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A GENERALIZATION OF LYAPUNOV'S EQUATION TO NONLINEAR SYSTEMS

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Abstract: In this paper a generalization of Lyapunov's equation for the stability of linear dynamical systems to globally asymptotically stable nonlinear systems is presented by embedding the system in a linear infinite-dimensional one on a tensor space by using Carleman linearization. This linear representation allows the definition and solution of a Lyapunov equation as in the usual linear case. The converse result is also discussed using the fact that globally asymptotically stable nonlinear systems are essentially linear. *Copyright ©2004 IFAC*

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1. INTRODUCTION

In the classical theory of linear system stability, a fundamental role is played by Lyapunov's equation

$$A^T P + PA = -Q \tag{1}$$

where Q is any positive definite symmetric matrix. If the equation (1) has a solution for P which is positive definite and symmetric, then the linear system

$$\dot{x} = Ax$$

is asymptotically stable and a Lyapunov function is given by

$$V = \langle x, Px \rangle$$

Moreover, the converse result is true. In this paper an analytic nonlinear system is considered

$$\dot{x} = f(x) \tag{2}$$

and by embedding the system in a linear, infinite-dimensional one, defined on the tensor space $T = \oplus_{l=0}^{\infty} \otimes_{k=1}^{l} \mathbb{R}^n$, a generalization of Lyapunov's theorem for linear systems is given.

Theorem The system (2) is globally asymptotically stable iff a certain linear equation

$$\mathcal{A}^T \mathcal{P} + \mathcal{P}\mathcal{A} = -\mathcal{Q} \tag{3}$$

has a symmetric solution on the subset $S = \{(x, x \otimes x, x \otimes x \otimes x, \ldots) : x \in \mathbb{R}^n\}$ of T. □

The $'\mathcal{A}'$ operator in (3) will be identified with an infinite matrix consisting essentially of the derivatives of f.

This approach has, of course, been used before (Banks, 1992) and is related to Carleman linearization (Kowalski and Steeb, 1992), which has been previously used to estimate domains of attraction of smooth nonlinear systems (Loparo and Blankenship , 1978). Other works related to finding Lyapunov functions and domains of attraction

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for nonlinear systems include the ones of Vanelly and Vidyasagar(1985). An earlier version of the Lyapunov eqn. (3) appears in (Banks, 1988), but it is only with the full tensor algebra formulation that the exact analogy with finite-dimensional linear theory becomes apparent. The result also depends on the fact that nonlinear, globally asymptotically stable systems are essentially linear (McCann, 1979).

2. THE TENSOR SPACE

Consider the tensor algebra of $\mathbb{R}^n$ over $\mathbb{R}$:

$$T = T(\mathbb{R}^n) = \oplus_{l=0}^{\infty} \otimes_{k=1}^{l} \mathbb{R}^n$$

and the subset

$$S = \{(x, x\otimes x, x\otimes x\otimes x, \dots) : x \in \mathbb{R}^n\}$$

(Identifying $(0, x, x\otimes x, \dots)$ with $(x, x\otimes x, \dots)$)

The usual topology on T is defined by the natural inner product

$$\left\langle (x^0, x^1, x^2, \dots), (y^0, y^1, \dots) \right\rangle_\infty = \sum_{i=0}^{\infty} \left\langle x^i, y^i \right\rangle_i$$

where $x^i, y^i \in \otimes_{k=1}^{i} \mathbb{R}^n$ and

$$\left\langle x^i, y^i \right\rangle_i = \prod_{j=1}^{i} \langle \xi_j^i, \eta_j^i \rangle$$

if $x^i = \xi_1^i \otimes \dots \otimes \xi_i^i$, $y^i = \eta_1^i \otimes \dots \otimes \eta_i^i$

are simple decomposable tensors. However, as later is shown, it is more useful to use the topology defined by the inner product

$$\begin{aligned} &\ll (x^0, x^1, x^2, \dots), (y^0, y^1, \dots) \gg \\ &\triangleq \sum_{i=0}^{\infty} \frac{1}{i!} \left\langle x^i, y^i \right\rangle_i \end{aligned}$$

