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Linear approximations to nonlinear dynamical systems with applications to stability and spectral theory

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In this paper we study a recently introduced technique for nonlinear dynamical systems in which the equation is replaced by a sequence of linear, time-varying approximations. This allows many of the classical results in linear systems theory to be applied to nonlinear systems. Here we study two particular areas—stability theory and the frequency-domain theory of nonlinear systems.

Keywords: nonlinear systems; linear approximations; stability; spectral theory.

1. Introduction

There are many approaches to the study of nonlinear dynamical systems, including local linearizations in phase space (Perko, 1991), global linear representations involving the Lie series solution (Banks & Iddir, 1992; Banks, 1992; Banks *et al.*, 1996), Lie algebraic methods (Banks, 2001) and global results based on topological indices (McCaffrey & Banks, 2002; Perko, 1991). Linear systems, on the other hand are very well understood and there is, of course, a vast literature on the subject (see, for example, Banks, 1986). The simplicity of linear mathematics relative to nonlinear theory is evident and forms the basis of much of classical mathematics and physics. It is therefore attractive to try to attack nonlinear problems by linear methods, which are not local in their applicability. In this paper we study a recently introduced approach to nonlinear dynamical systems based on a representation of the system as the limit of a sequence of linear, time-varying approximations which converge in the space of continuous functions to the solution of the nonlinear system, under a very mild local Lipschitz condition. This approach has already been used in optimal control theory (Banks & Dinesh, 2000), in the theory of nonlinear delay systems (Banks, 2002) and in the theory of chaos (Banks & McCaffrey, 1998). In these papers, however, only a local (in time) proof of convergence was given—here we shall give a global proof for dynamical systems which do not have finite escape time. We shall also study two further applications of the method: to the study of stability of nonlinear systems and the definition of a spectral theory for nonlinear systems. In the former case, we shall define a generalized Lyapunov function, the existence of which is equivalent to the stability of the system. In the second case we first study the spectral theory of a linear, time-varying system as the perturbation of a time-invariant system and then use the approximation scheme to extend it to nonlinear systems.

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The techniques described here are computationally simple as they are based on sequences of linear systems, so the usual linear techniques are available. It is also possible to extend many of the ideas developed for linear, time-varying systems to nonlinear systems by using these ideas. For example, in a future paper we shall apply them to the study of exponential dichotomies as discussed in Sacker & Sell (1978).

2. Linear, time-varying approximations

In this section we study the nonlinear differential equation

$$\dot{x} = A(x)x, \ \ x(0) = x_0 \in \mathbb{R}^n. \tag{2.1}$$

We do this by introducing the following sequence of linear, time-varying approximations:

$$\dot{x}^{[0]}(t) = A(x_0)x^{[0]}(t), \ \ x^{[0]}(0) = x_0 \tag{2.2}$$
$$\dot{x}^{[i]}(t) = A(x^{[i-1]}(t))x^{[i]}(t), \ \ x^{[i]}(0) = x_0 \tag{2.3}$$

for $i \geqslant 1$. We first prove a local convergence result for this system. This proof appears in a slightly different form in Banks (2001); we include it here for the convenience of the reader and to set the notation for the global result.

LEMMA 2.1 Suppose that $A : \mathbb{R}^n \to \mathbb{R}^{n^2}$ is locally Lipschitz. Then the sequence of functions $x^{[i]}(t)$ defined by (2.2), (2.3) converges uniformly on $[0, T]$, *for some* $T > 0$, in the space $C([0, T]; \mathbb{R}^n)$.

Proof. Let $\Phi^{[i-1]}(t, t_0)$ denote the transition matrix of $A(x^{[i-1]}(t))$ so that we have (Brauer, 1967):

$$\left\| \Phi^{[i-1]}(t, t_0) \right\| \leqslant \exp\left(\int_{t_0}^{t} \mu(A(x^{[i-1]}(\tau)))\mathrm{d}\tau \right),$$

where $\mu(A)$ is the logarithmic norm of A. By the local Lipschitz condition on $A(x)$, we have

$$\|A(x) - A(y)\| \leqslant \alpha(K)\, \|x - y\|\, ,$$

for $x, y \in B(K, x_0)$ (the ball, centre x_0, radius K) for some $K > 0$ and some $\alpha(K)$. We have

$$\begin{aligned} x^{[i]}(t) - x_0 = {} & \exp(A(x_0)t)x_0 - x_0 \\ & + \int_0^t \exp[A(x_0)(t-s)][A(x^{[i-1]}(s)) - A(x_0)]\mathrm{d}s \end{aligned}$$

and so, for any $T > 0$,

$$\begin{aligned} \left\| x^{[i]}(t) - x_0 \right\| \leqslant {} & \sup_{t \in [0,T]} \|\exp[A(x_0)t] - I\| \cdot \|x_0\| \\ & + \sup_{t \in [0,T]} \{\exp[\|A(x_0)t\|]\alpha(K)\} \times T \sup_{t \in [0,T]} \left\| x^{[i-1]}(t) - x_0 \right\|. \end{aligned}$$

