STABILITY OF NONLINEAR DISTRIBUTED PARAMETER SYSTEMS

S. P. Banks

University of Warwick, Coventry, England

ABSTRACT

The stability problem for the equation

$$\dot{x} = Ax + Bx,$$

where A generates a semigroup T(t) and B is a nonlinear operator , is studied by using the Lyapunov functional

$$V(x) = \left\langle \int_0^\infty T^*(t)QT(t)x\,dt, x \right\rangle$$

for some positive selfadjoint operator Q.

1. INTRODUCTION

In this paper we shall consider the stability of nonlinear distributed parameter systems which may be written in the form of an abstract evolution equation

$$\begin{aligned} \dot{x} &= Ax + Bx \\ x(0) &= x_0 \in \mathcal{D}(A), \end{aligned} \tag{1.1}$$

defined on a Banach space X, where A is a linear operator with dense domain $\mathcal{D}(A)$ which generates a strongly continuous (C_0) semigroup of operators, and B is some nonlinear operator with certain special properties to be introduced later.

In particular , we shall be interested in the generalisation to infinite dimensions of the well-known finite dimensional Liapunov theorem which states that the linear system $\dot{x} = Ax$, is asymptotically stable iff $\exists$, for any positive definite Hermitian matrix Q , a positive definite Hermitian matrix P such that

$$PA + A^*P = -Q\ . \tag{1.2}$$

Having obtained this generalisation (as in Datko[1], [2]) we shall consider to what extent it can be used to study the stability of the equation (1.1).

Of course , closely related to the stability problem for (1.1) is the question of existence and uniqueness of solutions of equation (1.1). We shall discuss some types of equations which have unique solutions in section 3.

First , however , we shall give a short survey of some of the existing literature on the stability of distributed parameter systems.

2. SURVEY OF THE LITERATURE

Many papers have been written on the stability of special types of partial differential equations obtained from elasticity and hydrodynamics , for example , using some form of approximation . We mention just two works by Bolotin [3] and Eckhaus [4] . Usually , when such an approximation is made , there is little or no rigorous justification for the method.

One of the first attempts to use Lyapunovs direct method for infinite dimensional systems was made by Massera [5] , who considered a countable number of ordinary differential equations

A general stability theory for metric spaces which is applicable to certain types of partial differential equations was given by Zubov [6] , who's results are applied to a specific problem of a nuclear reactor by Hsu [7] . There are also many other papers which use Lyapunov functions directly to study special problems , see for example Movchan [8] , Parks [9].

An approach to a completely rigorous and abstract theory of Lyapunov stability of infinite dimensional systems was studied by Buis , Vogt , Eisen [10], Pau [11] in which a Lyapunov function is defined by $\langle x,Sx\rangle$ where S is a bounded positive selfadjoint operator, for the Hilbert space case, and as $[x,Sx]$ for the Banach space case , where $[.,.]$ is a semi-inner product . By the use of linear or nonlinear semigroups , it was then possible to study the stability of systems of the form

$$\dot{x} = Ax ,$$

where A is the generator of a linear or nonlinear semigroup.

A theory for linear perturbed equations of the form (1.1) with B bounded or unbounded has been developed by Blakeley , Pritchard [12] .

Although we shall not be concerned with frequency domain methods, it is interesting to note the stability results of Freedman , Falb , Zames [13] obtained by an abstract transformation method . Here the circle criterion is generalised to infinite-dimensional systems.

3. EXISTENCE AND UNIQUENESS OF SOLUTIONS

The second method of Lyapunov does not require specific knowledge of the solutions of an equation for which the stability is being studied . However , it is , of course , interesting and important to know that the system under consideration has solutions and that they are unique . For example , if one can show the existence of a 'mild' solution of (1.1) then it is still possible to use Lyapunov's method , since one can justify a formal differentiation of the Lyapunov function (see [12] , for more details) . We shall therefore give , in this , section , some existence results for equation (1.1) with various nonlinearities B.

There are many existence results for equations of type (1.1) , but the most useful for our purposes is the following result of Webb [14] .

