Definition 1. We define...

A comment to myself from Andrew
A note from Laura
A comment from howald

Example 2. Here's one...

Definition 3. Let A be an $n \times n$ matrix with coefficients in a commutative unital ring R. We define the *determinant* of A to be:

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_i a_{i,\sigma(i)}$$

Theorem 4. *(Properties of the determinant)*

(1) *(calibration)* $\det(I_n) = 1$
(2) *(multilinearity)* $\det$ *is R-linear in each row. That is:*

$$\det \begin{pmatrix} v_1 \\ \vdots \\ av + bw \\ \vdots \\ v_n \end{pmatrix} = a \det \begin{pmatrix} v_1 \\ \vdots \\ v \\ \vdots \\ v_n \end{pmatrix} + b \det \begin{pmatrix} v_1 \\ \vdots \\ w \\ \vdots \\ v_n \end{pmatrix}$$

(3) *(alternating)* $\det$ *is alternating on the rows: (Here v and w are meant to represent arbitrary distinct row positions, including row 1 or row n.*

$$\det \begin{pmatrix} \vdots \\ v \\ \vdots \\ w \\ \vdots \end{pmatrix} = -\det \begin{pmatrix} \vdots \\ w \\ \vdots \\ v \\ \vdots \end{pmatrix}$$

(4) *(zero rows) If A has a zero row, then* $\det(A) = 0$.
(5) *(repeated rows) If A has a repeated row, then* $\det(A) = 0$.
(6) *(Permutation Matrix) Let P_σ be the permutation matrix associated to a permutation σ so that $P_{i,\sigma(i)} = 1$ but P is otherwise zero. Then* $\det(P_\sigma) = \operatorname{sgn}(\sigma)$
(7) *(Symmetry)* $\det(A) = \det(A^T)$
(8) *(Columns)* $\det$ *is linear on each column, and alternating on columns.*
(9) *(Laplace expansion on row i, $n > 1$) For a matrix A, write a_{ij} for its entries, and write A_{ij} for the matrix (A with row i and column j removed). Fix $1 \leq i \leq n$. Then*

$$\det(A) = \sum_j (-1)^{i+j} a_{ij} \det(A_{ij})$$

(10) *(Cramer's rule, $n > 1$) For a matrix A, let C, the cofactor matrix of A, be the matrix whose i, j entry is $C_{ij} = (-1)^{i+j} \det(A_{ij})$, as above. Then*

$$AC^T = \det(A)I = C^T A$$

(11) *(Homomorphism)* $\det(AB) = \det(A)\det(B)$

Proof. For 1, the identity permutation is the only contributing term to the sum $\det(A)$. The proof of property 6 is basically the same.

To prove that determinant satisfies property 2, fix k and let $(a_{k1}, \ldots, a_{kn})$ be a row of A. Note that the expression $\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_i a_{i,\sigma(i)}$ is a sum of products, each of which has exactly one a_{kj}. So it can be regrouped in the form $c_1 a_{k1} + \ldots + c_n a_{kn}$, which is clearly a linear function of the row vector.

The alternating property (property 3) is an exercise in reindexing: If B is obtained from A by switching rows i_1 and i_2, and τ is the corresponding transposition, then

$$\det(B) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma\tau) \prod_i b_{i,\sigma\tau(i)} = \sum_{\sigma \in S_n} -\operatorname{sgn}(\sigma) \prod_i b_{\tau i,\sigma(i)} = \sum_{\sigma \in S_n} -\operatorname{sgn}(\sigma) \prod_i a_{i,\sigma(i)} = -\det(A)$$

Property 4 follows from 2.

Property 5 follows from property 3 unless 2 is a zerodivisor. For a general proof, assume that rows i_1 and i_2 of A are identical, and let $\tau = (i_1 i_2)$. Then

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_i a_{i,\sigma(i)} = \sum_{\sigma \in A_n} \left[\operatorname{sgn}(\sigma) \prod_i a_{i,\sigma(i)} + \operatorname{sgn}(\sigma\tau) \prod_i a_{i,\sigma\tau(i)} \right]$$

The last product may be rewritten $\prod_i a_{\tau(i),\sigma(i)} = \prod_i a_{i,\sigma(i)}$, and everything cancels.

Properties 1-5 (just 1-3 if 2 is a nonzerodivisor) together imply the formula for $\det(A)$ given in the definition. In some contexts these first three properties are taken to be the definition of the determinant function.

The symmetry property (7) follows from the definition:

$$\det(A^T) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_i a_{\sigma(i),i} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma^{-1}) \prod_i a_{i,\sigma(i)} = \det(A)$$

Antilinearity in columns (8) follows from symmetry.

