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Stable controller design for T–S fuzzy systems based on Lie algebras

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Abstract

In this paper, we study the stability of fuzzy control systems of Takagi–Sugeno-(T–S) type based on the classical theory of Lie algebras. T–S fuzzy systems are used to model nonlinear systems as a set of rules with consequents of the type $\dot{x}(t) = A_l x(t) + B_l u(t)$. We conduct the stability analysis of such T–S fuzzy models using the Lie algebra L_A generated by the A_l matrices of these subsystems for each rule in the rule base. We first develop our approach of stability analysis for a commuting algebra L_A, where all the consequent state matrices A_l's commute. We then generalize our results to the noncommuting case. The basic idea here is to approximate the noncommuting Lie algebra with a commuting one, such that the approximation error is minimum. The results of this approximation are extended to the most general case using the Levi decomposition of Lie algebras. The theory is applied to the control of a flexible-joint robot arm, where we also present the decomposition procedure.

Keywords: Fuzzy systems; Stability; Lie algebras; Flexible-joint robot arm

1. Introduction

Fuzzy systems have been used as controllers in many applications due to their ability to use all sources of information from human experts, either numerical or linguistic. The most important issue in using fuzzy

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systems as controllers is the stability of these systems; thus, finding systematic ways for this stability analysis has been a major field of interest.

Stability analysis using Lyapunov's direct method is commonly used in the literature. Tanaka and Sugeno [5] have used a Takagi–Sugeno (T–S) fuzzy model, where the consequent parts of the rules form a set of linear models, and have examined the stability of such systems in terms of Lyapunov's direct method generating a sufficient condition for stability in terms of the existence of a common positive-definite matrix P for all the subsystems in the consequent parts of the rules. Cao et al. [1] have used the same model as in [5] together with a feedback control law for each linear subsystem, and using uncertain linear system theory, have determined a condition to guarantee the global stability of the closed-loop system. This global stability analysis is also based on the Lyapunov's method, but is less conservative than in the analysis of [5] since they have relaxed the condition on finding a common P matrix.

Wang et al. [9] have modelled a nonlinear plant using T–S fuzzy model and designed a controller using parallel distributed compensation scheme. The stability analysis in their work is based on the Lyapunov theorem; and they have turned the problem of finding the common P matrix into a linear matrix inequality (LMI) problem and used convex programming techniques for the solution. Park et al. [4] have developed a variety of LMI-based controller design methods, where the stability analysis depends on finding a common P matrix as in [9]. Zak [11] has proposed a Lyapunov-based method for the design of state feedback controllers that guarantee global stability for the systems modelled by T–S fuzzy systems.

Thathachar et al. [7] have shown the equivalence of stability properties of the fuzzy systems and linear time invariant switching systems, and carried out the stability analysis based on Lyapunov method. Cuesta et al. [2] have analyzed the complex behavior phenomena such as multiple equilibria and limit cycles in T–S fuzzy control systems via the use of MIMO frequency domain methods. They have presented a more general stability index to perform bifurcation analysis of fuzzy control systems. Kiriakidis et al. [3] have analyzed the T–S system with offset terms as a perturbed linear system, and derived a sufficient condition on the robust stability of the system against nonlinear perturbations to guarantee quadratic stability.

In this paper, we develop a novel approach for the stability analysis of T–S fuzzy systems using the Lie Algebra generated by the linear subsystems used in the T–S model. In general, stability analysis of such systems are performed using Lyapunov's method [5,9,11,7]. Such an analysis suffers from the difficulty of finding the common P matrix for each subsystem matrices. The use of Lie algebras has the advantage of relaxing the condition of finding the common P matrix for the subsystems. In [2], describing function method is used, and in the calculation of the describing function the method is experimental, not analytical. Our analysis provides analytical results that can easily be implemented.

Our approach is based on the Levi decomposition of the Lie algebra generated by the linear subsystems used in the T–S model. In this decomposition, Cartan subalgebra forms an Abelian subalgebra, where the matrices in this subalgebra are simultaneously diagonalizable by the same matrix. This has the advantage over the other analysis methods, which try to find a common diagonalizing matrix, developed in the literature. Once the Lie algebra is decomposed using Levi decomposition, we have a Cartan subalgebra that is Abelian. In order to derive the results for the general case, we start with two simpler cases. First, we consider commuting fuzzy systems as the simplest case, and then we generalize the results to noncommuting cases, where there is an Abelian algebra that can be used to approximate the noncommuting one. Finally, we give the results for the general case by using Levi decomposition.

The theory is applied to the control and stability of a flexible-joint robot arm for illustrative purposes. Section 2 provides a brief introduction to the fuzzy model we use, and introduces our approach to the stability analysis of commuting fuzzy systems, followed by the noncommuting cases. Section 3 illustrates the proposed methodology on the application example of the control of a flexible-joint robot arm. The Appendix provides a brief overview of Lie algebras.

2. Stability analysis

This section introduces our Lie algebra-based stability analysis of T–S fuzzy control systems where rules have consequents that can be represented in the form of Eq. (1) below. We thus begin by a review of such fuzzy systems; while subsequent subsections introduce our approach to the stability analysis, where we first start with a simpler case where all the consequent system matrices commute forming an Abelian Lie algebra. The results are then generalized to noncommuting system matrices.

2.1. Takagi–Sugeno fuzzy systems

The Takagi–Sugeno (T–S) fuzzy system used to model a nonlinear system has the following rule structure:

$$R^{(l)} : \text{IF } x_1 \text{ is } F_1^l \text{ and } \ldots \text{and } x_n \text{ is } F_n^l, \text{ THEN } \dot{x}(t) = A_l x(t) + B_l u(t), \tag{1}$$

where F_i^l $(i = 1, 2, \ldots, n)$ are the antecedent fuzzy sets, $x(t) = [x_1, x_2, \ldots, x_n]^{\mathrm{T}}$ the time-varying state vector, and $u(t)$ the control input. A_l and B_l are, respectively, the state and control input matrices of the linearized subsystem around the operation point of rule l.

The closed form of the output of the T–S fuzzy system with center average defuzzifier, product-inference rule and singleton fuzzifier is

$$\dot{x}(t) = \frac{\sum_{l=1}^{M} (A_l x(t) + B_l u(t))(\prod_{i=1}^{n} N_{F_i^l}(x_i))}{\sum_{l=1}^{M} (\prod_{i=1}^{n} N_{F_i^l}(x_i))}, \tag{2}$$

where $N_{F_i^l}(x_i)$'s are the membership functions assigned to the fuzzy sets in the antecedent parts of the rules, and M is the rule number. Letting $w_l(x(t)) = \prod_{i=1}^{n} N_{F_i^l}(x_i(t))$, Eq. (2) can be simplified as follows [9]:

$$\dot{x}(t) = \left[\frac{\sum_{l=1}^{M} (A_l w_l(x(t))}{\sum_{l=1}^{M} w_l(x(t))} \right] x(t) + \left[\frac{\sum_{l=1}^{M} (B_l w_l(x(t))}{\sum_{l=1}^{M} w_l(x(t))} \right] u(t). \tag{3}$$

For a simpler representation, $\mu_l(x)$ is defined to be $\frac{w_l(x(t))}{\sum_{l=1}^{M} w_l(x(t))}$. Then, Eq. (3) becomes

$$\dot{x}(t) = \left(\sum_{l=1}^{M} \mu_l(x(t)) A_l \right) x(t) + \left(\sum_{l=1}^{M} \mu_l(x(t)) B_l \right) u(t). \tag{4}$$

We use the closed form of Eq. (4) in the development of our approach for the Lie algebra-based stability analysis of T–S fuzzy control system, as introduced in the next subsections.

2.2. Commuting fuzzy systems

First, we assume not only that L_A is Abelian, so that all the A_i's commute, but also that all the A_i's are diagonalizable. Such matrices are generic, so this is not a particularly strong assumption, and the results easily generalize.

There exists a common diagonalizing matrix P, namely the modal matrix, in the singularity transformation of A_i:

$$P^{-1}A_i P = \Lambda_i,$$

where $\Lambda_i = diag(\lambda_1^i, \ldots, \lambda_n^i)$, λ_j^i being the eigenvalues of A_i. Hence, considering $y = P^{-1}x$, we transform Eq. (4) into

$$\dot{y}(t) = \left(\sum_{l=1}^{M} \mu_l(Py(t))\Lambda_l\right) y(t) + \left(\sum_{l=1}^{M} \mu_l(Py(t))P^{-1}B_l\right) u(t), \tag{5}$$

which in a more simplified representation, is

$$\dot{y}_i(t) = \alpha_i y_i(t) + \beta_i u(t), \quad 1 \leqslant i \leqslant n, \tag{6}$$

where

$$\alpha_i = \sum_{l=1}^{M} \mu_l(Py(t))\lambda_i^l, \quad \beta_i = \sum_{l=1}^{M} \mu_l(Py(t))(P^{-1}B_l)_i. \tag{7}$$

The subscript $(.)_i$ denotes the ith element of the corresponding vector $(.)$.

