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POLE PLACEMENT FOR NONLINEAR SYSTEMS

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Abstract: In this paper a pole-placement method for nonlinear systems is proposed. It is based on a recently introduced technique which approximates the solution of a nonlinear system by a sequence of linear time-varying systems and on the classical method of pole-placement for linear systems. The original nonlinear system is replaced by a sequence of linear time-varying equations and for each of this equation a pole-placement method is applied, giving stability of the system as a consequence if some condition is satisfied. *Copyright © 2004 IFAC*

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1. INTRODUCTION

In this paper the classical theory of pole placement controllers to linear, time invariant systems is extended to nonlinear systems.

The main idea is based on the fact that a nonlinear system can be approximated by a sequence of linear time-varying ones whose solutions converge to the solution of the nonlinear system provided the nonlinear matrix $A(x)$ satisfies a mild local Lipschitz condition,

$$||A(x) - A(y)|| \leq \alpha ||x - y||.$$

That is, given a nonlinear system of the form:

$$\dot{x}(t) = A[x(t)]x(t) + B[x(t)]u(t), \quad x(0) = x_0 \quad (1)$$

were the matrices $A(x)$ and $B(x)$ satisfy the Lipschitz condition, the system (1) can be representedby a sequence of linear time-varying control systems whose solutions converge to the solution of the nonlinear system:

$$\dot{x}^{(0)}(t) = A[x_0]x^{(0)}(t) + B[x_0]u(t), \qquad (2)$$
$$\dot{x}^{(1)}(t) = A[x^{(0)}(t)]x^{(1)}(t) + B[x^{(0)}]u(t),$$
$$\dot{x}^{(2)}(t) = A[x^{(1)}(t)]x^{(2)}(t) + B[x^{(1)}]u(t),$$
$$\vdots$$
$$\dot{x}^{(i)}(t) = A[x^{(i-1)}(t)]x^{(i)}(t) + B[x^{(i-1)}]u(t),$$

with initial conditions $x^{(0)}(0) = x^{(1)}(0) = x^{(2)}(0) = \cdots = x^{(i)}(0) = x_0$ at each iteration.

It is known that the sequence of solutions $x^{(i)}(t)$ converge uniformly on any compact time interval to the nonlinear solution $x(t)$, see (Tomas-Rodriguez M. *et. al.*, 2003). For this reason this study can be reduced to the case of linear, time-varying systems of the form:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \qquad (3)$$

for which a linear feedback shall be chosen:

$$u(t) = -Fx(t) \qquad (4)$$

so that the next closed loop system is obtained:

$$\dot{x}(t) = [A(t) - B(t)F]x(t). \qquad (5)$$

Thus, once the sequence (2) of time-varying systems has been obtained, the usual pole-placement

method will be applied to each of these equations with the objective of designing a constant feedback gain F such that the closed-loop time-varying plants have stable eigenvalues.
Due to the time dependency of these systems, the fact of having negative eigenvalues does not guarantee stability, so in this paper, the authors have derived an additional condition which each of the time-varying systems should satisfy in order to ensure stability by assuming the linear time-varying matrices $A(t)$ and $B(t)$ for each iteration can be written as a linear combination of a constant matrix A_1, B_1 and a time dependant perturbation $A_2(t)$, $B_2(t)$ respectively. This will be introduced in section 3.
In section 4, a practical example on how this technique works is presented, proving effectiveness in stabilising the plant.
The iteration technique for nonlinear systems is a recently introduced method that has been applied before to other areas of control of nonlinear systems with satisfactory results such the design of nonlinear observers (Navarro- Hernandez *et. al.*), control of vibrations (Banks S.P. *et. al.*, 2002) or the study of Lyapunov stability for MIMO multiperiodic repetitive control systems (Owens D.H *et. al.*) and optimal control for nonlinear systems (Banks S.P. *et. al.*, 2001). This iteration technique, is a general method that can be applied to any nonlinear system of the form $\dot{x} = A(x)x$ in several areas of research where this type of systems appear; i.e: for an example in its application to nonlinear parameter identification in physics see (Tomas-Rodriguez M. *et. al.*, 2004)