Hence, if $x^{[i-1]}(t) \in B(K, x_0)$, then $x^{[i]}(t) \in B(K, x_0)$ (for $t \in [0, T]$) if T is small enough (by the continuity of $\exp(A(x_0)t)$ in t). Since $x^{[0]}(t) \in B(K, x_0)$ for small enough T, we see that all the solutions $x^{[i]}(t)$ are bounded for $i \geqslant 0$ and $t \in [0, T]$. Also,

$$\left\| A(x^{[i-1]}(t)) \right\| \leqslant \alpha(K) \left\| x^{[i-1]}(t) - x_0 \right\| + \| A(x_0) \|$$

and since

$$\mu(A) = \frac{1}{2} \max[\sigma(A + A^T)]$$

in the standard matrix norm, we have that $\mu(A(x^{[i-1]}(t)))$ is bounded for all i, say

$$\mu(A(x^{[i-1]}(t))) \leqslant \mu, \text{ for all } t \in [0, T] \text{ and all } i.$$

Hence, by (2.3),

$$\begin{aligned} \dot{x}^{[i]}(t) - \dot{x}^{[i-1]}(t) &= A(x^{[i-1]}(t))x^{[i]}(t) - A(x^{[i-2]}(t))x^{[i-1]}(t) \\ &= A(x^{[i-1]}(t))(x^{[i]}(t) - x^{[i-1]}(t)) \\ &\quad + (A(x^{[i-1]}(t)) - A(x^{[i-2]}(t)))x^{[i-1]}(t) \end{aligned}$$

and so if we put

$$\xi^{[i]}(t) = \sup_{s \in [0,T]} \left\| x^{[i]}(s) - x^{[i-1]}(s) \right\|,$$

then

$$\xi^{[i]}(t) \leqslant \int_0^t \left\| \Phi^{[i-1]}(t, s) \right\| \cdot \left\| A(x^{[i-1]}(s)) - A(x^{[i-2]}(s)) \right\| \cdot \left\| x^{[i-1]}(s) \right\| \mathrm{d}s.$$

Hence,

$$\xi^{[i]}(t) \leqslant \int_0^t \exp[\mu(t-s)] \alpha(K) \xi^{[i-1]}(s) K \, \mathrm{d}s$$

so

$$\begin{aligned} \xi^{[i]}(T) &\leqslant \sup_{s \in [0,T]} \{\exp[\mu(T-s)]\} \alpha(K) T K \xi^{[i-1]}(T) \\ &\leqslant \lambda \xi^{[i-1]}(T), \end{aligned}$$

where $\lambda = \sup_{s \in [0,T]} \{\exp[\mu(T-s)]\} \alpha(K) T K$. If T is small enough, then $\lambda < 1$. In this case we have, for any $i \geqslant j$,

$$\begin{aligned} \left\| x^{[i]}(s) - x^{[j]}(s) \right\| \leqslant{} & \left\| x^{[i]}(s) - x^{[i-1]}(s) \right\| + \left\| x^{[i-1]}(s) - x^{[i-2]}(s) \right\| \\ & + \cdots + \left\| x^{[j+1]}(s) - x^{[j]}(s) \right\| \end{aligned}$$

so

$$\sup_{s\in[0,T]} \left\| x^{[i]}(s) - x^{[j]}(s) \right\| \leqslant \lambda^{i-j}\xi^{[j]}(T) + \lambda^{i-j-1}\xi^{[j]}(T) + \cdots + \lambda\xi^{[j]}(T)$$
$$= \lambda\left(\frac{1-\lambda^{i-j}}{1-\lambda}\right)\xi^{[j]}(T).$$

Hence, if N is a fixed positive integer and $i \geqslant j > N$, then

$$\sup_{s\in[0,T]} \left\| x^{[i]}(s) - x^{[j]}(s) \right\| \leqslant \lambda^{j-N+1}\left(\frac{1-\lambda^{i-j}}{1-\lambda}\right)\xi^{[N]}(T).$$

Since $\xi^{[N]}(T)$ is bounded, the right-hand side is arbitrarily small if j is large and so $\{x^{[i]}(t)\}$ is a Cauchy sequence in $C([0,T];\mathbb{R}^n)$. □

Having shown the local convergence of the sequence $\{x^{[i]}(t)\}$ in $C([0,T];\mathbb{R}^n)$ we now proceed to prove the global convergence in the sense that if the solution of the nonlinear equation exists and is bounded in the interval $[0,\tau] \subseteq \mathbb{R}$,then the sequence of approximations converges uniformly on $[0,\tau]$ to the solution of the nonlinear equation.

THEOREM 2.1 Suppose that the nonlinear equation (2.1) has a unique solution on the interval $[0,\tau]$ and assume that $A : \mathbb{R}^n \to \mathbb{R}^{n^2}$ is locally Lipschitz. Then the sequence of functions $\{x^{[i]}(t)\}$ defined in (2.2) and (2.3) converges uniformly on $[0,\tau]$.

Proof. We know from the previous lemma that, given any initial state x_0, the sequence (2.2), (2.3) converges uniformly on some interval $[0,T]$, where T may depend on x_0. However, it is clear from the proof of the lemma that T can be chosen to be locally constant; i.e. for any $\bar{x}$ there exists a neighbourhood $B_{\bar{x}}$ of $\bar{x}$ such that the sequence in (2.2), (2.3) with initial state $x_0 \in B_{\bar{x}}$ converges uniformly on some interval $[0, T_{\bar{x}}]$, where $T_{\bar{x}}$ is independent of x_0.