To achieve stability for the general system of the form in Eq. (6), our approach is to generate a control u which satisfies the inequalities

$$\alpha_i y_i^2(t) + \beta_i u(t) y_i(t) \leqslant -\frac{1}{2}\varepsilon y_i^2(t), \quad 1 \leqslant i \leqslant n \tag{8}$$

for some $\varepsilon > 0$, and the α_i's, β_i's and y_i's are real.

The reason in our selection of the control can be easily seen from the derivation that follows:

If $y_i(t) = 0$, then the inequality in Eq. (8) is readily satisfied, forming a trivial solution for the inequality where we can choose any u. If $y_i(t) \neq 0$, for α_i real if we multiply both sides of Eq. (6) by y_i we get an expression of y_i^2 that begins to act as a Lyapunov function:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} y_i^2(t) = y_i(t)\dot{y}_i(t) = \alpha_i y_i^2(t) + \beta_i u(t) y_i(t) \leqslant -\frac{1}{2}\varepsilon y_i^2(t) \tag{9}$$

so that,

$$y_i^2(t) \leqslant \mathrm{e}^{-\varepsilon t} y_i^2(0), \tag{10}$$

which shows that y_i's are stable.

The stability condition for $y_i \neq 0$ with real α_i's can be obtained by dividing Eq. (9) by y_i as

$$\begin{cases} \alpha_i y_i(t) + \beta_i u(t) \leqslant -\frac{1}{2}\varepsilon y_i(t) \text{ if } y_i(t) > 0, \\ \alpha_i y_i(t) + \beta_i u(t) \geqslant -\frac{1}{2}\varepsilon y_i(t) \text{ if } y_i(t) < 0, \end{cases} \tag{11}$$

for $1 \leqslant i \leqslant n$.

Remark. If some α_i's are complex, then

$$y_i(t) = \xi_i(t) + i\eta_i(t), \quad \alpha_i = a_i + ib_i. \tag{12}$$

Eq. (6) becomes:

$$\begin{aligned} \dot{\xi}_i &= a_i\xi_i - b_i\eta_i + (Re(\beta_i))u(t), \\ \dot{\eta}_i &= b_i\xi_i + a_i\eta_i + (Im(\beta_i))u(t). \end{aligned} \tag{13}$$

This time we choose u so that

$$\begin{aligned} & a_i\xi_i^2 - b_i\xi_i\eta_i + \xi_i(Re(\beta_i))u(t) \leqslant -\tfrac{1}{2}\varepsilon\xi_i^2, \\ & a_i\eta_i^2 + b_i\xi_i\eta_i + \eta_i(Im(\beta_i))u(t) \leqslant -\tfrac{1}{2}\varepsilon\eta_i^2. \end{aligned} \tag{14}$$

The above results lead to the following theorem.

Theorem 1. *The system in Eq.* (4) *with Abelian Lie algebra L_A is stabilizable* (*in the case of real eigenvalues and with a similar condition in the complex case*) *if the inequalities in Eq.* (11) (*and Eq.* (14) *for the complex case*) *are solvable for u, where* (A_l, B_l) *pairs in Eq.* (4) *form stabilizable pairs.*

In order to find a control u, the equations in 11 are directly solved. First, we chop the space into orthants, e.g. in 3 dimensions there are $2^3 = 8$ orthants. In general, there will be 2^n orthants. For $k \in \{1, 2, \ldots, 2^n\}$ the orthants are defined in the following form:

$$Q_k : \begin{cases} y_i > 0, \ i \in \{i_1, \ldots, i_{m_k}\} : \underline{m}_k, \\ y_i < 0, \ i \in \underline{n}\backslash\underline{m}_k, \end{cases} \tag{15}$$

where $\underline{n} = \{1, 2, \ldots, n\}$. In Q_k, the equations in 11 are as follows:

$$\begin{aligned} & \alpha_i y_i(t) + \beta_i u(t) \leqslant -\tfrac{1}{2}\varepsilon y_i(t), \ i \in \underline{m}_k, \\ & \alpha_i y_i(t) + \beta_i u(t) \geqslant -\tfrac{1}{2}\varepsilon y_i(t), \ i \in \underline{n}\backslash\underline{m}_k. \end{aligned} \tag{16}$$

We solve for u in Q_k directly:

$$\begin{aligned} & \left.\begin{aligned} u(t) &\leqslant \frac{(-\frac{1}{2}\varepsilon - \alpha_i)}{\beta_i}\, y_i(t), \beta_i > 0 \\ u(t) &\geqslant \frac{(-\frac{1}{2}\varepsilon - \alpha_i)}{\beta_i}\, y_i(t), \beta_i < 0 \end{aligned}\right\} i \in \underline{m}_k, \\ & \left.\begin{aligned} u(t) &\geqslant \frac{(-\frac{1}{2}\varepsilon - \alpha_i)}{\beta_i}\, y_i(t), \beta_i > 0 \\ u(t) &\leqslant \frac{(-\frac{1}{2}\varepsilon - \alpha_i)}{\beta_i}\, y_i(t), \beta_i < 0 \end{aligned}\right\} i \in \underline{n}\backslash\underline{m}_k. \end{aligned} \tag{17}$$

Eq. (17) gives a selection for u in each orthant Q_k, and will also define regions within Q_k where these are solvable.

Here, we will give a simple numerical example to illustrate how the above results apply.

Example. We consider a numerical example, where we assume that all the state matrices A_l's commute, i.e. they are diagonalizable with the same P matrix, and (A_l, B_l) pairs form stabilizable pairs. The system in this example is a second order system, and the rules are defined in the following form:

$$\begin{aligned} & R^{(l)} : \text{IF } x_1 \text{ is } F_1^l \text{ and } x_2 \text{ is } F_2^l, \text{ THEN} \\ & \dot{x}(t) = A_l x(t) + B_l u(t) \quad \text{for } l = 1, 2, 3, 4, \end{aligned} \tag{18}$$

where F_i^l are the fuzzy sets in the antecedent parts of the rules. The F_i^l are chosen to be 'small' and 'big' for each state, so we have four rules in the rule base as:

$$
\begin{aligned}
&R^{(1)}: \text{IF } x_1 \text{ is small and } x_2 \text{ is small, THEN } \dot{x}(t) = A_1x(t) + B_1u(t),\\
&R^{(2)}: \text{IF } x_1 \text{ is small and } x_2 \text{ is big, THEN } \dot{x}(t) = A_2x(t) + B_2u(t),\\
&R^{(3)}: \text{IF } x_1 \text{ is big and } x_2 \text{ is small, THEN } \dot{x}(t) = A_3x(t) + B_3u(t),\\
&R^{(4)}: \text{IF } x_1 \text{ is big and } x_2 \text{ is big, THEN } \dot{x}(t) = A_4x(t) + B_4u(t).
\end{aligned} \tag{19}
$$

The state and input matrices for the rules in Eq. (19) are

$$
\begin{aligned}
A_1 &= \begin{bmatrix} -2.8 & 3.6 \\ -2.4 & 3.8 \end{bmatrix}, \quad b_1 = \begin{bmatrix} -1.2 \\ -1.6 \end{bmatrix},\\
A_2 &= \begin{bmatrix} -6.2 & 8.4 \\ -5.6 & 9.2 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -0.6 \\ -0.8 \end{bmatrix},\\
A_3 &= \begin{bmatrix} -4.3 & 3.6 \\ -2.4 & 2.3 \end{bmatrix}, \quad b_3 = \begin{bmatrix} -0.3 \\ -0.4 \end{bmatrix},\\
A_4 &= \begin{bmatrix} -1.7 & 2.4 \\ -1.6 & 2.7 \end{bmatrix}, \quad b_4 = \begin{bmatrix} -0.9 \\ -1.2 \end{bmatrix}.
\end{aligned} \tag{20}
$$

The closed form for the fuzzy system is given in Eq. (4). The membership functions for the linguistic variables 'small' (S) and 'big' (B) are given by Eq. (21), and they are shown in Fig. 1.

