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Exact Boundary Controllability and Optimal Control for a Generalised Korteweg de Vries Equation

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Abstract
A generalized Korteweg de Vries system is studied and shown to be controllable by boundary control and a nonlinear optimal control problem is replaced by a linear, time-varying sequence of problems which can be solved by classical methods. This is achieved by showing the controllability of a linearized version and then by applying a fixed point argument. A subtle smoothing property is required which is more complex than the usual smoothing produced by the semigroup of the system. A new approximation technique is then applied to solve the optimal control problem, reducing it to a sequence of linear, time-varying systems.

1. Introduction

The original analytical approach to solitary waves developed in [3] can be considered from a more abstract position in the following way (see [4]). For a linear wave equation

$$\phi_{tt} - c^2\phi_{xx} = 0$$

we have a set of solutions of the form $\phi = \exp i(\omega t - kx)$, where the wave number k and the frequency ω are related by $\omega^2 = c^2k^2$. For general linear wave equations with higher order derivatives we may have a dispersion relation of the form

$$\omega^2 = f(k^2)$$

where f is not necessarily linear. For example, the linear wave equation

$$\phi_t + c\phi_x - \frac{\varepsilon}{2c}\phi_{xxx} = 0$$

has the dispersion relation

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$$\omega = ck + \frac{1}{2}\frac{\varepsilon k^3}{c}$$

(or $\omega^2 \approx c^2k^2 + \varepsilon k^4$ for small ε). If the speed of the wave depends on the amplitude ϕ then the equation becomes nonlinear. If we assume that the dependence is linear we obtain the equation

$$\phi_t + (c - a\phi)\phi_x = 0$$

and if we also include the dispersion term above we have

$$\phi_t + c\phi_x - a\phi\phi_x - (\varepsilon/2c)\phi_{xxx} = 0.$$

Choosing coordinates moving to the right with speed c gives the Korteweg-de Vries equation

$$\phi_t = 6\phi\phi_x - \phi_{xxx}$$

up to scaling constants, since the term ϕ_x drops out.

In this paper we shall study more general equations in which the wave speed does not depend linearly on the amplitude and hence we consider systems of the form

$$\phi_t + \phi_x + k(\phi)\phi_x + \phi_{xxx} = 0$$

(normalising c to 1). In particular, we shall be interested in generalising the results of [5] on exact boundary controllability for such systems and to study the optimal control problem for the suppression of these nonlinear waves.

Hence we shall consider the generalized Korteweg-de Vries (KdV) equation

$$\phi_t + \phi_x + k(\phi)\phi_x + \phi_{xxx} = 0\ ,\ x \in (\alpha, \beta) \tag{1.1}$$

with the boundary controls

$$\phi(\alpha, t) = u_1(t)\ ,\ \phi(\beta, t) = u_2(t)\ ,\ \phi_x(\beta, t) = u_3(t).$$

Then the exact boundary control problem can be stated in the following way:

$$\left\{\begin{array}{l} \text{Let } T > 0 \text{ and } s \geq 0. \text{ For any } f, g \in H^s(\alpha, \beta), \text{ find} \\ \text{boundary controls } u_j,\ j = 1, 2, 3 \text{ such that the solution} \\ \phi \in C([0, T]; H^s(\alpha, \beta)) \text{ satisfies} \\ \qquad \phi(x, 0) = f(x),\ \phi(x, T) = g(x) \\ \text{(in the distributional sense) on the interval } (\alpha, \beta). \end{array}\right.$$

We write the system in the form of an abstract equation in some Hilbert space X:

$$\frac{dy}{dt} = Ay + F(y) + Bu \tag{1.2}$$

where A generates a C^0-semigroup $W(t)$ in X and u is the control. To prove exact controllability, we then first prove exact controllability of the linearised system

$$\frac{dy}{dt} = Ay + Bu$$

by showing the existence of a bounded linear operator $G : X \times X \to L^2(0,T;Y)$ such that, for all $f, g \in X$, the unique solution of

$$\frac{dy}{dt} = Ay + BG(f,g)\ ,\ y(0) = f$$

satisfies $y(T) = g$. Then, writing the nonlinear system in the integral form

$$y(t) = W(t)y(0) + \int_0^t W(t-\tau)F(y(\tau))d\tau + \int_0^t W(t-\tau)Bu(\tau)d\tau$$

and setting

$$w(T,u) = \int_0^t W(t-\tau)F(u(\tau))d\tau$$

we define the operator Γ by

$$\Gamma(u) = W(t)f + \int_0^t W(t-\tau)F(u(\tau))d\tau + \int_0^t W(t-\tau)BG(f, g - w(T,u))(\tau)d\tau.$$

