The Tight Spectrum of an Inverse Semigroup

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Abstract

This paper provides the reader with an introduction to the theory of
filters on inverse semigroups, in particular providing a characterization
for the tight spectrum. Next it focuses on potentially infinite directed
graphs, identifying the tight spectrum of the graph inverse semigroup
and introducing its relation to the graph C*-algebra.

1   Introduction

The theory of inverse semigroups was founded independently by Wagner and
Preston in the 1950s through the study of compositions of partial bijections.
The purpose of this paper is to provide an introduction into inverse semi-
groups, their idempotent elements, and the theory of filters on the semilattice
of idempotents. In section 3, we introduce the notion of a filter, establish a
topology on the set of filters, and characterize the tight spectrum. In sec-
tion 4, we introduce an example: the graph inverse semigroup. We provide a
complete characterization of the tight spectrum of a graph inverse semigroup,

then in section 5 we relate it to graph C*-algebras, first studied in 1980 by Cuntz and Krieger. Our primary references were [2], [3], and [6].

2 Preliminaries

First we must clarify a few frequently used definitions and results.

Definition 2.1. We call a set S a *semigroup* if it has an associative binary operation, typically referred to as *multiplication*, that sends an ordered pair of elements $(a, b) \mapsto ab$ for all $a, b \in S$.

Definition 2.2. We call a semigroup S an *inverse semigroup* if for every $s \in S$, there exists a unique element $s^* \in S$ such that $s = ss^*s$ and $s^* = s^*ss^*$. We call s^* the *inverse* of s. An element $0 \in S$ such that $0s = s0 = 0 \; \forall s \in S$ is called a *zero element*. If S contains a zero element, we say S is an *inverse semigroup with zero*.

Definition 2.3. An element e of an inverse semigroup S is called an *idempotent* if $e = e^2$. Note that for all idempotents, $e = e^*$, and for all elements s in an inverse semigroup, ss^* and s^*s are idempotent. Furthermore, the product of idempotents is idempotent. Proofs of these remarks can be found in [2, Section 3]. We denote the set of idempotents on S by $E(S)$.

Remark 2.4. *If S is an inverse semigroup, for e, $f \in E(S)$, $ef = fe$.*

In other words, the idempotents of an inverse semigroup commute. This is proven in [2, Theorem 3.2].

Definition 2.5. Let S be a set. A relation $\leq$ on S is called a *partial order* if

- It is reflexive; $a \leq a \quad \forall a \in S$

- It is antisymmetric; $a \leq b$ and $b \leq a \implies a = b \quad \forall a, b \in S$

- It is transitive; $a \leq b$ and $b \leq c \implies a \leq c \quad \forall a, b, c \in S$

A set S paired with a *partial order* is called a partially ordered set, or a *poset*.

3 The Tight Spectrum

Now that we have a basic set of definitions in place, we can start to explore the set of idempotents on inverse semigroups. For this section, let S be an inverse semigroup with zero. We can define a partial order on $E(S)$, saying that $e \leq f \iff e = ef$. This can be extended to the entire set S, but in this paper we are only concerned with idempotents. An interesting property of $E(S)$ is that under this partial order, it is a meet semilattice, where $e \wedge f = ef$. This is explored in more depth in [5].

Definition 3.1. A nonempty proper subset $\xi \subset E(S)$ is called a *filter* on $E(S)$ if

1. For $e, f \in \xi$, $ef \in \xi$ i.e. ξ is *downward directed*

2. For $x \in \xi$, $e \in E(S)$, $x \leq e \implies e \in \xi$ i.e. ξ is *upward closed*

Subsets that only satisfy the first condition are called *prefilters*. We denote the set of filters on S by $\hat{E}_0^S$. An important remark is that the zero element of S is not contained inside any filters. If $0 \in \xi$, condition (2) would imply that $\xi = S$, which violates our definition. We can consider $\hat{E}_0^S$ a subset of $\{0,1\}^{E(S)}$, and thus endow it with the product topology inherited from $\{0,1\}^{E(S)}$. Equivalently, let X, Y be finite subsets of $E(S)$. Define $\mathcal{U}(X,Y) := \{\xi \in \hat{E}_0 \colon X \subseteq \xi \text{ and } Y \cap \xi = \emptyset\}$. Sets of the form $\mathcal{U}(X,Y)$ form a basis for the topology on the set of filters. With this topology, the topological space $\hat{E}_0^S$ is called the *Exel spectrum* of S. Note that if the inverse semigroup is S obvious, we need not superscript it when describing the spectrum.

