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Geometric existence theory for the control-affine nonlinear optimal regulator ☆

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Abstract

For infinite horizon nonlinear optimal control problems in which the control term enters linearly in the dynamics and quadratically in the cost, well-known conditions on the linearised problem guarantee existence of a smooth globally optimal feedback solution on a certain region of state space containing the equilibrium point. The method of proof is to demonstrate existence of a stable Lagrangian manifold M and then construct the solution from M in the region where M has a well-defined projection onto state space. We show that the same conditions also guarantee existence of a nonsmooth viscosity solution and globally optimal set-valued feedback on a much larger region. The method of proof is to extend the construction of a solution from M into the region where M no-longer has a well-defined projection onto state space.

Keywords: Nonlinear optimal regulator; Lagrangian manifold; Hamilton–Jacobi–Bellman equation; Viscosity solution

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1. Introduction

This paper addresses a gap in the literature concerning the existence of solutions to infinite horizon nonlinear optimal control problems in which the control term enters linearly in the dynamics and quadratically in the cost function. We show that the well-known conditions which guarantee existence of a smooth feedback solution on a certain region containing the equilibrium point, also guarantee existence of a nonsmooth viscosity solution and set-valued feedback on a much larger region.

This class of problems can be formulated as follows. Let $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $z \in \mathbb{R}^p$ and f, g, h be C^2 functions of the appropriate dimensions with $h(0) = 0$ and $h(x) \neq 0$ for $x \neq 0$. Consider the dynamical system

$$\begin{aligned} &\dot{x} = f(x) + g(x)u, \quad x(0) = \xi, \\ &z = h(x) \end{aligned} \tag{1}$$

and assume that there is an equilibrium at $x = 0$, i.e. $f(0) = 0$. Define the set of control functions by

$$\Psi = \bigl\{u : [0, \infty) \to \mathbb{R}^m\colon\ u(.) \in L_2[0, T] \text{ for all } T < \infty\bigr\}.$$

Given an initial point $\xi \in \mathbb{R}^n$, denote by $x_\xi(.; u)$ or simply $x_\xi(.)$ or $x(.)$ the unique solution to (1) corresponding to the choice of control $u \in \Psi$. Let $r : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ be a C^2 function such that $r(x)$ is positive definite for all x and define the following cost function:

$$J\bigl(u(.), \xi, T\bigr) = \int_0^T \frac{1}{2}\Bigl(\bigl|h\bigl(x(t)\bigr)\bigr|^2 + u(t)^T r\bigl(x(t)\bigr)u(t)\Bigr)\,dt \tag{2}$$

on solution trajectories $x_\xi(.; u)$ to (1). Given an open set $0 \in \Omega \subset \mathbb{R}^n$ and an initial point $\xi \in \Omega$, define the set of admissable controls in Ω to be

$$\Delta_\Omega = \bigl\{u \in \Psi\colon\ x_\xi(t; u) \in \Omega \text{ for all } t \geqslant 0\bigr\}. \tag{3}$$

Then the infinite horizon optimal control problem on the set Ω is to maximise J with respect to $T > 0$ and minimise it with respect to $u \in \Delta_\Omega$. In particular, a solution is said to exist to this problem on the set Ω if there exists a finite continuous value function

$$\hat{V}(\xi) = \inf_{u\in\Delta_\Omega} \sup_{T>0} J\bigl(u(.), \xi, T\bigr) \tag{4}$$

for all $\xi \in \Omega$.

In order for this problem to have a solution, the standard assumption is that the linearisation of the dynamics (1) at $x = 0$ is stabilisable and detectable. Under this condition, which we call assumption (A), it is well known (see, for instance, [16]) that the linearisation of the above problem has a solution on a small neighbourhood U of $x = 0$ in state space. Clearly we can take $\hat{V}(0) = 0$ and, if we let $P = \partial^2\hat{V}/\partial x^2|_{x=0}$ then on U we have $\hat{V}(x) = \frac{1}{2}x^T P x$, where P satisfies the well-known algebraic Riccati equation. An optimal feedback control exists in the form $\hat{u}(x) = -r^{-1}(0)g^T(0)\partial\hat{V}/\partial x$. The existence of this stationary solution is proved directly by showing that the value functions for the corresponding sequence of linearised finite horizon problems converges to an explicit limit as $T \to \infty$.

