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Global optimal feedback control for general nonlinear systems with nonquadratic performance criteria

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Abstract

Optimal control of general nonlinear nonaffine controlled systems with nonquadratic performance criteria (that permit state- and control-dependent time-varying weighting parameters), is solved classically using a sequence of linear- quadratic and time-varying problems. The proposed method introduces an "approximating sequence of Riccati equations" (ASRE) to explicitly construct nonlinear time-varying optimal state-feedback controllers for such nonlinear systems. Under very mild conditions of local Lipschitz continuity, the sequences converge (globally) to nonlinear optimal stabilizing feedback controls. The computational simplicity and effectiveness of the ASRE algorithm is an appealing alternative to the tedious and laborious task of solving the Hamilton–Jacobi–Bellman partial differential equation. So the optimality of the ASRE control is studied by considering the original nonlinear-nonquadratic optimization problem and the corresponding necessary conditions for optimality, derived from Pontryagin's maximum principle. Global optimal stabilizing state-feedback control laws are then constructed. This is compared with the optimality of the ASRE control by considering a nonlinear fighter aircraft control system, which is nonaffine in the control. Numerical simulations are used to illustrate the application of the ASRE methodology, which demonstrate its superior performance and optimality.

Keywords: Optimal control; Nonlinear non-affine control systems; Stabilization/regulation; Global convergence; Riccati equation; Maximum principle; Continuous-time systems; Time-varying systems; Approximation theory; Aircraft control

1. Introduction

Over the past decades, the nonlinear optimal control problem for affine control systems has received a great deal of attention in the literature. Banks and Mhana [7] provided a computationally simple and efficient nonlinear design method by extending the principles of linear-quadratic regulator (LQR) theory to control-affine nonlinear systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u}, \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state, $\mathbf{u} \in \mathbb{R}^m$ is the control input and $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. Locally asymptotically stabilizing, near-optimal, nonlinear feedback controllers were designed at fixed points $\mathbf{x} = \bar{\mathbf{x}}$ by applying the standard

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infinite-time horizon LQR control *pointwise* along the trajectory. In the literature, many authors have considered this approximation to nonlinear optimal control based on solving a Riccati equation at each point $\bar{\mathbf{x}}$ (see, for instance, [16,18,19,21]), and the algorithm is often referred to as the "*state-dependent Riccati equation*" or *SDRE* feedback control.

In a recent paper [6], Banks and McCaffrey proposed a *universal* theory which gives general results on solutions of nonlinear differential equations. The theory introduces linear, time-varying (LTV) approximations which are arbitrarily close to the true system. This approximation theory has been applied in [4,5,12] to solve the finite-time horizon *control-affine* nonlinear optimal control problem. The proposed algorithm uses the globally converged solution of an "*approximating sequence of Riccati equations*" (*ASRE*) to explicitly construct time-varying feedback controllers for the original control-affine nonlinear problem (1). The ASRE feedback algorithm for nonlinear optimal control provides outstanding performance in many practical applications, in particular, nonlinear solitary wave motion [4], the inverted pendulum system [11] (which is stabilized from any initial state including its unstabilizable points, unlike other methods), and optimal maneuvering of super-tankers at high speeds [12]. However, optimality has not been proved.

Many existing algorithms for nonlinear optimal control, including the ones mentioned above, only handle nonlinear systems having *affine* control inputs (linear in the manipulated variable $\mathbf{u}$), that is, systems of form (1). However, many applications of practical importance have the nonlinear structure

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u}), \tag{2}$$

which is *nonaffine* in the control input (nonlinear in $\mathbf{u}$). Consider, for example, the initial disturbances in angle of attack of an F-8 in a level trim, unaccelerated flight at $Mach = 0.85$ and an altitude of 30,000 ft (9000 m), for which the nonlinear equations of motion representing the dynamics of the aircraft become [15]

$$\begin{aligned}
\dot{x}_1 &= -0.877x_1+x_3-0.088x_1x_3+0.47x_1^2-0.019x_2^2-x_1^2x_3+3.846x_1^3-0.215u+0.28x_1^2u+0.47x_1u^2+0.63u^3,\\
\dot{x}_2 &= x_3,\\
\dot{x}_3 &= -4.208x_1-0.396x_3-0.47x_1^2-3.564x_1^3-20.967u+6.265x_1^2u+46x_1u^2+61.4u^3,
\end{aligned} \tag{3}$$

where x_1 is the *angle of attack* (rad), x_2 is the *pitch angle* (rad), x_3 is the *pitch rate* (rad s^{-1}) and u is the control input (manipulated variable) provided by the *tail deflection* (or *elevator*) *angle* (rad). Clearly, (3) is not of form (1) since it is *not affine* in u, and has the *control-nonaffine* nonlinear structure (2). Several other well-known processes are control-nonaffine in nature, such as the temperature effect of a reacting system, since temperature enters the model via the nonlinear Arrhenius relationship, which is used as a manipulated variable [22]. Another well-known example involves active magnetic bearing systems in industry, for which the primary problem is the strong coupling in the magnetic flux between magnetic poles. Consequently, the system is strongly nonlinear, not only in the states but also in the control inputs, resulting in a nonaffine nonlinear system (see [17]).

In principal, optimal control of the general nonlinear problem (2) can be solved by the use of Lie series and infinite-dimensional bilinear systems theory (see [2,3,8]). However, the solution is complex and difficult to implement. Therefore, to be able to employ existing algorithms, the nonlinear nonaffine dynamics is usually approximated as being linear in the control, and often over the entire operating range. However, by acknowledging and accounting for the presence of even slight nonlinearities, superior performance improvement can be achieved over neglected dynamics. This has already been illustrated by the authors for nonlinear functions of the state, using the inverted pendulum model in [11] and a real-world super-tanker model in [11,12]. Therefore, in this paper, the proposed ASRE synthesis approach is extended to nonlinear systems with general structure (2) so as to handle control-nonaffine dynamics. A computationally simple and systematic synthesis approach is proposed for nonlinear nonaffine controlled systems, which easily incorporates both state and control nonlinearities under very mild conditions of *local Lipschitz continuity*. The ASRE is used to explicitly construct a stabilizing nonlinear time-varying state-feedback control law that solves the optimal control problem for nonaffine nonlinear systems (2) with nonquadratic performance criteria (refer to [12] for the ASRE framework of the *nonlinear optimal tracking control* problem). The sequences will be shown to *globally converge* under certain conditions in an appropriate space. The optimality of the ASRE feedback algorithm will be examined by considering the full necessary equations derived from Pontryagin's maximum principle. This involves deriving the Hamiltonian function, which provides the necessary conditions for optimality. The resulting nonquadratic optimization problem with nonlinear dynamics will then be replaced with a sequence of time-varying linear-quadratic problems, which can be solved classically. This latter method is proposed to find the *global* optimal feedback control of nonlinear systems, in cases where such a solution exists, and will be used as a basis for comparison with the optimality of the ASRE method.

The paper is organized as follows. Classical linear-quadratic optimal control theory is reviewed briefly in Section 2. In Section 3, the ASRE algorithm is presented for the general finite-time nonlinear optimal state-regulator control problem together with its proof of global convergence. The necessary conditions for optimality of the nonlinear optimal control problem are established in Section 4 from the maximum principle. In order to verify the effectiveness and optimality of the proposed ASRE control-design scheme presented in this paper, the control law is simulated against a realistic model of a real-world application. The nonaffine nonlinear equations (3), describing the longitudinal motion of the F-8 Crusader aircraft, are chosen for this study because of the readily available data, which has been repeatedly used in the literature by various authors in an attempt to solve the control problem (see [13,15,24]). So, in Section 5, the performance of the ASRE control system is evaluated by applying these to the F-8 Crusader to design stabilizing nonlinear feedback controllers. The ASRE solutions are compared against that achieved by the necessary conditions (optimal solution), as well as LQR control and the popular SDRE method. Concluding remarks are given in Section 6.

2. Linear-quadratic optimal control theory

Let us first revisit the classical fixed finite-time horizon LQR optimal control problem where a linear system with full-state information $\mathbf{x}\in\mathbb{R}^n$ and control input $\mathbf{u}\in\mathbb{R}^m$, described by the equation

$$\dot{\mathbf{x}}(t)=\mathbf{A}(t)\mathbf{x}(t)+\mathbf{B}(t)\mathbf{u}(t),\qquad \mathbf{x}(t_0)=\mathbf{x}_0 \tag{4}$$

is considered with the finite-time, linear-quadratic cost functional

$$J(\mathbf{u})=\tfrac{1}{2}\mathbf{x}^{\mathrm{T}}(t_{\mathrm{f}})\mathbf{F}\mathbf{x}(t_{\mathrm{f}})+\frac{1}{2}\int_{t_0}^{t_{\mathrm{f}}}\{\mathbf{x}^{\mathrm{T}}(t)\mathbf{Q}(t)\mathbf{x}(t)+\mathbf{u}^{\mathrm{T}}(t)\mathbf{R}(t)\mathbf{u}(t)\}\,\mathrm{d}t, \tag{5}$$

where $\mathbf{F},\mathbf{Q}\in\mathbb{R}^{n\times n}$ are positive-semidefinite, $\mathbf{R}\in\mathbb{R}^{m\times m}$ is positive-definite, $\mathbf{u}$ is unconstrained, $\mathbf{A}\in\mathbb{R}^{n\times n}$, $\mathbf{B}\in\mathbb{R}^{n\times m}$, and the objective is to keep the state $\mathbf{x}(t)$ near zero without excessive control-energy expenditure. It is well-known (see, for example, [1,14,23]) that the maximum principle provides a way of finding the optimal control $\mathbf{u}(t)$ by minimizing the cost functional (5) subject to the control system constraint (4). Therefore, strictly speaking, the "minimum" principle provides the necessary conditions for optimality, where the corresponding Hamiltonian is minimized among admissible controls. For the finite-time LQR problem, the minimum principle results in the coupled two-point boundary value problem

$$\begin{pmatrix}\dot{\mathbf{x}}(t)\\ \dot{\boldsymbol{\lambda}}(t)\end{pmatrix}=\begin{pmatrix}\mathbf{A}(t) & -\mathbf{S}(t)\\ -\mathbf{Q}(t) & -\mathbf{A}(t)\end{pmatrix}\begin{pmatrix}\mathbf{x}(t)\\ \boldsymbol{\lambda}(t)\end{pmatrix}, \tag{6}$$

where

$$\mathbf{S}(t)\triangleq\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(t) \tag{7}$$

and

$$\mathbf{x}(t_0) = \mathbf{x}_0, \qquad \boldsymbol{\lambda}(t_f) = \mathbf{F}\mathbf{x}(t_f). \tag{8}$$

Assuming that the co-state $\boldsymbol{\lambda}(t)$ and the state $\mathbf{x}(t)$ are related by $\boldsymbol{\lambda}(t) = \mathbf{P}(t)\mathbf{x}(t)$ for some $n \times n$ time-varying positive-definite symmetric matrix $\mathbf{P}(t)$, the necessary conditions are satisfied by the nonlinear matrix Riccati differential equation

$$\dot{\mathbf{P}}(t) = -\mathbf{Q}(t) - \mathbf{P}(t)\mathbf{A}(t) - \mathbf{A}^{\mathrm{T}}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{S}(t)\mathbf{P}(t), \quad \mathbf{P}(t_f) = \mathbf{F} \tag{9}$$

yielding the LTV optimal feedback control law

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(t)\mathbf{P}(t)\mathbf{x}(t), \tag{10}$$

where the superscript * is used to indicate the optimal solution. This feedback control law is verified in the following theorem.

