14th Triennial World Congress, Beijing, P.R. China

E-2c-14-4

STABILIZATION OF NONLINEAR SYSTEMS USING THE ASSOCIATED ANGULAR SYSTEM

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Abstract: The stabilizability of general nonlinear systems is studied by considering the associated angular system, which leads to a simple stabilizing controller in many cases. Bilinear systems are studied in detail and general nonlinear systems are reduced to infinite dimensional bilinear form by using Lie theory. *Copyright © 1999 IFAC*

Keywords:Stabilization, Nonlinear Systems, Angular System.

1.INTRODUCTION

The control of nonlinear systems has been approached by a variety of techniques, including linearization (Banks, 1988), sliding mode control (Luk'yanov and Utkin, 1981;Utkin, 1992) and optimal state-space and H^{∞} control (Knobloch,*et al*, 1993). Morever, several authors have studied the frequency-domain theory of nonlinear systems with a view to establishing stabilizing controllers (Banks,1985;Banks,1997). In this paper we shall use the angular form of a differential equation to reduce the problem to the control of the radial component of the state. This representation has been used by a number of authors to study the Lyapunov spectrum of a nonlinear system (see, for example, (Colonius and Kliemann,1996) and the references cited there). In this paper we shall consider systems of the form

$$\dot{x} = A(x)x + B(x)u \qquad (1)$$

and write $z = x/\|x\|$ and $r = \|x\|$ which implies that we can obtain two equations of the form

$$\dot{z} = \overline{A}z + u\overline{B}z$$

$$\dot{r} = \lambda_A r + u\lambda_B r$$

from which we can obtain a control

$$u = -\frac{(\lambda_A + \alpha)}{\lambda_B} \qquad (2)$$

for any $\alpha > 0$ where $\lambda_B \neq 0$, which will drive $r = \|x\|$ to zero. If the set $\Sigma = \{x : \lambda_B(x) = 0\}$ is not just the origin then the control will drive the system to Σ while reducing r.

In section 2 we shall consider first the special case of bilinear systems and in section 3 the more general systems (1). In both cases we shall use the control (1) (or a simple modification) to drive the system to Σ. The results depend strongly on the choice of α as we shall see. Finally in section 4, we consider general nonlinear systems of the form

$$\dot{x} = f(x,u)$$

and use Lie series techniques (Banks, 1992) to reduce the problem to an infinite-dimensional bilinear system.

2. THE ANGULAR FORM AND CONTROL OF BILINEAR SYSTEMS

In this section we shall consider the bilinear system

$$\dot{x} = A(x)x + uBx \qquad (2)$$

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The first result is well-known, but we shall prove it here for the convenience of the reader and so that we can generalize it later.

Lemma 2.1 *The equation (2.1) can be written in the form*

$$\begin{aligned} \dot{z} &= \overline{A}(z)z + u\overline{B}(z)z \\ \dot{r} &= \lambda_A(z)r + u\lambda_B(z)r \end{aligned} \tag{3}$$

where

$$z = x/\|x\|, r = \|x\|,$$

$$\lambda_A(z) = z^T Az,\ \lambda_B(z) = z^T Bz$$

$$\overline{A}(z) = A - \lambda_A(z)I\ ,\ \overline{B}(z) = B - \lambda_B(z)I.$$

Proof We have

$$\dot{z} = \frac{\dot{x}}{\|x\|} - \frac{x\frac{d}{dt}\|x\|}{\|x\|^2}$$

and since

$$\frac{1}{2}\frac{d}{dt}(\|x\|^2) = \frac{1}{2}\frac{d}{dt}(x^T x) = x^T\dot{x}$$
$$= x^T(Ax + uBx)$$

it follows that

$$\frac{d}{dt}(\|x\|) = \frac{x^T Ax}{\|x\|} + u\frac{x^T Bx}{\|x\|}$$

and so

$$\dot{z} = Az + uBz - (z^T Az + uz^T Bz)z$$

and the result follows.

