

1 Introduction

2 Preliminaries

First we must clarify a few frequently used definitions and results.

Definition 2.1. We call a set S a *semigroup* if it has an associative binary operation that sends a pair of elements $(a, b) \mapsto ab$ for all $a, b \in S$.

Definition 2.2. We call a semigroup S an *inverse semigroup* if for every $s \in S$, there exists a unique element $s^* \in S$ such that $s = ss^*s$ and $s^* = s^*ss^*$. We call s^* the *inverse* of s . If S contains a zero element, we call it an *inverse semigroup with zero*.

Definition 2.3. An element e of an inverse semigroup S is called an *idempotent* if $e = e^2$. Note that for all idempotents, $e = e^*$, and for all elements s in an inverse semigroup, ss^* and s^*s are idempotent. We denote the set of idempotents on S by $E(S)$.

Remark 2.4. If S is an inverse semigroup, for $e, f \in E(S)$, $ef = fe$.

In other words, the idempotents of an inverse semigroup commute. (A proof of this can be found in Kyle's paper. cite this properly)

Definition 2.5. Let S be a set. A relation \leq on S is called a *partial order* if

- It is reflexive; $a \leq a \quad \forall a \in S$
- It is antisymmetric; $a \leq b$ and $b \leq a \implies a = b \quad \forall a, b \in S$
- It is transitive; $a \leq b$ and $b \leq c \implies a \leq c \quad \forall a, b, c \in S$

A set S paired with a *partial order* is called a partially ordered set, or a *poset*.

3 The Tight Spectrum of an Inverse Semigroup

Now that we have a basic set of definitions in place, we can start to explore the set of idempotents on inverse semigroups. For this section, let S be an inverse semigroup with zero. We can define a partial order on $E(S)$, saying that $e \leq f \iff e = ef$. This can be extended to the entire set S , but in this paper we are only concerned with idempotents. (mention it's a meet semilattice, point to ISG primer?)

Definition 3.1. A nonempty proper subset $\xi \subset E(S)$ is called a *filter* on S if

1. For $e, f \in \xi$, $ef \in \xi$
2. For $x \in \xi$, $e \in E(S)$, $x \leq e \implies e \in \xi$

Subsets that only satisfy the first condition are called *prefilters*. We denote the set of filters on S by \hat{E}_0^S . An important remark is that the zero element of S is not contained inside any filters. If $0 \in \xi$, condition (2) would imply that $\xi = S$, which violates our definition. We can consider \hat{E}_0^S a subset of $\{0, 1\}^{E(S)}$, and thus endow it with the product topology inherited from $\{0, 1\}^{E(S)}$. Equivalently, let X, Y be finite subsets of $E(S)$. Define $\mathcal{U}(X, Y) := \{\xi \in \hat{E}_0^S : X \subseteq \xi \text{ and } Y \cap \xi = \emptyset\}$. Sets of the form $\mathcal{U}(X, Y)$ form a basis for the topology on the set of filters. With this topology, the topological space \hat{E}_0^S is called the *Exel spectrum* of S .

Definition 3.2. A filter $\eta \in \hat{E}_0^S$ is called an *ultrafilter* if it is maximal with respect to set inclusion. In other words, it is not contained inside another filter.

We define the subspace $\hat{E}_\infty^S := \{\xi \in \hat{E}_0^S : \xi \text{ is an ultrafilter}\}$. Note that if the inverse semigroup is obvious, we need not superscript S when describing the spectrum.

Remark 3.3. For an ultrafilter $\eta \in \hat{E}_\infty^S$ and $e \in E(S)$, $e \notin \eta \implies ef = 0$ for some $f \in \eta$.

Theorem 3.4. Every filter is contained inside an ultrafilter.