Theorem 1 (Finite-time LQR optimal control). *Given the full-state observable linear system* (4) *and the quadratic cost functional* (5), *where* $\mathbf{u}(t)$ *is not constrained,* t_f *is specified,* $\mathbf{F}$ *and* $\mathbf{Q}(t)$ *are positive-semidefinite and* $\mathbf{R}(t)$ *is positive-definite, then an optimal control exists, it is unique, and is given by* (10) *where the* $n \times n$ *symmetric matrix* $\mathbf{P}(t)$ *is the unique solution of the Riccati equation* (9). *The corresponding optimal state trajectory then becomes the solution of the linear differential equation*

$$\dot{\mathbf{x}}^*(t) = [\mathbf{A}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(t)\mathbf{P}(t)]\mathbf{x}^*(t), \qquad \mathbf{x}^*(t_0) = \mathbf{x}_0.$$

3. ASRE feedback for optimal control of nonlinear systems

In this section, optimal control of general multi-input–multi-output, autonomous, nonlinear systems of form (2) is considered, where the control input $\mathbf{u}$ may not necessarily be linear-affine. Assuming the origin is an equilibrium point, that is $\mathbf{f}(\mathbf{0},\mathbf{0}) = \mathbf{0}$, consider the general control-nonaffine nonlinear dynamics (2) represented in state-space in the *factored* form

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{B}(\mathbf{x}(t), \mathbf{u}(t))\mathbf{u}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{11}$$

where $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a nonlinear matrix-valued function of $\mathbf{x}$, and $\mathbf{B} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times m}$ is a nonlinear matrix-valued function of both the state $\mathbf{x}$ and control input $\mathbf{u}$. The optimization problem is chosen to minimize the finite-time horizon, autonomous, nonlinear-nonquadratic cost functional

$$J(\mathbf{u}) = \tfrac{1}{2}\,\mathbf{x}^{\mathrm{T}}(t_f)\mathbf{F}(\mathbf{x}(t_f))\mathbf{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \{\mathbf{x}^{\mathrm{T}}(t)\mathbf{Q}(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{u}^{\mathrm{T}}(t)\mathbf{R}(\mathbf{x}(t))\mathbf{u}(t)\}\,\mathrm{d}t, \tag{12}$$

where $\mathbf{F}, \mathbf{Q} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $\mathbf{R} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ are state-dependent weighting matrices.

3.1. The ASRE methodology

The nonlinear-nonquadratic optimization problem of minimizing cost (12) subject to dynamics (11) can be transformed into an equivalent linear-quadratic, time-varying problem by introducing the LTV sequence (see [6])

$$\begin{aligned}
&\text{approx 0} \qquad \dot{\mathbf{x}}^{[0]}(t) = \mathbf{A}(\mathbf{x}_0)\mathbf{x}^{[0]}(t) + \mathbf{B}(\mathbf{x}_0, \mathbf{0})\mathbf{u}^{[0]}(t),\\
&\text{approx i} \qquad \dot{\mathbf{x}}^{[i]}(t) = \mathbf{A}(\mathbf{x}^{[i-1]}(t))\mathbf{x}^{[i]}(t) + \mathbf{B}(\mathbf{x}^{[i-1]}(t), \mathbf{u}^{[i-1]}(t))\mathbf{u}^{[i]}(t),\\
&\qquad\qquad\ \ \mathbf{x}^{[i]}(t_0) = \mathbf{x}_0, \quad i \geqslant 0
\end{aligned} \tag{13}$$

with the corresponding sequence of linear-quadratic and time-varying costs

$$J^{[0]}(\mathbf{u}) = \tfrac{1}{2}\,\mathbf{x}^{[0]\mathrm{T}}(t_\mathrm{f})\mathbf{F}(\mathbf{x}_0)\mathbf{x}^{[0]}(t_\mathrm{f}) + \frac{1}{2}\int_{t_0}^{t_\mathrm{f}} \{\mathbf{x}^{[0]\mathrm{T}}(t)\mathbf{Q}(\mathbf{x}_0)\mathbf{x}^{[0]}(t) + \mathbf{u}^{[0]\mathrm{T}}(t)\mathbf{R}(\mathbf{x}_0)\mathbf{u}^{[0]}(t)\}\,\mathrm{d}t,$$

$$J^{[i]}(\mathbf{u}) = \tfrac{1}{2}\mathbf{x}^{[i]\mathrm{T}}(t_\mathrm{f})\mathbf{F}(\mathbf{x}^{[i-1]}(t_\mathrm{f}))\mathbf{x}^{[i]}(t_\mathrm{f}) + \frac{1}{2}\int_{t_0}^{t_\mathrm{f}} \{\mathbf{x}^{[i]\mathrm{T}}(t)\mathbf{Q}(\mathbf{x}^{[i-1]}(t))\mathbf{x}^{[i]}(t) + \mathbf{u}^{[i]\mathrm{T}}(t)\mathbf{R}(\mathbf{x}^{[i-1]}(t))\mathbf{u}^{[i]}(t)\}\,\mathrm{d}t, \quad i \geqslant 1, \tag{14}$$

thus assuming the initial functions $\mathbf{x}^{[i-1]}(t) = \mathbf{x}_0$ and $\mathbf{u}^{[i-1]}(t) = \mathbf{0}$ for the first approximation $i = 0$, where the superscript $[i]$ represents the iteration process. Since each approximating problem in (13) and (14) is now LTV (with the exception of the first sequence) and quadratic, classical finite-time LQR optimal control theory (see Theorem 1, Section 2) can be applied to this sequence, which results in the LTV optimal feedback control sequence policy

$$\mathbf{u}^{[i]}(t) = -\mathbf{R}^{-1}(\mathbf{x}^{[i-1]}(t))\mathbf{B}^\mathrm{T}(\mathbf{x}^{[i-1]}(t), \mathbf{u}^{[i-1]}(t))\mathbf{P}^{[i]}(t)\mathbf{x}^{[i]}(t) \tag{15}$$

for $i \geqslant 0$, where the real, symmetric and positive-definite matrix $\mathbf{P}^{[i]}(t) \in \mathbb{R}^{n\times n}$ is the solution of the ASRE

$$\dot{\mathbf{P}}^{[i]}(t) = -\mathbf{Q}(\mathbf{x}^{[i-1]}(t)) - \mathbf{P}^{[i]}(t)\mathbf{A}(\mathbf{x}^{[i-1]}(t)) - \mathbf{A}^\mathrm{T}(\mathbf{x}^{[i-1]}(t))\mathbf{P}^{[i]}(t) + \mathbf{P}^{[i]}(t)\mathbf{S}(\mathbf{x}^{[i-1]}(t), \mathbf{u}^{[i-1]}(t))\mathbf{P}^{[i]}(t),$$

$$\mathbf{P}^{[i]}(t_\mathrm{f}) = \mathbf{F}(\mathbf{x}^{[i-1]}(t_\mathrm{f})) \tag{16}$$

with

$$\mathbf{S}(\mathbf{x}^{[i-1]}(t), \mathbf{u}^{[i-1]}(t)) \triangleq \mathbf{B}(\mathbf{x}^{[i-1]}(t), \mathbf{u}^{[i-1]}(t))\mathbf{R}^{-1}(\mathbf{x}^{[i-1]}(t))\mathbf{B}^\mathrm{T}(\mathbf{x}^{[i-1]}(t), \mathbf{u}^{[i-1]}(t)). \tag{17}$$

The optimal state trajectory then becomes the limit of the solution of the linear differential equation

$$\dot{\mathbf{x}}^{[i]}(t) = [\mathbf{A}(\mathbf{x}^{[i-1]}(t)) - \mathbf{S}(\mathbf{x}^{[i-1]}(t), \mathbf{u}^{[i-1]}(t))\mathbf{P}^{[i]}(t)]\mathbf{x}^{[i]}(t), \qquad \mathbf{x}^{[i]}(t_0) = \mathbf{x}_0. \tag{18}$$

Theorem 2 (Nonlinear optimal stabilizing ASRE feedback control). *Given the nonlinear-nonquadratic problem* (11) *and* (12), *the sequence of linear-quadratic and time-varying approximations* (13) *and* (14) *can be introduced. Provided that* $\mathbf{u}(t)$ *is unconstrained, the terminal time* t_f *is specified,* $\mathbf{R}(\mathbf{x})$ *is positive-definite* $\forall\mathbf{x}$, $\mathbf{F}(\mathbf{x})$ *and* $\mathbf{Q}(\mathbf{x})$ *are positive-semidefinite for* $\forall\mathbf{x}$, *and since each of the approximating systems* (13), (14) *are* (*time-varying*) *linear-quadratic, then the optimal control exists, is unique for each sequence of approximations, and is given by* (15). *Under bounded and local Lipschitz conditions of* $\mathbf{A}(\mathbf{x})$ *and* $\mathbf{B}(\mathbf{x},\mathbf{u})$ *with respect to their arguments, the sequence of approximations* $\mathbf{x}^{[i]}(t)$ *and* $\mathbf{u}^{[i]}(t)$ (*globally*) *converge in* $C([t_0, t_\mathrm{f}]; \mathbb{R}^n)$ *and* $C([t_0, t_\mathrm{f}]; \mathbb{R}^m)$, *respectively, to the functions* $\mathbf{x}(t)$ *and* $\mathbf{u}(t)$, *which minimize* (12) *over the set of nonautonomous feedback controls of the form*

$$-\mathbf{B}(\mathbf{x}(t), \mathbf{u}(t))\mathbf{R}^{-1}(\mathbf{x}(t))\mathbf{B}^\mathrm{T}(\mathbf{x}(t), \mathbf{u}(t))\mathbf{P}(t)\mathbf{x}(t). \tag{19}$$

Proof. The proof of Theorem 2 is presented in Section 3.2. □

Remark 1. From (16), note that the ASRE $\mathbf{P}^{[i]}(t) = \mathbf{P}^{[i]}(\mathbf{x}^{[i-1]}(t), \mathbf{u}^{[i-1]}(t), t)$. Therefore, the sequence of LTV feedback controllers given by (15), in fact, converge to state- and control-dependent, nonautonomous, nonlinear feedback controls of the form

$$-\mathbf{B}(\mathbf{x}(t), \mathbf{u}(t))\mathbf{R}^{-1}(\mathbf{x}(t))\mathbf{B}^\mathrm{T}(\mathbf{x}(t), \mathbf{u}(t))\mathbf{P}(\mathbf{x}(t), \mathbf{u}(t), t)\mathbf{x}(t).$$

Remark 2. The ASRE feedback control design strategy given by Theorem 2 can easily be generalized to *nonautonomous* nonlinear systems $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$ and, therefore, to include nonlinear “control” terms and “time” explicitly in both $\mathbf{A}$ and $\mathbf{B}$, in the form $\mathbf{A}(\mathbf{x}(t),\mathbf{u}(t),t)$ and $\mathbf{B}(\mathbf{x}(t),\mathbf{u}(t),t)$. The ASRE methodology can also be extended to account for time- and control-dependent, as well as state-dependent, weighting matrices in the form $\mathbf{Q}(\mathbf{x}(t),\mathbf{u}(t),t)$ and $\mathbf{R}(\mathbf{x}(t),\mathbf{u}(t),t)$. However, the proof in Section 3.2 will only be presented for the special case when the general nonlinear optimal control problem can be represented in form (11) and (12), and hence by (13) and (14), so as not to further complicate mathematical proofs.

Unlike most other methods, the ASRE technique is *not* local in its applicability, thus making it an attractive alternative to solve optimal control for nonlinear systems. The method is based on a representation of the nonlinear-nonquadratic optimal control problem as the limit of a sequence of linear-quadratic and time-varying problems, which *globally converge* in the space of continuous functions under a very mild *local Lipschitz condition*. The ASRE provides a *universal approximation scheme* to the nonlinear optimal control problem in that the method is applicable for all initial conditions. This does not, of course, guarantee that an optimal feedback control exists from any given initial state. The advantage of the ASRE scheme is that its application generally provides an extremely accurate solution after a relatively small number of iterations. The computational simplicity of the method is evident as it is based on a sequence of linear systems, which can be solved by classical means, since the usual linear techniques are available.