2. POLE PLACEMENT: BASIC IDEAS

In the classical theory of control, linear time-invariant systems of the form:

$$\dot{x} = Ax + Bu, \quad x(0) = x_0 \tag{6}$$

where the pair (A, B) is stabilizable, are often stabilized by designing a linear feedback control law $u = -Fx$ such that the matrix F is chosen so that the eigenvalues of the closed-loop matrix $[A - BF]$ are placed arbitrarily in the left-half plane,

$$Re\Big[eig\big[A - BF\big]\Big] < 0$$

So now, the feedback gain $F = [f_1, f_2]$ can be designed conveniently in order to place the poles anywhere in the left-half plane. Thus the control $u(t)$ will drive the system response from a set of initial conditions $x(0)$ to zero as $t \to 0$.
This approach will be utilised in the next section to stablish a control law $u(t)$ for each linear time-varying system of the sequence (2) so that each of their solutions $x^{(i)}(t)$ will converge to zero.

3. STABILIZATION PROBLEM

In this section, the stabilization of systems represented by equations of the form (3) will be analysed. Suppose that $A(t)$ and $B(t)$ can be separated into constant and time-varying parts:

$$A(t) = A_1 + A_2(t), \quad B(t) = B_1 + B_2(t)$$

then the expression (3) becomes:

$$\dot{x} = \Big[A_1x(t) + B_1u(t)\Big] + \Big[A_2(t)x(t) + B_2(t)u(t)\Big]$$

giving the closep loop expression of the form:

$$\dot{x} = \underbrace{\Big[A_1 + B_1F\Big]}_{M_1} x(t) + \underbrace{\Big[A_2(t) + B_2(t)F\Big]}_{M_2} x(t) \tag{7}$$

where the control $u(t) = -B(t)F$. The solution of (7) can be written as

$$x(t) = e^{M_1t}x_0 + \int_0^t Ce^{M_1(t-s)}\Big[A_2(s) - B_2(s)F\Big]x(s)ds$$

The objective here was to stabilise the system, the solution $x(t)$ should remain bounded, i.e.:

$$||x(t)|| \le C \cdot e^{-wt}||x_0|| +$$

$$+\int_0^t C \cdot e^{-w(t-s)}||x_0|| \cdot ||A_2(t) - B_2(t)F|| \cdot ||x(s)||ds$$

from $||e^{M_1(t)}x_0|| \le ||x_0|| \cdot e^{-wt}$ where w is the greatest of all the negative eigenvalues of the matrix A_1.

Multiplying both sides by e^{wt}:

$$\psi(t) = ||x(t)||e^{wt} \le C \cdot ||x_0|| +$$

$$+\int_0^t C \cdot e^{ws}||x_0|| \cdot ||A_2(t) - B_2(t)F|| \cdot ||x(s)||ds$$

then it follows that

$$\psi(t) \le C||x_0|| + \int_0^t C\psi(s)||x_0|| \cdot ||A_2(t) - B_2(t)F||ds$$

Thus,

$$\psi(t) \le C \cdot ||x_0|| + C\int_0^t W(s) \cdot \psi(s)||x_0|| \cdot ds$$

where $W(s) = ||A_2(t) - B_2(t)F||$.

Hence

$$\dot{\psi}(t) \le C \cdot W(t)\psi(t) \cdot |x_0|| \tag{8}$$

and so

$$\psi(t) \le e^{\int_0^t ||x_0||\cdot C\cdot W(s)\cdot ds}\psi(0). \qquad (9)$$

Since $\psi(t) = ||x(t)||e^{wt}$ and $\psi(0) = ||x_0||$, then (9) becomes

$$||x(t)|| \le e^{-wt}\cdot e^{\int_0^t ||x_0||\cdot C\cdot W(s)\cdot ds}||x_0||$$

which is an upperbound of the solution of the linear time-varying system (3).
It is assumed that $W(t) = ||A_2(t) - B_2(t)F||$ was bounded. This is one of the constrains of this method, as it is assumed this second time-varying term $||A_2(t) - B_2(t)F||$ to be 'small enough'. Now assume the upperbound of this term to be $||W(t)|| \le W$, then:

$$||x(t)|| \le e^{-wt+\int_0^t ||x_0||\cdot C\cdot W\cdot ds}\cdot ||x_0|| =$$
$$= e^{-wt||x_0||C\cdot Wt}\cdot ||x_0||$$

Thus,

$$||x(t)|| \le e^{(-w+C||x_0||W)t}\cdot ||x_0|| \qquad (10)$$

The exponent of (10) should be negative for stability:

$$-w + C||x_0||W < 0 \rightarrow w > C\cdot ||x_0||\cdot W$$

so then:

$$||x_0|| < \frac{w}{CW} \qquad (11)$$

This is, if condition (11) is satisfied, then the system (3) will be stable.

