NONLINEAR CONTROL DESIGN IN THE FREQUENCY DOMAIN

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Abstract. A first step is made towards a complete generalization of the classical linear frequency domain theory of feedback control. First, the theory of partial fraction expansions is extended to multi-dimensional complex rational functions. As may be expected, the theory is now much more complicated and requires the use of ideal theory and notions from algebraic geometry. It turns out that, as in the linear case, the coefficients of the expansion (which are now polynomials) are obtained by 'removing the given singularity' and evaluating on the singular variety, i.e. evaluating the rational function modulo the singularity in the coordinate ring of the singularity. Multi-dimensional residue theory is based on the use of homology groups of the space (in fact, the compactified version, S^{2n}) minus the singular variety T. We shall show that the inversion of the n-dimensional Laplace transform can be performed by finding a homology basis of $H_q(S^{2n}\backslash T)$, and a dual basis of $H_{r-1}(T \cup \{\infty\})$, where $r + q = n$. This will reduce the computation in many cases to a simple application of the n-dimensional version of Cauchy's theorem. The use of the theory in feedback control design is given with a particular study of a simple second-order bilinear system. We shall define an implicit closed-loop transfer function (which is nonseparable) and then apply norm inequalities in the time domain to complete the stability analysis.

Key Words. Nonlinear Control Design; Frequency Domain; Residues; Multi-Dimensional Laplace Transform .

1. INTRODUCTION

The classical theory of feedback control in the frequency domain is still widely applied in linear systems theory and although the development of Volterra series has led to generalized transfer functions and hence frequency response theory for nonlinear systems, there has been little use of these ideas in the control of such systems. One reason for this is that the classical method is based on partial fraction expansions and the inverse Laplace transform and these are difficult to apply in the multi-linear case. In this paper we shall give a generalization of both these techniques, the former being based on ideal theory and the latter on algebraic topology and n-dimensional residue theory. It turns out that if we attempt to write a rational function $R(s_1, \cdots, s_n) = P(s_1, \cdots, s_n)/Q(s_1, \cdots, s_n)$ of n complex variables in the form

$$R(s_1, \cdots, s_n) = \sum_{i=1}^{k} \frac{\alpha_i}{Q_i(s_1, \cdots, s_n)}$$

where Q_i are the irreducible components of Q, then we are faced with a number of difficulties. Firstly, this expansion is not always possible, even if α_i is a polynomial. Moreover, if it is possible then the polynomials $\alpha_i \in \mathbf{C}[s_1, \cdots, s_n]$ are not uniquely determined. Roughly speaking, they are determined modulo the coordinate ring of the singularity defined by Q_i.

For the inversion of the Laplace transform, we shall reduce the problem to integration over the basic homology cycles of the homology group of the compactified complex space minus the singular manifold. Using the Alexander-Poincare duality theorem can sometimes lead to a simplification of the integrals.

In the final section we shall study the effect of putting a simple separable system in a feedback system and it will be seen that nonanalytic operations naturally arise. (In the case of a second order bilinear system, it is necessary to take the square root of the output, which is shown to be positive.) We shall show that we are naturally led to nonseparable transfer functions, although in an implicit form. We shall combine the ideas of frequency domain transfer functions with time-domain norm bounds in order to prove input-output stability.

2. NONLINEAR SYSTEMS IN THE FREQUENCY DOMAIN

It is well known that a linear analytic system

$$\dot{x} = f(x) + ug(x) \quad , \quad x(0) = x_0 \in \mathbf{R}^n$$

gives rise to a Volterra series of the form

$$\begin{aligned} x(t) &= w_0(t) + \int_0^t w_1(t, \sigma_1, x_0) u(\sigma_1) d\sigma_1 \\ &+ \int_0^t \int_0^{\sigma 1} w_2(t, \sigma_1, \sigma_2, x_0) \times \\ &\quad u(\sigma_1) u(\sigma_2) d\sigma_1 d\sigma_2 + \cdots \end{aligned}$$

(see [1]). Moreover, by extending the kernels we can define, for the k^{th} order kernel, a k-dimensional 'transfer function' $W_k(s_1, \cdots, s_k)$ for $s_i \in \mathbf{C}$, $1 \leq i \leq k$.