Since $\Gamma(u)(0) = f$ and $\Gamma(u)(T) = w(T,u) + g - w(T,u) = g$, exact controllability will follow if we can show that Γ has a fixed point in $C([0,T], X)$.

We shall follow the approach of [5] and apply a subtle smoothing property of the linearised system to prove the existence of a fixed point. It turns out to be easier to consider an equivalent problem of initial value control of the equation on the whole of $\mathbb{R}$ rather than the boundary control problem on (α, β). Hence we consider the system

$$\begin{aligned} \phi_t + \phi_x + k(\phi)\phi_x + \phi_{xxx} &= 0 \\ \phi(x,0) &= f(x) \end{aligned} \tag{1.3}$$

and ask if, for any given $f, g \in H^s(\mathbb{R})$, we can find $\phi \in H^s(\mathbb{R})$ such that ϕ satisfies

$$\phi(x,0) = f(x),\ \phi(x,T) = g(x)$$

on the interval (α, β). We get a solution of the original problem by choosing the controls

$$u_1(t) = \phi(\alpha,t),\ u_2(t) = \phi(\beta,t),\ u_3(t) = \phi_x(\beta,t)$$

and take the restriction of ϕ to $[\alpha,\beta] \times [0,T]$.

Having proved local exact controllability, we consider the optimal control problem:

$$\min\ J = \langle \phi(T,\cdot), F(\phi)\phi(T,\cdot)\rangle_H + \int_0^T \left(\langle \phi(t,\cdot), Q(\phi)\phi(t,\cdot)\rangle_H + \sum_{i=1}^3 r_i u_i^2(t) \right) dt$$

subject to the dynamics (1.1). After writing the equation in the abstract form we shall find approximations to local optima by studying the series of approximate linear-quadratic problems:

$$\begin{aligned}
& \phi_t^{[i]} + \phi_x^{[i]} + k(\phi^{[i-1]})\phi_x^{[i]} + \phi_{xxx}^{[i]} = 0 \\
\phi^{[i]}(\alpha,t) &= u_1^{[i]}(t)\ ,\ \phi^{[i]}(\beta,t) = u_2^{[i]}(t)\ ,\ \phi_x^{[i]}(\beta,t) = u_3(t) \\
\min\ J &= \left\langle \phi^{[i]}(T,\cdot), F(\phi^{[i-1]})\phi^{[i]}(T,\cdot)\right\rangle_H + \\
& \int_0^T \left(\left\langle \phi^{[i]}(t,\cdot), Q(\phi^{[i-1]})\phi^{[i]}(t,\cdot)\right\rangle_H + \sum_{i=1}^{3} r_i u_i^{[i]2}(t)\right) dt.
\end{aligned}$$

2. Boundary Controllability

We shall require, as in [5], a number of smoothing properties of the linearisation. Let $W(t)$ be the unitary group generated by the operator

$$Af = -f' - f'''$$

from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$, with $\mathcal{D}(A) = H^3(\mathbb{R})$. Then the solution of the linear KdV equation

$$\begin{aligned}
\phi_t + \phi_x + \phi_{xxx} &= 0\ ,\ x,t \in \mathbb{R} \\
\phi(x,0) &= f(x)
\end{aligned}$$

is given by

$$\phi(t) = W(t)f.$$

If L_b^2 denotes the weighted Sobolev space $L^2(e^{2bx}dx)$ for any $b > 0$, then (see Kato [2]) A generates a semigroup $W_b(t)$ in this space, given by

