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Robust H^∞ control of uncertain nonlinear dynamical systems via linear time-varying approximations

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Abstract

In this paper, a novel mathematical technique is given for treating nonlinear systems in the presence of unstructured disturbances. A result is presented which guarantees asymptotic robust stabilization of the feedback system. The H^∞ feedback control law is obtained via direct application of the "Iteration Technique" that was considered for nonlinear optimal control by Banks and Dinesh (Ann. Oper. Res. 98(7) (2000) 19–44).

Keywords: Affine; Riccati equations; Disturbance; Feedback control; H^∞ control; Nonlinear control systems; Robust stability; State-space; Time varying plants; Uncertainty

1. Introduction

The H^∞ control problem was originally introduced in the frequency domain by Zames [11]. However, the more recent time-domain version has opened the way for nonlinear H^∞ designs (Refer to [9]). Consequently, stabilization of uncertain nonlinear time-varying plants has received the attention of a lot of theoreticians and engineers in this field over the last few decades [5,9,8].

Here a novel approach is given for devising control action for nonlinear uncertain systems. The novelty can be viewed as an extension to a simple and very effective method, originally introduced for optimal control by Banks and Dinesh [2] and subsequently revisited by Banks

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[1], called the "Iteration Technique". These sequences of approximations were shown to converge to the solution of the nonlinear plant on any compact time interval (see [10,4]). As such, these sets of decoupled linear equations enable the usage of standard linear control methods to robustly stabilize the nonlinear disturbed system.

Firstly, a problem statement is presented in Section 2. In Section 3, an H^∞ controller is derived, by solving the Riccati Operator Equation in the Hilbert Space, for controlling disturbed linear time-varying systems. A theorem is presented that ensures stabilization of this class of linear systems. Section 4 extends the previously stated theory to include non-linear disturbed dynamical plants by means of the "Iteration Technique"; it also includes an expanded theorem that robustly stabilizes the plant. A practical example is being considered in Section 5 followed by some simulations in Section 6. Finally some recommendations and some open questions that the reader might find of significance are given in Section 7.

2. Problem statement

Consider a nonlinear uncertain system having the following form:

$$\dot{x}(t) = A(x(t))x(t) + B(x(t))u(t) + E(x(t))w(t), \quad x(0) = x_0 \qquad \forall t \in \Re^+,$$

$$y(t) = C(x(t))x(t) + D(x(t))u(t).$$

where $x(t) \in X$ is the state, $u(t) \in U$ is the control, $w(t) \in W$ is the input disturbance and $y(t) \in Y$ is the observation output; with X, U, Y & W being real Hilbert spaces.

Then the H^∞ control problem is as follows:

Given a scalar $\gamma > 0$, find a linear feedback control law $u(t) = -B^*(t)P(t)x(t)$ such that:

1. the given disturbed nonlinear system is robustly stabilizable;
2. there exists a scalar $c_0 > 0$ such that

$$\sup \frac{\int_0^\infty \|y(t)\|^2 \, \mathrm{d}t}{c_0\|x_0\|^2 + \int_0^\infty \|w(t)\|^2 \, \mathrm{d}t} \leqslant \gamma$$

with the supremum taken over all $x_0 \in X$ and all non-zero admissible disturbances $w(t)$.

3. Linear systems

In this section an idea introduced by Phat [8] for semi-linear systems is being extended to design an H^∞ controller in Hilbert space for linear systems.

Initially consider the following dynamical linear system in the state-space representation:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) + E(t)w(t), \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned} \tag{1}$$

with unstructured disturbance $w(t)$ defined by $w \in L^2([0, \infty), W)$ denoting the set of all L^2 integrable and W-valued functions on $[0, t]$.

Assumption 3.1. The functions $B(.)u$, $E(.)w$, $C(.)x$ and $D(.)u$ are bounded and defined by $b=\sup_{t\in\Re^+}\|B(t)\|$, $e=\sup_{t\in\Re^+}\|E(t)\|$, $c=\sup_{t\in\Re^+}\|C(t)\|$ and $d=\sup_{t\in\Re^+}\|D(t)\|$ $\{\exists d, e, c, d>0\}$.

The feedback control law is

$$u(t) = -B^*(t)P(t)x(t) \tag{2}$$

with the operator $(.)^*$ referring to the adjoint.

