Systems & Control Letters 5 (1985) 327–333
North-Holland

April 1985

On a class of suboptimal controls for infinite-dimensional bilinear systems

S.P. BANKS and M.K. YEW

University of Sheffield, Department of Control Engineering, Mappin Street, Sheffield S1 3JD, U.K.

Received 7 September 1984
Revised 19 November 1984

The suboptimal control of a bilinear system is considered with respect to a quadratic cost criterion. The feedback control is in the space of formal power series on a Hilbert space.

Keywords: Bilinear systems, Suboptimal control, Tensor products on Hilbert space.

1. Introduction

Bilinear systems have been considered very extensively by many authors; see, for example, Brockett [5], Gutman [8], Banks [2]. The main reason for this is that they form the most simple generalization of linear systems. However it can also be shown that any system which is analytic has a bilinear approximation and so bilinear systems do not comprise too restrictive a class of nonlinearities.

Although a great deal of attention has been given to controllability and observability and stabilization of bilinear systems (see for example, Murthy [9], Ball and Slemrod [1] and Grasselli and Isidori [6]) there does not seem to have been much published work on the optimal control of bilinear systems subject to a quadratic cost functional. In this paper we shall consider the bilinear system

$$\dot{x} = Ax + uBx$$

for a scalar control u, where A and B are bounded operators on a separable Hilbert space H, together with the quadratic cost

$$J = \langle x, Gx \rangle + \int_0^{t_f} \{ \langle x, Mx \rangle + ru^2 \} \mathrm{d}t.$$

We shall determine the optimal control in a certain class of controls by extending the linear-quadratic dynamic programming argument. This will require the notion of tensors and tensor operators on H and so in the next section we shall give a brief introduction to these ideas.

The control will turn out to be given by a power series whose tensor coefficients can be determined recursively. When the series is truncated, we obtain a control which is suboptimal in the class of admissible controls.

2. Tensor theory in Hilbert space

In this section we shall give a brief introduction to the theory of tensors on a Hilbert space H. For more details see Greub [7]. First recall that if E and F are vector spaces and G is any vector space then the tensor product of E and F is defined as the pair $(E \otimes F, \otimes)$ ($\otimes$ a bilinear mapping) with the following universal property: if ϕ is a bilinear mapping then there exists a unique linear mapping $f: E \otimes F \to G$ such

0167-6911/85/$3.30 © 1985, Elsevier Science Publishers B.V. (North-Holland)

that the diagram

$$\begin{array}{ccc} E \times F & \xrightarrow{\phi} & G \\ \otimes \downarrow & \nearrow_{f} & \\ E \otimes F & & \end{array}$$

commutes. By induction we can define the tensor product of i copies of H, i.e. $H \otimes \cdots \otimes H$, which we shall denote by $\otimes_i H$. We can make $\otimes_i H$ into a Hilbert space by defining for x_j, $y_j \in H$,

$$\langle x_1 \otimes \cdots \otimes x_i,\ y_1 \otimes \cdots \otimes y_i \rangle_{\otimes_i H} = \prod_{j=1}^{i} \langle x_j, y_j \rangle_H, \tag{2.1}$$

and extending by linearity.

It is convenient to consider the space $H = \oplus_{i=1}^{\infty}(\otimes_i H)$ as a graded Hilbert space.

Let $\{e_k\}_{k \geqslant 1}$ be an orthonormal basis of H which will be fixed throughout the discussion. Then $\{e_{k_1} \otimes \cdots \otimes e_{k_i}\}$ $(1 \leqslant k_j < \infty,\ 1 \leqslant j \leqslant i)$ is an orthonormal basis of $\otimes_i H$ and so any tensor $\Xi \in \otimes_i H$ can be written in the form

$$\Xi = \sum_{k_1=1}^{\infty} \cdots \sum_{k_i=1}^{\infty} \xi_{k_1 \cdots k_i}(e_{k_1} \otimes \cdots \otimes e_{k_i}).$$

