1 Introduction

2 Preliminaries

First we must clarify a few frequently used definitions and results.

Definition 2.1. We call a set S a *semigroup* if it has an associative binary operation that sends a pair of elements $(a, b) \mapsto ab$ for all $a, b \in S$.

Definition 2.2. We call a semigroup S an *inverse semigroup* if for every $s \in S$, there exists a unique element $s^* \in S$ such that $s = ss^*s$ and $s^* = s^*ss^*$. We call s^* the *inverse* of s. If S contains a zero element, we call it an *inverse semigroup with zero.*

Definition 2.3. An element e of an inverse semigroup S is called an *idempotent* if $e = e^2$. Note that for all idempotents, $e = e^*$, and for all elements s in an inverse semigroup, ss^* and s^*s are idempotent. We denote the set of idempotents on S by $E(S)$.

Remark 2.4. *If S is an inverse semigroup, for $e,\ f \in E(S)$, $ef = fe$.*

In other words, the idempotents of an inverse semigroup commute. (A proof of this can be found in Kyle's paper. cite this properly)

Definition 2.5. Let S be a set. A relation $\leq$ on S is called a *partial order* if

- It is reflexive; $a \leq a \quad \forall a \in S$
- It is antisymmetric; $a \leq b$ and $b \leq a \implies a = b \quad \forall a, b \in S$
- It is transitive; $a \leq b$ and $b \leq c \implies a \leq c \quad \forall a, b, c \in S$

A set S paired with a *partial order* is called a partially ordered set, or a *poset.*

3 The Tight Spectrum of an Inverse Semigroup

Now that we have a basic set of definitions in place, we can start to explore the set of idempotents on inverse semigroups. For this section, let S be an inverse semigroup with zero. We can define a partial order on $E(S)$, saying that $e \leq f \iff e = ef$. This can be extended to the entire set S, but in this paper we are only concerned with idempotents. (mention it's a meet semilattice, point to ISG primer?)

Definition 3.1. A nonempty proper subset $\xi \subset E(S)$ is called a *filter* on S if

1. For $e, f \in \xi$, $ef \in \xi$
2. For $x \in \xi$, $e \in E(S)$, $x \leq e \implies e \in \xi$

Subsets that only satisfy the first condition are called *prefilters*. We denote the set of filters on S by $\hat{E}_0^S$. An important remark is that the zero element of S is not contained inside any filters. If $0 \in \xi$, condition (2) would imply that $\xi = S$, which violates our definition. We can consider $\hat{E}_0^S$ a subset of $\{0,1\}^{E(S)}$, and thus endow it with the product topology inherited from $\{0,1\}^{E(S)}$. Equivalently, let X, Y be finite subsets of $E(S)$. Define $\mathcal{U}(X,Y) := \{\xi \in \hat{E}_0\colon X \subseteq \xi \text{ and } Y \cap \xi = \emptyset\}$. Sets of the form $\mathcal{U}(X,Y)$ form a basis for the topology on the set of filters. With this topology, the topological space $\hat{E}_0^S$ is called the *Exel spectrum* of S.

Definition 3.2. A filter $\eta \in \hat{E}_0$ is called an *ultrafilter* if it is maximal with respect to set inclusion. In other words, it is not contained inside another filter.

We define the subspace $\hat{E}_\infty^S := \{\xi \in \hat{E}_0^S\colon \xi \text{ is an ultrafilter}\}$. Note that if the inverse semigroup is obvious, we need not superscript S when describing the spectrum.

Remark 3.3. *For an ultrafilter $\eta \in \hat{E}_\infty$ and $e \in E(S)$, $e \notin \eta \implies ef = 0$ for some $f \in \eta$.*

Theorem 3.4. *Every filter is contained inside an ultrafilter.*

Proof. Let $\xi \in \hat{E}_0$. To prove this, we hope to use Zorn's lemma. We define the set $\mathcal{P} := \{\mathcal{F} \in \hat{E}_0\colon \xi \subseteq \mathcal{F}\}$ and we order $\mathcal{P}$ by set inclusion. Let $\mathcal{C} \subseteq \mathcal{P}$ be a chain. If we take $\mathcal{F} := \bigcup_{\zeta \in \mathcal{C}} \zeta$, it is clear that $\mathcal{F}$ is an upper bound of $\mathcal{C}$ and $\xi \subseteq \mathcal{F}$. To show that $\mathcal{F} \in \mathcal{P}$, we need only show that it is a filter. Suppose f, $g \in \mathcal{F}$. Then $f \in \zeta_1$, $g \in \zeta_2$ for some ζ_1, $\zeta_2 \in \mathcal{C}$. Since $\mathcal{C}$ is totally ordered, without loss of generality we assume $\zeta_1 \subseteq \zeta_2$. Then $f, g \in \zeta_2 \implies fg \in \zeta_2 \subseteq \mathcal{F}$. So $\mathcal{F}$ is a prefilter. Now suppose $f \in \mathcal{F}$, $e \in E(S)$ with $f \leq e$. Then $f \in \zeta$ for some $\zeta \in \mathcal{C}$, but since ζ is a filter, $e \in \zeta \subseteq \mathcal{F}$. So $\mathcal{F}$ is upward closed, and hence a filter. By Zorn's lemma, there exists $\eta \in \mathcal{P}$ such that η is maximal with respect to set inclusion. This is our definition of an ultrafilter; we have shown that $\forall \xi \in \hat{E}_0$, $\exists \eta \in \hat{E}_\infty$ with $\xi \subseteq \eta$. $\square$