Therefore, consider applying the ASRE feedback algorithm in Theorem 2 to solve the nonlinear-nonquadratic regulator problem (11) and (12), that is, consider solving the sequence of approximations (15)–(18) for $i \geqslant 0$. The initial control is given from sequence (15) by

$$\mathbf{u}^{[0]}(t) = -\mathbf{R}^{-1}(\mathbf{x}_0)\mathbf{B}^{\mathrm{T}}(\mathbf{x}_0,\mathbf{0})\mathbf{P}^{[0]}(t)\mathbf{x}^{[0]}(t),$$

where $\mathbf{P}^{[0]}(t)$ is the solution of the ASRE (16) when $i=0$, and the initial functions $\mathbf{x}^{[i-1]}(t)$ and $\mathbf{u}^{[i-1]}(t)$ in the ASRE iteration technique have been fixed at $\mathbf{x}_0$ and $\mathbf{0}$, respectively, for $t \in [t_0, t_f]$. Hence, from (18), the first approximation is given by

$$\dot{\mathbf{x}}^{[0]}(t) = [\mathbf{A}(\mathbf{x}_0) - \mathbf{B}(\mathbf{x}_0,\mathbf{0})\mathbf{R}^{-1}(\mathbf{x}_0)\mathbf{B}^{\mathrm{T}}(\mathbf{x}_0,\mathbf{0})\mathbf{P}^{[0]}(t)]\mathbf{x}^{[0]}(t), \qquad \mathbf{x}^{[0]}(t_0) = \mathbf{x}_0,$$

which is linear and *time-invariant*. In solving (15)–(18) for $i \geqslant 1$, a sequence of *time-varying* and linear-quadratic equations is then obtained where each sequence is solved as a standard numerical problem. For each sequence, optimization has to be carried out at every numerical time-step resulting in LTV feedback control values. Therefore, using the ASRE methodology, nonlinearities in the state $\mathbf{x}$ and the manipulated variable $\mathbf{u}$ are compensated through computation of the sequences of LTV systems (15)–(18) at each iteration. Applying the feedback $\mathbf{P}^{[k]}(t)$ from the converged sequences $\mathbf{x}^{[k]}(t)$ and $\mathbf{u}^{[k]}(t)$ to the original nonlinear system (11) will yield the optimal state trajectory $\mathbf{x}(t)$ and control $\mathbf{u}(t)$, which must give *the same solutions* as $\mathbf{x}^{[k]}(t)$ and $\mathbf{u}^{[k]}(t)$, respectively, since *convergence* has been achieved.

Remark 3. Note that the factored representation in (11) is *not* unique. But, suppose a unique continuous nonlinear optimal feedback control exists on the interval $[t_0, t_f]$, minimized over a restricted set of nonlinear feedbacks of form (19). Then the ASRE feedback algorithm in Theorem 2 for the nonlinear optimal state-regulator control problem will converge to this unique feedback, regardless of the factorization used in the representation of $\mathbf{A}(\mathbf{x})$ (or $\mathbf{A}(\mathbf{x},\mathbf{u})$) and $\mathbf{B}(\mathbf{x},\mathbf{u})$. However, in general, a unique continuous nonlinear optimal feedback control in form (19) may not exist. In fact, there may not even be controls $\mathbf{u}$ which take $(\mathbf{x}_0, t_0)$ to the target set. On the other hand, there may be more than one optimal control, which may indeed be discontinuous.

3.2. Global convergence

In order to prove the sequence converges to a solution, it is required to show that

$$\lim_{i\to\infty} \|\mathbf{x}^{[i]}(t) - \mathbf{x}^{[i-1]}(t)\| = 0.$$

Let $\mathbf{\Phi}^{[i-1]}(t,t_0)$ and $\mathbf{\Phi}^{[i-2]}(t,t_0)$ denote the fundamental (transition) matrices generated by $\mathbf{A}(\mathbf{x}^{[i-1]}(t))$ and $\mathbf{A}(\mathbf{x}^{[i-2]}(t))$, respectively. Then, by Brauer's well-known inequality (see [9,10]),

$$\|\mathbf{\Phi}^{[i-1]}(t,t_0)\| \leqslant \exp\left[\int_{t_0}^{t} \mu(\mathbf{A}(\mathbf{x}^{[i-1]}(\tau)))\,\mathrm{d}\tau\right], \quad t \geqslant t_0, \tag{20}$$

where $\mathbf{\Phi}^{[i-1]}(t_0,t_0)=\mathbf{I}$, and $\mu(\mathbf{A})$ is used to denote the measure of the matrix $\mathbf{A}$, which is defined as the logarithmic norm of $\mathbf{A}$. Using (20), an estimate for $\mathbf{\Phi}^{[i-1]}(t,t_0)-\mathbf{\Phi}^{[i-2]}(t,t_0)$ can be obtained.

Lemma 1 (Çimen and Banks [12]). *Suppose the following conditions hold*:

(A1) $\mu(\mathbf{A}(\mathbf{x})) \leqslant \mu_0$, $\forall \mathbf{x}\in\mathbb{R}^n$, *and*
(A2) $\|\mathbf{A}(\mathbf{x}_1)-\mathbf{A}(\mathbf{x}_2)\| \leqslant \alpha\|\mathbf{x}_1-\mathbf{x}_2\|$, $\forall \mathbf{x}_1,\mathbf{x}_2\in\mathbb{R}^n$, *that is*, $\mathbf{A}(\mathbf{x})$ *is Lipschitz continuous*, *where* μ_0 *is some finite constant and* $\alpha>0$. *Then*

$$\|\mathbf{\Phi}^{[i-1]}(t,t_0)-\mathbf{\Phi}^{[i-2]}(t,t_0)\| \leqslant \alpha e^{\mu_0(t-t_0)}(t-t_0)\sup_{s\in[t_0,t]}\|\mathbf{x}^{[i-1]}(s)-\mathbf{x}^{[i-2]}(s)\|.$$

Consider the LTV state-feedback control (15), written in more compact form

$$\mathbf{u}^{[i]}(t)=\mathbf{D}(\mathbf{x}^{[i-1]}(t),\mathbf{u}^{[i-1]}(t))\mathbf{x}^{[i]}(t), \tag{21}$$

where $\mathbf{D}(\mathbf{x}^{[i-1]}(t),\mathbf{u}^{[i-1]}(t)) \triangleq -\mathbf{R}^{-1}(\mathbf{x}^{[i-1]}(t))\mathbf{B}^{\mathrm{T}}(\mathbf{x}^{[i-1]}(t),\mathbf{u}^{[i-1]}(t))\mathbf{P}^{[i]}(t)$, which yields the perturbed system (18), given in the general form by the approximating sequence

$$\dot{\mathbf{x}}^{[0]}(t)=\mathbf{A}(\mathbf{x}_0)\mathbf{x}^{[0]}(t)+\mathbf{E}(\mathbf{x}_0,\mathbf{0})\mathbf{x}^{[0]}(t),$$

$$\dot{\mathbf{x}}^{[i]}(t)=\mathbf{A}(\mathbf{x}^{[i-1]}(t))\mathbf{x}^{[i]}(t)+\mathbf{E}(\mathbf{x}^{[i-1]}(t),\mathbf{u}^{[i-1]}(t))\mathbf{x}^{[i]}(t),$$

$$\mathbf{x}^{[i]}(t_0)=\mathbf{x}_0, \quad i\geqslant 0, \tag{22}$$

corresponding to the perturbed nonlinear system

$$\dot{\mathbf{x}}(t)=\mathbf{A}(\mathbf{x}(t))\mathbf{x}(t)+\mathbf{E}(\mathbf{x}(t),\mathbf{u}(t))\mathbf{x}(t), \qquad \mathbf{x}(t_0)=\mathbf{x}_0, \tag{23}$$

where $\mathbf{E}(\mathbf{x}(t),\mathbf{u}(t)) \triangleq -\mathbf{S}(\mathbf{x}(t),\mathbf{u}(t))\mathbf{P}(t)$ and $\mathbf{S}(\mathbf{x}(t),\mathbf{u}(t))$ is given by the limit solution of (17). In addition to conditions (A1) and (A2) imposed on $\mathbf{A}(\mathbf{x})$, let us also suppose that $\mathbf{D}(\mathbf{x},\mathbf{u})$ and $\mathbf{E}(\mathbf{x},\mathbf{u})$ are bounded and Lipschitz continuous in their arguments $\mathbf{x}$ and $\mathbf{u}$, thus satisfying

(A3) $\|\mathbf{D}(\mathbf{x},\mathbf{u})\| \leqslant \delta_1$,
(A4) $\|\mathbf{D}(\mathbf{x}_1,\mathbf{u}_1)-\mathbf{D}(\mathbf{x}_2,\mathbf{u}_2)\| \leqslant \delta_2\|\mathbf{x}_1-\mathbf{x}_2\|+\delta_3\|\mathbf{u}_1-\mathbf{u}_2\|$,
(A5) $\|\mathbf{E}(\mathbf{x},\mathbf{u})\| \leqslant \varepsilon_1$,
(A6) $\|\mathbf{E}(\mathbf{x}_1,\mathbf{u}_1)-\mathbf{E}(\mathbf{x}_2,\mathbf{u}_2)\| \leqslant \varepsilon_2\|\mathbf{x}_1-\mathbf{x}_2\|+\varepsilon_3\|\mathbf{u}_1-\mathbf{u}_2\|$,

$\forall\mathbf{x}\in\mathbb{R}^n$, $\forall\mathbf{u}\in\mathbb{R}^m$, and for finite positive numbers $\delta_1,\delta_2,\delta_3,\varepsilon_1,\varepsilon_2$ and ε_3.

The approximating sequence (22), which is represented as an inhomogeneous differential equation, has a solution given by the variation of constants formula

$$\mathbf{x}^{[i]}(t)=\mathbf{\Phi}^{[i-1]}(t,t_0)\mathbf{x}^{[i]}(t_0)+\int_{t_0}^{t}\mathbf{\Phi}^{[i-1]}(t,s)\mathbf{E}(\mathbf{x}^{[i-1]}(s),\mathbf{u}^{[i-1]}(s))\mathbf{x}^{[i]}(s)\,\mathrm{d}s \tag{24}$$

so that

$$\|\mathbf{x}^{[i]}(t)\| \leqslant \|\mathbf{\Phi}^{[i-1]}(t,t_0)\|\|\mathbf{x}_0\|+\int_{t_0}^{t}\|\mathbf{\Phi}^{[i-1]}(t,s)\|\|\mathbf{E}(\mathbf{x}^{[i-1]}(s),\mathbf{u}^{[i-1]}(s))\|\|\mathbf{x}^{[i]}(s)\|\,\mathrm{d}s.$$