Conversely, if the function $W_k(s_1, \cdots, s_k)$ is rational, strictly proper and recognizable then there exists a finite-dimensional bilinear realization of this kernel (see [3]). (Here, recognizable means that if $W_k(s_1, \cdots, s_k) = P(s_1, \cdots, s_k)/Q(s_1, \cdots, s_k)$ for polynomials P and Q, then $Q(s_1, \cdots, s_k) = \Pi_{i=1}^k Q_i(s_i)$ for some polynomials Q_i.) The separability of Q makes the theory very easy; however, we shall see that introducing feedback naturally introduces nonseparable functions Q which do not have bilinear realizations. In this paper we shall develop a generalization of classical feedback control theory based on system poles and zeros, feedback compensation and root locus. Along the way we shall generalize the theory of partial fraction expansion and develop a general theory for the inversion of the k-dimensional Laplace transform based on Leray's theory of residues and Alexander-Poincare duality. Finally, we shall give a simple example to illustrate the theory.

3. INVERSION OF THE LAPLACE TRANSFORM

In this section we shall derive a method for inverting a k-dimensional Laplace transform by generalizing the one-variable case. Thus we show that partial fraction expansion can be extended to $\mathbf{C}^k$ and that the inversion of irreducible rational functions can be accomplished in many cases by k-dimensional residue theory (for more details, see [2]). In what follows we therefore consider a rational transfer function of the form

$$R(s_1 \cdots, s_k) = \frac{P(s)}{Q_1^{m_1}(s) \cdots Q_\ell^{m_\ell}(s)}$$

where $s = (s_1, \cdots, s_n)$ and each Q_i $(1 \leq i \leq \ell)$ is an irreducible polynomial on $\mathbf{C}^k$, and the Q_i are relatively prime. Thus, we would like to write R in the form

$$R(s_1 \cdots, s_k) = \sum_{i=1}^{\ell} \sum_{j=1}^{m_i} \frac{P_{ij}(s_1 \cdots, s_k)}{Q_i^j(s_1 \cdots, s_k)} \tag{1}$$

for some $P_{ij} \in \mathbf{C}[s_1 \cdots, s_k]$. It turns out that we can do this if and only if

$$(P) \subseteq (Q_1') + \cdots + (Q_\ell')$$

where (t) is the principal ideal generated by t and $Q_i' = \Pi_{j \neq i} Q_j^{m_j}$. From this it follows that a necessary condition that R can be written in the form (1) is

$$V(Q_i) \cap V(Q_j) \subseteq V(P) \ , \ \forall i, j \tag{2}$$

where $V(t)$ is the variety defined by the polynomial t. If the ideal $\sum_j (Q_j')$ is minimal, i.e. $\sqrt{\sum_j (Q_j')} = \sum_j (Q_j')$ where $\sqrt{()}$ is the radical ideal, then this condition is also sufficient.

Example

$$\frac{1}{s_1^2 - s_2^2} \neq \frac{a_1}{s_1 - s_2} + \frac{a_2}{s_1 + s_2}$$

for any $a_1, a_2 \in \mathbf{C}[s_1, s_2]$. Note that

$$V(s_1 - s_2) \cap V(s_1 + s_2) = 0$$

while $V(1) = \emptyset$ and condition (2) is violated. However, the rational function

$$\frac{s_1}{s_1^2 - s_2^2} = \frac{1}{2} \frac{1}{s_1 - s_2} + \frac{1}{2} \frac{1}{s_1 + s_2} \tag{3}$$

is separable since $V(s_1) = \{(s_1, s_2) : s_1 = 0\} \supseteq 0$. Note that, unlike the one-variable case, a_1 and a_2 are not unique, if they exist. In fact we can write (3) in the form

$$\begin{aligned} \frac{s_1}{s_1^2 - s_2^2} &= \frac{1}{2} \frac{1 + \alpha(s_1, s_2)(s_1 - s_2)}{s_1 - s_2} \\ &+ \frac{1}{2} \frac{1 - \alpha(s_1, s_2)(s_1 + s_2)}{s_1 + s_2} \end{aligned}$$

