1 Introduction

2 Preliminaries

First we must clarify a few frequently used definitions and results.

Definition 2.1. We call a set S a *semigroup* if it has an associative binary operation that sends a pair of elements $(a, b) \mapsto ab$ for all $a, b \in S$.

Definition 2.2. We call a semigroup S an *inverse semigroup* if for every $s \in S$, there exists a unique element $s^* \in S$ such that $s = ss^*s$ and $s^* = s^*ss^*$. We call s^* the *inverse* of s. If S contains a zero element, we call it an *inverse semigroup with zero.*

Definition 2.3. An element e of an inverse semigroup S is called an *idempotent* if $e = e^2$. Note that for all idempotents, $e = e^*$, and for all elements s in an inverse semigroup, ss^* and s^*s are idempotent. We denote the set of idempotents on S by $E(S)$.

Remark 2.4. *If S is an inverse semigroup, for $e,\ f \in E(S)$, $ef = fe$.*

In other words, the idempotents of an inverse semigroup commute. (A proof of this can be found in Kyle's paper. cite this properly)

Definition 2.5. Let S be a set. A relation $\leq$ on S is called a *partial order* if

- It is reflexive; $a \leq a \quad \forall a \in S$
- It is antisymmetric; $a \leq b$ and $b \leq a \implies a = b \quad \forall a, b \in S$
- It is transitive; $a \leq b$ and $b \leq c \implies a \leq c \quad \forall a, b, c \in S$

A set S paired with a *partial order* is called a partially ordered set, or a *poset.*

3 The Tight Spectrum of an Inverse Semigroup

Now that we have a basic set of definitions in place, we can start to explore the set of idempotents on inverse semigroups. For this section, let S be an inverse semigroup with zero. We can define a partial order on $E(S)$, saying that $e \leq f \iff e = ef$. This can be extended to the entire set S, but in this paper we are only concerned with idempotents. (mention it's a meet semilattice, point to ISG primer?)

Definition 3.1. A nonempty proper subset $\xi \subset E(S)$ is called a *filter* on S if

1. For $e, f \in \xi$, $ef \in \xi$
2. For $x \in \xi$, $e \in E(S)$, $x \leq e \implies e \in \xi$

Subsets that only satisfy the first condition are called *prefilters*. We denote the set of filters on S by $\hat{E}_0^S$. An important remark is that the zero element of S is not contained inside any filters. If $0 \in \xi$, condition (2) would imply that $\xi = S$, which violates our definition. We can consider $\hat{E}_0^S$ a subset of $\{0,1\}^{E(S)}$, and thus endow it with the product topology inherited from $\{0,1\}^{E(S)}$. Equivalently, let X, Y be finite subsets of $E(S)$. Define $\mathcal{U}(X,Y) := \{\xi \in \hat{E}_0\colon\ X \subseteq \xi \text{ and } Y \cap \xi = \emptyset\}$. Sets of the form $\mathcal{U}(X,Y)$ form a basis for the topology on the set of filters. With this topology, the topological space $\hat{E}_0^S$ is called the *Exel spectrum* of S.

Definition 3.2. A filter $\eta \in \hat{E}_0$ is called an *ultrafilter* if it is maximal with respect to set inclusion. In other words, it is not contained inside another filter.

We define the subspace $\hat{E}_\infty^S := \{\xi \in \hat{E}_0^S\colon\ \xi \text{ is an ultrafilter}\}$. Note that if the inverse semigroup is obvious, we need not superscript S when describing the spectrum.

Remark 3.3. *For an ultrafilter $\eta \in \hat{E}_\infty$ and $e \in E(S)$, $e \notin \eta \implies ef = 0$ for some $f \in \eta$.*