Now suppose that the result is false, so that there is a maximal time interval $[0,\bar{T})$ such that, for any $T < \bar{T}$, the sequence (2.2), (2.3) converges uniformly on $[0,T]$. Now consider the solution trajectory $x(t;x_0)$ of the original nonlinear system (2.1) on the interval $[0,\tau]$; define the set

$$S = \{x(t;x_0) : t \in [0,\tau]\}.$$

For each $\bar{x} \in S$, choose a neighbourhood $B_{\bar{x}}$ as above; i.e. the sequence of approximations converges uniformly on the interval $[0, T_{\bar{x}}]$ for any $x_0 \in B_{\bar{x}}$ for $T_{\bar{x}}$ independent of x_0. Since S is compact and $\cup_{\bar{x}\in S} B_{\bar{x}}$ is an open cover of S, there exists a finite subcover $\{B_{\bar{x}_1}, \ldots, B_{\bar{x}_p}\}$ with corresponding times $\{T_{\bar{x}_1}, \ldots, T_{\bar{x}_p}\}$. Let

$$T_m = \min\{T_{\bar{x}_1}, \ldots, T_{\bar{x}_p}\}.$$

Now the sequence (2.2), (2.3) converges uniformly on $[0, \bar{T} - T_m/2]$, by assumption. Let

$$x_{0,i} = x^{[i]}(\bar{T} - T_m/2).$$

Since these converge to $x(\bar{T} - T_m/2)$ we can assume that they belong to $B_{\bar{x}_p}$, so that we get a sequence of solutions given by (2.2), (2.3) from the initial states $x_{0,i}$ and converging

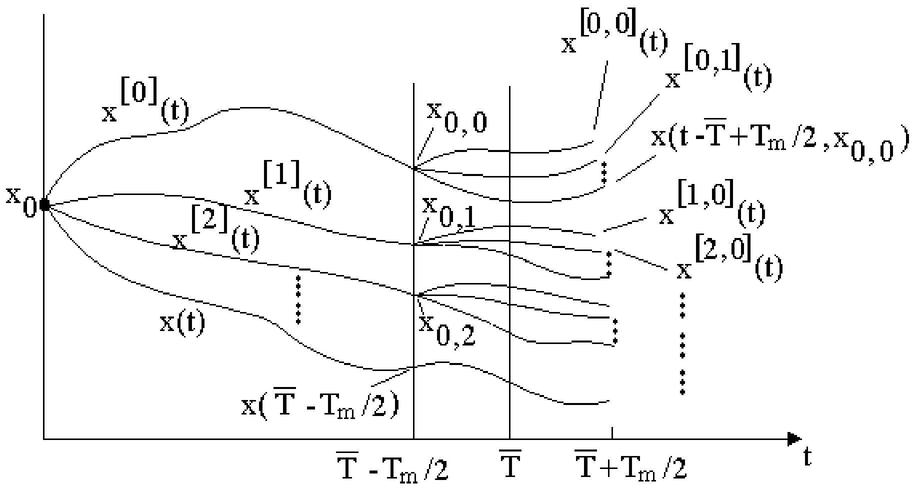

FIG. 1. Extending the solution sequence.

uniformly on the interval $[0, T_m]$ to the corresponding solutions of the nonlinear equation (2.1). (See Fig. 1.)

We shall denote these solutions by $x^{[i,j]}(t)$. Now we use a Cantor-like diagonal argument. Consider the functions

$$y^{[i]}(t) = \begin{cases} x^{[i]}(t), & 0 \leqslant t \leqslant \bar{T} - T_m/2 \\ x^{[i,i]}(t), & \bar{T} - T_m/2 \leqslant t \leqslant \bar{T} + T_m/2. \end{cases}$$

Then $y^{[i]}(t)$ converges uniformly to $x(t)$ on $[0, \bar{T} + T_m/2]$ and is arbitrarily close to $x^{[i]}(t)$ on $[0, \bar{T}]$ which contradicts the assumption that $\{x^{[i]}(t)\}$ is not uniformly convergent on $[0, \bar{T}]$. $\square$

EXAMPLE 2.1 Consider the Van der Pol oscillator given by the equations

$$\dot{x}_1 = x_1 - x_1^3 + x_2$$
$$\dot{x}_2 = -x_1.$$

We can write this system in the form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -x_1^2 + 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and hence obtain the sequence of linear, time-varying approximations

$$\begin{pmatrix} \dot{x}_1^{[i]}(t) \\ \dot{x}_2^{[i]}(t) \end{pmatrix} = \begin{pmatrix} -(x_1^{[i-1]}(t))^2 + 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1^{[i]}(t) \\ x_2^{[i]}(t) \end{pmatrix}.$$

The simulation in Fig. 2 shows the actual solution $x(t)$ (for a time 6.25 s) and the approximations $x^{[1]}(t)$, $x^{[8]}(t)$ and $x^{[14]}(t)$ only (for clarity). Note that we have used only a simple Euler integration routine here.