$$
\begin{aligned}
S &: \begin{cases} -\dfrac{x}{4} + 0.5, & -2 \leqslant x < 2, \\ 0, & x \geqslant 2, \\ 1, & x < -2. \end{cases}\\
B &: \begin{cases} \dfrac{x}{4} + 0.5, & -2 \leqslant x < 2, \\ 1, & x > 2, \\ 0, & x \leqslant -2. \end{cases}
\end{aligned} \tag{21}
$$

It is easy to check that the matrices A_1, A_2, A_3, A_4 commute and are diagonalized by the matrix:

$$P = \begin{bmatrix} -0.8944 & -0.6 \\ -0.4472 & -0.8 \end{bmatrix}.$$

The diagonalized matrices and transformed vectors $P^{-1}b_l$ are

$$
\begin{aligned}
\Lambda_1 &= \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \quad P^{-1}b_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix},\\
\Lambda_2 &= \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}, \quad P^{-1}b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\\
\Lambda_3 &= \begin{bmatrix} -2.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad P^{-1}b_3 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix},\\
\Lambda_4 &= \begin{bmatrix} -0.5 & 0 \\ 0 & 1.5 \end{bmatrix}, \quad P^{-1}b_4 = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}.
\end{aligned} \tag{22}
$$

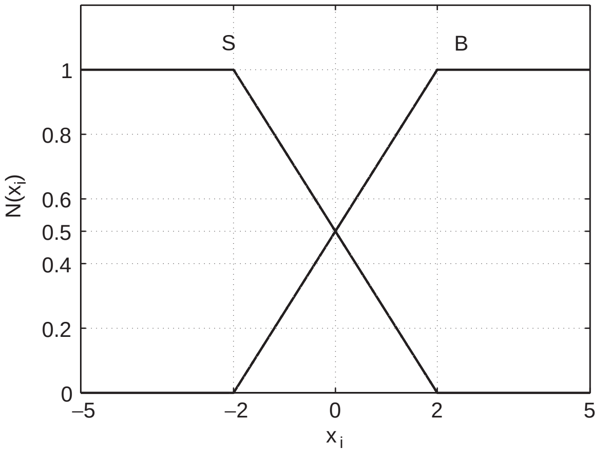

Fig. 1. Membership functions for Example.

The closed form of the fuzzy system after diagonalization is given by Eq. (5), where Λ_l's and $P^{-1}b_l$'s are shown in Eq. (22). In the design of the controller, we solve the equations in 17. We have $2^2 = 4$ orthants in this example. In the first orthant Q_1, we have $y_1 > 0$ and $y_2 > 0$. In this orthant, if $\beta_1 > 0$ and $\beta_2 > 0$, then:

$$\begin{aligned} u(t) &\leqslant \frac{(-\frac{1}{2}\varepsilon - \alpha_1)}{\beta_1}\, y_1(t), \\ u(t) &\leqslant \frac{(-\frac{1}{2}\varepsilon - \alpha_2)}{\beta_2}\, y_2(t). \end{aligned} \tag{23}$$

We choose u sufficiently small to satisfy the conditions in Eq. (23), in Q_1 when both $\beta_1 > 0$ and $\beta_2 > 0$. If $\beta_1 > 0$ and $\beta_2 < 0$, we must solve for:

$$\begin{aligned} u(t) &\leqslant \frac{(-\frac{1}{2}\varepsilon - \alpha_1)}{\beta_1}\, y_1(t), \\ u(t) &\geqslant \frac{(-\frac{1}{2}\varepsilon - \alpha_2)}{\beta_2}\, y_2(t). \end{aligned} \tag{24}$$

In this case, we choose u satisfying the above conditions. The rest of the conditions on u are derived similarly in all the four orthants. In the simulations, $\varepsilon = 1$, step size $h = 0.01$ and the initial conditions are $x = [3\ -4.5]^{\mathrm{T}}$. The results of the simulations are shown in Fig. 2 for the diagonalized states y_1, y_2, and the input u.

The simulation results for the system states x_1 and x_2 are given in Fig. 3.

2.3. Noncommuting fuzzy systems: general case

In this section, we generalize our results to the more generalized noncommuting case, i.e. where L_A generated by the linear subsystem matrices $\{A_1, A_2, \ldots, A_M\}$ of the rules (see Eq. 1) is not Abelian.

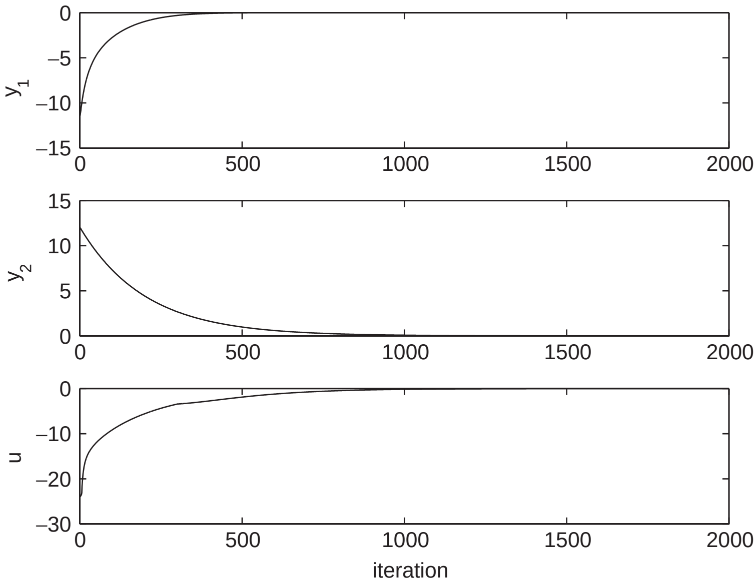

Fig. 2. Stabilized states (y) and control input for Example.

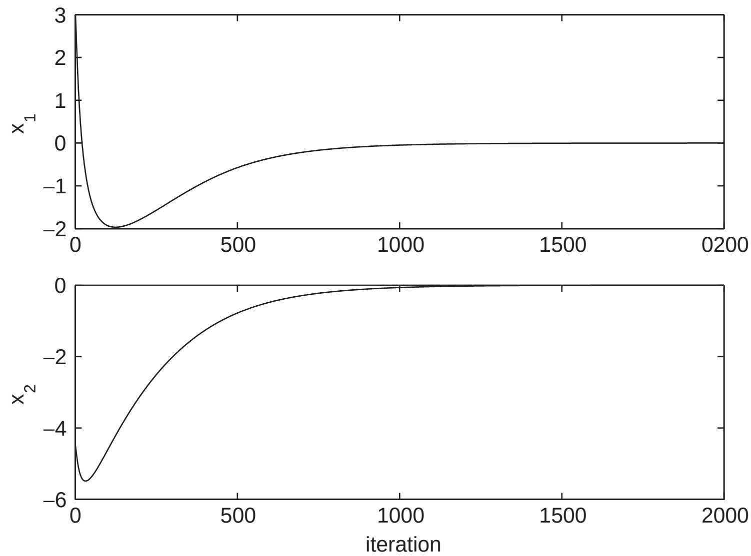

Fig. 3. Stabilized states (x) for Example.

First, we assume that there is an Abelian Lie algebra $L_{\tilde{A}}$ such that the error in the approximation of noncommuting system by this commuting Lie algebra is small. We give the results of this derivation in Theorem 2. Then, we generalize these results using the Levi decomposition reviewed in the Appendix. This decomposition guarantees to find an Abelian Lie algebra. The general results derived for this case are presented in Theorem 3.

Now, suppose that the Lie algebra L_A is not Abelian and consider the error of approximating the original set of matrices by the new commuting set of matrices $\{\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_M\}$:

$$\zeta = \max_{1 \leqslant l \leqslant M} \|A_l - \tilde{A}_l\| \tag{25}$$

and we aim at minimizing such an error. The matrix norm used in the equation, and in the rest of the paper is the induced infinity norm.

