1 Introduction

2 Preliminaries

First we must clarify a few frequently used definitions and results.

Definition 2.1. We call a set S a *semigroup* if it has an associative binary operation that sends a pair of elements $(a, b) \mapsto ab$ for all $a, b \in S$.

Definition 2.2. We call a semigroup S an *inverse semigroup* if for every $s \in S$, there exists a unique element $s^* \in S$ such that $s = ss^*s$ and $s^* = s^*ss^*$. We call s^* the *inverse* of s.

Definition 2.3. An element e of an inverse semigroup S is called an *idempotent* if $e = e^2$. Note that for all idempotents, $e = e^*$, and for all elements s in an inverse semigroup, ss^* and s^*s are idempotent. We denote the set of idempotents on S by $E(S)$.

Remark 2.4. *If S is an inverse semigroup, for e, $f \in E(S)$, $ef = fe$.*

In other words, the idempotents of an inverse semigroup commute. A proof of this can be found in Kyle's paper. cite this properly

Definition 2.5. Let S be a set. A relation $\leq$ on S is called a *partial order* if,

- It is reflexive; $a \leq a \quad \forall a \in S$.
- It is antisymmetric; $a \leq b$ and $b \leq a \implies a = b \quad \forall a, b \in S$.
- It is transitive; $a \leq b$ and $b \leq c \implies a \leq c \quad \forall a, b, c \in S$.

A set S paired with a *partial order* is called a partially ordered set, or a *poset.*

3 Filters

4 Directed Graphs

5 References

Lemma 5.1. *If $(\alpha, \alpha)(\beta, \beta) \neq 0$ and $|\alpha| = |\beta|$, then $\alpha = \beta$.*

Proof. Without loss of generality, suppose α is a prefix of β. Then $\alpha = \beta\beta'$ for some β. But $|\alpha| = |\beta|$, so $|\beta'| = 0$, meaning $\beta' = s(\beta)$. Thus $\alpha = \beta s(\beta) = \beta$. $\square$

An immediate consequence of this is that for any given length, a filter contains at most one element corresponding to a path of that length. Furthermore, if ζ is a filter, and $(\alpha, \alpha) \in \zeta$ such that $|\alpha| = n$, we can restrict α to any length $0 \leq m < n$ to produce a unique element of length m inside ζ.

Theorem 5.2. $\hat{E}_0 = \{\xi_\alpha \colon \alpha \in E^*\} \cup \{\eta_x \colon x \in E^\infty\}$

Proof. First we show that ξ_α and η_x are filters. Let α be a finite path and let x be an infinite path. Consider (β,β), $(\gamma,\gamma) \in \xi_\alpha$ with $|\beta| = m$, $|\gamma| = n$. Reading edges left to right, β contains the first m edges of α and γ contains the first n edges of α. Without loss of generality, assume $m \leq n$. We can extend β by the next $n - m$ edges of α to produce γ. Thus β is a prefix of γ, so $(\beta,\beta)(\gamma,\gamma) = (\gamma,\gamma) \in \xi_\alpha$. A similar argument shows that η_x is closed under multiplication. So both ξ_α and η_x are prefilters on $E(S(E))$. Next, let $(\beta,\beta) \in \xi_\alpha$, and let $(\delta,\delta) \in E(S(E))$ so that $(\beta,\beta) \leq (\delta,\delta)$. Then $\beta = \delta\delta'$ for some $\delta' \in E^*$. Thus $\alpha = \beta\beta' = \delta\delta'\beta'$. Therefore δ is a prefix of α, so $(\delta,\delta) \in \xi_\alpha$. Like before, a very similar argument works for η_x. So both ξ_α and η_x are filters. Going the other way, take a filter $\zeta \in \hat{E}_0$. We do this in cases.

Case 1. *ζ is finite.* Let (α,α) be the minimum element inside ζ. Since ζ is a filter, For $(\beta,\beta) \in E(S(E))$, $(\alpha,\alpha) \leq (\beta,\beta) \implies (\beta,\beta) \in \zeta$. However, by the definition of the minimal element, $(\beta,\beta) \in \zeta \implies (\alpha,\alpha) \leq (\beta,\beta)$. Thus $(\beta,\beta) \in \zeta$ if and only if $(\alpha,\alpha) \leq (\beta,\beta) \iff \beta$ is a prefix of α. So $\zeta = \xi_\alpha$.

Case 2. *ζ is infinite.* By the lemma above, we can get an idea of what elements of ζ look like. For each nonnegative integer, ζ contains precisely one element corresponding to a path of that length. Because we require nonzero product between elements, every path in the filter is a prefix of every longer path also contained inside the filter. All of this given, we can find an $x \in E^\infty$ such that every path inside ζ is a prefix of x. By the uniqueness of filter elements, it follows that $\zeta = \eta_x$.

We have shown that a nonempty subset $\zeta \subset E(S(E))$ is a filter if and only if it is of the form ξ_α or η_x. Thus $\hat{E}_0 = \{\xi_\alpha \colon \alpha \in E^*\} \cup \{\eta_x \colon x \in E^\infty\}$. $\square$

Now that we know what filters look like on the graph inverse semigroup, the task of identifying ultrafilters becomes much simpler.

