MIMO MULTI-PERIODIC REPETITIVE CONTROL SYSTEMS: A LYAPUNOV ANALYSIS

D.H.Owens, L.M.Li, S.P.Banks

*Department of Automatic Control and Systems Engineering,
University of Sheffield, Mappin Street, Sheffield S1 3JD, UK*
D.H.Owens@sheffield.ac.uk & L.Li@sheffield.ac.uk

Abstract: In this paper a new type of repetitive control problem where two or more periods exist in reference and disturbance signals is considered. A Lyapunov stability analysis under a positive real condition is provided and a new method of designing compensators in a negative feedback loop to create positive realness is outlined.

Keywords: MIMO, Repetitive Control, Lyapunov, Feedback Control.

1. INTRODUCTION

Signals associated with repetitive or rotating actions are often of periodic nature or are subjected to forms of periodic disturbances. In order to track or/and reject this kind of signals, an internal model that generates the corresponding periodic signals, should be included in the closed-loop (Francis and Wonham, 1975). This system is called a repetitive control system and has been widely used in control problems with robotic (Kaneko and Horowitz, 1997), motor (Kobayashi *et al.*, 1999), rolling process (Garimella and Srinivasan, 1996), rotating mechanisms (Fung, *et al.*, 2000), and much more (Tzou, *et al.*, 1999; Moon, 1998; Manayathara, *et al.*, 1996).

If the signal need to be tracked or rejected contains only one single frequency, a finite-dimensional model can serve as internal model (Bai and Wu, 1998). While more generally in practical situations a periodic signal contains many relevant frequencies—fundamental frequency and its harmonics within a certain bandwidth, and a (higher order) compensator is required but an infinite-dimensional internal model should be more appropriate. The one proposed is built on time-delay system. Perhaps the first infinite-dimensional repetitive controller was proposed by Inoue *et al.* (1981) and can be simplified in common form as illustrated in Figure 1(a), along with the poles of the controller. The infinitely many poles of the delay system on the imaginary axis are $ik\nu, \mathrm{k} = 0, \pm 1, \ldots$, where $\nu = 2\pi/\tau$ is called fundamental frequency, which make the system capable of generating signal containing frequency components of reference/disturbance. Hara *et al.* (1988) proposed a modified infinite-dimensional repetitive controller [see Figure 1(b)] by introducing a low-pass filter aiming at improving the system stability, at a cost of losing tracking accuracy at high frequencies.

In many cases the reference and/or disturbance signals may contain different fundamental frequencies. The ratio of these frequencies can be irrational. This leads to the idea of the so-called multi-periodic repetitive control, which has received very little attention (Weiss, 1997). In this paper the MIMO multi-periodic control problem is studied using Lyapunov techniques. It is proved that the stability of the multi-periodic repetitive control system is guaranteed if the plant satisfies a positive real condition. A new procedure to create positive realness by feedforward and feedback compensation is presented. A simulation example is also presented.

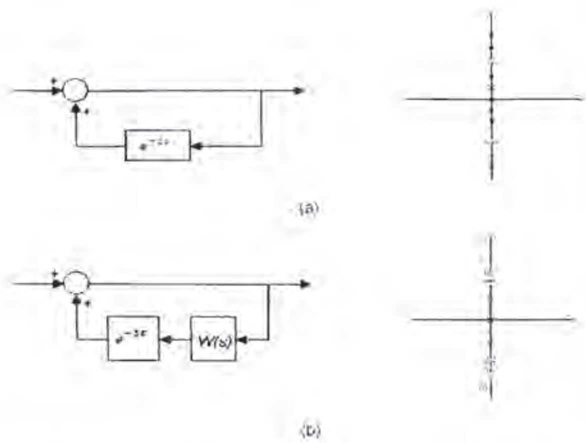

Fig. 1. Two common forms of infinite-dimensional repetitive controller and their poles location.

2. LYAPUNOV STABILITY ANALYSIS

The MIMO multi-periodic repetitive control system is shown in Figure 2.