Lemma 2.1 The norm $|||\cdot|||$ on $\mathcal{S}$ defined by the inner product $\ll \cdot \gg$ is given by

$$||| (x, x\otimes x, \dots) |||^2 = e^{\left(\|x\|^2\right)} - 1$$

where, of course, $\| x \|$ denotes the norm of x in $\mathbb{R}^n$

Proof This follows directly from the fact that

$$\begin{aligned} &\langle \underbrace{x\otimes\dots\otimes x}_{i}, \underbrace{x\otimes\dots\otimes x}_{i} \rangle_i \\ &= \Pi\langle x, x\rangle = \| x \|^{2i} \end{aligned}$$

Lemma 2.2 The map $t : \mathbb{R}^n \mapsto \mathcal{S}$ given by

$$t(x) = (x, x\otimes x, x\otimes x\otimes x, \dots)$$

where $\mathcal{S}$ has the above topology is a homeomorphism.

Proof Since the topology on $\mathcal{S}$ is given by the norm $|||\cdot|||$,this follows from lemma 2.1 and

$$||| \ t(x) - t(y) \ ||| = e^{\|x\|^2} + e^{\|y\|^2} - 2e^{\langle x, y\rangle}$$

and the k^{th} term in the Taylor expansion of the right hand side is

$$\| x \|^{2k} + \| y \|^{2k} - 2\langle x, y\rangle^k$$

which tends to zero iff $x \to y\Box$

Denoting the inverse map of t by p (the projection on the first component):

$$p(x, x\otimes x, \dots) = x$$

Now, let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be an analytic function and define the operator

$$A_f^l : \oplus_{k=1}^{l} \mathbb{R}^n \mapsto \mathbb{R}^n$$

by

$$\begin{aligned} &A_f^l(x\otimes x\otimes\dots\otimes x) \\ &= \sum_{i_1,\dots,i_l=1}^{n} \frac{1}{l!} \frac{\partial^l f(0)}{\partial x_{i_1}\cdot\partial x_{i_2}\cdots\partial x_{i_l}} x_{i_1}, x_{i_2}\cdots x_{i_l} \end{aligned}$$

Then the Taylor series of f can be written in the form

$$f(x) = \sum_{l=0}^{\infty} A_f^l(x\otimes\dots\otimes x)$$

where $A_f^0 = f(0)$. Thus, by writing

$$A_f = A_f^0 \oplus A_f^1 \oplus A_f^2 \oplus \dots$$

the Taylor series of f can be written as a map $A_f : \oplus\mathcal{S} \mapsto \mathbb{R}^n$ or

$$\begin{aligned} &\left(A_f^0 \oplus A_f^1 \oplus A_f^2 \oplus \dots\right)(1 \oplus x \oplus \\ &\oplus (x\otimes x) \oplus (x\otimes x\otimes x) \oplus \dots) \\ &= \sum_{l=0}^{\infty} A_f^l(x\otimes\dots\otimes x) \qquad (4) \end{aligned}$$

Note that if $f(0) = 0$, then $A_f : \mathcal{S} \mapsto \mathbb{R}^n$. If the function f is clear from the context, then the subscript $'f'$ is omitted and the operator A_f can be written as:

$$A = A^0 \oplus A^1 \oplus A^2 \oplus \dots$$

3. NONLINEAR DYNAMICAL SYSTEMS

In this section, consider a nonlinear dynamical system given by the differential equation:

$$\dot{x} = f(x), \qquad x(0) = x_0 \tag{5}$$

and assume that $f(0) = 0$ i.e. that the system has an equilibrium at the origin. The system can be written in terms of an infinite-dimensional linear operator using Carleman linearization (Kowalski and Steeb, 1992). Thus, define a new state in $\mathcal{S}$ by

$$\begin{aligned} w &= (x, x \otimes x, x \otimes x \otimes x, \ldots)^T \\ &= (x^1, x^2, x^3, \ldots)^T \end{aligned}$$

where $x^k = x \otimes x \otimes x \otimes \ldots \otimes x$ (k terms).