We can now begin to shift our focus towards the tight spectrum, which is the main topic of this section.

Definition 3.5. Let X, Y be finite subsets of $E(S)$. We define

$$E^{X,Y} := \{e \in E(S)\colon e \leq x \ \forall x \in x \text{ and } ey = 0 \ \forall y \in Y\}$$

Definition 3.6. Given $\mathcal{E} \subseteq E(S)$, we call $Z \subseteq E(S)$ an *outer cover* of $\mathcal{E}$ if $\forall e \neq 0 \in \mathcal{E}$, $\exists z \in Z$ with $ez \neq 0$. If is Z an outer cover of $\mathcal{E}$ and $Z \subseteq \mathcal{E}$, we say Z is a *cover* of $\mathcal{E}$.

Lemma 3.7. *Let X, Y be finite subsets of $E(S)$, and let $x = min(X)$. Then*

(i) $E^{X,Y} = E^{\{x\},Y}$

(ii) $\mathcal{U}(X,Y) = \mathcal{U}(\{x\},Y)$

Proof. (i)

$$\begin{aligned} E^{X,Y} &= \{e \in E(S) \colon e \le x \ \forall x \in x \text{ and } ey = 0 \ \forall y \in Y\} \\ &= \{e \in E(S) \colon e \le \min(X) \text{ and } ey = 0 \ \forall y \in Y\} \\ &= E^{\{x\},Y} \end{aligned}$$

(ii) Since $\{x\} \subseteq X$, $\mathcal{U}(X,Y) \subseteq \mathcal{U}(\{x\},Y)$. Now suppose $\xi \in \mathcal{U}(\{x\},Y)$. For $\chi \in X$, $x \le \chi \implies \chi \in \xi$. Thus $X \subseteq \xi$, so $\xi \in \mathcal{U}(X,Y)$. We have shown that $\mathcal{U}(X,Y)$ and $\mathcal{U}(\{x\},Y)$ are subsets of eachother, so they are equal. □

When we are working with $\mathcal{U}(X,Y)$ and $E^{X,Y}$, this lemma allows us the freedom of only considering the case where X is a singleton set.

Definition 3.8. Let $\xi \in \hat{E}_0$. We say that ξ is a *tight filter* if for all finite subsets X, $Y \subseteq E(S)$ and for all finite covers Z of $E^{X,Y}$, $\xi \in \mathcal{U}(X,Y) \implies Z \cap \xi \neq \emptyset$. We call the set of tight filters the *tight spectrum*, and denote it by $\hat{E}^S_{\text{tight}}$.

This definition may appear a bit contrived, but the next theorem hopes to hint at its significance.

Theorem 3.9. *$\hat{E}_{tight}$ is the closure of $\hat{E}_\infty$ in $\hat{E}_0$.*

Proof. We show that a filter $\xi \in \overline{\hat{E}_\infty} \iff \xi \in \hat{E}_{\text{tight}}$. First the forward implication. Let $\xi \in \overline{\hat{E}_\infty}$ and suppose $\xi \in \mathcal{U}(\{x\},Y)$. We prove the contrapositive of our definition for tightness. Let $Z \subseteq E^{\{x\},Y}$, and suppose $Z \cap \xi = \emptyset$. Then $\xi \in \mathcal{U}(\{x\}, Y \cup Z)$. Since $\xi \in \overline{\hat{E}_\infty}$, we can find an ultrafilter $\eta \in \mathcal{U}(\{x\}, Y \cup Z)$. By Remark 3.3, for every $f \in Y \cup Z$, we can find an idempotent $e_f \in \eta$ with $fe_f = 0$. Define $e := \left(\prod_{f \in Y \cup Z} e_f\right) x$. $e \le x$ and $ey = 0 \ \forall y \in Y$, so $e \in E^{\{x\},Y}$, but $ez = 0 \ \forall z \in Z$. So Z is not a finite cover of $E^{\{x\},Y}$, thus ξ is a tight filter, hence $\overline{\hat{E}_\infty} \subseteq \hat{E}_{\text{tight}}$.