Assuming (A1) and (A5) hold, Brauer's inequality (20) can be used to give

$$e^{-\mu_0 t}\|\mathbf{x}^{[i]}(t)\| \leqslant e^{-\mu_0 t_0}\|\mathbf{x}_0\| + \int_{t_0}^{t} \varepsilon_1 e^{-\mu_0 s}\|\mathbf{x}^{[i]}(s)\|\,ds$$

and therefore from Gronwall–Bellman's inequality

$$\|\mathbf{x}^{[i]}(t)\| \leqslant \|\mathbf{x}_0\| e^{(\mu_0+\varepsilon_1)(t-t_0)}, \tag{25}$$

which is bounded by a small interval time $t \in [t_0, t_f]$ or for small $\mathbf{x}_0$. Now, from (24), $\mathbf{x}^{[i]}(t) - \mathbf{x}^{[i-1]}(t)$ can be written in the form

$$\begin{aligned}
&\mathbf{x}^{[i]}(t) - \mathbf{x}^{[i-1]}(t) \\
&\quad = [\mathbf{\Phi}^{[i-1]}(t,t_0) - \mathbf{\Phi}^{[i-2]}(t,t_0)]\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}^{[i-1]}(t,s)\mathbf{E}(\mathbf{x}^{[i-1]}(s),\mathbf{u}^{[i-1]}(s))[\mathbf{x}^{[i]}(s) - \mathbf{x}^{[i-1]}(s)]\,ds \\
&\qquad + \int_{t_0}^{t} \mathbf{\Phi}^{[i-1]}(t,s)[\mathbf{E}(\mathbf{x}^{[i-1]}(s),\mathbf{u}^{[i-1]}(s)) - \mathbf{E}(\mathbf{x}^{[i-2]}(s),\mathbf{u}^{[i-2]}(s))]\mathbf{x}^{[i-1]}(s)\,ds \\
&\qquad + \int_{t_0}^{t} [\mathbf{\Phi}^{[i-1]}(t,s) - \mathbf{\Phi}^{[i-2]}(t,s)]\mathbf{E}(\mathbf{x}^{[i-2]}(s),\mathbf{u}^{[i-2]}(s))\mathbf{x}^{[i-1]}(s)\,ds.
\end{aligned}$$

Suppose that

$$\xi^{[i]}(t) \triangleq \sup_{s\in[t_0,t]} \|\mathbf{x}^{[i]}(s) - \mathbf{x}^{[i-1]}(s)\|, \tag{26}$$

$$\theta^{[i]}(t) \triangleq \sup_{s\in[t_0,t]} \|\mathbf{u}^{[i]}(s) - \mathbf{u}^{[i-1]}(s)\|. \tag{27}$$

Since $\|\mathbf{x}^{[i]}(t) - \mathbf{x}^{[i-1]}(t)\| \leqslant \xi^{[i]}(t)$, using (20), (25)–(27), assumptions (A1), (A2), (A5), (A6), and Lemma 1,

$$\begin{aligned}
\xi^{[i]}(t) &\leqslant \alpha\|\mathbf{x}_0\| e^{\mu_0(t-t_0)}(t-t_0)\xi^{[i-1]}(t) + \varepsilon_1 \int_{t_0}^{t} e^{\mu_0(t-s)}\xi^{[i]}(s)\,ds \\
&\quad + \|\mathbf{x}_0\| e^{\mu_0(t-t_0)} \int_{t_0}^{t} e^{\varepsilon_1(s-t_0)}[\varepsilon_2\xi^{[i-1]}(s) + \varepsilon_3\theta^{[i-1]}(s)]\,ds \\
&\quad + \alpha\varepsilon_1\|\mathbf{x}_0\| e^{\mu_0(t-t_0)} \int_{t_0}^{t} e^{\varepsilon_1(s-t_0)}(t-s)\xi^{[i-1]}(s)\,ds.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left(1 - \varepsilon_1\int_{t_0}^{t} e^{\mu_0(t-s)}\,ds\right)\xi^{[i]}(t) \\
&\quad \leqslant \left\{\alpha(t-t_0) + [\varepsilon_2 + \alpha\varepsilon_1(t-t_0)]\int_{t_0}^{t} e^{\varepsilon_1(s-t_0)}\,ds\right\}\|\mathbf{x}_0\| e^{\mu_0(t-t_0)}\xi^{[i-1]}(t) \\
&\qquad + \varepsilon_3\|\mathbf{x}_0\| e^{\mu_0(t-t_0)}\theta^{[i-1]}(t)\int_{t_0}^{t} e^{\varepsilon_1(s-t_0)}\,ds,
\end{aligned}$$

which gives

$$\xi^{[i]}(t) \leqslant \psi_1(t)\xi^{[i-1]}(t) + \psi_2(t)\theta^{[i-1]}(t), \tag{28}$$

where

$$\psi_1(t) \triangleq \frac{\{[\varepsilon_2/\varepsilon_1 + \alpha(t-t_0)]\mathrm{e}^{\varepsilon_1(t-t_0)} - \varepsilon_2/\varepsilon_1\}\|\mathbf{x}_0\|\mathrm{e}^{\mu_0(t-t_0)}}{1 + (\varepsilon_1/\mu_0)(1 - \mathrm{e}^{\mu_0(t-t_0)})},$$

$$\psi_2(t) \triangleq \frac{(\varepsilon_3/\varepsilon_1)\|\mathbf{x}_0\|\mathrm{e}^{\mu_0(t-t_0)}(\mathrm{e}^{\varepsilon_1(t-t_0)} - 1)}{1 + (\varepsilon_1/\mu_0)(1 - \mathrm{e}^{m}\mu_0(t-t_0))}.$$

Similarly, from (21), let us write

$$\begin{aligned}\mathbf{u}^{[i]}(t) - \mathbf{u}^{[i-1]}(t) &= \mathbf{D}(\mathbf{x}^{[i-1]}(t), \mathbf{u}^{[i-1]}(t))[\mathbf{x}^{[i]}(t) - \mathbf{x}^{[i-1]}(t)] \\ &\quad + [\mathbf{D}(\mathbf{x}^{[i-1]}(t), \mathbf{u}^{[i-1]}(t)) - \mathbf{D}(\mathbf{x}^{[i-2]}(t), \mathbf{u}^{[i-2]}(t))]\mathbf{x}^{[i-1]}(t),\end{aligned}$$

which, using (25)–(28) and assumptions (A3) and (A4), gives

$$\theta^{[i]}(t) \leqslant \psi_3(t)\xi^{[i-1]}(t) + \psi_4(t)\theta^{[i-1]}(t), \tag{29}$$

where

$$\psi_3(t) \triangleq \delta_1\psi_1(t) + \delta_2\|\mathbf{x}_0\|\mathrm{e}^{(\mu_0+\varepsilon_1)(t-t_0)},$$

$$\psi_4(t) \triangleq \delta_1\psi_2(t) + \delta_3\|\mathbf{x}_0\|\mathrm{e}^{(\mu_0+\varepsilon_1)(t-t_0)}.$$

Eqs. (28) and (29) are coupled, and together can be represented in the form

$$\mathbf{\Gamma}^{[i]}(t) \leqslant \mathbf{\Psi}(t)\mathbf{\Gamma}^{[i-1]}(t), \tag{30}$$

where

$$\mathbf{\Gamma}^{[i]}(t) \triangleq \begin{bmatrix} \xi^{[i]}(t) \\ \theta^{[i]}(t) \end{bmatrix} \quad \text{and} \quad \mathbf{\Psi}(t) \triangleq \begin{bmatrix} \psi_1(t) & \psi_2(t) \\ \psi_3(t) & \psi_4(t) \end{bmatrix}.$$

Lemma 2. *Let* $\mathbf{A}(\mathbf{x})$ *satisfy* (A1) *and* (A2), *and let* $\mathbf{D}(\mathbf{x},\mathbf{u})$ *and* $\mathbf{E}(\mathbf{x},\mathbf{u})$ *be bounded and Lipschitz continuous in their arguments such that* (A3)–(A6) *are also satisfied. Suppose that* $\|\mathbf{\Psi}(t)\| < 1$ *for* $t \in [t_0, t_\mathrm{f}]$ *for some* $t_\mathrm{f} > t_0$ *(possibly* ∞*). Then, for small enough* t *or* $\mathbf{x}_0$, *the limit of the solution of the approximating sequence* (22) *on* $C([t_0,t_\mathrm{f}];\mathbb{R}^n)$ *converges to the unique solution of* (23) *on* $[t_0,t_\mathrm{f}]$.

Proof. The proof follows directly from (30) since, by induction, $\mathbf{\Gamma}^{[i]}(t)$ satisfies

$$\mathbf{\Gamma}^{[i]}(t) \leqslant \mathbf{\Psi}^{i-1}(t)\mathbf{\Gamma}^{[1]}(t).$$

This implies that $\mathbf{x}^{[i]}(t)$ and $\mathbf{u}^{[i]}(t)$ are Cauchy sequences in the Banach spaces $C([t_0,t_\mathrm{f}];\mathbb{R}^n)$ and $C([t_0,t_\mathrm{f}];\mathbb{R}^m)$, respectively. Therefore, if $\|\mathbf{\Psi}(t)\| < 1$ for $t \in [t_0,t_\mathrm{f}]$, $\mathbf{x}^{[i]}(t) \to \mathbf{x}(t)$ on $C([t_0,t_\mathrm{f}];\mathbb{R}^n)$ and $\mathbf{u}^{[i]}(t) \to \mathbf{u}(t)$ on $C([t_0,t_\mathrm{f}];\mathbb{R}^m)$. □

Bounds on $\sup_{t\in[t_0,t_f]}\|\mathbf{x}(t)\|$ and $\sup_{t\in[t_0,t_f]}\|\mathbf{u}(t)\|$ can now be found where $\mathbf{x}(\cdot)=\lim_{i\to\infty}\mathbf{x}^{[i]}(\cdot)$ and $\mathbf{u}(\cdot)=\lim_{i\to\infty}\mathbf{u}^{[i]}(\cdot)$, so that if $\boldsymbol{\gamma}(t)\triangleq[\mathbf{x}(t)\ \mathbf{u}(t)]^{\mathrm{T}}$ then the limit is taken in $C([t_0,t_f];\mathbb{R}^{n+m})$ as follows:

$$
\begin{aligned}
\sup_{t\in[t_0,t_f]}\|\boldsymbol{\gamma}^{[i]}(t)\| &= \sup_{t\in[t_0,t_f]}\|\boldsymbol{\gamma}^{[i]}(t)-\boldsymbol{\gamma}^{[i-1]}(t)+\boldsymbol{\gamma}^{[i-1]}(t)-\cdots-\boldsymbol{\gamma}^{[0]}(t)+\boldsymbol{\gamma}^{[0]}(t)\|\\
&\leqslant \sum_{j=1}^{i}\|\boldsymbol{\Gamma}^{[j]}(t)\|+\sup_{t\in[t_0,t_f]}\|\boldsymbol{\gamma}^{[0]}(t)\|\\
&\leqslant \|\boldsymbol{\Gamma}^{[1]}(t)\|\sum_{j=1}^{i}\|\boldsymbol{\Psi}(t)\|^{j-1}+\sup_{t\in[t_0,t_f]}\|\boldsymbol{\gamma}^{[0]}(t)\|\\
&= \frac{1-\|\boldsymbol{\Psi}(t)\|^{i}}{1-\|\boldsymbol{\Psi}(t)\|}\|\boldsymbol{\Gamma}^{[1]}(t)\|+\sup_{t\in[t_0,t_f]}\|\boldsymbol{\gamma}^{[0]}(t)\|.
\end{aligned}
$$

Hence, if $\|\boldsymbol{\Psi}\|=\sup_{t\in[t_0,t_f]}\|\boldsymbol{\Psi}(t)\|$, as $i\to\infty$

$$
\sup_{t\in[t_0,t_f]}\|\boldsymbol{\gamma}(t)\|\leqslant\frac{1}{1-\|\boldsymbol{\Psi}\|}\|\boldsymbol{\Gamma}^{[1]}(t)\|+\sup_{t\in[t_0,t_f]}\|\boldsymbol{\gamma}^{[0]}(t)\|. \tag{31}
$$

Lemma 3. *If* $\bar{\mu}=\mu(\mathbf{A}(\mathbf{x}_0))$ *and* $\bar{\delta}=\|\mathbf{D}(\mathbf{x}_0,\mathbf{0})\|$, *under assumptions* (A1)–(A6), *bounds on* $\mathbf{u}^{[k]}(t)$ *and* $\mathbf{x}^{[k]}(t)$ *for* $k=0,1$ *are obtained from* (21) *and* (25), *respectively, which yield*

$$
\xi^{[1]}(t)=\sup_{s\in[t_0,t]}\|\mathbf{x}^{[1]}(t)-\mathbf{x}^{[0]}(t)\|\leqslant 2\|\mathbf{x}_0\|\mathrm{e}^{(\mu_0+\varepsilon_1)(t-t_0)},
$$

$$
\theta^{[1]}(t)=\sup_{s\in[t_0,t]}\|\mathbf{u}^{[1]}(t)-\mathbf{u}^{[0]}(t)\|\leqslant 2\delta_1\|\mathbf{x}_0\|\mathrm{e}^{(\mu_0+\varepsilon_1)(t-t_0)}
$$

since $\bar{\mu}\leqslant\mu_0$ *and* $\bar{\delta}\leqslant\delta_1$.