$$W_b(t) = \exp(-t(D-b)^3 - t(D-b))$$

in $L^2(\mathbb{R})$. Moreover, we have

$$\|W_b(t)\|_{L(H^s(\mathbb{R}),H^{s'}(\mathbb{R}))} \leq ct^{-(s'-s)/2}\exp(b^3 t)$$

for $s \leq s'$, and if $\phi_t + \phi_x + \phi_{xxx} = p$, $0 < t < T$, $e^{bx}\phi \in L^\infty([0,T];H^s(\mathbb{R}))$ and $e^{bx}p \in L^\infty([0,T];H^{s-1}(\mathbb{R}))$, then

$$e^{bx}\phi \in C([0,T];H^0) \cap C([0,T];H^s) \text{ for } s' < s+1.$$

Moreover, we have

$$e^{bx}\phi(t) = W_b(t)\phi(0) + \int_0^t W_b(t-\tau)e^{bx}p(\tau)d\tau.$$

We now require a better smoothing property so we define the space $Y_{s,b}$ $(s, b \in \mathbb{R})$ to be the completion of $S(\mathbb{R}^2)$ (tempered functions) with respect to the norm

$$\|f\|_{Y_{s,b}}^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+|\tau-\xi-\xi^3|)^{2b}(1+|\xi|)^{2s}|\hat{f}(\xi,\tau)|^2 d\xi d\tau,$$

where $\hat{f}$ is the Fourier transform of f. Then, if $s > -1$ and $b > 1/2$,

$$f \in Y_{s,b} \Rightarrow f \in C_{loc}^{[1+s],\alpha}(\mathbb{R}; L_t^2(\mathbb{R}))$$

for any $0 \le \alpha \le 1 + s - [1+s]$, and so

$$f \in L_{x,loc}^p(\mathbb{R}; L_t^2(\mathbb{R})),\ 1 \le p \le \infty.$$

The crucial smoothing property is (see [5])
Lemma 1 *Let $s > -3/4$, $\sigma \in C_0^\infty(\mathbb{R})$. Then there exists $\beta \in (1/2, 1)$ such that for all $b \in (1/2, \beta)$ there exists $c > 0$ such that*

$$\|\sigma(t)\partial_x(\phi\psi)\|_{Y_{s,b-1}} \le c\,\|\phi\|_{Y_{s,b}}\,\|\psi\|_{Y_{s,b}}$$

for all $\phi, \psi \in Y_{s,b}$. □

Since $Y_{s,b} \subset C(\mathbb{R}; H^s(\mathbb{R}))$ for $b > 1/2$, it follows that

$$\left\|\sigma_1(t)\int_0^t W(t-\tau)\sigma_2(\tau)(\partial_x(\phi\psi))(\cdot,\tau)d\tau\right\|_{Y_{s,b}} \le c\,\|\phi\|_{Y_{s,b}}\,\|\psi\|_{Y_{s,b}}$$

for functions $\sigma_1(t), \sigma_2(t) \in C_0^\infty(\mathbb{R})$. Using the method above and in [5], we therefore have the following boundary controllability result:
Theorem 1 *Suppose that $k(\phi)$ is of the form*

$$k(\phi) = p'(\phi)\phi + p(\phi)$$

where p is differentiable and

$$\|p(\phi)\|_{Y_{s,b}} \le \bar{c}\,\|\phi\|_{Y_{s,b}}$$

for some constant $\bar{c}$. Now let $T > 0$ and $s \ge 0$ be given and $[\alpha, \beta] \subseteq (\alpha_1, \beta_1)$. Suppose that

$$w(x,t) \in C^\infty[(\alpha_1, \beta_1) \times (-\varepsilon, T+\varepsilon)]$$

for some $\varepsilon > 0$ satisfies

$$w_t + w_x + k(w)w_x + w_{xxx} = 0,\ (x,t) \in (\alpha_1, \beta_1) \times (-\varepsilon, T+\varepsilon).$$