Now consider the Riccati operator equation (ROE):

$$\dot{P}(t) = -A^*(t)P(t) - P(t)A(t) + P(t)B(t)B^*(t)P(t) - Q(t)$$

with $P(t)$ being the solution of the ROE defined by $p=\sup_{t\in\Re^+}\|P(t)\|$; and where

$$Q(t) = C^*(t)C(t) + I. \tag{3}$$

Let

$$V = \langle P(t)x(t), x(t)\rangle, \tag{4}$$

with the inner product being defined over a complex or real field F as a map

$$\langle .,.\rangle : X \times X \to F.$$

Differentiating (3),

$$\dot{V}(t,x(t)) = \langle \dot{P}(t)x(t), x(t)\rangle + 2\langle P(t)\dot{x}(t), x(t)\rangle,$$

$$\begin{aligned}\dot{V}(t,x(t)) = {}& \langle(-A^*(t)P(t) - P(t)A(t) + P(t)B(t)B^*(t)P(t) - Q(t))x(t), x(t)\rangle\\ &+ 2\langle P(t)(A(t)x(t) + B(t)u(t) + E(t)w(t)), x(t)\rangle.\end{aligned} \tag{5}$$

And so from (5)

$$\begin{aligned}\dot{V}(t,x(t)) = {}& \langle -A^*(t)P(t)x(t) - P(t)A(t)x(t) + P(t)B(t)B^*(t)P(t)x(t)\\ &- Q(t)x(t), x(t)\rangle + 2\langle P(t)A(t)x(t) + P(t)B(t)u(t)\\ &+ P(t)E(t)w(t), x(t)\rangle.\end{aligned} \tag{6}$$

By substituting the control law (2) in (6) we obtain,

$$\begin{aligned}\dot{V}(t,x(t)) = {}& \langle -A^*(t)P(t)x(t) - P(t)A(t)x(t) + P(t)B(t)B^*(t)P(t)x(t)\\ &- Q(t)x(t), x(t)\rangle + \langle 2P(t)A(t)x(t) - 2P(t)B(t)B^*(t)P(t)x(t)\\ &+ 2P(t)E(t)w(t), x(t)\rangle\end{aligned}$$

$$\begin{aligned}\dot{V}(t,x(t)) = {}& \langle -(A^*(t)P(t) + P(t)A(t))x(t), x(t)\rangle + \langle P(t)B(t)B^*(t)P(t)x(t), x(t)\rangle\\ &- \langle Q(t)x(t), x(t)\rangle + \langle 2P(t)A(t)x(t), x(t)\rangle\\ &- \langle 2P(t)B(t)B^*(t)P(t)x(t), x(t)\rangle\\ &+ \langle 2P(t)E(t)w(t), x(t)\rangle.\end{aligned} \tag{7}$$

However, $P(t)=P^*(t)$ and consequently (7) becomes

$$\begin{aligned}\dot V(t,x(t)) = & -\langle (C^*(t)C(t)+I)x(t),x(t)\rangle - \langle P(t)B(t)B^*(t)P(t)x(t),x(t)\rangle \\ & + \langle 2P(t)E(t)w(t),x(t)\rangle,\end{aligned}$$

and by substituting (3) we have that

$$\begin{aligned}\dot V(t,x(t)) = & -\langle Ix(t),x(t)\rangle - \langle P(t)B(t)B^*(t)P(t)x(t),x(t)\rangle \\ & - \langle C^*(t)C(t)x(t),x(t)\rangle + \langle 2P(t)E(t)w(t),x(t)\rangle. \end{aligned} \tag{8}$$

By (8)

$$\begin{aligned}\dot V(t,x(t)) = & -\|x(t)\|^2 - \langle P(t)B(t)B^*(t)P(t)x(t),x(t)\rangle - \langle C^*(t)C(t)x(t),x(t)\rangle \\ & + \langle 2P(t)E(t)w(t),x(t)\rangle. \end{aligned} \tag{9}$$

But

$$\langle C^*(t)C(t)x(t),x(t)\rangle = \langle C(t)x(t),C(t)x(t)\rangle = \|C(t)x(t)\|^2 \geqslant 0,$$

and

$$\begin{aligned}\langle P(t)B(t)B^*(t)P(t)x(t),x(t)\rangle &= \langle B^*(t)P(t)x(t),B^*(t)P^*(t)x(t)\rangle \\ &= \|B^*(t)P(t)x(t)\|^2 \geqslant 0.\end{aligned} \tag{10}$$