Corollary 1. *Under assumptions* (A1)–(A6),

$$
\sup_{t\in[t_0,t_f]}\|\boldsymbol{\gamma}(t)\|\leqslant\left(\frac{2}{1-\|\boldsymbol{\Psi}\|}+1\right)\|\mathbf{x}_0\|\mathrm{e}^{(\mu_0+\varepsilon_1)(t-t_0)}\sqrt{1+\delta_1^2}.
$$

Hence, although these conditions have been assumed for all $\mathbf{x}\in\mathbb{R}^n$ *and* $\mathbf{u}\in\mathbb{R}^m$, *if* $\mu_0+\varepsilon_1<0$, *it is clear that* (A1)–(A6) *only require to hold in the ball*

$$
B=\left\{\boldsymbol{\gamma}:\|\boldsymbol{\gamma}\|\leqslant\left(\frac{2}{1-\|\boldsymbol{\Psi}\|}+1\right)\|\mathbf{x}_0\|\sqrt{1+\delta_1^2}\right\}.
$$

This means that the results apply to polynomial systems, which, of course, do not satisfy (A1)–(A6) *on the whole of* $\mathbb{R}^{n+m}$. *Therefore the results are not as restrictive as they might appear.*

Proof. Since $\|\boldsymbol{\Gamma}^{[1]}(t)\|=\|\xi^{[1]}(t)\ \theta^{[1]}(t)\|$ and $\sup_{t\in[t_0,t_f]}\|\boldsymbol{\gamma}^{[0]}(t)\|=\|\mathbf{x}^{[0]}(t)\ \mathbf{u}^{[0]}(t)\|$, bounds for $\boldsymbol{\Gamma}^{[1]}(t)$ and $\boldsymbol{\gamma}^{[0]}(t)$ are obtained using (21), (25) and Lemma 3. The result then follows from (31). □

In the case of the controlled sequence (18),

$$\|\mathbf{D}(\mathbf{x}^{[i-1]}(t),\mathbf{u}^{[i-1]}(t))\| \leqslant \|\mathbf{B}(\mathbf{x}^{[i-1]}(t),\mathbf{u}^{[i-1]}(t))\|\|\mathbf{R}^{-1}(\mathbf{x}^{[i-1]}(t))\|\|\mathbf{P}^{[i]}(t)\|,$$

$$\|\mathbf{E}(\mathbf{x}^{[i-1]}(t),\mathbf{u}^{[i-1]}(t))\| \leqslant \|\mathbf{B}(\mathbf{x}^{[i-1]}(t),\mathbf{u}^{[i-1]}(t))\|^2\|\mathbf{R}^{-1}(\mathbf{x}^{[i-1]}(t))\|\|\mathbf{P}^{[i]}(t)\|.$$

The following lemma provides bounds on $\|\mathbf{P}^{[i]}(t)\|$ and $\|\mathbf{P}^{[i]}(t)-\mathbf{P}^{[i-1]}(t)\|$.

Lemma 4 (Çimen and Banks [12]). *Provided* $\mathbf{A}(\mathbf{x})$ *satisfies* (A1) *and* (A2), *under some bounded growth and Lipschitz conditions of* $\mathbf{B}(\mathbf{x},\mathbf{u})$, $\mathbf{Q}(\mathbf{x})$ *and* $\mathbf{R}^{-1}(\mathbf{x})$, *the ASRE* $\mathbf{P}^{[i]}(t)$ *are bounded and Lipschitz continuous on* $[t_0,t_f]$ *for small enough* t_f.

So far, given any initial state $\mathbf{x}_0$, the sequence of LTV control systems obtained by classical linear-quadratic methods for the nonlinear-nonquadratic optimal state-regulator control problem has been shown to converge uniformly on some small interval $[t_0,t_f]$, where t_f may depend on $\mathbf{x}_0$. This has been shown in the space of continuous functions, under local Lipschitz continuity of the nonlinear dynamical operators $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x},\mathbf{u})$, and for small finite-time t_f. Recent results by Çimen and Banks [12] can be applied to conclude convergence in more general cases so that, if a unique solution to the nonlinear optimal control problem exists, for which the optimal cost is finite and the optimal trajectory is bounded in the interval $[t_0,\tau]\subseteq\mathbb{R}$, then the approximating sequences can be shown to converge uniformly on $[t_0,\tau]$.

Lemma 5 (Çimen and Banks [12]). *Suppose that the nonlinear optimal control problem has a unique continuous feedback control on the interval* $[t_0,\tau]$. *Then the controlled sequence of functions* $\{\mathbf{x}^{[i]}(t)\}$ *and feedback controls* $\{\mathbf{u}^{[i]}(t)\}$ *defined by the linear-quadratic, time-varying approximations converge uniformly on* $[t_0,\tau]$ *to this solution.*

The proof of Theorem 2 for the nonlinear optimal stabilizing ASRE feedback control law now follows directly from Lemmas 1–5. Since the sequence $\{\mathbf{x}^{[i]}(t)\}$ converges in $C([t_0,t_f];\mathbb{R}^n)$ and since the ASRE $\{\mathbf{P}^{[i]}(t)\}$ are bounded and Lipschitz continuous, the feedback controls $\{\mathbf{u}^{[i]}(t)\}$ also converge in $C([t_0,t_f];\mathbb{R}^m)$ when expressed in the feedback form (19).

4. Global optimal feedback control of nonlinear systems

This section outlines some very general necessary conditions for the optimality of a control $\mathbf{u}$ of the nonquadratic optimization problem (12) subject to the control-nonaffine nonlinear dynamical constraint (11). In order to develop necessary conditions, which an optimal control must satisfy, it is assumed that an optimal control $\mathbf{u}^*$ exists. (Indeed, there may be many or none at all.) Therefore, in analogy to classical LQR optimal control theory, from the maximum principle, the Hamiltonian for the nonlinear-nonquadratic regulator problem (11), (12) becomes

$$H=\tfrac{1}{2}(\mathbf{x}^{\mathrm{T}}\mathbf{Q}(\mathbf{x})\mathbf{x}+\mathbf{u}^{\mathrm{T}}\mathbf{R}(\mathbf{x})\mathbf{u})+\boldsymbol{\lambda}^{\mathrm{T}}(\mathbf{A}(\mathbf{x})\mathbf{x}+\mathbf{B}(\mathbf{x},\mathbf{u})\mathbf{u}),$$

which satisfies the equations $\dot{\boldsymbol{\lambda}}=-\partial H/\partial\mathbf{x}$ and $\partial H/\partial\mathbf{u}=\mathbf{0}$ along an optimal trajectory, giving

$$\dot{\boldsymbol{\lambda}}=-\left[\mathbf{Q}(\mathbf{x})\mathbf{x}+\tfrac{1}{2}\mathbf{x}^{\mathrm{T}}\frac{\partial\mathbf{Q}(\mathbf{x})}{\partial\mathbf{x}}\mathbf{x}+\tfrac{1}{2}\mathbf{u}^{\mathrm{T}}\frac{\partial\mathbf{R}(\mathbf{x})}{\partial\mathbf{x}}\mathbf{u}+\mathbf{A}^{\mathrm{T}}(\mathbf{x})\boldsymbol{\lambda}+\left(\frac{\partial\mathbf{A}(\mathbf{x})}{\partial\mathbf{x}}\mathbf{x}\right)^{\mathrm{T}}\boldsymbol{\lambda}+\left(\frac{\partial\mathbf{B}(\mathbf{x},\mathbf{u})}{\partial\mathbf{x}}\mathbf{u}\right)^{\mathrm{T}}\boldsymbol{\lambda}\right],$$

$$\mathbf{u}=-\mathbf{R}^{-1}(\mathbf{x})\mathbf{W}^{\mathrm{T}}(\mathbf{x},\mathbf{u})\boldsymbol{\lambda},$$

where

$$\mathbf{W}(\mathbf{x},\mathbf{u}) \triangleq \left(\frac{\partial \mathbf{B}(\mathbf{x},\mathbf{u})}{\partial \mathbf{u}}\mathbf{u}\right) + \mathbf{B}(\mathbf{x},\mathbf{u})$$

and, for instance, $\mathbf{x}^{\mathrm{T}}(\partial \mathbf{Q}(\mathbf{x})/\partial \mathbf{x})\mathbf{x}$ is a vector of quadratic forms represented by

$$\left(\mathbf{x}^{\mathrm{T}}\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial x_1}\mathbf{x},\ldots,\mathbf{x}^{\mathrm{T}}\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial x_n}\mathbf{x}\right)^{\mathrm{T}}.$$

Hence, substituting for $\mathbf{u}$ gives

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} - \mathbf{B}(\mathbf{x},\mathbf{u})\mathbf{R}^{-1}(\mathbf{x})\mathbf{W}^{\mathrm{T}}(\mathbf{x},\mathbf{u})\boldsymbol{\lambda}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{32}$$

together with the co-state (adjoint) equation

$$\begin{aligned}\dot{\boldsymbol{\lambda}} = &-\mathbf{Q}(\mathbf{x})\mathbf{x} - \tfrac{1}{2}\mathbf{x}^{\mathrm{T}}\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{x} - \tfrac{1}{2}\boldsymbol{\lambda}^{\mathrm{T}}\mathbf{W}(\mathbf{x},\mathbf{u})\mathbf{R}^{-1}(\mathbf{x})\frac{\partial \mathbf{R}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{R}^{-1}(\mathbf{x})\mathbf{W}^{\mathrm{T}}(\mathbf{x},\mathbf{u})\boldsymbol{\lambda}\\ &-\mathbf{A}^{\mathrm{T}}(\mathbf{x})\boldsymbol{\lambda} - \left(\frac{\partial \mathbf{A}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{x}\right)^{\mathrm{T}}\boldsymbol{\lambda} + \left(\frac{\partial \mathbf{B}(\mathbf{x},\mathbf{u})}{\partial \mathbf{x}}\mathbf{R}^{-1}(\mathbf{x})\mathbf{W}^{\mathrm{T}}(\mathbf{x},\mathbf{u})\boldsymbol{\lambda}\right)^{\mathrm{T}}\boldsymbol{\lambda},\end{aligned} \tag{33}$$

where the transversality condition $\boldsymbol{\lambda}(t_{\mathrm{f}}) = (\partial/\partial\mathbf{x})(\frac{1}{2}\mathbf{x}^{\mathrm{T}}(t_{\mathrm{f}})\mathbf{F}(\mathbf{x}(t_{\mathrm{f}}))\mathbf{x}(t_{\mathrm{f}}))$ therefore satisfies

$$\boldsymbol{\lambda}(t_{\mathrm{f}}) = \mathbf{F}(\mathbf{x}(t_{\mathrm{f}}))\mathbf{x}(t_{\mathrm{f}}) + \tfrac{1}{2}\mathbf{x}^{\mathrm{T}}(t_{\mathrm{f}})\frac{\partial \mathbf{F}(\mathbf{x}(t_{\mathrm{f}}))}{\partial \mathbf{x}}\mathbf{x}(t_{\mathrm{f}}).$$