Since $\otimes_i H$ is a Hilbert space we can consider linear operators defined on $\otimes_i H$. The space of all bounded linear operators on a space X will be denoted by $\mathscr{L}(X)$. Let $P \in \mathscr{L}(\otimes_i H)$. Then the matrix representation of P with respect to the above basis of $\otimes_i H$ will be written $P_{k_1 \cdots k_i}^{l_1 \cdots l_i}$, i.e.

$$P(e_{k_1} \otimes \cdots \otimes e_{k_i}) = \sum_{\substack{l_j=1 \\ (j=1,\dots,i)}}^{\infty} P_{k_1 \cdots k_i}^{l_1 \cdots l_i}(e_{l_1} \otimes \dots \otimes e_{l_i}).$$

(Since $\otimes_i H$ is a 'flat' space writing indices contra- or co-variantly makes no difference.)

The dual or adjoint P^* of $P \in \mathscr{L}(\otimes_i H)$ is defined in the usual way:

$$\langle P^*(x_1 \otimes \cdots \otimes x_i), (y_1 \otimes \cdots \otimes y_i)\rangle = \langle (x_1 \otimes \cdots \otimes x_i), P(y_1 \otimes \cdots \otimes y_i)\rangle$$

for all x_j, $y_j \in H$. Clearly, P is self-adjoint if

$$P_{k_1 \cdots k_i}^{l_1 \cdots l_i} = P_{l_1 \cdots l_i}^{k_1 \cdots k_i},$$

and such an operator P will be said to be symmetric. (This should not be confused with the usual definition of symmetric tensor.)

3. Optimal control of bilinear systems

We shall consider the bilinear system

$$\dot{x} = Ax + uBx \tag{3.1}$$

where $x \in H$ (a separable Hilbert space) and u is a scalar control (the latter assumption being purely for notational convenience – the general case presents no further difficulties). However, we shall assume here, for simplicity, that A and B are bounded operators. The generalisation to unbounded operators will be considered in a future paper. We shall determine the control u which minimises the quadratic cost functional

$$J = \langle x, Gx\rangle + \int_0^{t_f}\{\langle x, Mx\rangle + ru^2\}\mathrm{d}t \tag{3.2}$$

for controls which belong to a certain class, to be introduced shortly. In (3.2), G and M are nonnegative definite bounded linear operators on H and $r > 0$.

If $V(t, x)$ denotes the usual value function, then the dynamic programming equation for V is

$$\langle x, Mx\rangle + V_t + (\mathscr{F}_x V)Ax + \min_u \left(ru^2 + (\mathscr{F}_x V)Bxu \right) = 0 \tag{3.3}$$

where $\mathscr{F}_x V$ is the Fréchet derivative of V (which we assume for the moment exists). Now, as in the linear-quadratic regular problem, if $c = (\mathscr{F}_x V)Bx$, then

$$ru^2 + cu = \left(u + \tfrac{1}{2}r^{-1}c\right)^2 r - \tfrac{1}{4}c^2 r^{-1}$$

and so the minimum is attained when $u = -\frac{1}{2}r^{-1}c$. Then (3.3) becomes

$$V_t + \langle x, Mx\rangle + (\mathscr{F}_x V)Ax - \tfrac{1}{4}\langle (\mathscr{F}_x V)Bx, r^{-1}(\mathscr{F}_x V)Bx\rangle = 0. \tag{3.4}$$

Now let $\Phi = \mathbb{R}^e[[x]]$ denote the ring of formal power series in the indeterminate x $(\in H)$ which have only even order powers; i.e. we may write, for any $\phi \in \Phi$,

$$\phi = \sum_{i=1}^{\infty} \langle \otimes_i x, \phi_i \otimes_i x\rangle_{\otimes, H} \tag{3.5}$$

where $\otimes_i H$ is the tensor product of i copies of H, $\otimes_i x = x \otimes x \cdots \otimes x$ (i times) and $\phi_i \in \mathscr{L}(\otimes_i H)$. (Recall that the inner product on $\otimes_i H$ is given by (2.1).)