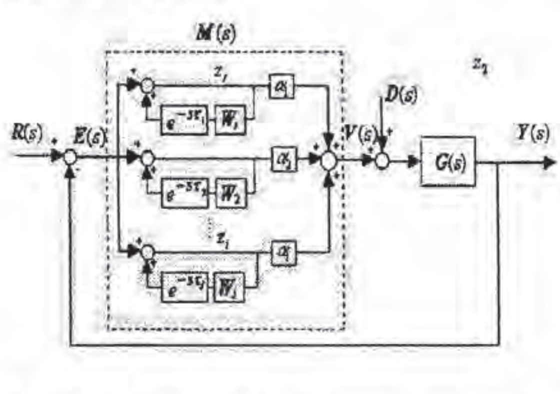

Fig. 2. MIMO multi-periodic repetitive control system

where the multi-periodic repetitive controller $M(s)=\sum_{i=1}^{m}\frac{\alpha_i I}{1-W_i e^{-s\tau_i}}$ is a convex linear combination of single-periodic repetitive control elements. That is, $\sum_{i=1}^{m}\alpha_i=1, \alpha_i>0$ and $\tau_i, i=1,\ldots,m$ (the periods of the components of the external signals reference r and disturbance d) are assumed known.

The plant Σ_G is finite-dimensional, linear time-invariant and its transfer function $G(s)$ is a $p\times p$ matrix. Initially the plant G is assumed to be positive real but later a procedure to create positive realness of a class of general plants is outlined.

Definition (Anderson and Vongpanitherd, 1973): The system Σ_G is said to be positive real(PR) if

1). All elements of its transfer function $G(s)$ are analytic in $\mathrm{Re}[s]>0$,

2). $G(s)$ is real for real positive s, and

3). $G(s)+G^*(s)\geq 0$ for $\mathrm{Re}[s]>0$.

where the superscript $*$ denotes complex conjugate transposition.

The following is the well-known positive real lemma.

Lemma (Anderson and Vongpanitherd, 1973): Assume

$$\dot{x}=Ax+Bu$$
$$y=Cx+Du, \quad x(0)=x_0$$

is a minimum realisation of $\Sigma_{G\,p\times q}$. *Then* $G(s)$ *is positive real if and only if there exist matrices* $Q\in R^{n\times n}$, $L\in R^{n\times k}$, and $W\in R^{k\times p}$ *with* $Q^T=Q>0$ *such that*

$$QA+A^TQ=-LL^T \tag{1}$$

$$QB=C^T-LW \tag{2}$$

$$D+D^T=H^TH \tag{3}$$

Here the superscript ^{T} denotes transposition. Note that LL^T is positive semidefinite.

It is obvious that if $G(s)$ is strictly proper, that is, $D=0$, which means $W=0$, then (1)~(3) become

$$QA+A^TQ=-LL^T \tag{4}$$

$$B=C^T \tag{5}$$

The main result of stability analysis can be stated below.

Theorem 1: Suppose that both reference r *and disturbance* d *are multi-periodic with components of period* $\tau_i, i=1,\ldots,m$. Suppose *that* r *can be exactly tracked by some choice of input and that* Σ_G *is positive real and strict proper. Then the MIMO multi-periodic repetitive system in Figure 2 is globally asymptotically stable in the sense that the error signal* $e\in L_2^m(0,\infty)\cap L_\infty^m(0,\infty)$, *the state* $x(\cdot)\in L_\infty^n(0,\infty)$ *and* $Lx(\cdot)\in L_2^k(0,\infty)$.

Proof: Without loss of generality, take $r=0$. The system Σ_G has the form

$$\dot{x}=Ax+B(v+d)$$
$$y=Cx \qquad x(0)=x_0$$

Let $d=\sum_{i=1}^{m}\alpha_i d_i$ where d_i has period τ_i. From Figure 2 it yields

$$v(t)=\sum_{i=1}^{m}\alpha_i z_i, \qquad \sum_{i=1}^{m}\alpha_i=1$$

$$z_i(t)=W_i z_i(t-\tau_i)+e(t)=W_i z_i(t-\tau_i)-y(t) \quad (6)$$

Noting that $v+d=\sum_{i=1}^{m}\alpha_i(z_i+d_i)$, it is easily verify that $\bar{z}_i=z_i+d_i$ satisfies the same evolution equation as z_i. Hence, without loss of generality, it is possible to assume that $d_i=0$, $1\le i\le m$, (i.e. $d=0$) for the rest of the proof.

