Date:- 12-06-23
Day:- Monday.

Sol:- Step1:- To check w

Unit-5

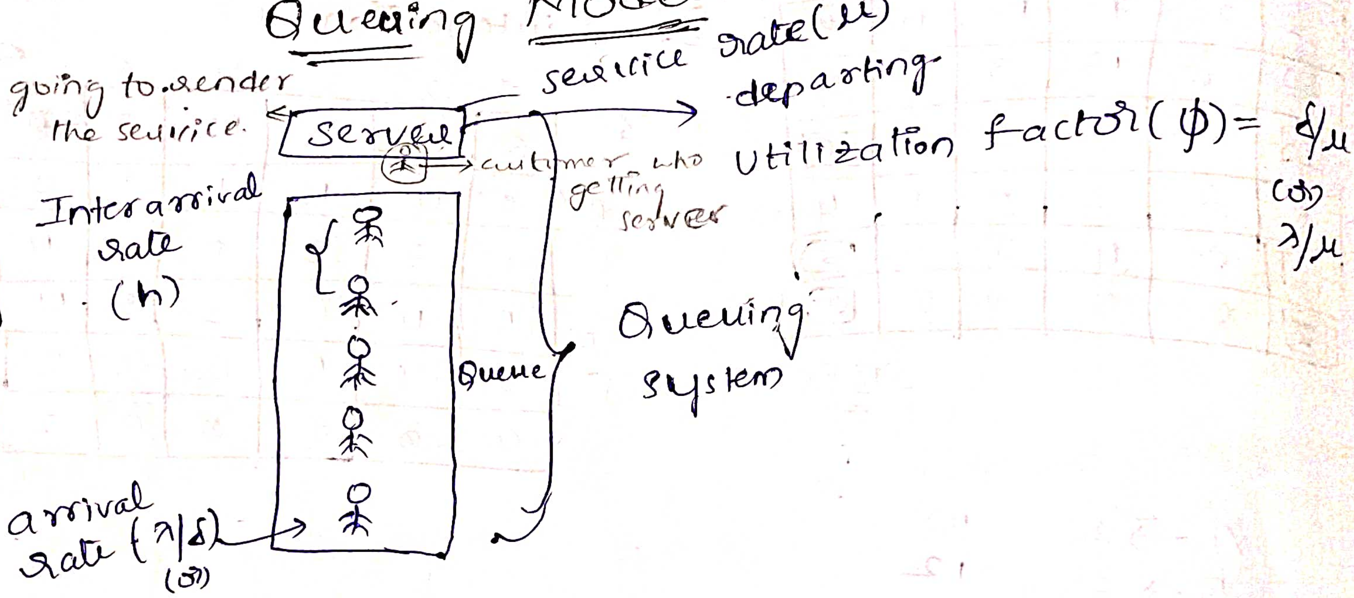
Assignment $\longrightarrow$ Transportation

$m + \boxed{n-1}\ \epsilon$s

In assignment problem $m \times m$
m assignments

* To solve this assignment problem by using transportation we need $n-1$ ϵ's.

* Queuing Models



λ = 55 customers are in Queue.

μ = 30 customers in hr

Inefficient System:-
If the service rate is < the arrival rate

$\mu < \lambda$

some customers are left over in that hour

* For any efficient system ϕ value should be < 1 means $\mu > \lambda$

* ϕ is an utilization factor

Interarrival Rate:-

* always inter arrival rate is very small
* The distribution is exponential (because arrivals are
* completely random)

