Lemma. *If* $(\alpha,\alpha)(\beta,\beta) \neq 0$ *and* $|\alpha| = |\beta|$*, then* $\alpha = \beta$.

Proof. Without loss of generality, suppose α is a prefix of β. Then $\alpha = \beta\beta'$ for some β. But $|\alpha| = |\beta|$, so $|\beta'| = 0$, meaning $\beta' = s(\beta)$. Thus $\alpha = \beta s(\beta) = \beta$. ☐

An immediate consequence of this is that for any given length, a filter contains at most one element corresponding to a path of that length. Furthermore, if ζ is a filter, and $(\alpha,\alpha) \in \zeta$ such that $|\alpha| = n$, we can restrict α to any length $0 \leq m < n$ to produce a unique element of length m inside ζ.

Theorem. $\hat{E}_0 = \{\xi_\alpha\colon \alpha \in E^*\} \cup \{\eta_x\colon x \in E^\infty\}$

Proof. First we show that ξ_α and η_x are filters. Let α be a finite path and let x be an infinite path. Consider $(\beta,\beta),\ (\gamma,\gamma) \in \xi_\alpha$ with $|\beta| = m,\ |\gamma| = n$. Reading edges left to right, β contains the first m edges of α and γ contains the first n edges of α. Without loss of generality, assume $m \leq n$. We can extend β by the next $n-m$ edges of α to produce γ. Thus β is a prefix of γ, so $(\beta,\beta)(\gamma,\gamma) = (\gamma,\gamma) \in \xi_\alpha$. A similar argument shows that η_x is closed under multiplication. So both ξ_α and η_x are prefilters on $E(S(E))$. Next, let $(\beta,\beta) \in \xi_\alpha$, and let $(\delta,\delta) \in E(S(E))$ so that $(\beta,\beta) \leq (\delta,\delta)$. Then $\beta = \delta\delta'$ for some $\delta' \in E^*$. Thus $\alpha = \beta\beta' = \delta\delta'\beta'$. Therefore δ is a prefix of α, so $(\delta,\delta) \in \xi_\alpha$. Like before, a very similar argument works for η_x. So both ξ_α and η_x are filters. Going the other way, take a filter $\zeta \in \hat{E}_0$. We do this in cases.

Case 1. ζ *is finite.* Let (α,α) be the minimum element inside ζ. Since ζ is a filter, For $(\beta,\beta) \in E(S(E))$, $(\alpha,\alpha) \leq (\beta,\beta) \implies (\beta,\beta) \in \zeta$. However, by the definition of the minimal element, $(\beta,\beta) \in \zeta \implies (\alpha,\alpha) \leq (\beta,\beta)$. Thus $(\beta,\beta) \in \zeta$ if and only if $(\alpha,\alpha) \leq (\beta,\beta) \iff \beta$ is a prefix of α. So $\zeta = \xi_\alpha$.

Case 2. ζ *is infinite.* By the lemma above, we can get an idea of what elements of ζ look like. For each nonnegative integer, ζ contains precisely one element corresponding to a path of that length. Because we require nonzero product between elements, every path in the filter is a prefix of every longer path also contained inside the filter. All of this given, we can find an $x \in E^\infty$ such that every path inside ζ is a prefix of x. By the uniqueness of filter elements, it follows that $\zeta = \eta_x$.

We have shown that a nonempty subset $\zeta \subset E(S(E))$ is a filter if and only if it is of the form ξ_α or η_x. Thus $\hat{E}_0 = \{\xi_\alpha\colon \alpha \in E^*\} \cup \{\eta_x\colon x \in E^\infty\}$. ☐

Theorem. *Let E be a directed graph, and $\alpha \in E^*$ such that $|r^{-1}\{s(\alpha)\}| = \infty$. Let $X, Y \subseteq_{fin} E(S(E))$, and Z be a finite cover of $E^{X,Y}$. If $\xi_\alpha \in \mathcal{U}(X,Y)$, then $\xi_\alpha \cap Z \neq \emptyset$.*

Proof. First note:

$$\begin{aligned} E^{X,Y} &= \{e \in E(S(E))\colon e \leq x\ \forall x \in x \text{ and } ey = 0\ \forall y \in Y\} \\ &= \{e \in E(S(E))\colon e \leq \min(X) \text{ and } ey = 0\ \forall y \in Y\} \\ &= E^{\{\min(X)\},Y} \end{aligned}$$

Letting $\min(X) = (x, x)$,

$$E^{X,Y} = \{(xx', xx') \colon x' \in E^*,\ r(x') = s(x) \text{ and } (xx', xx')y = 0\ \forall y \in Y\}$$

Consider the set $C := \{(\alpha b, \alpha b) \colon b \in E^1,\ s(\alpha) = r(b)\}$. By the assumption that $s(\alpha)$ is an infinite receiver, C is infinite. Given $y \in Y$, let ν be the path corresponding to y. Since $\xi_\alpha \in \mathcal{U}(X, Y)$, ν is not a prefix of α, and thus not a proper prefix of αb for any b. Thus, if $(\alpha b, \alpha b)y \neq 0$, αb is a prefix of ν. Then for $\beta \neq b$, $\alpha\beta$ cannot be a prefix of ν. So there is at most one element of C such that $(\alpha b, \alpha b)y \neq 0$. By the assumption that Y is finite, all but finitely many elements of C are inside $E^{\{(x,x)\},Y}$. Therefore, if Z is a cover of $E^{X,Y}$, Z is an outer cover of the infinite set $E^{X,Y} \cap C$. Because Z is finite, $\exists z \in Z$ with $(\alpha b, \alpha b)z \neq 0$ for infinitely many $(\alpha b, \alpha b) \in E^{X,Y} \cap C$. If υ is the path corresponding to z, then for every b, either υ is a prefix of αb, or αb is a prefix of υ. All the αb are the same length with a different starting edge, so if one is a prefix of υ, no other can be a prefix of υ. So υ is a prefix of αb for infinitely many b. Thus $|\upsilon| \leq |\alpha| + 1$. If $|\upsilon| = |\alpha| + 1$, we have a contradiction: $b = \beta$ for all $(\alpha b, \alpha b), (\alpha\beta, \alpha, \beta) \in C$. Thus $|\upsilon| \leq |\alpha|$, so υ is a prefix of α. Therefore $z = (\upsilon, \upsilon) \in \xi_\alpha$, so $\xi_\alpha \cap Z \neq \emptyset$. $\square$