Summarising:

- w was stated to be the nearest to the origin of the eigenvalues of the matrix $(A_1 - B_1F)$ chosen to be in left half plane as usual for pole-placement methods.

- An estimation of the value of the constant C can be calculated by using the norm of the constant part of the system (3) as follows:

$$||e^{(A_1-B_1F)t}|| = ||e^{\Omega t}|| \le C\cdot e^{-wt}$$

being $||e^{\Omega t}|| = ||P||\cdot||e^{\Lambda t}||\cdot||P^{-1}||$ and

$$||e^{\Lambda t}|| \le \left|\left|\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t}\end{pmatrix}\right|\right| \le e^{Re(\lambda_1)t}$$

where λ_1 is the biggest of the eigenvalues.

So this is finally $||P||\cdot||e^{\Lambda t}||\cdot||P^{-1}|| \le C\cdot e^{-wt}$ which gives an estimation of the value of C.

- The value of W:

$$||A_2(s) - B_2(s)F|| = W(s),$$
$$sup_{s\in[0,t]}[A_2(s) - B_2(s)F] = W$$

This is, W can be obtained as the supreme value of the norm of the time-varying matrix in the system (3).

This can be summarised as follows:

Lemma:

Given a linear time-varying control system of the form:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad u(t) = -Fx(t)$$

with some initial conditions x_0, such that:

- *$A(t) = A_1 + A_2(t)$ and $B(t) = B_1 + B_2(t)$.*
- *and F is the feedback control law designed to achieve negative closed-loop poles ($\lambda < w < 0$) for $\dot{x}(t) = [A_1 - B_1F]x(t)$*

Then if the condition

$$||x_0|| < \frac{w}{C\cdot W}$$

is satisfied, the system $\dot{x}(t) = A(t)x(t) - B(t)Fx(t)$ is stable.
In the next section, a practical example of this will be presented. It will be seen how this tecnique works on a linear time-varying system.

4. PRACTICAL EXAMPLES

4.1 Example 1

Consider a time-varying system of the form:

$$x'' + 5x' + g(x) = 2u, \quad g(x) = -3x - tx$$

where the state-space representation is:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ t+3 & -5 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 2 \end{pmatrix} u(t)$$

The system can be decomposed into two, one of them being time invariant, (A_1, B_1), and the second one containing the time-varying terms $(A_2(t), B_2(t))$, which in this case, for simplicity has been considered to consist of constant term B_1 only:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 3 & -5 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 2 \end{pmatrix} u(t)$$

This system (A_1, B_1), is unstable and controllable, so by applying the theory introduced in this paper, the usual pole-placement procedure is applied to design a feedback gain $F = [f_1 \quad f_2]$ such that the poles of the constant part remain on the left hand side of the plane, i.e: $s = -3$, $s = -8$,

$$\left[sI - \left(\begin{pmatrix} 0 & 1 \\ 3 & -5 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix}[f_1 f_2]\right)\right] = (s+3)(s+8)$$

this requires a feedback gain $F = [27 \quad 6]$.

Taking into account the condition of stability (11) for the time-varying part $(A_2(t), B_2(t))$ introduced in the previous section, this is, stablishing a compromise between the values of the initial conditions $||x_0||$, the biggest eigenvalue $w = -3$, the constant C obtained from the diagonal expression of the constant matrix $e^{(A_1 - B_1 F)t}$ and the upperbound of the norm

$$W = sup_{s \in [0,t]} ||A_2(s) - B_2(s)F||$$

such that:

$$||x_0|| < \frac{w}{C \cdot W}$$

is satisfied, and then simulating the system again with the control law $u = -Fx(t)$ it is shown in the next figure the efectivenes of this method to stabilise the plant:

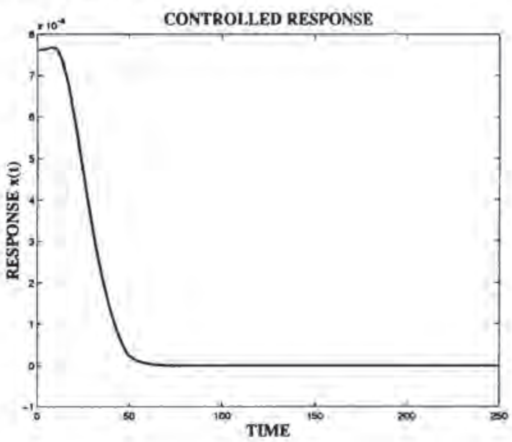

Fig. 1. Controlled system

4.2 Example 2

The system to be simulated in this last example, consists of a time-varying term affecting the control $u(t)$.
Given the dynamical system:

$$x'' + 2x' + g(x) = (3 + 3sint)u(t),$$

with $g(x) = (3 - cost)x$, in this case the state-space representation is:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -3 + cost & -2 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 3 - 3sin(t) \end{pmatrix} u(t)$$

Exactly as stated in the prevoius example, the system can be decomposed into two, one of them being time invariant, (A_1, B_1), and the second one containing the time-varying terms $(A_2(t), B_2(t))$:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 3 \end{pmatrix} u(t)$$

$$+ \begin{pmatrix} 0 & 0 \\ cost & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ -3sin(t) \end{pmatrix} u(t)$$

Applying the usual pole-placement method to design a feedback gain $F = [f_1 \quad f_2]$ so that the poles of the constant part remain on the left hand side of the plane, i.e: $s = -2$, $s = -4$,

$$\left[sI - \left(\begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix}[f_1 f_2]\right)\right] = (s+2)(s+4)$$

this requires a feedback gain $F = [1.6667 \quad 1.3333]$.

Taking into account the condition of stability (11) for the time-varying part $(A_2(t), B_2(t))$ introduced in the previous section, this is, establishing a compromise between the values of the initial conditions $||x_0||$, the biggest eigenvalue $w = -2$, the constant C obtained from the diagonal expression of the constant matrix $e^{(A_1 - B_1 F)t}$ and the upperbound of the norm

$$W = sup_{s \in [0,t]} ||A_2(s) - B_2(s)F||$$

such that:

$$||x_0|| < \frac{w}{C \cdot W}$$

is satisfied, and then simulating the system again with the control law $u = -Fx(t)$ it is shown in the next figure the efectivenes of this method to stabilise the plant:

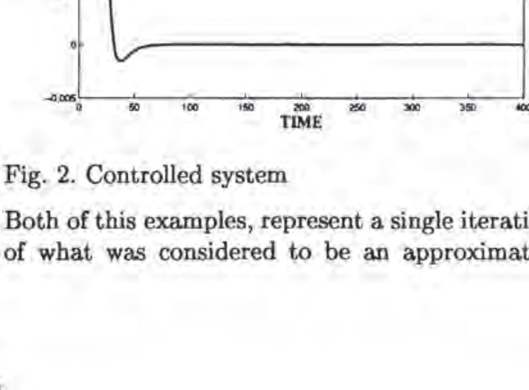

Fig. 2. Controlled system

Both of this examples, represent a single iteration of what was considered to be an approximated

sequence to the nonlinear system. In here only one has been included due to space restrictions, but it has been studied and proved that in the case of nonlinear dynamics, after approximating the system by a set of linear time-varing iterations, and obtainig the feedback gain $F^{(i)}$ for each of them, provided the condition (11) was satified at each iteration, the sequence of linear time-varying systems will produce a sequence of linear controlled responses that eventually will converge to the controlled output of nonlinear system.
In this examples, the term $g(x)$ was time-varying for convenience but it has been proof that in cases where this term was nonlinear in the variable x, the iteration technique could be applied as well to this term in order to approximate it by a sequence of linear time-varying terms.

5. CONCLUSIONS

In this paper, an alternative method for pole-placement for nonlinear systems has been presented. It is based on the fact that a nonlinear system can be approximated by a sequence of linear time-varying systems whose solutions do converge to the solution of the nonlinear system.

Having this sequence of linear time-varying systems, and assuming that each of them could be separated into two different parts, constant and time-varying part respectivelly, a pole-placement technique is developed relaying on the fact that for stability, the solution should remain bounded, this is, for the constant part of the system, the classical pole-placement method is applied and provided some condition between the constant and the time-varying part is satisfied, the overall system remains stable.

Numerical example 5, has been included in order to illustrate the theory introduced. The authors are currently working a way to relax the condition required for stability in order to develop a more flexible theory for this type of systems: Their current line of research is trying to apply Duhamel's principle to each of the linear-time varying iterations so a global control can be produced without the need of such a restrictive stability condition. These results will be published in a separate report.

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