The canonical equations (32), (33) can be combined to obtain the solution to the finite-time nonlinear-nonquadratic regulator problem (11) and (12), which is given by the coupled nonlinear two-point boundary value problem

$$\begin{pmatrix}\dot{\mathbf{x}}\\ \dot{\boldsymbol{\lambda}}\end{pmatrix} = \begin{pmatrix}\mathbf{A}(\mathbf{x}) & -\bar{\mathbf{S}}(\mathbf{x},\mathbf{u})\\ -\bar{\mathbf{Q}}(\mathbf{x},\mathbf{u},\boldsymbol{\lambda}) & -\mathbf{A}^{\mathrm{T}}(\mathbf{x})\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \boldsymbol{\lambda}\end{pmatrix}, \tag{34}$$

where

$$\bar{\mathbf{S}}(\mathbf{x},\mathbf{u}) \triangleq \mathbf{B}(\mathbf{x},\mathbf{u})\mathbf{R}^{-1}(\mathbf{x})\mathbf{W}^{\mathrm{T}}(\mathbf{x},\mathbf{u}),$$

$$\begin{aligned}\bar{\mathbf{Q}}(\mathbf{x},\mathbf{u},\boldsymbol{\lambda}) \triangleq &\ \mathbf{Q}(\mathbf{x}) + \tfrac{1}{2}\mathbf{x}^{\mathrm{T}}\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial \mathbf{x}}\tfrac{1}{2}\boldsymbol{\lambda}^{\mathrm{T}}\mathbf{W}(\mathbf{x},\mathbf{u})\mathbf{R}^{-1}(\mathbf{x})\frac{\partial \mathbf{R}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{R}^{-1}(\mathbf{x})\mathbf{W}^{\mathrm{T}}(\mathbf{x},\mathbf{u})\boldsymbol{\lambda}/\mathbf{x}\\ &+\left(\frac{\partial \mathbf{A}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{x}\right)^{\mathrm{T}}\boldsymbol{\lambda}/\mathbf{x} - \left(\frac{\partial \mathbf{B}(\mathbf{x},\mathbf{u})}{\partial \mathbf{x}}\mathbf{R}^{-1}(\mathbf{x})\mathbf{W}^{\mathrm{T}}(\mathbf{x},\mathbf{u})\boldsymbol{\lambda}\right)^{\mathrm{T}}\boldsymbol{\lambda}/\mathbf{x}.\end{aligned}$$

Here $\mathbf{L}(\mathbf{x},\mathbf{u},\boldsymbol{\lambda})/\mathbf{x}$ means that $\mathbf{L}(\mathbf{x},\mathbf{u},\boldsymbol{\lambda})$ is assumed to have a factor $\mathbf{x}$, which can be written as $\mathbf{L}(\mathbf{x},\mathbf{u},\boldsymbol{\lambda}) = \mathbf{M}(\mathbf{x},\mathbf{u},\boldsymbol{\lambda})\mathbf{x}$ for some function $\mathbf{M}(\mathbf{x},\mathbf{u},\boldsymbol{\lambda})$, so that $\mathbf{L}(\mathbf{x},\mathbf{u},\boldsymbol{\lambda})/\mathbf{x} = \mathbf{M}(\mathbf{x},\mathbf{u},\boldsymbol{\lambda})$.

In order to solve the nonlinear two-point boundary value problem (34), it is replaced by a sequence of LTV approximations [6]

$$\begin{pmatrix}\dot{\mathbf{x}}^{[i]}(t)\\ \dot{\boldsymbol{\lambda}}^{[i]}(t)\end{pmatrix} = \begin{pmatrix}\tilde{\mathbf{A}}(t) & -\tilde{\mathbf{S}}(t)\\ -\tilde{\mathbf{Q}}(t) & -\tilde{\mathbf{A}}^{\mathrm{T}}(t)\end{pmatrix}\begin{pmatrix}\mathbf{x}^{[i]}(t)\\ \boldsymbol{\lambda}^{[i]}(t)\end{pmatrix}, \tag{35}$$

where

$$
\begin{aligned}
\tilde{\mathbf{A}}(t) &\triangleq \mathbf{A}(\mathbf{x}^{[i-1]}(t)),\\
\tilde{\mathbf{S}}(t) &\triangleq \bar{\mathbf{S}}(\mathbf{x}^{[i-1]}(t),\mathbf{u}^{[i-1]}(t)) = \tilde{\mathbf{B}}(t)\tilde{\mathbf{R}}^{-1}(t)\tilde{\mathbf{W}}^{\mathrm{T}}(t),\\
\tilde{\mathbf{B}}(t) &\triangleq \mathbf{B}(\mathbf{x}^{[i-1]}(t),\mathbf{u}^{[i-1]}(t)),\\
\tilde{\mathbf{R}}(t) &\triangleq \mathbf{R}(\mathbf{x}^{[i-1]}(t)),\\
\tilde{\mathbf{W}}(t) &\triangleq \mathbf{W}(\mathbf{x}^{[i-1]}(t),\mathbf{u}^{[i-1]}(t)),\\
\tilde{\mathbf{Q}}(t) &\triangleq \bar{\mathbf{Q}}(\mathbf{x}^{[i-1]}(t),\mathbf{u}^{[i-1]}(t),\boldsymbol{\lambda}^{[i-1]}(t))
\end{aligned} \tag{36}
$$

with

$$
\mathbf{x}^{[i]}(t_0) = \mathbf{x}_0, \qquad \boldsymbol{\lambda}^{[i]}(t_{\mathrm{f}}) = \tilde{\mathbf{F}}\mathbf{x}^{[i]}(t_{\mathrm{f}}), \tag{37}
$$

where the matrix

$$
\begin{aligned}
\tilde{\mathbf{F}} &\triangleq \mathbf{F}(\mathbf{x}^{[i-1]}(t_{\mathrm{f}})) + \tfrac{1}{2}\,\mathbf{x}^{[i-1]\mathrm{T}}(t_{\mathrm{f}})\frac{\partial \mathbf{F}(\mathbf{x}^{[i-1]}(t_{\mathrm{f}}))}{\partial \mathbf{x}}\\
&= \mathbf{F}(\mathbf{x}^{[i-1]}(t_{\mathrm{f}})) + \tfrac{1}{2}\left(x_1^{[i-1]}(t_{\mathrm{f}})\,\frac{\partial \mathbf{F}(\mathbf{x}^{[i-1]}(t_{\mathrm{f}}))}{\partial x_1} + \cdots + x_n^{[i-1]}(t_{\mathrm{f}})\,\frac{\partial \mathbf{F}(\mathbf{x}^{[i-1]}(t_{\mathrm{f}}))}{\partial x_n}\right).
\end{aligned} \tag{38}
$$

It is well known that the linear two-point boundary value problem (6) with conditions (7) and (8) represents the classical optimal control of a linear system with quadratic cost (5) subject to the dynamics (4), the solution of which is given by Theorem 1. Therefore, system (35) with conditions (36)–(38) is equivalent to the classical optimal control of a linear system

$$
\dot{\mathbf{x}}(t) = \tilde{\mathbf{A}}(t)\mathbf{x}(t) + \tilde{\mathbf{B}}(t)\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{39}
$$

with quadratic cost

$$
\tilde{J}(\mathbf{u}) = \tfrac{1}{2}\,\mathbf{x}^{\mathrm{T}}(t_{\mathrm{f}})\tilde{\mathbf{F}}x(t_{\mathrm{f}}) + \frac{1}{2}\int_{t_0}^{t_{\mathrm{f}}} \{\mathbf{x}^{\mathrm{T}}(t)\tilde{\mathbf{Q}}(t)\mathbf{x}(t) + \mathbf{u}^{\mathrm{T}}(t)\tilde{\mathbf{R}}(t)\mathbf{u}(t)\}\,\mathrm{d}t \tag{40}
$$

and so the solution is given by the Riccati equation

$$
\dot{\mathbf{P}}(t) = -\tilde{\mathbf{Q}}(t) - \mathbf{P}(t)\tilde{\mathbf{A}}(t) - \tilde{\mathbf{A}}^{\mathrm{T}}(t)\mathbf{P}(t) + \mathbf{P}(t)\tilde{\mathbf{S}}(t)\mathbf{P}(t), \qquad \mathbf{P}(t_{\mathrm{f}}) = \tilde{\mathbf{F}} \tag{41}
$$

together with the optimal feedback control law

$$
\mathbf{u}(t) = -\tilde{\mathbf{R}}^{-1}(t)\tilde{\mathbf{W}}^{\mathrm{T}}(t)\mathbf{P}(t)\mathbf{x}(t), \tag{42}
$$

where it is assumed that the co-state $\boldsymbol{\lambda}(t) = \mathbf{P}(t)\mathbf{x}(t)$ as in classical linear optimal control theory. Using exactly the same techniques of Section 3.2, $\mathbf{x}^{[i]}(t)$ and $\mathbf{u}^{[i]}(t)$ can be shown to converge on $C([t_0,t_{\mathrm{f}}];\mathbb{R}^n)$ and $C([t_0,t_{\mathrm{f}}];\mathbb{R}^m)$, respectively. In addition, this would require $\tilde{\mathbf{W}}(t)$ and $\tilde{\mathbf{Q}}(t)$ are bounded and Lipschitz continuous, which must therefore satisfy that $\mathbf{A}(\mathbf{x})$, $\mathbf{B}(\mathbf{x},\mathbf{u})$, $\mathbf{F}(\mathbf{x})$, $\mathbf{Q}(\mathbf{x})$ and $\mathbf{R}(\mathbf{x})$ are also continuously differentiable matrix-valued functions with respect to their arguments, whose Jacobian matrices are subject to some bounded growth and Lipschitz conditions. The limit of the sequence of LTV systems (39)–(42) would then globally converge on $C([t_0,t_{\mathrm{f}}];\mathbb{R}^{2n})$ to the unique solution of (34) on $[t_0,t_{\mathrm{f}}]$. Hence the implication of the necessary conditions for a "global" optimal feedback solution of the nonlinear-nonquadratic regulator control problem (34) is given by the following theorem:

Theorem 3 (“Global” nonlinear optimal stabilizing feedback control). *Given the nonlinear system* (11) *and the cost functional* (12), *where* $\mathbf{u}(t)$ *is unconstrained, the terminal time* t_f *is specified,* $\mathbf{R}(\mathbf{x})$ *is positive-definite* $\forall\mathbf{x}$, $\mathbf{F}(\mathbf{x})$ *and* $\mathbf{Q}(\mathbf{x})$ *are positive-semidefinite for* $\forall\mathbf{x}$, *then an optimal feedback control is given by the limit of the sequence*

$$\mathbf{u}^{*[i]}(t) = -\mathbf{R}^{-1}(\mathbf{x}^{[i-1]}(t))\tilde{\mathbf{W}}^{\mathrm{T}}(t)\mathbf{P}^{[i]}(t)\mathbf{x}^{[i]}(t), \qquad i \geqslant 0,$$

where the $n\times n$ *symmetric matrix* $\mathbf{P}(t)$ *is the unique solution of the sequence of Riccati equations*

$$\begin{aligned}\dot{\mathbf{P}}^{[i]}(t) &= -\tilde{\mathbf{Q}}(t) - \mathbf{P}^{[i]}(t)\mathbf{A}(\mathbf{x}^{[i-1]}(t)) - \mathbf{A}^{\mathrm{T}}(\mathbf{x}^{[i-1]}(t))\mathbf{P}^{[i]}(t)\\ &\quad + \mathbf{P}^{[i]}(t)\mathbf{B}(\mathbf{x}^{[i-1]}(t),\mathbf{u}^{[i-1]}(t))\mathbf{R}^{-1}(\mathbf{x}^{[i-1]}(t))\tilde{\mathbf{W}}^{\mathrm{T}}(t)\mathbf{P}^{[i]}(t),\\ \mathbf{P}^{[i]}(t_f) &= \mathbf{F}(\mathbf{x}^{[i-1]}(t_f)) + \tfrac{1}{2}\,\mathbf{x}^{[i-1]\mathrm{T}}(t_f)\,\frac{\partial\mathbf{F}(\mathbf{x}^{[i-1]}(t_f))}{\partial\mathbf{x}}\end{aligned}$$

with

$$\begin{aligned}\tilde{\mathbf{W}}(t) &= \left(\frac{\partial\mathbf{B}(\mathbf{x}^{[i-1]}(t),\mathbf{u}^{[i-1]}(t))}{\partial\mathbf{u}}\,\mathbf{u}^{[i-1]}(t)\right) + \mathbf{B}(\mathbf{x}^{[i-1]}(t),\mathbf{u}^{[i-1]}(t)),\\ \tilde{\mathbf{Q}}(t) &= \mathbf{Q}(\mathbf{x}^{[i-1]}(t)) + \tfrac{1}{2}\,\mathbf{x}^{[i-1]\mathrm{T}}(t)\,\frac{\partial\mathbf{Q}(\mathbf{x}^{[i-1]}(t))}{\partial\mathbf{x}} + \left(\frac{\partial\mathbf{A}(\mathbf{x}^{[i-1]}(t))}{\partial\mathbf{x}}\,\mathbf{x}^{[i-1]}(t)\right)^{\mathrm{T}}\mathbf{P}^{[i-1]}(t)\\ &\quad + \tfrac{1}{2}\,\mathbf{x}^{[i-1]\mathrm{T}}(t)\mathbf{P}^{[i-1]}(t)\tilde{\mathbf{W}}(t)\mathbf{R}^{-1}(\mathbf{x}^{[i-1]}(t))\frac{\partial\mathbf{R}(\mathbf{x}^{[i-1]}(t))}{\partial\mathbf{x}}\,\mathbf{R}^{-1}(\mathbf{x}^{[i-1]}(t))\tilde{\mathbf{W}}^{\mathrm{T}}(t)\mathbf{P}^{[i-1]}(t)\\ &\quad - \left(\frac{\partial\mathbf{B}(\mathbf{x}^{[i-1]}(t),\mathbf{u}^{[i-1]}(t))}{\partial\mathbf{x}}\mathbf{R}^{-1}(\mathbf{x}^{[i-1]}(t))\tilde{\mathbf{W}}^{\mathrm{T}}(t)\mathbf{P}^{[i-1]}(t)\mathbf{x}^{[i-1]}(t)\right)^{\mathrm{T}}\mathbf{P}^{[i-1]}(t).\end{aligned}$$

The state of the optimal system is then given by the converged solution of the sequence of LTV differential equations

$$\dot{\mathbf{x}}^{*[i]}(t) = [\mathbf{A}(\mathbf{x}^{[i-1]}(t)) - \mathbf{B}(\mathbf{x}^{[i-1]}(t),\mathbf{u}^{[i-1]}(t))\mathbf{R}^{-1}(\mathbf{x}^{[i-1]}(t))\tilde{\mathbf{W}}^{\mathrm{T}}(t)\mathbf{P}^{[i]}(t)]\mathbf{x}^{*[i]}(t), \qquad \mathbf{x}^{*[i]}(t_0) = \mathbf{x}_0, \tag{43}$$

where the first approximation for $i=0$ *is obtained by evaluating the system at* $\mathbf{P}^{[i-1]}(t)=\mathbf{F}(\mathbf{x}_0)$, $\mathbf{x}^{[i-1]}(t)=\mathbf{x}_0$ *and* $\mathbf{u}^{[i-1]}(t)=\mathbf{0}$.

Remark 4. Any solution outlined in Theorem 3 (which may or may not exist), arising from Pontryagin's necessary conditions, is an extremal. In general, the sequence of LTV feedback controls may therefore converge to any extremal. The conditions are not sufficient to guarantee the existence of any optimal solution. However, if a unique continuous “global” optimal feedback control exists, the sequence will (globally) converge to this extremum.

5. Example: the F-8 Crusader

The objective of the automatic flight control system is to provide acceptable dynamics response over the entire range of angle of attack, which a modern high-performance aircraft may operate. At the specified flight condition, the F-8 stalls when the angle of attack is 0.41 rad. In an attempt to solve this problem, Garrard and Jordan [15] presented an approach for computing a nonlinear control law, and derived second- and third-order nonlinear feedback controllers by solving a truncated version of the Hamilton–Jacobi–Bellman partial differential equation. Their derivation was based on the nonlinear model (3) after eliminating nonlinear control terms in the aircraft dynamics. Since the Hamilton–Jacobi–Bellman equation cannot be solved analytically, perturbational procedures were used to obtain approximate solutions by representing the partial differential equation in a series form, leading to a tedious algebraic determination of many coefficients; 10 for the second-order controller, 15 for the third-order controller. Even though the range of recovery of the aircraft from stall was increased significantly through the use of the third-order controller, the determination of the nonlinear control was very laborious. The increasing complexity of implementing higher-order control terms and the increasing effort needed to drive them make the practicality of including higher-order control terms and using this technique questionable. Other attempts to solve the control problem for the aircraft system have also been made in the literature based on a simplified nonlinear model of the plant [13] as well as using the concept of extended linearization and gain scheduling [24]. However, these techniques are only effective locally due to linearization and approximation procedures, and cannot provide control that is optimal for the original nonlinear plant (3), since the model being used no longer represents the global motion of the nonlinear system. Even though nonlinear control terms have only a small effect on the F-8 aircraft dynamics as shown in [13,15], this is often not true for more complicated, highly unstable, nonlinear, high-performance aircraft, similar to F-16. The proposed ASRE feedback control design methodology can, however, account for any continuous nonlinearity in a given model, including nonaffine control inputs.

The purpose of the F-8 aircraft model is to apply the ASRE theory and study the effectiveness of this controller by simulation, so that the performance and the optimality of the ASRE nonlinear feedback control law can be compared with the "global" optimal feedback solution provided by the maximum principle (Theorem 3) and with that of the conventional LQR and the well-known SDRE feedback control laws. Therefore, consider representing the nonaffine nonlinear aircraft equations (3) in state-space in the factored form $\mathbf{A}(\mathbf{x})\mathbf{x}$ and $\mathbf{B}(\mathbf{x},u)u$, which, given the wide selection of possible representations, have been chosen as

$$\mathbf{A}(\mathbf{x}) = \begin{bmatrix} -0.877 + 0.47x_1 + 3.846x_1^2 - x_1x_3 & -0.019x_2 & 1 - 0.088x_1 \\ 0 & 0 & 1 \\ -4.208 - 0.47x_1 - 3.564x_1^2 & 0 & -0.396 \end{bmatrix},$$

and

$$\mathbf{B}(\mathbf{x},u) = \begin{bmatrix} -0.215 + 0.28x_1^2 + 0.47x_1u + 0.63u^2 \\ 0 \\ -20.967 + 6.265x_1^2 + 46x_1u + 61.4u^2 \end{bmatrix}. \tag{44}$$

The ASRE and "global" optimal regulator designs given by Theorems 2 and 3 can now be applied to (11) with dynamics (44). As a basis for comparison with the nonlinear ASRE controller, the conventional finite-time LQR control law is computed based on the linearized version of the nonlinear model (3). The SDRE feedback is also computed, which is based on the "*affinized*" version of the nonlinear dynamics (44), that is with $u=0$ in $\mathbf{B}$. As the LQR theory cannot account for state-dependent weighting matrices in the performance index (12), these are assigned constant values such that $\mathbf{F} = 0.1\mathbf{I}_{3\times 3}$, $\mathbf{Q} = 0.01\mathbf{I}_{3\times 3}$ and $R = 1$. Note that these weightings are chosen just to present the theory and they do not correspond to any specific design criteria. Simulations are carried out at the given constant operating point (trim condition) for both small and large initial conditions in the angle of attack, which represent small or large deviation from the trim conditions due to disturbances. The feedback control laws are then simulated with the original nonaffine nonlinear aircraft dynamics (3) in MATLAB® by numerically approximating the state-space model using Euler's method with a time-step of 0.005 s. Note that the "global" optimal feedback control algorithm of Theorem 3 yields at least a local extremal for (12) and, if it exists, the global optimal state trajectory $\mathbf{x}^*$ and control u^* for the nonlinear equations.

The first approximation in the ASRE algorithm is linear, time-invariant, and has been obtained by evaluating the nonlinear system at $\mathbf{x}=\mathbf{x}_0$ and $u=0$. After the first approximation, the recursive procedure becomes LTV.

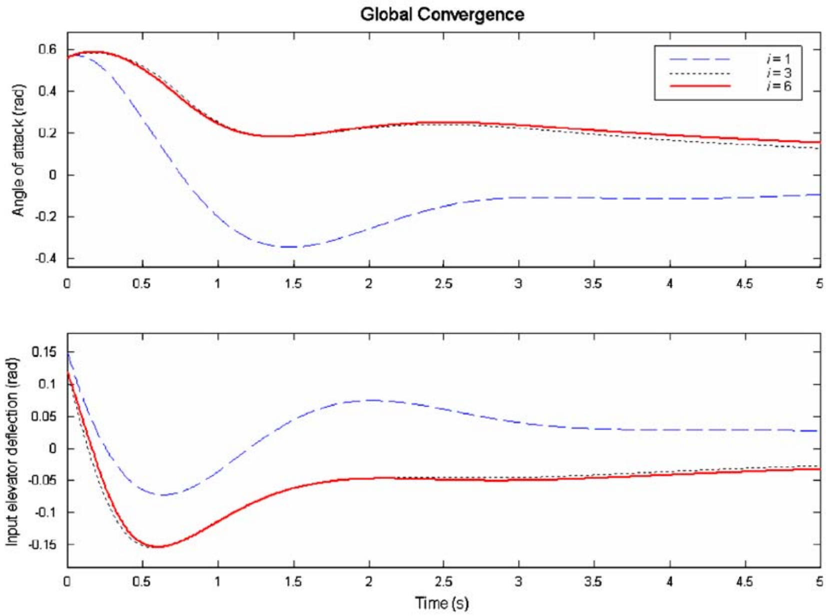

Fig. 1. ASRE feedback response of the LTV sequences $x_1^{[i]}(t)$ and $u^{[i]}(t)$ for the F-8 Crusader when $\mathbf{x}_0 = [0.56\ 0\ 0]^{\mathrm{T}}$.

In general, the convergence rate will vary for each application depending on the upper bound $\|\boldsymbol{\Psi}(t)\|$ in Lemma 2 (Section 3.2), which strongly depends on the particular choice of $\mathbf{A}(\mathbf{x},\mathbf{u})$ and $\mathbf{B}(\mathbf{x},\mathbf{u})$, the initial condition $\mathbf{x}_0$, and the finite time interval t_{f} (not necessarily in that order). Therefore, a larger $\mathbf{x}_0$ or t_{f} will certainly increase the rate of convergence of the sequence of LTV problems. Let us define the required error norm between the controlled sequence of LTV solutions by $\|\mathbf{x}^{[i]}(t) - \mathbf{x}^{[i-1]}(t)\| < \sigma$ (for some finite constant $\sigma > 0$), which needs to be satisfied if convergence is to be achieved. If, for example, $\sigma = 0.5$, $t_{\mathrm{f}} = 10$ s and the factorization is given by (44), the ASRE optimal regulator for the F-8 aircraft model requires six sequences to satisfy the required bound for converge from most initial conditions. Note that even a highly nonlinear and complex model such as a super-tanker requires only 20 iterations to converge from large initial conditions when $t_{\mathrm{f}} = 250$ s [12]. Therefore, a relatively small number of iterations generally yields an extremely accurate control solution.