<u>Theorem 3.1</u> Let A be a closed , densely defined , linear m-accretive operator from a Banach space X to itself , and let $T(t), t \geq 0$, be the semigroup of operators generated by $-A$. Let B be a continuous , everywhere defined , nonlinear accretive operator from X to itself . There exists , for each $x \in X$, a unique solution $U(t)x$ to the integral equation

$$U(t)x = T(t)x - \int_0^t T(t-s)BU(s)x\,ds , \qquad t \geq 0.$$

Moreover , $U(t)$, $t \geq 0$, is a strongly continuous semigroup of nonlinear contractions on X , $-(A + B)$ is the infinitesimal generator of $U(t)$, $t \geq 0$, and A + B is m-accretive on X. □

The theorem is true , in fact , if the condition that A + B is accretive replaces the condition that B is accretive .

If we assume that A generates an analytic semigroup , and that we can define the fractional power of A by

$$\left.\begin{array}{l} A^{-\alpha} = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^\infty e^{-sA} s^{\alpha-1} ds \\ A^{\alpha} = (A^{-\alpha})^{-1} \end{array}\right\} \alpha > 0 ,$$

(cf. Friedman [15]), then we can obtain the following result (Banks [16]):

Proposition 3.2 Let A be as in theorem 3.1 where $A^{-\alpha}$ is compact for some $\alpha > 0$ and suppose that B is an everywhere defined continuous nonlinear operator such that $B(\mathcal{D}(A^{\alpha})) \subseteq \mathcal{D}(A^{\alpha})$ and $A^{\alpha} BA^{-\alpha}$ maps bounded sets into bounded sets . Then the equation

$$u(t,x) = T(t)x - \int_0^t T(t-s)Bu(s,x)ds \tag{3.1}$$

has a unique solution for each $x \in \mathcal{D}(A^{2\alpha})$.

Moreover , we have the estimate

$$\| u(t,x)\| \leq \|A^{-\alpha}\|^2 \|A^{2\alpha} x\| , \qquad x \in \mathcal{D}(A^{2\alpha}),$$

so the solution is stable as a map from $[\mathcal{D}(A^{2\alpha})]$ to X . □

Here , $[\mathcal{D}(A^{2\alpha})]$ means $\mathcal{D}(A^{2\alpha})$ equipped with the graph norm . In order to obtain global existence we have to restrict ourselves to the case where X is a Hilbert space . Then we obtain (Banks [16]):

Theorem 3.3 Let A be as in proposition 3.2 and assume that B is an everywhere defined , locally Lipschitz continuous , nonlinear operator such that $A^{\alpha} BA^{-\alpha}$ is continuous and $A^{2\alpha} BA^{-2\alpha}$ maps bounded sets into bounded sets . Then equation (3.1) has a unique solution , defined for all $t \geq 0$, and for $x \in \mathcal{D}(A^{3\alpha})$. □

Finally , if B is not defined everywhere on X , but only on a dense subspace Y, where the injection $i: Y \hookrightarrow X$ is continuous , then we have the following result (Banks [17]):

Theorem 3.4 Let A be a closed , densely defined , linear m-accretive operator which generates a semigroup T(t) satisfying the assumption

$$T(t)X \subseteq Y \quad \text{and} \quad \|T(t)x\|_Y \leq g(t)\,\|x\|_X , \qquad x \in X,$$

where $g \in L^1_{loc}$.

Let B be a nonlinear accretive operator from X into X with domain Y , and suppose that B is continuous from Y_w to X. Then , for each $y \in Y$, the equation

$$u(t,y) = T(t)y - \int_0^t T(t-s)Bu(s,y)ds ,$$

has a unique solution which satisfies

$$u(t+t',y) = u(t',u(t,y)) , \qquad \text{for } t,t' \geq 0. \ \square$$

We note finally that there are many more existence results for nonlinear equations : cf. Barbu [18] .

4. STABILITY OF LINEAR SYSTEMS

We shall now study the stability of the linear system

$$\dot{x} = Ax , \qquad x(0) = x_0 , \tag{4.1}$$

by considering an infinite-dimensional version of equation (1.2) .

Let us first note the following theorem of Zubov [6] , which generalises Lyapunov's basic theorem to a general metric space .