We now prove the other direction. Let $\xi \in \hat{E}_{\text{tight}}$ and suppose $\xi \in \mathcal{U}(\{x\},Y)$. First we show that $E^{\{x\},Y} \neq \{0\}$, by way of contradiction. If $E^{\{x\},Y} = \{0\}$, then $Z = \emptyset$ is a finite cover. Since ξ is tight, $\xi \cap Z \neq \emptyset$, which is a contradiction. So we can find a nonzero idempotent $e \in E^{\{x\},Y}$. Construct a filter ζ by including all the idempotents at least as large as e and closing it under products. By Theorem 3.4, we can find an ultrafilter η with $\eta \supseteq \zeta \ni e$. Note that since $ey = 0 \ \forall y \in Y$ and $e \le x$, any filter containing e must not intersect Y and must contain x. Thus $\eta \in \mathcal{U}(\{x\},Y)$, and because open sets of this form are a basis for the topology on $\hat{E}_0$, it follows that every open neighbourhood of ξ contains an ultrafilter η. So $\hat{E}_{\text{tight}} \subseteq \overline{\hat{E}_\infty}$. Finally, by this and the work above, $\hat{E}_{\text{tight}} = \overline{\hat{E}_\infty}$. □

4 Directed Graphs

Lemma 4.1. *If* $(\alpha,\alpha)(\beta,\beta) \neq 0$ *and* $|\alpha| = |\beta|$*, then* $\alpha = \beta$*.*

Proof. Without loss of generality, suppose α is a prefix of β. Then $\alpha = \beta\beta'$ for some β. But $|\alpha| = |\beta|$, so $|\beta'| = 0$, meaning $\beta' = s(\beta)$. Thus $\alpha = \beta s(\beta) = \beta$. $\square$

An immediate consequence of this is that for any given length, a filter contains at most one element corresponding to a path of that length. Furthermore, if ζ is a filter, and $(\alpha,\alpha) \in \zeta$ such that $|\alpha| = n$, we can restrict α to any length $0 \leq m < n$ to produce a unique element of length m inside ζ.

Theorem 4.2. $\hat{E}_0 = \{\xi_\alpha\colon \alpha \in E^*\} \cup \{\eta_x\colon x \in E^\infty\}$

Proof. First we show that ξ_α and η_x are filters. Let α be a finite path and let x be an infinite path. Consider (β,β), $(\gamma,\gamma) \in \xi_\alpha$ with $|\beta| = m$, $|\gamma| = n$. Reading edges left to right, β contains the first m edges of α and γ contains the first n edges of α. Without loss of generality, assume $m \leq n$. We can extend β by the next $n - m$ edges of α to produce γ. Thus β is a prefix of γ, so $(\beta,\beta)(\gamma,\gamma) = (\gamma,\gamma) \in \xi_\alpha$. A similar argument shows that η_x is closed under multiplication. So both ξ_α and η_x are prefilters on $E(S(E))$. Next, let $(\beta,\beta) \in \xi_\alpha$, and let $(\delta,\delta) \in E(S(E))$ so that $(\beta,\beta) \leq (\delta,\delta)$. Then $\beta = \delta\delta'$ for some $\delta' \in E^*$. Thus $\alpha = \beta\beta' = \delta\delta'\beta'$. Therefore δ is a prefix of α, so $(\delta,\delta) \in \xi_\alpha$. Like before, a very similar argument works for η_x. So both ξ_α and η_x are filters. Going the other way, take a filter $\zeta \in \hat{E}_0$. We do this in cases.

Case 1. *ζ is finite.* Let (α,α) be the minimum element inside ζ. Since ζ is a filter, For $(\beta,\beta) \in E(S(E))$, $(\alpha,\alpha) \leq (\beta,\beta) \implies (\beta,\beta) \in \zeta$. However, by the definition of the minimal element, $(\beta,\beta) \in \zeta \implies (\alpha,\alpha) \leq (\beta,\beta)$. Thus $(\beta,\beta) \in \zeta$ if and only if $(\alpha,\alpha) \leq (\beta,\beta) \iff \beta$ is a prefix of α. So $\zeta = \xi_\alpha$.

Case 2. *ζ is infinite.* By the lemma above, we can get an idea of what elements of ζ look like. For each nonnegative integer, ζ contains precisely one element corresponding to a path of that length. Because we require nonzero product between elements, every path in the filter is a prefix of every longer path also contained inside the filter. All of this given, we can find an $x \in E^\infty$ such that every path inside ζ is a prefix of x. By the uniqueness of filter elements, it follows that $\zeta = \eta_x$.

We have shown that a nonempty subset $\zeta \subset E(S(E))$ is a filter if and only if it is of the form ξ_α or η_x. Thus $\hat{E}_0 = \{\xi_\alpha\colon \alpha \in E^*\} \cup \{\eta_x\colon x \in E^\infty\}$. $\square$

Now that we know what filters look like on the graph inverse semigroup, the task of identifying ultrafilters becomes much simpler.

