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A separation theorem for nonlinear systems[☆]

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ABSTRACT

In this paper we have obtained a nonlinear separation result for controlled stochastic systems. The result is based on a sequential technique, introduced by the second author, which has been applied with significant success for nonlinear deterministic systems.

1. Introduction

Consider the linear stochastic regulator problem: i.e. state X_t of a system is given by the stochastic differential equation

$$dX_t = (A_t X_t + B_t u_t)dt + \sigma_t d\mathcal{B}_t, \quad t \geq s; \qquad X_s = x \tag{1}$$

and the cost is of the form

$$J^u(s,x) = E^{s,x}\left[\int_s^{t_1} \left\{X_t^{\mathrm{T}} Q_t X_t + u_t^{\mathrm{T}} R_t u_t\right\} dt + X_{t_1}^{\mathrm{T}} F X_{t_1}\right], \quad s \leq t_1, \tag{2}$$

where $A_t \in R^{n\times n}$, $B_t \in R^{n\times m}$, $\sigma_t \in R^{n\times k}$, $Q_t \in R^{n\times n}$, $R_t \in R^{m\times m}$, $F \in R^{n\times n}$ are continuous. Here $\mathcal{B}_t$ is a standard Brownian motion. Then it is well known (Fleming & Rishel, 1975) that the optimal control is given by

$$u^*(t, X_t) = -R_t^{-1} B_t^{\mathrm{T}} P_t X_t, \tag{3}$$

where P_t satisfies the Riccati equation

$$\frac{dP_t}{dt} = -A_t^{\mathrm{T}} P_t - P_t A_t - Q_t + P_t B_t R_t^{-1} B_t^{\mathrm{T}} P_t, \qquad P_{t_1} = F. \tag{4}$$

If we do not have complete knowledge of X_t, but only a noisy observation

$$dZ_t = C_t X_t dt + \gamma_t d\widetilde{\mathcal{B}}_t \tag{5}$$

then the optimal control is given by

$$u^*(t, X_t) = -R_t^{-1} B_t^{\mathrm{T}} P_t \widehat{X}_t(\omega) \tag{6}$$

where $\widehat{X}_t$ is the filtered estimate of X_t given by the Kalman–Bucy filter

$$\begin{aligned} d\widehat{X}_t &= \left(A_t - \widetilde{P}_t C_t^{\mathrm{T}} (\gamma_t \gamma_t^{\mathrm{T}})^{-1} C_t\right) \widehat{X}_t dt + B_t u_t dt \\ &\quad + \widetilde{P}_t C_t^{\mathrm{T}} (\gamma_t \gamma_t^{\mathrm{T}})^{-1} dZ_t; \qquad \widehat{X}_0 = E[X_0] \end{aligned} \tag{7}$$

where $\widetilde{P}_t = E\left[(X_t - \widehat{X}_t)(X_t - \widehat{X}_t)^{\mathrm{T}}\right]$ satisfies the Riccati equation

$$\frac{d\widetilde{P}_t}{dt} = A_t \widetilde{P}_t + \widetilde{P}_t A_t^{\mathrm{T}} - \widetilde{P}_t C_t^{\mathrm{T}} (\gamma_t \gamma_t^{\mathrm{T}})^{-1} C_t \widetilde{P}_t + \sigma_t \sigma_t^{\mathrm{T}};$$

$$\widetilde{P}_0 = E\left[(X_0 - E[X_0])\,(X_0 - E[X_0])^{\mathrm{T}}\right]. \tag{8}$$

This is the *linear separation principle*, i.e. we can separate control and filtering.

In this paper we shall consider a method of generalizing this result to nonlinear systems of the form

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$$\mathrm{d}X_t = (A_t(X_t)X_t + B_t(X_t, u_t)u_t)\,\mathrm{d}t + \sigma_t \mathrm{d}\mathscr{B}_t \tag{9}$$

$$\mathrm{d}Z_t = C_t(X_t)X_t \mathrm{d}t + \gamma_t \mathrm{d}\widetilde{\mathscr{B}}_t \tag{10}$$

together with a non-quadratic cost functional

$$J^u(s,x) = E^{s,x}\left[\int_s^{t_1} \{X_t^{\mathrm{T}} Q_t(X_t) X_t + u_t^{\mathrm{T}} R_t(X_t) u_t\}\,\mathrm{d}t + X_{t_1}^{\mathrm{T}} F X_{t_1}\right]. \tag{11}$$

This technique has been extensively used in deterministic control and systems (Banks & Dinesh, 2000; Cimen & Banks, 2004a,b; Tomas-Rodriguez & Banks, 2003). To illustrate the idea, consider the nonlinear-quadratic control problem

$$\min J = \frac{1}{2}x^{\mathrm{T}}(t_f)Fx(t_f) + \frac{1}{2}\int_0^{t_f} \{x^{\mathrm{T}}Q(t)x + u^{\mathrm{T}}R(t)u\}\,\mathrm{d}t \tag{12}$$

subject to the dynamics

$$\dot{x} = A(x)x + B(x)u, \qquad x(0) = x_0. \tag{13}$$

We replace this problem with a sequence of linear, time varying ones:

$$\min J = \frac{1}{2}x^{[i]T}(t_f)Fx^{[i]}(t_f) + \frac{1}{2}\int_0^{t_f} \{x^{[i]T}Q(t)x^{[i]} + u^{[i]T}R(t)u^{[i]}\}\,\mathrm{d}t \tag{14}$$

subject to the dynamics

$$\dot{x}^{[1]}(t) = A(x_0)x^{[1]}(t) + B(x_0)u(t), \qquad x^{[1]}(0) = x_0 \tag{15}$$

$$\dot{x}^{[i]}(t) = A(x^{[i-1]}(t))x^{[i]}(t) + B(x^{[i-1]}(t))u(t), \quad i \geq 2,$$
$$x^{[i]}(0) = x_0. \tag{16}$$

The solution of (14)–(16) for $i \geq 1$ gives a sequence of controls $u^{[i]}(t)$ and corresponding states $x^{[i]}(t)$ which can be shown to converge uniformly (Tomas-Rodriguez & Banks, 2003). With some convexity assumptions, one can also prove optimality of the solution, giving an optimal solution to the nonlinear problem (Cimen & Banks, 2004b). Note that nonquadratic costs and terms $A(x,u)$, $B(x,u)$ depending on u can also be considered (see the references above). For approaches to nonlinear filtering and control problems see Arslan and Basar (2003), Deng and Krstic (1997), Deng and Krstic (1999), Deng and Krstic (2000); Deng, Krstic, and Williams (2001), Germani, Manes, and Palumbo (2005), Germani, Manes, and Palumbo (2007), Ito and Xiong (2000), Kushner and Budhiraja (2000), Pan and Basar (1999), Petersen, James, and Dupuis (2000), and Petersen (2006).