We shall need the following lemma, whose proof is trivial:

Lemma 3.1. *Let* $P \in \mathscr{L}(\otimes_i H)$, $Q \in \mathscr{L}(\otimes_j H)$. *Then*

$$[\mathscr{F}_x\langle \otimes_i x, P \otimes_i x\rangle]x = 2i\langle \otimes_i x, P \otimes_i x\rangle, \tag{3.6}$$

(*if* P *is symmetric*) *and* [1]

$$\langle \otimes_i x, P \otimes_i x\rangle\langle \otimes_j x, Q \otimes_j x\rangle = \langle \otimes_{i+j} x, (P \otimes Q) \otimes_{i+j} x\rangle. \tag{3.7}$$

Moreover, we have

$$\|P \otimes Q\|_{\mathscr{L}(\otimes_{i+j} H)} \leqslant \|P\|_{\mathscr{L}(\otimes_i H)} \|Q\|_{\mathscr{L}(\otimes_j H)}. \qquad \square \tag{3.8}$$

It follows from this lemma that, for a bounded operator $C \in \mathscr{L}(H)$, we have for any symmetric P,

$$\mathscr{F}_x\langle \otimes_i x, P \otimes_i x\rangle Cx = 2\langle \otimes_i x, (PC) \otimes_i x\rangle \tag{3.9}$$

where PC $(\in \otimes_i H)$ is defined by

$$PC = \sum_{j=1}^{i} \left(\sum_{k_j=1}^{\infty} P^{\kappa_1 \cdots \kappa_i}_{k_1 \cdots k_j \cdots k_i} C_{k_j l_1} \right)_{(1 \leqslant l_1 < \infty, 1 \leqslant k_1 < \infty, \dots, 1 \leqslant k_i < \infty, 1 \leqslant \kappa_\alpha < \infty)}$$

and $P^{\kappa_1 \cdots \kappa_i}_{k_1 \cdots k_i}$, $C_{k_j l_1}$ are the components of the tensors P, C with respect to some (fixed) orthonormal basis of H. A similar definition can be given for CP. Also, it is clear that

$$\begin{aligned}\|(PC) \otimes_i x\|_{\otimes_i H} &= \|P(Cx \otimes x \cdots \otimes x) + P(x \otimes Cx \cdots \otimes x) + \cdots + P(x \otimes \cdots \otimes Cx)\|_{\otimes_i H} \\ &\leqslant i\|P\|_{\mathscr{L}(\otimes_i H)} \|Cx\|_H \|x\|_H^{i-1}\end{aligned}$$

[1] Note that $P \otimes Q$ is defined by $(P \otimes Q)(\xi \otimes \eta) = P\xi \otimes Q\eta$, where $\xi \in \otimes_i H$, $\eta \in \otimes_j H$.

and so

$$\|PC\|_{\mathscr{L}(\otimes_i H)} \leqslant i\|P\|_{\mathscr{L}(\otimes_i H)}\|C\|_{\mathscr{L}(H)}. \tag{3.10}$$

Now let $V=\langle \otimes_i x,\ P \otimes_i x\rangle$ and substitute V into (3.4):

$$\sum_{i=1}^{\infty}\langle \otimes_i x,\ \dot{P}_i(t) \otimes_i x\rangle_{\otimes_i H} + \sum_{i=1}^{\infty} 2\langle \otimes_i x,\ (P_iA) \otimes_i x\rangle_{\otimes_i H} + \langle x,\ Mx\rangle_H$$
$$-\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} r^{-1}\langle \otimes_{i+j}x,\ (P_iB \otimes P_jB) \otimes_{i+j}x\rangle_{\otimes_{i+j}H} = 0 \tag{3.11}$$

with the final conditions $P_1(t_f)=G$, $P_i(t_f)=0$, $i>1$ (using (3.7), (3.9)). Equating like 'powers' in x in (3.11) we obtain the equations