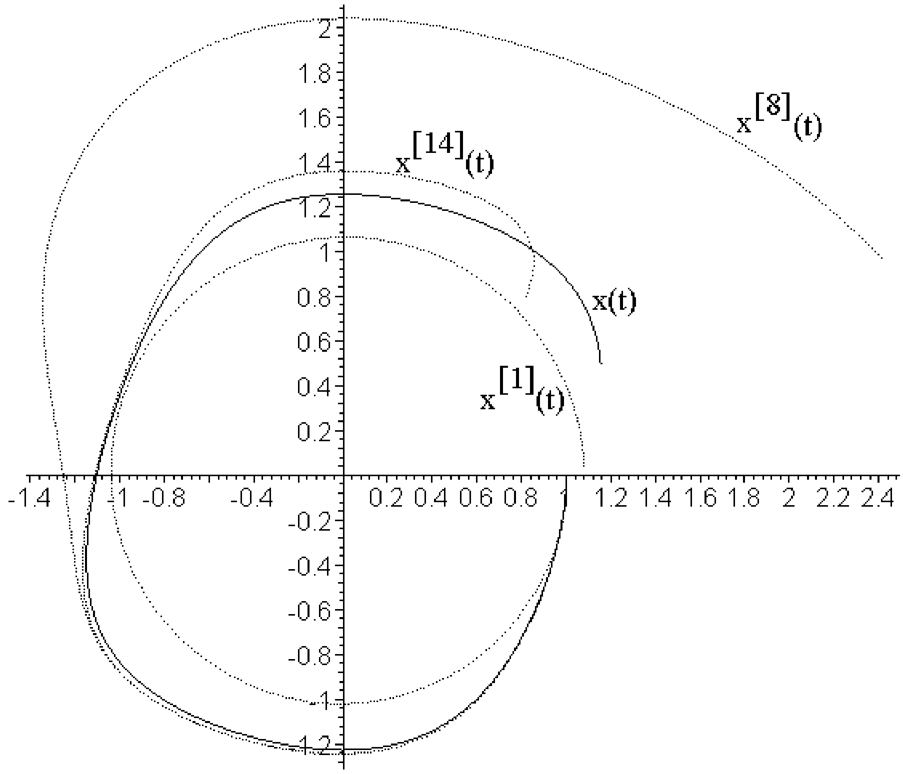

FIG. 2. The Van der Pol oscillator and approximations 1, 8 and 14.

3. Stability theory

In this section we apply the approximation sequence introduced in Section 2 to the problem of nonlinear stability. We use the following well known result from the stability theory of linear, nonautonomous systems (see Kalman & Bertram, 1960).

THEOREM 3.1 Consider the dynamical system

$$\dot{x} = A(t)x \tag{3.1}$$

where $A(t)$ is a sufficiently smooth and uniformly bounded matrix function. Then the equilibrium state $x = 0$ is uniformly asymptotically stable iff the system admits a quadratic Lyapunov function, i.e. there exists a symmetric matrix function $P(\cdot) : \mathbb{R} \to \mathbb{R}^{n^2}$ such that

(i) $P(\cdot)$ is continuous
(ii) $\alpha I \leqslant P(t) \leqslant \beta I$, for all $t \in \mathbb{R}^+$ and some $\alpha, \beta > 0$
(iii) $x^T[P(t)A(t) + A^T(t)P(t) + \dot{P}(t)]x \leqslant -\lambda \|x\|^2$ holds for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}^+$ and some positive constant λ.

In order to find P we consider a fixed positive definite symmetric matrix Q and obtain P as the solution of the differential equation

$$\dot{P}(t) = -A^T(t)P(t) - P(t)A(t) - Q\ ,\ \ P(0) = P_0.$$

Thus,

$$P(t) = \Phi^T(t;0)P_0\Phi(t;0) - \int_0^t \Phi^T(t;s)Q\Phi(t;s)\mathrm{d}s \tag{3.2}$$

where $\Phi(t;s)$ is the transition matrix generated by $-A(t)$. Recall that Φ satisfies the inequalities

$$\|x_0\|\exp\left\{-\int_s^t \mu(A(t'))\mathrm{d}t'\right\} \leqslant \|\Phi(t;s)x_0\| \leqslant \|x_0\|\exp\left\{\int_s^t \mu(-A(t'))\mathrm{d}t'\right\} \tag{3.3}$$

where $\mu(A)$ is the logarithmic norm of A. From (3.2) we have

$$x^TP(t)x = x^T\Phi^T(t;0)P_0\Phi(t;0)x - \int_0^t x^T\Phi^T(t;s)Q\Phi(t;s)x\mathrm{d}s$$

for any $x \in \mathbb{R}^n$. Since P_0 is positive definite we have

$$\begin{aligned} x^T\Phi^T(t;0)P_0\Phi(t;0)x &\geqslant \alpha\,\|\Phi(t;0)x\|^2 \\ &\geqslant \alpha\,\|x\|^2\exp\left\{-\int_s^t \mu(A(t'))\mathrm{d}t'\right\} \end{aligned}$$

for some $\alpha > 0$, by (3.3). Also,

$$\begin{aligned} \int_0^t x^T\Phi^T(t;s)Q\Phi(t;s)x\mathrm{d}s &= \int_0^t \left\|Q^{\frac{1}{2}}\Phi(t;s)x\right\|^2\mathrm{d}s \\ &\leqslant \left\|Q^{\frac{1}{2}}\right\|^2\|x\|^2\int_0^t \exp\left\{\int_s^t \mu(-A(t'))\mathrm{d}t'\right\}\mathrm{d}s \end{aligned}$$

again by (3.3). Hence we have proved the following lemma.

LEMMA 3.1 If

$$\alpha > \left\|Q^{\frac{1}{2}}\right\|^2 \sup_{t\geqslant 0}\left(\exp\left\{\int_s^t \mu(A(t'))\mathrm{d}t'\right\}\int_0^t \exp\left\{\int_s^t \mu(-A(t'))\mathrm{d}t'\right\}\mathrm{d}s\right)$$

then $P(t)$ is positive definite uniformly for all t.

