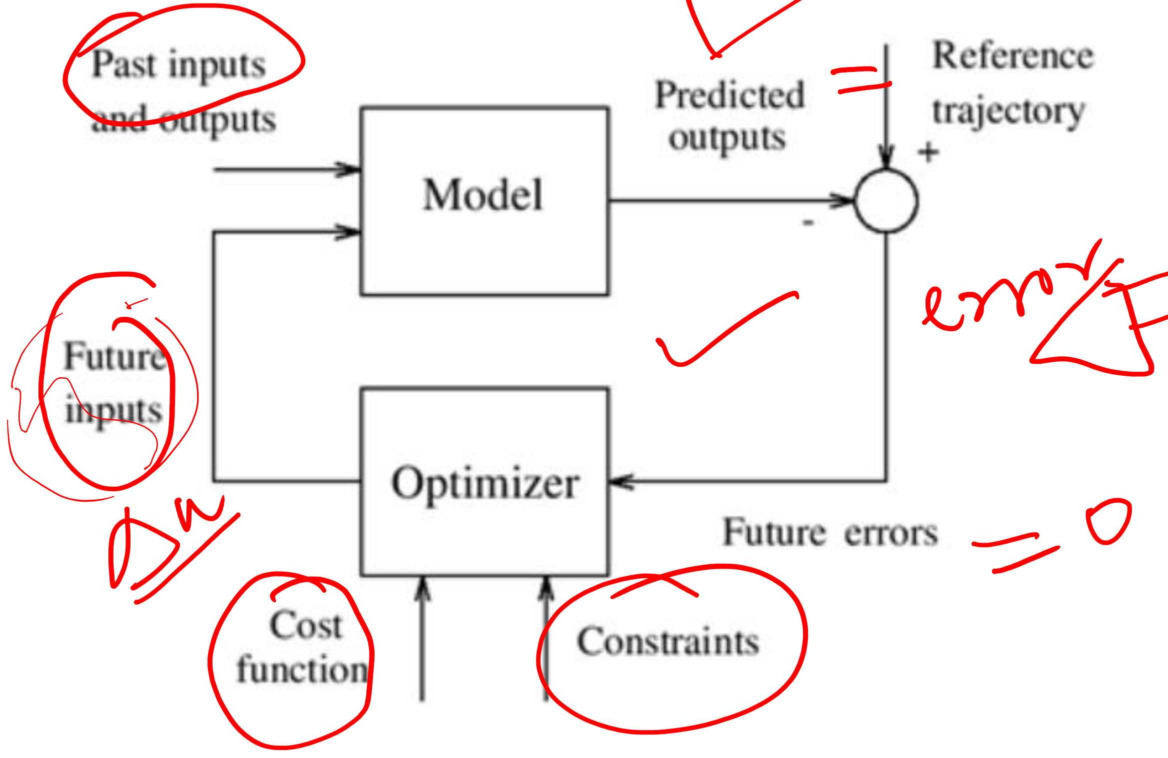
Modern Control Theory

Model Predict Control - MPC – Lecture 8

Unconstrained Control move calculation

RECAP

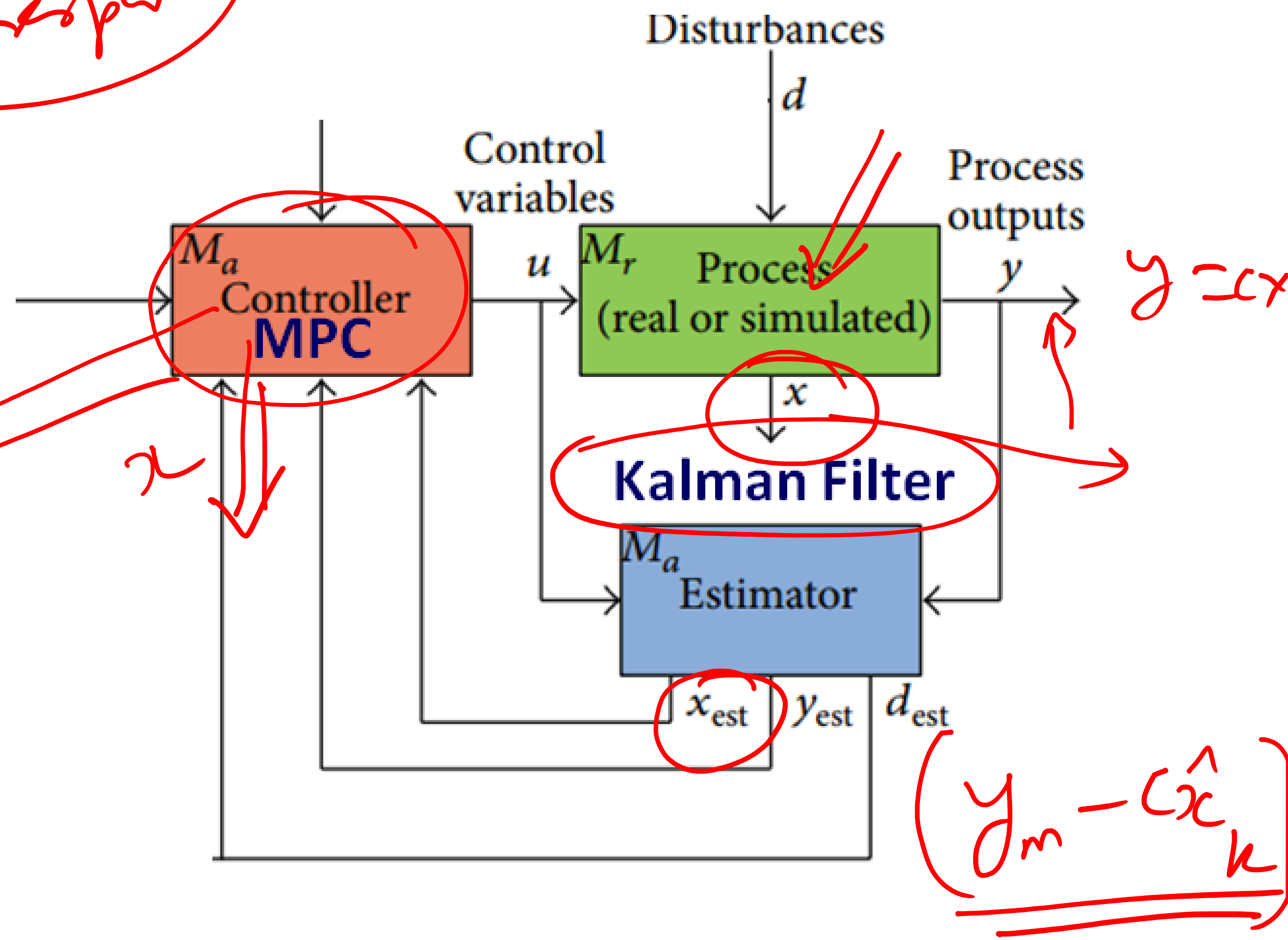
Free response + Forced response



error

= 0

Δu



x

$y = Cx$

$y_m - C\hat{x}_k$

M_a: assumed model used in simulations
M_r: real model used in simulations

Prediction using State space models (6/6)

System representation

$$x_m(k+1) = A_m x_m(k) + B_m u(k)$$
$$y(k) = C_m x_m(k)$$

SS

System in incremental form

$$\boxed{x(k+1) = Ax(k) + B\Delta u(k)}$$

$$\boxed{y(k+1) = C\,x(k)}$$

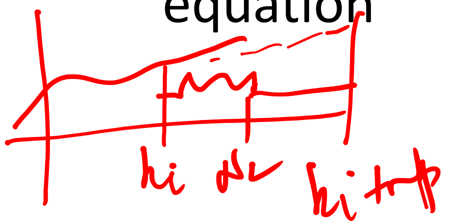
$$x(k) = \begin{bmatrix} \Delta x_m(k) \\ y(k) \end{bmatrix} \qquad A = \begin{bmatrix} A_m & 0_m^T \\ C_m A_m & 1 \end{bmatrix} \qquad B = \begin{bmatrix} B_m \\ C_m B_m \end{bmatrix} \qquad C = \begin{bmatrix} 0_m & 1 \end{bmatrix}$$

