

Lemma. If $(\alpha, \alpha)(\beta, \beta) \neq 0$ and $|\alpha| = |\beta|$, then $\alpha = \beta$.

Proof. Without loss of generality, suppose α is a prefix of β . Then $\alpha = \beta\beta'$ for some β' . But $|\alpha| = |\beta|$, so $|\beta'| = 0$, meaning $\beta' = s(\beta)$. Thus $\alpha = \beta s(\beta) = \beta$. \square

An immediate consequence of this is that for any given length, a filter contains at most one element corresponding to a path of that length. Furthermore, if ζ is a filter, and $(\alpha, \alpha) \in \zeta$ such that $|\alpha| = n$, we can restrict α to any length $0 \leq m < n$ to produce a unique element of length m inside ζ .

Theorem. $\hat{E}_0 = \{\xi_\alpha: \alpha \in E^*\} \cup \{\eta_x: x \in E^\infty\}$

Proof. First we show that ξ_α and η_x are filters. Let α be a finite path and let x be an infinite path. Consider $(\beta, \beta), (\gamma, \gamma) \in \xi_\alpha$ with $|\beta| = m, |\gamma| = n$. Reading edges left to right, β contains the first m edges of α and γ contains the first n edges of α . Without loss of generality, assume $m \leq n$. We can extend β by the next $n - m$ edges of α to produce γ . Thus β is a prefix of γ , so $(\beta, \beta)(\gamma, \gamma) = (\gamma, \gamma) \in \xi_\alpha$. A similar argument shows that η_x is closed under multiplication. So both ξ_α and η_x are prefilters on $E(S(E))$. Next, let $(\beta, \beta) \in \xi_\alpha$, and let $(\delta, \delta) \in E(S(E))$ so that $(\beta, \beta) \leq (\delta, \delta)$. Then $\beta = \delta\delta'$ for some $\delta' \in E^*$. Thus $\alpha = \beta\beta' = \delta\delta'\beta'$. Therefore δ is a prefix of α , so $(\delta, \delta) \in \xi_\alpha$. Like before, a very similar argument works for η_x . So both ξ_α and η_x are filters. Going the other way, take a filter $\zeta \in \hat{E}_0$. We do this in cases.

Case 1. ζ is finite. Let (α, α) be the minimum element inside ζ . Since ζ is a filter, For $(\beta, \beta) \in E(S(E))$, $(\alpha, \alpha) \leq (\beta, \beta) \implies (\beta, \beta) \in \zeta$. However, by the definition of the minimal element, $(\beta, \beta) \in \zeta \implies (\alpha, \alpha) \leq (\beta, \beta)$. Thus $(\beta, \beta) \in \zeta$ if and only if $(\alpha, \alpha) \leq (\beta, \beta) \iff \beta$ is a prefix of α . So $\zeta = \xi_\alpha$.

Case 2. ζ is infinite. By the lemma above, we can get an idea of what elements of ζ look like. For each nonnegative integer, ζ contains precisely one element corresponding to a path of that length. Because we require nonzero product between elements, every path in the filter is a prefix of every longer path also contained inside the filter. All of this given, we can find an $x \in E^\infty$ such that every path inside ζ is a prefix of x . By the uniqueness of filter elements, it follows that $\zeta = \eta_x$.

We have shown that a nonempty subset $\zeta \subset E(S(E))$ is a filter if and only if it is of the form ξ_α or η_x . Thus $\hat{E}_0 = \{\xi_\alpha: \alpha \in E^*\} \cup \{\eta_x: x \in E^\infty\}$. \square

Now that we know what filters look like on the graph inverse semigroup, the task of identifying ultrafilters becomes much simpler.

Theorem. $\hat{E}_\infty = \{\eta_x: x \in E^\infty\} \cup \{\xi_\alpha: \alpha \in E^* \text{ and } |r^{-1}\{s(\alpha)\}| = 0\}$

Proof. We need only consider various kinds of filters, determining whether they are ultrafilters or not. We do this in cases.

Case 1. ξ_α , $|r^{-1}\{s(\alpha)\}| > 0$. Take $e \in E^1$ such that $r(e) = s(\alpha)$. It is clear that $\xi_\alpha \subset \xi_{\alpha e}$. So ξ_α is not an ultrafilter.