Then,

$$\begin{aligned} \dot{x}^k &= \sum_{i=1}^{k} x \otimes x \otimes \ldots \otimes \underbrace{\dot{x}}_{i} \otimes x \otimes \ldots \otimes x \\ &= \sum_{i=1}^{k} x \otimes x \otimes \ldots \otimes f(x) \otimes x \otimes \ldots \otimes x \end{aligned}$$

Now, applying the Taylor series in (4) for f:

$$\begin{aligned} \dot{x}^{[k]} &= \sum_{i=1}^{k} x \otimes x \otimes \ldots \otimes \left[\sum_{l=1}^{\infty} A^l \underbrace{(x \otimes \ldots \otimes x)}_{l} \right] \otimes \\ &\otimes x \otimes \ldots \otimes x \\ &= \sum_{l=1}^{\infty} \sum_{i=1}^{k} (\underbrace{I \otimes \ldots \otimes I}_{i-1} \otimes A^l \otimes \underbrace{I \otimes \ldots \otimes I}_{k-i})(x^{[k-1+l]}) \\ &= \sum_{l=1}^{\infty} A_k^l \left(x^{[k-1+l]} \right) \end{aligned}$$

where $A_k^l : \otimes_{i=1}^{k-1+l} \mathbb{R}^n \mapsto \otimes_{i=1}^{k} \mathbb{R}^n$ is the operator given by

$$A_k^l = \sum_{i=1}^{k} \underbrace{I \otimes I \otimes \ldots \otimes I}_{i-1} \otimes A^l \otimes \underbrace{I \otimes \ldots \otimes I}_{k-i} \tag{6}$$

From (6),

$$\begin{aligned} \dot{w} &= \mathcal{A}w, \\ w_0 &= (x_0, x_0 \otimes x_0, x_0 \otimes x_0 \otimes x_0, \ldots)^T \end{aligned} \tag{7}$$

where $\mathcal{A} : \mathcal{S} \mapsto \mathcal{S}$ is the operator with matrix representation

$$\mathcal{A} = \begin{pmatrix} A_1^1 & A_1^2 & A_1^3 & \ldots & \ldots & \ldots & \ldots \\ 0 & A_2^1 & A_2^2 & A_2^3 & \ldots & \ldots & \ldots \\ 0 & 0 & A_3^1 & A_3^2 & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & A_r^1 & A_r^2 \ldots \end{pmatrix} \tag{8}$$

Note that the action of the operator $\mathcal{A}$ can be written in the form

$$\mathcal{A}w = \left\{ \sum_{j=1}^{\infty} A_i^j x^{[i-1+j]} \right\}_{i \geq 1} \tag{9}$$

where $w = (x, x \otimes x, x \otimes x \otimes x, \ldots)^T$

In the next section, the exponential of the operator $\mathcal{A}$ is examined.

4. THE OPERATOR $EXP(\mathcal{A}T)$

It is not entirely straightforward to define the operator $e^{\mathcal{A}t}$, since as it will be shown, it is not bounded on $\mathcal{S}$. Of course, intuitively the solution of (7) is given by

$$w(t) = e^{\mathcal{A}t} w_0 \tag{10}$$

but first, $e^{\mathcal{A}t}$ must be rigorously defined. Let B_Δ be the ball

$$B_\Delta = \{ t(x) \in \mathcal{S} : |||t(x)||| \leq \Delta, x \in \mathbb{R}^n \}$$

in $\mathcal{S}$. The $\Delta - norm$ of $\mathcal{A}$ is defined as

$$|||\mathcal{A}|||_\Delta = \sup_{t(x) \in B_\Delta} \frac{|||\mathcal{A}(t(x))|||}{|||t(x)|||}$$

Lemma 4.1 The following inequality holds:

$$|||\mathcal{A}|||_\Delta \leq \alpha(\Delta) \left(\sum_j j! \|A^j\|^2 \right)$$

where

$$\alpha(\Delta) = \sup_{t(x) \in B_\Delta} \left(\sum_i \frac{i}{(i-1)!} \|x\|^{2(i-1)} \right)$$

Proof Using (9),

$$|||\mathcal{A}|||_\Delta^2 = \sup_{t(x)\in B_\Delta} \frac{\left|\left|\left|\left\{\sum_j A_i^j x^{[i-1+j]}\right\}\right|\right|\right|^2}{|||\{x\}|||^2}$$

$$= \sup_{t(x)\in B_\Delta} \frac{\sum_i \frac{1}{i!}\left\|\sum_j A_i^j x^{[i-1+j]}\right\|^2}{\sum_i \frac{1}{i!}\|x\|^{2i}}$$

$$= \sup_{t(x)\in B_\Delta} \frac{\sum_i \frac{1}{i!}\left\|\sum_j (j!)^{1/2} A_i^j \frac{x^{[i-1+j]}}{(j!)^{\frac{1}{2}}}\right\|^2}{\sum_i \frac{1}{i!}\|x\|^{2i}}$$

$$\leq \sup_{t(x)\in B_\Delta} \frac{\sum_i \frac{1}{i!}\left(\sum_j j!\left\|A_i^j\right\|^2\right)\left(\sum_j \frac{1}{j!}\|x\|^{2(i-1+j)}\right)}{\sum_i \frac{1}{i!}\|x\|^{2i}}$$