$$\begin{aligned}&\dot{P}_1(t)+P_1(t)A+A^{\mathrm{T}}P_1(t)+M=0, \quad P_1(t_f)=G,\\ &\dot{P}_m(t)+P_m(t)A+A^{\mathrm{T}}P_m(t)-r^{-1}\sum_{\substack{i+j=m\\ i,j\geqslant 1}}\{P_iB \otimes P_jB\}=0, \quad P_m(t_f)=0,\end{aligned} \tag{3.12}$$

for $m>1$. Note that the latter equation can also be written in the form

$$\dot{P}_m(t)+P_m(t)A+A^{\mathrm{T}}P_m(t)-\frac{1}{2r}\left\{\sum_{\substack{i+j=m\\ i,j\geqslant 1}}\left(B^{\mathrm{T}}P_i \otimes B^{\mathrm{T}}P_j+P_iB \otimes P_jB\right)\right\}=0$$

since clearly $(P_iB \otimes P_jB)^{\mathrm{T}}=B^{\mathrm{T}}P_i \otimes B^{\mathrm{T}}P_j$, so that P_i is indeed symmetric.

Consider the operators $\mathscr{A}_i$ defined on the Banach spaces $\mathscr{L}(\otimes_i H)$ by [2]

$$\mathscr{A}_iP_i=P_iA, \quad P_i\in\mathscr{L}(\otimes_i H), \qquad i\geqslant 1,$$

where P_iA is defined as in (3.9). Then $\mathscr{A}_i$ is clearly a bounded operator and $\|\mathscr{A}_iP_i\| \leqslant i\|P_i\|\cdot\|A\|$ by (3.10), whence

$$\|\mathscr{A}_i\|_{\mathscr{L}(\mathscr{L}(\otimes_i H))} \leqslant i\|A\|_{\mathscr{L}(H)}.$$

Hence we can define the operator $\mathrm{e}^{\mathscr{A}_it}\in\mathscr{L}(\mathscr{L}(\otimes_i H))$ and the solution of (3.12(1)) is then

$$P_1(t)=\mathrm{e}^{\mathscr{A}_1(t_f-t)}G\,\mathrm{e}^{\mathscr{A}_1^{\mathrm{T}}(t_f-t)}+\int_0^{t_f-t}\mathrm{e}^{\mathscr{A}_1(t_f-t-s)}M\,\mathrm{e}^{\mathscr{A}_1^{\mathrm{T}}(t_f-t-s)}\mathrm{d}s. \tag{3.13}$$

Similarly, from (3.12(m)), we have

$$P_m(t)=-r^{-1}\sum_{\substack{i+j=m\\ i,j\geqslant 1}}\int_0^{t_f-t}\mathrm{e}^{\mathscr{A}_m(t_f-t-s)}P_i(t_f-s)B \otimes P_j(t_f-s)B\,\mathrm{e}^{\mathscr{A}_m^{\mathrm{T}}(t_f-t-s)}\mathrm{d}s. \tag{3.14}$$

The optimal control is then formally

$$u(t)=-\tfrac{1}{2}r^{-1}(\mathscr{F}_xV)Bx=-r^{-1}\sum_{i=1}^{\infty}\langle \otimes_i x,\ (P_iB) \otimes_i x\rangle. \tag{3.15}$$

[2] The operator $\mathscr{A}_i$ is well defined. It is a simple generalisation of the operator $\mathscr{B}$ defined on the space of bounded operators $\mathscr{L}(H)$ as follows: If $B\in\mathscr{L}(H)$ put $\mathscr{B}A=AB$, $A\in\mathscr{L}(H)$.

However this series may not converge and so we propose the following suboptimal controls:

$$u_m(t) = -r^{-1} \sum_{i=1}^{m} \langle \otimes_i x, (P_i B) \otimes_i x \rangle. \tag{3.16}$$

These controls have been shown to be effective for finite-dimensional systems (see Banks and Yew [4]), for example in stabilising unstable bilinear control systems.