From (9) and (10) we have that

$$\dot V(t,x(t)) \leqslant -\|x(t)\|^2 + \langle 2P(t)E(t)w(t),x(t)\rangle,$$

$$\dot V(t,x(t)) \leqslant -\|x(t)\|^2 + 2\|P(t)E(t)w(t)\|\|x(t)\|,$$

$$\dot V(t,x(t)) \leqslant -\|x(t)\|^2 + 2\|P(t)\|\|E(t)\|\|w(t)\|\|x(t)\|.$$

Conversely,

$$\dot V(t,x(t)) \leqslant -\|x(t)\|^2 + 2pe\|w(t)\|\|x(t)\|. \tag{11}$$

Integrating both sides of (11) from 0 to t, yields

$$\int_0^t \dot V(s,x(s)\,\mathrm{d}s \leqslant \int_0^t -\|x(s)\|^2\,\mathrm{d}s + \int_0^t 2pe\|w(s)\|\|x(s)\|\,\mathrm{d}s,$$

or

$$\begin{aligned}\langle P(t)x(t),x(t)\rangle - \langle P(0)x(0),x(0)\rangle \leqslant & -\delta_1\int_0^t \|x(s)\|^2\,\mathrm{d}s \\ & + 2\delta_2\int_0^t \|w(s)\|\|x(s)\|\,\mathrm{d}s;\end{aligned} \tag{12}$$

with $\delta_1=1$ and $\delta_2=pe$.

Note that

$$\int_0^t \|w(s)\|\|x(s)\|\,\mathrm{d}s \leqslant \left\{\int_0^t \|w(s)\|^2\,\mathrm{d}s\right\}^{1/2}\left\{\int_0^t \|x(s)\|^2\,\mathrm{d}s\right\}^{1/2}.$$

Given that $w \in L^2([0,\infty), W)$, then by rearranging (12) we obtain

$$\int_0^t \|x(s)\|^2\,\mathrm{d}s \leqslant \frac{2\delta_2}{\delta 1}\int_0^t \|w(s)\|\|x(s)\|\,\mathrm{d}s - \frac{1}{\delta_1}\langle P(t)x(t), x(t)\rangle + \frac{1}{\delta_1}\langle P(0)x(0), x(0)\rangle. \tag{13}$$

Defining $\delta_3 = 1/\delta_1\langle P(0)x(0), x(0)\rangle$, and $\delta_4 = \delta_2/\delta_1\{\int_0^\infty \|w(s)\|^2\,\mathrm{d}s\}^{1/2}$; subsequently, (13) can be rewritten as

$$\int_0^t \|x(s)\|^2\,\mathrm{d}s \leqslant \delta_3 + 2\delta_4\left\{\int_0^t \|x(s)\|^2\,\mathrm{d}s\right\}^{1/2} - \frac{1}{\delta_1}\langle P(t)x(t), x(t)\rangle. \tag{14}$$

Now set

$$\alpha = \left\{\int_0^t \|x(s)\|^2\,\mathrm{d}s\right\}^{1/2}. \tag{15}$$

So from (14) & (15)

$$\alpha^2 \leqslant \delta_3 + 2\delta_4\alpha - \frac{1}{\delta_1}\langle P(t)x(t), x(t)\rangle. \tag{16}$$

However,

$$\langle P(t)x(t), x(t)\rangle = \langle\sqrt{P(t)}x(t), \sqrt{P(t)}x(t)\rangle = \|\sqrt{P(t)}x(t)\|^2 \geqslant 0.$$

and (16) reduces to $\alpha^2 - 2\delta_4\alpha \leqslant \delta_3$. By completing the square $(\alpha-\delta_4)^2 - \delta_4^2 \leqslant \delta_3$, equivalently

$$\alpha^2 \leqslant \delta_4 + \sqrt{\delta_3 + \delta_4^2}. \tag{17}$$

From (15) in (17)

$$\int_0^t \|x(s)\|^2\,\mathrm{d}s \leqslant \delta_4 + \sqrt{\delta_3 + \delta_4^2} \quad (\forall t \in \Re^+). \tag{18}$$

