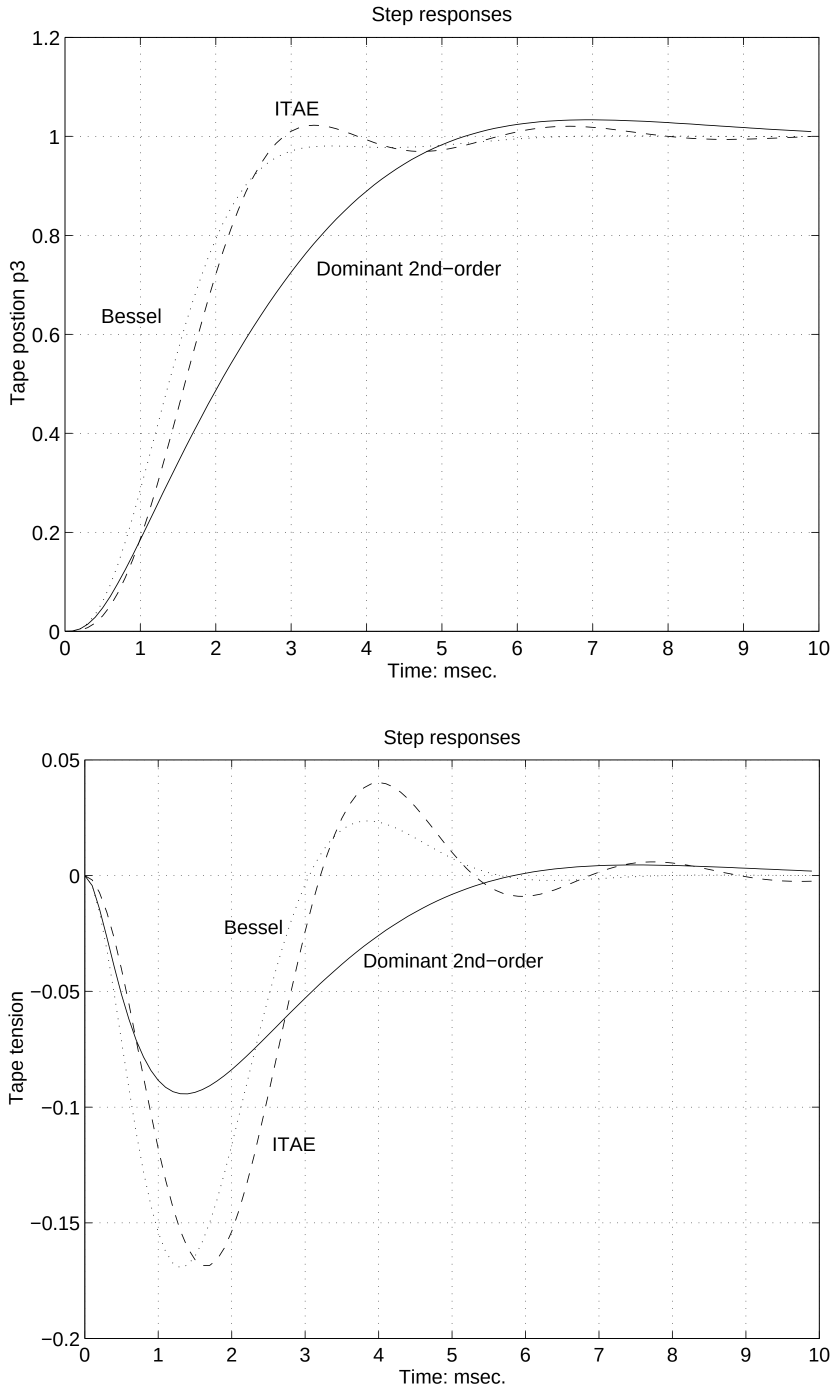
Check the step responses of the Bessel prototypes, it appears that the rise time for the 5th order system is about 6 sec. So let $\alpha = 6/4 = 1.5$. From the Bessel poles table, the following poles are selected:

$(-1.3896, -0.8859 \pm 1.3608j, -1.2774 \pm 0.6641j) \times 1.5.$

$\Rightarrow K = \left[\begin{array}{ccccc} 3.9492 & 9.1131 & 2.3792 & 5.2256 & 2.9662 \end{array}\right].$

$$u = -Kx + 6.3284r.$$

Step responses:



Matlab Functions

poly

real

polyvalm

acker

lyap

place