If we incorporate this approximation within our system given by Eq. (4), we get

$$\begin{aligned}\dot{x}(t) = {} & \left(\sum_{l=1}^{M} \mu_l(x(t))\tilde{A}_l\right) x(t) + \left(\sum_{l=1}^{M} \mu_l(x(t))B_l\right) u(t) \\ & + \left(\sum_{l=1}^{M} \mu_l(x(t))(A_l - \tilde{A}_l)\right) x(t).\end{aligned} \tag{26}$$

Hence, if $\tilde{P}$ is a common diagonalizing matrix for the set $\{\tilde{A}_1, \dots, \tilde{A}_M\}$, then we have

$$\tilde{P}^{-1}\tilde{A}_l\tilde{P} = \tilde{\Lambda}_l = diag(\tilde{\lambda}_1^l, \dots, \tilde{\lambda}_n^l) \tag{27}$$

and

$$\begin{aligned}\dot{y}(t) = {} & \left(\sum_{l=1}^{M} \mu_l(\tilde{P}y(t))\tilde{\Lambda}_l\right) y(t) + \left(\sum_{l=1}^{M} \mu_l(\tilde{P}y(t))\tilde{P}^{-1}B_l\right) u(t) \\ & + \left(\sum_{l=1}^{M} \mu_l(\tilde{P}y(t))\tilde{P}^{-1}(A_l - \tilde{A}_l)\tilde{P}\right) y(t).\end{aligned} \tag{28}$$

In a simpler and compact representation

$$\begin{aligned}\dot{y}_i(t) = {} & \tilde{\alpha}_i y_i(t) + \tilde{\beta}_i u(t) + \left\{\left[\sum_{k=1}^{M} \mu_k(\tilde{P}y(t))\tilde{P}^{-1}(A_k - \tilde{A}_k)\tilde{P}\right] y(t)\right\}_i, \\ & 1 \leqslant i \leqslant n,\end{aligned} \tag{29}$$

where

$$\tilde{\alpha}_i = \sum_{l=1}^{M} \mu_l(\tilde{P}y(t))\tilde{\lambda}_i^l, \quad \tilde{\beta}_i = \sum_{l=1}^{M} \mu_l(\tilde{P}y(t))(\tilde{P}^{-1}B_l)_i. \tag{30}$$

As before, since we are dealing with an approximating linear commuting system, we choose a control u for the real case such that

$$\sum_i (\tilde{\alpha}_i y_i^2(t) + \tilde{\beta}_i u(t) y_i(t)) \leqslant -\frac{1}{2}\varepsilon \sum_i y_i^2(t). \tag{31}$$

The solution for these equations are also given by Eq. (17), where α_i's should be replaced with $\tilde{\alpha}_i$'s and β_i's with $\tilde{\beta}_i$'s. We have in the case of real eigenvalues:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_i y_i^2(t) &= \sum_i y_i(t)\dot{y}_i(t)\\ &= \sum_i (\tilde{\alpha}_i y_i^2(t) + \tilde{\beta}_i u(t) y_i(t))\\ &\quad + \sum_i \left\{\left[\sum_{k=1}^{M} \mu_k(\tilde{P}y(t))\tilde{P}^{-1}(A_k - \tilde{A}_k)\tilde{P}\right] y(t)\right\}_i y_i(t)\\ &\leqslant -\frac{1}{2}\varepsilon\sum_i y_i^2(t) + \sum_i \left\{\left[\sum_{k=1}^{M} \mu_k(\tilde{P}y(t))\tilde{P}^{-1}(A_k - \tilde{A}_k)\tilde{P}\right] y(t)\right\}_i y_i(t). \end{aligned}\tag{32}$$

So,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|y(t)\|^2 \leqslant -\frac{1}{2}\varepsilon\|y(t)\|^2 + \|y(t)\|^2\|\tilde{P}^{-1}\|\cdot\|\tilde{P}\|\cdot\zeta, \tag{33}$$

ζ is the approximating error of Eq. (25).

We therefore have stability if

$$\|\tilde{P}^{-1}\|\cdot\|\tilde{P}\|\cdot\zeta < \tfrac{1}{2}\varepsilon. \tag{34}$$

Remark. If some $\tilde{\alpha}_i$'s are complex, then expressing

$$y_i(t) = \xi_i(t) + i\eta_i(t), \quad \tilde{\alpha}_i = \tilde{a}_i + i\tilde{b}_i, \tag{35}$$

we have

$$\begin{aligned}&\sum_i \tilde{a}_i\xi_i^2 - \tilde{b}_i\xi_i\eta_i + \xi_i Re(\tilde{\beta}_i)u(t)\\ &\quad + \sum_i Re\left\{\left[\sum_{k=1}^{M} \mu_k(\tilde{P}y(t))\tilde{P}^{-1}(A_k - \tilde{A}_k)\tilde{P}\right] y(t)\right\}_i \xi_i(t) \leqslant -\frac{1}{2}\varepsilon\sum_i \xi_i^2(t)\\ &\quad + \sum_i Re\left\{\left[\sum_{k=1}^{M} \mu_k(\tilde{P}y(t))\tilde{P}^{-1}(A_k - \tilde{A}_k)\tilde{P}\right] y(t)\right\}_i \xi_i(t),\\ &\quad \sum_i \tilde{a}_i\eta_i^2 + \tilde{b}_i\xi_i\eta_i + \eta_i Im(\tilde{\beta}_i)u(t)\\ &\quad + \sum_i Im\left\{\left[\sum_{k=1}^{M} \mu_k(\tilde{P}y(t))\tilde{P}^{-1}(A_k - \tilde{A}_k)\tilde{P}\right] y(t)\right\}_i \eta_i(t) \leqslant -\frac{1}{2}\varepsilon\sum_i \eta_i^2(t)\\ &\quad + \sum_i Im\left\{\left[\sum_{k=1}^{M} \mu_k(\tilde{P}y(t))\tilde{P}^{-1}(A_k - \tilde{A}_k)\tilde{P}\right] y(t)\right\}_i \eta_i(t). \end{aligned}\tag{36}$$

So,

$$\sum_i \tilde{a}_i \xi_i^2 - \tilde{b}_i \xi_i \eta_i + \xi_i Re(\tilde{\beta}_i) u(t)$$
$$+ \sum_i Re \left\{ \left[\sum_{k=1}^{M} \mu_k(\tilde{P} y(t)) \tilde{P}^{-1} (A_k - \tilde{A}_k) \tilde{P} \right] y(t) \right\}_i \xi_i(t) \leqslant -\frac{1}{2} \varepsilon \|\xi\|^2$$
$$+ \|\xi\|^2 \|\tilde{P}^{-1}\| \|\tilde{P}\| \zeta \sum_i \tilde{a}_i \eta_i^2 + \tilde{b}_i \xi_i \eta_i + \eta_i Im(\tilde{\beta}_i) u(t)$$
$$+ \sum_i Im \left\{ \left[\sum_{k=1}^{M} \mu_k(\tilde{P} y(t)) \tilde{P}^{-1} (A_k - \tilde{A}_k) \tilde{P} \right] y(t) \right\}_i \eta_i(t) \leqslant -\frac{1}{2} \varepsilon \|\eta\|^2$$
$$+ \|\eta\|^2 \|\tilde{P}^{-1}\| \|\tilde{P}\| \zeta. \tag{37}$$

We again have stability if

$$\|\tilde{P}^{-1}\| \cdot \|\tilde{P}\| \cdot \zeta < \tfrac{1}{2} \varepsilon. \tag{38}$$

The above results form the proof of the following generalizing theorem.

Theorem 2. *The system in Eq.* (4) *with Lie algebra L_A is stabilizable* (*in the case of real eigenvalues—with a similar condition in the complex case*) *if there is an Abelian Lie algebra $L_{\tilde{A}}$ such that inequalities* (31) *are solvable for u and inequality* (34) *is satisfied where $\zeta = max_{1 \leqslant l \leqslant M} \|A_l - \tilde{A}_l\|$, and $(\tilde{A}_l, B_l)$ pairs in Eq.* (26) *form stabilizable pairs.*

We will now consider the general case using the Levi decomposition given by Eq. (62). For any choice of the Levi subalgebra *m* and any choice of Cartan subalgebra h_m of *m* (for review of these concepts, please refer to the Appendix), the subsystem matrices A_l's of Eq. (4) can be written as

$$A_l = A_{l(r)} + \bar{A}_{l(h_m)} + \sum_{\varphi \in \Sigma} A_{l(m^{\varphi})}, \tag{39}$$

where $A_{l(r)} \in r$, $\bar{A}_{l(h_m)} \in h_m$ and $A_{l(m^{\varphi})} \in m^{\varphi}$ $(\varphi \in \Sigma)$. Note that the set of matrices $\{\bar{A}_{1(h_m)}, \ldots, \bar{A}_{M(h_m)}\}$ is commutative.

If we now apply Theorem 2 with $\tilde{A}_l = \bar{A}_{l(h_m)}$, then the system is stable and this result can be put immediately in the form of a theorem.