Theorem 4.1 (Zubov). Let M be a closed invariant set for a semidynamical system $\phi_t(x)$ defined on a metric space (X,ρ) . In order that M be asymptotically stable , in the sense of Lyapunov it is necessary and sufficient that , in a sufficiently small neighbourhood

$S(M,r) = \{x \in X: \rho(x,M) \leq r\}$ of M, $\exists$ a functional V with the following properties:

(1). For any sufficiently small quantity $c_1 > 0$, $\exists$ $c_2 > 0$ such that $V(p) > c_2$ for $p \in S(M,r)$ and $\rho(p,M) > c_1$.

(2). For any $\gamma_2 > 0$, $\exists$ $\gamma_1 > 0$ such that $V(p) \leq \gamma_2$, when $\rho(p,M) < \gamma_1$.

(3). $V(\phi_t(p))$ is a nondecreasing function $\forall\, t \geq 0$ and p such that $\phi_t(p) \in S(M,r)$.

(4). $V(\phi_t(p)) \to 0$ as $t \to +\infty$, for any motion $\phi_t(p) \in S(M,r)$ $(0 \leq t < \infty)$. □

Here, we shall only be concerned with critical points and so M will be a singleton consisting of the critical point. In the case of a finite-dimensional normed space, it is well known that condition (1) in theorem 4.1 may be replaced by the equivalent condition

(1'). $V(p) > 0$, except at a critical point p_c, where $V(p_c) = 0$.

In the infinite-dimensional case, this is no longer sufficient, and for the nonlinear case we shall need a different stability theorem (cf. §5).

In [1], Datko considers a complex Banach space X with nondegenerate Hermitian forms from $X \times X$ into C, i.e. there exists a map $H: X \times X \to C$ such that if $x,y,z \in X$, $\lambda \in C$, then the following axioms hold

(i). $H(x+y) = H(x,z) + H(y,z)$

(ii). $H(x,y) = \overline{H(y,x)}$

(iii). $H(\lambda x,y) = \lambda H(x,y)$

(iv). $H(x,x) > 0$ if $x \neq 0$

(v). $\exists$ a constant $0 < M < \infty$ such that

$$|H(x,y)| \leq M\|x\| \cdot \|y\| .$$

Clearly, the space X has a nondegenerate Hermitian form iff $\exists$ a positive selfadjoint operator $Q: X \to X^*$. This can be seen by writing $Qx \triangleq H(x,.)$. We can therefore state the result (cf. Datko [1]):

Theorem 4.3 Suppose that $T(t)$ is a strongly continuous semigroup of operators and that there exist constants $\chi > 0$ and $\mu > 0$ such that $\|T(t)\| \leq \chi e^{-\mu t}$ for all $t \geq 0$. Then, given any positive selfadjoint operator $Q: X \to X^*$, $\exists$ a unique positive selfadjoint operator $P: X \to X^*$ such that

$$Px = \int_0^\infty T^*(t) Q T(t) x\, dt$$

and

$$PAx + A^*Px = -Qx \qquad \forall\, x \in \mathcal{D}(A). \;\square$$

We call a selfadjoint operator $Q: X \to X^*$, (and the associated nondegenerate Hermitian form) strictly positive if it satisfies (the analogues of) conditions (i)-(v) above and also $\exists$ an increasing function $\gamma: R^+ \to R^+$ with $\gamma(0) = 0$ such that

$$\langle Qx, x\rangle_{X^*,X} \geq \gamma(\|x\|) \; ; \quad \forall\, x \in X .$$

Then, using theorem 4.1, we obtain

Proposition 4.4 Let A be a closed operator on a Banach space X which has a strictly positive Hermitian form. Suppose that if $Q: X \to X^*$ is a strictly positive selfadjoint operator, then $\exists$ a strictly positive selfadjoint operator P such that

$$PAx + A^*Px = -Qx , \qquad x \in \mathcal{D}(A).$$

Then the system

$$\dot{x} = Ax$$

is asymptotically stable . □

It seems , therefore , that in the case of an infinite-dimensional Banach space , it is only possible to give a very weak converse to theorem 4.2 . (Note , however , that for a bounded operator A one can obtain a complete converse , cf. Daleckii and Krein [19].) The situation in a Hilbert space is much better and we have (Datko [2])