Now consider the following relation:

$$\int_0^\infty [\|y(t)\|^2 - \gamma\|w(t)\|^2]\,\mathrm{d}t = \int_0^\infty [\|y(t)\|^2 - \gamma\|w(t)\|^2 + \dot{V}(t, x(t))]\,\mathrm{d}t - \int_0^\infty \dot{V}(t, x(t))\,\mathrm{d}t.$$

Equally,

$$\int_0^\infty [\|y(t)\|^2 - \gamma\|w(t)\|^2]\,\mathrm{d}t = \int_0^\infty [\|y(t)\|^2 - \gamma\|w(t)\|^2 + \dot{V}(t, x(t))]\,\mathrm{d}t - \langle P(t)x(t), x(t)\rangle + \langle P(0)x(0), x(0)\rangle. \tag{19}$$

Since the initial condition $P(0)$ is chosen such that $P(0) \neq 0$, (19) can be written as

$$\int_0^\infty [\|y(t)\|^2 - \gamma\|w(t)\|^2]\,\mathrm{d}t \leqslant \int_0^\infty [\|y(t)\|^2 - \gamma\|w(t)\|^2 + \dot{V}(t, x(t))]\,\mathrm{d}t + \langle P(0)x(0), x(0)\rangle. \tag{20}$$

Recall that the closed-loop state-space representation of the output is

$$y(t) = C(t)x(t) - D(t)B^*(t)P(t)x(t).$$

Therefore

$$\|y(t)\|^2 = \langle C(t)x(t) - D(t)B^*(t)P(t)x(t), C(t)x(t) - D(t)B^*(t)P(t)x(t)\rangle$$

$$\begin{aligned}\|y(t)\|^2 = {} & \langle C(t)x(t), C(t)x(t)\rangle - \langle C(t)x(t), D(t)B^*(t)P(t)x(t)\rangle \\ & - \langle D(t)B^*(t)P(t)x(t), C(t)x(t)\rangle + \langle D(t)B^*(t)P(t)x(t), \\ & D(t)B^*(t)P(t)x(t)\rangle,\end{aligned}$$

$$\begin{aligned}\|y(t)\|^2 = {} & \langle C^*(t)C(t)x(t), x(t)\rangle - \langle x(t), C^*(t)D(t)B^*(t)P(t)x(t)\rangle \\ & - \langle C^*(t)D(t)B^*(t)P(t)x(t), x(t)\rangle \\ & + \langle P(t)B(t)D^*(t)D(t)B^*(t)P(t)x(t), x(t)\rangle.\end{aligned} \tag{21}$$

Assuming that

$$C^*(t)D(t) = 0 \quad \text{and} \quad D^*(t)D(t) = I \quad \forall t \geqslant 0$$

as it is common practice in modern control (see for example [3,6–8]), so Eq. (21) reduces to

$$\|y(t)\|^2 = \langle C^*(t)C(t)x(t), x(t)\rangle + \langle P(t)B(t)B^*(t)P(t)x(t), x(t)\rangle. \tag{22}$$

Substituting (22) and (8) in (20)

$$\int_0^\infty [\|y(t)\|^2 - \gamma\|w(t)\|^2]\,\mathrm{d}t \leqslant \int_0^\infty \begin{bmatrix} \langle C^*(t)C(t)x(t), x(t)\rangle \\ +\langle P(t)B(t)B^*(t)P(t)x(t), x(t)\rangle \\ -\gamma\langle w(t), w(t)\rangle - \langle Ix(t), x(t)\rangle \\ -\langle P(t)B(t)B^*(t)P(t)x(t), x(t)\rangle \\ -\langle C^*(t)C(t)x(t), x(t)\rangle \\ +\langle 2P(t)E(t)w(t), x(t)\rangle \end{bmatrix} \mathrm{d}t + \langle P(0)x(0), x(0)\rangle;$$

which leads to

$$\int_0^\infty [\|y(t)\|^2 - \gamma\|w(t)\|^2]\,\mathrm{d}t \leqslant \int_0^\infty [-\gamma\|w(t)\|^2 - \|x(t)\|^2 + 2\|P(t)\|\|E(t)\| \times \|w(t)\|\|x(t)\|]\,\mathrm{d}t + \langle P(0)x(0), x(0)\rangle, \tag{23}$$

and we finally obtain

$$\int_0^\infty [\|y(t)\|^2 - \gamma\|w(t)\|^2]\,\mathrm{d}t \leqslant \int_0^\infty [-\gamma\|w(t)\|^2 - \|x(t)\|^2 + 2pe\|w(t)\|\|x(t)\|]\,\mathrm{d}t + \|P(0)\|\|x(0)\|^2. \tag{24}$$