Under the same assumption (A), existence of a solution to the full nonlinear problem on a larger region Ω_0 containing the equilibrium point $x = 0$ was proved in [3,11] over thirty years ago. This proof is less direct than the argument used in the linear case and applies a theorem of global topology to deduce the existence of a certain differential manifold in phase space and then constructs the solution from this manifold.

The modern viewpoint on this proof is that of symplectic geometry and is set out in [18,19], where the proof is generalised to solve the nonlinear H_∞ control problem with affine control and disturbance terms. This viewpoint is fundamental to the current paper and the basic idea as it applies to the optimal control problem (4) is as follows. We refer the reader to standard references such as [12,15,21] for background on symplectic geometry and Lagrangian manifolds. The maximum principle applied to our control problem gives the following Hamiltonian:

$$\begin{aligned} H(x,y) &= \max_{u\in\mathbb{R}^n}\left\{ y^T\big(f(x)+g(x)u\big) - \frac{1}{2}\big|h(x)\big|^2 - \frac{1}{2}u^T r(x)u \right\} \\ &= \frac{1}{2}y^T g(x)r(x)^{-1}g(x)^T y + y^T f(x) - \frac{1}{2}\big|h(x)\big|^2 \end{aligned} \tag{5}$$

on $\mathbb{R}^{2n}$ phase space, where $y \in \mathbb{R}^n$ is the adjoint variable and $x \in \mathbb{R}^n$ is the state variable. Then assumption (A) implies that the Hamiltonian dynamics

$$\dot{x} = \partial H/\partial y, \qquad \dot{y} = -\partial H/\partial x \tag{6}$$

have a hyperbolic equilibrium point at $x = y = 0$. The stable manifold theorem then says that there exists a global stable manifold M^+ in $\mathbb{R}^{2n}$ for these dynamics. This manifold is n-dimensional, Lagrangian and H vanishes on it. Also, there exists a simply connected region M_0 of M^+ which contains the point $x = y = 0$ and which has a well-defined projection onto a region Ω_0 in state space containing the point $x = 0$. If we let $\pi : \mathbb{R}^{2n} \to \mathbb{R}^n$ denote the canonical projection, then this means that $\pi|_{M_0}$ is nonsingular, and so y can be expressed as a function of x for $(x,y) \in M_0$. If we define $S(x)$ for $x \in \Omega_0$ to be the function satisfying $dS = y\,dx$ on M_0, with $S(0) = 0$, then M_0 is the graph $\{x, \partial S/\partial x\}$ in phase space. It follows that $\hat{V}(x)$ defined by (4) exists on Ω_0 and equals $-S(x)$. Furthermore, $\hat{V}(x)$ is a smooth solution to the Hamilton–Jacobi–Bellman (HJB) equation

$$H(x, -\partial\hat{V}/\partial x) = 0 \tag{7}$$

on Ω_0 and an optimal feedback control exists in the form

$$\hat{u}(x) = r^{-1}(x)g^T(x)y(x) = -r^{-1}(x)g^T(x)\partial\hat{V}/\partial x \tag{8}$$

on Ω_0. The function S is called a generating function for M_0, and it can be seen that the solution to the linearised problem above is given by the generating function for the tangent plane to M_0 at $x = 0$.

The existence of a smooth solution to (7) breaks down at points where $\pi|_{M^+}$ becomes singular. These correspond to points where asymptotically stable optimal trajectories in state space start to cross one another as we go backwards in time from the equilibrium point. However, the manifold M^+ exists globally in phase space and, in general, covers a region of state space strictly larger than Ω_0. On this larger region, M^+ becomes multi-valued when thought of as a section of the co-tangent bundle over state space.

In the next section we show how to construct from M^+ a locally Lipschitz single valued function $V(x)$ which gives a viscosity solution to (7) on a region Ω strictly larger than Ω_0, and which reduces to the above smooth solution $-S(x)$ on Ω_0. This result combines a topological technique for constructing a global Lipschitz function from M^+, with a local proof of the viscosity property put forward by Marty Day. The existence of this nonsmooth solution, on the larger region Ω, follows, with no extra hypotheses, from the same assumption (A) already required for the smooth case. We refer the reader to standard references such as [6,9] for background on viscosity solutions.