The response of the sixth approximated LTV control system (*converged solution*), therefore, represents the response of the nonlinear dynamics (3) of the F-8 to the sixth approximated time-varying ASRE feedback, $\mathbf{P}^{[6]}(t)$. To illustrate the convergence of these controlled LTV systems, Fig. 1 represents the angle-of-attack response from the first, third, and sixth approximated systems and the control input generated from these when $x_1(0) = 0.56$ rad. Maximum angle of attack the aircraft could recover from stall is 0.64 rad through the use of the nonlinear nonaffine controlled ASRE feedback. Note that the range of recovery is increased significantly when compared to previous attempts [13,15,24] in the literature or the nonlinear "affinized" SDRE feedback-control system, for which maximum angle of attack before stall is 0.59 rad. The ASRE controller can survive large deviation in angle of attack and recover quickly. When $0.57\ \mathrm{rad} < x_1(0) \leqslant 0.64$ rad, the ASRE algorithm converges to several solutions, which oscillate between iterations. For example, 2 different solutions are obtained when $x_1(0) = 0.57$ or 0.58 rad, and 4 solutions are obtained when $x_1(0) = 0.59$ or 0.60 rad. This phenomenon is currently under investigation using several examples and is still an open problem for further research, but seems to be related to the existence of multiple optima.

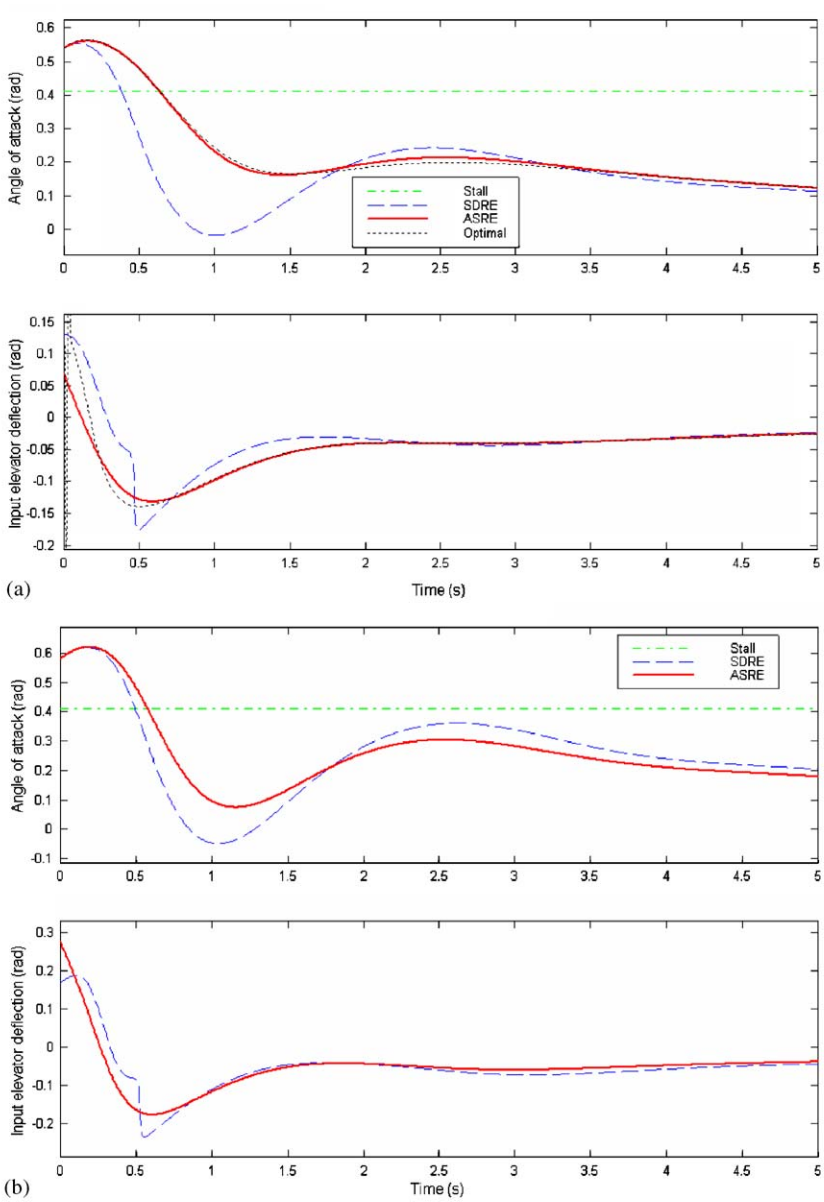

Fig. 2. Controlled trajectory x_1 and control input u using different feedback controllers when (a) $\mathbf{x}_0 = [0.54\ 0\ 0]^{\mathrm{T}}$, and (b) $\mathbf{x}_0 = [0.58\ 0\ 0]^{\mathrm{T}}$.

The approximating systems in Theorem 3 for the "global" optimal feedback control strategy require various partial derivatives of the system matrices, which results in a complex algorithm, making it harder to implement and converging to the optimal solution at a slower rate than the ASRE algorithm, for $i \geqslant 8$. Even though Theorem 3 may now converge to the globally optimal solution, the algorithm only provides control that stabilizes the aircraft for initial angles $x_1(0) \leqslant 0.55$ rad. Beyond this, problems arise related to the convergence of the approximations for the necessary conditions, which is caused by numerical problems since Theorem 3

Table 1
Comparison of costs using different methods for various initial disturbances in angle of attack of the F-8 aircraft

$x_1(0)$ (rad)	0.40	0.45	0.50	0.51	0.52	0.53	0.54	0.55	0.56
LQR ($\times 10^{-3}$)	3.56	6.06	13.5	18.4	40.6	—	—	—	—
SDRE ($\times 10^{-3}$)	3.52	6.59	11.5	13.0	14.7	16.8	19.4	22.7	26.9
ASRE ($\times 10^{-3}$)	3.52	5.80	10.4	11.8	13.6	15.7	18.3	21.5	25.6
Optimal ($\times 10^{-3}$)	3.49	5.63	9.63	10.9	12.3	14.3	16.6	21.6	—

provides an approximation to a more complex system than the ASRE algorithm (that is, a nonlinear two-point boundary value problem), causing the algorithm to eventually blow up. Fig. 2 compares the (*converged*) ASRE controlled trajectory x_1 and the corresponding ASRE controller u against those obtained using the SDRE and "global" optimal feedback control laws. The corresponding costs are also evaluated (within the time interval the system achieves steady-state behavior) and compared in Table 1 for various initial values of angle of attack. It is clear that the lowest costs are provided using the ASRE and "global" optimal feedback control algorithms, which are indeed very similar in magnitude. The slight difference in the results may have been caused by the error build-up in the Euler routine since the solutions are only numerical approximations. Therefore, intuitively, it can be stated that the ASRE algorithm provides the optimal control to the nonlinear system (3), which can also be verified from Fig. 2(a).

6. Conclusions

In this paper, global optimal feedback control of a very general class of multi-input–multi-output nonlinear control systems has been considered, which do not necessarily need to be affine in the control inputs. A new method has been proposed to solve the general nonlinear finite-time horizon optimal state-regulator control problem associated with the continuous-time, deterministic, nonautonomous, nonlinear, and nonaffine controlled dynamic system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) = \mathbf{A}(\mathbf{x}, \mathbf{u}, t)\mathbf{x}(t) + \mathbf{B}(\mathbf{x}, \mathbf{u}, t)\mathbf{u}(t)$$

and a nonautonomous, nonlinear-nonquadratic Bolza-type performance functional, which offers great design flexibility via state-, control- and time-dependent weighting matrices $\mathbf{F}(\mathbf{x}(t_f))$, $\mathbf{Q}(\mathbf{x}(t), \mathbf{u}(t), t)$ and $\mathbf{R}(\mathbf{x}(t), \mathbf{u}(t), t)$. The requirement is that $\mathbf{f}(\mathbf{0}, \mathbf{0}, t) = \mathbf{0}$, that is, there is an equilibrium at the origin, at $\mathbf{x} = \mathbf{0}$ and $\mathbf{u} = \mathbf{0}$, so that factorization is possible, which results in nonunique representations of nonlinear nonaffine matrix-valued functions $\mathbf{A}(\mathbf{x}, \mathbf{u}, t)$ and $\mathbf{B}(\mathbf{x}, \mathbf{u}, t)$. Provided $\mathbf{A}(\mathbf{x}, \mathbf{u}, t)$ and $\mathbf{B}(\mathbf{x}, \mathbf{u}, t)$ are jointly continuous in their arguments, the method introduces a sequence of linear-quadratic, time-varying problems, and requires solving an "*approximating sequence of Riccati equations*" (*ASRE*) by classical methods, which *globally converge* to the nonlinear systems considered, leading to an optimal control when one exists. It has been shown that the *local Lipschitz continuity* of the nonlinear operators $\mathbf{A}(\mathbf{x}, \mathbf{u}, t)$ and $\mathbf{B}(\mathbf{x}, \mathbf{u}, t)$ can be used to prove the convergence of the optimal controls and the ASRE feedback operators in the weak form in the interval $t \in [t_0, t_f]$. If the problem is coercive, so that a unique optimum exists, the limit of the ASRE gives this optimal feedback control, regardless of the factorization used to represent $\mathbf{A}(\mathbf{x}, \mathbf{u}, t)$ and $\mathbf{B}(\mathbf{x}, \mathbf{u}, t)$.

The theory for the nonlinear optimal ASRE control-design strategy has been illustrated on a fighter aircraft automatic control system. The ASRE controller is simple to implement, since the method is developed using classical linear-quadratic solutions, and provides very effective control, which results in superior performance in many real-world nonlinear applications. Provided that enough computational power is available, the proposed design can be used to solve an on-line optimization problem, which admittedly may be computationally infeasible for real-time applications. However, the method can be applied in a receding-horizon sense in the usual way to make real-time implementation easier, but at the expense of optimality. Therefore, a suboptimal algorithm can be derived to improve computational efficiency for real-time applications.

In general, the ASRE feedback systems may only converge to some local optimum. So in order to verify the global optimality of the ASRE feedback algorithm, in cases where such a solution exists, the necessary conditions for optimality of the finite-time nonlinear-nonquadratic optimal control problem have been established from Pontryagin's maximum principle. The ASRE feedback is seen to give solutions very close to the optimal one when simulated using the nonlinear aircraft model from any given initial state. This strongly suggests that the limit control of the ASRE feedback algorithm is very likely to be optimal in general. However, optimality has not been proved. It will, therefore, be of interest when these approximation schemes converge weakly in the viscosity sense thus providing a link with the well-known viscosity interpretation of nonlinear optimal control problems [20]. The major challenge now is to prove that the control obtained by the ASRE method of linear-quadratic, time-varying approximations is optimal. This will require a detailed study of the Hamilton–Jacobi–Bellman equation from the viewpoint of viscosity solutions (see [20]). The basic problem is, of course, existence of solutions of this equation, which is a nonlinear partial differential equation. Hence, after a detailed examination of the optimal control of general (nonaffine) nonlinear systems by considering the proposed ASRE scheme, future research shall be concerned with the study of viscosity solutions of the Hamilton–Jacobi–Bellman equation and their representation as limits of solutions of linear and time-varying approximations of the equation.

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