Theorem 5.3. $\hat{E}_\infty = \{\eta_x \colon x \in E^\infty\} \cup \{\xi_\alpha \colon \alpha \in E^* \textit{ and } |r^{-1}\{s(\alpha)\}| = 0\}$

Proof. We need only consider various kinds of filters, determining whether they are ultrafilters or not. We do this in cases.

Case 1. ξ_α, $|r^{-1}\{s(\alpha)\}| > 0$. Take $e \in E^1$ such that $r(e) = s(\alpha)$. It is clear that $\xi_\alpha \subset \xi_{\alpha e}$. So ξ_α is not an ultrafilter.

Case 2. ξ_α, $|r^{-1}\{s(\alpha)\}| = 0$. Suppose we can find $\zeta \in \hat{E}_0$ such that $\xi_\alpha \subset \zeta$. For $0 \leq i \leq |\alpha|$, ζ inherits its idempotent with the path corresponding to length i from ξ_α. So we can find an element $(\beta,\beta) \in \zeta$ with $|\beta| > |\alpha|$ and $(\alpha,\alpha)(\beta,\beta) \neq 0$. Thus $\beta = \alpha\alpha'$ for some $\alpha' \in E^*$. But no edges go to $s(\alpha)$, so we cannot construct α' with $r(\alpha') = s(\alpha)$. This is a contradiction, so ξ_α is an ultrafilter.

Case 3. η_x, $x \in E^\infty$. Suppose there exists $\zeta \in \hat{E}_0$ such that $\eta_x \subset \zeta$. Then we can find $(\beta, \beta) \in \zeta$ such that $(\beta, \beta) \notin \eta_x$. But η_x, as we showed before, contains an element z corresponding to a path of length $|\beta|$. As a superset of η_x, ζ also contains z. By the lemma at the start, it follows that $(\beta, \beta) = z \in \eta_x$, which is a contradiction. So η_x is an ultrafilter for all $x \in E^\infty$.

We have identified which filters are ultrafilters, showing that $\hat{E}_\infty = \{\xi_\alpha \colon \alpha \in E^*$ and $|r^{-1}\{s(\alpha)\}| = 0\} \cup \{\eta_x \colon x \in E^\infty\}$. □

Theorem 5.4. *Let E be a directed graph, and $\alpha \in E^*$ such that $|r^{-1}\{s(\alpha)\}| = \infty$. Let $X, Y \subseteq_{fin} E(S(E))$, and Z be a finite cover of $E^{X,Y}$. If $\xi_\alpha \in \mathcal{U}(X, Y)$, then $\xi_\alpha \cap Z \neq \emptyset$.*

Proof. First note:

$$\begin{aligned} E^{X,Y} &= \{e \in E(S(E)) \colon e \leq x \ \forall x \in x \text{ and } ey = 0 \ \forall y \in Y\} \\ &= \{e \in E(S(E)) \colon e \leq \min(X) \text{ and } ey = 0 \ \forall y \in Y\} \\ &= E^{\{\min(X)\}, Y} \end{aligned}$$

Letting $\min(X) = (x, x)$,

$$E^{X,Y} = \{(xx', xx') \colon x' \in E^*, \ r(x') = s(x) \text{ and } (xx', xx')y = 0 \ \forall y \in Y\}$$

Consider the set $C := \{(\alpha b, \alpha b) \colon b \in E^1, \ s(\alpha) = r(b)\}$. By the assumption that $s(\alpha)$ is an infinite receiver, C is infinite. Given $y \in Y$, let ν be the path corresponding to y. Since $\xi_\alpha \in \mathcal{U}(X, Y)$, ν is not a prefix of α, and thus not a proper prefix of αb for any b. Thus, if $(\alpha b, \alpha b)y \neq 0$, αb is a prefix of ν. Then for $\beta \neq b$, $\alpha\beta$ cannot be a prefix of ν. So there is at most one element of C such that $(\alpha b, \alpha b)y \neq 0$. By the assumption that Y is finite, all but finitely many elements of C are inside $E^{\{(x,x)\}, Y}$. Therefore, if Z is a cover of $E^{X,Y}$, Z is an outer cover of the infinite set $E^{X,Y} \cap C$. Because Z is finite, $\exists z \in Z$ with $(\alpha b, \alpha b)z \neq 0$ for infinitely many $(\alpha b, \alpha b) \in E^{X,Y} \cap C$. If υ is the path corresponding to z, then for every b, either υ is a prefix of αb, or αb is a prefix of υ. All the αb are the same length with a different starting edge, so if one is a prefix of υ, no other can be a prefix of υ. So υ is a prefix of αb for infinitely many b. Thus $|\upsilon| \leq |\alpha| + 1$. If $|\upsilon| = |\alpha| + 1$, we have a contradiction: $b = \beta$ for all $(\alpha b, \alpha b), (\alpha\beta, \alpha, \beta) \in C$. Thus $|\upsilon| \leq |\alpha|$, so υ is a prefix of α. Therefore $z = (\upsilon, \upsilon) \in \xi_\alpha$, so $\xi_\alpha \cap Z \neq \emptyset$. □