Theorem 3.4. *Every filter is contained inside an ultrafilter.*

Proof. Let $\xi \in \hat{E}_0$. To prove this, we hope to use Zorn's lemma. We define the set $\mathcal{P} := \{\mathcal{F} \in \hat{E}_0\colon\ \xi \subseteq \mathcal{F}\}$ and we order $\mathcal{P}$ by set inclusion. Let $\mathcal{C} \subseteq \mathcal{P}$ be a chain. If we take $\mathcal{F} := \bigcup_{\zeta \in \mathcal{C}} \zeta$, it is clear that $\mathcal{F}$ is an upper bound of $\mathcal{C}$ and $\xi \subseteq \mathcal{F}$. To show that $\mathcal{F} \in \mathcal{P}$, we need only show that it is a filter. Suppose f, $g \in \mathcal{F}$. Then $f \in \zeta_1$, $g \in \zeta_2$ for some ζ_1, $\zeta_2 \in \mathcal{C}$. Since $\mathcal{C}$ is totally ordered, without loss of generality we assume $\zeta_1 \subseteq \zeta_2$. Then $f, g \in \zeta_2 \implies fg \in \zeta_2 \subseteq \mathcal{F}$. So $\mathcal{F}$ is a prefilter. Now suppose $f \in \mathcal{F}$, $e \in E(S)$ with $f \leq e$. Then $f \in \zeta$ for some $\zeta \in \mathcal{C}$, but since ζ is a filter, $e \in \zeta \subseteq \mathcal{F}$. So $\mathcal{F}$ is upward closed, and hence a filter. By Zorn's lemma, there exists $\eta \in \mathcal{P}$ such that η is maximal with respect to set inclusion. This is our definition of an ultrafilter; we have shown that $\forall \xi \in \hat{E}_0$, $\exists \eta \in \hat{E}_\infty$ with $\xi \subseteq \eta$. □

We can now begin to shift our focus towards the tight spectrum, which is the main topic of this section.

Definition 3.5. Let X, Y be finite subsets of $E(S)$. We define

$$E^{X,Y} := \{e \in E(S)\colon\ e \leq x \ \forall x \in x \text{ and } ey = 0 \ \forall y \in Y\}$$

Definition 3.6. Given $\mathcal{E} \subseteq E(S)$, we call $Z \subseteq E(S)$ an *outer cover* of $\mathcal{E}$ if $\forall e \neq 0 \in \mathcal{E}$, $\exists z \in Z$ with $ez \neq 0$. If is Z an outer cover of $\mathcal{E}$ and $Z \subseteq \mathcal{E}$, we say Z is a *cover* of $\mathcal{E}$.

Definition 3.7. Let $\xi \in \hat{E}_0$. We say that ξ is a *tight filter* if for all finite subsets X, $Y \subseteq E(S)$ and for all finite covers Z of $E^{X,Y}$, $\xi \in \mathcal{U}(X,Y) \implies Z \cap \xi \neq \emptyset$. We call the set of tight filters the *tight spectrum*, and denote it by $\hat{E}_{\text{tight}}^S$.

This definition may appear a bit contrived, but the next theorem hopes to hint at its significance.

Lemma 3.8. *Let X, Y be finite subsets of $E(S)$, and let $x = min(X)$. Then*

(i) $E^{X,Y} = E^{\{x\},Y}$

(ii) $\mathcal{U}(X,Y) = \mathcal{U}(\{x\},Y)$

Proof. (i)

$$\begin{aligned} E^{X,Y} &= \{e \in E(S)\colon\ e \leq x\ \forall x \in x \text{ and } ey = 0\ \forall y \in Y\} \\ &= \{e \in E(S)\colon\ e \leq \min(X) \text{ and } ey = 0\ \forall y \in Y\} \\ &= E^{\{x\},Y} \end{aligned}$$

(ii) Since $\{x\} \subseteq X$, $\mathcal{U}(X,Y) \subseteq \mathcal{U}(\{x\},Y)$. Now suppose $\xi \in \mathcal{U}(\{x\},Y)$. For $\chi \in X$, $x \leq \chi \implies \chi \in \xi$. Thus $X \subseteq \xi$, so $\xi \in \mathcal{U}(X,Y)$. We have shown that $\mathcal{U}(X,Y)$ and $\mathcal{U}(\{x\},Y)$ are subsets of eachother, so they are equal.

□

When we are working with $\mathcal{U}(X,Y)$ and $E^{X,Y}$, this lemma allows us the freedom of only considering the case where X is a singleton set.