In order to study the stability of nonlinear systems of the form (2.1) by applying Theorem 3.1 and the sequence of approximating systems (2.2), (2.3), note that the approximating solutions of a stable system may not be stable. For example, there is no reason why $A(x_0)$ should be a stable matrix. Hence, rather than consider the sequence (2.2), (2.3) we shall consider a regularized sequence of equations defined in the following way. Consider the linear, time-varying system

$$\dot{x} = A(t)x \tag{3.4}$$

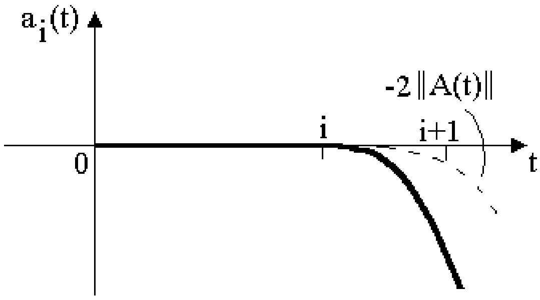

FIG. 3. A typical function $a_i(t)$.

and let $a_i(t)$ be a function which satisfies the properties

$$\begin{array}{ll} \text{(i)} & a_i(t) = 0,\ 0 \leqslant t \leqslant i \\ \text{(ii)} & a_i(t) \leqslant -2\,\|A(t)\|\,,\ i+1 \leqslant t < \infty \\ \text{(iii)} & a_i \text{ is } C^{\infty}. \end{array} \tag{3.5}$$

Clearly such functions exist (see Fig. 3).

LEMMA 3.2 The function $\phi_i(t) = \exp\left(\int_0^t a_i(s)\mathrm{d}s\right)$, where a_i is given by (3.5), satisfies the properties

(i) $\phi_i(t) = 1, 0 \leqslant t \leqslant i$
(ii) $x(t)\phi_i(t) \to 0$ as $t \to \infty$, where $x(t)$ is the solution of (3.4).

Proof. The property (i) is obvious. For (ii) note that $x\phi_i$ satisfies the equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(x(t)\phi_i(t)) &= A(t)x(t)\phi_i(t) + x(t)\dot{\phi}_i(t) \\ &= A(t)x(t)\phi_i(t) + x(t)a_i(s)\phi_i(t) \end{aligned} \tag{3.6}$$

so that, if $y(t) = x(t)\phi_i(t)$, we have

$$\dot{y}(t) = A(t)y(t) + a_i(t)y(t)$$

so that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|y(t)\|^2 &= y^T(t)A(t)y(t) + a_i(t)\,\|y(t)\|^2 \\ &\leqslant \|A(t)\|\,\|y(t)\|^2 + a_i(t)\,\|y(t)\|^2 \\ &\leqslant -\,\|A(t)\|\,\|y(t)\|^2\,, \end{aligned}$$

so

$$\|y(t)\| \to 0 \text{ as } t \to \infty.$$

□

Note that

$$x(t)\phi_i(t) = x(t) \text{ for } 0 \leqslant t \leqslant i$$

and that $\dot{\phi}_i(t) = 0$, $t \in [0, i]$, so that (3.6) satisfied by $x(t)\phi_i(t)$ is the same as the original equation on $[0, i]$. The new equation (3.6) simply 'regularizes' $x(t)$ after $t = i$ so that the new equation is stable. The next result thus follows from the above remarks and Theorem 2.2:

THEOREM 3.2 Suppose that the system (2.1) is asymptotically stable for the initial condition x_0. Then the sequence $y^{[i]}(t) = x^{[i]}(t)\phi_i(t)$, where $\phi_i(t)$ is defined as in Lemma 3.2 (and $a_i(t)$ is related to $A(x^{[i-1]}(t))$ as in (3.5)) converges uniformly on any compact interval to the solution $x(t; x_0)$ of (2.1). Moreover, the approximating systems

$$\begin{aligned} \dot{y}^{[i]}(t) &= (A(x^{[i-1]}(t)) + a_i(t)I)y^{[i]}(t), \\ y^{[i]}(0) &= x_0 \end{aligned} \tag{3.7}$$

are all asymptotically stable.

We shall now consider the stability of general nonlinear systems and derive a general Lyapunov theory which will include the classical one. In order to motivate the definition of a generalized Lyapunov function which we shall define, consider a two-dimensional system with an equilibrium point at the origin and phase-plane trajectories as in Fig. 4(a) (and other equilibria, two of which are shown).

For systems such as this, which are far from being hyperbolic, the set of points which are on trajectories which are asymptotically stable to the origin form a set which is not even a manifold in general (it will, of course, be a union of solution manifolds defined by each trajectory). The set of such points is shown for the example above in Fig. 4(b) (the inner boundary is in the set, but not the outer one). In classical Lyapunov theory, a Lyapunov function is defined in some neighbourhood of an equilibrium point, so the stability domain is an open set. In the general case, such as the example in Fig. 4, we cannot expect a single function to be useful as a Lyapunov function, since it would have to be defined in some neighbourhood of 0. Thus we shall introduce the following generalized notion of the Lyapunov function.

DEFINITION 3.1 A Lyapunov function for the system (2.1) over a subset $S \subseteq \mathbb{R}^n$ is a continuous map $x_0 \to x^T P(t; x_0)x$ from S to the space of positive definite quadratic forms which depend on a continuously differentiable matrix-valued function $P(\cdot, x_0) : \mathbb{R}^+ \to \mathbb{R}^{n^2}$ such that, for each $x_0 \in S$, we have

$$\dot{P}(t, x_0) + A^T(x(t; x_0))P(t, x_0) + P(t, x_0)A(x(t; x_0)) \leqslant -\alpha I$$

for some $\alpha > 0$.

Then we have the following theorem.

