Minima.

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$f: \mathbb{R}^n \to \mathbb{R}$ continuous, 2^{nd} diff

$$\nabla f \in \mathbb{R}^n = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$ steepest ascent

$-\nabla f$: steepest descent

∇^2 $(n \times n)$ symmetric

$$f(x) = c_0 + c^T x + \frac{1}{2} x^T \underbrace{H}_{sym} x$$

$$\nabla f = c + Hx$$

$$\nabla^2 = H \quad \text{(constant)}$$

$$f(x + h) \approx f(x) + \nabla f(x)^T h + \frac{1}{2} h^T \nabla^2(x) h$$

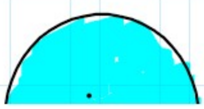
local minimum :

x^* is called a local minimum if $\exists$

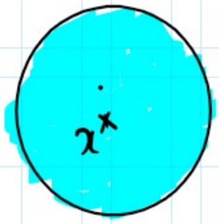
a 'neighbourhood' around x^* s.t $f(x^*) \leq f(x)$

$\forall x$ in the neighbourhood

$\forall x$ in blue shaded region



$f(x^*) \leq f(x)$



$f(x^*) \leq f(x)$

neighbourhood $|x - x^*| \leq r$

Strict minimum

x^* is called a local minimum if -
a 'neighbourhood' around x^* s.t $f(x^*) < f(x)$
$\forall x \neq x^*$ in the neighbourhood

Global minimum x^* is a global minimum
if $\forall x$ $f(x^*) \leq f(x)$

Global (strict) minimum x^* is a strict global minimum
if $\forall x \neq x^*$ $f(x^*) < f(x)$

Maxima:

Local maximum

x^* is called a local minimum if $\exists$
'neighbourhood' around x^* s.t $f(x^*) \geq f(x)$
$\forall x$ in the neighbourhood

Strict maximum

x^* is called a local minimum if

a 'neighbourhood' around x^* s.t. $f(x^*) > f(x)$

$\forall x \neq x^*$ in the neighbourhood

Global maximum: x^* is a global maximum

if $\forall x$ $f(x^*) \geq f(x)$

Global (strict) maximum x^* is a strict global maximum

if $\forall x \neq x^*$ $f(x^*) > f(x)$

max $f(x)$ is equivalent to finding $\min(-f)$

Hence, we work with minimization problems

$f(x) = c_0$ Every x is a minimum! Global minimum

Is there a minimum? Is it strict? Is it global

Yes. No YES

$\forall x \; |x - x^*| < \varepsilon$

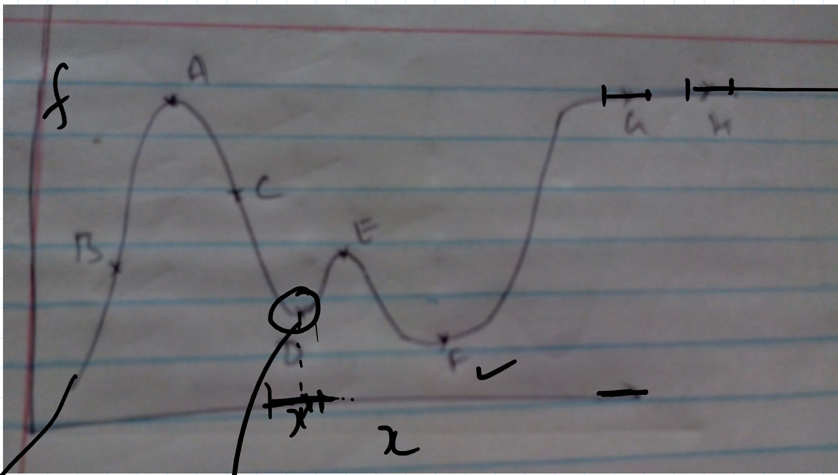
$f(x) - f(x^*) = 0$

$f(x^*) = f(x)$

Is there a maximum? "NO" "YES"

Yes

Every x is a maximum



maxima.

