Let $E$ be a directed graph, and $\alpha \in E^*$ such that $|r^{-1}\{s(\alpha)\}| = \infty$. Let $X, Y \subseteq_{\text{fin}} E(S(E))$, and $Z$ be a finite cover of $E^{X,Y}$. If $\xi_\alpha \in \mathcal{U}(X,Y)$, then $\xi_\alpha \cap Z \neq \emptyset$. Proof:

First note:

$$\begin{aligned} E^{X,Y} &= \{e \in E(S(E))\colon\ e \leq x\ \forall x \in x \text{ and } ey = 0\ \forall y \in Y\} \\ &= \{e \in E(S(E))\colon\ e \leq\ \min(X) \text{ and } ey = 0\ \forall y \in Y\} \\ &= E^{\{\min(X)\},Y} \end{aligned}$$

Letting $\min(X) = (x, x)$,

$$E^{X,Y} = \{(xx', xx')\colon\ x' \in E^*,\ r(x') = s(x) \text{ and } (xx', xx')y = 0\ \forall y \in Y\}$$

Consider the set $C := \{(\alpha b, \alpha b)\colon\ b \in E^1,\ s(\alpha) = r(b)\}$. By the assumption that $s(\alpha)$ is an infinite receiver, $C$ is infinite. Given $y \in Y$, let $\nu$ be the path corresponding to $y$. Since $\xi_\alpha \in \mathcal{U}(X,Y)$, $\nu$ is not a prefix of $\alpha$, and thus not a prefix of $\alpha b$ for any $b$. Thus, if $(\alpha b, \alpha b)y \neq 0$, $\alpha b$ is a prefix of $\nu$. Then for $\beta \neq b$, $\alpha\beta$ cannot be a prefix of $\nu$. So there is at most one element of $C$ such that $(\alpha b, \alpha b)y \neq 0$. By the assumption that $Y$ is finite, all but finitely many elements of $C$ are inside $E^{\{(x,x)\},Y}$. Therefore, if $Z$ is a cover of $E^{X,Y}$, $Z$ is an outer cover of the infinite set $E^{X,Y} \cap C$. Because $Z$ is finite, $\exists z \in Z$ with $(\alpha b, \alpha b)z \neq 0$ for infinitely many $(\alpha b, \alpha b) \in E^{X,Y} \cap C$. If $\upsilon$ is the path corresponding to $z$, then $\forall b$, either $\upsilon$ is a prefix of $\alpha b$, or $\alpha b$ is a prefix of $\upsilon$. All the $\alpha b$ are the same length with a different starting edge, so if one is a prefix of $\upsilon$, no other can be a prefix of $\upsilon$. So $\upsilon$ is a prefix of $\alpha b$ for infinitely many $b$. Thus $|\upsilon| \leq |\alpha| + 1$. If $|\upsilon| = |\alpha| + 1$, we have a contradiction: $b = \beta\ \forall (\alpha b, \alpha b), (\alpha\beta, \alpha, \beta) \in E^{X,Y}$. Thus $|\upsilon| \leq |\alpha|$, so $\upsilon$ is a prefix of $\alpha$. Thus $z = (\upsilon, \upsilon) \in \xi_\alpha$, so $\xi_\alpha \cap Z \neq \emptyset$.