Then there exists a $\delta > 0$ such that for any $\phi, \psi \in H^s(\alpha, \beta)$ satisfying

$$\|f(\cdot) - w(\cdot,0)\|_{H^s(\alpha,\beta)} \le \delta \text{ and } \|g(\cdot) - w(\cdot,0)\|_{H^s(\alpha,\beta)} \le \delta$$

one can find controls u_1, u_2, u_3 *in* $L^2(0,T)$ *(*$u_j \in C[0,T]$, $j = 1,2,3$ *if* $s > 3/2$*) such that the system has the solution*

$$\phi \in C([0,T]; H^s(\alpha,\beta)) \cap L^2(0,T; H^{s+1}(\alpha,\beta))$$

satisfying

$$\phi(x,0) = f(x)\ ,\ \phi(x,T) = g(x)$$

on the interval (α,β). □

The proof of theorem 1 follows from the next proposition, which is a generalised form of proposition 4.1 in [5].

Proposition 1 *Consider the nonlinear KdV equation*

$$\begin{aligned} \phi_t + \phi_x + k(\phi)\phi_x + (a(x,t)\phi)_x + \phi_{xxx} &= 0,\ x,t \in R \\ \phi(x,0) &= h(x). \end{aligned} \tag{2.1}$$

where $a(x,t) \in Y_{s,b}$ *and* k *is of the form in theorem 1, and let* $s \geq 0, T > 0$ *and* $b > 0$ *be as in lemma 1. Then there exists* $\delta > 0$ *such that if* $f, g \in H^s(\alpha,\beta)$ *with*

$$\|f\|_{H^s(\alpha,\beta)} \leq \delta,\ \ \|g\|_{H^s(\alpha,\beta)} \leq \delta$$

there exists $h \in H^s(\mathbb{R})$ *such that the solution of (2.1) satisfies*

$$\phi(x,0) = f(x)\ ,\ \phi(x,T) = g(x),\ x \in (\alpha,\beta).$$

Proof We have

$$\phi(t) = W_a(t)h - \int_0^t W_a(t-\tau)(k(\phi)\phi_x)(\tau)d\tau \tag{2.2}$$

where W_a is the C^0-group introduced above. Let

$$w(T,\phi) = \int_0^T W_a(t-\tau)(k(\phi)\phi_x)(\tau)d\tau.$$

Then we can choose

$$h = G(f, g + w(T,\phi))$$

in (2.2) to give

$$\phi(t) = W_a(t)G(f, g + w(T,\phi)) - \int_0^t W_a(t-\tau)(k(\phi)\phi_x)(\tau)d\tau$$

and then

$$\phi(x,0) = f(x)\ ,\ \phi(x,T) = g(x).$$

All that remains is to show that the map

$$\Gamma(\phi) = W_a(t)G(f, g + w(T,\phi)) - \int_0^t W_a(t-\tau)(k(\phi)\phi_x)(\tau)d\tau$$

has a fixed point in $Y_{s,b}$ by demonstrating that it is a contraction. This follows as in [5] from the inequality

$$\begin{aligned}
&\left\| \int_0^t W_a(t-\tau)(k(\phi)\phi_x)(\tau)d\tau \right\|_{H^s(\mathbb{R})} \\
\leq\ & \sup_{t\in\mathbb{R}} \left\| \sigma_1(t) \int_0^T W_a(T-\tau)\sigma_2(\tau)(k(\phi)\phi_x)(\tau)d\tau \right\|_{H^s(\mathbb{R})} \\
=\ & \sup_{t\in\mathbb{R}} \left\| \sigma_1(t) \int_0^T W_a(T-\tau)\sigma_2(\tau)(\partial_x(p(\phi)\phi)\phi_x)(\tau)d\tau \right\|_{H^s(\mathbb{R})} \\
\leq\ & c\|\phi\|^2_{Y_{s,b}} \ \square
\end{aligned}$$

3. Local Optimal Control

Now consider the optimisation problem:

$$\min\ J = \langle \phi(T,\cdot), F(\phi)\phi(T,\cdot)\rangle_H + \int_0^T \left(\langle \phi(t,\cdot), Q(\phi)\phi(t,\cdot)\rangle_H + \sum_{i=1}^3 r_i u_i^2(t) \right) dt \tag{3.1}$$

subject to

$$\begin{aligned}
\phi_t + \phi_x + k(\phi)\phi_x + \phi_{xxx} = 0 \qquad (3.2)\\
\phi(\alpha,t) = u_1(t)\ ,\ \phi(\beta,t) = u_2(t)\ ,\ \phi_x(\beta,t) = u_3(t).
\end{aligned}$$