Theorem 4.5 A necessary and sufficient condition that a strongly continuous semigroup of operators $T(t)$ of class C_0 with generator A defined on a complex Hilbert space H satisfy the condition $\|T(t)\| \le M_0 e^{-\alpha t}$, where $1 \le M_0 < \infty$, and $0 < \alpha < \infty$, is the existence of a positive selfadjoint operator B such that

$$2(BAx,x) = -\|x\|^2 , \qquad x \in \mathcal{D}(A) . \square$$

5. NONLINEAR SYSTEMS

In this section , we shall try to obtain stability results for nonlinear systems by using the theory of section 4 . In particular , we shall find conditions under which we may use the Lyapunov functional

$$V(x) = \int_0^\infty \|T_t x\|^2 dt ,$$

where T_t is the semigroup generated by the linear unperturbed equation . The difficulty here is that we only have the estimate

$$V(x) > 0 , \ x \neq 0 , \tag{5.1}$$

which (together with $\dot{V}(x) \le -\gamma(\|x\|)$) is not sufficient for stability . However , we have the following result:

Lemma 5.1 If A is a nonlinear dissipative operator on $\mathcal{D}(A)$, with $A0 = 0$ (i.e. $\mathrm{Re}[Ax-Ay,x-y] \le 0$, $x,y \in \mathcal{D}(A)$), then any weak solution of the system

$$\dot{x} = Ax , \qquad x(0) = x_0 \in \mathcal{D}(A)$$

is stable at x=0.

Remarks. (1). By a weak solution we mean a continuous function $x: R^+ \to \mathcal{D}(A)$ which is weakly differentiable and satisfies

$$\langle \dot{x}(t), f\rangle = \langle Ax(t), f\rangle , \ \forall f \in X^* \ , t \ge 0.$$

(2). By $[\cdot,\cdot]: X \times X \to R^+$ we mean a semi-inner product (cf. Yosida [20]) . Let $F: X \to X^*$ be the duality map , then we will use the semi-inner product

$$[x,y] = \langle x, f_y \rangle \qquad \text{where } f_y \in F(y).$$

Proof. We have , for each weak solution $u(t)$,

$$\|u(s)\|\,\|u(s)\| - \|u(s-h)\|\,\|u(s)\| \le [u(s) - u(s-h) , u(s)] ,$$

and so

$$\bar{D}^- \|u(s)\| = \limsup_{h \to 0+} \left(\frac{\|u(s)\| - \|u(s-h)\|}{h}\right)\|u(s)\| \le [u'(s), u(s)] ,$$

where $\bar{D}^-$ denotes the Dini derivative . Thus , for $s > 0$,

$$(\bar{D}^- \|u(s)\|)\,\|u(s)\| \le [Au(s), u(s)] = [Au,u] \le 0 .$$

Thus , (Yosida, [20] , p.240),

Note that in the preceding proof, we have effectively used the Lyapunov function

$$V(x) = \|x\|^2 ,$$

even though $\|x(t)\|$ may not be differentiable.

We can now get round the difficulty of requiring a stronger condition than (5.1) in the following result:

<u>Lemma 5.2</u>. Let the system

$$\dot{x} = Ax \quad , \qquad x(0) = x_0 \in \mathcal{D}(A) \tag{5.2}$$

be stable at $x = 0$, where A is a nonlinear operator such that $A0=0$, and suppose that $\exists$ a function $V: X \to R^+$ such that if $x(t)$ is a (strong) solution of (5.2) then $V(x(\cdot)): R^+ \to R^+$ is differentiable. Suppose, moreover, that

$$\dot{V}(x(t)) \leq -\gamma(\|x(t)\|)$$

where $x(t)$ is a (strong) solution of (5.2) and $\gamma: R^+ \to R^+$ is an increasing function with $\gamma(0) = 0$. Then the system is asymptotically stable at $x = 0$.

