# OPTIMAL CONTROL OF NONHOMOGENEOUS CHAOTIC SYSTEMS

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Abstract: An optimal control approach for controlling chaotic dynamics is developed. This is possible by using an approximation technique that solves the nonlinear optimal control problem by reducing it to a sequence of linear time-varying systems. The theory is implemented to direct chaotic trajectories of the Duffing equation to a period-$n$ orbit. 

Keywords: Chaos, nonlinear optimal control, linear time-varying systems.

## 1. INTRODUCTION

Over the past decades, the control of chaotic systems has received a great deal of attention, for instance the reader is referred to the surveyed literature (Chen and Dong, 1993*a*), (Andrievskii and Fradkov, 2003) and (Chen and Dong, 1998).
The seminal work of (Ott *et al.*, 1990) gave the breakthrough in the theory of control of chaos. Their method is a linear controller that is switched on as soon as the system state enters a prescribed small neighbourhood of the fixed point. No control is applied before. Due to the recurrence property of chaotic systems, the trajectory will come close to a small area where the fixed point is; but it may take a long time for this event to occur. Several authors have dealt with this problem (e.g. (Shinbrot *et al.*, 1990) and (Vincent, 1997)).
Controlling chaotic systems using optimal control has been studied in (Abarbanel *et al.*, 1997), where the Pontryagin maximum principle is locally applied, and a simple explicit one-step control formula is derived. This control formula directs the system to a chosen set of points in phase space, and it maintains the system there. In (Chen, 1994), optimal control is used to control chaos by obtaining minimum parameter perturbation sequences that are applied in the Ott-Grebogi-Yorke (OGY) method.
In this paper, an alternative approach based on optimal control theory is considered. An approximation technique developed in (Banks and McCaffrey, 1998) for the analysis of nonlinear systems is used. Here the solution of a nonlinear system is approximated by the solution of linear time-varying systems which are arbitrarily close to the true solution. In (Banks and Dinesh, 2000) the above-mentioned technique is applied to optimal control, where the problem is solved explicitly, and it is shown that the sequence converges.
The aforementioned methods are employed in this article to stabilize nonhomogeneous chaotic systems. This is accomplished by steering the trajectories to an unstable period-$n$ orbit of the chaotic system. The use of optimal control is an effective tool to address the following issues; a) to reduce the response time; i.e., to avoid the need of waiting for the system to come close to a small neighbourhood. This is because the system is not locally linearized in the proposed method and therefore, the controller can be turned on at any time, b) the controller is not unrealistically changed for getting stability.

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The paper is organized in the following way. A brief introduction to the approximation technique is introduced in Section 2. Section 3 recalls in short, linear optimal control theory and the linear tracking problem. In Section 4, the nonlinear optimal control problem of nonhomogeneous chaotic systems is presented. Here the solution is approximated by linear time-varying sequences and the dynamics can be directed to an unstable periodic orbit of the chaotic system. An example is demonstrated in Section 5 by controlling a chaotic trajectory of the well known Duffing oscillator to one of its period-1 and period-2 orbits. Final remarks are given in Section 6.

## 2. THE APPROXIMATING SEQUENCE

Consider a nonlinear differential equation

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{B}(\mathbf{x}(t))\mathbf{u}(t), \qquad (1) \\ \mathbf{x}(t_0) &= \mathbf{x}_0, \end{aligned}$$

where $\mathbf{x}(t)$ is a $n$-dimensional state vector, $\mathbf{A}\in\mathbb{R}^{n\times n}$ and $\mathbf{B}\in\mathbb{R}^{n\times m}$ are nonlinear matrix-valued functions of $\mathbf{x}(t)$ that satisfy a mild Lipschitz condition, and $\mathbf{u}(t)$ is a $m$-dimensional unconstrained control vector.

Then a sequence of linear time-varying approximations can replace the above system where the solutions converge to the solution of the original nonlinear system. The sequence is

$$\begin{aligned} \dot{\mathbf{x}}^{[1]}(t) &= \mathbf{A}(\mathbf{x}_0)\mathbf{x}^{[1]}(t) + \mathbf{B}(\mathbf{x}_0)\mathbf{u}^{[1]}(t), \qquad (2) \\ \dot{\mathbf{x}}^{[2]}(t) &= \mathbf{A}(\mathbf{x}^{[1]}(t))\mathbf{x}^{[2]}(t) + \mathbf{B}(\mathbf{x}^{[1]}(t))\mathbf{u}^{[2]}(t) \\ &\vdots \\ \dot{\mathbf{x}}^{[i]}(t) &= \mathbf{A}(\mathbf{x}^{[i-1]}(t))\mathbf{x}^{[i]}(t) + \mathbf{B}(\mathbf{x}^{[i-1]}(t))\mathbf{u}^{[i]}(t), \end{aligned}$$

where the initial conditions are: $\mathbf{x}^{[1]}(t_0)=\mathbf{x}^{[2]}(t_0)=\ldots=\mathbf{x}^{[i]}(t_0)=\mathbf{x}_0$.

