ASYMPTOTICALLY OPTIMAL NONLINEAR FILTERING

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Abstract: In this paper, a theoretical investigation of the state-dependent Riccati equation (SDRE) filter is carried out, which is derived by constructing the dual of the well-known SDRE nonlinear regulator control design technique. The SDRE filter has been studied in various papers, with mainly practical investigations of the filter. However, the theoretical aspects of the filter have not been fully investigated and there remain many unanswered questions, such as stability and convergence of the filter. This paper investigates conditions under which the state estimate given by this algorithm converges asymptotically to the first order minimum variance estimate given by the extended Kalman filter (EKF). Conditions for determining a region of stability for the SDRE filter are also investigated. The analysis is based on stable manifold theory and Hamilton-Jacobi-Bellman (HJB) equations. The motivation for introducing HJB equations is given by reference to the maximum likelihood approach to deriving the EKF. The application of the SDRE filter will be demonstrated on a simple pendulum problem to illustrate the theory. The behavioral differences and similarities between the SDRE filter, the linearized Kalman filter (LKF) and the EKF are also discussed using this example.

Keywords: Nonlinear Filtering, SDRE Control, Hamilton-Jacobi-Bellman Equation, Extended Kalman Filter.

1. INTRODUCTION

During the past decade, SDRE feedback control for nonlinear regulator systems has become well-known within the control community (Mracek and Cloutier, 1998). Following the duality between linear-quadratic optimal regulation and linear-quadratic Gaussian estimation, SDRE filters have naturally been suggested in the literature (see Mracek et al., 1996; Pappano and Friedland, 1997; Haessig and Friedland, 1997) for continuous-time nonlinear systems. The algorithm is relatively well-known and involves solving, at a given point in state space, an algebraic (state-dependent) Riccati equation, or SDRE. The coefficients of this equation vary with the given point in state space. The resulting SDRE filter has the same structure as the infamous continuous steady-state linear Kalman filter. In contrast to the linearized Kalman filter (LKF) and the extended Kalman filter (EKF), which are based on linearization, the SDRE filter fully captures the nonlinearities of the system using parameterization, and bringing the nonlinear system to a (nonunique) linear structure having state-dependent coefficients. The nonuniqueness of the parameterization creates extra degrees of freedom, which are not available in traditional filtering methods. These additional degrees of freedom can be used to enhance filter performance, avoid singularities, and/or avoid loss of observability.

The SDRE filter has been studied in various papers, with mainly practical investigations of the filter (Harman and Bar-Itzhack, 1999; Bar-Itzhack et al, 2002). However, the theoretical aspects of the filter have not been fully investigated and there remain many unanswered questions, such as stability and convergence of the filter (see, Ewing, 2000; Jaganath, 2005). This paper investigates conditions under which the state estimate given by this algorithm converges asymptotically to the first order minimum variance estimate given by the EKF. Conditions for determining a region of stability for the SDRE filter are also investigated. The analysis is based on stable manifold theory and Hamilton-Jacobi-Bellman (HJB) equations. The motivation for introducing HJB equations is given by reference to the maximum likelihood approach to deriving the EKF. The application of the SDRE filter is demonstrated on a simple pendulum problem to illustrate the theory. A comparative study is also carried out, outlining the behavioral differences and similarities between the SDRE filter, the LKF and the EKF.

The paper is thus organized as follows. First, SDRE control is reviewed in Sections 2. The filtering counterpart is then stated in Section 3. In Section 4, asymptotic minimum variance filter is studied. The theoretical results are illustrated numerically using a simple pendulum example in Section 5. Concluding remarks are given in Section 6.

2. SDRE CONTROL

SDRE control represents a systematic way of designing nonlinear feedback controllers for a broad class of nonlinear regulator problems. It guarantees a locally asymptotically optimal and stabilizing solution to the time-invariant infinite-horizon nonlinear regulator problem, where the objective is to minimize the cost functional

$$J = \tfrac{1}{2}\int_{t_0}^{\infty}\left\{\mathbf{x}^T(t)\mathbf{Q}(\mathbf{x})\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}(\mathbf{x})\mathbf{u}(t)\right\}dt \quad (1)$$

with respect to the state $\mathbf{x}(t) \in \mathbb{R}^n$ and control $\mathbf{u}(t) \in \mathbb{R}^m$ $\mathbf{x}(t)$ subject to the input-affine nonlinear differential constraint

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad (2)$$

where $\mathbf{f}:\mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{B}:\mathbb{R}^n \to \mathbb{R}^{n\times m}$, $\mathbf{Q}:\mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $\mathbf{R}:\mathbb{R}^n \to \mathbb{R}^{m\times m}$. Assuming that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, $\mathbf{B}(\mathbf{x}) \neq \mathbf{0} \quad \forall \mathbf{x} \neq \mathbf{0}$, $\mathbf{Q}(\mathbf{x}) \geq \mathbf{0}$ and $\mathbf{R}(\mathbf{x}) > \mathbf{0} \quad \forall \mathbf{x}$, recall that the SDRE approach for obtaining a suboptimal solution of the nonlinear optimization problem (1) and (2) is given as follows:

Algorithm 1 (*SDRE Control*)**.**

1. Use direct parameterization to bring the nonlinear dynamics (2) to the state-dependent coefficient form
$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad (3)$$
where
$$\mathbf{f}(\mathbf{x}) = \mathbf{A}(\mathbf{x})\mathbf{x}$$
such that $\mathbf{A}:\mathbb{R}^n \to \mathbb{R}^{n\times n}$, which is *nonunique* when $n > 1$.
2. Solve the (steady-state) State-Dependent Riccati Equation (that is, SDRE)
$$\begin{aligned}&\mathbf{Q}(\mathbf{x}) + \mathbf{P}(\mathbf{x})\mathbf{A}(\mathbf{x}) + \mathbf{A}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) \\ &\quad - \mathbf{P}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) = \mathbf{0}\end{aligned} \quad (4)$$
for $\mathbf{P}(\mathbf{x}) \geq \mathbf{0}$.
3. Construct the nonlinear feedback controller via
$$\mathbf{u}(\mathbf{x}) = -\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x})\mathbf{x} \quad (5)$$
such that the SDRE-control trajectory becomes the solution of
$$\dot{\mathbf{x}}(t) = [\mathbf{A}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x})]\mathbf{x}(t).$$

Hence, the time-invariant infinite-horizon nonlinear SDRE regulator problem (1)-(5) is a generalization of the time-invariant infinite-horizon *linear-quadratic* regulator problem, where all of the coefficients matrices are state-dependent. When these coefficient matrices are constant, the nonlinear regulator problem (1) and (2) collapses to the linear-quadratic regulator problem and the SDRE control method collapses to the steady-state linear regulator.

3. THE SDRE FILTER

The filtering counterpart of the SDRE Control Algorithm 1 can be obtained by taking the dual of the steady-state linear-regulator and then allowing the coefficient matrices of the dual to be state-dependent. The dual of the steady-state linear regulator is the steady-state continuous Kalman observer, which in the absence of control reduces to the steady-state continuous Kalman filter. This leads to the following formulation. Consider the white-noise driven nonlinear system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\mathbf{w}(t) \quad (6)$$

together with white-noise-corrupted observations

$$\mathbf{z}(t) = \mathbf{h}(\mathbf{x}) + \mathbf{v}(t), \quad (7)$$

where $\mathbf{w}(t)$ and $\mathbf{z}(t)$ are zero-mean, white and Gaussian noise vectors, which are uncorrelated with themselves or with $\mathbf{x}(t_0)$, such that for $t > t_0$,

$$\operatorname{cov}\{\mathbf{w}(t), \mathbf{w}(\tau)\} = \mathbf{W}\delta(t-\tau)$$
$$\operatorname{cov}\{\mathbf{v}(t), \mathbf{v}(\tau)\} = \mathbf{V}\delta(t-\tau)$$

where $\mathbf{W} \geq \mathbf{0}$ and $\mathbf{V} > \mathbf{0}$. Let us define the processes $d\boldsymbol{\omega}(t) \triangleq \mathbf{w}(t)dt$, $d\boldsymbol{\upsilon}(t) \triangleq \mathbf{v}(t)dt$ and $d\mathbf{y}(t) \triangleq \mathbf{z}(t)dt$. Then the system of equations (6) and (7) can be written more properly as the following Itô-sense stochastic differential equations

$$d\mathbf{x}(t) = \mathbf{f}(\mathbf{x})dt + \mathbf{G}(\mathbf{x})d\boldsymbol{\omega}(t) \quad (8)$$
$$d\mathbf{y}(t) = \mathbf{h}(\mathbf{x})dt + d\boldsymbol{\upsilon}(t) \quad (9)$$

where $\boldsymbol{\omega}(t)$ and $\boldsymbol{\upsilon}(t)$ are independent Brownian motions uncorrelated with $\mathbf{x}(t_0)$, such that

$$\operatorname{cov}\{\boldsymbol{\omega}(t), \boldsymbol{\omega}(\tau)\} = \mathbf{W}\min(t, \tau)$$
$$\operatorname{cov}\{\boldsymbol{\upsilon}(t), \boldsymbol{\upsilon}(\tau)\} = \mathbf{V}\min(t, \tau).$$

As with (3), let us bring Eqs. (8) and (9) to the state-dependent coefficient form

$$d\mathbf{x}(t) = \mathbf{F}(\mathbf{x})\mathbf{x}dt + \mathbf{G}(\mathbf{x})d\boldsymbol{\omega}(t) \quad (10)$$
$$d\mathbf{y}(t) = \mathbf{H}(\mathbf{x})\mathbf{x}dt + d\boldsymbol{\upsilon}(t) \quad (11)$$