Then in Section 3, under the additional assumption that $V(x) > 0$ for $x \neq 0$, we show that $V(x)$ equals the value function $\hat{V}(x)$ defined by (4) on Ω, and that an optimal feedback exists in set-valued form which reduces to (8) on Ω_0. The main result of this paper is the proof (in Proposition 3.2) that this feedback is well defined—in particular that any multi-valued points can only occur at the start of controlled trajectories. These are the points of nondifferentiability of V at which controlled trajectories given by the feedback lose global optimality as we go backwards in time. Similar results have been proved for finite horizon problems in [4] and, more recently, in [10] for manifolds containing just fold and cusp type singularities.

2. Existence of a nonsmooth solution to the HJB equation

To establish the existence of V, let Ω be the largest open region in state space containing 0 with the following properties:

(1) Ω is covered by M^+, i.e. for every $x \in \Omega$ there is some $(x, y) \in M^+$,
(2) Ω is forward invariant with respect to the dynamics (6) on M^+, i.e. for every $(x, y) \in M^+$ with $x \in \Omega$, the integral curve γ_y for (6) with $\gamma_y(0) = (x, y)$ satisfies $\pi(\gamma_y(t)) \in \Omega$ for all $t \geqslant 0$.

Note that Ω will in general be strictly larger than Ω_0. Consider the submanifold M of M^+ consisting of those $(x, y) \in M^+$ with $x \in \Omega$. Now, as noted in [18,19], M is simply connected. So for $x \in \Omega$, we can define a smooth function $S(x, y)$ on phase space which satisfies $dS = y\,dx$ on M. This function reduces to $S(x)$ over Ω_0 and is the generating function of M, i.e. $M = \{(x, d_x S(x, y))\colon\ x \in \Omega,\ d_y S(x, y) = 0\}$. Now define V to be the following function on Ω:

$$V(x) = \inf\bigl\{-S(x, y)\colon\ y \text{ such that } (x, y) \in M\bigr\}. \tag{9}$$

Over the region Ω_0 where M is just the single branch M_0, V clearly equals $-S(x)$ and so coincides with the above smooth solution to (7). Over the larger region Ω, we can apply results in the recent literature to state the following theorem. For background details on the following proof see [14].

Theorem 2.1. *$V(x)$ is a locally Lipschitz viscosity solution of Eq.* (7) *for all $x \in \Omega$.*

Proof. M is simply connected and Lagrangian isotopic to the zero section of the cotangent bundle over Ω, i.e. to the Lagrangian submanifold of $T^*\Omega = \Omega \times \mathbb{R}^n$ given by Ω itself. So Ω can be lifted to a closed manifold X and M to a Lagrangian submanifold of T^*X satisfying the required exactness and transversality conditions for existence of a global generating function quadratic at infinity—see the results of [5,17,20]. It is further shown in these references how to apply a Lusternik–Schnirelman type minimax procedure to construct from M a global Lipschitz continuous function V over X. This function is smooth on a subset X_0 of X of full measure, and is called a graph selector for M because $(x, dV(x)) \in M$ for $x \in X_0$. Since H is convex, it follows that V has the local expression (9) over Ω. We can then apply results of [7] to show that this local function V is a viscosity solution of (7) in Ω. $\square$

3. Optimality of the nonsmooth solution to the HJB equation

In the previous section we established the existence of a viscosity solution $V(x)$ to (7) for $x \in \Omega$. We now show that $V(x) = \hat{V}(x)$ for $x \in \Omega$.

The first step is to show that the set Δ_Ω of admissable controls on Ω is nonempty. Recall that, for $(x, y) \in M$, the maximum in (5) is achieved by $u^*(y) = r^{-1}(x)g^T(x)y$. Define a potentially multi-valued feedback control for all $x \in \Omega$ as follows:

$$\hat{u}(x) = r^{-1}(x)g^T(x)\hat{y}(x), \tag{10}$$

where

$$\hat{y}(x) \in \hat{Y}(x) = \arg\min\bigl\{-S(x, y):\ y \text{ s.t. } (x, y) \in M\bigr\}. \tag{11}$$

It is shown in [7] that $\hat{Y}(x) \neq \emptyset$ for each $x \in \Omega$.

For $x \in \Omega_0$, $S(x, y) = S(x)$ and there is only one $y \in \mathbb{R}^n$ such that $(x, y) \in M$, namely $y = \partial S/\partial x$. By default, this is the minimising argument for $-S(x, .)$ on M and so (10) reduces to (8) on Ω_0.