2. The formal solution

In this section we shall construct the problem formally, and in the next section prove the convergence of the approximations. Suppose, therefore, that we have a stochastic system of the form (9) and (10), together with a cost functional of the form (11). Generalizing the Eqs. (12)–(16) leads to the systems of equations

$$\mathrm{d}\widehat{X}_t^{[i]} = \left(A_t\left(\widehat{X}_t^{[i-1]}\right)\widehat{X}_t^{[i]} + B_t\left(\widehat{X}_t^{[i-1]}, u_t^{[i-1]}\right)u_t^{[i]}\right)\mathrm{d}t + \widetilde{P}_t^{[i]}C_t^{\mathrm{T}}\left(\widehat{X}_t^{[i-1]}\right)\widetilde{R}_t^{-1}\left[\mathrm{d}Z_t - C_t\left(\widehat{X}_t^{[i-1]}\right)\widehat{X}_t^{[i]}\right],$$
$$x^{[i]}(0) = x_0 \tag{17}$$

$$\dot{\widetilde{P}}_t^{[i]} = A_t\left(\widehat{X}_t^{[i-1]}\right)\widetilde{P}_t^{[i]} + \widetilde{P}_t^{[i]}A_t^{\mathrm{T}}\left(\widehat{X}_t^{[i-1]}\right) + \widetilde{Q}_t - \widetilde{P}_t^{[i]}C_t^{\mathrm{T}}\left(\widehat{X}_t^{[i-1]}\right)\widetilde{R}_t^{-1}C_t\left(\widehat{X}_t^{[i-1]}\right)\widetilde{P}_t^{[i]}, \qquad \widetilde{P}^{[i]}(t_0) = \widetilde{P}_0 \tag{18}$$

$$u_t^{[i]} = -R_t^{-1}\left(\widehat{X}_t^{[i-1]}\right)B_t^{\mathrm{T}}\left(\widehat{X}_t^{[i-1]}, u_t^{[i-1]}\right)P_t^{[i]}\widehat{X}_t^{[i]} \tag{19}$$

$$\dot{P}_t^{[i]} = -A_t^{\mathrm{T}}\left(\widehat{X}_t^{[i-1]}\right)P_t^{[i]} - P_t^{[i]}A_t\left(\widehat{X}_t^{[i-1]}\right) - Q_t\left(\widehat{X}_t^{[i-1]}\right) + P_t^{[i]}B_t\left(\widehat{X}_t^{[i-1]}, u_t^{[i-1]}\right) \times R_t^{-1}\left(\widehat{X}_t^{[i-1]}\right)B_t^{\mathrm{T}}\left(\widehat{X}_t^{[i-1]}, u_t^{[i-1]}\right)P_t^{[i]},$$
$$P^{[i]}(t_f) = F \tag{20}$$

where

$$\widetilde{R}_t = \gamma_t\gamma_t^{\mathrm{T}} \tag{21}$$

$$\widetilde{Q}_t = \sigma_t\sigma_t^{\mathrm{T}}. \tag{22}$$

If the sequence of functions $\left\{\widehat{X}_t^{[i]}, \widetilde{P}_t^{[i]}, u_t^{[i]}, P_t^{[i]}\right\}_{i\geq 1}$ converges in some sense (to be made precise in the next section), we denote the limit functions by $\left\{\widehat{X}_t^{[\infty]}, \widetilde{P}_t^{[\infty]}, u_t^{[\infty]}, P_t^{[\infty]}\right\}_{i\geq 1}$. The controlled dynamics then becomes

$$\mathrm{d}X_t = \left(A_t(X_t)X_t + B_t\left(X_t, u_t^{[\infty]}\right)u_t^{[\infty]}\right)\mathrm{d}t + \sigma_t\mathrm{d}\mathscr{B}_t \tag{23}$$

and the question is: what sense does the sequence of systems

$$\mathrm{d}X_t^{[1]} = \left(A_t(x_0)X_t^{[1]} + B_t(x_0, 0)u_t^{[1]}\right)\mathrm{d}t + \sigma_t\mathrm{d}\mathscr{B}_t^{[1]} \tag{24}$$

$$\mathrm{d}X_t^{[i]} = \left(A_t\left(X_t^{[i-1]}\right)X_t^{[i]} + B_t\left(X_t^{[i-1]}, u_t^{[i-1]}\right)u_t^{[i]}\right)\mathrm{d}t + \sigma_t\mathrm{d}\mathscr{B}_t^{[i]}, \quad i \geq 2 \tag{25}$$

converge to the solution of (23)? (Here $u_t^{[i]}$ can be chosen to be the optimal control of a standard linear regulator, and we assume that $\mathscr{B}_t^{[i]}$, $i \geq 1$ are independent Ito processes.)

3. Convergence of the approximating sequences

The first result follows directly from the convergence theory developed for deterministic control systems (Tomas-Rodriguez & Banks, 2003), since the Eqs. (17)–(20) are deterministic.

Remark. In the following results, we are assuming that all the Riccati equations involved have unique solutions on the horizon interval $[0, t_f]$. This will be valid for small enough t_f; the general case will be considered in a future paper.