Definition 3.2. A filter $\eta \in \hat{E}_0$ is called an *ultrafilter* if it is maximal with respect to set inclusion. In other words, it is not contained inside another filter. We define the subspace $\hat{E}_\infty^S := \{\xi \in \hat{E}_0 \colon \xi \text{ is an ultrafilter}\} \subseteq \hat{E}_0$.

Remark 3.3. *For an ultrafilter $\eta \in \hat{E}_\infty$ and $e \in E(S)$, $e \notin \eta \implies ef = 0$ for some $f \in \eta$.*

This follows from a characterization for the ultrafilters given in [3, Lemma 12.3].

Theorem 3.4. *Every filter is contained inside an ultrafilter.*

Proof. Let $\xi \in \hat{E}_0$. To prove this, we hope to use Zorn's lemma. We define the set $\mathcal{P} := \{\mathcal{F} \in \hat{E}_0 \colon \xi \subseteq \mathcal{F}\}$ and we order $\mathcal{P}$ by set inclusion. Let $\mathcal{C} \subseteq \mathcal{P}$

be a chain. If we take $\mathcal{F} := \bigcup_{\zeta \in \mathcal{C}} \zeta$, it is clear that $\mathcal{F}$ is an upper bound of $\mathcal{C}$ and $\xi \subseteq \mathcal{F}$. To show that $\mathcal{F} \in \mathcal{P}$, we need only show that it is a filter. Suppose $f, g \in \mathcal{F}$. Then $f \in \zeta_1$, $g \in \zeta_2$ for some $\zeta_1, \zeta_2 \in \mathcal{C}$. Since $\mathcal{C}$ is totally ordered, without loss of generality we assume $\zeta_1 \subseteq \zeta_2$. Then f, $g \in \zeta_2 \implies fg \in \zeta_2 \subseteq \mathcal{F}$. So $\mathcal{F}$ is a prefilter. Now suppose $f \in \mathcal{F}$, $e \in E(S)$ with $f \leq e$. Then $f \in \zeta$ for some $\zeta \in \mathcal{C}$, but since ζ is a filter, $e \in \zeta \subseteq \mathcal{F}$. So $\mathcal{F}$ is upward closed, and hence a filter. By Zorn's lemma, there exists $\eta \in \mathcal{P}$ such that η is maximal with respect to set inclusion. This is our definition of an ultrafilter; we have shown that $\forall \xi \in \hat{E}_0$, $\exists \eta \in \hat{E}_\infty$ with $\xi \subseteq \eta$. $\qquad \square$

We can now begin to shift our focus towards the tight spectrum, which is the main topic of this section.

Definition 3.5. Let X, Y be finite subsets of $E(S)$. We define

$$E^{X,Y} := \{e \in E(S) \colon e \leq x \; \forall x \in x \text{ and } ey = 0 \; \forall y \in Y\}$$

Definition 3.6. Given $\mathcal{E} \subseteq E(S)$, we call $Z \subseteq E(S)$ an *outer cover* of $\mathcal{E}$ if $\forall e \neq 0 \in \mathcal{E}$, $\exists z \in Z$ with $ez \neq 0$. If is Z an outer cover of $\mathcal{E}$ and $Z \subseteq \mathcal{E}$, we say Z is a *cover* of $\mathcal{E}$.