<u>Proof</u>. We have

$$\dot{V}(x(t)) \leq -\gamma(\|x(t)\|)$$

and so

$$V(x(t)) \leq V(x(0)) - \int_0^t \gamma(\|x(s)\|)ds.$$

Hence,

$$\int_0^\infty \gamma(\|x(s)\|)ds \leq V(x(0)) - V(x(\infty))$$

$$< +\infty \quad .$$

Thus, if $\varepsilon > 0$, $\exists$ a point $t \geq 0$ such that

$$\gamma(\|x(t)\|) \leq \gamma(\varepsilon),$$

i.e.

$$\|x(t)\| \leq \varepsilon \quad .$$

The result now follows since the system is stable. □

We shall now study the perturbed equation

$$\dot{x} = Ax + Bx \, , \qquad x(0) = x_0 \, , \tag{5.3}$$

where A is linear and generates a stable C_0 semigroup $T(t)$ and B is some nonlinear operator. We shall use the Lyapunov function

$$V(x) = \left\langle \int_0^\infty T^*(t)QT(t)x\,dt, x \right\rangle .$$

However, as we have remarked, we do not have an inequality of the form

$$\dot{V}(x) \geq -\gamma(\|x\|)$$

for some increasing function γ and so we shall have to base our stability results on lemmas 5.1, 5.2. As we have seen in section 3, in some cases the existence and uniqueness theorem gives a stability result and so we may either combine this result with lemma 5.2 or assume **A + B is dissipative** and apply lemmas 5.1, 5.2.

We shall assume that $\|T(t)\| \leq Me^{-\omega t}$. Then we have

<u>Lemma 5.3</u>. Let A generate the semigroup $T(t)$ which satisfies the above estimate, and let $B: X \to X$ be a nonlinear operator for which there exists an increasing function $f: R^+ \to R^+$ such that

$$\|Bx\| \leq f(\|x\|)$$

and suppose that $B0=0$, so that $x = 0$ is a critical point of the system (5.3). Then, if

$Q: X \to X^*$ is any strictly positive selfadjoint (linear) operator , the system (5.2) is asymptotically stable at the origin if it is stable and if

$$\frac{\|Q\|}{\omega} f(\|x\|) \leq \xi \frac{\langle Qx, x\rangle}{\|x\|} \quad , \; x \neq 0,$$

for x belonging to some set $\mathcal{D} \subseteq X$ such that $0 \in \mathcal{D}$ and where ξ is some number $< M^{-2}$. Moreover , $\mathcal{D}$ is a subset of the domain of attraction of $\{0\}$.

Proof. Since the linear system

$$\dot{x} = Ax$$

is stable , a Lyapunov functional is given by

$$V(x) = \left\langle \int_0^\infty T^*(t)QT(t)x\,dt, x \right\rangle .$$

Thus , $PA + A^*P = -Q$, where $P = \int_0^\infty T^*(t)QT(t)dt$. Now,

$$\|P\| \leq \int_0^\infty M^2 e^{-2\omega t} \|Q\| dt = \frac{M^2}{2\omega}\|Q\| ,$$

and

$$\begin{aligned} \langle PBx, x\rangle &\leq \|PBx\| \cdot \|x\| \\ &\leq \|P\| \cdot \|Bx\| \cdot \|x\| \\ &\leq \frac{M^2}{2\omega}\|Q\| f(\|x\|)\,\|x\| \\ &\leq \frac{M^2 \xi}{2} \langle Qx, x\rangle \quad . \end{aligned}$$

Hence ,

$$\begin{aligned} \dot{V}(x) &= \langle \dot{x}, Px\rangle + \langle x, P\dot{x}\rangle & (*) \\ &= \langle Ax + Bx, Px\rangle + \langle x, PAx + PBx\rangle \\ &= -\langle Qx, x\rangle + \langle PBx, x\rangle + \langle x, PBx\rangle \\ &= -\langle Qx, x\rangle + 2\langle PBx, x\rangle \\ &= (\xi M^2 - 1)\langle Qx, x\rangle . \end{aligned}$$

The result now follows from lemma 5.2 . □

(Note that , even if we only have a mild solution to equation (5.2) , the formal differentiation in (*) can be justified rigorously , cf. Blakeley , Pritchard [12] .)

The last lemma is of little practical use in view of the generality of Q . However , if we specialise to the case of a Hilbert space , we obtain the following result :

Corollary 5.4. Assume that X is a Hilbert space and that A and B satisfy the same conditions as in lemma 5.3 . Then , if the equation (5.3) is stable at the origin and

$$\frac{1}{\omega \xi} f(r) \leq r \qquad r \in [0, \alpha] \subseteq R^+$$

it is asymptotically stable at the origin and the ball $B(0, \alpha) \subseteq X$ is contained in the domain of attraction .