Corollary 2.2 *The system is stable if we choose* u *to satisfy*

$$\int_0^t (\lambda_A(z) + u\lambda_B(z))dt \to -\infty$$

as $t \to \infty$.

Proof We simply integrate the scalar equation for r in (4) to give

$$r(t) = e^{\int_0^t (\lambda_A(z)+u\lambda_B(z))dt} r_0 .$$

The above results suggest the use of the control

$$u = -\frac{(\lambda_A(z) + \alpha)}{\lambda_B(z)} \tag{5}$$

for any $\alpha > 0$ provided $\lambda_B(z) \neq 0$. In fact, we have

Theorem 2.3 *If* $\lambda_B(z) \neq 0$ *for* $z \neq 0$, *then the control (5) will drive the system (3) to the origin. The system with this control is*

$$\dot{x} = A(x)x - \frac{(\lambda_A(x) + \alpha\|x\|^2)}{\lambda_B(x)}B(x)\ . \tag{6}$$

Proof We note that

$$\dot{r} = \lambda_A(z)r - \frac{(\lambda_A(z) + \alpha)}{\lambda_B(z)}\lambda_B(z)r$$
$$= -\alpha r$$

and so

$$r(t) = e^{-\alpha t}r_0.$$

Remark 2.4 In the case where $\lambda_B(z) = 0$ on some set Σ with nonzero elements, we can use the control

$$u = -\frac{(\lambda_A(z) + \alpha)}{(\lambda_B(z) + \varepsilon\lambda_B(z))/|\lambda_B(z)|}\ , z \notin \Sigma,$$

for some $\varepsilon > 0$. Then we have

$$\dot{r} = \lambda_A(z)r -$$
$$\frac{(\lambda_A(z) + \alpha)}{(\lambda_B(z) + \varepsilon\lambda_B(z))/|\lambda_B(z)|}\lambda_B(z)r$$
$$= -\frac{(\alpha\lambda_B(z) - \varepsilon\sigma(z))}{\lambda_B(z) + \varepsilon\sigma(z)}r$$

where

$$\sigma(z) = \frac{\lambda_B(z)}{|\lambda_B(z)|} = \operatorname{sgn}(\lambda_B(z))$$

and r will decay for all z for which

$$\alpha|\lambda_B(z)| > \varepsilon\ . \tag{7}$$

Example 2.5 Consider the system

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + u\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} \tag{8}$$
$$= Ax + uBx$$

Then,

$$\lambda_A(z) = 2z_1z_2 + z_2^2$$
$$\lambda_B(z) = z_2^2$$

and the controlled system becomes

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} - \begin{pmatrix}0\\ (2x_1x_2 + x_2^2 + \alpha(x_1^2 + x_2^2))/x_2\end{pmatrix} \tag{9}$$

This system is not globally stabilizable, but we can drive the velocity $x_2 = \dot{x}_1$ by using this control. By choosing α to depend on the initial state, we get effective control reducing the state to a neighbourhood of zero for many initial states. Results for $\alpha = 0.84$ and $\alpha = 1$ are shown in figures 1,2. Of course, if we have a small control effect in the x_1 direction, then λ_B is nonsingular and we get global control as in figures 3,4. These correspond to the systems

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + u\begin{pmatrix}0.1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} \tag{10}$$

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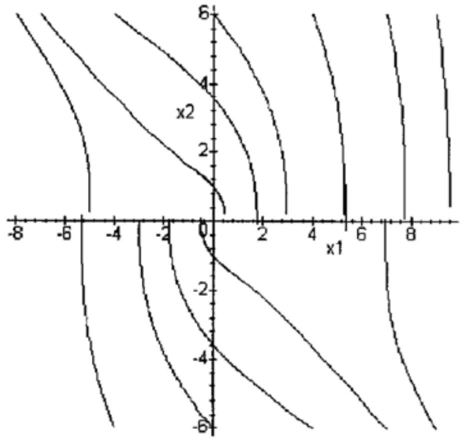