Definition 3.7. Let $\xi \in \hat{E}_0$. We say that ξ is a *tight filter* if for all finite subsets X, $Y \subseteq E(S)$ and for all finite covers Z of $E^{X,Y}$, $\xi \in \mathcal{U}(X,Y) \implies Z \cap \xi \neq \emptyset$. We call the set of tight filters the *tight spectrum*, and denote it by $\hat{E}^S_{\text{tight}}$.

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This definition may appear a bit contrived, but the next theorem hopes to hint at its significance.

Lemma 3.8. *Let X, Y be finite subsets of $E(S)$, and let $x = \prod_{e \in X} e$. Then*

(i) $E^{X,Y} = E^{\{x\},Y}$

(ii) $\mathcal{U}(X,Y) = \mathcal{U}(\{x\},Y)$

Proof. (i)

$$
\begin{aligned}
E^{X,Y} &= \{e \in E(S) \colon e \le x \; \forall x \in X \text{ and } ey = 0 \; \forall y \in Y\} \\
&= \{e \in E(S) \colon e = ex \; \forall x \in X \text{ and } ey = 0 \; \forall y \in Y\} \\
&= \{e \in E(S) \colon e = e \prod_{x \in X} x \text{ and } ey = 0 \; \forall y \in Y\} \\
&= E^{\{x\},Y}
\end{aligned}
$$

(ii) Since $\{x\} \subseteq X$, $\mathcal{U}(X,Y) \subseteq \mathcal{U}(\{x\},Y)$. Now suppose $\xi \in \mathcal{U}(\{x\},Y)$. For $\chi \in X$, $x \le \chi \implies \chi \in \xi$. Thus $X \subseteq \xi$, so $\xi \in \mathcal{U}(X,Y)$. We have shown that $\mathcal{U}(X,Y)$ and $\mathcal{U}(\{x\},Y)$ are subsets of eachother, so they are equal.

$\square$

When we are working with $\mathcal{U}(X,Y)$ and $E^{X,Y}$, this lemma allows us the freedom of only considering the case where X is a singleton set.

Theorem 3.9. *$\hat{E}_{tight}$ is the closure of $\hat{E}_\infty$ in $\hat{E}_0$.*

Proof. We show that a filter $\xi \in \overline{\hat{E}_\infty} \iff \xi \in \hat{E}_{\text{tight}}$. First the forward implication. Let $\xi \in \overline{\hat{E}_\infty}$ and suppose $\xi \in \mathcal{U}(\{x\}, Y)$. We prove the contrapositive of our definition for tightness. Let $Z \subseteq E^{\{x\},Y}$, and suppose $Z \cap \xi = \emptyset$. Then $\xi \in \mathcal{U}(\{x\}, Y \cup Z)$. Since $\xi \in \overline{\hat{E}_\infty}$, we can find an ultrafilter $\eta \in \mathcal{U}(\{x\}, Y \cup Z)$. By Remark 3.3, for every $f \in Y \cup Z$, we can find an idempotent $e_f \in \eta$ with $fe_f = 0$. Define $e := \left(\prod_{f \in Y \cup Z} e_f \right) x$. $e \leq x$ and $ey = 0 \ \forall y \in Y$, so $e \in E^{\{x\},Y}$, but $ez = 0 \ \forall z \in Z$. So Z is not a finite cover of $E^{\{x\},Y}$, thus ξ is a tight filter, hence $\overline{\hat{E}_\infty} \subseteq \hat{E}_{\text{tight}}$.

Conversely, let $\xi \in \hat{E}_{\text{tight}}$ and suppose $\xi \in \mathcal{U}(\{x\}, Y)$. First we show that $E^{\{x\},Y} \neq \{0\}$, by way of contradiction. If $E^{\{x\},Y} = \{0\}$, then $Z = \emptyset$ is a finite cover. Since ξ is tight, $\xi \cap Z \neq \emptyset$, which is a contradiction. So we can find a nonzero idempotent $e \in E^{\{x\},Y}$. We note that $\{e\}$ is a prefilter; we close this set upwards to produce a filter ζ, called the *principal filter at e*. By Theorem 3.4, we can find an ultrafilter η with $\zeta \subseteq \eta$. Note that since $ey = 0 \ \forall y \in Y$ and $e \leq x$, any filter containing e must not intersect Y and must contain x. Thus $\eta \in \mathcal{U}(\{x\}, Y)$, and because open sets of this form are a basis for the topology on $\hat{E}_0$, it follows that every open neighbourhood of ξ contains an ultrafilter η. So $\hat{E}_{\text{tight}} \subseteq \overline{\hat{E}_\infty}$. Finally, by this and the work above, $\hat{E}_{\text{tight}} = \overline{\hat{E}_\infty}$. $\qquad \square$