for any $\alpha \in \mathbf{C}[s_1, s_2]$. Note that $\alpha(s_1 - s_2) \in (Q_1)$ where $Q_1 = s_1 - s_2$ and $\alpha(s_1 + s_2) \in (Q_2)$ where $Q_2 = s_1 + s_2$. Here, (.) denotes the principal ideal. It turns out that this is generally true, i.e. the P_{ij} in (1) are determined modulo an appropriate ideal depending on the $Q_i's$. To see how this works suppose

$$R(s_1 \cdots, s_k) = \sum_{i=1}^{\ell} \sum_{j=1}^{m_i} \frac{\overline{P}_{ij}(s_1 \cdots, s_k)}{Q_i^j(s_1 \cdots, s_k)}$$

is another representation of R. Then,

$$\sum_{i=1}^{\ell}\sum_{j=1}^{m_i}(P_{ij}-\overline{P}_{ij})/Q_i^j = 0$$

and so

$$\sum_{i=1}^{\ell}\sum_{j=1}^{m_i}(P_{ij}-\overline{P}_{ij})Q_1^{m_1}\cdots Q_i^{m_i-j}\cdots Q_k^{m_k} = 0$$

Then the next result follows:

Lemma

3.1 In the expansion (3.1), P_{ij} is uniquely determined modulo the ideal $(Q_i^{m_i-j+1})$ so that P_{ij} is well-defined in $\mathbf{C}[s_1\cdots,s_k]/(Q_i^{m_i-j+1})$. □

Note that we have the sequence of (generalized) coordinate rings

$$\begin{aligned}\mathbf{C}[s_1\cdots,s_k]/(Q_i) &\subseteq \mathbf{C}[s_1\cdots,s_k]/(Q_i^2)\subseteq\\ \cdots &\subseteq \mathbf{C}[s_1\cdots,s_k]/(Q_i^{m_i})\end{aligned}$$

for each i and so repeated factors are associated with a corresponding coordinate ring.

In order to determine values of P_{ij} (mod the ideal $(Q_i^{m_i-j+1})$), consider the special case

$$\frac{P(s_1,s_2)}{Q_1^{m_1}(s_1,s_2)Q_2^{m_2}(s_1,s_2)} = \frac{P_{11}}{Q_1}+\cdots+\frac{P_{1m_1}}{Q_1^{m_1}}+\frac{P_{21}}{Q_2}+\cdots+\frac{P_{2m_2}}{Q_2^{m_2}}$$

(The general case is similar.) Then, if such an expression exists, we have

$$P = vQ_2^{m_2}+wQ_1^{m_1}$$

for some polynomials v and w, where

$$\begin{aligned}v &\in \mathbf{C}[s_1\cdots,s_k]/(Q_1^{m_1})\ ,\\ w &\in \mathbf{C}[s_1\cdots,s_k]/(Q_2^{m_2}).\end{aligned}$$

Now,

$$v = P_{11}Q_1^{m_1-1}+P_{12}Q_1^{m_1-2}+\cdots+P_{1m_1}.$$

Using the division lemma we have

$$\begin{aligned}v &= \alpha_1Q_1+\beta_1\\ &= (\alpha_2Q_1+\beta_2)Q_1+\beta_1=\cdots\\ &= \alpha_{m_1-1}Q_1^{m_1-1}+\beta_{m_1-1}Q_1^{m_1-2}+\cdot+\beta_1\end{aligned}$$

and the P_{ij}'s are determined. If we write the polynomials in s_1 with coefficients in $\mathbf{C}[s_2]$ and each polynomial has highest-order coefficient independent of s_2 (i.e. a unit in $\mathbf{C}[s_2]$) then it can be shown that the P_{ij}'s are unique.