THEOREM 3.3 The system (2.1) is asymptotically stable over the set S (the equilibrium point $0 \in S$) iff there exists a Lyapunov function over S in the sense of Definition 3.1.

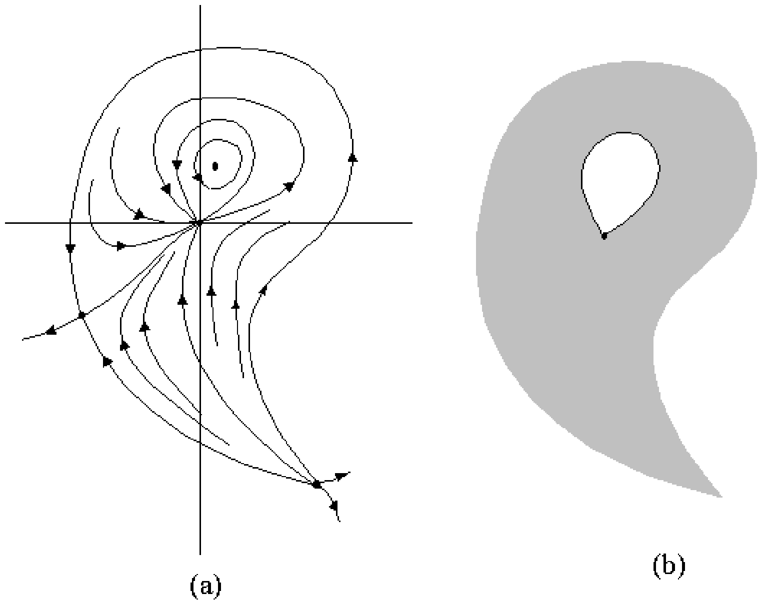

FIG. 4. A system with a non-simply connected stability region.

Proof. Sufficiency is obvious. To prove necessity we simply note that if $\xi(t)$ is the solution of (2.1) through x_0 at $t = 0$, then $\xi(t)$ is also the solution of the linear, time-varying system

$$\dot{x}(t) = A(\xi(t))x(t), \;\; x(0) = x_0.$$

Hence if x_0 lies in the stable subset S then the linear system is stable and so a positive definite matrix $P(t, x_0)$ exists by Theorem 3.1. □

REMARK 1 One can compute the function $P(t, x_0)$ as the limit of the corresponding solutions to the approximating problems (2.2), (2.3). Thus, if $y^{[i]}(t)$ is the solution of the 'regularized' system (3.7) in Theorem 3.2, i.e.

$$\dot{y}^{[i]}(t) = \bar{A}^{[i]}(t)y^{[i]}(t), \;\; y^{[i]}(0) = x_0,$$

where

$$\bar{A}^{[i]}(t) = A(x^{[i]}(t)) + a_i(t)I,$$

then since this system is stable, there exists a function $P^{[i]}(t; x_0)$ such that

$$\dot{P}^{[i]}(t; x_0) = \Phi^{[i]T}(t; 0)P_0\Phi^{[i]}(t; 0) - \int_0^t \Phi^{[i]T}(t; s)Q\Phi^{[i]}(t; s)\mathrm{d}s$$

where $\Phi^{[i]}$ is generated by $\bar{A}^{[i]}(t)$. The convergence of $P^{[i]}(t; x_0)$ now follows from the earlier results.

REMARK 2 In the case of analytic hyperbolic (Hamiltonian-like) systems, the assignment $x_0 \to P^{[i]}(t; x_0)$ on the stable manifold is a section of the structure sheaf of the manifold, i.e. the sheaf of germs of locally analytic functions (see Bredon, 1998). Thus, in the general case, we see that a generalized Lyapunov function is a kind of generalized section.

REMARK 3 To see why this notion generalizes the usual one, consider the system

$$\dot{x} = A(x)x$$

and suppose that $x^T \tilde{P}(x)x$ is an ordinary Lyapunov function for the equilibrium point 0. Then

$$x^T P(t; x_0)x = x^T \tilde{P}(x(t; x_0))x$$

is the 'generalized' Lyapunov function at x_0 where $x(t; x_0)$ is the solution of the system through x_0.

4. The spectrum of a nonlinear differential equation

As another example of the application of the approximation technique in Section 2, we shall develop a general spectral theory for nonlinear differential equations, which extends the linear theory. In Banks *et al.* (1997) we have given a general theory for nonlinear systems based on the Lie series. However, this only gives a local result and it is difficult to apply. The approximation technique will provide a much more effective approach to this problem. We begin by considering the linear, time-varying system

$$\dot{x}(t) = A(t)x(t), \ x(0) = x_0. \tag{4.1}$$

We denote by $X(\mathrm{i}\omega)$ the Fourier transform of the solution $x(t; x_0) \cdot u(t)$ of (4.1), where $u(t)$ is the step function; thus,

$$X(\mathrm{i}\omega) = \int_0^\infty x(t; x_0)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t$$

and

$$x(t; x_0) = \frac{1}{2\pi}\int_{-\infty}^\infty X(\mathrm{i}\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega.$$