Theorem 3. *The system in Eq.* (4) *with Lie algebra L_A is stabilizable* (*in the case of real eigenvalues and with a similar condition in the complex case*) *if there is a Levi and Cartan decomposition of L_A such that inequalities* (31) *are solvable for u, inequality* (34) *is satisfied where $\zeta = max_{1 \leqslant l \leqslant M} \|A_l - \bar{A}_{l(h_m)}\|$, $\tilde{P}$ diagonalizes the Cartan subalgebra of L_A, and $(\bar{A}_{l(h_m)}, B_l)$ pairs form stabilizable pairs.*

3. Application example

In this section, we apply the results to the control of a flexible-joint robot arm system. First, we represent the system as a T–S fuzzy system, modelling our system with rules having linear subsystems as consequents. We then discuss a systematic way on how to find a Levi decomposition of the Lie algebra L_A generated by the A matrices of the linear subsystems of this model. In this decomposition, the matrices in the semisimple part should form a stabilizable pair with the B_l matrices of the system. Finally, we design the controller so that the system is stable.

3.1. T–S representation of the system

The flexible-joint robot arm system used in this paper is shown in Fig. 4.

The system is described by the following equations [6]:

$$\begin{aligned} &I_1\ddot{\theta}_1 + mgl\ \sin(\theta_1) + k(\theta_1 - \theta_2) = 0, \\ &I_2\ddot{\theta}_2 + k(\theta_2 - \theta_1) = u. \end{aligned} \tag{40}$$

In the equations, u is the torque input, I_1 the link inertia, I_2 the motor inertia, m the mass, g the gravity constant, l the link length, k the stiffness, θ_1 joint1 angular position, and θ_2 joint2 angular position.

The state equations of the system are:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{mgl}{I_1}\ \sin(x_1) + \frac{k}{I_1}(x_3 - x_1), \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= \frac{k}{I_2}(x_1 - x_3) + \frac{u}{I_2}, \end{aligned} \tag{41}$$

where $x_1 = \theta_1$, $x_2 = \dot{\theta}_1$, $x_3 = \theta_2$ and $x_4 = \dot{\theta}_2$.

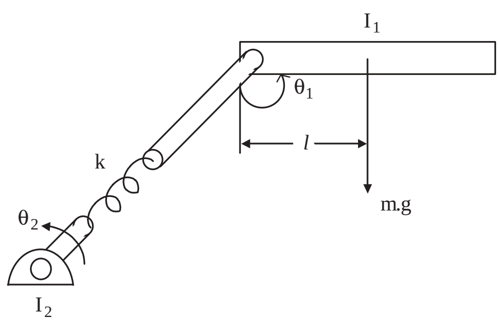

Fig. 4. Flexible-joint robot arm system.

First, we put the system into the $\dot{x} = A(x)x + B(x)u$ form. This can easily be achieved using the state equations in (41):

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{mgl}{I_1}\dfrac{\sin(x_1)}{x_1} - \dfrac{k}{I_1} & 0 & \dfrac{k}{I_1} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k}{I_2} & 0 & -\dfrac{k}{I_2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{I_2} \end{bmatrix} u, \tag{42}$$

where $x = [x_1, x_2, x_3, x_4]^{\mathrm{T}}$.

The only nonlinear term in $A(x)$ is $sinc(x_1) = \frac{\sin(x_1)}{x_1}$, so when we form the rule base, we use the exact value of the *sinc* function at the operation point of the rules. According to this, the rules acquire a linear form around the point of operation:

$R^{(l)}$: IF x_1 is F_1^l, THEN

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{mgl}{I_1}\dfrac{\sin(x_1)}{x_1}\Big|_{x_1^l} - \dfrac{k}{I_1} & 0 & \dfrac{k}{I_1} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k}{I_2} & 0 & -\dfrac{k}{I_2} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{I_2} \end{bmatrix} u(t). \tag{43}$$

We choose two rules for the system as follows:

$R^{(1)}$: IF x_1 is around 0, THEN

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{mgl}{I_1} - \dfrac{k}{I_1} & 0 & \dfrac{k}{I_1} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k}{I_2} & 0 & -\dfrac{k}{I_2} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{I_2} \end{bmatrix} u(t), \tag{44}$$

$R^{(2)}$: IF x_1 is around $\mp\frac{\pi}{2}$, THEN

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{mgl}{I_1}\dfrac{2}{\pi} - \dfrac{k}{I_1} & 0 & \dfrac{k}{I_1} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k}{I_2} & 0 & -\dfrac{k}{I_2} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{I_2} \end{bmatrix} u(t). \tag{45}$$

The system parameters are taken to be $m = 0.01\,\mathrm{kg}$, $I_1 = I_2 = 1\,\mathrm{kg\,m}^2$, $k = 0.05\,\mathrm{N.m/rad}$, $l = 1\,\mathrm{m}$ and $g = 9.81\,\mathrm{m/s}^2$ for illustrative purposes, so the system matrices become

$$A_l = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.0981\dfrac{\sin(x_1)}{x_1}\Big|_{x_1^l} - 0.05 & 0 & 0.05 & 0 \\ 0 & 0 & 0 & 1 \\ 0.05 & 0 & -0.05 & 0 \end{bmatrix}, \quad B_l = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{46}$$

If we set $c_l = -0.0981 \frac{\sin(x_1)}{x_1}|_{x_1^l} - 0.05$, then $A_l = \begin{bmatrix} 0 & 1 & 0 & 0 \\ c_l & 0 & 0.05 & 0 \\ 0 & 0 & 0 & 1 \\ 0.05 & 0 & -0.05 & 0 \end{bmatrix}$.

Next, we define how to decompose the Lie algebra generated by these A_l matrices.

3.2. Decomposition procedure

In order to decompose the A_l matrix into semisimple and solvable bits, we need to look at the constant matrices that would generate A_l. The choice of these matrices forms the first step of the procedure. It should be noted that the choice of these matrices are not unique.

We start with an arbitrary decomposition of the A_l matrix as follows:

$$A_l = \underbrace{c_l \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A_{l1}} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \\ 0 & 0 & 0 & 1 \\ 0.05 & 0 & -0.05 & 0 \end{bmatrix}}_{A_{l2}}. \tag{47}$$

We now consider the Lie algebra L_A generated by the matrices $\{A_{l1}, A_{l2}\}$. In order to find the Levi decomposition of L_A, we need to separate the semisimple bit and the rest will form the solvable part. From the stability point of view, the semisimple bit can be considered to contain the controllable modes of the system and the solvable bit to have the uncontrollable modes, so we want the solvable bit to be small enough to be able to stabilize the system by stabilizing the semisimple bit.

The procedure for this decomposition is based on the Cartan's criterion (see Theorem A.4), and follow the four steps described below.

Step 1: Form the basis for the Lie algebra: In order to do this, first find the basis elements for L_A generated by $\{A_{l1}, A_{l2}\}$. Then, find the commutators ($[G, H] = GH - HG$ is the commutator of two matrices G, H) of the basis elements, and form the set of matrices composed of the basis elements and the commutators. Finally, find the basis elements of this new set of matrices, which form the Lie algebra basis.

Step 2: Obtain the Killing form: The calculation of the Killing form is illustrated by an example in the Appendix.

Step 3: Obtain the Killing matrix, *K*. The Killing matrix *K* is found by the Killing form equality $kf = gKh^{\mathrm{T}}$, where $g = [g_1, g_2, \ldots, g_n]$, $h = [h_1, h_2, \ldots, h_n]$, *n* is the dimension of the Lie algebra L_A and k_{ij}'s are the coefficients of the terms $g_i h_j$ in the Killing form such that $k_{ij} = k_{ji}$. Diagonalize the Killing matrix *K*, i.e. $\Lambda_K = P^{-1}KP$, where *P* is the modal matrix. Transform the basis elements by the same *P*. The semisimple bit is taken to be the transformed basis elements corresponding to the large eigenvalues of the matrix *K*. This is deduced from the fact that eigenvectors corresponding to eigenvalues that are close to zero result in a degenerate Killing form, meaning that the corresponding basis elements belong to the solvable part.

Step 4: Decompose A_l with respect to the transformed basis. This gives the semisimple part (that corresponds to the basis elements for the semisimple part found in Step 3) and the solvable part (that corresponds to the rest of the basis elements).