Theorem 1. *The sequence of functions* $\left\{\widehat{X}_t^{[i]}, \widetilde{P}_t^{[i]}, u_t^{[i]}, P_t^{[i]}\right\}_{i\geq 1}$ *converges uniformly on* $[0, t_f]$. $\square$

Hence we concentrate on the system (24) and (25) ($i \geq 1$) which we can write in this form

$$\mathrm{d}X_t^{[1]} = \left(A_t(x_0)X_t^{[1]} + V_t^{[1]}(x_0)\right)\mathrm{d}t + \sigma_t d\mathscr{B}_t^{[1]} \tag{26}$$

$$\mathrm{d}X_t^{[i]} = \left(A_t\left(X_t^{[i-1]}\right)X_t^{[i]} + V_t^{[i]}\left(X_t^{[i-1]}\right)\right)\mathrm{d}t + \sigma_t\mathrm{d}\mathscr{B}_t^{[i]}, \quad i \geq 2, \qquad X_0^{[i]} = X_0 \tag{27}$$

where each $V_t^{[i]}(\cdot)$ is a (local) Lipschitz continuous function. From the standard theory of Ito stochastic differential equations, we see that each equation in (26) and (27) has a unique solution (Oksendal, 2007).

Lemma 2. *The solutions of Eqs. (26) and (27) are given by*

$$X_t^{[1]} = \Phi\left(A_t(x_0), t\right)\left[X_0 + \Phi\left(-A_t(x_0), t\right)\sigma_t\mathscr{B}_t^{[1]}\right] + \int_0^t \Phi\left(A_t(x_0), t-s\right)\left\{V_s^{[1]}(x_0) + A_s(x_0)\sigma_s\mathscr{B}_s^{[1]}\right\}\mathrm{d}s \tag{28}$$

and

$$X_t^{[i]} = \Phi\left(A_t\left(X_t^{[i-1]}\right), t\right)\left[X_0 + \Phi\left(-A_t\left(X_t^{[i-1]}\right), t\right)\sigma_t \mathcal{B}_t^{[i]}\right] + \int_0^t \Phi\left(A_t\left(X_t^{[i-1]}\right), t-s\right) \times \left\{V_s^{[i]}\left(X_s^{[i-1]}\right) + A_s\left(X_s^{[i-1]}\right)\sigma_s \mathcal{B}_s^{[i]}\right\} \mathrm{d}s, \quad i \geq 2 \tag{29}$$

where $\Phi(A_t, t)$ is the fundamental matrix of A_t.

Proof. First we prove (28) then (29). From (26) we have

$$\Phi(-A_t(x_0), t) X_t^{[1]} - X_0 = \int_0^t \Phi(-A_t(x_0), s) V_s^{[1]}(x_0)\mathrm{d}s + \int_0^t \Phi(-A_t(x_0), s)\sigma_s \mathrm{d}\mathcal{B}_s^{[1]} \tag{30}$$

$$= \int_0^t \Phi(-A_t(x_0), s) V_s^{[1]}(x_0)\mathrm{d}s + \Phi(-A_t(x_0), t)\sigma_t \mathcal{B}_t^{[1]} + \int_0^t \Phi(-A_t(x_0), s) A_s(x_0)\sigma_s \mathcal{B}_s^{[1]}\mathrm{d}s. \tag{31}$$

Similarly from (27) we have

$$\Phi\left(-A_t\left(X_t^{[i-1]}\right), t\right) X_t^{[i]} - X_0 = \int_0^t \Phi\left(-A_t\left(X_t^{[i-1]}\right), s\right) V_s^{[i]}\left(X_t^{[i-1]}\right)\mathrm{d}s + \int_0^t \Phi\left(-A_t\left(X_t^{[i-1]}\right), s\right)\sigma_s d\mathcal{B}_s^{[i]} \tag{32}$$

$$= \int_0^t \Phi\left(-A_t\left(X_t^{[i-1]}\right), s\right) V_s^{[i]}\left(X_t^{[i-1]}\right)\mathrm{d}s + \Phi\left(-A_t\left(X_t^{[i-1]}\right), t\right)\sigma_t \mathcal{B}_t^{[i]} + \int_0^t \Phi\left(-A_t\left(X_t^{[i-1]}\right), s\right) A_s\left(X_t^{[i-1]}\right)\sigma_s \mathcal{B}_s^{[i]}\mathrm{d}s, \quad i \geq 2 \tag{33}$$

by the Ito integration by parts formula (Oksendal, 2007). □

We now prove the main results that the sequence of stochastic systems (24) and (25) converges in the mean.

Theorem 3. *The sequence of systems (24) and (25) converges uniformly in the mean on compact time intervals.*

Proof. From Lemma 3.2, we have

$$\Phi\left(-A_t\left(X_t^{[i-1]}\right), t\right) X_t^{[i]} = \left[X_0 + \Phi\left(-A_t\left(X_t^{[i-1]}\right), t\right)\sigma_t \mathcal{B}_t^{[i]}\right] + \int_0^t \Phi\left(-A_t\left(X_t^{[i-1]}\right), s\right) \times \left\{V_s^{[i]}\left(X_s^{[i-1]}\right) + A_s\left(X_s^{[i-1]}\right)\sigma_s \mathcal{B}_s^{[i]}\right\}\mathrm{d}s \tag{34}$$

and

$$\Phi\left(-A_t\left(X_t^{[i-2]}\right), t\right) X_t^{[i-1]} = \left[X_0 + \Phi\left(-A_t\left(X_t^{[i-2]}\right), t\right)\sigma_t \mathcal{B}_t^{[i-1]}\right] + \int_0^t \Phi\left(-A_t\left(X_t^{[i-2]}\right), s\right) \times \left\{V_s^{[i-1]}\left(X_s^{[i-2]}\right) + A_s\left(X_s^{[i-2]}\right)\sigma_s \mathcal{B}_s^{[i-1]}\right\}\mathrm{d}s \tag{35}$$

and so

$$\Phi\left(-A_t\left(X_t^{[i-1]}\right), t\right) X_t^{[i]} - \Phi\left(-A_t\left(X_t^{[i-2]}\right), t\right) X_t^{[i-1]} = \Phi\left(-A_t\left(X_t^{[i-1]}\right), t\right)\sigma_t \mathcal{B}_t^{[i]} - \Phi\left(-A_t\left(X_t^{[i-2]}\right), t\right)\sigma_t \mathcal{B}_t^{[i-1]} + \int_0^t \left\{\Phi\left(-A_t\left(X_t^{[i-1]}\right), s\right) \times \left\{V_s^{[i]}\left(X_s^{[i-1]}\right) + A_s\left(X_s^{[i-1]}\right)\sigma_s \mathcal{B}_s^{[i]}\right\} - \Phi\left(-A_t\left(X_t^{[i-2]}\right), s\right) \times \left\{V_s^{[i-1]}\left(X_s^{[i-2]}\right) + A_s\left(X_s^{[i-2]}\right)\sigma_s \mathcal{B}_s^{[i-1]}\right\}\right\}\mathrm{d}s. \tag{36}$$