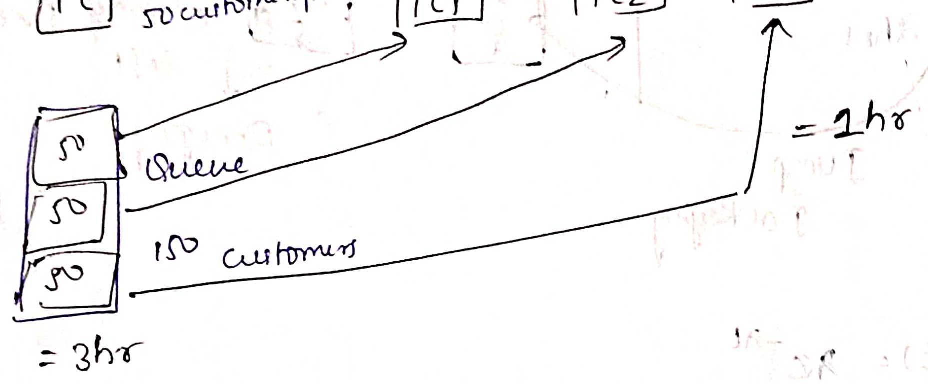
$f(t) = \lambda e^{-\lambda t}$

Departing Rate = Service Rate.

because only after getting service the customer will departure.

* As no of servers increases the waiting time of the customer will decreases. (operating cost also increases)

Ex: $\boxed{TC}$ service rate = 50 customers/hr, $\boxed{TC_1}$ service rate = 50 customers/hr, $\boxed{TC_2}$, $\boxed{TC_3}$



Two models of Queuing System.

Pure Birth

$\boxed{\text{Server}}$

'n' no of customers are present.

h is very small

At t=0
arrivals.
no departure

Pure Death

$\boxed{\text{Server}}$

N = 50

'n' no of departures

$\boxed{0 < n < N}$

N
N-1
N-2

At t=0
departures
no arrivals.

N-n no of customers are present

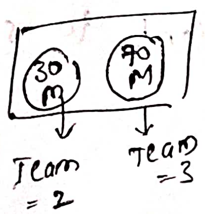
M1 Infinite no of customers permitted. ∞/∞

M2 finite no of customers permitted. Ex: car shed. here also N/∞

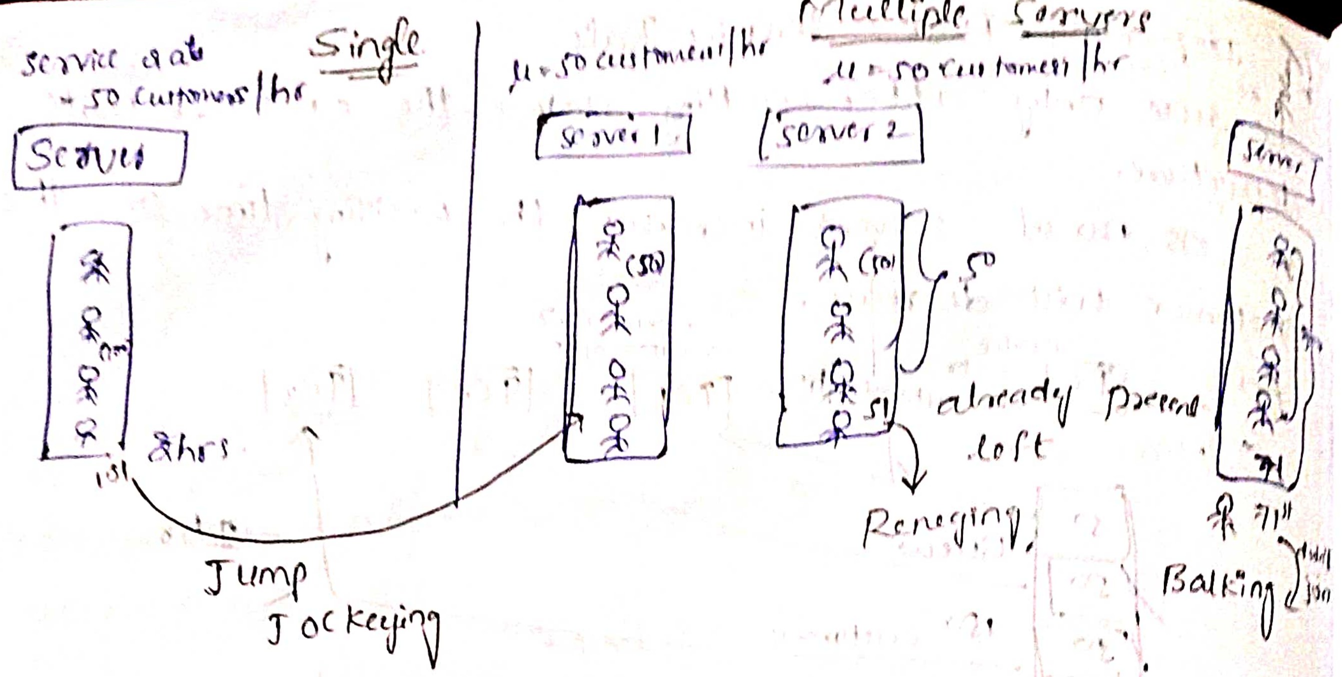
M3 restricted service to particular no of customers. (finite no of customers are permitted)