This approximation technique, developed in (Banks and McCaffrey, 1998), is a method that reduces nonlinear systems to sequences of linear time-varying equations, which are arbitrarily close to the solution of the original system. The first approximation, $i=1$, is a linear time-invariant system whose solution is stored for computing the next approximation. For $i\geq 2$, the subsequent approximations are linear time-varying solutions of the nonlinear system. At each iteration $i$, the procedure uses the solution of the previous iteration, $i-1$, to calculate the current one. The initial value of all the approximations is $\mathbf{x}_0$. The nonlinear system can be considered as the limit of the sequence of linear time-varying approximations whose solution converges, on any compact time interval, to the true solution of the nonlinear system. Hence, linear control techniques can be applied to each of the differential equations defined in (2).

## 3. LINEAR-QUADRATIC REGULATOR PROBLEM

In this section, the linear-quadratic regulator system, an important class of optimal control problems, is revisited. The plant is described by the linear state equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \qquad (3)$$

which has time-varying coefficients. The linear-quadratic cost functional or performance measure to be minimized is

$$\begin{aligned} J = &\frac{1}{2}\mathbf{x}^T(t_f)\mathbf{F}\mathbf{x}(t_f) \qquad (4) \\ &+\frac{1}{2}\int_{t_0}^{t_f} \left\{\mathbf{x}^T(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t)\right\} dt; \end{aligned}$$

the final time is fixed, $\mathbf{F}$ and $\mathbf{Q}$ are real symmetric positive semi-definite matrices, and $\mathbf{R}$ is a real symmetric positive definite matrix. It is assumed that the states and controls are not bounded, and $\mathbf{x}(t_f)$ is free. The purpose is to keep the state vector close to the origin without an excessive expenditure of control effort.

The optimal control is a linear time-varying function of the system states and is given by

$$\mathbf{u}(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t)\mathbf{x}(t). \qquad (5)$$

where the real, symmetric and positive-definite matrix $\mathbf{P}\in\mathbb{R}^{n\times n}$ is the solution of the Riccati-type matrix differential equation

$$\begin{aligned} \dot{\mathbf{P}}(t) = &-\mathbf{Q}(t) - \mathbf{A}^T(t)\mathbf{P}(t) - \mathbf{P}(t)\mathbf{A}(t) \\ &+\mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t). \qquad (6) \end{aligned}$$

In order to find $\mathbf{P}(t)$, $n(n+1)/2$ first-order differential equations must be solved. These equations can be integrated numerically by starting the integration at $t=t_f$ and proceeding backwards in time to $t=t_o$, satisfying the final condition $\mathbf{P}(t_f)=\mathbf{F}$.

Then the corresponding optimal state trajectory becomes the solution of the differential equation

$$\begin{aligned} \dot{\mathbf{x}}^*(t) &= [\mathbf{A}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t)]\mathbf{x}^*(t), \\ \mathbf{x}^*(t_0) &= \mathbf{x}_0. \qquad (7) \end{aligned}$$

The weighting factor $\mathbf{F}$ penalizes the system at the terminal time $t_f$. The matrices $\mathbf{Q}$ and $\mathbf{R}$ regulate the trajectories of the state and the control over the time span $[t_0, t_f]$. If $\mathbf{Q}$ is large and $\mathbf{R}$ small, then the optimal system will apply a large control effort. It will direct the system to the origin in a fast way in order to avoid a large penalty due to large $\mathbf{Q}$. On the other hand, if $\mathbf{Q}$ is small and $\mathbf{R}$ large, then the expenditure of control effort will be smaller and the system will converge to zero slowly.