1 Fig. 1. Controlled Dynamics for System (8) with α=0.84

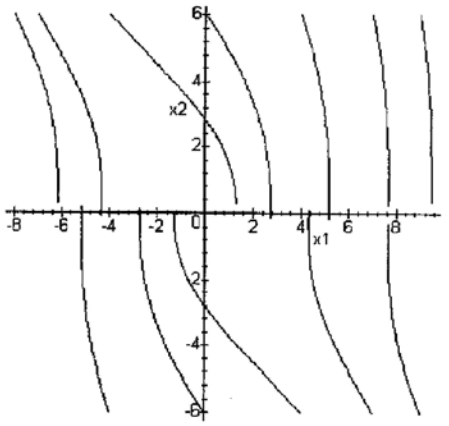

Fig. 2 Controlled Dynamics for System (8) with α=1

and

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + u\begin{pmatrix} 0.01 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (11)$$

with the respective controls

$$u = -(2x_1x_2 + x_2^2 + \alpha(x_1^2 + x_2^2))/(0.1x_1^2 + x_2^2)$$

$$u = -(2x_1x_2 + x_2^2 + \alpha(x_1^2 + x_2^2))/(0.01x_1^2 + x_2^2)$$

In figure 4, the crossing trajectories are due to inaccuracies in the numerical computation of the solution and is not present in the actual system.

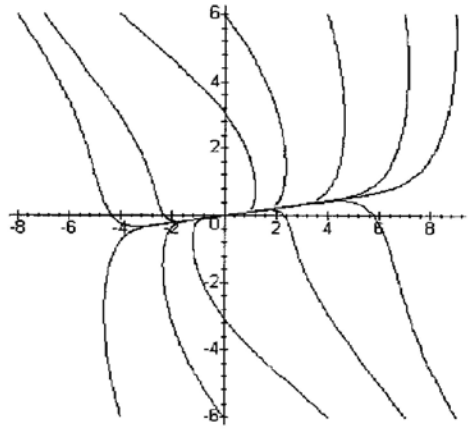

Fig. 3. Dynamics for the Controlled System (10), α=1

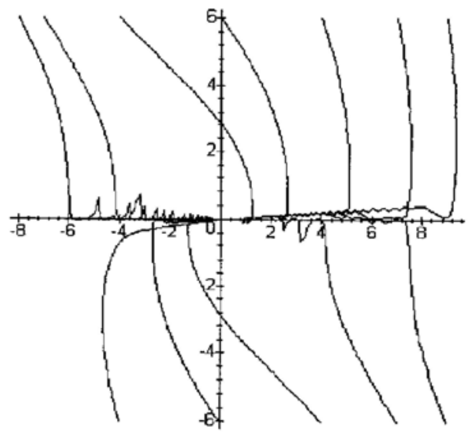

Fig. 4. Dynamics for the Controlled System (11), α=1

PSEUDO-LINEAR SYSTEMS

Now consider systems of the form

$$\dot{x} = A(x)x + uB(x)x \quad (12)$$

Lemma 2.1 can be generalized to

Lemma 3.1 *The equation (12) can be written in the form*

$$\dot{z} = \overline{A}(z)z + u\overline{B}(z)z$$

$$\dot{r} = \lambda_A(z)r + u\lambda_B(z)r$$

where

$$z = x/\|x\|,\ r = \|x\|$$

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$$\lambda_A(z) = z^T A(rz)z \, , \, \lambda_B(z) = z^T B(rz)z$$
$$\overline{A}(z) = A(rz) - \lambda_A(z)I,$$
$$\overline{B}(z) = B(rz) - \lambda_B(z)I \quad .$$