First we write the abstract form of the state equation by applying Green's identity. Thus, in the linear case, suppose ϕ satisfies

$$\begin{aligned}
\phi_t + \phi_x + \phi_{xxx} = 0 \qquad (3.3)\\
\phi(\alpha,t) = u_1(t)\ ,\ \phi(\beta,t) = u_2(t)\ ,\ \phi_x(\beta,t) = u_3(t)
\end{aligned}$$

and let ψ be a test function. Then,

$$\int_0^T \int_\alpha^\beta (\phi_t\psi + \phi_x\psi + \phi_{xxx}\psi)dxdt = 0$$

and so

$$\begin{aligned}
&\left[\int_\alpha^\beta \phi\psi dx\right]_0^T - \int_0^T\int_\alpha^\beta \phi\psi_t dxdt + \int_0^T [\phi\psi]_\alpha^\beta\, dt - \int_0^T\int_\alpha^\beta \phi\psi_x dxdt \\
&+ \int_0^T [\phi_{xx}\psi]_\alpha^\beta\, dt - \int_0^T [\phi_x\psi_x]_\alpha^\beta\, dt + \int_0^T [\phi\psi_{xx}]_\alpha^\beta\, dt - \int_0^T\int_\alpha^\beta \phi\psi_{xxx} dxdt = 0.
\end{aligned}$$

Hence if ψ satisfies

$$\begin{aligned}
\psi_t + \psi_x + \psi_{xxx} &= 0, \qquad (3.4)\\
[\phi_{xx}\psi]_\alpha^\beta &= [\phi\psi_{xx}]_\alpha^\beta = \psi_x(\alpha,t) = 0
\end{aligned}$$

and ϕ satisfies the equation

$$\begin{aligned} \phi_t + \phi_x + \phi_{xxx} + u_1\delta(x-\alpha) + u_2\delta(x-\beta) + u_3\delta_x(x-\beta) &= 0 \\ \phi(\alpha,t) = \phi(\beta,t) = \phi_x(\beta,t) &= 0 \end{aligned} \tag{3.5}$$

we obtain formally the same solution as the equation (3.2). Hence we say that ϕ is a *weak solution* of (3.2) if it satisfies (3.5) in the sense of distributions with ψ as test functions satisying (3.4).

However, note that lemma 1 only gives the required smoothing for $s > -3/4$. Since $\delta \in H^{-1/2+\varepsilon}$ this is satisfactory for the terms u_1 and u_2, but the linear dynamics may not smooth out the term $u_3\delta_x$. Thus, we shall consider the approximate problem

$$\begin{aligned} \phi_t + \phi_x + \phi_{xxx} + u_1\delta(x-\alpha) + u_2\delta(x-\beta) + u_3\delta(x-\beta-\varepsilon) &= 0 \\ \phi(\alpha,t) = \phi(\beta,t) = \phi_x(\beta,t) &= 0 \end{aligned} \tag{3.6}$$

for some new u_2, u_3 and some $\varepsilon > 0$. Hence we shall consider the optimal control problem of minimising (3.1) subject to the dynamics (3.6). To do this we introduce the approximate optimal control problem consisting of an approximate quadratic cost functional and a sequence of nonautonomous linear systems, given by

$$\begin{aligned} \min\ J^{[i]} &= \left\langle \phi^{[i]}(T,\cdot), F(\phi^{[i-1]})\phi^{[i]}(T,\cdot)\right\rangle_H + \\ &\quad \int_0^T \left(\left\langle \phi^{[i]}(t,\cdot), Q(\phi^{[i-1]})\phi^{[i]}(t,\cdot)\right\rangle_H + \sum_{j=1}^{3} r_j u_j^{[i]2}(t)\right) dt \end{aligned}$$

$$\begin{aligned} \phi_t^{[i]} + \phi_x^{[i]} + \phi_{xxx}^{[i]} + k(\phi_x^{[i-1]})\phi_x^{[i]} + u_1^{[i]}\delta(x-\alpha) + & \\ u_2^{[i]}\delta(x-\beta) + u_3^{[i]}\delta(x-\beta-\varepsilon) &= 0 \\ \phi^{[i]}(\alpha,t) = \phi^{[i]}(\beta,t) = \phi_x^{[i]}(\beta,t) &= 0. \end{aligned} \tag{3.7}$$

The following result now follows easily from [1]; we shall give a brief outline of the proof.