When we apply this procedure to our example system, we obtain the following computational results:
Step 1: The Lie algebra basis for A_{l1}, A_{l2} is

$$L_{\text{basis}} = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \\ 0 & 0 & 0 & 1 \\ 0.05 & 0 & -0.05 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$
$$\begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & -0.05 & 0 \\ 0 & 0 & 0 & 0 \\ -0.05 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -0.15 & 0 \\ 0 & 0 & 0 & 0.05 \\ -0.05 & 0 & 0 & 0 \\ 0 & 0.15 & 0 & 0 \end{bmatrix},$$
$$\begin{bmatrix} 0 & -4 & 0 & 0 \\ 0 & 0 & -0.05 & 0 \\ 0 & 0 & 0 & 0 \\ -0.05 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0.2 \\ -0.005 & 0 & 0.0025 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0.0025 & 0 & -0.015 & 0 \end{bmatrix},$$
$$\begin{bmatrix} 0 & 0 & 0.15 & 0 \\ 0 & 0 & 0 & 0.05 \\ -0.05 & 0 & 0 & 0 \\ 0 & -0.15 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0.1 \\ 0.005 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.015 & 0 \end{bmatrix},$$
$$\left. \begin{bmatrix} -0.015 & 0 & 0.0125 & 0 \\ 0 & 0.015 & 0 & -0.0025 \\ 0.0025 & 0 & -0.025 & 0 \\ 0 & -0.0125 & 0 & 0.025 \end{bmatrix} \right\}$$

Step 2: The Killing form for this example is

$$\begin{aligned} kf = {}& 12h_4g_1 - 24h_6g_1 + 0.18h_5g_5 - 0.18h_4g_7 - 0.18h_9g_6 + 0.18h_9g_2 - 0.18h_7g_4 \\ & +0.003h_7g_9 + 0.012h_7g_7 - 0.18h_8g_8 + 0.18h_{10}g_3 + 0.01095h_{10}g_{10} - 0.003h_{10}g_8 \\ & +0.003h_9g_7 - 0.18h_6g_9 + 6h_2g_1 - 0.6h_2g_2 + 12h_3g_3 - 0.012h_{10}g_5 \\ & -0.012h_5g_{10} + 6h_1g_2 + 12h_1g_4 - 24h_1g_6 - 0.003h_8g_{10} + 0.18h_3g_{10} + 0.18h_2g_9. \end{aligned} \tag{48}$$

Step 3: The Killing matrix is

$$K_M = [\, k_1 \quad k_2 \quad k_3 \quad k_4 \quad k_5 \quad k_6 \quad k_7 \quad k_8 \quad k_9 \quad k_{10} \,], \tag{49}$$

where $k_1 = (0, 6, 0, 12, 0, -24, 0, 0, 0, 0)^{\mathrm{T}}$, $k_2 = (6, -0.6, 0, 0, 0, 0, 0, 0, 0.18, 0)^{\mathrm{T}}$, $k_3 = (0, 0, 12, 0, 0, 0, 0, 0, 0, 0.18)^{\mathrm{T}}$, $k_4 = (12, 0, 0, 0, 0, 0, -0.18, 0, 0, 0)^{\mathrm{T}}$, $k_5 = (0, 0, 0, 0, 0.18, 0, 0, 0, 0, -0.012)^{\mathrm{T}}$, $k_6 = (-24, 0, 0, 0, 0, 0, 0, 0, -0.18, 0)^{\mathrm{T}}$, $k_7 = (0, 0, 0, -0.18, 0, 0, 0.012, 0, 0.003, 0)^{\mathrm{T}}$, $k_8 = (0, 0, 0, 0, 0, 0, 0, -0.18, 0, -0.003)^{\mathrm{T}}$, $k_9 = (0, 0.18, 0, 0, 0, -0.18, 0.003, 0, 0, 0)^{\mathrm{T}}$, and $k_{10} = (0, 0, 0.18, 0, -0.012, 0, 0, -3 \times 10^{-3}, 0, 12 \times 10^{-3})^{\mathrm{T}}$. When we diagonalize K_M, we get:

$$\begin{aligned} \Lambda_{K_M} = {}& diag(-0.1678, 0.0232, 0.1906, -0.6054, 0.18, \\ & 0.0075, 0.1808, -27.5109, 27.4823, 12.0027). \end{aligned} \tag{50}$$

The modal matrix P is

$$P = [\, p_1 \quad p_2 \quad p_3 \quad p_4 \quad p_5 \quad p_6 \quad p_7 \quad p_8 \quad p_9 \quad p_{10}\,], \tag{51}$$

$p_1 = (6\times10^{-4}, 0.11, 0, 0.64, 0, 0.35, 0.63, 0, 0.24, 0)^{\mathrm{T}}$, $p_2 = (6.6\times10^{-3}, -0.19, 0, -0.04, 0, -0.07, 0.44, 0, -0.87, 0)^{\mathrm{T}}$, $p_3 = (-4\times10^{-4}, 0.08, 0, -0.62, 0, -0.29, 0.63, 0, 0.36, 0)^{\mathrm{T}}$, $p_4=(6\times10^{-3}, 0.95, 0, -0.13, 0, 0.17, -0.04, 0, -0.23, 0)^{\mathrm{T}}$, $p_5 = (0, 0, 2\times10^{-4}, 0, -5\times10^{-4}, 0, 0, -1, 0, -0.02)^{\mathrm{T}}$, $p_6 = (0, 0, 0.01, 0, -0.07, 0, 0, 0.02, 0, -0.99)^{\mathrm{T}}$, $p_7 = (0, 0, 1\times10^{-3}, 0, 1, 0, 0, 6\times10^{-4}, 0, -0.07)^{\mathrm{T}}$, $p_8 = (0.71, -0.16, 0, -0.31, 0, 0.62, -2\times10^{-3}, 0, 5\times10^{-3}, 0)^{\mathrm{T}}$, $p_9 = (0.71, 0.15, 0, 0.31, 0, -0.62, 0, 0, 5\times10^{-3}, 0)^{\mathrm{T}}$, $p_{10} = (0, 0, -1, 0, 0, 0, 0, 0, 0, -0.01)^{\mathrm{T}}$.

When we compare the eigenvalues, we conclude that apart from the last three, the others can be considered as 0, so we take the last three transformed basis elements as the semisimple part.

The three basis elements after transformation become:

$$T_{\text{basis}} = \left\{ \begin{bmatrix} 0 & -3.2412 & 0 & 0.0001 \\ 0.7069 & 0 & -0.0233 & 0 \\ 0 & 0.0001 & 0 & -0.1576 \\ -0.0233 & 0 & 0.0080 & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 0 & 3.2395 & 0 & 0.0001 \\ 0.7073 & 0 & 0.023 & 0 \\ 0 & 0.0001 & 0 & 0.1511 \\ 0.023 & 0 & -0.0075 & 0 \end{bmatrix}, \begin{bmatrix} 1.0001 & 0 & -0.0002 & 0 \\ 0 & -1.0001 & 0 & 0 \\ 0 & 0 & 0.0004 & 0 \\ 0 & 0.0002 & 0 & -0.0004 \end{bmatrix} \right\}. \tag{52}$$

The transformation of all the basis elements is performed by $New_{\text{basis}} = Pe^{\mathrm{T}}$, where $e = [e_1, e_2, \ldots, e_{10}]$ are the basis elements. In T_{basis}, the last three of the transformed basis elements are taken.

For stability, each of the three matrices should form a stabilizable pair with the B_l in Eq. (46). However, we see that the last matrix violates this condition, i.e. it does not form a stabilizable pair with B_l. To solve this problem, one way is to find a similarity transformation so that the result is a stabilizable set of matrices, but this is not an easy task.

Another way is to start with a different decomposition of A_l than the one in Eq. (47). Antisymmetric matrices form semisimple Lie algebras, whereas a lower or upper triangular matrix forms a solvable Lie algebra. Using this fact, we decompose A_l into its antisymmetric and lower triangular matrices, which form the semisimple and solvable parts, respectively. The decomposition is as follows:

$$A_l = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0.05 & 0 \\ 0 & -0.05 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}}_{A_{l1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ c_l+1 & 0 & 0 & 0 \\ 0 & 0.05 & 0 & 0 \\ 0.05 & 0 & 0.95 & 0 \end{bmatrix}}_{A_{l2}}. \tag{53}$$

The first part of this matrix decomposition, i.e. A_{l1}, is certainly one dimensional and therefore Abelian, and also a simple algebra. We can think of this one-dimensional Abelian Lie algebra as a kind of ‘degenerate’ semisimple algebra (we cannot directly say that it is semisimple because one-dimensional algebras are not semisimple). It is Abelian, so the Cartan subalgebra is the whole algebra and the roots are all zero.