Single Server

Ex:



N/N



* $f(t) = \lambda e^{-\lambda t}$

λ is arrival rate.

$P(t \le T) = \int_0^T f(t)\,dt$ [probability of finding no of customers]

$\because \int e^{ax}dx = \frac{e^{ax}}{a}$

$= \int_0^T \lambda e^{-\lambda t}dt$

$\because e^{ax} = 1 + ax + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!}$

$= \left[\frac{\lambda e^{-\lambda t}}{-\lambda}\right]_0^T$

$= -e^{-\lambda T} + e^{-\lambda(0)}$

$= 1 - e^{-\lambda T}$

$P(t > T) = 1 - p(t \le T)$

$= 1 - 1 + e^{-\lambda T}$

$= e^{-\lambda T}$

In steady state

* The departure rate is equivalent to service rate

Pure Birth Model

* $P_0(t) = e^{-\lambda t}$

$P_0(h) = e^{-\lambda h}$

$= 1 - \lambda h + \cancel{\frac{(\lambda h)^2}{2!}} - \cancel{\frac{(\lambda h)^3}{3!}} + \text{---}$

$= 1 - \lambda h$

$P_1(h) = 1 - P_0(h)$

$= \lambda h$

$P_n(t+h) = P_n(t)(1-\lambda h) + P_{n-1}(t)\lambda h;$

$P_0(t+h) = P_0(t)(1-\lambda h) +$

$$P_n'(t) = \lim_{h \to 0} \frac{P_n(t+h) - P_n(t)}{h}$$

$$= \frac{P_n \cancel{(t)} - P_n(t \cancel{+h}) + P_n\cancel{(t+h)} - \cancel{P_n(t)}}{h}$$

$P_0'(t) = \lambda \left[P_{n-1}(t) - P_n(t) \right]$

$P_0'(t) = -\lambda P_0(t)$

$= -\lambda e^{-\lambda t}$

$P_0(t) = \lambda t e^{-\lambda t}$

$$\boxed{P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}}$$ poisson. $n = 0, 1, 2, \ldots$

Pure ~~Death~~ Death Model

$P_1(t) = e^{-\mu t}$

$P_n(t+h) = P_n(t)(1-\mu h) + P_{n+1}(t)\mu h$

$$P_n^*(t) = \lim_{h \to 0} \frac{P_n(t+h) - P_n(t)}{h}$$

$= -P_n(t)\mu h + \mu h P_{n+1}(t)$

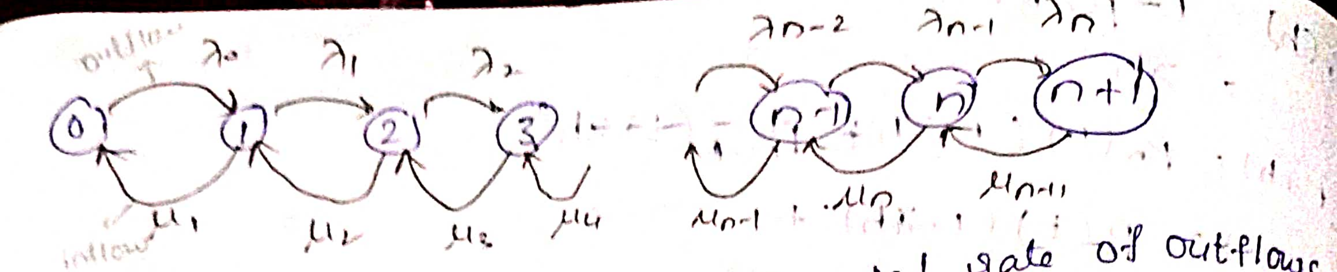
$= \mu [P_{n+1}(t) - P_n(t)]$

$P_0'(t) = \mu P_1(t)$

$$\boxed{P_0(t) = \frac{(\mu t)^{N-n} e^{-\mu t}}{(N-n)!}}$$ → poisson.