Theorem 3.9. *$\hat{E}_{tight}$ is the closure of $\hat{E}_\infty$ in $\hat{E}_0$.*

Proof. We show that a filter $\xi \in \overline{\hat{E}_\infty} \iff \xi \in \hat{E}_{\text{tight}}$. First the forward implication. Let $\xi \in \overline{\hat{E}_\infty}$ and suppose $\xi \in \mathcal{U}(\{x\},Y)$. We prove the contrapositive of our definition for tightness. Let $Z \subseteq E^{\{x\},Y}$, and suppose $Z \cap \xi = \emptyset$. Then $\xi \in \mathcal{U}(\{x\}, Y \cup Z)$. Since $\xi \in \overline{\hat{E}_\infty}$, we can find an ultrafilter $\eta \in \mathcal{U}(\{x\}, Y \cup Z)$. By Remark 3.3, for every $f \in Y \cup Z$, we can find an idempotent $e_f \in \eta$ with $fe_f = 0$. Define $e := \left(\prod_{f \in Y \cup Z} e_f\right) x$. $e \leq x$ and $ey = 0\ \forall y \in Y$, so $e \in E^{\{x\},Y}$, but $ez = 0\ \forall z \in Z$. So Z is not a finite cover of $E^{\{x\},Y}$, thus ξ is a tight filter, hence $\overline{\hat{E}_\infty} \subseteq \hat{E}_{\text{tight}}$.

We now prove the other direction. Let $\xi \in \hat{E}_{\text{tight}}$ and suppose $\xi \in \mathcal{U}(\{x\},Y)$. First we show that $E^{\{x\},Y} \neq \{0\}$, by way of contradiction. If $E^{\{x\},Y} = \{0\}$, then $Z = \emptyset$ is a finite cover. Since ξ is tight, $\xi \cap Z \neq \emptyset$, which is a contradiction. So we can find a nonzero idempotent $e \in E^{\{x\},Y}$. Construct a filter ζ by including all the idempotents at least as large as e and closing it under products. By Theorem 3.4, we can find an ultrafilter η with $\eta \supseteq \zeta \ni e$. Note that since $ey = 0\ \forall y \in Y$ and $e \leq x$, any filter containing e must not intersect Y and must contain x. Thus $\eta \in \mathcal{U}(\{x\},Y)$, and because open sets of this form are a basis for the topology on $\hat{E}_0$, it follows that every open neighbourhood of ξ contains an ultrafilter η. So $\hat{E}_{\text{tight}} \subseteq \overline{\hat{E}_\infty}$. Finally, by this and the work above, $\hat{E}_{\text{tight}} = \overline{\hat{E}_\infty}$.

□

4 Directed Graphs

Now that we have introduced filters and the tight spectrum, we consider an example: the graph inverse semigroup. We require a few preliminary definitions.

Definition 4.1. A *directed graph* is a 4-tuple $E = (E^0, E^1, r, s)$, where we call E^0 the set of *vertices*, and E^1 the set of *edges*. $r : E^1 \to E^0$ and $s : E^1 \to E^0$ are called the *range* and *source* maps, respectively. A vertex can be thought of as a point in a plane, where an edge e can be thought of as an arrow pointing from $s(e)$ to $r(e)$.

For the rest of the section, we will let E be a directed graph.

Definition 4.2. If $\alpha_1, \alpha_2, ..., \alpha_n \in E^1$, where for $1 \leq i \leq n-1$, $r(\alpha_{i+1}) = s(\alpha_i)$, we can concatenate the edges to form a *finite path* $\alpha = \alpha_1 ... \alpha_n$ in E. We say α has length n, and denote it by $|\alpha|$.

Definition 4.3. For $n \in \mathbb{N}$, we define

$$E^n := \{\alpha\text{: } \alpha \text{ is a finite path in E, } |\alpha| = n\}$$

This provides some intuition about our naming convention for the set of vertices; a vertex can be considered a "path of length zero". We say two finite paths are equal if and only if they have the same length and consist of exactly the same edges. Next, we extend the range and source maps to finite paths; if $\alpha = \alpha_1 ... \alpha_n$, we say $r(\alpha) = r(\alpha_1)$, and $s(\alpha) = s(\alpha_n)$. We can now concatenate paths under the same rule mentioned in Definition 4.2. Some important terminology is the notion of a prefix; we say α is a *prefix* of β if $\beta = \alpha\alpha'$ for some $\alpha' \in E^*$. Furthermore, We say α is a *proper prefix* of β if α is a prefix of β and $\alpha \neq \beta$.