Denote the right hand side of (36) as Ψ. Hence,

$$\Phi\left(-A_t\left(X_t^{[i-1]}\right), t\right) X_t^{[i]} - \Phi\left(-A_t\left(X_t^{[i-1]}\right), t\right) X_t^{[i-1]} = \Phi\left(-A_t\left(X_t^{[i-2]}\right), t\right) X_t^{[i-1]} - \Phi\left(-A_t\left(X_t^{[i-1]}\right), t\right) X_t^{[i-1]} + \Psi \tag{37}$$

and so

$$X_t^{[i]} - X_t^{[i-1]} = \Phi\left(A_t\left(X_t^{[i-1]}\right), t\right)\Phi\left(-A_t\left(X_t^{[i-2]}\right), t\right) X_t^{[i-1]} - X_t^{[i-1]} + \Phi\left(A_t\left(X_t^{[i-1]}\right), t\right)\Psi. \tag{38}$$

Hence

$$E\left(\|X_t^{[i]} - X_t^{[i-1]}\|^2\right) \leq E\left(2\left\|\Phi\left(A_t\left(X_t^{[i-1]}\right), t\right) \times \Phi\left(-A_t\left(X_t^{[i-2]}\right), t\right) X_t^{[i-1]} - X_t^{[i-1]}\right\|^2\right) + 2E\left(\left\|\Phi\left(A_t\left(X_t^{[i-1]}\right), t\right)\Psi\right\|^2\right). \tag{39}$$

To estimate the first term on the right of this inequality

$$\Phi\left(A_t\left(X_t^{[i-1]}\right), t\right)\Phi\left(-A_t\left(X_t^{[i-2]}\right), t\right) X_t^{[i-1]} - X_t^{[i-1]} = \Phi\left(A_t\left(X_t^{[i-1]}\right), t\right)\left[\Phi\left(-A_t\left(X_t^{[i-2]}\right), t\right) - \Phi\left(-A_t\left(X_t^{[i-1]}\right), t\right)\right] X_t^{[i-1]} \tag{40}$$

and this can be bounded by $\|X_t^{[i-1]} - X_t^{[i-2]}\|$ as in Tomas-Rodriguez and Banks (2003). To bound the second term in the inequality note that

$$E(\|\Psi\|^2) \leq 3E\left(\left\|\Phi\left(-A_t\left(X_t^{[i-1]}\right), t\right)\sigma_t \mathcal{B}_t^{[i]} - \Phi\left(-A_t\left(X_t^{[i-2]}\right), t\right)\sigma_t \mathcal{B}_t^{[i-1]}\right\|^2\right) + 3\int_0^t E\left(\left\|\Phi\left(-A_t\left(X_t^{[i-1]}\right), s\right) \times \left\{V_s^{[i]}\left(X_s^{[i-1]}\right) + A_s\left(X_s^{[i-1]}\right)\sigma_s \mathcal{B}_s^{[i]}\right\}\right\|^2\right)\mathrm{d}s + 3\int_0^t E\left(\left\|\Phi\left(A_t\left(X_t^{[i-2]}\right), s\right)\left\{V_s^{[i-1]}\left(X_s^{[i-2]}\right) + A_s\left(X_s^{[i-2]}\right)\sigma_s \mathcal{B}_s^{[i-1]}\right\}\right\|^2\right)\mathrm{d}s. \tag{41}$$

Applying the Ito's isometry to the second two terms and using the independence of the $\mathcal{B}^{[i]}$'s and $\mathcal{B}^{[i-1]}$'s, the proof of local convergence then follows, as in Tomas-Rodriguez and Banks

(2003). The global convergence proof is identical to that in Tomas-Rodriguez and Banks (2003). □

4. Optimality

In this section we shall briefly discuss the optimality of the control strategy developed above leaving the general theory of global optimality to a future paper, along with a general consideration of the existence of unique solutions of the Riccati equations. We shall use the notation and ideas of Oksendal (2007). The basic idea is to show that the Hamilton–Jacobi–Belman (HJB) equation is satisfied (as a necessary and sufficient condition for (local) optimality. Then, for convex problems we will have global optimality). Thus, for the system (9), i.e.

$$\mathrm{d}X_t = (A_t(X_t)x_t + B_t(X_t, u_t)u_t)\,\mathrm{d}t + \sigma_t \mathrm{d}\mathcal{B}_t \tag{42}$$

we have the linear operator L^v, where v is a Markov control given by

$$(L^v f)(x) = \frac{\partial f(x)}{\partial s} + \sum_{i=1}^{n} (A(x)x + B(x,v)v)\frac{\partial f}{\partial x_i} + \sum_{i,j=1}^{n} a_{ij}(x,v)\frac{\partial^2 f}{\partial x_i \partial x_j} \tag{43}$$

where $a_{ij} = \frac{1}{2}(\sigma\sigma^{\mathrm{T}})_{ij}$. Then, a necessary condition for optimality with the cost functional (2) is that the HJB equation

$$X_t^{\mathrm{T}} Q_t X_t + u_t^{\mathrm{T}} R_t u_t + (L^{u(x)}\Psi)(t,x) = 0 \quad \text{for } t < t_1 \tag{44}$$

and

$$\Psi(t,x) = x^{\mathrm{T}} F x \tag{45}$$

$$a_{t_1} = 0 \tag{46}$$

is satisfied for some functions u and Ψ.