* Combining arrivals & departures in the real time Queuing System.

λ_n = arrival rate, μ_n = departure rate, P_n = probability of having 'n' no of customers in steady state system



* Expected rate of inflows = Expected rate of outflows.

State '0':

$$\mu_1 P_1 = \lambda_0 P_0$$

$$P_1 = \left(\frac{\lambda_0}{\mu_1}\right) P_0$$

State '1':-

$$\lambda_0 P_0 + \cancel{\mu_2} + \mu_2 P_2 = (\lambda_1 + \mu_1) P_1$$

$$\cancel{\lambda_0 P_0} + \mu_2 P_2 = (\lambda_1 + \mu_1)\left(\frac{\lambda_0}{\mu_1}\right) P_0$$

$$= \frac{\lambda_1 \lambda_0 P_0}{\mu_1} + \cancel{\lambda_0 P_0}$$

$$P_2 = \left(\frac{\lambda_1 \lambda_0}{\mu_1 \mu_2}\right) P_0$$

State 'n':

$$\lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1} = (\lambda_n + \mu_n) P_n$$ [∵ from P_1 and P_2]

$$P_n = P_0 \left(\frac{\lambda_0 . \lambda_1 . \lambda_2 \cdots \lambda_{n-1}}{\mu_1 . \mu_2 . \mu_3 \cdots \mu_n}\right)$$

Kendall's Notation :- According to this which the various characteristics of a queing model are identified. There are 6 parameters.

$$P \,|\, Q \,|\, R : X \,|\, Y \,|\, Z$$

P = no of customers.

Q = service time distribution

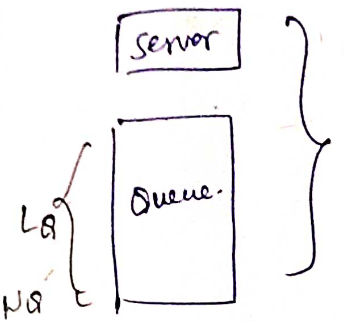
R = no of servers.

X = system capacity (queue discipline) (General discipline | FC

Y = population size (no of customers permitted)

Z = service discipline.

M_1 : $M|M|1 : GD|\infty|\infty$
M_2 : $M|M|1 : GD|\cancel{N}|\infty$
M_3 : $M|M|1 : GD|N|(C)N.$



$L_{sys} > L_q$
$W_{sys} > W_q$

L_{sys}
W_{sys}

* Avg no of customers waiting in a system (L_s)
* Avg no of customers waiting in a queue (L_q)
* Avg waiting time of customers in a system (W_s)
* Avg waiting time of customer in a queue (W_q)
* Avg waiting time

$\underline{M_1}$: $\phi = \frac{\delta}{\mu}$

$L_{sys} = \frac{\phi}{1-\phi}$ customer

$\boxed{P_n = (1-\phi)\phi^n}$

$L_q = \frac{\phi^2}{1-\phi}$ customers
$= L_s - \phi$

$W_{sys} = \frac{L_{sys}}{\delta}$ hours

$W_q = \frac{L_q}{\delta}$ hours.

Problems Related to Model 1:

P_1: The number of ~~arrivals~~ of car in the shed is 45 per hour. Service rate is 60 cars per hour. Find the avg no of car in the system, queue, waiting time in system & queue. of having $n = 0, 15, 20$ cars in the system. P_0, P_{15}, P_{20}

Assuming $N = \infty$

$\delta = 45$ cars/hr

$\mu = 60$ cars/hr

$\phi = \frac{45}{60} = \frac{9}{12} = 0.75$

$L_{sys} = \frac{0.75}{1-0.75} = \frac{0.75}{0.25} = 3$ cars

$L_q = \frac{\phi^2}{1-\phi}$

$= \frac{(0.75)^2}{1-0.25} = \frac{0.5625}{0.25}$

$= 2.25$ cars

$W_{sys} = \frac{L_{sys}}{\delta}$

$= \frac{L_{sys}}{\delta} = \frac{3}{45} = 0.066 \times 60 \times 60$ seconds

$= 240$ seconds.