Definition 4.4. We define the *set of finite paths*, and denote it by E^*. Note that the vertex set is included.

$$E^* := \bigcup_{n=0}^{\infty} E^n$$

With these preliminaries developed, we can now define the graph inverse semigroup.

Definition 4.5. Let $S(E) = \{(\alpha, \beta) \in E^* \times E^*\text{: } s(\alpha) = s(\beta)\} \cup \{0\}$. This is called the *graph inverse semigroup* corresponding to the graph E, and we define multiplication as follows:

$$(\alpha, \beta)(\gamma, \delta) = \begin{cases} (\alpha, \delta\gamma') & \text{if } \beta = \gamma\gamma' \\ (\alpha\beta', \delta) & \text{if } \gamma = \beta\beta' \\ 0 & \text{otherwise} \end{cases}$$

In (Kyle's paper, cited), it is shown that this is indeed an inverse semigroup, where $(\alpha, \beta)^* = (\beta, \alpha)$, and the set of idempotents, $E(S(E))$, contains elements of the form (α, α). Note that the product $(\alpha, \alpha)(\beta, \beta)$ is nonzero exactly when α is a prefix of β, or β is a prefix of α. The rest of this section explores the spectrum of $S(E)$.

Lemma 4.6. *If $(\alpha, \alpha)(\beta, \beta) \neq 0$ and $|\alpha| = |\beta|$, then $\alpha = \beta$.*

Proof. Without loss of generality, suppose α is a prefix of β. Then $\alpha = \beta\beta'$ for some β. But $|\alpha| = |\beta|$, so $|\beta'| = 0$, meaning $\beta' = s(\beta)$. Thus $\alpha = \beta s(\beta) = \beta$. $\square$

An immediate corollary of this is that for any given length, a filter contains at most one element corresponding to a path of that length. Furthermore, if ζ is a filter, and $(\alpha, \alpha) \in \zeta$ such that $|\alpha| = n$, we can restrict α to any length $0 \leq m < n$ to produce a unique element of length m inside ζ.

Definition 4.7. If $x = x_1 x_2 x_3 ...$ is a sequence of edges in E, where for $i \in \mathbb{N}$, $r(x_{i+1}) = s(x_i)$, we call x an *infinite path*, and denote the *set of infinite paths*

$$E^\infty := \{x\colon\ x \text{ is an infinite path in } E\}$$

For $x \in E^\infty$, we say $\alpha \in E^*$ is a prefix of x if $x = \alpha x'$ for some infinite path x'. Our goal is now to identify filters on $E(S(E))$, with the ultimate goal of providing a characterization for the tight spectrum $\hat{E}_{\text{tight}}$.

Definition 4.8. Let $\alpha \in E^*$, $x \in E^\infty$. We define

- $\xi_\alpha = \{(\beta, \beta) \in E(S(E))\colon\ \beta \text{ is a prefix of } \alpha\}$
- $\eta_x = \{(\beta, \beta) \in E(S(E))\colon\ \beta \text{ is a prefix of } x\}$

Theorem 4.9. $\hat{E}_0 = \{\xi_\alpha\colon\ \alpha \in E^*\} \cup \{\eta_x\colon\ x \in E^\infty\}$

Proof. First we show that ξ_α and η_x are filters. First note that they are both nonempty. Let α be a finite path and let x be an infinite path. Consider $(\beta, \beta),\ (\gamma, \gamma) \in \xi_\alpha$ with $|\beta| = m$, $|\gamma| = n$. Reading edges left to right, β contains the first m edges of α and γ contains the first n edges of α. Without loss of generality, assume $m \leq n$. We can extend β by the next $n-m$ edges of α to produce γ. Thus β is a prefix of γ, so $(\beta, \beta)(\gamma, \gamma) = (\gamma, \gamma) \in \xi_\alpha$. A similar argument shows that η_x is closed under multiplication. So both ξ_α and η_x are prefilters on $E(S(E))$. Next, let $(\beta, \beta) \in \xi_\alpha$, and let $(\delta, \delta) \in E(S(E))$ so that $(\beta, \beta) \leq (\delta, \delta)$. Then $\beta = \delta\delta'$ for some $\delta' \in E^*$. Thus $\alpha = \beta\beta' = \delta\delta'\beta'$. Therefore δ is a prefix of α, so $(\delta, \delta) \in \xi_\alpha$. Like before, a very similar argument works for η_x. So both ξ_α and η_x are filters. Going the other way, take a filter $\zeta \in \hat{E}_0$. We do this in cases.