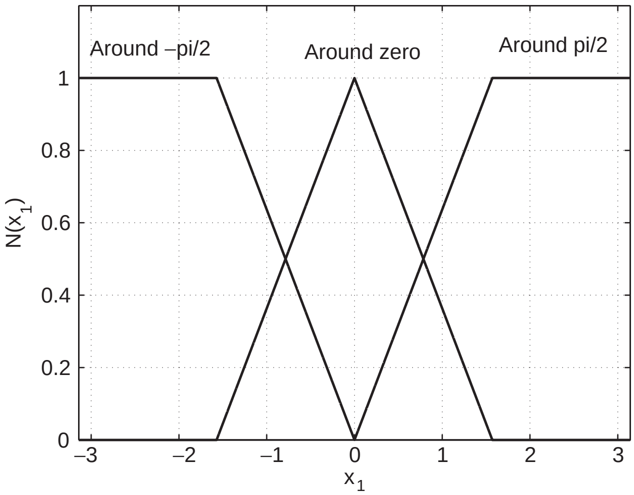

Fig. 5. Membership functions for the Application Example.

The second part, i.e. A_{l2}, generates a solvable Lie algebra for different c_l's, since they are lower triangular matrices. Since we have found a direct decomposition of the system into semisimple and solvable bits, we do not need to apply the decomposition steps here. This concludes our decomposition of A_l, and we see that A_{l1} forms a stabilizable pair with B_l.

After decomposition, the rules in Eqs. 44 and 45 become:

$R^{(1)}$: IF x_1 is around 0, THEN

$$\dot{x}(t) = \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0.05 & 0 \\ 0 & -0.05 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.8519 & 0 & 0 & 0 \\ 0 & 0.05 & 0 & 0 \\ 0.05 & 0 & 0.95 & 0 \end{bmatrix} \right) x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t). \qquad (54)$$

$R^{(2)}$: IF x_1 is around $\mp\frac{\pi}{2}$, THEN

$$\dot{x}(t) = \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0.05 & 0 \\ 0 & -0.05 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.8875 & 0 & 0 & 0 \\ 0 & 0.05 & 0 & 0 \\ 0.05 & 0 & 0.95 & 0 \end{bmatrix} \right) x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t). \qquad (55)$$

The membership functions for x_1 are given in Eq. (56) and Fig. 5.

$$\text{Around zero}: \begin{cases} \frac{2x}{\pi} + 1, & -\frac{\pi}{2} \leqslant x < 0, \\ -\frac{2x}{\pi} + 1, & 0 \leqslant x \leqslant \frac{\pi}{2}, \\ 0, & \text{o/w}, \end{cases}$$

$$\text{Around } \mp\frac{\pi}{2}: \begin{cases} \frac{2x}{\pi}, & 0 \leqslant x \leqslant \frac{\pi}{2}, \\ -\frac{2x}{\pi}, & -\frac{\pi}{2} \leqslant x < 0, \\ 1, & \text{o/w}. \end{cases} \qquad (56)$$

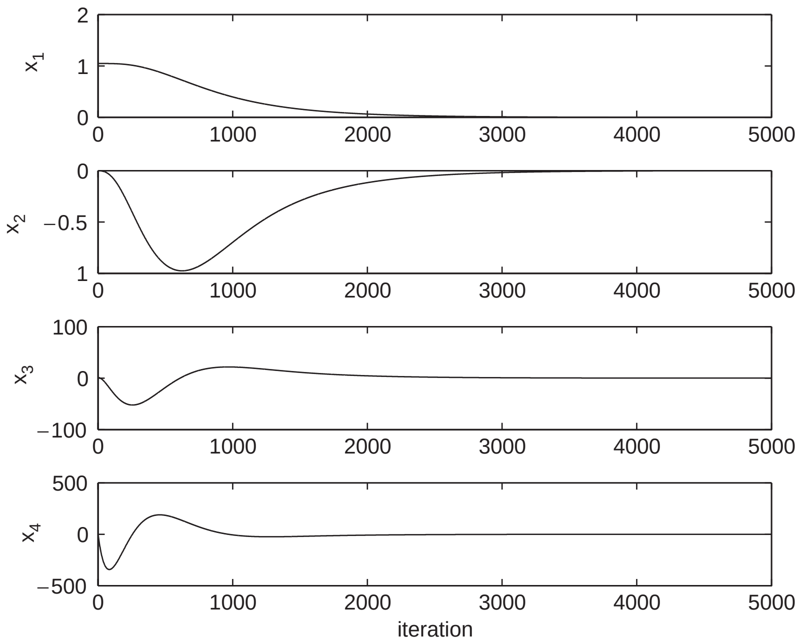

Fig. 6. The states of the controlled system.

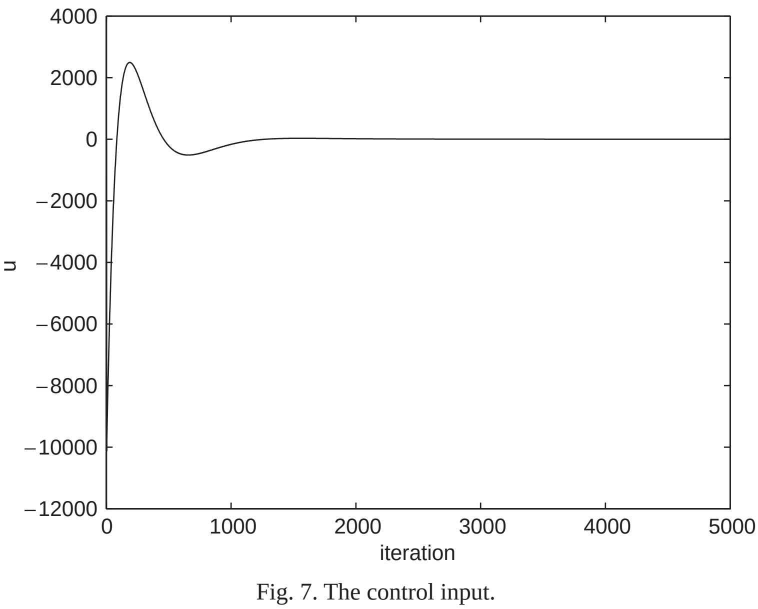

Fig. 7. The control input.

For stability, we need to check the condition in Eq. (34), that is $\|\tilde{P}^{-1}\| \cdot \|\tilde{P}\| \cdot \zeta < \frac{1}{2}\varepsilon$. For our application example, $\tilde{P} = \begin{bmatrix} 0.4937j & -0.4937j & -0.5062j & 0.5062j \\ -0.5062 & -0.5062 & 0.4937 & 0.4937 \\ -0.5062j & 0.5062j & -0.4937j & 0.4937j \\ 0.4937 & 0.4937 & 0.5062 & 0.5062 \end{bmatrix}$, $\zeta = 1$, so when $\varepsilon = 10$ is chosen the stability is guaranteed. We have the same semisimple part for both rules 1 and 2, and we design a controller to control this semisimple part. This satisfies the criterion in Eq. (31). The closed form of the system is given by Eq. (4). The controller is of the form: $u = -Kx(t)$ where $K = \begin{bmatrix} 9520 & 9599 & 148 & 20 \end{bmatrix}$. Fig. 6 shows an example of stabilized states for initial condition $(\pi/3, 0, \pi/3, 0)$. The control input is shown in Fig. 7.

4. Conclusion

We have developed a novel approach for the stability analysis of the T–S fuzzy systems using Lie algebra L_A generated by the consequent subsystem matrices of the T–S fuzzy rules. We have first considered the case where L_A is Abelian, then we have generalized our results to noncommuting cases. We have applied the theory to the control of flexible-joint robot arm system after modelling it as a T–S fuzzy system, and designed a controller using the developed theory. We have seen that the results demonstrate the successful stability performance of the designed controller after being concluded as stabilizable by our developed Lie algebra-based stability analysis approach.

Appendix A. Lie algebras

In this overview appendix, we go through the basic definitions and results from Lie algebra theory that we make use in our approach, but omitting their proofs, which can be found in [8].

Definition A.1. A Lie algebra g is a vector space which has a bilinear product map $[.,.]$: $g \times g$ satisfying:

(i) $[X, Y] = -[Y, X]$ for all $X, Y \in g$,
(ii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in g$.
Two elements $X, Y \in g$ satisfying $[X, Y] = 0$ are said to be commuting, and g naturally becomes an Abelian Lie algebra.

Definition A.2. A subspace h of g is said to be a subspace if $[h, h] \subseteq h$, i.e. $X, Y \in h$ implies that $[X, Y] \in h$.