Next we prove the Laplace expansion 9. Fix row index k. Grouping the terms of the sum $\det(A)$ by the value $\sigma(k)$ gives $\det(A) = c_1 a_{k1} + \ldots + c_n a_{kn}$, as in the proof of linearity. Now

$$c_j = \sum_{\sigma \in S_n, \sigma(k)=j} \operatorname{sgn}(\sigma) \prod_{i \neq k} a_{i,\sigma(i)}$$

Note that this is a sum of products from the submatrix A_{kj}. Now if $\pi \in S_{n-1} \subset S_n$, we can associate to π the permutation $\sigma_\pi = (j, j+1, \ldots, n)\pi(n, \ldots, k+1, k) \in S_n$, which sends k to j. The permutations σ for which $\sigma(k) = j$ are exactly the permutations σ_π for $\pi \in S_n$, and

$$\operatorname{sgn}(\sigma) \prod_{i \neq k} a_{i,\sigma(i)} = (-1)^{k+j} \operatorname{sgn}(\pi) \prod_{i=1}^{n-1} [A_k j]_{i,\pi(i)}$$

Therefore $c_j = \det(A_{kj})$ as desired.

Next we prove Cramer's rule. Let A be a matrix and let C be the cofactor matrix of A. The (i,i) entry of AC^T is the Laplace expansion for $\det(A)$. The (i,j) entry $(i \neq j)$ is a Laplace expansion for a matrix like A but with row j replaced with row i, which forces zero determinant. For the product $C^T A$, we use a column variant of the Laplace expansion, by symmetry.

Finally we prove multiplicativity. If R is a field, things are much easier. Since we need the result more generally, we choose a strategy of direct computation. Let A and B be $n \times n$ matrices.

$$
\begin{aligned}
\det(AB) &= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_i (AB)_{i,\sigma(i)} \\
&= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_i \sum_k A_{ik} B_{k,\sigma(i)} \\
&= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \sum_{f:[n]\to[n]} \prod_i A_{i,f(i)} B_{f(i),\sigma(i)} \\
&= \sum_{f:[n]\to[n]} \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_i A_{i,f(i)} B_{f(i),\sigma(i)} \\
&= \sum_{\pi\in S_n} \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_i A_{i,\pi(i)} B_{\pi i,\sigma(i)} \qquad \text{(see below)} \\
&= \sum_{\pi\in S_n} \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_i A_{i,\pi(i)} \prod_j B_{\pi j,\sigma(j)} \qquad \text{(reindex j product)} \\
&= \sum_{\pi\in S_n} \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_i A_{i,\pi(i)} \prod_j B_{j,\sigma\pi^{-1}(j)} \qquad (\sigma = \tau\pi) \\
&= \sum_{\pi\in S_n} \sum_{\tau\in S_n} \operatorname{sgn}(\tau\pi) \prod_i A_{i,\pi(i)} \prod_j B_{j,\tau(j)} \\
&= \left(\sum_{\pi\in S_n} \operatorname{sgn}(\pi) \prod_i A_{i,\pi(i)} \right) \left(\sum_{\tau\in S_n} \operatorname{sgn}(\tau) \prod_j B_{j,\tau(j)} \right) \\
&= \det(A)\det(B)
\end{aligned}
$$

□

It remains to explain the conversion of $\sum_{f:[n]\to[n]}$ to $\sum_{\pi\in S_n}$. We argue that when f is not a permutation, it contributes 0 to the sum. If f is not a permutation, it is not injective. Say $f(c) = f(d) = e$, with $c < d$. Then any $\sigma \in S_n$ yields the product

$$
\operatorname{sgn}(\sigma) \prod_i A_{i,f(i)} B_{f(i),\sigma i} = \operatorname{sgn}(\sigma) \ldots \cdot A_{c,e} B_{e,\sigma(c)} \cdot \ldots \cdot A_{d,e} B_{e,\sigma(d)} \cdot \ldots
$$

The permutation $\sigma \cdot (cd)$ yields:

$$
\operatorname{sgn}(\sigma(cd)) \prod_i A_{i,f(i)} B_{f(i),\sigma i} = -\operatorname{sgn}(\sigma) \ldots \cdot A_{c,e} B_{e,\sigma(d)} \cdot \ldots \cdot A_{d,e} B_{e,\sigma(c)} \cdot \ldots
$$

Terms hidden in "…" are identical, and terms shown are equal. These terms cancel in the sum $\sum_{\sigma\in S_n}$. Pairing every permutation σ with its partner $\sigma(cd)$ shows that the entire sum $\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_i A_{i,f(i)} B_{f(i),\sigma i}$ is zero when f is not a permutation.