So far we have been working under the assumption that S is an inverse semigroup with zero. In the case where S does not contain a zero element, we can extend the definitions introduced in this section by considering the

inverse semigroup $S \cup \{0\}$, but the tight spectrum is always just a single point, and thus not very interesting.

4 Directed Graphs

Now that we have introduced filters and the tight spectrum, we consider an example: the graph inverse semigroup. We require a few preliminary definitions.

Definition 4.1. A *directed graph* is a 4-tuple $E = (E^0, E^1, r, s)$, where we call E^0 the set of *vertices*, and E^1 the set of *edges*. $r : E^1 \rightarrow E^0$ and $s : E^1 \rightarrow E^0$ are called the *range* and *source* maps, respectively. A vertex can be thought of as a point in a plane, where an edge e can be thought of as an arrow pointing from $s(e)$ to $r(e)$.

For the rest of the section, we will let E be a directed graph.

Definition 4.2. If $\alpha_1, \alpha_2, ..., \alpha_n \in E^1$, where for $1 \leq i \leq n-1$, $r(\alpha_{i+1}) = s(\alpha_i)$, we can concatenate the edges to form a *finite path* $\alpha = \alpha_1...\alpha_n$ in E. We say α has length n, and denote it by $|\alpha|$.

Definition 4.3. For $n \in \mathbb{N}$, we define

$$E^n := \{\alpha \colon \alpha \text{ is a finite path in } E, |\alpha| = n\}$$

This provides some intuition about our naming convention for the set of vertices; a vertex can be considered "a path of length zero". We say two

finite paths are equal if and only if they have the same length and consist of exactly the same edges in exactly the same order. Next, we extend the range and source maps to finite paths; if $\alpha = \alpha_1...\alpha_n$, we say $r(\alpha) = r(\alpha_1)$, and $s(\alpha) = s(\alpha_n)$. We can now concatenate paths under the same rule mentioned in Definition 4.2. Some important terminology is the notion of a prefix; we say α is a *prefix* of β if $\beta = \alpha\alpha'$ for some finite path α'. Furthermore, We say α is a *proper prefix* of β if α is a prefix of β and $\alpha \neq \beta$.

Definition 4.4. We define the *set of finite paths*, and denote it by E^*. Note that the vertex set is included.

$$E^* := \bigcup_{n=0}^{\infty} E^n$$

With these preliminaries developed, we can now define the graph inverse semigroup.

Definition 4.5. Let $S(E) = \{(\alpha, \beta) \in E^* \times E^* : s(\alpha) = s(\beta)\} \cup \{0\}$. This is called the *graph inverse semigroup* corresponding to the graph E, and we define multiplication as follows:

$$(\alpha, \beta)(\gamma, \delta) = \begin{cases} (\alpha, \delta\gamma') & \text{if } \beta = \gamma\gamma' \\ (\alpha\beta', \delta) & \text{if } \gamma = \beta\beta' \\ 0 & \text{otherwise} \end{cases}$$

In [2, Section 3], it is shown that this is indeed an inverse semigroup, where

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$(\alpha, \beta)^* = (\beta, \alpha)$, and the set of idempotents, $E(S(E))$, consists of elements of the form (α, α), along with 0. Note that the product $(\alpha, \alpha)(\beta, \beta)$ is nonzero exactly when α is a prefix of β, or β is a prefix of α. The rest of this section explores the spectrum of $S(E)$.