Now

$$\left\|A_i^j\right\| \leq \sum_{l=1}^{i} \|I\otimes\cdots\otimes I\otimes A^j\otimes I\otimes\ldots\otimes I\|$$
$$= i\left\|A^j\right\|$$

so

$$|||\mathcal{A}|||_\Delta^2$$
$$\leq \sup_{t(x)\in B_\Delta}\left(\sum_i \frac{i}{(i-1)!}\|x\|^{2(i-1)}\right)\left(\sum_j j!\left\|A^j\right\|^2\right)$$

and the result follows. □

Even though $\mathcal{A}$ is not bounded, it is 'upper triangular' and so $e^{\mathcal{A}t}$ can be formally defined in the usual way, i.e.

$$e^{\mathcal{A}t} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\mathcal{A}^n$$

since each element of $\mathcal{A}^n$ (for all n) is a finite sum of products of operators. In fact, since

$$(\mathcal{A})_{ij} = \begin{cases} A_i^{j-i+1} & ,j\geq 1\\ 0 & ,j<1\end{cases}$$

then

$$(\mathcal{A}^n)_{ij} = \sum_{\substack{k_l=1\\1\leq l\leq n-1}}^{\infty}(\mathcal{A})_{ik_1}(\mathcal{A})_{k_1k_2}(\mathcal{A})_{k_2k_3}\ldots(\mathcal{A})_{k_{n-1}j}$$

$$= \sum_{\substack{k_l=1\\1\leq l\leq n-1}}^{\infty} A_i^{k_1-i+1}A_{k_1}^{k_2-k_1+1}A_{k_2}^{k_3-k_2+1}\ldots A_{k_{n-1}}^{j-k_{n-1}+1} \tag{11}$$

Note that each term is well-defined as a finite sum of finite products of operators. Thus,

$$e^{\mathcal{A}t} = I + \mathcal{A}t + $$
$$+\sum_{n=2}^{\infty}\frac{t^n}{n!}\sum_{\substack{k_l=1\\1\leq l\leq n-1}}^{\infty} A_i^{k_1-i+1}\ldots A_{k_{n-1}}^{j-k_{n-1}+1}$$

Instead of trying to bound $e^{\mathcal{A}t}$ in the same way as $\mathcal{A}$ in lemma 4.1, it is more convenient to note that only the solution $\mathbf{x}$ to the original nonlinear problem (5) is required. Hence, consider only the first element of the infinite vector $e^{\mathcal{A}t}\{x, x\otimes x,\ldots\}^T$. Hence

$$\left\|\left\{e^{\mathcal{A}t}\begin{pmatrix}x\\ x\otimes x\\ \ldots\end{pmatrix}\right\}_1\right\| = \|x + \sum_{j=1}^{\infty}(A_1^j t x^{[j]} +$$
$$+\sum_{n=2}^{\infty}\frac{t^n}{n!}\sum_{\substack{k_l=1\\1\leq l\leq n-1}}^{\infty} A_1^{k_1}\ldots A_{k_{n-1}}^{j-k_{n-1}+1}x^{[j]})\|$$
$$\leq \|x\| + \sum_{j=1}^{\infty}(\|A\|^j t\|x\|^j +$$
$$+\sum_{n=2}^{\infty}\frac{t^n}{n!}\sum_{\substack{k_l=1\\1\leq l\leq n-1}}^{\infty}\|A_1^{k_1}\ldots A_{k_{n-1}}^{j-k_{n-1}+1}\|\|x\|^j)$$
$$\leq \|x\| + \sum_{j=1}^{\infty}(\|A\|^j t\|x\|^j +$$
$$+\sum_{n=2}^{\infty}\frac{t^n}{n!}\sum_{\substack{k_l=1\\1\leq l\leq n-1}}^{\infty} k_1k_2\ldots k_{n-1}\|A^{k_1}\|\|A^{k_2-k_1+1}\|$$
$$\|A^{k_3-k_2+1}\|\ldots\|A^{j-k_{n-1}+1}\|\|x\|^j) \tag{12}$$