For $x \in \Omega \setminus \Omega_0$, where there exist multiple y such that $(x, y) \in M$, there can also be multiple $\hat{y}(x) \in \hat{Y}(x)$. We therefore interpret the resulting controlled system $\dot{x} = f(x) + g(x)\hat{u}(x)$ in the sense of Filippov [8], namely as an almost sure differential inclusion $\dot{x} \in F(x)$, where $F(x)$ is a set-valued extension of the vector field $f(x) + g(x)\hat{u}(x)$ satisfying certain compactness and continuity conditions. For our purposes these are satisfied by taking

$$F(x) = f(x) + g(x)U(x), \tag{12}$$

where

$$U(x) = r^{-1}(x)g^T(x)Y(x) \tag{13}$$

and

$$Y(x) = \mathrm{co}\bigl\{\hat{Y}(x)\bigr\}. \tag{14}$$

Here co denotes convex hull. Note that in general for a Hamiltonian such as (5) which is convex in y and for $\dim M \geqslant 2$, $Y(x)$ is strictly contained in $\mathrm{co}\{y:\ (x, y) \in M\}$.

We require some definitions in order to state the results of this section. We will show that $U(x)$ is a weakly admissable multi-valued feedback in the sense of Definition 2.59 of Chapter III of [2] and weakly globally optimal in the sense of Definition 2.60 of Chapter III of [2]. We will also show that $U(x)$ is weakly asymptotically stable with weak Lyapunov function V in the sense of Section 15, Chapter 3 of [8]. These definitions mean that, for all $x_0 \in \Omega$, there exists at least one solution $x(t;\hat{u})$ to the differential inclusion $\dot{x} \in f(x) + g(x)U(x)$, $x(0) = x_0$ (i.e. a solution $x(t;\hat{u})$ satisfying $\hat{u}(t) \in U(x(t))$ for a.e. $t > 0$) with the following properties:

- $x(t;\hat{u})$ is an admissable solution, i.e. $\hat{u}(.) \in \Delta_\Omega$,
- $x(t;\hat{u})$ is asymptotically stable,
- the minimum value of the cost functional $\sup_{T>0} J(u, x_0, T)$ over $u \in \Delta_\Omega$ is achieved along $x(t;\hat{u})$.

It will be shown that, in fact, that these properties are satisfied by any choice of feedback term $\hat{u}(t) \in \hat{U}(x(t))$, where $\hat{U}(x)$ is the subset of $U(x)$ defined by

$$\hat{U}(x) = r^{-1}(x) g^T(x) \hat{Y}(x). \tag{15}$$

Note that $\hat{u}(t) \in \hat{U}(x(t))$ is the control corresponding to some choice of minimising argument $\hat{y}(x(t))$ for $-S(x(t), .)$ on M. Note also that the stronger notion of full optimality, which means that every choice of feedback term from $U(x)$ is optimal, does not hold for this problem.

We start by showing that a point $x \in \Omega$ at which $\hat{Y}(x)$ is multi-valued can only occur as the initial point on a controlled trajectory $x(t;\hat{u})$ with $\hat{u}(.) \in \hat{U}(x(.))$. It follows that the feedback $\hat{U}(x)$ is single valued along controlled trajectories, with the possible exception of the initial point. This requires the following technical lemma.

Lemma 3.1. *Let $x_0 \in \Omega$ and $y_0 \in \hat{Y}(x_0)$. Then there exists a open neighbourhood U of (x_0, y_0) on M such that $\pi(U)$ is an open neighbourhood of x_0 in state space.*

Proof. If $\pi|_M$ is nonsingular at (x_0, y_0) then M is locally a graph over state space in a small neighbourhood of (x_0, y_0) and so the result is immediate. Suppose then that $\pi|_M$ is singular at (x_0, y_0). Since (x_0, y_0) is a minimising point for $-S(x_0, .)$ over M, it follows from Theorem 5.27 of [13] that (x_0, y_0) is a nonfolded singularity. This means (see Definition 5.18 ibid) that given any sequence $x_n \to x_0$ in $\mathbb{R}^n$, there exists a corresponding sequence y_n in $\mathbb{R}^n$ such that $(x_n, y_n) \in M$ for all n and $(x_n, y_n) \to (x_0, y_0)$ as $n \to \infty$. Since $\pi(x_n, y_n) = x_n$ the result again follows. □