Theorem 4.3. $\hat{E}_\infty = \{\eta_x\colon x \in E^\infty\} \cup \{\xi_\alpha\colon \alpha \in E^*$ *and* $|r^{-1}\{s(\alpha)\}| = 0\}$

Proof. We need only consider various kinds of filters, determining whether they are ultrafilters or not. We do this in cases.

Case 1. ξ_α, $|r^{-1}\{s(\alpha)\}| > 0$. Take $e \in E^1$ such that $r(e) = s(\alpha)$. It is clear that $\xi_\alpha \subset \xi_{\alpha e}$. So ξ_α is not an ultrafilter.

Case 2. ξ_α, $|r^{-1}\{s(\alpha)\}| = 0$. Suppose we can find $\zeta \in \hat{E}_0$ such that $\xi_\alpha \subset \zeta$. For $0 \leq i \leq |\alpha|$, ζ inherits its idempotent with the path corresponding to length i from ξ_α. So we can find an element $(\beta, \beta) \in \zeta$ with $|\beta| > |\alpha|$ and $(\alpha, \alpha)(\beta, \beta) \neq 0$. Thus $\beta = \alpha\alpha'$ for some $\alpha' \in E^*$. But no edges go to $s(\alpha)$, so we cannot construct α' with $r(\alpha') = s(\alpha)$. This is a contradiction, so ξ_α is an ultrafilter.

Case 3. η_x, $x \in E^\infty$. Suppose there exists $\zeta \in \hat{E}_0$ such that $\eta_x \subset \zeta$. Then we can find $(\beta, \beta) \in \zeta$ such that $(\beta, \beta) \notin \eta_x$. But η_x, as we showed before, contains an element z corresponding to a path of length $|\beta|$. As a superset of η_x, ζ also contains z. By the lemma at the start, it follows that $(\beta, \beta) = z \in \eta_x$, which is a contradiction. So η_x is an ultrafilter for all $x \in E^\infty$.

We have identified which filters are ultrafilters, showing that $\hat{E}_\infty = \{\xi_\alpha\colon \alpha \in E^*$ and $|r^{-1}\{s(\alpha)\}| = 0\} \cup \{\eta_x\colon x \in E^\infty\}$. □

Theorem 4.4. *Let E be a directed graph, and $\alpha \in E^*$ such that $|r^{-1}\{s(\alpha)\}| = \infty$. Let $X, Y \subseteq_{fin} E(S(E))$, and Z be a finite cover of $E^{X,Y}$. If $\xi_\alpha \in \mathcal{U}(X, Y)$, then $\xi_\alpha \cap Z \neq \emptyset$.*

Proof. Letting $\min(X) = (x, x)$,

$$E^{X,Y} = \{(xx', xx')\colon x' \in E^*,\ r(x') = s(x) \text{ and } (xx', xx')y = 0\ \forall y \in Y\}$$

Consider the set $C := \{(\alpha b, \alpha b)\colon b \in E^1,\ s(\alpha) = r(b)\}$. By the assumption that $s(\alpha)$ is an infinite receiver, C is infinite. Given $y \in Y$, let ν be the path corresponding to y. Since $\xi_\alpha \in \mathcal{U}(X, Y)$, ν is not a prefix of α, and thus not a proper prefix of αb for any b. Thus, if $(\alpha b, \alpha b)y \neq 0$, αb is a prefix of ν. Then for $\beta \neq b$, $\alpha\beta$ cannot be a prefix of ν. So there is at most one element of C such that $(\alpha b, \alpha b)y \neq 0$. By the assumption that Y is finite, all but finitely many elements of C are inside $E^{\{(x,x)\},Y}$. Therefore, if Z is a cover of $E^{X,Y}$, Z is an outer cover of the infinite set $E^{X,Y} \cap C$. Because Z is finite, $\exists z \in Z$ with $(\alpha b, \alpha b)z \neq 0$ for infinitely many $(\alpha b, \alpha b) \in E^{X,Y} \cap C$. If υ is the path corresponding to z, then for every b, either υ is a prefix of αb, or αb is a prefix of υ. All the αb are the same length with a different starting edge, so if one is a prefix of υ, no other can be a prefix of υ. So υ is a prefix of αb for infinitely many b. Thus $|\upsilon| \leq |\alpha| + 1$. If $|\upsilon| = |\alpha| + 1$, we have a contradiction: $b = \beta$ for all $(\alpha b, \alpha b), (\alpha\beta, \alpha, \beta) \in C$. Thus $|\upsilon| \leq |\alpha|$, so υ is a prefix of α. Therefore $z = (\upsilon, \upsilon) \in \xi_\alpha$, so $\xi_\alpha \cap Z \neq \emptyset$. □

5 References