Theorem 2 *The sequence of systems (3.7) with the standard linear feedback control (given by the Riccati solution of a standard nonautonomous, linear-quadratic problem) converges in $C([0,T];H)$ for some $T > 0$. If the problem is coercive then the limit of the control gives the optimal control of the nonlinear problem.*

Proof (Outline-see [1] for more details.) First write the problem in the abstract form

$$J = x^T(t_f)F(x(t_f))x(t_f) + \int_0^{t_f} (x^TQ(x)x + u^TRu)dt.$$

$$\dot{x} = A(x)x + B(x)u\ ,\ x(0) = x_0$$

and introduce the associated sequence of problems

$$\begin{aligned} \dot{x}^{[0]} &= A(x_0)x^{[0]} + B(x_0)u^{[0]}\ ,\ x^{[0]}(0) = x_0 \\ J^{[0]} &= x^{[0]T}(t_f)Fx^{[0]}(t_f) + \int_0^{t_f} (x^{[0]T}Qx^{[0]} + u^{[0]T}Ru^{[0]})dt \end{aligned}$$

and for $k \geq 1$,

$$\begin{aligned} \dot{x}^{[k]} &= A(x^{[k-1]}(t))x^{[k]} + B(x^{[k-1]}(t))u^{[k]} \ , \ x^{[k]}(0) = x_0 \\ J^{[k]} &= x^{[k]T}(t_f)Fx^{[k]}(t_f) + \int_0^{t_f} (x^{[k]T}Qx^{[k]} + u^{[k]T}Ru^{[k]})dt \ . \end{aligned} \tag{3.8}$$

Since each approximating problem in (3.8) is linear (time-varying), quadratic we can write the optimal control in the form

$$u^{[k]} = -R^{-1}B^T(x^{[k-1]}(t))P^{[k]}x^{[k]}(t) \tag{3.9}$$

where $P^{[k]}$ is the solution of the usual Riccati equation

$$\begin{aligned} \dot{P}^{[k]}(t) &= -Q - P^{[k]}A(x^{[k-1]}(t)) - A(x^{[k-1]}(t))^T P^{[k]} + \\ &\quad P^{[k]}B(x^{[k-1]}(t))R^{-1}B^T(x^{[k-1]}(t))P^{[k]} \\ P^{[k]}(t_f) &= F \end{aligned} \tag{3.10}$$

and the k^{th} dynamical system becomes

$$\dot{x}^{[k]} = A(x^{[k-1]}(t))x^{[k]} - B(x^{[k-1]}(t))R^{-1}B^T(x^{[k-1]}(t))P^{[k]}x^{[k]}(t) \ . \tag{3.11}$$

Then the local Lipschitz continuity of A and B can be used to prove the convergence of the controls (3.9) and the Riccati operators (in the weak form) given in (3.10). Hence the dynamical systems in (3.11) converge in $C([0,T];H)$. If the problem is coercive, so that a unique optimum exists, it is easy to show that the limit of the controls in (3.9) gives this optimal control. □

It follows from the theorem that nonlinear waves produced by generalized KdV systems can be suppressed optimally by the use of boundary control. This may become important in the future for controlling nonlinear water waves in channels.

4. Conclusions

In this paper we have shown that a generalised Korteweg de Vries system is exactly boundary controllable by using a subtle a priori estimate and the fixed point approach. Then, in order to solve the general nonlinear optimal control problem associated with this system, we have introduced a sequence of approximating linear, nonautonomous systems which can be solved by classical methods, leading to an optimal control when one exists. (In general these systems may only converge to some local optimum.)

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