*3.1 Linear-quadratic optimal tracking control*

In the regulator problem, it is desired to maintain the state vector, $\mathbf{x}(t)$, close to the origin while using the minimum control. For the tracking problem, it is required to make $\mathbf{x}(t)$ to follow the input $\mathbf{r}(t)$. Then the following cost functional is considered

$$\begin{aligned} J = &\frac{1}{2}\left[\mathbf{x}(t_f)-\mathbf{r}(t_f)\right]^T \mathbf{F}\left[\mathbf{x}(t_f)-\mathbf{r}(t_f)\right] \qquad (8)\\ &+\frac{1}{2}\int_{t_0}^{t_f}\Big\{\left[\mathbf{x}(t)-\mathbf{r}(t)\right]^T\mathbf{Q}(t)\left[\mathbf{x}(t)-\mathbf{r}(t)\right]\\ &+\mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t)\Big\}dt. \end{aligned}$$

This functional together with the linear dynamics (3) constitute the tracking problem where the results obtained for the linear regulator are generalized to solve this problem.
The optimal control for the tracking problem is

$$\mathbf{u}(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t)\mathbf{x}(t)-\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{s}(t), \qquad (9)$$

where $\mathbf{P}$ is given in (6) and $\mathbf{s}(t)$ is a feedforward vector that is obtained by the unique solution of the equation

$$\begin{aligned} \dot{\mathbf{s}}(t) = &-[\mathbf{A}^T(t)-\mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)]\mathbf{s}(t)\\ &+\mathbf{Q}(t)\mathbf{r}(t), \qquad (10) \end{aligned}$$

with the terminal condition $\mathbf{s}(t_f)$=$-\mathbf{F}\mathbf{r}(t_f)$.
The optimal trajectory is the solution of the linear differential equation

$$\begin{aligned} \dot{\mathbf{x}}^*(t) = &[\mathbf{A}(t)-\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t)]\mathbf{x}^*(t) \quad (11)\\ &-\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{s}(t),\\ \mathbf{x}^*(t_0) = &\,\mathbf{x}_0. \end{aligned}$$

## 4. CONTROL OF NONHOMOGENEOUS NONLINEAR SYSTEMS

Consider a nonhomogeneous nonlinear system of the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x}(t))\mathbf{x}(t)+\mathbf{B}(\mathbf{x}(t))\mathbf{u}(t)+\mathbf{z}(t), \quad (12)$$

where $\mathbf{x}(t)$ is the state vector, $\mathbf{A}\in\mathbb{R}^{n\times n}$, $\mathbf{B}\in\mathbb{R}^{n\times m}$, $\mathbf{u}(t)$ is unconstrained and $\mathbf{z}(t)$ is a $n$-dimensional forcing vector.
Due to the fact that the sequence of linear time-varying equations of the approximation technique in (2) converges to the nonlinear solution, then the linear-quadratic optimal tracking control can be applied to each of the equations of the sequence. Thus, the the procedure in Section 2 can be implemented in (12). That is,

$$\begin{aligned} \dot{\mathbf{x}}^{[1]}(t) &= \mathbf{A}(\mathbf{x}_0)\mathbf{x}^{[1]}(t)+\mathbf{B}(\mathbf{x}_0)\mathbf{u}^{[1]}(t)+\mathbf{z}(t) \quad (13)\\ \dot{\mathbf{x}}^{[2]}(t) &= \mathbf{A}(\mathbf{x}^{[1]}(t))\mathbf{x}^{[2]}(t)+\mathbf{B}(\mathbf{x}^{[1]}(t))\mathbf{u}^{[2]}(t)\\ &\quad+\mathbf{z}(t),\\ &\;\;\vdots\\ \dot{\mathbf{x}}^{[i]}(t) &= \mathbf{A}(\mathbf{x}^{[i-1]}(t))\mathbf{x}^{[i]}(t)+\mathbf{B}(\mathbf{x}^{[i-1]}(t))\mathbf{u}^{[i]}(t)\\ &\quad+\mathbf{z}(t), \end{aligned}$$

and the initial conditions are $\mathbf{x}^{[1]}(t_0)$= $\mathbf{x}^{[2]}(t_0)$= $\ldots$=$\mathbf{x}^{[i]}(t_0)$=$\mathbf{x}_0$. So at each of the above linear equations, the linear-quadratic optimal tracking control is employed until the $i$th solution converges to a desired reference value $\mathbf{r}(t)$. This is explained as follows.
Consider the $i$th approximation, which is a linear time-varying nonhomogeneous system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)+\mathbf{B}(t)\mathbf{u}(t)+\mathbf{z}(t), \qquad (14)$$