$W_q = \frac{L_q}{\lambda} = \frac{2.25}{45} hr = 0.05 \times 60 \times 60 = 180$ seconds

$P_0 = (1-\phi)\phi^0 = 1-\phi = 0.25$

$P_{15} = (1-\phi)\phi^{15} = (0.25)(0.75)^{15}$
$= 0.00334$

$P_{20} = (1-\phi)\phi^{20} = (0.25)(0.75)^{15}$
$= 0.00079$

Idle time $= 1-\phi$ (having no cars)
$= 0.25$

Note:-
* If Idle time is < 25% then changes happen in the system it is Justifiable required.

P2:- The number of ships anchored for service in a harbour is 8 ships/week. The service rate of the server is 14 ship/week. Find L_q, L_s, W_q, W_s.
(Assuming $N=\infty$)

$\lambda = 8$ ships/week

$\mu = 14$ ships/week.

$\phi = \frac{\lambda}{\mu} = \frac{8}{14} = 0.571$

$L_s = \frac{\phi}{1-\phi} = \frac{0.571}{1-0.571} = 1.331$ ships

$L_q = \frac{\phi^2}{1-\phi} = \frac{(0.571)^2}{1-0.571} = 0.760$ ships

$W_s = \frac{L_s}{\lambda} = \frac{1.331}{8} = 0.166$ weeks $= 0.166 \times 7$ days $= 0.166 \times 7 \times 24$ hrs
$= 0.166 \times 7 \times 24 \times 60 \times 60$
$= 1,00,396.8$ seconds

$W_q = \frac{L_q}{\lambda} = \frac{0.760}{14} = 0.095$ weeks $= 0.095 \times 7 \times 24 \times 60 \times 60$ seconds
$= 57456$

$P_0 = (1-\phi)\phi^0 = 0.429$

$P_2 = (1-\phi)\phi^2 = 0.139$

~~of children at toll gate in an~~

problems related to Model 2:-

$M/M/1 : GD/N/\infty$

$\boxed{N}$

effective arrival

$\lambda_{eff} = \lambda(1-P_N)$, $\phi = \cancel{\frac{\mu}{\lambda}}\ \frac{\lambda}{\mu}$

$$P_N = \frac{(1-\phi)\phi^{n}}{(1-\phi^{n+1})}, \quad \phi \neq 1$$ (arrival rate ≠ service rate)

$$= \frac{1}{N+1}, \quad \phi = 1$$ $(\lambda = \mu)$ (arrival rate = service rate)

$$L_{sys} = \frac{\phi\left(1-(N+1)\phi^{N}+N\phi^{N+1}\right)}{(1-\phi)(1-\phi^{N+1})}, \quad \phi \neq 1$$

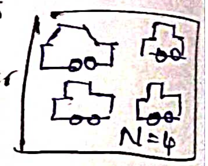
$$= N/2, \quad \phi = 1$$ customers

$$L_{queue} = L_{sys} - \frac{\lambda_{eff}}{\mu}$$ customers

$W_{sys} = \frac{L_{sys}}{\lambda_{eff}}$ hr, $W_q = \frac{L_q}{\lambda_{eff}}$ hr.

problems

P1: The number of arrivals of cars into a shed is 24 cars per hour. The service rendered is 20 cars per hour. 4 no of cars are permitted in the system. Find out the avg no of cars present in the system, queue waiting time in system & queue