Theorem 3.1. *Suppose the assumption* 3.1 *holds then the* H^∞ *optimal control problem has a solution if*

$$1-\frac{p^2e^2}{\gamma}>0. \tag{25}$$

Proof. Recall from (23) that

$$\int_0^\infty [\|y(t)\|^2-\gamma\|w(t)\|^2]\,\mathrm{d}t \leqslant \int_0^\infty [-\gamma\|w(t)\|^2-\|x(t)\|^2+2pe\|w(t)\|\|x(t)\|]\,\mathrm{d}t + \|P(0)\|\|x(0)\|^2.$$

By completion of the square

$$\int_0^\infty [-\gamma\|w(t)\|^2-\|x(t)\|^2+2pe\|w(t)\|\|x(t)\|]\,\mathrm{d}t - \int_0^\infty [(\sqrt{\gamma}\|w(t)\|-\frac{pe}{\sqrt{\gamma}}\|x(t)\|)^2]\,\mathrm{d}t + \int_0^\infty \left[\frac{p^2e^2}{\gamma}\|x(t)\|^2-\|x(t)\|^2\right]\mathrm{d}t.$$

Then (23) reduces to

$$\int_0^\infty [\|y(t)\|^2-\gamma\|w(t)\|^2]\,\mathrm{d}t \leqslant \int_0^\infty \left[\left[-1+\frac{p^2e^2}{\gamma}\right]\|x(t)\|^2\right]\mathrm{d}t + \|P(0)\|\|x(0)\|^2.$$

Therefore,

$$\int_0^\infty [\|y(t)\|^2-\gamma\|w(t)\|^2]\,\mathrm{d}t < \|P(0)\|\|x(0)\|^2.$$

Dividing both sides by γ and rearranging

$$\frac{1}{\gamma}\int_0^\infty [\|y(t)\|^2]\,\mathrm{d}t < \int_0^\infty \gamma\|w(t)\|^2\,\mathrm{d}t + \frac{\|P(0)\|}{\gamma}\|x(0)\|^2,$$

and setting $c_0=\frac{\|P(0)\|}{\gamma}$,

$$\frac{1}{\gamma}\int_0^\infty [\|y(t)\|^2]\,\mathrm{d}t < \int_0^\infty \|w(t)\|^2\,\mathrm{d}t + c_0\|x(0)\|^2.$$

We finally obtain

$$\frac{\int_0^\infty \|y(t)\|^2\,\mathrm{d}t}{c_0\|x(0)\|^2+\int_0^\infty \|w(t)\|^2\,\mathrm{d}t} < \gamma. \qquad \square \tag{26}$$

4. Nonlinear systems

In this section a sequence of linear time-varying approximations are introduced which converge to the solution of the nonlinear H^∞ control problem.

Consider the following nonlinear dynamical system under the presence of disturbance

$$\begin{aligned}\dot{x}(t) &= A(x(t))x(t) + B(x(t))u(t) + E(x(t))w(t) \\ y(t) &= C(x(t))x(t) + D(x(t))u(t)\end{aligned}; \quad x(t_0) = x_0. \tag{27}$$

Now the sequence of linear time-varying approximations can be introduced as follows:

$$\dot{x}^{[0]}(t) = A(x_0(t))x^{[0]}(t) + B(x_0(t))u^{[0]}(t) + E(x_0(t))w^{[0]}(t), \quad x^{[0]}(t_0) = x_0. \tag{28}$$

$$\begin{aligned}\dot{x}^{[i]}(t) &= A(x^{[i-1]}(t))x^{[i]}(t) + B(x^{[i-1]}(t))u^{[i]}(t) + E(x^{[i-1]}(t))w^{[i]}(t), \\ &\quad x^{[i]}(t_0) = x_0,\end{aligned} \tag{29}$$

with the index "i" referring to the iteration.