Proposition 3.2. *Let $x_0 \in \Omega$ be such that $\hat{Y}(x_0)$ is multi-valued. Let $\hat{y}_0 \in \hat{Y}(x_0)$. Let $\gamma(t) = (x(t), y(t))$ be the integral curve for* (6) *which lies on M and satisfies $x(0) = x_0$ and $y(0) = \hat{y}_0$. Then for all $t > 0$, $\hat{Y}(x(t)) = \{y(t)\}$ while for all $t < 0$, $y(t) \notin \hat{Y}(x(t))$.*

Remark 3.3. The above proposition says that if $y(t)$ is the adjoint half of the Hamiltonian trajectory $\gamma(t)$ on M and $y(0) = \hat{y}_0$ is one of multiple minimising arguments for $-S(x(0), .)$ over M, then for all $t > 0$, $y(t)$ is the unique minimising argument for

$-S(x(t), .)$ over M, while for all $t < 0$, $y(t)$ does not minimise $-S(x(t), .)$. Now for $t \geqslant 0$, $\pi(\gamma(t))$ coincides with a controlled trajectory $x(t; \hat{u})$ with $x(0) = x_0$ and $\hat{u}(t) \in \hat{U}(x(t))$. It follows that $\hat{U}(x(t))$ is single valued along this trajectory, except at the initial point x_0. Thus the controlled trajectory $x(t; \hat{u})$ is uniquely defined apart from at the initial point where one can choose between a number of trajectories.

Proof. Let Λ be an index set for the branches of M lying over x_0 on which the minimising arguments for $-S(x_0, .)$ occur. Let $\lambda = 0$ be the index of the branch containing the point $(x_0, \hat{y}_0)$. So we can write $\hat{Y}(x_0) = \{\hat{y}_\lambda\colon\ \lambda \in \Lambda\}$ where $(x_0, \hat{y}_\lambda) \in M$ and $-S(x_0, \hat{y}_\lambda) = -S(x_0, \hat{y}_0)$ for all $\lambda \in \Lambda$, this being the minimum value of $-S(x_0, .)$ over all $(x_0, y) \in M$. Consider the integral curve $\gamma(t) = (x(t), y(t))$ lying on M with $\gamma(0) = (x_0, \hat{y}_0)$. The projection $\pi(\gamma(t))$ of this curve in state space has tangent $\dot{x}(0) = f(x_0) + g(x_0)r^{-1}(x_0)g^T(x_0)\hat{y}_0$ at x_0. This corresponds to an initial choice of feedback term $\hat{u}_0 = r^{-1}(x_0)g^T(x_0)\hat{y}_0$ from the multi-valued set $\hat{U}(x_0)$. However, note that $x(t)$ is uniquely and well-defined independent of $\hat{u}(t)$ since it is the state space projection of an integral curve for the Hamiltonian dynamics on M. By the above lemma, for each $\lambda \in \Lambda$ there is a neighbourhood U_λ of $(x_0, \hat{y}_\lambda)$ on M such that $\pi(U_\lambda)$ is a neighbourhood of x_0. So, for t in a small interval around 0, there is a trajectory of points $(x(t), y_\lambda(t))$ lying on the branch of M indexed by λ which projects onto the curve $x(t)$ in state space and satisfies $y_\lambda(0) = \hat{y}_\lambda$.

Now for $t > 0$, the value of S along the trajectory $(x(t), y_\lambda(t))$ on M is given by

$$S\big(x(t), y_\lambda(t)\big) = \int_0^t y_\lambda(s)\dot{x}(s)\,ds + S(x_0, \hat{y}_\lambda). \tag{16}$$

Since $S(x_0, \hat{y}_\lambda) = S(x_0, \hat{y}_0)$ for all $\lambda \in \Lambda$, the minimum value of $-S(x(t), y_\lambda(t))$ over $\lambda \in \Lambda$ occurs at that λ which maximises $y_\lambda(0)\dot{x}(0)$ over all $\lambda \in \Lambda$. Now

$$y_\lambda(0)\dot{x}(0) = \hat{y}_\lambda f(x_0) + \hat{y}_\lambda g(x_0)r^{-1}(x_0)g^T(x_0)\hat{y}_0.$$

Since $(x_0, \hat{y}_\lambda) \in M$, we have $H(x_0, \hat{y}_\lambda) = 0$ and so from (5),

$$\begin{aligned} y_\lambda(0)\dot{x}(0) &= -\frac{1}{2}\hat{y}_\lambda g r^{-1} g^T \hat{y}_\lambda + \frac{1}{2}|h|^2 + \hat{y}_\lambda g r^{-1} g^T \hat{y}_0 \\ &= -\frac{1}{2}(\hat{y}_\lambda - \hat{y}_0) g r^{-1} g^T (\hat{y}_\lambda - \hat{y}_0) + \frac{1}{2}\hat{y}_0 g r^{-1} g^T \hat{y}_0 + \frac{1}{2}|h|^2. \end{aligned} \tag{17}$$