Now introducing a positive definite Lyapunov function V of the form

$$V=x^TQx+\sum_{i=1}^{m}\alpha_i\int_{t-\tau_i}^{t}\|z_i(\theta)\|^2d\theta \quad (7)$$

By differentiating V along solutions and using the positive real lemma

$$\begin{aligned}\dot V&=\dot x^TQx+x^TQ\dot x+\sum_{i=1}^{m}\alpha_i[\|z_i(t)\|^2-\|z_i(t-\tau_i)\|^2]\\&=-x^TLL^Tx+2y^Tv(t)\\&\quad+\sum_{i=1}^{m}\alpha_i[\|z_i(t)\|^2-\|z_i(t-\tau_i)\|^2]\\&=-x^TLL^Tx-\|y\|^2\\&\quad+\sum_{i=1}^{m}\alpha_i[\|y\|^2+2y^Tz_i(t)+\|z_i(t)\|^2-\|z_i(t-\tau_i)\|^2]\end{aligned} \quad (8)$$

Notice that from (6) the last term in (8) is $-\sum_{i=1}^{m}\alpha_i(1-W_i^2)\|z_i(t-\tau_i)\|^2$ and notice that $0\le W_i\le 1$, thus (8) becomes

$$\begin{aligned}\dot V&=-x^TLL^Tx-\|y\|^2\\&\quad-\sum_{i=1}^{m}\alpha_i(1-W_i^2)\|z_i(t-\tau_i)\|^2<0\end{aligned} \quad (9)$$

Integrating (9) and using (7) and the positivity of V yield

$$\begin{aligned}0<V(0)&=V+\int_0^t x^TLL^Txdt+\int_0^t\|y\|^2\\&\quad+\int_0^t\sum_{i=1}^{m}\alpha_i(1-W_i)\|z_i(t-\tau_i)\|^2<\infty\end{aligned}$$

from which

$Lx(\cdot)\in L_2^k(0,\infty)$, $x(\cdot)\in L_\infty^n(0,\infty)$

and $y(\cdot)\in L_2^m(0,\infty)$ which proves the result. ■

Note: Extension to this result is possible to include low-pass filtering actions. There are omitted for brevity but will be included in the final version of the paper if space permits.

3. CREATE POSITIVE REALNESS BY FEEDFORWARD AND FEEDBACK COMPENSATIONS

It has been proved in section 2 that stability of closed-loop is guaranteed if the plant is positive real. This is, however, not necessarily true in most practical situations. Therefore the question arises: how to achieve positive realness by the use of compensators. In the literature this problem can be called 'positive real control' and efforts have been made by numerous authors. In this section a new and very simple algorithm to achieve positive realness for a class of strict proper plant by feedforward and output feedback compensation is introduced to indicate the potential of the approach.

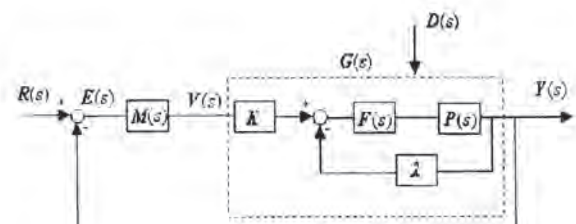

Fig. 3. PR compensation configuration

First the criterion for testing PR is reviewed.

Theorem 2 (Anderson and Vongpanitherd, 1973): Let $G(s)$ be a real rational matrix of functions of s. Then $G(s)$ is PR if and only if

1). No element of $G(s)$ has a pole in $\mathrm{Re}[s]>0$,

2). $G(j\omega)+G^*(j\omega)\ge 0$ for all real ω, with $j\omega$ not a pole of any elements of $G(s)$.

Suppose that non-PR plant P is minimum phase and strictly proper with relative degree 1.

If define P as the state space model $P\sim(A,B,C,0)$

then $\qquad P=C(sI-A)^{-1}B$

where we assume that that $(CB)=(CB)^{\mathrm T}>0$.

The inverse of P can be expressed in the form

$$L=P^{-1}=(CB)^{-1}s-\underbrace{(CB)^{-1}(CAB)}_{\text{Constant}}+\underbrace{H(s)}_{\substack{\text{Strict Proper and}\\ \text{asymptotically stable}}} \quad (10)$$

Then

$$\tilde{L} = L(j\omega) + L^*(j\omega)$$
$$= j\omega[(CB)^{-1} - ((CB)^{-1})^T] \qquad (11)$$
$$-2(CB)^{-1}(CAB) + H(j\omega) + H^*(j\omega)$$

If we denote

$$F_0 = -\frac{1}{2}[-2(CB)^{-1}(CAB) + H(j\omega) + H^*(j\omega)] \qquad (12)$$

then from (11)

$$\tilde{L} + 2F_0 = j\omega[(CB)^{-1} - ((CB)^{-1})^T] = 0 \qquad (13)$$

Therefore choosing $\lambda = \lambda I$, with $\lambda I \geq F_0$, it follows that

$$\tilde{L} + 2\lambda I \geq 0,\ \omega > 0 \qquad (14)$$

which implies that G is PR where G is given by

$$G = (I + P\lambda)^{-1} PK \geq 0 \qquad (15)$$

This procedure can be graphically interpreted in Figure 4 for a SISO system. Equation (12) also gives a simple method of determining a least value of λ for P^{-1} in the case of SISO control.