which together with the finite-time linear quadratic cost functional (8) minimize the error given by

$$\mathbf{e}(t) = \mathbf{x}(t)-\mathbf{r}(t). \qquad (15)$$

The objective is to control the system (14) to the desired vector $\mathbf{r}(t)$. The problem of minimizing the cost functional (8) subject to the constraint (14), that makes the state $\mathbf{x}(t)$ follow a desired time-varying vector $\mathbf{r}(t)$ is called the Linear Quadratic Optimal Tracking Problem that can be solved using the Pontryagin Maximum Principle.
Having (14) and (8), the Hamiltonian for this problem is

$$\begin{aligned} \mathbf{H}\,\big(\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t),t\big) = &\frac{1}{2}\left[\mathbf{x}(t)-\mathbf{r}(t)\right]^T \quad (16)\\ \mathbf{Q}(t)\left[\mathbf{x}(t)-\mathbf{r}(t)\right] &+ \frac{1}{2}\mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t)\\ +\boldsymbol{\lambda}^T(t)\Big\{\mathbf{A}(t)\mathbf{x}(t) &+\mathbf{B}(t)\mathbf{u}(t)+\mathbf{z}(t)\Big\}. \end{aligned}$$

Assume that $\boldsymbol{\lambda}(t)$ can be written in the following form:

$$\boldsymbol{\lambda}(t) = \mathbf{P}(t)\mathbf{x}(t)+\mathbf{s}(t)+\mathbf{w}(t), \qquad (17)$$

the necessary conditions for optimality are

$$\begin{aligned} \dot{\boldsymbol{\lambda}}(t) &= -\frac{\partial\mathbf{H}}{\partial\mathbf{x}}\\ &= -\mathbf{Q}(t)\mathbf{x}(t)-\mathbf{A}^T(t)\boldsymbol{\lambda}(t)+\mathbf{Q}(t)\mathbf{r}(t) \quad (18)\\ 0 &= \frac{\partial\mathbf{H}}{\partial\mathbf{u}} = \mathbf{R}(t)\mathbf{u}(t)+\boldsymbol{\lambda}^T(t)\mathbf{B}(t), \qquad (19) \end{aligned}$$

the latter equation gives the optimal control law

$$\mathbf{u}(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\boldsymbol{\lambda}(t). \qquad (20)$$

Substituting (17) into (20) yields

$$\begin{aligned} \mathbf{u}(t) = &-\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t)\mathbf{x}(t) \qquad (21)\\ &-\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{s}(t)-\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{w}(t). \end{aligned}$$

Taking the derivative of $\boldsymbol{\lambda}(t)$ in (17) and substituting (14) into the result gives

$$
\begin{aligned}
\dot{\boldsymbol{\lambda}}(t) &= \dot{\mathbf{P}}(t)\mathbf{x}(t) + \mathbf{P}(t)\dot{\mathbf{x}}(t) + \dot{\mathbf{s}}(t) + \dot{\mathbf{w}}(t) \qquad (22)\\
&= \dot{\mathbf{P}}(t)\mathbf{x}(t) + \mathbf{P}(t)\Big(\mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)\\
&\quad +\mathbf{z}(t)\Big) + \dot{\mathbf{s}}(t) + \dot{\mathbf{w}}(t).
\end{aligned}
$$

Now, substituting (17) in (18) and equating to the substitution of (21) into (22) produces,

$$
\begin{aligned}
&-\mathbf{Q}(t)\mathbf{x}(t) - \mathbf{A}^T(t)\mathbf{P}(t)\mathbf{x}(t) - \mathbf{A}^T(t)\mathbf{s}(t) \qquad (23)\\
&-\mathbf{A}^T(t)\mathbf{w}(t) + \mathbf{Q}(t)\mathbf{r}(t) = \dot{\mathbf{P}}(t)\mathbf{x}(t)\\
&+\mathbf{P}(t)\mathbf{A}(t)\mathbf{x}(t) - \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t)\mathbf{x}(t)\\
&-\mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{s}(t)\\
&-\mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{w}(t) + \mathbf{P}(t)\mathbf{z}(t)\\
&+\dot{\mathbf{s}}(t) + \dot{\mathbf{w}}(t);
\end{aligned}
$$

by rearranging the previous equation yields

$$
\begin{aligned}
&\Big[\Big(\dot{\mathbf{P}}(t) + \mathbf{Q}(t) + \mathbf{A}^T(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}(t) \qquad (24)\\
&-\mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t)\Big)\mathbf{x}(t)\Big] + \Big[\dot{\mathbf{s}}(t)\\
&-\mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{s}(t) - \mathbf{Q}(t)\mathbf{r}(t)\\
&+\mathbf{A}^T(t)\mathbf{s}(t)\Big] + \Big[\dot{\mathbf{w}}(t) + \mathbf{A}^T(t)\mathbf{w}(t)\\
&-\mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{w}(t) + \mathbf{P}(t)\mathbf{z}(t)\Big] = 0.
\end{aligned}
$$