Using the theory of Section 2 for each linear time-varying system, the sequence of linear feedback control laws can be derived and are given by

$$u^{[i]}(t) = -B^*(x^{[i-1]}(t))P^{[i]}(x^{[i-1]}(t))x^{[i]}(t). \tag{30}$$

These sequences of de-coupled solutions are known to converge uniformly to the solution of the nonlinear problem on any compact time interval (see [10,4]).

Theorem 4.1. *Suppose that the following conditions*: $b = \sup_{t\in\mathfrak{R}^+} \|B(t)\|$, $e = \sup_{t\in\mathfrak{R}^+} \|E(t)\|$, $c = \sup_{t\in\mathfrak{R}^+} \|C(t)\|$ *and* $d = \sup_{t\in\mathfrak{R}^+} \|D(t)\|$ $\{\exists d, e, c, d > 0\}$ *hold*; *then the* H^∞ *optimal control problem has a solution if*

$$1 - \frac{p^2 e^2}{\gamma} > 0. \tag{31}$$

Proof. This result directly follows by direct application of Theorem 3.1, and the convergence holds. □

5. Application

The previously presented theory requires very mild conditions (local Lipschitz) for its applicability and hence the range of applications is very wide. However, for illustrative purposes a well-known physical model of an inverted pendulum on a cart depicted in Fig. 1 is considered (see for instance [4]), where the *control objective* is to move the cart to a specified position while maintaining the pendulum arm in a vertical position.

The nonlinear mathematical model describing the dynamics of the system is

$$(M+m)\ddot{x}(t) + mr\ddot{\theta}(t)\cos\theta(t) + f\dot{x}(t) - mr\dot{\theta}^2(t)\sin\theta(t) = F, \tag{32}$$

$$Mr\ddot{x}(t)\cos\theta(t) + (j + mr^2)\ddot{\theta}(t) + c\dot{\theta}(t) - mgr\sin\theta(t) = 0. \tag{33}$$

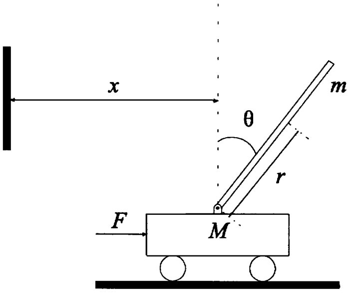

Fig. 1. The inverted pendulum on a cart system.

Initially, the state vector $x(t)$ of the inverted pendulum can be defined as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} \triangleq \begin{bmatrix} x(t) \\ \theta(t) \\ \dot{x}(t) \\ \dot{\theta}(t) \end{bmatrix}.$$

Excluding disturbances for the time being, model {(32) and (33)} can be represented in the following factored form $A(x)x$ as

$$\dot{x}(t) = f(x, u) = A(x)x(t) + B(x)u(t), \tag{34}$$

where t is an independent time variable, $u = F$, and A & B are nonlinear time-invariant matrices functions in x given by

$$A(x) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a_{32}(x) & a_{33}(x) & a_{34}(x) \\ 0 & a_{42}(x) & a_{43}(x) & a_{44}(x) \end{bmatrix} \quad \text{and} \quad B(x) = \begin{bmatrix} 0 \\ 0 \\ b_3(x) \\ b_4(x) \end{bmatrix},$$

where the parameters $a_{ij}(x)$ and $b_i(x)$ are

$$a_{32}(x) = -\Gamma m^2 r^2 g \cos x_2 \ \operatorname{sinc} x_2,$$

$$a_{33}(x) = -\Gamma(j + mr^2) f,$$

$$a_{34}(x) = \Gamma[c \cos x_2 + (j + mr^2)x_4 \sin x_2]mr,$$

$$a_{42}(x) = \Gamma(M + m)mgr \operatorname{sinc} x_2,$$

$$a_{43}(x) = \Gamma mrf \cos x_2,$$

$$a_{44}(x) = -\Gamma[(M + m)c + \tfrac{1}{2} m^2 r^2 x_4 \sin(2x_2)],$$

$$b_3(x) = \Gamma(j + mr^2),$$

$$b_4(x) = -\Gamma mr \cos x_2$$

Table 1
Nomenclature

M	mass of the cart in kg
m	mass of the pendulum arm in kg
r	distance between the centre of the hinge and the pendulum arm's centre of gravity in m
x	the cart's position in m
θ	the pendulum arm's deflection from the vertical axis (clockwise direction [positive]) in rad
j	the pendulum arm's moment of inertia in kg m^2
f	the cart's coefficient of friction in kg/s
c	the coefficient of the viscous rotational friction in the hinge supporting the pendulum arm kg m^2/s
g	The acceleration due to gravity in m/s^2
F	The input control force applied to the cart in N