D = 0

Causal System

estimated state

Output Prediction equation



$$Y = \begin{bmatrix} y(k_i+1 \mid k_i) \\ y(k_i+2 \mid k_i) \\ y(k_i+3 \mid k_i) \\ \vdots \\ y(k_i+N_p \mid k_i) \end{bmatrix} \qquad \Delta U = \begin{bmatrix} \Delta u(k_i) \\ \Delta u(k_i+1) \\ \Delta u(k_i+2) \\ \vdots \\ \Delta u(k_i+N_c-1) \end{bmatrix}$$

x(k-1) =>

$$\boxed{Y = F\,\hat{x}(k_i) + \Phi\,\Delta U}$$

Free response

Forced resp

x(ki)

$$F = \begin{bmatrix} CA \\ CA^2 \\ CA^3 \\ \vdots \\ CA^{N_p} \end{bmatrix} \qquad \Phi = \begin{bmatrix} CB & 0 & 0 & \cdots & 0 \\ CAB & CB & 0 & \cdots & 0 \\ CA^2B & CAB & CB & \cdots & 0 \\ \vdots \\ CA^{N_p-1}B & CA^{N_p-2}B & CA^{N_p-3}B & \cdots & CA^{N_p-N_c}B \end{bmatrix}$$

Np x 1

Np x Nc

Cost function

- Minimize the error between set-point and predicted response at all steps
  - This leads faster response

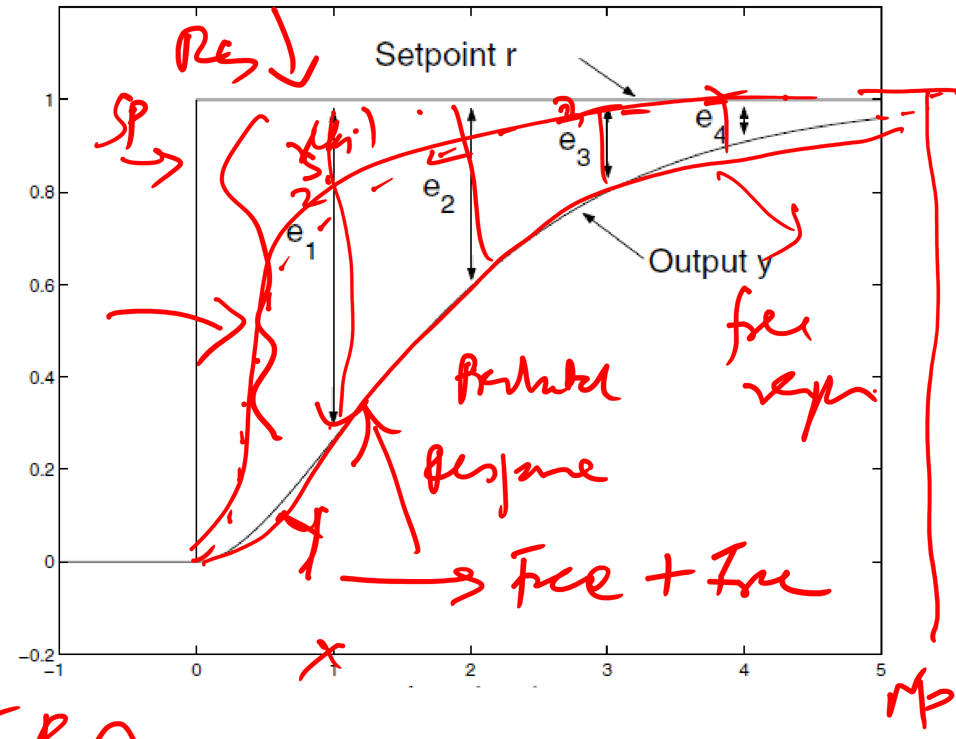
- Minimize the control that minimized the error between set-point and predicted response at all steps

  This leads trade off between faster response and use of input energy/ } Cost of energy

  constraints on rate of change of input } physical limitation

$$J = (\overbrace{R_s - Y}^{E})^T(\overbrace{R_s - Y}^{E}) + \Delta U^T \bar{R} \Delta U$$

$= \Delta u^2 R \Delta u$



For SISO system:

Y = Np step prediction for output (Np x 1)

Rs = Set-point trajectory into the future (Np x 1) $= 1 \times [1\ 1\ 1 \dots N_p]^T$

Δu = Nc future control moves (Nc x 1)

R is scalar – penalty on control move

Unconstrained solution – SISO (1/2)

Objective function $$J = (R_s - Y)^t (R_s - Y) + \Delta U^t R \Delta U \quad (1)$$

The ΔU that minimized J can be obtained from the solution of $\boxed{\dfrac{\partial J}{\partial \Delta U} = 0}$ $\neq 0$