Using the iteration scheme, each linear, time varying system in the scheme introduced in Section 2 satisfies a HJB equation of the form

$$Y_t^{\mathrm{T}} Q_t Y_t + v_t^{\mathrm{T}} R_t v_t + (L^{v(y)}\widetilde{\Psi})(t,y) = 0 \quad \text{for } t < t_1 \tag{47}$$

and

$$\widetilde{\Psi}(t,y) = y^{\mathrm{T}} F y \tag{48}$$

$$a_{t_1} = 0 \tag{49}$$

and

$$\widetilde{\Psi}(t,y) = y^{\mathrm{T}} P y + a_t. \tag{50}$$

The form of the functional (43) and the cost functional show that the solutions of the HJB equations for the iterated systems converge to that of the nonlinear problem in (44), the technical details being similar to those of the general convergence proof. It follows that the filtered control system is locally optimal and we therefore have a nonlinear separation theorem.

5. Example

In this section, we shall illustrate the theory by using a nonlinear oscillator with negative damping given by the equation

$$m\ddot{y} - \varepsilon\dot{y} + k(y + \alpha y^3) = u + gw. \tag{51}$$

The dynamics of the noisy system can be written in state space form as follows:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m}(1+\alpha x_1^2) & \frac{\varepsilon}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u + \begin{pmatrix} 0 \\ g \end{pmatrix} w. \tag{52}$$

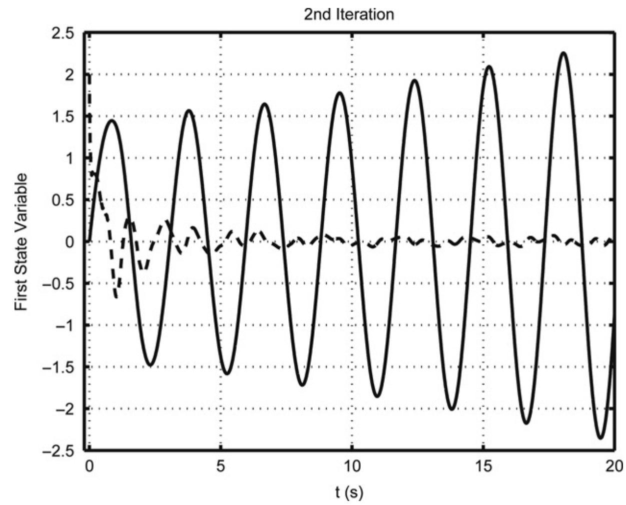

Fig. 1. First state variable (actual: solid line, estimated: dashed line).

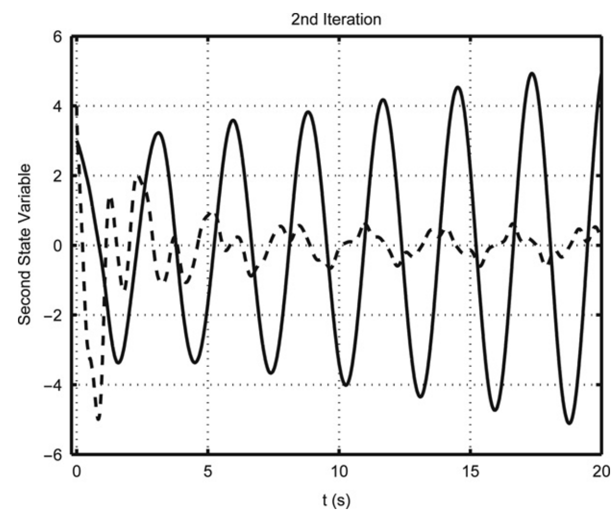

Fig. 2. Second state variable (actual: solid line, estimated: dashed line).

The measurement equation can be given as

$$z = Cx + v. \tag{53}$$

In the above equations, $m, \varepsilon, k, \alpha$, and g are the system parameters, y is the variable, u is the control input, w and v are the plant and measurement noises, respectively. Euler–Maruyama (Kloeden & Platen, 1992) method is used for the numerical integration of the rate of state variables. Euler's method is for numerical integration of the other variables.

For the numerical simulations, system parameters are taken as $m = 2, k = 10, \alpha = 1, \varepsilon = 0.1, g = 0.02$. Measurement sensitivity matrix is taken as $C = [1\ 0]$. For the control part of the system, weighting coefficient for state and control input variables are taken as $Q = I_{2\times 2}$ and $R = 1$, respectively. Final value of matrix Riccati equation variable is taken as $F = I_{2\times 2}$. Plant and measurement noises are considered as Gaussian white noise with zero mean and variance of 0.2. For the estimation part of the system, the initial value of matrix Riccati equation variable is taken as $\widetilde{P} = 5I_{2\times 2}$. Initial value of actual and estimated state variables are taken as $x(t_0) = [0\ 3]^{\mathrm{T}}, \widehat{x}(t_0) = [2\ 4]^{\mathrm{T}}$. The simulation time step is taken as 0.0025 s. The simulation results are given in Figs. 1–18.

The first approximation is linear, time-invariant, and has been obtained by evaluating the nonlinear system at $x = x_0$ and $u = 0$. After the first approximation, the recursive procedure becomes linear, time-varying (LTV). The response of the eighth

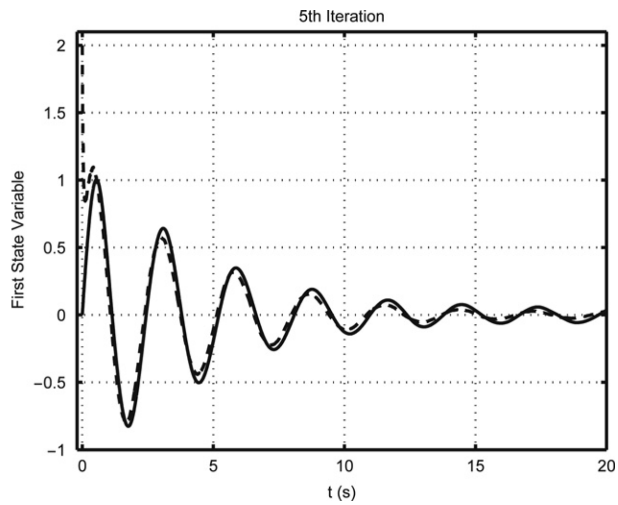

Fig. 3. First state variable (actual: solid line, estimated: dashed line).