Lemma 4.6. *If $(\alpha, \alpha)(\beta, \beta) \neq 0$ and $|\alpha| = |\beta|$, then $\alpha = \beta$.*

Proof. Without loss of generality, suppose α is a prefix of β. Then $\alpha = \beta\beta'$ for some β. But $|\alpha| = |\beta|$, so $|\beta'| = 0$, meaning $\beta' = s(\beta)$. Thus $\alpha = \beta s(\beta) = \beta$. $\square$

An immediate corollary of this is that for any given length, a filter contains at most one element corresponding to a path of that length. Furthermore, if ζ is a filter, and $(\alpha, \alpha) \in \zeta$, we may take a $0 \leq m < |\alpha|$ length prefix of α to produce a new unique element of length m inside ζ.

Definition 4.7. If $x = x_1 x_2 x_3...$ is a sequence of edges in E, where for $i \in \mathbb{N}$, $r(x_{i+1}) = s(x_i)$, we call x an *infinite path*, and denote the *set of infinite paths*

$$E^\infty := \{x \colon x \text{ is an infinite path in } E\}$$

For $x \in E^\infty$, we say $\alpha \in E^*$ is a prefix of x if $x = \alpha x'$ for some infinite path x'. Our goal is now to identify filters on $E(S(E))$, with the ultimate goal of providing a characterization for the tight spectrum $\hat{E}_{\text{tight}}$.

Definition 4.8. Let $\alpha \in E^*$, $x \in E^\infty$. We define

- $\xi_\alpha = \{(\beta, \beta) \in E(S(E)): \beta$ is a prefix of $\alpha\}$

- $\eta_x = \{(\beta, \beta) \in E(S(E)): \beta$ is a prefix of $x\}$

Theorem 4.9. $\hat{E}_0 = \{\xi_\alpha: \alpha \in E^*\} \cup \{\eta_x: x \in E^\infty\}$

Proof. First we show that ξ_α and η_x are filters. First note that they are both nonempty. Let α be a finite path and let x be an infinite path. Consider (β, β), $(\gamma, \gamma) \in \xi_\alpha$ with $|\beta| = m$, $|\gamma| = n$. Reading edges left to right, β contains the first m edges of α and γ contains the first n edges of α. Without loss of generality, assume $m \leq n$. We can extend β by the next $n - m$ edges of α to produce γ. Thus β is a prefix of γ, so $(\beta, \beta)(\gamma, \gamma) = (\gamma, \gamma) \in \xi_\alpha$. A similar argument shows that η_x is closed under multiplication. So both ξ_α and η_x are prefilters on $E(S(E))$. Next, let $(\beta, \beta) \in \xi_\alpha$, and let $(\delta, \delta) \in E(S(E))$ so that $(\beta, \beta) \leq (\delta, \delta)$. Then $\beta = \delta\delta'$ for some $\delta' \in E^*$. Thus $\alpha = \beta\beta' = \delta\delta'\beta'$. Therefore δ is a prefix of α, so $(\delta, \delta) \in \xi_\alpha$. Like before, a very similar argument works for η_x. So both ξ_α and η_x are filters.

Conversely, take a filter $\zeta \in \hat{E}_0$. We do this in cases. *Case 1.* ζ *is finite.* Let $\zeta \ni (\alpha, \alpha) = \prod_{e \in \zeta} e$. Since ζ is a filter, for $(\beta, \beta) \in E(S(E))$, $(\alpha, \alpha) \leq (\beta, \beta) \implies (\beta, \beta) \in \zeta$. However, $(\beta, \beta) \in \zeta \implies (\alpha, \alpha) \leq (\beta, \beta)$, since $(\alpha, \alpha) = (\beta, \beta)f$ for the idempotent $f := \prod_{e \in \zeta \setminus \{(\beta, \beta)\}} e$. Thus $(\beta, \beta) \in \zeta$ if and only if $(\alpha, \alpha) \leq (\beta, \beta) \iff \beta$ is a prefix of α. So $\zeta = \xi_\alpha$.