Hence, the radius of convergence of this series in t is at least

$$R = \frac{1}{limsup(\mathcal{C})^{1/n}}$$

where

$$\mathcal{C} = \frac{\sum_{\substack{k_l=1\\1\leq l\leq n-1}}^{\infty} k_1\ldots k_{n-1}\|A^{k_1}\|\ldots\|A^{j-k_{n-1}+1}\|\|x\|^j}{n!}$$

5. LYAPUNOV THEORY

In this section a generalized Lyapunov's theorem for the stability of a linear system is given, i.e. a linear dynamical system

$$\dot{x} = Ax$$

is asymptotically stable iff for each positive definite matrix Q, $\exists$ a positive definite matrix P such that Lyapunov's equation

$$A^TP + PA = -Q \tag{13}$$

is satisfied. In this case, a Lyapunov function is given by

$$V = x^T P x$$

Now consider the general nonlinear system

$$\dot{x} = f(x) \tag{14}$$

and assume $f(0) = 0$. From section 3, the system can be written in the form

$$\dot{w} = \mathcal{A} w \tag{15}$$

where

$$w = (x, x \otimes x, x \otimes x \otimes x, \ldots)^T$$

Consider the Lyapunov equation

$$\mathcal{A}^T \mathcal{P} + \mathcal{P} \mathcal{A} = -\mathcal{I} \tag{16}$$

where $\mathcal{P}$ is a positive definite symmetric operator on $\mathcal{S}$ i.e.

$$\begin{aligned} &\langle (x, x \otimes x, x \otimes x \otimes x, \ldots), \mathcal{P}(y, y \otimes y, \ldots) \rangle \\ &= \langle \mathcal{P}(x, x \otimes x, \ldots), (y, y \otimes y, y \otimes y \otimes y, \ldots) \rangle \end{aligned}$$

and $\langle (x, x \otimes x, x \otimes x \otimes x, \ldots), \mathcal{P}(x, x \otimes x, x \otimes x \otimes x, \ldots) \rangle > 0$ for $x \neq 0$

Suppose that such an operator $\mathcal{P}$ exists and consider the function

$$V = \langle (x, x \otimes x, \ldots), \mathcal{P}(x, x \otimes x, \ldots) \rangle$$

Then,

$$\begin{aligned} \dot{V} &= \langle \mathcal{A}(x, x \otimes x, \ldots), \mathcal{P}(x, x \otimes x, \ldots) \rangle + \\ &+ \langle (x, x \otimes x, \ldots), \mathcal{P}\mathcal{A}(x, x \otimes x, \ldots) \rangle \\ &= \langle (\mathcal{A}^T \mathcal{P} + \mathcal{P}\mathcal{A})(x, x \otimes x, \ldots), (x, x \otimes x, \ldots) \rangle \\ &= -|||(x, x \otimes x, \ldots)|||^2 \\ &= -(e^{\|x\|^2} - 1) \end{aligned}$$

and so V is a Lyapunov function for (14) and so it is globally asymptotically stable.

To prove the converse statement, i.e. that if the system (14) is stable then there exists a positive definite solution of (16), the following theorem of McCann is used:

Theorem 5.1 (McCann, 1979) Globally asymptotically stable dynamical systems on topological spaces are topologically equivalent to linear systems.

This means that if a continuous nonlinear dynamical system is defined by the map

$$\sigma : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$$

such that

$$\sigma(x, t+s) = \sigma(\sigma(x, t), s)$$

and

$$\sigma(x, 0) = x$$

then there is a homeomorphism.

$$h : \mathbb{R}^n \mapsto \mathbb{R}^{2n}$$

such that

$$h(\sigma(x, t) = \rho(h(x), t))$$

where ρ can be taken to be a diffeomorphism for analytic dynamical systems.