4. Examples

(i) As a simple example of the above theory we shall consider the system

$$\dot{x} = Ax + ux, \quad x \in \ell^2, \tag{4.1}$$

where $B = I$ and A is the left shift operator. This is not too restrictive on the operator A since any bounded operator has the left shift as a model on some Hilbert space (see Rota [10] and Banks and Abbasi-Ghelmansarai [3]). Recall that the left shift operator has the matrix representation

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & \\ \vdots & & & & & \end{pmatrix}$$

on ℓ^2. Before considering this particular system let us examine the operator $e^{\mathscr{A}_m t}$ in more detail. Recall that $\mathscr{A}_m$ is defined on $\mathscr{L}(\otimes_m H)$ by

$$\mathscr{A}_m P = PA, \quad P \in \mathscr{L}(\otimes_m H),$$

where

$$PA = \sum_{j=1}^{m} \left\{ \sum_{k_j=1}^{\infty} \left(P_{k_1 \cdots k_j \cdots k_m}^{\kappa_1 \cdots \kappa_m} A_{k_j l_1} \right) \right\}_{(1 \leqslant l_1 < \infty, 1 \leqslant k_1 < \infty, \ldots, 1 \leqslant k_i < \infty, 1 \leqslant \kappa_n < \infty)}$$

Write

$$\mathscr{A}_m^j P = \sum_{k_j=1}^{\infty} P_{k_1 \cdots k_j \cdots k_m}^{\kappa_1 \cdots \kappa_m} A_{k_j l_1}. \tag{4.2}$$

Then $\mathscr{A}_m P = (\Sigma \mathscr{A}_m^j) P$. Note that $\mathscr{A}_m^j$, $\mathscr{A}_m^k$ commute for all j, k and so $e^{\mathscr{A}_m t} = e^{\mathscr{A}_m^1 t} e^{\mathscr{A}_m^2 t} \cdots e^{\mathscr{A}_m^m t}$. Also,

$$e^{\mathscr{A}_j t} = \sum_{k_j=1}^{\infty} P_{k_1 \cdots k_j \cdots k_m}^{\kappa_1 \cdots \kappa_m} (e^{At})_{k_j l_1}$$

and it is easy to see that

$$e^{\mathscr{A}_m t} P = P(e^{At} \otimes \cdots \otimes e^{At}) \quad (m \text{ times}). \tag{4.3}$$

From (3.14) with $B = I$ we have

$$\begin{aligned} e^{\mathscr{A}_m t} P_m(t)\, e^{\mathscr{A}_m^{\mathrm{T}} t} &= -r^{-1} \sum_{\substack{i+j=m \\ i,j \geqslant 1}} \int_0^{t_f - t} e^{\mathscr{A}_m (t_f - s)} P_i(t_f - s) \otimes P_j(t_f - s)\, e^{\mathscr{A}_m^{\mathrm{T}} (t_f - s)} \mathrm{d}s \\ &= -r^{-1} \sum_{\substack{i+j=m \\ i,j \geqslant 1}} \int_0^{t_f - t} \left\{ e^{\mathscr{A}_i (t_f - s)} P_i(t_f - s)\, e^{\mathscr{A}_i^{\mathrm{T}} (t_f - s)} \right\} \otimes \left\{ e^{\mathscr{A}_j (t_f - s)} P_j(t_f - s)\, e^{\mathscr{A}_j^{\mathrm{T}} (t_f - s)} \right\} \mathrm{d}s \end{aligned}$$

where the latter equality follows from (4.3), and the fact that $(A \otimes B)\cdot(P \otimes Q) = (AP \otimes BQ)$ for any operators A, B. Hence, writing $Q_m = e^{\mathscr{A}_m t} P_m(t)\, e^{\mathscr{A}_m^{\mathrm{T}} t}$ we have

$$Q_m(t) = -r^{-1} \sum_{\substack{i+j=m \\ i,j \geqslant 1}} \int_0^{t_f - t} Q_i(t_f - s) \otimes Q_j(t_f - s)\,\mathrm{d}s,$$
$$Q_1(t) = e^{\mathscr{A}_1 t_f} G\, e^{\mathscr{A}_1^{\mathrm{T}} t_f} + \int_0^{t_f - t} e^{\mathscr{A}_1 (t_f - s)} M\, e^{\mathscr{A}_1^{\mathrm{T}}(t_f - s)}\mathrm{d}s. \tag{4.4}$$