Table 2
The inverted pendulum's specifications

Variable	Value
M	1 kg
m	0.1 kg
r	0.5 kg
g	9.81 $\mathrm{m/s}^2$
j	0 kg m^2
f	0 kg/s
c	0 kg $\mathrm{m/s}^2$

with

$$\Gamma \triangleq \frac{1}{(M + m\sin^2 x_2)mr^2 + (M+m)j} \qquad \text{and} \qquad \operatorname{sinc} x_2 = \begin{cases} 1, & x_2 = 0, \\ \dfrac{\sin x_2}{x_2}, & x_2 \neq 0. \end{cases}$$

6. Simulations and results

The simulations and results therein were carried out using MATLAB®. Including the disturbance term and recalling the inverted pendulum's model (31) and (32) in its factored form (33)

$$\dot{x}(t) = A(x)x(t) + B(x)u(t) + E(x)w(t); \tag{35}$$

with specifications (in SI units) represented in Tables 1 and 2. The inputs to the system are shown in Table 3.

By introducing the sequence of linear time-varying approximations (with $i = 5$) as discussed in Section 4 to the nonlinear system (35) with the parameters of Table 2 and the inputs of Table 3, and initially setting the disturbance term to zero yields Fig. 2. Now under the same inputs and parameters let us consider the more general case with the inclusion of the $\alpha \times Ew$ term; with α being some scalar.

Table 3
The inputs to the system

$x_1(0)$	Initial value of the cart's position $= 0\,\mathrm{m}$
$x_2(0)$	Initial pendulum's angle $= \pi/3$
$x_3(0)$	Initial value of the cart's velocity $= 0\,\mathrm{m/s}$
$x_4(0)$	Initial value of the pendulum's angular velocity $= 0\,\mathrm{rad/s}$

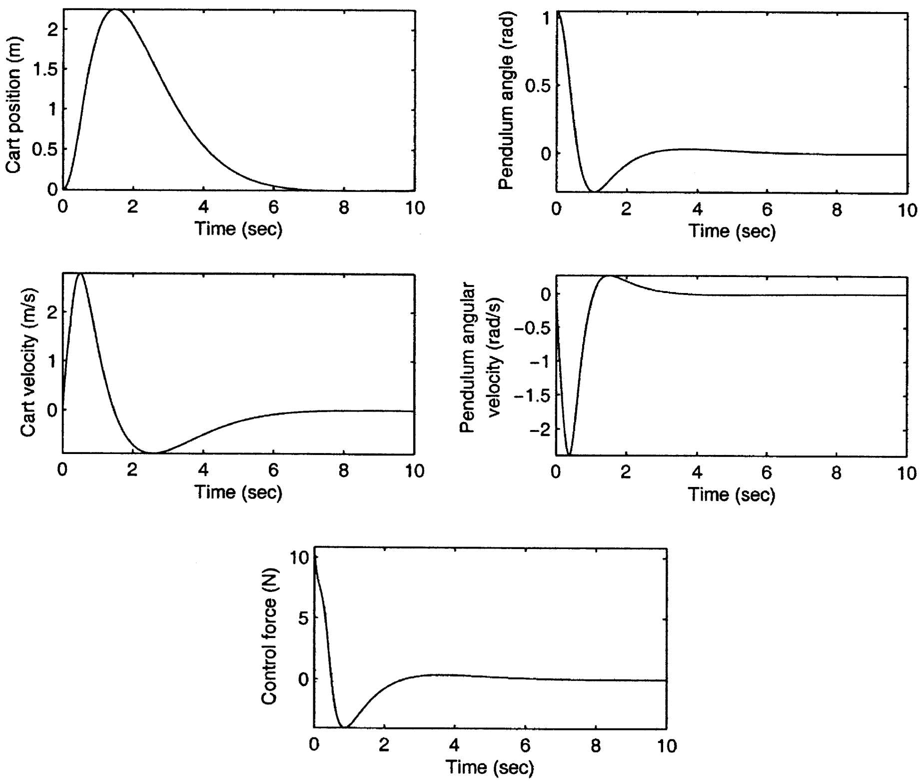

Fig. 2. The undisturbed inverted pendulum on a cart system.