This has a unique maximum at $\lambda = 0$. Now the integral curve $\gamma(t) = (x(t), y(t))$ is, by definition, the trajectory of points $(x(t), y_0(t))$ lying over $x(t)$ on the branch of M indexed by $\lambda = 0$. So for some small interval of $t > 0$, $y(t)$ is the unique minimising argument for $-S(x(t), .)$ over M, i.e. $\hat{Y}(x(t)) = \{y(t)\}$ for $t \in (0, \delta_1)$ for some $\delta_1 > 0$.

Note, it is sufficient to consider only those branches containing minimising arguments for $-S(x_0, .)$ in the above optimisation, i.e. to only minimise $-S(x(t), y_\lambda(t))$ over $\lambda \in \Lambda$. To see this let $(x_0, y_\mu) \in M$ be such that $y_\mu \notin \hat{Y}(x_0)$ and suppose that the corresponding branch with index μ contains a trajectory of points $(x(t), y_\mu(t))$ lying over $x(t)$ with $y_\mu(0) = y_\mu$. Then repeating the argument in (16), $S(x_0, y_\mu) < S(x_0, \hat{y}_0)$, while

$H(x_0, y_\mu) = 0$, so the calculation in (17) can also be repeated to show that $y_\mu(0)\dot{x}(0) < y_0(0)\dot{x}(0)$.

For $t < 0$, the value of S along the trajectory $(x(t), y_\lambda(t))$ on M is given by the following relationship:

$$S(x_0, \hat{y}_\lambda) = \int_t^0 y_\lambda(s)\dot{x}(s)\,ds + S\big(x(t), y_\lambda(t)\big). \tag{18}$$

Since $S(x_0, \hat{y}_\lambda) = S(x_0, \hat{y}_0)$ for all $\lambda \in \Lambda$, the minimum value of $-S(x(t), y_\lambda(t))$ over $\lambda \in \Lambda$ occurs at that λ which minimises $y_\lambda(0)\dot{x}(0)$ over all $\lambda \in \Lambda$. The above calculation (17) shows that, provided as in this case that there is at least one element in $\hat{Y}(x_0)$ in addition to $\hat{y}_0$, then the minimum value of $y_\lambda(0)\dot{x}(0)$ does not occur on the branch indexed by $\lambda = 0$. So for some small interval of $t < 0$, $y(t)$ is not the minimising argument for $-S(x(t), .)$ over M, i.e. $y(t) \notin \hat{Y}(x(t))$ for $t \in (-\delta_2, 0)$ for some $\delta_2 > 0$.

To extend the above result to all $t > 0$, there are two possibilities to be excluded. The first possibility, which we will denote $(*)$, is that there exists some $t_1 \geqslant \delta_1$ such that $y(t_1) \in \hat{Y}(x(t_1))$ but $\hat{Y}(x(t_1))$ is multi-valued. This situation cannot occur because it produces an integral curve $\gamma(t)$ which passes through a point $(x(t_1), y(t_1))$ at which $\hat{Y}(x(t_1))$ is multi-valued, but which also satisfies $y(t) \in \hat{Y}(x(t))$ for $t < t_1$. This contradicts the previous paragraph.

The second possibility, which we will denote $(**)$, is that there exists some $t_2 > \delta_1$ such that $y(t_2) \notin \hat{Y}(x(t_2))$. For this to occur, there must exist some t_1 with $t_2 > t_1 \geqslant \delta_1$ at which the minimising argument for $-S(x(t), .)$ along the trajectory $x(t)$ jumps from the branch with index $\lambda = 0$ to some other branch with index $\lambda = \lambda_1$ say. Continuing with the notation used earlier in the proof, let $(x(t_1), y(t_1))$ denote the point lying over $x(t_1)$ on the $\lambda = 0$ branch. Let $(x(t_1), y_{\lambda_1}(t_1))$ denote the point on the $\lambda = \lambda_1$ branch. Then we claim that both $y(t_1)$ and $y_{\lambda_1}(t_1)$ are in $\hat{Y}(x(t_1))$ and we have already shown, in $(*)$, that this situation cannot occur.