Case 2. ξ_α , $|r^{-1}\{s(\alpha)\}| = 0$. Suppose we can find $\zeta \in \hat{E}_0$ such that $\xi_\alpha \subset \zeta$. For $0 \leq i \leq |\alpha|$, ζ inherits its idempotent with the path corresponding to length i from ξ_α . So we can find an element $(\beta, \beta) \in \zeta$ with $|\beta| > |\alpha|$ and $(\alpha, \alpha)(\beta, \beta) \neq 0$. Thus $\beta = \alpha\alpha'$ for some $\alpha' \in E^*$. But no edges go to $s(\alpha)$, so we cannot construct α' with $r(\alpha') = s(\alpha)$. This is a contradiction, so ξ_α is an ultrafilter.

Case 3. η_x , $x \in E^\infty$. Suppose there exists $\zeta \in \hat{E}_0$ such that $\eta_x \subset \zeta$. Then we can find $(\beta, \beta) \in \zeta$ such that $(\beta, \beta) \notin \eta_x$. But η_x , as we showed before, contains an element z corresponding to a path of length $|\beta|$. As a superset of η_x , ζ also contains z . By the lemma at the start, it follows that $(\beta, \beta) = z \in \eta_x$, which is a contradiction. So η_x is an ultrafilter for all $x \in E^\infty$.

We have identified which filters are ultrafilters, showing that $\hat{E}_\infty = \{\xi_\alpha: \alpha \in E^* \text{ and } |r^{-1}\{s(\alpha)\}| = 0\} \cup \{\eta_x: x \in E^\infty\}$. \square

Theorem. Let E be a directed graph, and $\alpha \in E^*$ such that $|r^{-1}\{s(\alpha)\}| = \infty$. Let $X, Y \subseteq_{\text{fin}} E(S(E))$, and Z be a finite cover of $E^{X,Y}$. If $\xi_\alpha \in \mathcal{U}(X, Y)$, then $\xi_\alpha \cap Z \neq \emptyset$.

Proof. First note:

$$\begin{aligned} E^{X,Y} &= \{e \in E(S(E)): e \leq x \ \forall x \in X \text{ and } ey = 0 \ \forall y \in Y\} \\ &= \{e \in E(S(E)): e \leq \min(X) \text{ and } ey = 0 \ \forall y \in Y\} \\ &= E^{\{\min(X)\}, Y} \end{aligned}$$

Letting $\min(X) = (x, x)$,

$$E^{X,Y} = \{(xx', xx'): x' \in E^*, r(x') = s(x) \text{ and } (xx', xx')y = 0 \ \forall y \in Y\}$$

Consider the set $C := \{(\alpha b, \alpha b): b \in E^1, s(\alpha) = r(b)\}$. By the assumption that $s(\alpha)$ is an infinite receiver, C is infinite. Given $y \in Y$, let ν be the path corresponding to y . Since $\xi_\alpha \in \mathcal{U}(X, Y)$, ν is not a prefix of α , and thus not a proper prefix of αb for any b . Thus, if $(\alpha b, \alpha b)y \neq 0$, αb is a prefix of ν . Then for $\beta \neq b$, $\alpha\beta$ cannot be a prefix of ν . So there is at most one element of C such that $(\alpha b, \alpha b)y \neq 0$. By the assumption that Y is finite, all but finitely many elements of C are inside $E^{\{(x,x)\}, Y}$. Therefore, if Z is a cover of $E^{X,Y}$, Z is an outer cover of the infinite set $E^{X,Y} \cap C$. Because Z is finite, $\exists z \in Z$ with $(\alpha b, \alpha b)z \neq 0$ for infinitely many $(\alpha b, \alpha b) \in E^{X,Y} \cap C$. If v is the path corresponding to z , then for every b , either v is a prefix of αb , or αb is a prefix of v . All the αb are the same length with a different starting edge, so if one is a prefix of v , no other can be a prefix of v . So v is a prefix of αb for infinitely many b . Thus $|v| \leq |\alpha| + 1$. If $|v| = |\alpha| + 1$, we have a contradiction: $b = \beta$ for all $(\alpha b, \alpha b), (\alpha\beta, \alpha, \beta) \in C$. Thus $|v| \leq |\alpha|$, so v is a prefix of α . Therefore $z = (v, v) \in \xi_\alpha$, so $\xi_\alpha \cap Z \neq \emptyset$. \square