Example Consider the rational function

$$\frac{s_1^2}{(s_1-s_2^2)^2(s_1+2s_2^2)} = \frac{P_1}{s_1-s_2^2}+\frac{P_2}{(s_1-s_2^2)^2}+\frac{P_3}{s_1+2s_2^2}.$$

Note that

$$V(s_1-s_2^2)\cap V(s_1+2s_2^2)=\{0\}\subseteq V(s_1^2),$$

and

$$\sqrt{(s_1^2)}=(s_1^2)\subseteq\sqrt{(s_1-s_2^2)^2+(s_1+2s_2^2)}$$

and so a partial fraction expansion exists. Then,

$$\begin{aligned}P_3 &= \left.\frac{s_1^2}{(s_1-s_2^2)^2}\right|_{s_1+2s_2^2=0}=\frac{4}{9}\\ &\quad(\in\mathbf{C}[s_1,s_2]/(s_1+2s_2^2)).\end{aligned}$$

Write

$$s_1^2=r(s_1+2s_2^2)+\frac{4}{9}(s_1-s_2^2)^2.$$

Then,

$$r=\frac{5}{9}s_1-\frac{2}{9}s_2^2$$

and so

$$r=P_1(s_1-s_2^2)+P_2$$

whence,

$$P_1=\frac{5}{9}\ ,\ \ P_2=\frac{1}{3}s_2^2.$$

We come next to the inversion of the Laplace transform. The general expression for the inverse transform is, of course,

$$\begin{aligned}f(t_1,\cdots,t_k) &= \frac{1}{(2\pi i)^k}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\cdots\int_{\sigma_k-i\infty}^{\sigma_k+i\infty}\\ &\quad F(s_1,\cdots,s_k)\times\\ &\quad e^{(s_1t_1+\cdots+s_kt_k)}ds_1\cdots ds_k \qquad (4)\end{aligned}$$

as in the one-variable case. Computation of this integral is, however, generally difficult and we search for a generalization of the one-variable residue theory. To do this we shall use singular homology and duality, and so we first recall some basic notions. If X is a differentiable manifold, let $C_p(X)$ denote the singular p-chains and $Z_p(X), B_p(X)$ the singular p cycles and boundaries, respectively. If ω is a p-form we define

$$\int_{c_p}\omega=\sum_{i=1}^{r}m_i\int_{\sigma_k^i}\omega$$

where $c_p\in C_p(X)$ is given by $c_p=\sum_{i=1}^r m_i\sigma_p^i$

for singular simplexes σ_p^i. If $c_p^1, c_p^2 \in Z_p(X)$ and $c_p^1 \cong c_p^2$, i.e. $c_p^1 = c_p^2 + \ell b_p^2$ for some $b_p^2 \in B_p(X)$, then by Stoke's theorem we have

$$\int_{c_p^1} \omega = \int_{c_p^2} (\omega + d\phi) \tag{5}$$

for any ϕ. Hence, if $H_p(X)$ and $H^p(X)$ denote the (singular) homology and (de Rham) cohomology groups, respectively, we can define

$$\int_{[c_p]} [c^p] = \int_{c_p} \omega$$

where $c_p \in [c_p] \in H_p(X)$ and $\omega \in [c^p] \in H^p(X)$. The main result we need is the well-known Alexander-Poincare duality theorem:

Theorem 3.1 Let S^n be an n-manifold homeomorphic to the n-dimensional sphere and T a polyhedral submanifold of S^n. Then, if $r+q=n$, the homology groups $H_{r-1}(T)$ and $H_q(S^n \backslash T)$ are isomorphic. Moreover, if $\{c_1, \cdots, c_p\}$ is an $(r-1)$-dimensional homology basis of T, then there is a corresponding dual basis $\{d_1, \cdots, d_p\}$ of $S^n \backslash T$ such that

$$v(c_i, d_j) = \delta_{ij} \quad (i,j = 1, \cdots, p).$$

□

Here, v is the linking coefficient. (For a proof of this and a definition of the intersection index and linking coefficients, see [4].) The linking coefficients are linear in the arguments and so if F is analytic in $\mathbf{C}^n \backslash T$ and $\hat{T} = T \cup \{\infty\}$, then for any cycle $c \in Z_n(\mathbf{C}^n \backslash T)$ we have

$$\int_c F(z) dz = (2\pi i)^n \sum_{j=1}^{p} h_j R_j$$

where

$$h_j = v(\sigma_j, c)$$

for some $(n-1)$-dimensional homology basis $\{\sigma_j\}$ of $\hat{T}$, and R is the 'residue' given by

$$R_j = \frac{1}{(2\pi i)^n} \int_{c_j} F(z) dz$$

in which $\{c_j\}$ is an n-dimensional homology basis of $\mathbf{C}^n \backslash T$ dual to $\{\sigma_j\}$.