To prove the claim that both $y(t_1)$ and $y_{\lambda_1}(t_1)$ are in $\hat{Y}(x(t_1))$, let $t_n \to t_1$ be a sequence converging to t_1 with $t_n < t_1$ for all n. Each $y(t_n) \in \hat{Y}(x(t_n))$, so $-S(x(t_n), y(t_n)) = V(x(t_n))$. Now V is locally Lipschitz continuous, so

$$V\big(x(t_n)\big) \to V\big(x(t_1)\big).$$

Also, S is smooth and thus continuous on M, and $(x(t_n), y(t_n)) \to (x(t_1), y(t_1))$ on M, so

$$-S\big(x(t_n), y(t_n)\big) \to -S\big(x(t_1), y(t_1)\big).$$

It follows that $-S(x(t_1), y(t_1)) = V(x(t_1))$ and so $y(t_1) \in \hat{Y}(x(t_1))$. A similar argument with $t_m \to t_1$ and $t_m > t_1$ for all m shows that $-S(x(t_1), y_{\lambda_1}(t_1)) = V(x(t_1))$ and so $y_{\lambda_1}(t_1) \in \hat{Y}(x(t_1))$ also.

Since both the above possibilities $(*)$ and $(**)$ can be excluded, it thus follows that $\hat{Y}(x(t)) = \{y(t)\}$ for all $t > 0$. A similar argument shows that $y(t) \notin \hat{Y}(x(t))$ for all $t < 0$. $\square$

Corollary 3.4. *$U(x)$ is a weakly admissable, weakly asymptotically stable multi-valued feedback in the sense defined above.*

Proof. For any initial point $x_0 \in \Omega$, consider the controlled trajectory $x(t;\hat{u})$ with $x(0) = x_0$ and $\hat{u}(t) \in \hat{U}(x(t))$. By the previous proposition, the set $\hat{U}(x(t))$ is single valued along this trajectory, except possibly at the initial point x_0. Thus $x(t;\hat{u})$ is uniquely defined apart from, possibly, at the initial point where one can choose between a number of trajectories. Also $x(t;\hat{u})$ is the projection of an integral curve $\gamma(t) = (x(t), y(t))$ lying on a branch of the stable manifold M for the dynamics (6), the particular choice of branch being determined by the initial choice of feedback term $\hat{u}(0) \in \hat{U}(x_0)$. It follows that $x(t;\hat{u}) \to 0$ as $t \to \infty$, establishing weak asymptotic stability for U. Also $x(t;\hat{u}) \in \Omega$ for all $t \geqslant 0$, since by construction Ω is forward invariant with respect to the dynamics (6). Thus $\hat{u}(.) \in \Delta_\Omega$ which establishes weak admissability for U. □

Theorem 3.5. *Suppose $V(x) > 0$ for all $0 \neq x \in \Omega$. Suppose that for all $\varepsilon > 0$, there exists $\delta > 0$ with $|h(x)| \geqslant \delta$ for all $x \in \Omega \setminus B_\varepsilon(0)$. Then $V(x) = \hat{V}(x)$ for all $x \in \Omega$, i.e. V is the value function for this problem, and $U(x)$ is weakly (globally) optimal, with any choice of feedback term $\hat{u}(t) \in \hat{U}(x(t))$ giving rise to an optimal controlled trajectory. In particular, if $\hat{U}(x_0)$ is multi-valued, then $V(x_0)$ is the value of the cost functional* (2) *along any of the controlled trajectories $x(t;\hat{u})$, $x(0) = x_0$ for different initial choices $\hat{u}(0) \in \hat{U}(x_0)$. Also, V is a weak Lyapunov function for U, again corresponding to any choice of feedback term $\hat{u}(t) \in \hat{U}(x(t))$.*

Proof. Note, by the assumptions on the linearised problem at the origin, that $V(0) = -S(0,0) = 0$. Also, by hypothesis $V(x) > 0$ for $0 \neq x \in \Omega$, and so $S(x,y) < 0$ for all $(x,y) \in M$ with $x \neq 0$.