Case 1. *ζ is finite.* Let (α, α) be the minimum element inside ζ. Since ζ is a filter, For $(\beta, \beta) \in E(S(E))$, $(\alpha, \alpha) \leq (\beta, \beta) \implies (\beta, \beta) \in \zeta$. However, by the definition of the minimal element, $(\beta, \beta) \in \zeta \implies (\alpha, \alpha) \leq (\beta, \beta)$. Thus $(\beta, \beta) \in \zeta$ if and only if $(\alpha, \alpha) \leq (\beta, \beta) \iff \beta$ is a prefix of α. So $\zeta = \xi_\alpha$.

Case 2. *ζ is infinite.* Using Lemma 4.6, we can get an idea of what elements of ζ look like. For each nonnegative integer, ζ contains precisely one element corresponding to a path of that length. Because we require nonzero product between elements, every path in the filter is a prefix of every longer path also contained inside the filter. All of this given, we can find an $x \in E^\infty$ such that every path inside ζ is a prefix of x. By the uniqueness of filter elements, it follows that $\zeta = \eta_x$.

We have shown that a nonempty subset $\zeta \subset E(S(E))$ is a filter if and only if it is of the form ξ_α or η_x. Thus $\hat{E}_0 = \{\xi_\alpha \colon \alpha \in E^*\} \cup \{\eta_x \colon x \in E^\infty\}$. □

Now that we know what filters look like on the graph inverse semigroup, the task of identifying ultrafilters becomes much simpler.

Theorem 4.10. $\hat{E}_\infty = \{\eta_x \colon x \in E^\infty\} \cup \{\xi_\alpha \colon \alpha \in E^* \textit{ and } |r^{-1}\{s(\alpha)\}| = 0\}$

Proof. We need only consider various kinds of filters, determining whether they are ultrafilters or not. We do this in cases.

Case 1. ξ_α, $|r^{-1}\{s(\alpha)\}| > 0$. Take $e \in E^1$ such that $r(e) = s(\alpha)$. It is clear that $\xi_\alpha \subset \xi_{\alpha e}$. So ξ_α is not an ultrafilter.

Case 2. ξ_α, $|r^{-1}\{s(\alpha)\}| = 0$. Suppose we can find $\zeta \in \hat{E}_0$ such that $\xi_\alpha \subset \zeta$. For $0 \leq i \leq |\alpha|$, ζ inherits its idempotent with the path corresponding to length i from ξ_α. So we can find an element $(\beta, \beta) \in \zeta$ with $|\beta| > |\alpha|$ and $(\alpha, \alpha)(\beta, \beta) \neq 0$. Thus $\beta = \alpha\alpha'$ for some $\alpha' \in E^*$. But no edges go to $s(\alpha)$, so we cannot construct α' with $r(\alpha') = s(\alpha)$. This is a contradiction, so ξ_α is an ultrafilter.

Case 3. η_x, $x \in E^\infty$. Suppose there exists $\zeta \in \hat{E}_0$ such that $\eta_x \subset \zeta$. Then we can find $(\beta, \beta) \in \zeta$ such that $(\beta, \beta) \notin \eta_x$. But η_x, as we showed before, contains an element z corresponding to a path of length $|\beta|$. As a superset of η_x, ζ also contains z. By Lemma 4.6, it follows that $(\beta, \beta) = z \in \eta_x$, which is a contradiction. So η_x is an ultrafilter for all $x \in E^\infty$.