A, E, G, H

For x in this segment

$f(x^*) < f(x)$

D, F, G, H are minima!

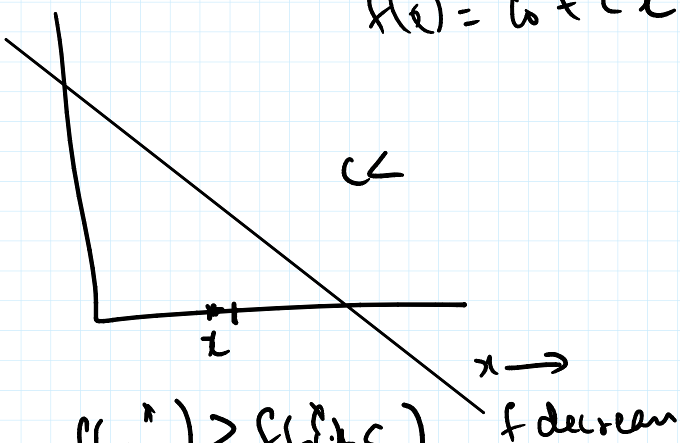
D, F strict

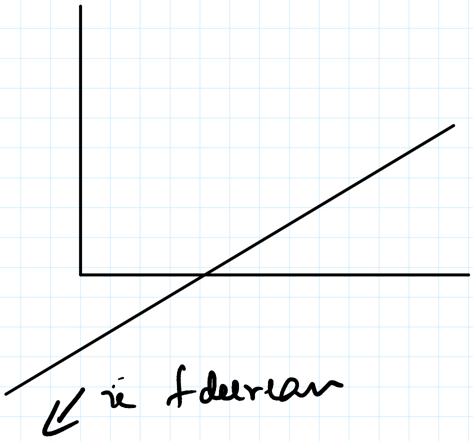
G, H are not strict.

$f(x) = c_0 + c^T x \qquad x \in \mathbb{R}^n. \qquad (c \neq 0)$

$f(x) = c_0 + cx$

$n=1$





$\exists \varepsilon, \quad f(x^*) > f(x^* + \varepsilon)$

$n \geq 1$

Say x^* is a minimum.

$\exists \; r$ s.t $|x - x^*| < r, \quad f(x^*) \leq f(x)$

$x = x^* - \frac{r}{2} c$

$f(x) = c_0 + c^T x = c_0 + c^T \left[x^* - \frac{r}{2} c\right]$

$= \underbrace{c_0 + c^T x^*} - \frac{r}{2} c^T c$

$= f(x^*)$

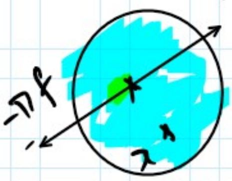
$f(x) - f(x^*) = -\underbrace{\frac{r}{2}} c^T c \quad < 0$

$f(x) < f(x^*)$

$\therefore \nexists$ minimum does not exist!

$f(x) = c_0 + c^T x + \frac{1}{2} x^T H x.$

Let us assume x^* is a minimum (local)



$f(x^*) \le f(x)$ in this neighborhood.

$$f(x) \geq f(x^*).$$

$-\nabla f$ is a decent direction

If you go along $-\nabla f$ f has to decrease

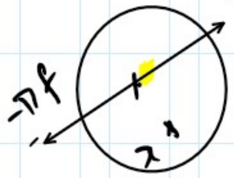
This implies $\nabla f = 0$

at x^* $\nabla f = 0$

First Order Necessary Condition $\nabla f = 0$

(FONC)

x^* is a maximum



$f(x^*) \geq f(x)$ in this neighborhood.

∇f is an ascent direction

if you go along ∇f f has to increase

this implies $\nabla f = 0$

at x^* $\nabla f = 0$