We first show that $V(x) \geqslant \hat{V}(x)$ for all $x \in \Omega$. Let $x_0 \in \Omega$ and let

$$\hat{u}(0) = r^{-1}(x_0) g^T(x_0) \hat{y}_0 \in \hat{U}(x_0)$$

be any initial choice of feedback term. Then, as shown above, the resulting controlled trajectory $x(t) = x(t;\hat{u})$ is asymptotically stable with $x(t) \in \Omega$ for all $t \geqslant 0$. Furthermore, there exists an integral curve $\gamma(t) = (x(t), y(t))$ lying over $x(t)$ on M such that $\hat{Y}(x(t)) = \{y(t)\}$ for all $t > 0$. So by definition, $V(x_0) = -S(x_0, \hat{y}_0)$ and $V(x(t)) = -S(x(t), y(t))$. Now $H(x(t), y(t)) = 0$ for all $t > 0$ so along the trajectory $x(t)$, we have $y(t)\dot{x}(t) = l(x(t), \hat{u}(t))$, where $l(x,u) = \frac{1}{2}(|h(x)|^2 + u^T r(x) u)$ and $\hat{U}(x(t)) = \{\hat{u}(t)\}$. Then since $dS = y\,dx$, we have

$$-V\big(x(t)\big) + V(x_0) = \int_0^t l\big(x(s), \hat{u}(s)\big)\,ds. \tag{19}$$

Now $x(t) \to 0$ as $t \to \infty$. So, since $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$,

$$V(x_0) = \sup_t \int_0^t l\big(x(s), \hat{u}(s)\big)\,ds \geqslant \inf_{u \in \Delta_\Omega} \sup_t \int_0^t l\big(x(s), u(s)\big)\,ds = \hat{V}(x_0). \tag{20}$$

To prove the converse, we apply an argument from [1] which was stated for the case where V is a classical solution of (7) but works also in the viscosity setting. Note first that V is a subsolution of (7), so for all $p \in D^+V$,

$$\max_u \bigl\{-p(f+gu) - l(x,u)\bigr\} \leqslant 0.$$

So for any admissable control $u = u(.) \in \Delta_\Omega$, the inequality

$$-p(f+gu) - l(x,u) \leqslant 0$$

holds true at any point $x(t)$ along the solution trajectory to $\dot{x} = f + gu(t)$, $x(0) = x_0$. Then by Theorem I.14 of [6],

$$-V\bigl(x(t)\bigr) + V(x_0) \leqslant \int_0^t l\bigl(x(s), u(s)\bigr)\,ds.$$

Now, if 0 is a limit point of $x(t)$, take a sequence t_n with $x(t_n) \to 0$ as $n \to \infty$. Then, since $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$,

$$V(x_0) \leqslant \sup_t \int_0^t l\bigl(x(s), u(s)\bigr)\,ds.$$

On the other hand, if 0 is not a limit point of $x(t)$, then there exists $\varepsilon > 0$ and some $T > 0$ such that $x(t) \notin B_\varepsilon(0)$ for all $t > T$, from which it follows that $|h(x(t))|^2 \geqslant \delta$ for some $\delta > 0$ and all $t > T$. So in this case also we have

$$V(x_0) \leqslant \sup_t \int_0^t l\bigl(x(s), u(s)\bigr)\,ds = +\infty.$$

Since this holds for all controls $u(.) \in \Delta_\Omega$, we have that

$$V(x_0) \leqslant \inf_{u \in \Delta_\Omega} \sup_t \int_0^t l\bigl(x(s), u(s)\bigr)\,ds = \hat{V}(x_0).$$

Thus $V(x_0) = \hat{V}(x_0)$ for all $x_0 \in \Omega$. Furthermore, it follows from (20), that the infimum in (4) is achieved by any choice of feedback term $\hat{u}(.) \in \hat{U}(x(.)) \subseteq U(x(.))$. So $U(x)$ is a weak globally optimal set-valued feedback.

Lastly, note from (19) that V is monotonic decreasing along trajectories $x(t) = x(t;\hat{u})$ corresponding to any $\hat{u} \in \hat{U}(x)$. It follows from Theorem 2, Section 15, Chapter 3 of [8] that V is a weak Lyapunov function for the set-valued feedback $U(x)$. □

Note, the condition on h in the above theorem can be removed by restricting the set of admissable controls Δ_Ω to those which are asymptotically stable, in addition to remaining within Ω for all $t \geqslant 0$.

4. Conclusion

For the above optimal control problem, we have used the conditions which already guarantee existence of a smooth solution on a region Ω_0 of the equilibrium point, to prove the existence of a viscosity solution V on a larger region Ω. We have further shown that V is the value function for the optimal control problem and constructed a set-valued feedback from M which achieves the optimal value in a weak sense.

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