Thus, suppose the system (14) is globally asymptotically stable (with unique equilibrium point x=0). Then by McCann's theorem there exists a diffeomorphism $\Phi : \mathbb{R}^n \mapsto \mathbb{R}^{2n}$ (into) such that

$$\Phi(\sigma(x, t)) = c^t y$$

where $c = e^k$ and $y = \Phi(x)$. This means that the dynamical system $c^t y$ is the set of solutions of the linear equation

$$\frac{dy}{dt} = ky, \qquad k < 0$$

Hence, it is possible to suppose that our nonlinear system is embedded in a stable linear system

$$\dot{y} = Ay, \qquad y \in \mathbb{R}^{2n} \tag{17}$$

where $y = \Phi(x)$. Then, by using Taylor series of y,

$$y = \sum_{i=0}^{2n} h_i x^{[i]}$$

or

$$y = Lw \tag{18}$$

with $L = \sum_{i=0}^{2n} h_i$ and $w = (x, x \otimes x, x \otimes x \otimes x, \ldots)$

Define a Lyapunov function for the system (17)

$$V = \langle Lw, PLw \rangle = \langle w, \mathcal{P} w \rangle \tag{19}$$

where $\mathcal{P} = L^T P L$

Then,

$$\dot{V} = \langle \mathcal{A}(x, x \otimes x, \ldots), \mathcal{P}(x, x \otimes x, \ldots) \rangle + \langle (x, x \otimes x, \ldots), \mathcal{P}\mathcal{A}(x, x \otimes x, \ldots) \rangle = \langle (x, x \otimes x, \ldots), (\mathcal{A}^T \mathcal{P} + \mathcal{P}\mathcal{A})(x, x \otimes x, \ldots) \rangle \tag{20}$$

and the Lyapunov equation

$$\mathcal{A}^T \mathcal{P} + \mathcal{P}\mathcal{A} = -\mathcal{Q} \tag{21}$$

will have a solution iff $L^T L = \mathcal{Q}$

Example 1. Consider the simple example

$$\dot{x} = -x + x^3$$

In this case, the $\mathcal{A}$ matrix is

$$\mathcal{A} = \begin{pmatrix} -1 & 0 & 1 & 0 & \ldots & \ldots & \ldots \\ 0 & -2 & 0 & 2 & 0 & \ldots & \ldots \\ 0 & 0 & -3 & 0 & 3 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

and we wish to solve the equation

$$\mathcal{A}^T \mathcal{P} + \mathcal{P}\mathcal{A} = -\mathcal{I}$$

Truncating to $3x3$ matrices, the following equation is obtained

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix} \mathcal{P} + \mathcal{P} \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

which has solution

$$\mathcal{P} = \begin{pmatrix} 1/2 & 0 & 1/8 \\ 0 & 1/4 & 0 \\ 1/8 & 0 & 5/24 \end{pmatrix}$$

so that

$$V = \frac{1}{2}x^2 + \frac{1}{2}x^4 + \frac{5}{24}x^6$$

and

$$\dot{V} = (x^2 - 1)(x^2 + 2x^4 + 54x^6)$$

which is negative for $|x| < 1$, as expected.

Truncating to a $7x7$ system

$$V = \frac{1}{2}x^2 + \frac{1}{2}x^4 + \frac{1}{2}x^6 + \frac{1}{2}x^8 + \frac{253}{640}x^{10} + \frac{1087}{3840}x^{12} + \frac{1343}{10752}x^{14}$$

and

$$\dot{V} = (x^2 - 1)\left(x^2 + \frac{7680}{3840}x^4 + \frac{11520}{3840}x^6 + \frac{15360}{3840}x^8 + \frac{15180}{3840}x^{10} + \frac{13044}{3840}x^{12} + \frac{6715}{3840}x^{14}\right)$$

In this case, any finite-dimensional approximation will give a Lyapunov function. In general, the use of a computer package,such as Maple, is necessary to solve the successive approximations to the basin of attraction.

6. CONCLUSIONS

In this paper, Lyapunov's theorem for linear system stability is generalized to globally asymptotically stable nonlinear systems by embedding the system in a linear infinite-dimensional one on a tensor space using Carleman Linearization. It is then possible to define a Lyapunov equation as in the usual linear case and if a Lyapunov function exists, then the nonlinear system is globally asymptotically stable.

The converse theorem is also discussed by the application of a result which states that all globally asymptotically stable nonlinear systems are essentially linear.

Further work in this subject will include the study of control problems for nonlinear systems by using the linear infinite-dimensional representation proposed in this paper.

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