Lemma: If $(\alpha,\alpha)(\beta,\beta) \neq 0$ and $|\alpha| = |\beta|$, then $\alpha = \beta$. Proof: WOLOG suppose α is a prefix of β. Then $\alpha = \beta\beta'$ for some β. But $|\alpha| = |\beta|$, so $|\beta'| = 0$, meaning $\beta' = s(\beta)$. Thus $\alpha = \beta s(\beta) = \beta$. An immediate consequence of this is that a proper filter contains at most one path of any given length.

Theorem: $\hat{E}_0 = \{\xi_\alpha\colon \alpha \in E^*\} \cup \{\eta_x\colon x \in E^\infty\}$. Proof:

First we show that ξ_α and η_x are filters. Let α be a finite path and let x be an infinite path. Consider $(\beta,\beta),\ (\gamma,\gamma) \in \xi_\alpha$ with $|\beta| = m$, $|\gamma| = n$. Reading edges left to right, β contains the first m edges of α and γ contains the first n edges of α. Without loss of generality, assume $m \leq n$. We can extend β by the next $n-m$ edges of α to produce γ. Thus β is a prefix of γ, so $(\beta,\beta)(\gamma,\gamma) = (\gamma,\gamma) \in \xi_\alpha$. A similar argument shows that η_x is closed under multiplication. So both ξ_α and η_x are prefilters on $E(S(E))$. Next, let $(\beta,\beta) \in \xi_\alpha$, and let $(\delta,\delta) \in E(S(E))$ so that $(\beta,\beta) \leq (\delta,\delta)$. Then $\beta = \delta\delta'$ for some $\delta' \in E^*$. Thus $\alpha = \beta\beta' = \delta\delta'\beta'$. δ is a prefix of α, so $(\delta,\delta) \in \xi_\alpha$. Like before, a very similar argument works for η_x. Thus both ξ_α and η_x are filters.

Let E be a directed graph, and $\alpha \in E^*$ such that $|r^{-1}\{s(\alpha)\}| = \infty$. Let $X, Y \subseteq_{\text{fin}} E(S(E))$, and Z be a finite cover of $E^{X,Y}$. If $\xi_\alpha \in \mathcal{U}(X,Y)$, then $\xi_\alpha \cap Z \neq \emptyset$. Proof:

First note:

$$\begin{aligned} E^{X,Y} &= \{e \in E(S(E))\colon e \leq x\ \forall x \in x \text{ and } ey = 0\ \forall y \in Y\} \\ &= \{e \in E(S(E))\colon e \leq \min(X) \text{ and } ey = 0\ \forall y \in Y\} \\ &= E^{\{\min(X)\},Y} \end{aligned}$$

Letting $\min(X) = (x,x)$,

$$E^{X,Y} = \{(xx',xx')\colon x' \in E^*,\ r(x') = s(x) \text{ and } (xx',xx')y = 0\ \forall y \in Y\}$$

Consider the set $C := \{(\alpha b,\alpha b)\colon b \in E^1,\ s(\alpha) = r(b)\}$. By the assumption that $s(\alpha)$ is an infinite receiver, C is infinite. Given $y \in Y$, let ν be the path corresponding to y. Since $\xi_\alpha \in \mathcal{U}(X,Y)$, ν is not a prefix of α, and thus not a proper prefix of αb for any b. Thus, if $(\alpha b,\alpha b)y \neq 0$, αb is a prefix of ν. Then for $\beta \neq b$, $\alpha\beta$ cannot be a prefix of ν. So there is at most one element of C such that $(\alpha b,\alpha b)y \neq 0$. By the assumption that Y is finite, all but finitely many elements of C are inside $E^{\{(x,x)\},Y}$. Therefore, if Z is a cover of $E^{X,Y}$, Z is an outer cover of the infinite set $E^{X,Y} \cap C$. Because Z is finite, $\exists z \in Z$ with $(\alpha b,\alpha b)z \neq 0$ for infinitely many $(\alpha b,\alpha b) \in E^{X,Y} \cap C$. If υ is the path corresponding to z, then for every b, either υ is a prefix of αb, or αb is a prefix of υ. All the αb are the same length with a different starting edge, so if one is a prefix of υ, no other can be a prefix of υ. So υ is a prefix of αb for infinitely many b. Thus $|\upsilon| \leq |\alpha| + 1$. If $|\upsilon| = |\alpha| + 1$, we have a contradiction: $b = \beta$ for all $(\alpha b,\alpha b),(\alpha\beta,\alpha,\beta) \in C$. Thus $|\upsilon| \leq |\alpha|$, so υ is a prefix of α. Therefore $z = (\upsilon,\upsilon) \in \xi_\alpha$, so $\xi_\alpha \cap Z \neq \emptyset$.