Returning to (4), it follows from (5) that if γ is any cycle in $C_n(S^{2k} \backslash T)$ which is weakly homologous to the cycle $S^k \backslash \{\infty\} \in C_k(S^{2k} \backslash \{\infty\})$ then

$$\begin{aligned} & f(t_1, \cdots, t_k) = \mathcal{L}^{-1}(F(s_1, \cdots, s_k)) \\ = \; & \frac{1}{(2\pi i)^k} \int_\gamma F(s_1, \cdots, s_k) \times \\ & \exp\left(\sum_{i=1}^{k} s_i t_i\right) ds_1 \wedge \cdots \wedge ds_k. \end{aligned}$$

Hence, by theorem 3.1, if $\{c^j\}_{1 \le j \le p}$ is a basis of $H_{k-1}(\hat{T})$ and $\{c_j\}_{1 \le j \le p}$ is the dual basis of $H_k(\mathbf{C}^k \backslash T)$, we have

$$\begin{aligned} \mathcal{L}^{-1}(F(s_1, \cdots, s_k)) &= \frac{1}{(2\pi i)^k} \sum_{j=1}^{p} h_j \int_{c_j} \\ & F(s_1, \cdots, s_k) \exp\left(\sum_{i=1}^{k} s_i t_i\right) ds_1 \wedge \cdots \wedge ds_k \end{aligned}$$

where

$$h_j = v(c^j, \gamma).$$

Example Consider the inverse Laplace transform of the function

$$F(s_1, s_2) = \frac{1}{1 + s_1 s_2}$$

The singular variety is

$$T = \{(s_1, s_2) \in \mathbf{C}^2 : s_1 s_2 = -1\}.$$

Since $\hat{T} = T \cup \{\infty\}$ is the 2-sphere with two points identified, the homology groups $H_1(\hat{T}) \cong H_2(\mathbf{C}^2 \backslash T)$ have dimension 1. Dual homology basic cycles are given by

$$\begin{aligned} c \;=\; & \{(s_1, s_2) : Im\ s_1 = Im\ s_2 = 0, \\ & Re\ s_2 = 1/Re\ s_1, \;\; 0 \le Re\ s_1 < \infty\} \end{aligned}$$

(in $Z_1(\hat{T})$ and

$$d = \{(s_1, s_2) : s_1 = 2e^{it}, s_2 = 2e^{i\tau}, t, \tau \in [0, 2\pi]\}$$

(in $Z_2(\mathbf{C}^2 \backslash T)$). Now it can be shown that

$$v(c, d) = 1$$

and so

$$\begin{aligned} & (\mathcal{L}^{-1} F)(t_1, t_2) \\ = \; & \frac{1}{(2\pi i)^2} \int_d \frac{e^{s_1 t_1 + s_2 t_2}}{1 + s_1 s_2} ds_1 \wedge ds_2 \\ = \; & \sum_{k=0}^{\infty} \frac{1}{(2\pi i)^2} (-1)^k \int_d \frac{e^{s_1 t_1 + s_2 t_2}}{(s_1 s_2)^{k+1}} ds_1 \wedge ds_2 \\ = \; & \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \frac{\partial^{2k}}{\partial s_1^k \partial s_2^k} (e^{s_1 t_1 + s_2 t_2}) \big|_{(s_1, s_2) = (0,0)} \\ = \; & \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} (t_1 t_2)^k \\ = \; & J_0(2(t_1 t_2)^{1/2}). \end{aligned}$$