We write (4.1) in the form

$$\dot{x}(t) = (A_0 + A'(t))x(t) \tag{4.2}$$

for some nominal time-invariant A_0. Then, taking Fourier transforms, we have

$$\mathrm{i}\omega X(\mathrm{i}\omega) - x_0 = \left(A_0 + \frac{1}{2\pi}\tilde{A}'(\mathrm{i}\omega)*\right)X(\mathrm{i}\omega) \tag{4.3}$$

where $\tilde{\cdot}$ denotes Fourier transform and $*$ denotes convolution. Consider the linear equation given by

$$(\Gamma X)(\omega) \stackrel{\Delta}{=} \mathrm{i}\omega X(\mathrm{i}\omega) - \left(\frac{1}{2\pi}\tilde{A}' * X\right)(\omega) = S(\omega)$$

for X in terms of S, i.e. in the time domain

$$\dot{x} - A(t)x = s(t), \;\; x(0) = 0.$$

Thus,

$$x(t) = \int_0^t \Phi(t;\tau)s(\tau)\mathrm{d}\tau$$

where Φ is the evolution operator generated by $A(t)$. We assume that

$$\|\Phi(t;\tau)\|_{\mathcal{L}(\mathbb{R}^n)} \leqslant C\mathrm{e}^{\theta(t-\tau)}$$

so that if $\varepsilon > \theta$, we have

$$\mathrm{e}^{-\varepsilon t}\,\|x(t)\| \leqslant C\int_0^t \mathrm{e}^{(-\varepsilon+\theta)(t-\tau)}\mathrm{e}^{-\varepsilon\tau}\,\|s(\tau)\|\,\mathrm{d}\tau. \tag{4.4}$$

Let $w = \mathrm{e}^{-\varepsilon t}$ and put $H_w \overset{\Delta}{=} L^2_w([0,\infty);\mathbb{R}^n)$ as the space of all measurable functions $x(t)$ such that

$$\|x\|_{H_w} = \|x\|_{L^2_w([0,\infty);\mathbb{R}^n)} \overset{\Delta}{=} \left(\int_0^\infty \mathrm{e}^{-2\varepsilon t}\,\|x(t)\|^2\,\mathrm{d}t\right)^{\frac{1}{2}} < \infty.$$

Then the Fourier transform $\mathcal{F}$ is an isomorphism of H_w onto $\tilde{H}_w$ (the homomorphic image of H_w under $\mathcal{F}$) and by Parseval's theorem,

$$\|x\|_{H_w} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X_\varepsilon^2(\mathrm{i}\omega)\mathrm{d}\omega\right)^{\frac{1}{2}} \overset{\Delta}{=} \|X_\varepsilon\|_{\tilde{H}_w}$$

where

$$X_\varepsilon(\mathrm{i}\omega) = \int_0^\infty \mathrm{e}^{-\varepsilon t}x(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t.$$

LEMMA 4.1 If

$$\|\Phi(t;\tau)\|_{\mathcal{L}(\mathbb{R}^n)} \leqslant C\mathrm{e}^{\theta(t-\tau)} \tag{4.5}$$

then Γ^{-1} is bounded by

$$\left\|\Gamma^{-1}\right\|_{\mathcal{L}(\tilde{H}_w)} \leqslant C\int_0^\infty \mathrm{e}^{(-\varepsilon+\theta)t}\mathrm{d}t = \frac{C}{\varepsilon-\theta}$$

(for $\varepsilon > \theta$).

Proof. We have seen in (4.4) that

$$\mathrm{e}^{-\varepsilon t}\,\|x(t)\| \leqslant CE * (\mathrm{e}^{-\varepsilon t}\,\|s(t)\|)$$

where $E(t) = \mathrm{e}^{(-\varepsilon+\theta)t}$. We recall Young's inequality: if $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$ then $f * g \in L^r(\mathbb{R}^n)$ where $1 \leqslant p, q, r \leqslant \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. If $r = 2$ and $q = 2$ then $p = 1$ and the result follows. □

In particular, if $A(t) = A_0$ (constant) and

$$\left\| e^{A_0(t-\tau)} \right\| \leqslant C e^{\theta(t-\tau)}$$

then Γ is given by

$$(\Gamma X)(\omega) = \mathrm{i}\omega X(\mathrm{i}\omega) - A_0 X(\mathrm{i}\omega)$$

and

$$\left\| \Gamma^{-1} \right\|_{\mathcal{L}(\tilde{H}_w)} \leqslant \frac{C}{\varepsilon - \theta}.$$

Hence, from (4.2), we have

$$\mathrm{i}\omega X(\mathrm{i}\omega) - x_0 = A_0 X(\mathrm{i}\omega) + \frac{1}{2\pi} \tilde{A}'(\mathrm{i}\omega) * X(\mathrm{i}\omega)$$

and so

$$\left(I - \frac{1}{2\pi} (\mathrm{i}\omega I - A_0)^{-1} \tilde{A}'(\mathrm{i}\omega) * \right) X(\mathrm{i}\omega) = \Gamma^{-1} x_0.$$

Now define the operator L on $\tilde{H}_w$ by

$$\begin{aligned}(LX)(\mathrm{i}\omega) &= \frac{1}{2\pi} (\mathrm{i}\omega I - A_0)^{-1} [\tilde{A}'(\mathrm{i}\omega) * X(\mathrm{i}\omega)] \\ &= \left(\frac{1}{2\pi} \Gamma^{-1} \tilde{A}'(\mathrm{i}\omega) * \right) X(\mathrm{i}\omega).\end{aligned}$$

Then,

$$\|LX\|_{\tilde{H}_w} \leqslant \frac{1}{2\pi} \left\| \Gamma^{-1} \right\|_{\mathcal{L}(\tilde{H}_w)} \left\| \tilde{A}'_{-\varepsilon} \right\|_{L^1(-\infty,\infty;\mathbb{R}^{n^2})} \|X\|_{\tilde{H}_w}$$

and so if

$$\frac{1}{2\pi} \frac{C}{\varepsilon - \theta} \left\| \tilde{A}'_{-\varepsilon} \right\|_{L^1(-\infty,\infty;\mathbb{R}^{n^2})} < 1$$