Proof. This follows directly from lemma 5.3 by taking $Q = I$. □

Up to now we have been concerned with the case where B is defined everywhere on the space X . It is important , however , to consider the case where B is defined only on a dense subspace of X . Therefore , let V,X be Banach spaces such that $V \subseteq X$ with continuous injection and suppose $\overline{V} = X$. We therefore have , for some $\beta > 0$ and $\forall\, x \in V$,

$$\|x\|_X \leq \beta \|x\|_V \,.$$

We shall also assume that B is a continuous nonlinear operator from V into X . Then , we have the following result:

Lemma 5.5. Suppose that A generates the semigroup $T(t)$ and that $T(t) : X \to V$ $(t>0)$, and assume that the estimates

$$\|T(t)x\|_V \leq g_1(t)\|x\|_X \quad , \quad \|T^*(t)y\|_{V^*} \leq g_2(t)\|y\|_{V^*} \quad .$$

for $x \in X$, $y \in V^*$ are valid , where $g_1 g_2 \in L(0,\infty)$. Put $G = \|g_1 g_2\|_L$.

Suppose , moreover , that $B : V \to X$ is a continuous nonlinear operator with $B(0) = 0$ for which there exists an increasing function $f : R^+ \to R^+$ such that

$$\|B(x)\|_X \leq f(\|x\|_V) \,.$$

Let $Q : V \to V^*$ be a strictly positive selfadjoint operator and suppose that

$$2G\|Q\|f(\|x\|_V) \leq \xi \frac{\langle Qx,x\rangle_{V^*,V}}{\|x\|_V} \quad ,$$

for some $\xi<1$. Then if the system (5.3) is stable at the origin , it is asymptotically stable there .

Proof. Consider the selfadjoint operator

$$P = \int_0^\infty T^*(t)QT(t)dt \; : X \to X^* \,.$$

We have , for $x \in V$,

$$\begin{aligned}
\langle PBx,x\rangle_{V,V^*} &\leq \int_0^\infty \|T^*(t)QT(t)Bx\|_{V^*}\, dt.\, \|x\|_V \\
&\leq \int_0^\infty \|Q\|\, g_1 g_2 \,\|Bx\|_X dt.\, \|x\|_V \\
&\leq G\|Q\|\, f(\|x\|_V)\|x\|_V \\
&\leq \frac{\xi}{2}\langle Qx,x\rangle_{V^*,V} \,.
\end{aligned}$$

Thus , if $x \in V \cap \mathcal{D}(A)$ (in the V^*,V duality)

$$\begin{aligned}
\dot{V}(x) &= \langle \dot{x},Px\rangle + \langle x,P\dot{x}\rangle \\
&= \langle x,A^*Px + PAx\rangle + 2\langle PBx,x\rangle \\
&= -\langle x,Qx\rangle + 2\langle PBx,x\rangle \\
&\leq (\xi-1)\langle x,Qx\rangle \,.
\end{aligned}$$

The result follows from lemma 5.2. □

If we specialise again to the case where V is a Hilbert space , we have

Corollary 5.6. Let $T(t),T^*(t),B$ satisfy the conditions of lemma 5.5 , where V is a Hilbert space . Then if the system (5.3) is stable at the origin , it is asymptotically stable there if

$$2Gf(r) \leq \xi r \;, \quad \xi<1 \;, \; r \in [0,\alpha] \subseteq R^+$$

and $B(0,\alpha) \subseteq V$ is contained in the domain of attraction . □

So far . we have obtained only local asymptotic stability of solutions in a neighbourhood

Theorem 5.7. Suppose that A is a bounded operator on a Hilbert space X and that $\|Bx\| \le f(\|x\|)$ where f is an increasing function and $B0 = 0$. Then if the system

$$\dot{x} = Ax + Bx \tag{5.3}$$

is stable at the origin and

$$f\left(\frac{M^2\|Q\|\cdot\|y\|}{2\omega}\right) \le \xi \frac{\langle Qy,y\rangle}{\|y\|} \qquad \xi < \tfrac{1}{2} \tag{5.4}$$

for $y \in \mathfrak{D} \subseteq X$ and some strictly positive selfadjoint operator Q , it is asymptotically stable and $\mathfrak{D} \subseteq$ domain of attraction . (Again , $\|e^{At}\| \le Me^{-\omega t}$.)