Notice that if $(CB) \neq (CB)^T$ then it is always possible to choose $F = (CB)^{-1}$ to achieve a symmetry condition. In this case

$$L_{new} = (PF)^{-1} = sI - \underbrace{(CAB)}_{\text{Constant}} + \underbrace{(CB)H(s)}_{\text{Strict Proper}} \qquad (16)$$

which leads to the choice of λ_{new} so that

$$\tilde{L}_{new} + 2\lambda_{new} I \geq 0 \qquad (17)$$

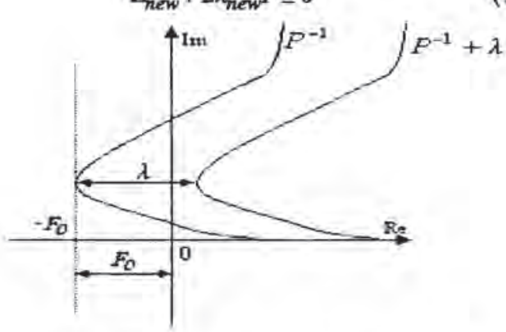

Fig. 4. PR interpretation in inverse polar plane

4. ISSUES ON THE CHOICES OF COMPENSATION FACTORS F, λ AND GAIN K

The design of a repetitive control system consists of two procedures, first classical stabilising controllers are designed independently from repetitive controller in order to obtain general performances, then the multi-periodic repetitive controller is added on to achieve high tracking/rejecting accuracy. Because there exists time-delays within the repetitive controller, the classical controllers play an important role in system transient performance before the repetitive controller cuts in and during the time-delay intervals. In this paper, apart from enforcing PR, which guarantees stability, controllers $F(s)$, λ and K can be also designed for robustness, poles assignment or other improvement, etc. Here just brief discussions are given on the choice of these controllers in terms of steady-state error. For simplicity, SISO form is used.

From Figure 3

$$G = \frac{KPF}{1+\lambda PF} = \frac{KP'}{1+\lambda P'} \qquad (18)$$

where $\lambda \geq -\inf_{\omega} \mathrm{Re}[P'^{-1}]$ is chosen in order to achieve a PR condition.

The steady state error for step reference can be determined as

$$E_{ss} = \lim_{s\to 0}\frac{1}{1+G} = \frac{1}{1+G(0)} \qquad (19)$$

By assuming $F(0) = 1$, (19) becomes

$$E_{ss} = \frac{1}{1+\frac{KP(0)}{\lambda P(0)}} \qquad (21)$$

therefore the steady state error can be reduced either by increasing gain K or by reducing λ. The advantage of enforcing PR is that the value of K will never affect the stability of the system, but increasing K tends to reduce the relative stability margin, so K should be properly increased to achieve faster response and smaller steady state error. The feedforward compensator $F(s)$ could be designed in a manner to reduce the feedback compensating factor λ. This point can be better illustrated in a simulation example below.

5. SIMULATION EXAMPLES

For the sake of simplicity, a SISO system is examined to illustrate the control system performance. W_i is set to be constants close to one throughout the simulation. The plant under control is

$$P = \frac{4s^2 + 2s + 300}{s(s^2 + 4s + 40)}$$

The property of multi-periodic control system will be shown in two ways, namely, typical application to different fundamental frequencies, and an extension to the single fundamental frequency case.

5.1. Two Fundamental Frequencies Case

The reference is a mix of harmonics of two fundamental frequencies 0.5Hz and 1.7Hz, that is,

$$r = r_1 + r_2$$

where $r_1 = \sin\pi t + 2\sin 3\pi t + 0.7\sin 5\pi t$

$$r_2 = \sin 3.4\pi t + 3\sin 6.8\pi t$$

therefore the two fundamental periods are

$$\tau_1 = 1/0.5 = 2, \quad \tau_2 = 1/1.7$$

For simplicity, choose $\alpha_1 = 0.5$ and $\alpha_2 = 0.5$, which means equal weighting in the controller on two components of reference signal. The gain K is chosen as 100.

Two simulation experiments are conducted.

(1). Let $F \equiv 1$

In this case, $\hat{\lambda}$ can be determined as

$$\hat{\lambda} = -\inf_{\omega} \mathrm{Re}[(PF)^{-1}] \cong 20.27$$

so choose $\lambda = 22$ to satisfy the positive real condition. The simulation result of the convergent error $e(t)$ is shown in Figure 5.