We have identified which filters are ultrafilters, showing that $\hat{E}_\infty = \{\xi_\alpha \colon \alpha \in E^*$ and $|r^{-1}\{s(\alpha)\}| = 0\} \cup \{\eta_x \colon x \in E^\infty\}$. □

Theorem 4.11.

$$\hat{E}_{tight} = \{\eta_x \colon x \in E^\infty\} \cup \{\xi_\alpha \colon \alpha \in E^*,\ |r^{-1}\{s(\alpha)\}| = 0 \textit{ or } |r^{-1}\{s(\alpha)\}| = \infty\}$$

Proof. From Theorem 3.9, it follows that $\hat{E}_\infty \subseteq \hat{E}_{\text{tight}}$. To provide a complete characterization of the tight spectrum, all we have left to do is consider which elements of $\hat{E}_0 \setminus \hat{E}_\infty$ are tight filters. We first let $\alpha \in E^*$ such that $|r^{-1}\{s(\alpha)\}| = \infty$. Here $s(\alpha)$ is called an *infinite receiver.* Let $\{(x, x)\}, Y \subseteq_{\text{fin}} E(S(E))$, and Z be a finite cover of $E^{\{(x,x)\},Y}$. Suppose $\xi_\alpha \in \mathcal{U}(\{(x, x)\}, Y)$. We first note:

$$E^{\{(x,x)\},Y} = \{(xx', xx') \colon x' \in E^*,\ r(x') = s(x) \text{ and } (xx', xx')y = 0\ \forall y \in Y\}$$

Consider the set $C := \{(\alpha b, \alpha b) \colon b \in E^1,\ s(\alpha) = r(b)\}$. By the assumption that $s(\alpha)$ is an infinite receiver, C is infinite. Given $y \in Y$, let ν be the path corresponding to y. Since $\xi_\alpha \in \mathcal{U}(\{(x, x)\}, Y)$, ν is not a prefix of α, and thus not a proper prefix of αb for any b. Thus, if $(\alpha b, \alpha b)y \neq 0$, αb is a prefix of ν. Then for $\beta \neq b$, $\alpha\beta$ cannot be a prefix of ν. So there is at most one element of C such that $(\alpha b, \alpha b)y \neq 0$. By the assumption that Y is finite, all but finitely many elements of C are inside $E^{\{(x,x)\},Y}$. Therefore, if Z is a cover of $E^{\{(x,x)\},Y}$, Z is an outer cover of the infinite set $E^{\{(x,x)\},Y} \cap C$. Because Z is finite, $\exists z \in Z$ with $(\alpha b, \alpha b)z \neq 0$ for infinitely many $(\alpha b, \alpha b) \in E^{\{(x,x)\},Y} \cap C$. If υ is the path corresponding to z, then for every b, either υ is a prefix of αb, or αb is a prefix of υ. All the αb are the same length with a different starting edge, so if one is a prefix of υ, no other can be a prefix of υ. So υ is a prefix of αb for infinitely many b. Thus $|\upsilon| \leq |\alpha| + 1$. If $|\upsilon| = |\alpha| + 1$, we have a contradiction: $b = \beta$ for all $(\alpha b, \alpha b), (\alpha\beta, \alpha, \beta) \in C$. Thus $|\upsilon| \leq |\alpha|$, so υ is a prefix of α. Therefore $z = (\upsilon, \upsilon) \in \xi_\alpha$, thus $\xi_\alpha \cap Z \neq \emptyset$, so ξ_α is tight.

In our final case, we let $\alpha \in E^*$, and suppose $0 < |r^{-1}\{s(\alpha)\}| < \infty$. Then there exists a nonempty, finite collection of edges $e_1, ..., e_n$ with $r(e_i) = s(\alpha)$ for $1 \leq i \leq n$. Let $Y \subseteq_{\text{fin}} E(S(E))$ such that $\xi_\alpha \in \mathcal{U}(\{(\alpha, \alpha)\}, Y)$. Then $Z = \{(\alpha e_1, \alpha e_1), ..., (\alpha e_n, \alpha e_n)\}$ is a finite cover of $E^{\{(\alpha,\alpha)\},Y}$, but $Z \cap \xi_\alpha = \emptyset$. So ξ_α is not tight. Thus a filter $\xi_\alpha \in \hat{E}_0 \setminus \hat{E}_\infty$ is tight if and only if it starts at an infinite receiver. This result combined with Theorem 3.9 completes the proof. □

5 References