$\lambda = 24$ cars/hr

$\mu = 20$ cars/hr

Model 2:- $N = 4$

$$\phi = \frac{\lambda}{\mu} = \frac{24}{20} = 1.2$$

$$P_N = \frac{(1-1.2)(1.2)^4}{(1-(1.2)^5)} = 0.278$$

$\lambda_{eff} = \lambda(1-P_N) = 24(1-0.278) = 17.328$ cars/hr

$$L_{sys} = \frac{1.2}{(1-1.2)} \cdot \frac{(1 - 5(1.2)^4 + 4(1.2)^5)}{(1-(1.2)^5)} \text{ cars}$$

$$= 2.36 \text{ cars}$$

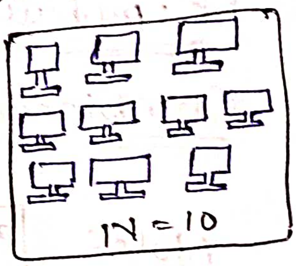
$$L_q = L_{sys} - \frac{\lambda_{eff}}{\mu} = 2.36 - \frac{17.328}{20} = 1.4496 \text{ cars}$$

$$W_{sys} = \frac{L_{sys}}{\lambda_{eff}} = \frac{2.36}{17.328} \text{ hr} = 0.136 \text{ hr} \times 3600 = 310.32 \text{ seconds}$$

$$W_q = \frac{1.494}{17.328} \text{ hr} = 0.0862 \text{ hr} = 310.38 \text{ seconds}$$

* All the L_q, L_{sys}, W_{sys} & W_q values are more taken when compared to model 1 com in model 2 problem in the showroom.

P_2:- 10 number of desktops are permitted into 45 desktops per hour. The no of arrivals of desktops in an hour service is rendered to 55 desktops in an hour find L_{sys}, L_q, W_{sys} & W_q



λ = 45 desktops / hr

μ = 55 desktops / hr

$$\rho = \frac{45}{55} = 0.818$$

Model 2:- $N = 10$

$\lambda_{eff} = \lambda(1-P_N)$

$$P_{10} = \frac{(1-0.818)(0.818)^{10}}{(1-0.818^{11})} = \frac{0.0244}{0.8902} = 0.027$$

P_{10} =

$\lambda_{eff} = \lambda(1-P_N)$

= ~~0.027~~

= $45(1-0.027)$

= 43.785 desktops / hr

$$L_{sys} = \frac{0.818}{(1-0.818)} \times \frac{(1 - 11(0.818)^{10} + 10(0.818)^{11})}{(1-(0.818)^{11})}$$

$$= \frac{0.818}{0.182} \times \frac{0.6217}{0.8902}$$

$$= 4.494 \times 0.698$$

$$= 3.137 \text{ desktops}$$

$$L_q = 3.137 - \frac{43.785}{55}$$

$$= 2.341 \text{ desktops}$$

$$W_{sys} = \frac{3.138}{43.785} \text{ hr} = 0.071 \times 3600 \text{ sec} = 258 \text{ seconds}$$

$$W_q = \frac{2.341}{43.785} \text{ hr} = 0.053 \times 3600 \text{ sec} = 190.8 \text{ seconds}$$

Permutations:- (Arrangement order) $\quad {}^nP_r = \frac{n!}{(n-r)!}$

Combinations:- $\quad {}^nC_r = \frac{n!}{n(n-r)!}$

Note:- $\boxed{{}^nC_r = {}^nC_{n-r}}$ where $n > r$.

Model 3

$$M/M/1 : GD/N/N^c$$

$$\phi = \frac{\lambda}{\mu}$$

Assumptions: $\lambda_n = (N-n)\lambda \qquad 0 \le n \le N$

$$= 0 \qquad n > N$$

$$\mu_n = n\mu, \quad 0 \le n < 1$$

$$= \mu, \quad 1 \le n \le N$$

$$= 0, \quad n > N$$

$\lambda_{eff} = \mu(1-P_0)$ customers/hr

$P_n = {}^NC_n\, \phi^n P_0 \qquad 0 \le n \le c$

$= {}^NC_n \dfrac{n!\, \phi^n}{c!\, c^{n-c}} \cdot P_0 \qquad c \le n \le N$

$$P_0 = \left[\sum_{n=0}^{c} {}^NC_n \phi^n + \sum_{n=c+1}^{N} {}^NC_n \frac{n!\phi^n}{c!\, c^{n-c}} \right]^{-1}$$

$L_q = N - \left(1 + \frac{1}{\phi}\right)(1 - P_0)$ customers

$L_s = N - \dfrac{(1-P_0)}{\phi}$ customers

$W_s = \dfrac{L_s}{\lambda_{eff}}$ hr

$W_q = \dfrac{L_q}{\lambda_{eff}}$ hr.