Case 2. ζ *is infinite.* Using Lemma 4.6, we can get an idea of what elements

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of ζ look like. For each nonnegative integer, ζ contains precisely one element corresponding to a path of that length. Because we require nonzero product between elements, every path in the filter is a prefix of every longer path also contained inside the filter. All of this given, we can find an $x \in E^\infty$ such that every path inside ζ is a prefix of x. By the uniqueness of filter elements, it follows that $\zeta = \eta_x$.

We have shown that a nonempty subset $\zeta \subset E(S(E))$ is a filter if and only if it is of the form ξ_α or η_x. Thus $\hat{E}_0 = \{\xi_\alpha \colon \alpha \in E^*\} \cup \{\eta_x \colon x \in E^\infty\}$. $\qquad\square$

Now that we know what filters look like on the graph inverse semigroup, the task of identifying ultrafilters becomes much simpler.

Theorem 4.10. $\hat{E}_\infty = \{\eta_x \colon x \in E^\infty\} \cup \{\xi_\alpha \colon \alpha \in E^* \text{ and } |r^{-1}\{s(\alpha)\}| = 0\}$

Proof. We need only consider various kinds of filters, determining whether they are ultrafilters or not. We do this in cases.

Case 1. ξ_α, $|r^{-1}\{s(\alpha)\}| > 0$. Take $e \in E^1$ such that $r(e) = s(\alpha)$. It is clear that $\xi_\alpha \subset \xi_{\alpha e}$. So ξ_α is not an ultrafilter.

Case 2. ξ_α, $|r^{-1}\{s(\alpha)\}| = 0$. Suppose we can find $\zeta \in \hat{E}_0$ such that $\xi_\alpha \subset \zeta$. For $0 \leq i \leq |\alpha|$, ζ inherits its idempotent with the path corresponding to length i from ξ_α. So we can find an element $(\beta, \beta) \in \zeta$ with $|\beta| > |\alpha|$ and $(\alpha, \alpha)(\beta, \beta) \neq 0$. Thus $\beta = \alpha\alpha'$ for some $\alpha' \in E^*$. But no edges go to $s(\alpha)$, so we cannot construct α' with $r(\alpha') = s(\alpha)$. This is a contradiction, so ξ_α is an ultrafilter.

Case 3. η_x, $x \in E^\infty$. Suppose there exists $\zeta \in \hat{E}_0$ such that $\eta_x \subset \zeta$. Then

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we can find $(\beta, \beta) \in \zeta$ such that $(\beta, \beta) \notin \eta_x$. But η_x, as we showed before, contains an element z corresponding to a path of length $|\beta|$. As a superset of η_x, ζ also contains z. By Lemma 4.6, it follows that $(\beta, \beta) = z \in \eta_x$, which is a contradiction. So η_x is an ultrafilter for all $x \in E^\infty$.

We have identified which filters are ultrafilters, showing that $\hat{E}_\infty = \{\xi_\alpha \colon \alpha \in E^* \text{ and } |r^{-1}\{s(\alpha)\}| = 0\} \cup \{\eta_x \colon x \in E^\infty\}$. $\qquad\square$

Theorem 4.11.

$$\hat{E}_{tight} = \{\eta_x \colon x \in E^\infty\} \cup \{\xi_\alpha \colon \alpha \in E^*, \ |r^{-1}\{s(\alpha)\}| = 0 \ or \ |r^{-1}\{s(\alpha)\}| = \infty\}$$

Proof. From Theorem 3.9, it follows that $\hat{E}_\infty \subseteq \hat{E}_{tight}$. To provide a complete characterization of the tight spectrum, all we have left to do is consider which elements of $\hat{E}_0 \setminus \hat{E}_\infty$ are tight filters. We first let $\alpha \in E^*$ such that $|r^{-1}\{s(\alpha)\}| = \infty$. Here $s(\alpha)$ is called an *infinite receiver*. Let $\{(x, x)\}, Y \subseteq_{\text{fin}} E(S(E))$, and Z be a finite cover of $E^{\{(x,x)\},Y}$. Suppose $\xi_\alpha \in \mathcal{U}(\{(x, x)\}, Y)$. We first note:

$$E^{\{(x,x)\},Y} = \{(xx', xx') \colon x' \in E^*, \ r(x') = s(x) \text{ and } (xx', xx')y = 0 \ \forall y \in Y\}$$