Proof. Let $\xi \in \hat{E}_0^S$. To prove this, we hope to use Zorn's lemma. We define the set $\mathcal{P} := \{\mathcal{F} \in \hat{E}_0^S : \xi \subseteq \mathcal{F}\}$ and we order \mathcal{P} by set inclusion. Let $\mathcal{C} \subseteq \mathcal{P}$ be a chain. If we take $\mathcal{F} := \bigcup_{\zeta \in \mathcal{C}} \zeta$, it is clear that \mathcal{F} is an upper bound of \mathcal{C} and $\xi \subseteq \mathcal{F}$. To show that $\mathcal{F} \in \mathcal{P}$, we need only show that it is a filter. Suppose $f, g \in \mathcal{F}$. Then $f \in \zeta_1, g \in \zeta_2$ for some $\zeta_1, \zeta_2 \in \mathcal{C}$. Since \mathcal{C} is totally ordered, without loss of generality we assume $\zeta_1 \subseteq \zeta_2$. Then $f, g \in \zeta_2 \implies fg \in \zeta_2 \subseteq \mathcal{F}$. So \mathcal{F} is a prefilter. Now suppose $f \in \mathcal{F}, e \in E(S)$ with $f \leq e$. Then $f \in \zeta$ for some $\zeta \in \mathcal{C}$, but since ζ is a filter, $e \in \zeta \subseteq \mathcal{F}$. So \mathcal{F} is upward closed, and hence a filter. By Zorn's lemma, there exists $\eta \in \mathcal{P}$ such that η is maximal with respect to set inclusion. This is our definition of an ultrafilter; we have shown that $\forall \xi \in \hat{E}_0^S, \exists \eta \in \hat{E}_\infty^S$ with $\xi \subseteq \eta$. \square

We can now begin to shift our focus towards the tight spectrum, which is the main topic of this section.

Definition 3.5. Let X, Y be finite subsets of $E(S)$. We define

$$E^{X,Y} := \{e \in E(S) : e \leq x \ \forall x \in X \text{ and } ey = 0 \ \forall y \in Y\}$$

Definition 3.6. Given $\mathcal{E} \subseteq E(S)$, we call $Z \subseteq E(S)$ an *outer cover* of \mathcal{E} if $\forall e \in \mathcal{E}, \exists z \in Z$ with $ez \neq 0$. If Z is an outer cover of \mathcal{E} and $Z \subseteq \mathcal{E}$, we say Z is a *cover* of \mathcal{E} .

Lemma 3.7. Let X, Y be finite subsets of $E(S)$, and let $x = \min(X)$. Then

- (i) $E^{X,Y} = E^{\{x\},Y}$
- (ii) $\mathcal{U}(X, Y) = \mathcal{U}(\{x\}, Y)$

Proof. (i)

$$\begin{aligned} E^{X,Y} &= \{e \in E(S) : e \leq x \ \forall x \in X \text{ and } ey = 0 \ \forall y \in Y\} \\ &= \{e \in E(S) : e \leq \min(X) \text{ and } ey = 0 \ \forall y \in Y\} \\ &= E^{\{x\},Y} \end{aligned}$$

(ii) Since $\{x\} \subseteq X$, $\mathcal{U}(X,Y) \subseteq \mathcal{U}(\{x\},Y)$. Now suppose $\xi \in \mathcal{U}(\{x\},Y)$. For $\chi \in X$, $x \leq \chi \implies \chi \in \xi$. Thus $X \subseteq \xi$, so $\xi \in \mathcal{U}(X,Y)$. We have shown that $\mathcal{U}(X,Y)$ and $\mathcal{U}(\{x\},Y)$ are subsets of each other, so they are equal. \square

When we are working with $\mathcal{U}(X,Y)$ and $E^{X,Y}$, this lemma allows us the freedom of only considering the case where X is a singleton set.

Definition 3.8. Let $\xi \in \hat{E}_0$. We say that ξ is a *tight filter* if for all finite subsets $X, Y \subseteq E(S)$ and for all finite covers Z of $E^{X,Y}$, $\xi \in \mathcal{U}(X,Y) \implies Z \cap \xi \neq \emptyset$. We call the set of tight filters the *tight spectrum*, and denote it by \hat{E}_{tight}^S .

This definition may appear a bit contrived, but the next theorem hopes to hint at its significance.

Theorem 3.9. \hat{E}_{tight} is the closure of \hat{E}_∞ in \hat{E}_0 .

Proof. We show that a filter $\xi \in \widehat{\hat{E}_\infty} \iff \xi \in \hat{E}_{\text{tight}}$. First the forward implication. Let $\xi \in \widehat{\hat{E}_\infty}$ and suppose $\xi \in \mathcal{U}(\{x\},Y)$. We prove the contrapositive of our definition for tightness. Let $Z \subseteq E^{\{x\},Y}$, and suppose $Z \cap \xi = \emptyset$. Then $\xi \in \mathcal{U}(\{x\}, Y \cup Z)$. Since $\xi \in \widehat{\hat{E}_\infty}$, we can find an ultrafilter $\eta \in \mathcal{U}(\{x\}, Y \cup Z)$. By Remark 3.3, for every $f \in Y \cup Z$, we can find an idempotent $e_f \in \eta$ with $fe_f = 0$. Define $e := \left(\prod_{f \in Y \cup Z} e_f\right)x$. $e \leq x$ and $ey = 0 \ \forall y \in Y$, so $e \in E^{\{x\},Y}$, but $ez = 0 \ \forall z \in Z$. So Z is not a finite cover of $E^{\{x\},Y}$, thus ξ is a tight filter, hence $\widehat{\hat{E}_\infty} \subseteq \hat{E}_{\text{tight}}$. \square