(2). Choose $F = \dfrac{1+s}{1+0.1s}$, a phase-lead compensator. It is easy to verify that $F(0) = 1$.

In this case, $\hat{\lambda} = -\inf_{\omega} \mathrm{Re}[(PF)^{-1}] \cong 1.749$

which is significantly reduced compared with the above case. Choose $\lambda = 1.76$ in this case.

The simulation result is given in Figure 6(solid). It is clear that the accuracy in the second case has been almost improved by an order of magnitude compared to the first one during transient process, by simply designing feedforward compensator.

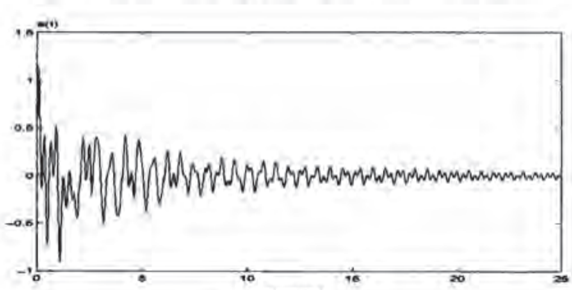

Fig. 5. The error when feedforward compensator $F \equiv 1$

Simulations show that this repetitive controller is also capable of effectively rejecting a periodic disturbance with the same frequency components.

Suppose in the followings that the disturbance is a square wave at a period of 3.4Hz and at peak value ± 10. A square wave is chosen to indicate that the scheme can cope with signals with an infinite (Fourier) frequency content. The tracking error is also demonstrated in Figure 6(dashed) and is almost identical to that of disturbance-free case (Figure 6, solid). The simulation result suggests that despite the significant existence of the disturbance signal, its influence is negligible.

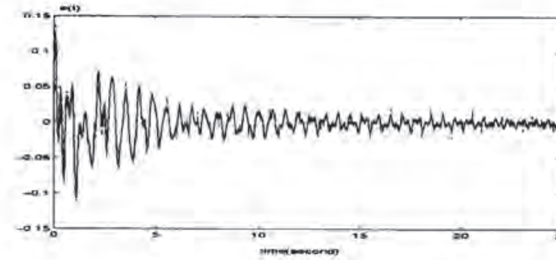

Fig. 6. The error when F is a phase-lead compensator—Solid for disturbance-free, and Dashed for square wave disturbance involved.

5.2. Single Fundamental Frequencies Case

It is interesting to notice that certain kinds of traditional single-periodic repetitive control systems can be extended to multi-periodic repetitive control with improved performance. The following example is shown based on above plants P, with controller parameters used in experiment (2).

Reference signal used in first simulation is taken as,

$$r = \sin\pi t + 2\sin 3\pi t + 0.7\sin 5\pi t$$

It can be considered either as a single period at fundamental period $\tau = 1/0.5$, or as a multi-periodic at three fundamental periods $\tau_1 = 1/0.5, \ \tau_2 = 1/1.5, \ \tau_3 = 1/2.5$. The simulation results for these two situations are shown in Figure 7.

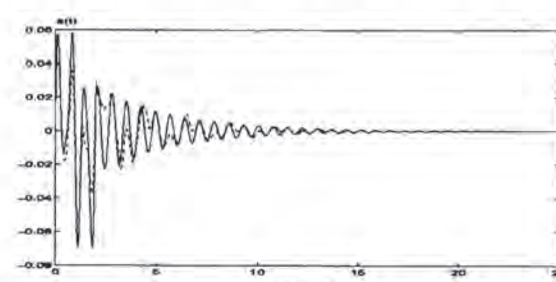

Fig. 7. Tracking errors for $r = \sin\pi t + 2\sin 3\pi t + 0.7\sin 5\pi t$ when using single-periodic repetitive control (_solid) and multi-periodic repetitive control (--dashed).

The Integral of square error are ISE1=0.3410(single-periodic) and ISE2=0.2437 (multi-periodic), which indicates that a better tracking performance is possible by reconfiguring the highest frequency content.

6. CONCLUSIONS

A new kind of MIMO multi-periodic repetitive control system is studied. These results are extensions of single periodic repetitive control problems. The stability is analysed in the sense of Lyapunov stability and it has been shown that asymptotic stability is guaranteed if the plant is positive real. While most real plants are not necessarily positive real, procedures can be employed to build positive realness by feedforward and feedback compensations. The simulation example shows that the new control system can be very effective.

Extensions to the work to relax the PR condition, prove exponential stability and include frequency dependent weighting $W_i(s)$ will be reported separately.

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