Lemma: If $(\alpha,\alpha)(\beta,\beta)\neq 0$ and $|\alpha|=|\beta|$, then $\alpha=\beta$. Proof: WOLOG suppose α is a prefix of β. Then $\alpha=\beta\beta'$ for some β. But $|\alpha|=|\beta|$, so $|\beta'|=0$, meaning $\beta'=s(\beta)$. Thus $\alpha=\beta s(\beta)=\beta$. An immediate consequence of this is that a proper filter contains at most one path of any given length.

Let E be a directed graph, and $\alpha\in E^*$ such that $|r^{-1}\{s(\alpha)\}|=\infty$. Let $X,Y\subseteq_{\text{fin}} E(S(E))$, and Z be a finite cover of $E^{X,Y}$. If $\xi_\alpha\in\mathcal{U}(X,Y)$, then $\xi_\alpha\cap Z\neq\emptyset$. Proof:

First note:

$$\begin{aligned}E^{X,Y}&=\{e\in E(S(E))\colon\ e\leq x\ \forall x\in x \text{ and } ey=0\ \forall y\in Y\}\\&=\{e\in E(S(E))\colon\ e\leq\ \min(X)\text{ and } ey=0\ \forall y\in Y\}\\&=E^{\{\min(X)\},Y}\end{aligned}$$

Letting $\min(X)=(x,x)$,

$$E^{X,Y}=\{(xx',xx')\colon\ x'\in E^*,\ r(x')=s(x)\text{ and }(xx',xx')y=0\ \forall y\in Y\}$$

Consider the set $C:=\{(\alpha b,\alpha b)\colon\ b\in E^1,\ s(\alpha)=r(b)\}$. By the assumption that $s(\alpha)$ is an infinite receiver, C is infinite. Given $y\in Y$, let ν be the path corresponding to y. Since $\xi_\alpha\in\mathcal{U}(X,Y)$, ν is not a prefix of α, and thus not a proper prefix of αb for any b. Thus, if $(\alpha b,\alpha b)y\neq 0$, αb is a prefix of ν. Then for $\beta\neq b$, $\alpha\beta$ cannot be a prefix of ν. So there is at most one element of C such that $(\alpha b,\alpha b)y\neq 0$. By the assumption that Y is finite, all but finitely many elements of C are inside $E^{\{(x,x)\},Y}$. Therefore, if Z is a cover of $E^{X,Y}$, Z is an outer cover of the infinite set $E^{X,Y}\cap C$. Because Z is finite, $\exists z\in Z$ with $(\alpha b,\alpha b)z\neq 0$ for infinitely many $(\alpha b,\alpha b)\in E^{X,Y}\cap C$. If υ is the path corresponding to z, then for every b, either υ is a prefix of αb, or αb is a prefix of υ. All the αb are the same length with a different starting edge, so if one is a prefix of υ, no other can be a prefix of υ. So υ is a prefix of αb for infinitely many b. Thus $|\upsilon|\leq|\alpha|+1$. If $|\upsilon|=|\alpha|+1$, we have a contradiction: $b=\beta$ for all $(\alpha b,\alpha b),(\alpha\beta,\alpha,\beta)\in C$. Thus $|\upsilon|\leq|\alpha|$, so υ is a prefix of α. Therefore $z=(\upsilon,\upsilon)\in\xi_\alpha$, so $\xi_\alpha\cap Z\neq\emptyset$.