4 Directed Graphs

Lemma 4.1. If $(\alpha, \alpha)(\beta, \beta) \neq 0$ and $|\alpha| = |\beta|$, then $\alpha = \beta$.

Proof. Without loss of generality, suppose α is a prefix of β . Then $\alpha = \beta\beta'$ for some β' . But $|\alpha| = |\beta|$, so $|\beta'| = 0$, meaning $\beta' = s(\beta)$. Thus $\alpha = \beta s(\beta) = \beta$. \square

An immediate consequence of this is that for any given length, a filter contains at most one element corresponding to a path of that length. Furthermore, if ζ is a filter, and $(\alpha, \alpha) \in \zeta$ such that $|\alpha| = n$, we can restrict α to any length $0 \leq m < n$ to produce a unique element of length m inside ζ .

Theorem 4.2. $\hat{E}_0 = \{\xi_\alpha : \alpha \in E^*\} \cup \{\eta_x : x \in E^\infty\}$

Proof. First we show that ξ_α and η_x are filters. Let α be a finite path and let x be an infinite path. Consider $(\beta, \beta), (\gamma, \gamma) \in \xi_\alpha$ with $|\beta| = m, |\gamma| = n$. Reading edges left to right, β contains the first m edges of α and γ contains the first n edges of α . Without loss of generality, assume $m \leq n$. We can extend β by the next $n - m$ edges of α to produce γ . Thus β is a prefix of γ , so $(\beta, \beta)(\gamma, \gamma) = (\gamma, \gamma) \in \xi_\alpha$. A similar argument shows that η_x is closed under multiplication. So both ξ_α and η_x are prefilters on $E(S(E))$. Next, let $(\beta, \beta) \in \xi_\alpha$, and let $(\delta, \delta) \in E(S(E))$ so that $(\beta, \beta) \leq (\delta, \delta)$. Then $\beta = \delta\delta'$ for some $\delta' \in E^*$. Thus $\alpha = \beta\beta' = \delta\delta'\beta'$. Therefore δ is a prefix of α , so $(\delta, \delta) \in \xi_\alpha$. Like before, a very similar argument works for η_x . So both ξ_α and η_x are filters. Going the other way, take a filter $\zeta \in \hat{E}_0$. We do this in cases.

Case 1. ζ is finite. Let (α, α) be the minimum element inside ζ . Since ζ is a filter, For $(\beta, \beta) \in E(S(E))$, $(\alpha, \alpha) \leq (\beta, \beta) \implies (\beta, \beta) \in \zeta$. However, by the definition of the minimal element, $(\beta, \beta) \in \zeta \implies (\alpha, \alpha) \leq (\beta, \beta)$. Thus $(\beta, \beta) \in \zeta$ if and only if $(\alpha, \alpha) \leq (\beta, \beta) \iff \beta$ is a prefix of α . So $\zeta = \xi_\alpha$.

Case 2. ζ is infinite. By the lemma above, we can get an idea of what elements of ζ look like. For each nonnegative integer, ζ contains precisely one element corresponding to a path of that length. Because we require nonzero product between elements, every path in the filter is a prefix of every longer path also contained inside the filter. All of this given, we can find an $x \in E^\infty$ such that every path inside ζ is a prefix of x . By the uniqueness of filter elements, it follows that $\zeta = \eta_x$.

We have shown that a nonempty subset $\zeta \subset E(S(E))$ is a filter if and only if it is of the form ξ_α or η_x . Thus $\hat{E}_0 = \{\xi_\alpha: \alpha \in E^*\} \cup \{\eta_x: x \in E^\infty\}$. \square

Now that we know what filters look like on the graph inverse semigroup, the task of identifying ultrafilters becomes much simpler.

Theorem 4.3. $\hat{E}_\infty = \{\eta_x: x \in E^\infty\} \cup \{\xi_\alpha: \alpha \in E^* \text{ and } |r^{-1}\{s(\alpha)\}| = 0\}$

Proof. We need only consider various kinds of filters, determining whether they are ultrafilters or not. We do this in cases.