If all of the bracketed terms of (24) are zero; then from the first one, the matrix Riccati differential equation is obtained. Its solution is $\mathbf{P}(t)$, a real symmetric positive-definite $n{\times}n$ matrix that is solved backwards in time with the terminal condition $\mathbf{P}(t_f)$=$\mathbf{F}$.
After finding $\mathbf{P}(t)$, the result is substituted into the second and third bracketed terms of (24) to produce

$$
\begin{aligned}
\dot{\mathbf{s}}(t) &= -\Big(\mathbf{A}^T(t) - \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\Big)\mathbf{s}(t)\\
&\quad +\mathbf{Q}(t)\mathbf{r}(t), \qquad (25)\\
\dot{\mathbf{w}}(t) &= \Big(\mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t) - \mathbf{A}^T(t)\Big)\mathbf{w}(t)\\
&\quad -\mathbf{P}(t)\mathbf{z}(t); \qquad (26)
\end{aligned}
$$

then $\mathbf{s}(t)$ and $\mathbf{w}(t)$ can be obtained by integrating (25) and (26) backwards in time respectively. The final conditions are $\mathbf{s}(t_f)$=$-\mathbf{Fr}(t_f)$ and $\mathbf{w}(t_f)$=0. Once the vectors $\mathbf{s}(t)$ and $\mathbf{w}(t)$ are determined, the results are stored together with $\mathbf{P}(t)$ and substituted in (21) to produce the controller $\mathbf{u}(t)$. This indicates that the optimal control law is a linear time-varying system. Then the control law is substituted into (14) to produce the optimized controlled system

$$
\begin{aligned}
\dot{\mathbf{x}}^*(t) &= \Big(\mathbf{A}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{P}(t)\Big)\mathbf{x}^*(t) \qquad (27)\\
&\quad -\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\Big(\mathbf{s}(t) + \mathbf{w}(t)\Big) + \mathbf{z}(t)\\
\mathbf{x}^*(t_0) &= \mathbf{x}_0.
\end{aligned}
$$

Note that this procedure must be performed at each iteration until the $i$th solution produces the desired result. In other words, for $i$=1 the linear time-invariant equations are the following:

$$
\begin{aligned}
\dot{\mathbf{P}}^{[1]}(t) &= \mathbf{P}^{[1]}(t)\mathbf{B}(x_0)\mathbf{R}^{-1}(t)\mathbf{B}^T(x_0)\mathbf{P}^{[1]}(t) \qquad (28)\\
&\quad -\mathbf{Q}(t) - \mathbf{A}^T(x_0)\mathbf{P}^{[1]}(t) - \mathbf{P}^{[1]}(t)\mathbf{A}(x_0),\\
\dot{\mathbf{s}}^{[1]}(t) &= -\Big(\mathbf{A}^T(x_0) - \mathbf{P}^{[1]}(t)\mathbf{B}(x_0)\mathbf{R}^{-1}(t) \qquad (29)\\
&\quad \mathbf{B}^T(x_0)\Big)\mathbf{s}^{[1]}(t) + \mathbf{Q}(t)\mathbf{r}(t),\\
\dot{\mathbf{w}}^{[1]}(t) &= \Big(\mathbf{P}^{[1]}(t)\mathbf{B}(x_0)\mathbf{R}^{-1}(t)\mathbf{B}^T(x_0) \qquad (30)\\
&\quad -\mathbf{A}^T(x_0)\Big)\mathbf{w}^{[1]}(t) - \mathbf{P}^{[1]}(t)\mathbf{z}(t),
\end{aligned}
$$

and the boundary conditions are $\mathbf{P}^{[1]}(t_f) = \mathbf{F}$, $\mathbf{s}^{[1]}(t_f)$=$-\mathbf{Fr}(t_f)$ and $\mathbf{w}^{[1]}(t_f)$=0.
Then the optimal state trajectory is