Consider the set $C := \{(\alpha b, \alpha b) \colon b \in E^1, \ s(\alpha) = r(b)\}$. By the assumption that $s(\alpha)$ is an infinite receiver, C is infinite. Given $y \in Y$, let ν be the path corresponding to y. Since $\xi_\alpha \in \mathcal{U}(\{(x, x)\}, Y)$, ν is not a prefix of α, and thus not a proper prefix of αb for any b. Thus, if $(\alpha b, \alpha b)y \neq 0$, αb is a prefix of ν.

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Then for any edge $c \neq b$, αc cannot be a prefix of ν. So there is at most one element of C such that $(\alpha b, \alpha b)y \neq 0$. By the assumption that Y is finite, all but finitely many elements of C are inside $E^{\{(x,x)\},Y}$. Therefore, if Z is a cover of $E^{\{(x,x)\},Y}$, Z is an outer cover of the infinite set $E^{\{(x,x)\},Y} \cap C$. Because Z is finite, $\exists z \in Z$ with $(\alpha b, \alpha b)z \neq 0$ for infinitely many $(\alpha b, \alpha b) \in E^{\{(x,x)\},Y} \cap C$. If υ is the path corresponding to z, then for every b, either υ is a prefix of αb, or αb is a prefix of υ. All the αb are the same length with a different starting edge, so if one is a prefix of υ, no other can be a prefix of υ. So υ is a prefix of αb for infinitely many b. Thus $|\upsilon| \leq |\alpha| + 1$. If $|\upsilon| = |\alpha| + 1$, we have a contradiction: $b = c$ for all $(\alpha b, \alpha b), (\alpha c, \alpha, c) \in C$. Thus $|\upsilon| \leq |\alpha|$, so υ is a prefix of α. Therefore $z = (\upsilon, \upsilon) \in \xi_\alpha$, thus $\xi_\alpha \cap Z \neq \emptyset$, so ξ_α is tight. In our final case, we let $\alpha \in E^*$, and suppose $0 < |r^{-1}\{s(\alpha)\}| < \infty$. Then there exists a nonempty, finite collection of edges $e_1, ..., e_n$ with $r(e_i) = s(\alpha)$ for $1 \leq i \leq n$. Let $Y \subseteq_{\text{fin}} E(S(E))$ such that $\xi_\alpha \in \mathcal{U}(\{(\alpha, \alpha)\}, Y)$. Then $Z = \{(\alpha e_1, \alpha e_1), ..., (\alpha e_n, \alpha e_n)\}$ is a finite cover of $E^{\{(\alpha,\alpha)\},Y}$, but $Z \cap \xi_\alpha = \emptyset$. So ξ_α is not tight. Thus a filter $\xi_\alpha \in \hat{E}_0 \setminus \hat{E}_\infty$ is tight if and only if it starts at an infinite receiver. This result along with Theorem 3.9 completes the proof. $\qquad \square$

5   C*-Algebras

We provide a brief introduction into the theory of C*-algebras, and state the main result found in the literature linking the tight spectrum to this field of

study.

Definition 5.1. A Banach algebra A over $\mathbb{C}$ endowed with a unary map $x \mapsto x^*$ for $x \in A$ is called a *C*-algebra* if, $\forall x, y \in A$, $\lambda \in \mathbb{C}$,

- $x = (x^*)^*$

- $(xy)^* = y^* x^*$

- $(x + \lambda y)^* = x^* + \overline{\lambda} y^*$

- $\|xx^*\| = \|x\|^2$ called the *C*-condition*

We call A a *commutative C*-algebra* if $xy = yx$ $\forall x, y \in A$.

