Lemma: If $(\alpha,\alpha)(\beta,\beta)\neq 0$ and $|\alpha|=|\beta|$, then $\alpha=\beta$. Proof: WOLOG suppose α is a prefix of β. Then $\alpha=\beta\beta'$ for some β. But $|\alpha|=|\beta|$, so $|\beta'|=0$, meaning $\beta'=s(\beta)$. Thus $\alpha=\beta s(\beta)=\beta$. An immediate consequence of this is that a proper filter contains at most one path of any given length.

Theorem: $\hat{E}_0=\{\xi_\alpha\colon \alpha\in E^*\}\cup\{\eta_x\colon x\in E^\infty\}$. Proof:

First we show that ξ_α and η_x are filters. Let α be a finite path and let x be an infinite path. Consider (β,β), $(\gamma,\gamma)\in\xi_\alpha$ with $|\beta|=m$, $|\gamma|=n$. Reading edges left to right, β contains the first m edges of α and γ contains the first n edges of α. Without loss of generality, assume $m\leq n$. We can extend β by the next $n-m$ edges of α to produce γ. Thus β is a prefix of γ, so $(\beta,\beta)(\gamma,\gamma)=(\gamma,\gamma)\in\xi_\alpha$. A similar argument shows that η_x is closed under multiplication. So both ξ_α and η_x are prefilters on $E(S(E))$. Next, let $(\beta,\beta)\in\xi_\alpha$, and let $(\delta,\delta)\in E(S(E))$ so that $(\beta,\beta)\leq(\delta,\delta)$. Then $\beta=\delta\delta'$ for some $\delta'\in E^*$. Thus $\alpha=\beta\beta'=\delta\delta'\beta'$. Therefore δ is a prefix of α, so $(\delta,\delta)\in\xi_\alpha$. Like before, a very similar argument works for η_x. So both ξ_α and η_x are filters. Going the other way, take a proper filter $\zeta\in\hat{E}_0$. We do this in cases.

Case 1: ζ is finite. Let (α,α) be the minimum element inside ζ. Since ζ is a filter, For $(\beta,\beta)\in E(S(E))$, $(\alpha,\alpha)\leq(\beta,\beta)\implies(\beta,\beta)\in\zeta$. However, by the definition of the minimal element, $(\beta,\beta)\in\zeta\implies(\alpha,\alpha)\leq(\beta,\beta)$. Thus $(\beta,\beta)\in\zeta$ if and only if $(\alpha,\alpha)\leq(\beta,\beta)\iff\beta$ is a prefix of α. So $\zeta=\xi_\alpha$.

Let E be a directed graph, and $\alpha\in E^*$ such that $|r^{-1}\{s(\alpha)\}|=\infty$. Let $X,Y\subseteq_{\text{fin}}E(S(E))$, and Z be a finite cover of $E^{X,Y}$. If $\xi_\alpha\in\mathcal{U}(X,Y)$, then $\xi_\alpha\cap Z\neq\emptyset$. Proof:

First note:

$$\begin{aligned}E^{X,Y}&=\{e\in E(S(E))\colon e\leq x\ \forall x\in x \text{ and } ey=0\ \forall y\in Y\}\\&=\{e\in E(S(E))\colon e\leq\ \min(X)\text{ and } ey=0\ \forall y\in Y\}\\&=E^{\{\min(X)\},Y}\end{aligned}$$

Letting $\min(X)=(x,x)$,

$$E^{X,Y}=\{(xx',xx')\colon x'\in E^*,\ r(x')=s(x)\text{ and }(xx',xx')y=0\ \forall y\in Y\}$$

Consider the set $C:=\{(\alpha b,\alpha b)\colon b\in E^1,\ s(\alpha)=r(b)\}$. By the assumption that $s(\alpha)$ is an infinite receiver, C is infinite. Given $y\in Y$, let ν be the path corresponding to y. Since $\xi_\alpha\in\mathcal{U}(X,Y)$, ν is not a prefix of α, and thus not a proper prefix of αb for any b. Thus, if $(\alpha b,\alpha b)y\neq 0$, αb is a prefix of ν. Then for $\beta\neq b$, $\alpha\beta$ cannot be a prefix of ν. So there is at most one element of C such that $(\alpha b,\alpha b)y\neq 0$. By the assumption that Y is finite, all but finitely many elements of C are inside $E^{\{(x,x)\},Y}$. Therefore, if Z is a cover of $E^{X,Y}$, Z is an outer cover of the infinite set $E^{X,Y}\cap C$. Because Z is finite, $\exists z\in Z$

with $(\alpha b, \alpha b)z \neq 0$ for infinitely many $(\alpha b, \alpha b) \in E^{X,Y} \cap C$. If υ is the path corresponding to z, then for every b, either υ is a prefix of αb, or αb is a prefix of υ. All the αb are the same length with a different starting edge, so if one is a prefix of υ, no other can be a prefix of υ. So υ is a prefix of αb for infinitely many b. Thus $|\upsilon| \leq |\alpha| + 1$. If $|\upsilon| = |\alpha| + 1$, we have a contradiction: $b = \beta$ for all $(\alpha b, \alpha b), (\alpha\beta, \alpha, \beta) \in C$. Thus $|\upsilon| \leq |\alpha|$, so υ is a prefix of α. Therefore $z = (\upsilon, \upsilon) \in \xi_\alpha$, so $\xi_\alpha \cap Z \neq \emptyset$.