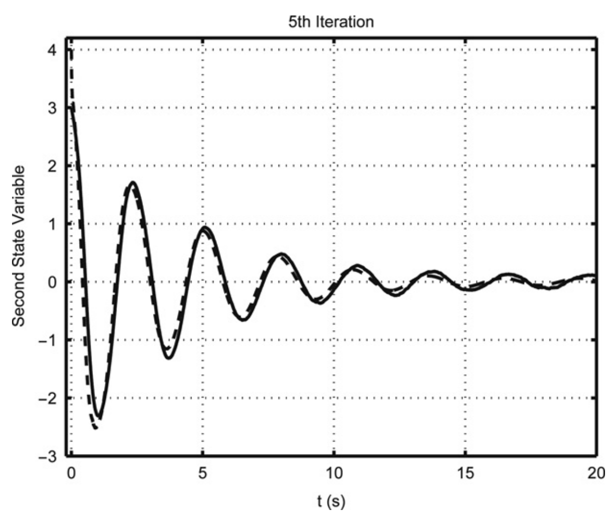

Fig. 4. Second state variable (actual: solid line, estimated: dashed line).

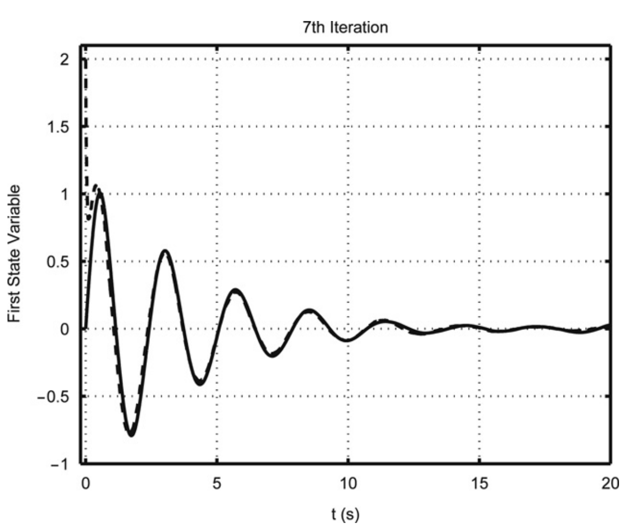

Fig. 5. First state variable (actual: solid line, estimated: dashed line).

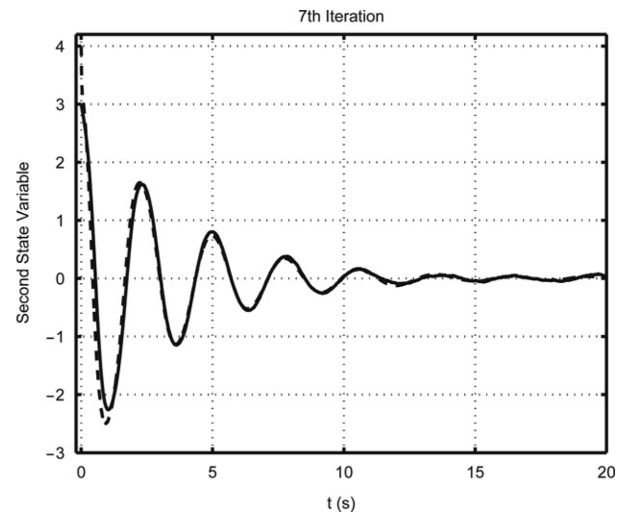

Fig. 6. Second state variable (actual: solid line, estimated: dashed line).

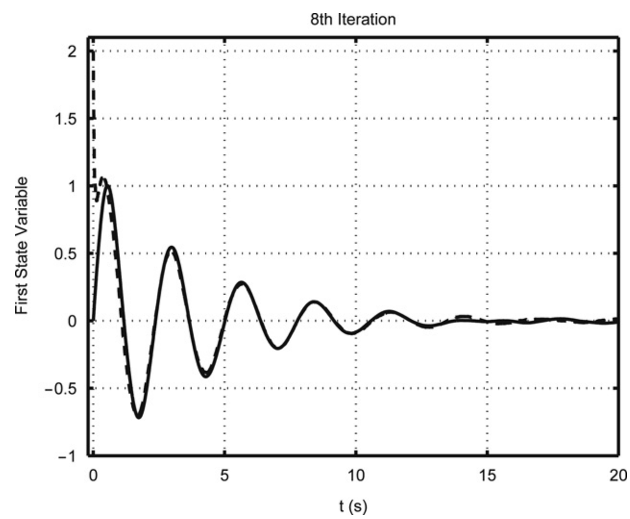

Fig. 7. First state variable (actual: solid line, estimated: dashed line).

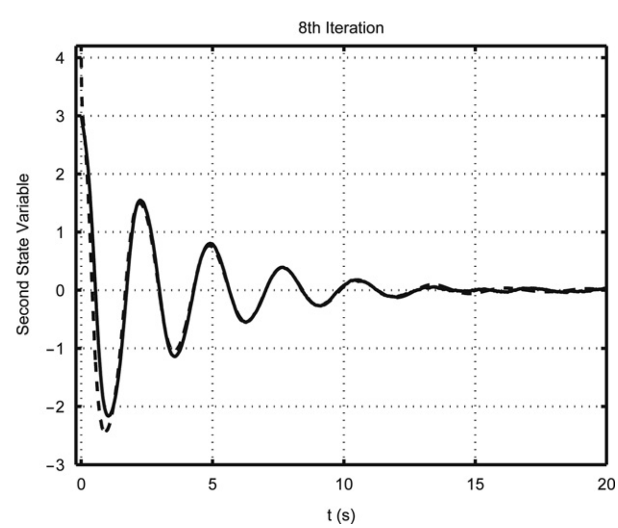

Fig. 8. Second state variable (actual: solid line, estimated: dashed line).

approximated LTV control system (converged solution), therefore, represents the response of the nonlinear dynamics (52) of the oscillator with negative damping to the eighth approximated time-varying system. To illustrate the convergence of these controlled LTV systems, response of second, fifth, seventh, and eighth approximated systems are given in Figs. 1–8. The corresponding