Now, if A is the left shift operator, then

$$e^{At} = \begin{pmatrix} 1 & t & t^2/2! & t^3/3! & t^4/4! & \dots \\ 0 & 1 & t & t^2/2! & t^3/3! & \\ 0 & 0 & 1 & t & t^2/2! & \\ \vdots & & & & & \end{pmatrix}.$$

Equations (4.4) can be solved recursively for Q_m. Of course we must terminate at some finite value of M and thus obtain a suboptimal control. If $G = M = I$, we clearly have, for example,

$$Q_{1,i}^{j}(t) = \sum_{n=0}^{\infty} \frac{t_f^{2n+|i-j|}}{(n+|i-j|!)n!} + \sum_{n=0}^{\infty} \frac{(t_f - t)^{2n+|i-j|+1}}{(2n+|i-j|+1)(n+|i-j|!)n!}$$

for the (i, j)-th component of Q_1 when A is the left shift operator.

(ii) Consider the system defined on $L^2[0, \infty]$ by the equation

$$\dot{\phi}(t, x) = \int_0^x k(x-y)\phi(t, y)\mathrm{d}y + u\phi(t, x)$$

where k is some Laplace-transformable function with $k(x) = 0$, $x < 0$. This is of the form

$$\dot{\Phi}(t) = A\Phi(t) + u\Phi(t), \quad \Phi(t) \in L^2[0, \infty],$$

where

$$\Phi(t)(x) = \phi(t, x) \quad \text{and} \quad (A\Phi(t))(x) = \int_0^x k(x-y)\phi(t, y)\mathrm{d}y.$$

It is obvious that e^{At} is given by

$$(e^{At}\Phi)(x) = \{(\mathscr{L}^{-1}\, e^{K(s)t}) * \Phi\}(x) = (E(t) * \Phi)(x),$$

say, where $\mathscr{L}$ denotes the Laplace operator and $*$ denotes convolution. Hence, by (4.4), we have

$$Q_1(t) = E(t_f) * (E(t_f) * (\cdot)) + \int_0^{t_f - t} E(t_f - s) * E(t_f - s) * (\cdot)\mathrm{d}s.$$

(Note that $A^* = A$.)

5. Conclusions

In this paper we have derived a class of suboptimal controls for a bilinear system subject to a quadratic cost criterion. The control is a nonlinear feedback which is a power series in the state and can be calculated recursively. We have considered the case of bounded operators here – the unbounded case will be considered in a future paper.

References

[1] J.M. Ball and M. Slemrod, Feedback stabilization of distributed semilinear control systems, *Appl. Math. Optim.* **5** (1979) 169–179.

[2] S.P. Banks, On the generation of infinite dimensional bilinear systems on Volterra series, *Internat. J. Systems Sci.*, to appear.

[3] S.P. Banks and F. Abbasi-Ghelmansarai, Realization theory and the infinite-dimensional root locus, *Internat. J. Control* **38** (1983) 589–606.

[4] S.P. Banks and M.K. Yew, Suboptimal control of bilinear systems – some examples, in preparation (1984).

[5] R.W. Brockett, Volterra series and geometric control theory, *Automatica* **12** (1976) 167–176.

[6] O.M. Grasselli and A. Isidori, An existence theorem for observers of bilinear systems, *IEEE Trans. Automat. Control* **26** (1981) 1299–1300.

[7] W. Greub, *Multilinear Algebra*, 2nd Ed. (Springer, Berlin-New York, 1978).

[8] P.O. Gutman, Stabilizing controllers for bilinear systems, *IEEE Trans. Automat. Control* **26** (1981) 917–922.

[9] D.N.P. Murthy, Controllability of a discrete-time bilinear system, *IEEE Trans. Automat. Control* **24**, 974–975.

[10] G.-C. Rota, Universal models for linear operators, *Comm. Pure Appl. Math.* **13** (1960) 469–472.