(where $A'_{-\varepsilon}(t) = e^{\varepsilon t} A'(t)$), we have

$$\|L\|_{\mathcal{L}(\tilde{H}_w)} < 1.$$

Hence, by the Neumann series, we have

$$\begin{aligned}X &= (I + L + L^2 + \cdots)(\Gamma^{-1} x(0)) \\ &= (I + L + L^2 + \cdots)((\mathrm{i}\omega I - A_0)^{-1} x(0)) \\ &= \Bigg((\mathrm{i}\omega I - A_0)^{-1} + \frac{1}{2\pi} (\mathrm{i}\omega I - A_0)^{-1} \int_{-\infty}^{\infty} \tilde{A}'(\mathrm{i}\omega')(\mathrm{i}(\omega - \omega')I - A_0)^{-1} \mathrm{d}\omega' \\ &\quad + \left(\frac{1}{2\pi}\right)^2 (\mathrm{i}\omega I - A_0)^{-1} \int_{-\infty}^{\infty} \tilde{A}'(\mathrm{i}\omega'')(\mathrm{i}(\omega - \omega'')I - A_0)^{-1} \int_{-\infty}^{\infty} \tilde{A}'(\mathrm{i}\omega') \cdot \\ &\quad (\mathrm{i}(\omega - \omega'' - \omega')I - A_0)^{-1} \mathrm{d}\omega' \mathrm{d}\omega'' + \cdots \Bigg) x(0).\end{aligned}$$

Hence we have proved the following theorem.

THEOREM 4.1 Given a linear, time-varying system of the form

$$\dot{x}(t) = (A_0 + A'(t))x(t), \ x(0) = x_0$$

for which the perturbation A' satisfies

$$\frac{1}{2\pi}\frac{C}{\varepsilon-\theta}\left\|\tilde{A}'_{-\varepsilon}\right\|_{L^1(-\infty,\infty;\mathbb{R}^{n^2})} < 1$$

where C, ε and θ are as above, the frequency spectrum of $x(t; x_0)$ is given by

$$X(\mathrm{i}\omega) = \sum_{k=0}^{\infty}\frac{1}{2\pi}\left((\mathrm{i}\omega I - A_0)^{-1}\tilde{A}'(\mathrm{i}\omega)*\right)^k(\mathrm{i}\omega I - A_0)^{-1}x(0).$$

Consider now the nonlinear system

$$\dot{x}(t) = (A_0 + A'(x(t)))x(t), \ x(0) = x_0 \tag{4.6}$$

and replace it by the sequence of linear, time-varying approximations

$$\dot{x}^{[i]}(t) = (A_0 + A'(x^{[i-1]}(t)))x^{[i]}(t), \ x^{[i]}(0) = x_0. \tag{4.7}$$

We know that this system of equations converges in $C([0, T]; \mathbb{R}^n)$ for any finite T for which (4.6) has a unique solution, by Theorem 2.1. Hence it converges in $L^2([0, T]; \mathbb{R}^n)$ and so, by Parseval's theorem and by a simple truncation argument similar to that in Section 3, we see that the spectrum $X^{[i]}(\mathrm{i}\omega)$ of $x^{[i]}(t; x_0)$ converges to the spectrum $X(\mathrm{i}\omega)$ of the solution $x(t; x_0)$ of (4.6) on $L^2([0, \Omega]; \mathbb{R}^n)$ for any finite Ω. Hence we have the following theorem.

THEOREM 4.2 The spectrum $X(\mathrm{i}\omega)$ $(= X(\mathrm{i}\omega; x_0)$, since it depends on $x_0)$ of the solution $x(t; x_0)$ of (4.6) is given by the limit (in $L^2([0, \Omega]; \mathbb{R}^n)$ for any finite Ω)

$$\lim_{k\to\infty} X^{[k]}(\mathrm{i}\omega)$$

where

$$X^{[k]}(\mathrm{i}\omega) = \sum_{k=0}^{\infty}\frac{1}{2\pi}\left((\mathrm{i}\omega I - A_0)^{-1}\tilde{A}'^{[k]}(\mathrm{i}\omega)*\right)^k(\mathrm{i}\omega I - A_0)^{-1}x(0),$$

where

$$\tilde{A}'^{[k]}(\mathrm{i}\omega) = \mathcal{F}(A'(x^{[k-1]}(t))).$$

REMARK 4 The spectrum of the system (4.6) can be regarded as a s section of the trivial Hilbert bundle $\mathbb{R}^n \times L^2([0, \infty]; \mathbb{R}^n)$ where $s(x_0) = X(\mathrm{i}\omega; x_0)$. The case of nontrivial bundles on a manifold M will be considered in a future paper.

5. Conclusions

In this paper we have studied a recently introduced approach to nonlinear dynamical systems based on a representation of the system as the limit of a sequence of linear, time-varying approximations which converge in the space of continuous functions to the solution of the nonlinear system, under a very mild local Lipschitz condition. The use of linear approximations allows us to apply the existing techniques for linear systems and we have seen how to generalize the ideas of stability and frequency spectrum to nonlinear systems. The method clearly has applications to many aspects of nonlinear systems theory for which there is a corresponding theory for linear, time-varying systems and in a future paper we shall study the generalization of exponential dichotomies to nonlinear dynamical systems.

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