Problems on Model-3 $C=1$

P1:- In a factory, incase of breakdowns & repairer a crew is assigned to 5 machines. Arrival rate is 2 machines per hour. Service rate is 1.5 machines/hr. Find L_s, L_q, W_s, W_q

Sol:- Model 3 :- $N=5,\ C=1$

$\phi = \dfrac{2}{1.5} = 1.33$

$$P_0 = \left[\sum_{n=0}^{1} 5C_n (1.33)^n + \sum_{n=2}^{5} 5C_n \frac{n!(1.33)^n}{1!\, 1^{n-1}} \right]^{-1}$$

$P_0 = 5C_0 (1.33)^0 + 5C_1 (1.33)^1 + 5C_2 \dfrac{2!(1.33)^2}{1} + 5C_3 \dfrac{3!(1.33)^3}{1} + 5C_4 \dfrac{4!(1.33)^4}{1} + 5C_5 \dfrac{5!(1.33)^5}{1}$

$= \left[1 + 5(1.33) + [10 \times 2 \times (1.33)^2] + [10 \times 6 \times (1.33)^3] + [5 \times 24 (1.33)^4] + [120 (1.33)^5]\right]^{-1}$

$= [1 + 6.65 + \ldots$

$= 0.00844$

$\lambda_{eff} = 1.5(1 - 0.00844) = 1.48734$ machines/hr.

$$L_{sys} = \frac{5-(1-0.00844)}{1.33} = 4.2544 \text{ machines}$$

$$L_q = 5 - \left(1+\frac{1}{1.33}\right)(1-0.00844) = 3.262 \text{ machines.}$$

$$W_{sys} = \frac{4.254}{1.487} \text{ hr} = 2.860 \text{ hr} \times 3600 = 7897.24 \text{ seconds}$$

$$W_q = \frac{3.262}{1.487} \text{ hr} = 2.193 \times 3600 =$$

Rough

$L_{server} = 1 - P_0$ (or) $L_{server} = L_{system} - L_q$

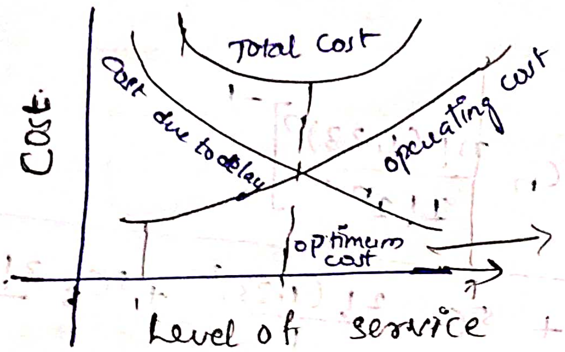
$$L_q = N - (1-P_0) - \frac{(1-P_0)}{\phi}$$

$$L_{sys} = N - \frac{(1-P_0)}{\phi}$$

$$= L_q + L_{server}$$

* Always $L_q < L_{sys}$ & $W_q < W_{sys}$.

Queuing Costs :-



→ where operating cost & cost due to delay are balanced.

→ Expected total cost = Expected operating cost + Expected waiting costs

$$ETC = EOC + EWC$$

$$= C_1(x) + C_2 L_{sys}$$

x = mean or service level, C_1, C_2 are costs of operating & delay respectively.

L_{sys} = Avg no of customers waiting in system $= \frac{\phi}{1-\phi}$ (for M)

P_1:- Keen co publishing in the process of purchasing a high speed commerical Copier.

Four models are given.

Copier model	operating cost C_1 ($/hr)	speed (sheets/min)
1	15	30
2	20	36
3	24	50
4	27	66

Jobs arrive at keen co according to poisson distribution with a mean of four jobs (λ) per 24 hours a day (x). Job size is random but averages about 10,000 sheets / job. penality cost of $80 jobs ($C_2$) per day is specified. which model to purchase?