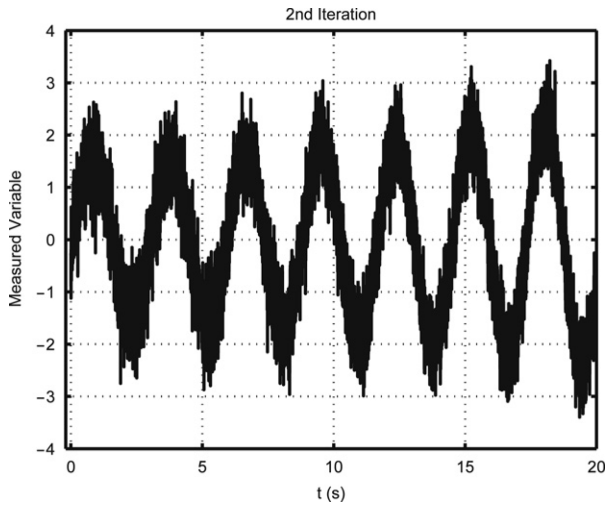

Fig. 9. Measured variable.

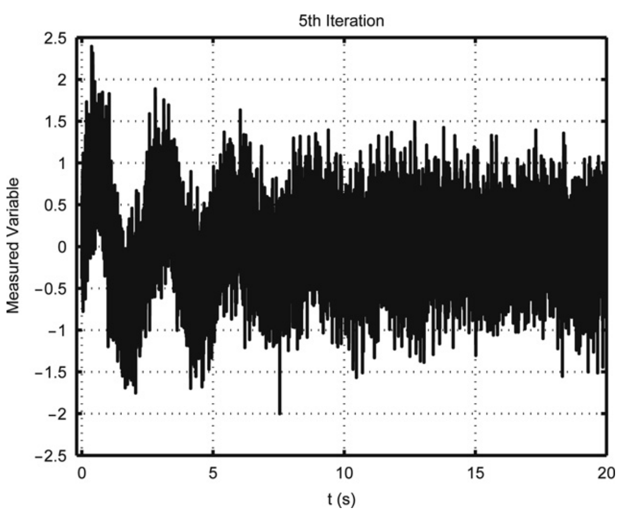

Fig. 10. Measured variable.

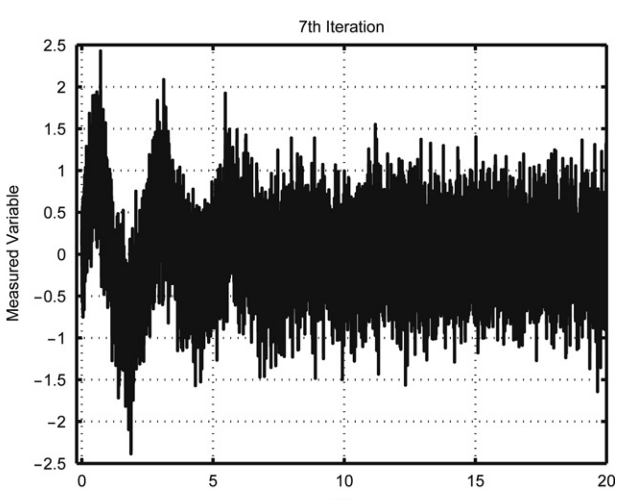

Fig. 11. Measured variable.

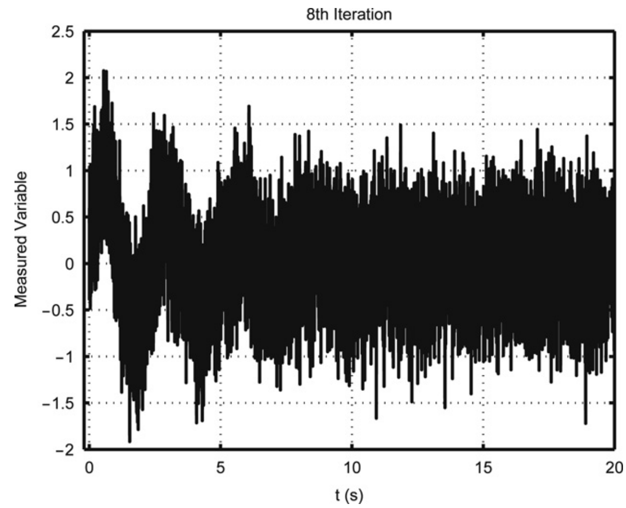

Fig. 12. Measured variable.

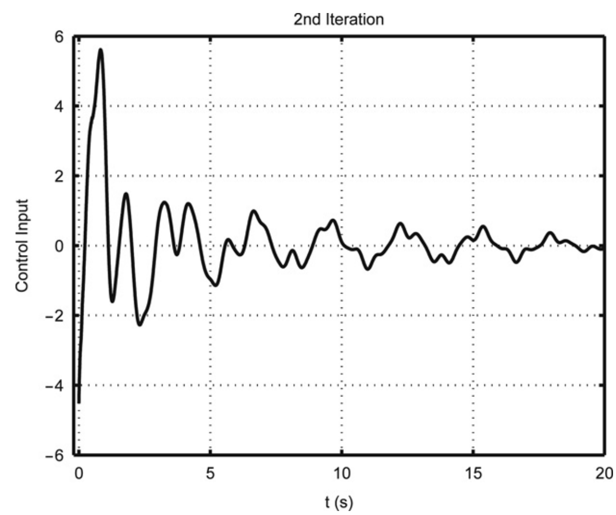

Fig. 13. Control input.