Case 1. ξ_α , $|r^{-1}\{s(\alpha)\}| > 0$. Take $e \in E^1$ such that $r(e) = s(\alpha)$. It is clear that $\xi_\alpha \subset \xi_{\alpha e}$. So ξ_α is not an ultrafilter.

Case 2. ξ_α , $|r^{-1}\{s(\alpha)\}| = 0$. Suppose we can find $\zeta \in \hat{E}_0$ such that $\xi_\alpha \subset \zeta$. For $0 \leq i \leq |\alpha|$, ζ inherits its idempotent with the path corresponding to length i from ξ_α . So we can find an element $(\beta, \beta) \in \zeta$ with $|\beta| > |\alpha|$ and $(\alpha, \alpha)(\beta, \beta) \neq 0$. Thus $\beta = \alpha\alpha'$ for some $\alpha' \in E^*$. But no edges go to $s(\alpha)$, so we cannot construct α' with $r(\alpha') = s(\alpha)$. This is a contradiction, so ξ_α is an ultrafilter.

Case 3. η_x , $x \in E^\infty$. Suppose there exists $\zeta \in \hat{E}_0$ such that $\eta_x \subset \zeta$. Then we

can find $(\beta, \beta) \in \zeta$ such that $(\beta, \beta) \notin \eta_x$. But η_x , as we showed before, contains an element z corresponding to a path of length $|\beta|$. As a superset of η_x , ζ also contains z . By the lemma at the start, it follows that $(\beta, \beta) = z \in \eta_x$, which is a contradiction. So η_x is an ultrafilter for all $x \in E^\infty$.

We have identified which filters are ultrafilters, showing that $\hat{E}_\infty = \{\xi_\alpha: \alpha \in E^* \text{ and } |r^{-1}\{s(\alpha)\}| = 0\} \cup \{\eta_x: x \in E^\infty\}$. \square

Theorem 4.4. *Let E be a directed graph, and $\alpha \in E^*$ such that $|r^{-1}\{s(\alpha)\}| = \infty$. Let $X, Y \subseteq_{fin} E(S(E))$, and Z be a finite cover of $E^{X,Y}$. If $\xi_\alpha \in \mathcal{U}(X, Y)$, then $\xi_\alpha \cap Z \neq \emptyset$.*

Proof. Letting $\min(X) = (x, x)$,

$$E^{X,Y} = \{(xx', xx'): x' \in E^*, r(x') = s(x) \text{ and } (xx', xx')y = 0 \ \forall y \in Y\}$$

Consider the set $C := \{(\alpha b, \alpha b): b \in E^1, s(\alpha) = r(b)\}$. By the assumption that $s(\alpha)$ is an infinite receiver, C is infinite. Given $y \in Y$, let ν be the path corresponding to y . Since $\xi_\alpha \in \mathcal{U}(X, Y)$, ν is not a prefix of α , and thus not a proper prefix of αb for any b . Thus, if $(\alpha b, \alpha b)y \neq 0$, αb is a prefix of ν . Then for $\beta \neq b$, $\alpha\beta$ cannot be a prefix of ν . So there is at most one element of C such that $(\alpha b, \alpha b)y \neq 0$. By the assumption that Y is finite, all but finitely many elements of C are inside $E^{\{(x,x)\}, Y}$. Therefore, if Z is a cover of $E^{X,Y}$, Z is an outer cover of the infinite set $E^{X,Y} \cap C$. Because Z is finite, $\exists z \in Z$ with $(\alpha b, \alpha b)z \neq 0$ for infinitely many $(\alpha b, \alpha b) \in E^{X,Y} \cap C$. If v is the path corresponding to z , then for every b , either v is a prefix of αb , or αb is a prefix of v . All the αb are the same length with a different starting edge, so if one is a prefix of v , no other can be a prefix of v . So v is a prefix of αb for infinitely many b . Thus $|v| \leq |\alpha| + 1$. If $|v| = |\alpha| + 1$, we have a contradiction: $b = \beta$ for all $(\alpha b, \alpha b), (\alpha\beta, \alpha, \beta) \in C$. Thus $|v| \leq |\alpha|$, so v is a prefix of α . Therefore $z = (v, v) \in \xi_\alpha$, so $\xi_\alpha \cap Z \neq \emptyset$. \square

5 References