Generalizing theorem 2.3, we have

Theorem 3.2 *If* $\lambda_B(z) \neq 0$ *for* $z \neq 0$, *then the control*

$$u = -\frac{(\lambda_A(z) + \alpha)}{\lambda_B(z)}$$

will drive the system (12) to the origin and the controlled system is

$$\dot{x} = Ax - \frac{(\lambda'_A(x) + \alpha\|x\|^2)}{\lambda'_B(x)} B(x)x$$

where

$$\lambda'_A(x) = x^T A(x)x \, , \, \lambda'_B(x) = x^T B(x)x \, .$$

Example 3.3 Consider the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + u \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x_1^2 + x_2^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The controlled system is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ c(x) \end{pmatrix} \tag{13}$$

where

$$c(x) = x_1 x_2 - (x_1 x_2 + x_2 x_3 + x_1 x_2 x_3 + \alpha(x_1^2 + x_2^2 + x_3^2)) / x_3 \, .$$

Again, by a judicious choice of α depending on the initial state, we can obtain regulation of x_3 and even for x_1, x_2 in certain regions (fig. 5).

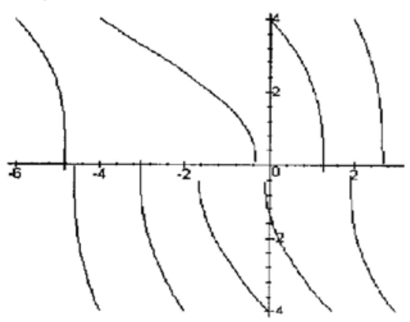

Fig. 5. Controlled Trajectories for (13) in the x_1, x_2 plane.

As before, if we have a component of control in the x_1, x_2 directions, then the control is globally stabilizing; e.g. the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + u \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & x_1^2 + x_2^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

with control

$$u = -\frac{(x_1 x_2 + x_2 x_3 + x_1 x_2 x_3 + \alpha(x_1^2 + x_2^2 + x_3^2))}{((x_1^2 + x_2^2)x_3^2 + 0.1x_1^2 + 0.1x_2^2)} \tag{14}$$

for any $\alpha > 0$ (fig. 6).

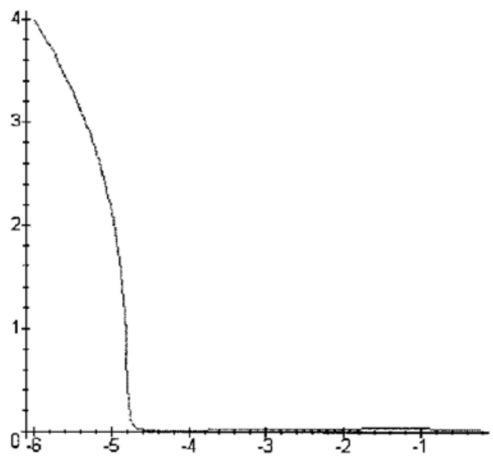

Fig. 6. Typical Trajectory for System (14)

For the case of pseudo-linear systems

$$\dot{x} = A(x)x + b(x)u \tag{15}$$

(where $b(x)$ is a vector and u is a scalar), suppose that $b(x)$ can be written in the form

$$b(x) = B(x)x.$$

Then from theorem 3.2, we have

Corollary 3.4 *If* $x^T b(x) \neq 0$ *for* $x \neq 0$, *then the control*

$$u = -\frac{(\lambda_A(z) + \alpha)}{z^T b(rz)}$$

will drive the system (15) to the origin.

Example 3.5 Consider the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \sin x_2 & x_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ x_1^2 + 3x_2^4 \end{pmatrix} u$$
$$= \begin{pmatrix} 0 & 1 \\ \sin x_2 & x_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + u \begin{pmatrix} 0 & 0 \\ x_1 & 3x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

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Then we have the control

$$u = -\frac{(x_1x_2 + x_1x_2\sin x_2 + x_1x_2^2 + \alpha(\|x\|^2))}{x_2(x_1^2 + 3x_2^4)} \tag{16}$$

Some of trajectories of the controlled system are shown in fig. 7.