Definition A.3. An ideal h in a Lie algebra g is a subspace such that $[h, g] \subseteq h$, where $[h, g]$ denotes the subspace spanned by the set of all elements of the form $[X, Y]$, $X \in h$, $Y \in g$. An ideal h in g is minimal if $\{0\}$ is the only ideal of g contained in h.

$h_1 + h_2$ denotes the subspace spanned by all elements of the form $X + Y$, $X \in h_1$, $Y \in h_2$, for any subsets $h_1, h_2 \subseteq g$. If $h \in g$ is an ideal, then g/h denotes the quotient Lie algebra which is the quotient of the vector spaces g and h with brackets $[\overline{X}, \overline{Y}] = \overline{[X, Y]}$, $X, Y \in g$, where $\overline{X}$ is the coset of X. The projection map $g \to g/h$ is a homomorphism of Lie algebras with kernel h. A homomorphism of Lie algebras is a homomorphism of the underlying vector spaces which preserves the brackets.

Definition A.4. A Lie algebra g is said to be simple if g and $\{0\}$ are the only ideals of g.

Definition A.5. If $g = g_1 \oplus g_2 \oplus \cdots \oplus g_k$ (vector space direct sum) and each g_h is an ideal, then g is called the direct sum of $g_1, \ldots, g_k$. For $i \neq j$, $g_i \cap g_j = [g_i, g_j] = \{0\}$.

Definition A.6. The ideal $\mathscr{D}g = [g, g]$ of a Lie algebra g is called the derived algebra of g. The derived series of g is

$$g \supseteq \mathscr{D}g \supseteq \mathscr{D}^2 g \supseteq \cdots \supseteq \mathscr{D}^n g \supseteq \cdots,$$

where $\mathscr{D}^n g = \mathscr{D}(\mathscr{D}^{n-1} g)$. Each term in the series is ideal.

Definition A.7. If $\mathscr{D}^k g = \{0\}$ for some $k > 0$, then g is said to be a solvable Lie algebra.

Definition A.8. If g does not contain any solvable ideal apart from $\{0\}$, then g is said to be semisimple.

Theorem A.1. *Every semisimple Lie algebra is the direct sum of all its minimal ideals.*

Theorem A.2. *Every Lie algebra g has a unique maximal solvable ideal r called the radical of g. Then g/r is semisimple.*

Another important class of Lie algebras is the nilpotent class.

Definition A.9. If g is a Lie algebra, let $C^{(0)}g = g$, $C^{(1)}g = [g, C^{(0)}g], \ldots, C^{(n+1)}g = [g, C^{(n)}g], \ldots$. Then, all $C^{(n)}g$ $(n = 0, 1, 2, \ldots)$ are ideals of g. The descending central series is obtained as:

$$C^{(0)}g \supseteq C^{(1)}g \supseteq \cdots \supseteq C^{(n)}g \supseteq \cdots$$

If $C^{(k)}g = 0$ for some $k > 0$, then g is called nilpotent. It can be proved that $\mathscr{D}^n g \subseteq C^{(n)}g$ for each n, so if g is nilpotent, then it is solvable. In fact, g is solvable if and only if $\mathscr{D}g$ is nilpotent.

The adjoint map “ad” is an important linear operator acting on any Lie algebra g, and is defined for each $X \in g$ as:

$$(adX)Y = [X, Y].$$

Using this map, a geometric structure on a Lie algebra is defined in terms of a symmetric bilinear form (., .) called the Killing form. The Killing form is defined to be:

$$(X, Y) = Tr(adXadY)$$

Example (*Calculation of Killing form*). Consider the three matrices M_1, M_2 and M_3 as follows:

$$M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{57}$$

It is clear that this is a three-dimensional Lie algebra g_3 and consists of all skew-symmetric matrices:

$$\begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} = x_1 M_1 + x_2 M_2 + x_3 M_3. \tag{58}$$

For any $X \in g_3$, $X = x_1M_1 + x_2M_2 + x_3M_3$, then: $(adX)M_1 = [x_1M_1 + x_2M_2 + x_3M_3, M_1] = -x_2M_3 + x_3M_2$, and similarly $(adX)M_2 = x_1M_3 - x_3M_1$ and $(adX)M_3 = -x_1M_2 + x_2M_1$. Hence,

$$X = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

is actually the matrix representation of *adX* in the basis $\{M_1, M_2, M_3\}$. If $Y = y_1M_1 + y_2M_2 + y_3M_3$ is another such element, then:

$$\begin{aligned} Tr(adXadY) &= Tr\left(\begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix}\right) \\ &= -2(x_1y_1 + x_2y_2 + x_3y_3). \end{aligned} \tag{59}$$

Theorem 3. *A Lie algebra g is solvable if and only if* $(X, X) = 0$ *for all* $X \in \mathscr{D}g$.

Theorem 4 (*Cartan's criterion*). *A Lie algebra g is semisimple if and only if the Killing form of g is nondegenerate, meaning that* $(X, Y) = 0$ *for all* $Y \in g$ *implies* $X = 0$.

We also need to review, the decomposition of Lie algebras that we make use in our approach. As in the case of the decomposition of a vector space into the generalized eigenspaces of any given linear operator, any nilpotent linear Lie algebra *h* acting on a vector space *V* defines a decomposition of *V* in the following way.

For any given linear function $\alpha : h \to \mathscr{C}$, the set $V^\alpha = \{v \in V : [H - \alpha(H)I]^k v = 0$, for some $k > 0$ and all $H \in h\}$, is the generalized eigenspace for all $H \in h$ with eigenvalue $\alpha(H)$. If $V^\alpha \neq \emptyset$, it is said that α is a weight or a root of *h* in *V* and V^α is a weight (root) subspace of *V*. Then,

$$V = \bigoplus_{\alpha \in \Delta} V^\alpha, \tag{60}$$

where Δ is the set of all weights of *h* in *V*.

If *g* is a Lie algebra and *h* is a nilpotent subalgebra, then $adh = \{adH : H \in h\}$ is a nilpotent linear Lie algebra acting on *g*; so if Eq. (60) is applied with $V = g$, and *h* replaced by *adh*, the following decomposition of *g* is obtained:

$$g = \bigoplus_{\alpha \in \Delta} g^\alpha, \tag{61}$$

where $g^\alpha = \{G \in g : [adH - \alpha(adH)I]^k G = 0$, for some $k > 0$ and all $H \in h\}$.

Definition A.10. If $h = g^0$, then h is called a Cartan subalgebra of g.

It can be shown that every Lie algebra has a Cartan subalgebra and each such subalgebra is a maximal nilpotent subalgebra. Any two Cartan subalgebras are conjugate under a certain group of automorphisms of the algebra.

In the case of a semisimple Lie algebra, the root space decomposition (61) takes the form:

$$g = h \oplus \bigoplus_{\alpha \in \Sigma} g^{\alpha},$$

where Σ is the set of nonzero roots of h in g, and the Cartan subalgebra h is a maximal Abelian subalgebra of g. The Killing form $(.,.)$ is nondegenerate on h, each g^{α} is one-dimensional for $\alpha \neq 0$ and there are $dimh$ (dimension of h) linearly independent roots.

Any Lie algebra g can be written in the form:

$$g = r + m, \quad r \cap m = \emptyset, \tag{62}$$

where r is solvable and m is semisimple. This is called a Levi decomposition of g. However, this decomposition is not a direct sum, so it is not unique. Each Lie algebra has many Levi decompositions.

Two Levi subalgebras m_1 and m_2 are related by

$$R^{-1} m_1 R = m_2, \tag{63}$$

where $R = exp(adS)$, $S \in [r, g]$.

A Levi subalgebra m is semisimple, since $m \cong g/r$ and so m can be decomposed in terms of a Cartan subalgebra h_m:

$$m = h_m \oplus \sum_{\varphi \in \Sigma} m^{\varphi}, \tag{64}$$

where Σ is the set of nonzero roots with respect to h_m. Cartan subalgebras are not unique, but any two Cartan subalgebras h_1 and h_2 of m are conjugate under the group of automorphisms of m generated by $exp(adX)$ where $X \in m$ and adX is nilpotent. Thus,

$$h_1 = \sigma^{-1}(X) h_2 \sigma(X) \tag{65}$$

for some $X \in m$ with adX nilpotent, where $\sigma(X) = exp(adX)$. Note that a Cartan subalgebra is a maximal Abelian subalgebra of m. Combining (62) and (64), any Lie algebra g may be written in the form:

$$g = r + m = r + \left(h_m \oplus \sum_{\varphi \in \Sigma} m^{\varphi} \right), \tag{66}$$

where h_m is an Abelian Cartan subalgebra.

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