Example Consider the function

$$F(s_1,s_2)=\frac{2}{(1-s_1s_2)(1+s_1s_2)}.$$

It is easy to check that we can write

$$F(s_1,s_2)=\frac{1}{1-s_1s_2}+\frac{1}{1+s_1s_2}$$

so that

$$f(t_1,t_2)=-I_0(2(t_1t_2)^{1/2})+J_0(2(t_1t_2)^{1/2}).$$

4. APPLICATION TO CONTROL THEORY

In this section we shall consider a very simple application of the previous theory to feedback control design, generalizing the familiar linear theory. Consider a second order bilinear system, i.e. one with only a second order kernel $H(s_1,s_2)$ which is recognizable. In the first place we shall assume that H is also symmetric, i.e. $H(s_1,s_2)=H(s_2,s_1)$. Suppose we put H in a feedback loop of the form shown in fig. 1.

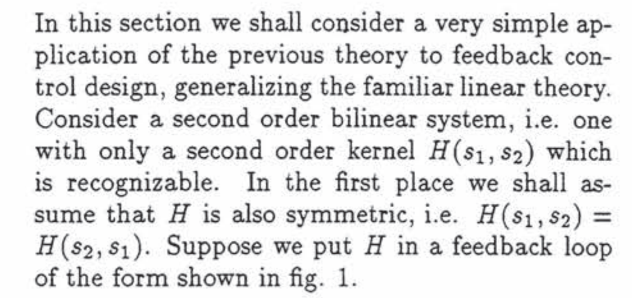

Fig. 1. A Simple Nonlinear Feedback System

Then, the general input-output relation for H is

$$H(s_1,s_2)=\frac{\Xi(s_1,s_2)}{U(s_1)U(s_2)} \tag{6}$$

where $\Xi(s_1,s_2)$ is the output of H such that the actual output ξ is given by $\xi(t)=\mathcal{L}^{-1}(\Xi)(t_1,t_2)|_{t_1=t_2=t}$. Since H is symmetric and separable it follows from (6) that $\Xi(s_1,s_2)$ is also symmetric and separable. Hence,

$$\Xi(s_1,s_2)=Y(s_1)Y(s_2)$$

where $\xi(t)=y^2(t)$. Hence, chasing the system around the loop, we get

$$\begin{aligned}&(R(s_1)-E(s_1))(R(s_2)-E(s_2))\\&=Y(s_1)Y(s_2)\\&=H(s_1,s_2)U(s_1)U(s_2)\\&=H(s_1,s_2)G(s_1)E(s_1)G(s_2)E(s_2).\end{aligned}$$

Thus,

$$E(s_1)E(s_2)=\frac{1}{1-H(s_1,s_2)G(s_1)G(s_2)}\times$$
$$(-R(s_1)R(s_2)+E(s_1)R(s_2)+E(s_2)R(s_1)).$$

Define the **closed loop 'transfer function'** $K(s_1,s_2)$ by

$$K(s_1,s_2)=\frac{1}{1-H(s_1,s_2)G(s_1)G(s_2)}.$$

Then we have

$$\begin{aligned}E(s_1)E(s_2) &= K(s_1,s_2)(-R(s_1)R(s_2)+\\&\quad E(s_1)R(s_2)+E(s_2)R(s_1)).\end{aligned}$$

Hence,

$$\begin{aligned}e(t_1)e(t_2) &= -(k(t_1,t_2)**(r(t_1)r(t_2)))+\\&\quad (k(t_1,t_2)**(e(t_1)r(t_2)))+\\&\quad (k(t_1,t_2)**(e(t_2)r(t_1)))\end{aligned}$$

where $**$ denotes the double convolution. It $f(t_1,t_2)$ is a function of two time variables, we denote by $||f(t_1,t_2)||_{L^p[0,T]}$ the norm

$$\left(\int_0^T\int_0^T|f(t_1,t_2)|^p dt_1dt_2\right)^{1/p}$$

and if $h(t)$ is a function of a sungle variable, $||h||_{L^p[0,T]}$ denotes the usual p-norm. Thus,

$$\begin{aligned}&||e||^2_{L^r[0,T]}\le||k||_{L^p[0,T]}||r||^2_{L^q[0,T]}+\\&2||k||_{L^p[0,T]}||e||_{L^q[0,T]}||r||_{L^q[0,T]}\end{aligned} \tag{7}$$

by Young's inequality where $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1$.