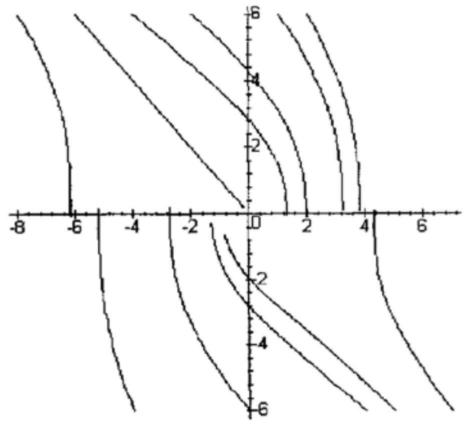

Fig. 7. Controlled Trajectories for System (16)

GENERAL NONLINEAR SYSTEMS

In this section we shall consider the system

$$\dot{x} = f(x,u) \tag{17}$$

for scalar control u and some analytic function f. We can write the system in the form

$$\dot{x} = f(x,u)$$
$$\dot{u} = v$$

i.e.

$$\dot{y} = F(y) + \begin{pmatrix}0\\1\end{pmatrix}v. \tag{18}$$

Hence there is no loss of generality in considering the linear-analytic system

$$\dot{x} = f(x) + ug(x). \tag{19}$$

As is well-known, this system can be written in the form of an infinite-dimensional bilinear system in the following way (see Banks, 1992):

Put $\phi_1(x) = x$ and

$$\phi_i(x) = \frac{\partial\phi_{i/2}(x)}{\partial x}\cdot f(x) \stackrel{\Delta}{=} L_f\phi_{i/2}(x)$$

if i even

$$\phi_i(x) = \frac{\partial\phi_{(i-1)/2}(x)}{\partial x}\cdot g(x) \stackrel{\Delta}{=} L_f\phi_{(i-1)/2}(x)$$

if i odd. Then,

$$\dot{\phi}_i = \frac{\partial\phi_i}{\partial x}\dot{x}$$
$$= \frac{\partial\phi_i}{\partial x}f + u\frac{\partial\phi_i}{\partial x}g$$
$$= \phi_{2i}(x) + u\phi_{2i+1}(x)$$

and so, if $\Phi = (\phi_1, \phi_2, \cdots)^T$, we have

$$\dot{\Phi} = A\Phi + uB\Phi$$

where

$$A = (a_{ij}),\ B = (b_{ij})$$

with

$$a_{ij} = \delta_{2i,j} \quad , \quad b_{ij} = \delta_{2i+1,j}$$

i.e.

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

In a similar way to (Banks, 1992) we can define a map $N_{f,g} : R^n \to \widetilde{S}$, where $\widetilde{S}$ is the set of all (unrestricted) sequences, by

$$N_{f,g}(x) = (L_f^0x, L_f^1x, L_g^1x, L_f^2x, L_gL_fx, L_fL_gx, L_g^2x, \cdots)^T$$

where $L_f^i = L_fL_f\cdots L_f$ (i times) and the inner product and norm on $\widetilde{S}$ by

$$\langle\sigma,\tau\rangle_S = \sum_{i=0}^{\infty}\frac{\langle\sigma_i,\tau_i\rangle}{(i!)^2}$$
$$\|\sigma\|_S = \left(\langle\sigma,\sigma\rangle_S\right)^{1/2}$$

where $\langle\cdot,\cdot\rangle$ is the usual inner product on R^n and

$$\sigma = (\sigma_0, \sigma_1, \sigma_2, \cdots)^T,$$
$$\tau = (\tau_0, \tau_1, \tau_2, \cdots)^T \quad .$$

Then it can be shown, as in (Banks, 1992), that $S = \{\sigma \in \widetilde{S} : \|\sigma\|_S < \infty\}$ is a Banach space.