$$
\begin{aligned}
\dot{\mathbf{x}}^{*[1]}(t) &= \Big(\mathbf{A}(x_0) - \mathbf{B}(x_0)\mathbf{R}^{-1}(t)\mathbf{B}^T(x_0)\mathbf{P}^{[1]}(t)\Big)\\
&\quad \times\Big(\mathbf{x}^{*[1]}(t)\Big) - \mathbf{B}(x_0)\mathbf{R}^{-1}(t)\mathbf{B}^T(x_0)\\
&\quad \times\Big(\mathbf{s}^{[1]}(t) + \mathbf{w}^{[1]}(t)\Big) + \mathbf{z}(t),\\
\mathbf{x}^{*[1]}(t_0) &= \mathbf{x}_0. \qquad (31)
\end{aligned}
$$

From (21), the optimal control for $i$=1 is

$$
\begin{aligned}
\mathbf{u}^{[1]}(t) &= -\mathbf{R}^{-1}(t)\mathbf{B}^T(\mathbf{x}^{[1]}(t)) \times \qquad (32)\\
&\quad \Big(\mathbf{P}^{[1]}(t)\mathbf{x}^{[1]}(t) + \mathbf{s}^{[1]}(t) + \mathbf{w}^{[1]}(t)\Big).
\end{aligned}
$$

For $i{\geq}2$ the linear time-varying equations are:

$$
\begin{aligned}
\dot{\mathbf{P}}^{[i]}(t) &= \mathbf{P}^{[i]}(t)\mathbf{B}(\mathbf{x}^{[i-1]}(t))\mathbf{R}^{-1}(t)\mathbf{B}^T(\mathbf{x}^{[i-1]}(t))\\
&\quad \times\mathbf{P}^{[i]}(t) - \mathbf{Q}(t) - \mathbf{A}^T(\mathbf{x}^{[i-1]}(t))\mathbf{P}^{[i]}(t)\\
&\quad -\mathbf{P}^{[i]}(t)\mathbf{A}(\mathbf{x}^{[i-1]}(t)), \qquad (33)\\
\dot{\mathbf{s}}^{[i]}(t) &= -\Big(\mathbf{A}^T(\mathbf{x}^{[i-1]}(t)) - \mathbf{P}^{[i]}(t)\mathbf{B}(\mathbf{x}^{[i-1]}(t))\\
&\quad \mathbf{R}^{-1}(t)\mathbf{B}^T(\mathbf{x}^{[i-1]}(t))\Big)\mathbf{s}^{[i]}(t) \qquad (34)\\
&\quad +\mathbf{Q}(t)\mathbf{r}(t),\\
\dot{\mathbf{w}}^{[i]}(t) &= \Big(\mathbf{P}^{[i]}(t)\mathbf{B}(\mathbf{x}^{[i-1]}(t))\mathbf{R}^{-1}(t) \qquad (35)\\
&\quad \mathbf{B}^T(\mathbf{x}^{[i-1]}(t)) - \mathbf{A}^T(\mathbf{x}^{[i-1]}(t))\Big)\mathbf{w}^{[i]}(t)\\
&\quad -\mathbf{P}^{[i]}(t)\mathbf{z}(t),
\end{aligned}
$$

with the terminal conditions $\mathbf{P}^{[i]}(t_f) = \mathbf{F}$, $\mathbf{s}^{[i]}(t_f)$=$-\mathbf{Fr}(t_f)$ and $\mathbf{w}^{[i]}(t_f)$=0.
The optimal state trajectory then becomes the limit of the solution of the linear differential equation

$$
\begin{aligned}
\dot{\mathbf{x}}^{*[i]}(t) &= \Big(\mathbf{A}(\mathbf{x}^{[i-1]}(t)) - \mathbf{B}(\mathbf{x}^{[i-1]}(t))\mathbf{R}^{-1}(t) \\
&\quad \mathbf{B}^T(\mathbf{x}^{[i-1]}(t))\mathbf{P}^{[i]}(t)\Big)\mathbf{x}^{*[i]}(t) - \\
&\quad \mathbf{B}(\mathbf{x}^{[i-1]}(t))\mathbf{R}^{-1}(t)\mathbf{B}^T(\mathbf{x}^{[i-1]}(t)) \times \\
&\quad \Big(\mathbf{s}^{[i]}(t) + \mathbf{w}^{[i]}(t)\Big) + \mathbf{z}(t), \\
\mathbf{x}^{*[i]}(t_0) &= \mathbf{x}_0. \qquad (36)
\end{aligned}
$$