Theorem 4.1 For the feedback system in fig.1 with a separable H of second order, we have

$$||e||_{L^2[0,T]}\le\frac{a+\sqrt{a^2+4b}}{2} \tag{8}$$

where

$$\begin{aligned}a &= 2||k||_{L^1[0,T]}||r||_{L^2[0,T]},\\ b &= ||k||_{L^1[0,T]}||r||^2_{L^2[0,T]}.\end{aligned}$$

In particular, if $k\in L^p[0,T]$ then the system $(r\longrightarrow e)$ is $(L^p\cap L^q,L^r)$-input-output stable.

Proof From (7) we have $\eta^2-a\eta-b\le 0$ where $\eta=||e||_{L^2[0,T]}$ and a,b are as above. The result now follows easily from the elementary theory of quadratic inequalities. □

Example Consider the bilinear system given by $H(s_1,s_2)=-\frac{1}{s_1s_2}$. Then $K(s_1,s_2)=-\frac{s_1s_2}{\gamma^2+s_1s_2}$

where we have simply chosen G to be a gain γ. By theorem 4.1 we see that

$$||e||_{L^r[0,T]} \leq \frac{a + \sqrt{a^2 + 4b}}{2}$$

where

$$\begin{aligned} a &= 2||J_0(2\gamma(t_1t_2)^{1/2})||_{L^1[0,T]}||r||_{L^2[0,T]}, \\ b &= ||J_0(2\gamma(t_1t_2)^{1/2})||_{L^1[0,T]}||r||^2_{L^2[0,T]}. \end{aligned}$$

Remark (a). For a fixed T in the last example $||J_0(2\gamma(t_1t_2)^{1/2})||_{L^1[0,T]} \longrightarrow 0$ as $\gamma \longrightarrow \infty$. Thus we see that the error can be made small for large gain.

(b) If H is not symmetric, but is invertible then we can consider the symmetrized system $H + \overline{H}$ where $\overline{H}(s_1, s_2) = H(s_2, s_1)$. A desired output from H determines an input u so that $(H + \overline{H})U$ can be used as the desired output from the symmetrized system. The use of such techniques coupled with a generalization of the root locus are currently under investigation.

5. Conclusions

In this paper we have begun a generalized frequency domain feedback control systems theory for nonlinear systems. Firstly, the classical partial fraction expansion technique has been generalized to the case of rational functions of n complex variables using ideal theory. It has been shown that such an expansion does not always exist and when it does is only determined modulo a (generalized) coordinate ring in the sense of algebraic geometry. Next the n-dimensional Laplace transform has been considered and, by using de Rham cohomology theory, a method of computing the inverse has been presented. Finally, it has been shown that by including separable rational transfer functions in feedback systems leads naturally to nonseparable 'transfer functions' and a technique for studying the stability of the feedback system has been given. There is clearly a great deal more work to be done in this area before anything like a classical design technique exists, but these results certainly demonstrate that such a generalization of classical feedback theory is quite possible. It appears that functions like $J_0(t)$ may well take the place of exponentials for linear systems, a conjecture which is currently under investigation.

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6. REFERENCES

[1] S.P.Banks, 'Mathematical Theories of Nonlinear Systems', (London:Prentice Hall), 1988.

[2] S.P.Banks and A.Moser, 'On the Inversion of the n-dimensional Laplace Transform', to appear in IMA J. Math. Control and Inf., 1994.

[3] G.E.Mitzel, S.J.Clancy and W.J.Rugh, 'On Transfer Function Representations for Homogeneous Nonlinear Systems', IEEE Trans. Aut. Contr., **AC-24**, 242-249, 1979.

[4] L.A.Aizenberg and A.P.Yuzhakov, 'Integral Representations and Residues in Multi-Dimensional Complex Analysis', AMS Translation, 1983.