As before, put $\psi = \phi/\|\phi\|_S$, $\rho = \|\phi\|_S$ and we then obtain the system

$$\begin{aligned} \dot{\psi} &= \overline{A}(\psi)\psi + u\overline{B}(\psi)\psi \\ \dot{\rho} &= \lambda_A(\psi)\rho + u_B(\psi)\rho \end{aligned} \tag{20}$$

where

$$\lambda_A(\psi) = \langle\psi, A\psi\rangle_S \ , \ \lambda_B(\psi) = \langle\psi, B\psi\rangle_S$$
$$\overline{A}(\psi) = A - \lambda_A(\psi)I \ , \ \overline{B}(\psi) = A - \lambda_B(\psi)I \ .$$

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The control is then

$$u = -\frac{(\lambda_A(\psi) + \alpha)}{\lambda_B(\psi)}$$
$$= -\frac{(\lambda_A(\phi) + \alpha\|\phi\|^2)}{\lambda_B(\phi)} \quad .$$

Typical trajectories for this control applied to system (16) with only two terms in the Lie series included are shown in fig. 8. These should be compared with those in fig. 7.

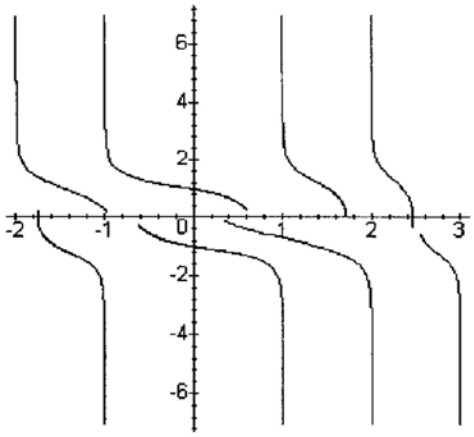

Fig. 8. Lie Series Control for System (16)

The controller described here will be improved with the order of the Lie series. However, a small number of terms will usually be satisfactory, especially in the region where the Lie series converges rapidly.

CONCLUSIONS

In this paper we have studied the control of nonlinear systems via the angular system associated with the given system. The controls are seen to be effective for the controllable part of the system and provide global stabilizing controls when the system is globally stabilizable. The Lie series approach to general nonlinear systems has been shown to be particularly effective. The main drawback with the latter approach is that the computation time is significantly greater than with the other methods.

REFERENCES

Banks, S.P. (1985). On Nonlinear Systems and Algebraic Geometry, *Int. J. Control*, **42**, 333-352.

Banks, S.P. (1988). *Mathematical Theories of Nonlinear Systems*,:Prentice-Hall, London.

Banks, S.P.(1992). Nonlinear Systems, the Lie Series and the Left-Shift Operator: Applications to Nonlinear Optimal Control,*IMA J. Math. Control and Inf.*, **9**, 23-34.

Banks, S.P.(1997). The Schwartz Kernel Theorem and the Frequency-Domain Theory of Nonlinear Systems, *Arch. Cont. Sci.*, **6**(XLII),29-45.

Colonius, F.and Kliemann, W.(1996). The Lyapunov Spectrum of Families of Time-Varying Matrices, *Trans. Amer. Math. Soc.*, **348**, 4389-4407.

Knobloch, H.W., Isidori, A. and Flockerzi, D. (1993) *Topics in Control Theory*, DMV Seminar Band 22,Birkhauser, Berlin.

Luk'yanov, A.G.and Utkin, V.I.. (1981) Methods of Reducing Equations for Dynamic Systems to a Regular Form, Automation and Remote Control, **42**, 413-420.

Utkin, V.I..(1992) *Sliding Modes in Control and Optiimization*,Springer-Verlag, Berlin.

 ISBN: 0 08 043248 4