The optimal control law is

$$
\begin{aligned}
\mathbf{u}^{[i]}(t) &= -\mathbf{R}^{-1}(t)\mathbf{B}^T(\mathbf{x}^{[i]}(t)) \times \qquad (37) \\
&\quad \Big(\mathbf{P}^{[i]}(t)\mathbf{x}^{[i]}(t) + \mathbf{s}^{[i]}(t) + \mathbf{w}^{[i]}(t)\Big).
\end{aligned}
$$

This controller is the one that stabilizes the nonlinear system (12). That is, when the $i$th trajectory uniformly converges to the desired reference signal $r(t)$, the iteration process is stopped. Then the $i$th controller is replaced into the original nonlinear system in order to stabilize its dynamics. So, in the proposed method, a controller for chaotic dynamics is designed. In the next section an example is shown to explain the procedure.

## 5. EXAMPLE

The Duffing's mechanical oscillator is used as an example because it constitutes one of the best-known models in the studies of chaos, bifurcations and nonlinear oscillations. The dynamics of this oscillator are described by a nonhomogeneous nonlinear differential equation.

The modified Duffing equation employed in (Chen and Dong, 1993*b*) shows complex dynamics, chaos and bifurcations and is described in the following form

$$\ddot{x} + \chi\dot{x} + \psi x + x^3 = \kappa\cos(\omega t), \qquad (38)$$

where $\chi>0$, $\psi$, $\kappa$ and $\omega$ are constants and $t$ is the time variable. Writing (38) into two first-order equations by introducing $y=\dot{x}$, gives

$$
\begin{aligned}
\dot{x} &= y, \qquad (39)\\
\dot{y} &= -\chi y - \psi x - x^3 + \kappa\cos(\omega t).
\end{aligned}
$$

By fixing the parameters $\chi$=0.4, $\psi$=−1.1, $\kappa$=2.1 and $\omega$=1.8, the trajectories display a chaotic behaviour. The purpose is to control a chaotic trajectory of the modified Duffing equation to one of its unstable periodic orbits.

For showing how to apply optimal control to chaos, the quadratic cost functional and a sequence of linear time-varying approximations to the nonlinear problem, explained in the previous section, is used to direct the chaotic dynamics of the modified Duffing equation to one of its period-1 and period-2 orbits.

In order to control the system trajectory to a periodic orbit, a formulation of the latter is required.

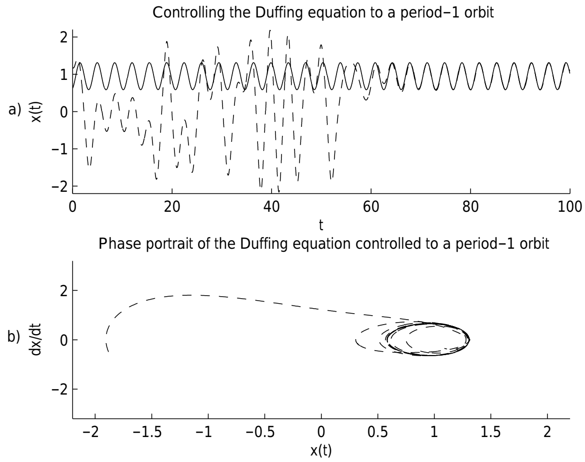


Fig. 1. Controlling the chaotic trajectory to a period-1 orbit. (- -) optimal trajectory, (−) reference signal.

The explicit analytic approximations of the orbits of period-1 and period-2 are found in (Chen and Dong, 1993*b*).

The modified Duffing equation has a period-1 solution with the parameters $\chi$=0.4, $\psi$=−1.1, $\kappa$=0.62 and $\omega$=1.8; and the corresponding explicit analytic approximation is

$$
\begin{aligned}
r_{p_1}(t) &= 0.9488 - 0.3305\cos(1.8t) \qquad (40)\\
&\quad +0.1546\sin(1.8t);
\end{aligned}
$$

and it has a period-2 solution with the parameters $\chi$=0.4, $\psi$=−1.1, $\kappa$=1.46 and $\omega$=1.8, where the corresponding approximation is

$$
\begin{aligned}
r_{p_2}(t) &= 0.7482 - 0.5745\cos(1.8t) \qquad (41)\\
&\quad -0.3270\sin(1.8t) - 1.1870\cos(0.9t)\\
&\quad -0.9810\sin(0.9t).
\end{aligned}
$$

These two periodic orbits are unstable; i.e. all trajectories that are close to them are repelled, unless control is applied. Equations (40) and (41) are used as the reference signal $\mathbf{r}(t)$ that must be tracked by the $\mathbf{x}(t)$ state.