Definition 5.2. Let X be a locally compact Hausdorff space. We say a function $f : X \to \mathbb{C}$ *vanishes at infinity* if $\forall \varepsilon > 0$, $\exists$ a compact $K \subseteq X$ such that $|f(x)| < \varepsilon$ $\forall x \in X \setminus K$. This generalizes the notion of a function of a real variable approaching zero as one extends further along the real line. We define the set

$$C_0(X) = \{f : X \to \mathbb{C}: f \text{ is continuous and } f \text{ vanishes at infinity}\}$$

For a fixed X, $C_0(X)$ is a commutative C*-algebra under the following op-

erations.

$$(f + g)(x) = f(x) + g(x)$$

$$(fg)(x) = f(x)g(x)$$

$$(f^*)(x) = \overline{f(x)}$$

$$\|f\|_\infty = \sup_{x \in X} |f(x)|$$

Theorem 5.3 (Gelfand-Naimark).

- *Every commutative C*-algebra is isomorphic to $C_0(X)$ for some X.*

- $C_0(X) \cong C_0(Y) \iff X$ *is homeomorphic to Y.*

Definition 5.4. For a C*-algebra A, we define a *projection* as an element $p \in A$ such that $p = p^2 = p^*$. We define a *partial isometry* as an element $s \in A$ such that $ss^*s = s$. Note that for any partial isometry s, ss^* and s^*s are projections. These translate to the characterizations of projections and partial isometries as linear operators between Hilbert spaces.

Definition 5.5. [7, Chapter 5] We say a graph E is *countable* if both E^0 and E^1 are countable. For a countable graph E and a Hilbert space $\mathcal{H}$, we define a *Cuntz-Krieger E-family* on $\mathcal{H}$, consisting of a set $\{p_v \colon v \in E^0\}$ of mutually orthogonal projections on $\mathcal{H}$ and a set $\{s_e \colon e \in E^1\}$ of partial isometries with

mutually orthogonal ranges on $\mathcal{H}$ such that $\forall v \in E^0$, $e \in E^1$,

(CK1) $s_e^* s_e = p_{s(e)}$

(CK2) $p_v = \displaystyle\sum_{e \in r^{-1}\{v\}} s_e s_e^*$ whenever $0 < |r^{-1}\{v\}| < \infty$

(CK3) $p_{r(e)} s_e s_e^* = s_e s_e^*$

Conditions (CK1) and (CK2) are called the *Cuntz-Krieger relations*. In [4, Definition 1], it is stated that for a countable graph E, it is possible to associate a C*-algebra $C^*(E)$, generated by the corresponding Cuntz-Krieger family of operators on $\mathcal{H}$.

For a finite path $\alpha = \alpha_1...\alpha_n$, we define $s_\alpha = s_{\alpha_1}...s_{\alpha_n}$. One can identify a commutative subalgebra $D_E = \overline{\text{span}}\{s_\mu s_\mu^*\colon \mu \in E^*\} \subset C^*(E)$. This is called the *diagonal C*-algebra* of E.

Definition 5.6. In [8, Theorem 2.1], the set $E^* \cup E^\infty$ is endowed with a locally compact, Hausdorff topology. We define the *boundary path space* $\partial E = E^\infty \cup \{\alpha \in E^*\colon |r^{-1}\{s(\alpha)\}| \in \{0, \infty\}\} \subseteq E^* \cup E^\infty$. Note that these are the paths corresponding to tight filters.

Theorem 5.7. ∂E *is homeomorphic to* $\hat{E}_{tight}$. This is proven in [1, Example 6.8]

Theorem 5.8. $D_E \cong C_0(\partial E)$. A proof of this is given in [8, Theorem 3.7].

This leads us to our final result: the relationship between the tight spectrum of the graph inverse semigroup and the graph C*-algebra.

Corollary 5.9. $D_E \cong C_0(\hat{E}_{tight})$

Proof. This falls out when one combines the second part of Theorem 5.3 with Theorems 5.7 and 5.8, considering that isomorphism is an equivalence relation. $\qquad\square$

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