Proof. Since A is bounded , the operator

$$P = \int_0^\infty T^*(t)QT(t)dt$$

has a bounded inverse $P^{-1}: X \to X$. If x(t) is a solution of (5.3) , then we write $y(t) = P^{-1}x(t)$ so that

$$\dot{y} = P^{-1}APy + P^{-1}BPy.$$

Then , if $V = \langle y, Py\rangle$,

$$\begin{aligned} \dot{V} &= \langle \dot{y}, Py\rangle + \langle y, P\dot{y}\rangle \\ &= \langle P^{-1}APy + P^{-1}BPy, Py\rangle + \langle y, APy+BPy\rangle \\ &= \langle A^*Py + PAy, y\rangle + 2\langle BPy, y\rangle \\ &= -\langle Qy,y\rangle + 2\langle BPy, y\rangle . \end{aligned}$$

Now ,

$$\begin{aligned} \langle BPy, y\rangle &\le \|BPy\|\,\|y\| \\ &\le f(\|Py\|)\,\|y\| \\ &\le f\left(\frac{M^2\|Q\|\,\|y\|}{2\omega}\right)\|y\| \quad . \end{aligned}$$

and so , by the stated condition ,

$$\dot{V} \quad -(1-2\xi)\langle Qy, y\rangle . \ \square$$

Consider now those functions $f: R^+ \to R^+$ which satisfy the following properties:

(i). $f(0)=0$, f increasing , and $f(ab) \le f(a)f(b)$, $\forall\, a,b \in R^+$.

(ii). for any $b,c \in R^+$, $\exists\, a \in R^+$ such that $f(ab) \le ac$.

(iii). $r/f(r)$ decreases as r increases .

Then we have

Corollary 5.8. Let A be a bounded operator on a Hilbert space X and suppose that $\|Bx\| \le f(\|x\|)$ $(B0=0)$, where f is a function satisfying (i),(ii),(iii) . Then, if system (5.3) is stable , it is globally asymptotically stable .

Proof. Let $0 < \xi < 1$ and consider the ball $B(0,r) \subseteq X$. We shall show that B(0,r) is contained in the domain of attraction for each $r \ge 0$. Let $r_1 > 0$, therefore , and choose α (by (ii)) such that

$$f\left(\frac{M^2}{2\omega} r_1\right) f(\alpha) \le \xi \alpha r_1 \quad .$$

Then ,

$$f\left(\frac{M^2}{2\omega} r_1 \alpha\right) \le \xi \alpha r_1$$

and if $r_2 \le r_1$,

$$f(\alpha) \le \frac{\xi \alpha r_1}{f\left(\frac{M^2}{2\omega} r_1\right)} \le \frac{\xi \alpha r_2}{f\left(\frac{M^2}{2\omega} r_2\right)}$$

and so

$$f\left(\frac{M^2}{2\omega} r_2 \alpha\right) \le \xi \alpha r_2 \ .$$

Hence , condition (5.4) is satisfied on $B(0,r_1)$ if we take $Q = \alpha I$. Since this is true for all $r_1 > 0$ the result follows . □

We remark , in conclusion , that corollary 5.8 can be easily extended to the case where B is not everywhere defined .

6. FINAL REMARKS

In this paper , we have given a method for studying the stability of a nonlinearly perturbed linear stable system

$$\dot{x} = Ax + Bx$$

where A generates a semigroup and B is a nonlinear operator defined on the whole space X , or on a dense subspace . We have seen that the stability results are intimately connected with the existence and uniqueness of solutions of the equation .

In the case when A is a bounded operator we have been able to obtain a much more decisive result than in the general case , assuming that the function f related to B satisfies conditions (i),(ii),(iii) above . (These conditions are satisfied , for example , if $f(s) = s^r, r>1$.)

7. REFERENCES

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