Writing (39) in the form (12), produces

$$
\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{y}}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \psi - x^2(t) & -\chi \end{bmatrix} \begin{bmatrix} \mathbf{x}(\mathbf{t}) \\ \mathbf{y}(\mathbf{t}) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ \kappa\cos(\omega t) \end{bmatrix}. \qquad (42)
$$

The controller $\mathbf{u}(t)$ only affects the $\mathbf{y}(t)$ state. For controlling the modified Duffing equation to a period-1 orbit, the procedure of Section 4 is employed. In order to solve the differential equations, a simple Euler numerical method is used, where the step length is 0.005.

Figure 1 shows the control of the modified Duffing equation to a period-1 orbit. It displays in (*a*)

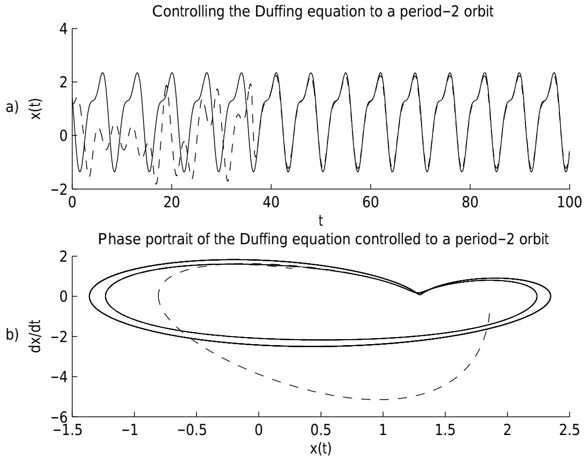


Fig. 2. Controlling the chaotic trajectory to a period-2 orbit. (- -) optimal trajectory, (−) reference signal.

the optimal trajectory after most of the transients have decayed and in ($b$) the phase portrait. The values of each of the weighting matrices are: $\mathbf{F}$=$diag\{1, 1\}$, $\mathbf{Q}$=$diag\{10, 1\}$ and $\mathbf{R}$=0.215. The initial condition of the system for designing the controller is $x_0 = (-1.8816, -0.4896)^T$. Using the procedure, the optimal trajectory is obtained at the $3^{rd}$ iteration. Then the designed controller for the $3^{rd}$ iteration is substituted into the original nonlinear system (42). The controller is switched on at $t$=52.

It can be seen that the state converges to the desired reference signal rapidly and stays there for future time. This ability is one of the advantages of using the approximation technique. Another one is that the produced controller is feasible.

Because of the trade-off between the weighting matrices is, $\mathbf{Q}$ large and $\mathbf{R}$ small; then the optimal system applies a large control effort directing the trajectory to the reference signal quickly.

For controlling to a period-2 orbit, the values of each of the weighting matrices are: $\mathbf{F}$=$diag\{1, 1\}$, $\mathbf{Q}$=$diag\{22, 1\}$ and $\mathbf{R}$=0.1. The initial condition for the design is $x_0 = (1.8559, -0.8482)^T$. The results are shown in Fig. 2, where ($a$) depicts the time-series of the optimal trajectory, which is found at the $3^{rd}$ iteration, that tracks the period-2 orbit and ($b$) displays the phase portrait. The controller is turned on at $t$=22. It shows that it follows the reference signal rapidly due to the compromise between the matrices $\mathbf{Q}$ and $\mathbf{R}$.

This shows how the approximation technique applied to nonlinear optimal control can be employed to stabilize chaotic dynamics by directing the trajectories to one of its period-$n$ orbits.

## 6. CONCLUSIONS

A general technique for studying nonlinear systems is revisited. This technique replaces their defining equations by sequences of linear time-varying systems. This procedure is then applied to optimal control for controlling nonhomogeneous chaotic systems. The stabilization is achieved by sending the chaotic trajectories to one of the period-$n$ orbits of the chaotic system. The advantage of using this method is to avoid the linearization of the nonlinear system. Consequently, the controller can be turned on at any time in order to stabilize the chaotic dynamics.

The implementation of this technique in high-dimensional phase spaces is considered as an open problem for future research.

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