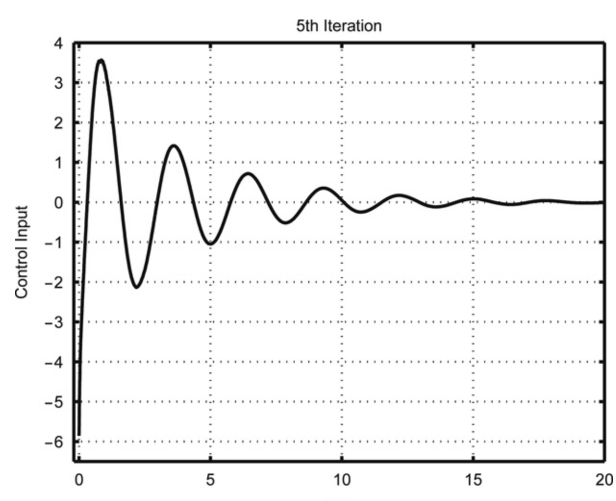

Fig. 14. Control input.

control inputs are given in Figs. 13–16. To see the convergence easily, state variables for second, fifth, seventh, and eighth nonlinear iterations are plotted in Figs. 17 and 18. It can be seen from the Figs. 1–8 that solutions converge and oscillations are suppressed. The measured variable for second, fifth, seventh, and eighth approximated systems are given in Figs. 9–12.

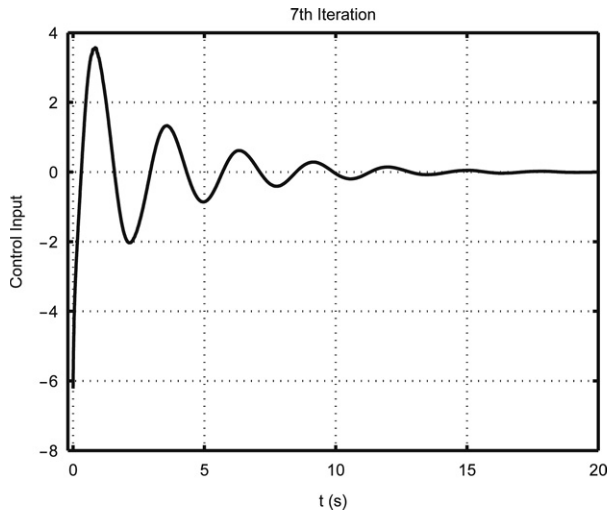

Fig. 15. Control input.

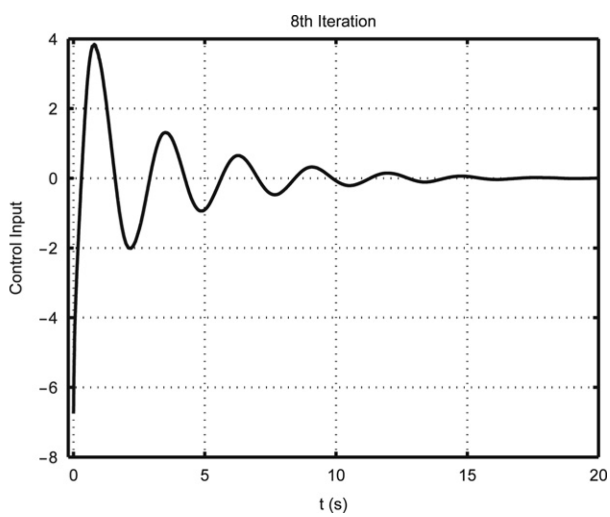

Fig. 16. Control input.

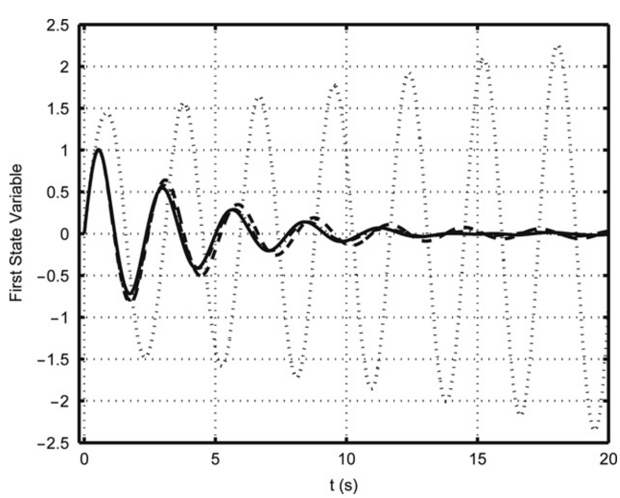

Fig. 17. Global convergence of first state variable (2nd iteration: dotted line, 5th iteration: dashed line, 7th iteration: dash-dot line, 8th iteration: solid line).

6. Conclusions

We have given a nonlinear separation result for controlled stochastic systems. Using a sequence of linear, time-varying Ito systems we have shown that the controlled systems with the approximate estimated state feedback converges to a stabilizing control for the nonlinear system. A nonlinear oscillator with negative damping is used to illustrate the method. Since the example is purely simulated, we have generated the measured outputs artificially. Of course, in a real system, we would use the actual measurements from the output devices. In a future paper, we will study the global optimality of this technique along with a general consideration of the existence of unique solutions of the Riccati equations.

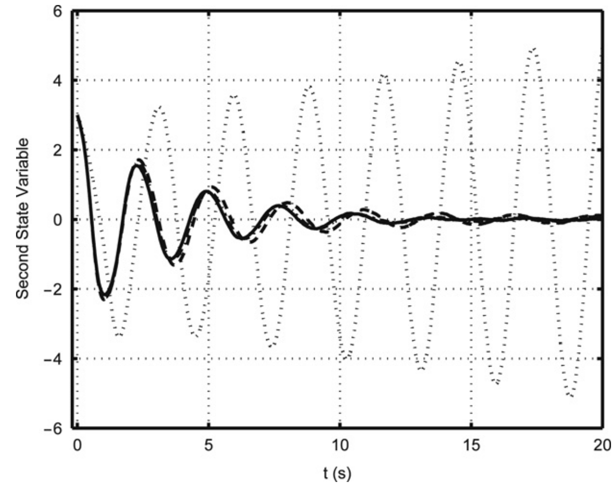

Fig. 18. Global convergence of second state variable (2nd iteration: dotted line, 5th iteration: dashed line, 7th iteration: dash-dot line, 8th iteration: solid line).

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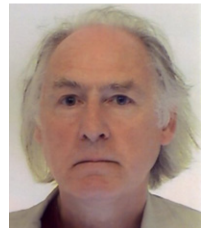

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