Initially the following case is considered:
Case 1:

$$\alpha = 0.5 \quad \text{and} \quad Ew = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

the controlled nonlinear system is shown in Fig. 3; however due to the large disturbance the pendulum angle is only controlled over the closed time interval (2,4].

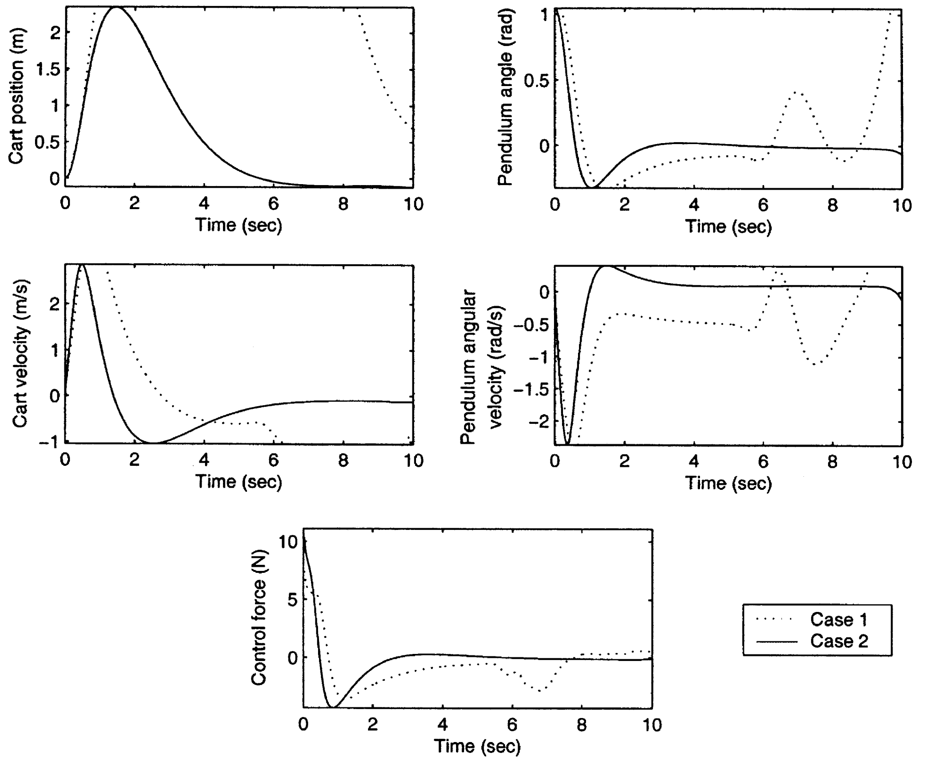

Fig. 3. The disturbed inverted pendulum on a cart system.

Hence by decreasing the disturbance size to; say:
Case 2:

$$\alpha = 0.1 \quad \text{and} \quad Ew = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix},$$

a better response that matches the control objective is obtained and is shown in Fig. 3.

7. Conclusions and open questions

In this paper an H^∞ control problem has been considered for nonlinear uncertain systems by solving the state-dependent Riccati equation in Hilbert space while taking the measurement equation to be the same as the controlled variable one, and applying the Approximation Theory. The method was successfully validated via direct application to the highly nonlinear dynamical model of an inverted pendulum on a cart.

The main advantage of this simple and systematic approach over its rivals is that it requires very mild conditions. Note that although there is still some ambiguity arising in explicitly designing the weights for multi-input multi-output (MIMO) systems; such weights were chosen, in this context, to follow a regular H^{∞} optimal controller setting. At this point it is worth pointing out that most real-life systems can be represented via system (27), and the limitation, however, lies on being capable of promptly applying this technique to real-time systems without the use of lookup tables. Furthermore, the number of iterations is directly proportional to the complexity of any given nonlinear system—a fact which undeniably amplifies the computational burden.

Naturally, there are still several open questions in this field especially since the research in this area is partially complete; and as an extension to this far-reaching work one can consider the case of invariant zeros on the imaginary axis for continuous-time systems and the real mixed LQG/H^{∞} control problem.

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