CB

CAB CB

Substitute $\boxed{Y = F\,\hat{x}(k_i) + \Phi\,\Delta U}$ In (1)

$$J = (R_s - F\hat{x}(k_i) - \Phi\Delta U)^T (R_s - F\hat{x}(k_i) - \Phi\Delta U) + \Delta U^T R \Delta U$$

$$J = (R_s - F\hat{x}(k_i))^T (R_s - F\hat{x}(k_i)) - (R_s - F\hat{x}(k_i))^T \Phi\Delta U - (\Phi\Delta U)^T (R_s - F\hat{x}(k_i)) + \Delta U^T \Phi^T \Phi \Delta U + \Delta U^T R \Delta U$$

$$J = (R_s - F\hat{x}(k_i))^t (R_s - F\hat{x}(k_i)) - 2\Delta U^t \Phi^T (R_s - F\hat{x}(k_i)) + \Delta U^t (\Phi^T \Phi + R) \Delta U$$

Unconstrained solution – SISO (2/2)

$$x^2 \quad 2x$$

$$J = (R_s - F\hat{x}(k_i))^t(R_s - F\hat{x}(k_i)) - 2\Delta U^t\Phi^T(R_s - F\hat{x}(k_i)) + \Delta U^t(\Phi^T\Phi + R)\Delta U$$

$= 0 \qquad -2x$

Now $\dfrac{\partial J}{\partial \Delta U} = -2\Phi^T(R_s - F\hat{x}(k_i)) + 2(\Phi^T\Phi + R)\Delta U = 0$

$$(\Phi^T\Phi + R)\Delta U = \Phi^T(R_s - F\hat{x}(k_i))$$

(Hessian matrix)

Red = forc + free

(Red − free) = forc (Δu)

(free response)

Solving for ΔU we get

$$\boxed{\Delta U = (\Phi^T\Phi + R)^{-1}\Phi^T(R_s - F\hat{x}(k_i))}$$

(x_0)

$= N_C \times 1$

$(N_C \times 1)$

$u_{ki} = u_{(ki-1)} + \Delta u_{ki}$ (first value in ΔU)

- *Implementation steps ONLINE*

1. *Obtain the current measurements of states,*
2. *Estimate the states using Kalman filter*
3. *Compute the optimal finite horizon control sequence ($u_k, u_{k+1}, \ldots u_M$)*
4. *Implement first move u_k*
5. *Repeat from Step 1*

$$\Delta U(k_i) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}(\Phi^T\Phi + \bar{R})^{-1}(\Phi^T\bar{R}_s r(k_i) - \Phi^T F\hat{x}(k_i))$$

Example problem

Step response — transient

- Double integrator system

$$x_m(k+1) = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{A_m} x_m(k) + \underbrace{\begin{bmatrix} 0.5 \\ 1 \end{bmatrix}}_{B_m} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C_m} x_m(k).$$

Bm → DC gain

$$\Delta U = (\Phi^T\Phi + R)^{-1}\Phi^T (R_s - F\hat{x}(k_i))$$

Matrices to compute:

Φ, F,

$\Phi^T\Phi$,

$\Phi^T R_s$

$\Phi^T F$

$$F = \begin{bmatrix} CA \\ CA^2 \\ CA^3 \\ \vdots \\ CA^{N_p} \end{bmatrix}; \quad \Phi = \begin{bmatrix} CB & 0 & 0 & \cdots & 0 \\ CAB & CB & 0 & \cdots & 0 \\ CA^2B & CAB & CB & \cdots & 0 \\ \vdots \\ CA^{N_p-1}B & CA^{N_p-2}B & CA^{N_p-3}B & \cdots & CA^{N_p-N_c}B \end{bmatrix}$$

Np x 1   Np x Nc

$$A = \begin{bmatrix} A_m & 0_m^T \\ C_m A_m & 1 \end{bmatrix} \quad B = \begin{bmatrix} B_m \\ C_m B_m \end{bmatrix} \quad C = \begin{bmatrix} 0_m & 1 \end{bmatrix}$$

$$\boldsymbol{x}(k) = \begin{bmatrix} \Delta \boldsymbol{x}_m(k) \\ y(k) \end{bmatrix}$$

$$\boxed{\Delta \boldsymbol{x}_m(k+1) = A_m \Delta \boldsymbol{x}_m(k) + B_m \Delta u(k)}$$