Let $\mathbf{Y}(t) \triangleq \{\mathbf{y}(\tau): t_0 \leq \tau \leq t\}$ denote the observations up to time t. Let $\hat{\mathbf{x}}(t) \triangleq E\{\mathbf{x}(t)|\mathbf{Y}(t)\}$ denote the *conditional mean*, that is, the *minimum variance optimal estimate*, and $\mathbf{P}(t) \triangleq \operatorname{var}\{\mathbf{x}(t) - \hat{\mathbf{x}}(t)|\mathbf{Y}(t)\}$ denote the *conditional error variance*. The SDRE filter for estimating the state $\mathbf{x}$ is then given by

$$d\hat{\mathbf{x}}(t) = \mathbf{f}(\hat{\mathbf{x}})dt + \mathbf{K}(\hat{\mathbf{x}})[d\mathbf{y}(t) - \mathbf{h}(\hat{\mathbf{x}})dt] \quad (12)$$

where

$$\mathbf{K}(\hat{\mathbf{x}}) = \mathbf{P}(\hat{\mathbf{x}})\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{V}^{-1} \quad (13)$$

is the filter gain, and $\mathbf{P}$ is the positive-definite solution of the SDRE

$$\begin{aligned}&\mathbf{G}(\hat{\mathbf{x}})\mathbf{W}\mathbf{G}^T(\hat{\mathbf{x}}) + \mathbf{F}(\hat{\mathbf{x}})\mathbf{P}(\hat{\mathbf{x}}) + \mathbf{P}(\hat{\mathbf{x}})\mathbf{F}^T(\hat{\mathbf{x}}) \\ &\quad - \mathbf{P}(\hat{\mathbf{x}})\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{V}^{-1}\mathbf{H}(\hat{\mathbf{x}})\mathbf{P}(\hat{\mathbf{x}}) = \mathbf{0}\end{aligned} \quad (14)$$

4. ASYMPTOTIC MINIMUM VARIANCE FILTER

To a first order approximation, the solution to the filtering problem is given by the EKF (Sage and Melsa, 1971, Chapter 9)

$$d\hat{\mathbf{x}}(t) = \mathbf{f}(\hat{\mathbf{x}}(t))dt + \mathbf{P}(t)\tfrac{\partial \mathbf{h}^T(\hat{\mathbf{x}}(t))}{\partial \hat{\mathbf{x}}(t)}\mathbf{V}^{-1}\{d\mathbf{y}(t) - \mathbf{h}(\hat{\mathbf{x}}(t))dt\} \quad (15)$$

$$\begin{aligned} d\mathbf{P}(t) = \Big\{ & \tfrac{\partial \mathbf{f}(\hat{\mathbf{x}}(t))}{\partial \hat{\mathbf{x}}(t)}\mathbf{P}(t) + \mathbf{P}(t)\tfrac{\partial \mathbf{f}^T(\hat{\mathbf{x}}(t))}{\partial \hat{\mathbf{x}}(t)} + \mathbf{G}(\hat{\mathbf{x}}(t))\mathbf{W}\mathbf{G}^T(\hat{\mathbf{x}}(t)) \\ & -\mathbf{P}(t)\tfrac{\partial \mathbf{h}^T(\hat{\mathbf{x}}(t))}{\partial \hat{\mathbf{x}}(t)}\mathbf{V}^{-1}\tfrac{\partial \mathbf{h}(\hat{\mathbf{x}}(t))}{\partial \hat{\mathbf{x}}(t)}\mathbf{P}(t)\Big\}dt. \end{aligned} \quad (16)$$

The initial conditions for the EKF are $\hat{\mathbf{x}}(t_0) = E\{\mathbf{x}(t_0)\}$ and $\mathbf{P}(t_0) = \mathrm{var}\{\mathbf{x}(t_0)|\mathbf{Y}(t_0)\}$.

Hypotheses 1. There is an equilibrium at $\mathbf{x} = \mathbf{0}$, such that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{h}(\mathbf{0}) = \mathbf{0}$. Defining

$$\mathbf{F}(\mathbf{0}) \triangleq \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{0}), \quad \mathbf{H}(\mathbf{0}) \triangleq \frac{\partial \mathbf{h}}{\partial \mathbf{x}}(\mathbf{0})$$

the linear system $(\mathbf{F}(\mathbf{0}), \mathbf{G}(\mathbf{0}), \mathbf{H}(\mathbf{0}))$ is then assumed completely controllable and completely observable.

If the state variable $\mathbf{x}$ is close to the equilibrium $\mathbf{x} = \mathbf{0}$, then the EKF reduces to the linear Kalman filter around $\mathbf{x} = \mathbf{0}$, that is,

$$\begin{aligned} d\hat{\mathbf{x}}(t) = {} & \mathbf{F}(\mathbf{0})\hat{\mathbf{x}}(t)dt \\ & + \mathbf{P}(t)\mathbf{H}^T(\mathbf{0})\mathbf{V}^{-1}\{d\mathbf{y}(t) - \mathbf{H}(\mathbf{0})\hat{\mathbf{x}}(t)dt\} \end{aligned} \quad (17)$$

$$\begin{aligned} d\mathbf{P}(t) = \Big\{ & \mathbf{F}(\mathbf{0})\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^T(\mathbf{0}) + \mathbf{G}(\mathbf{0})\mathbf{W}\mathbf{G}^T(\mathbf{0}) \\ & -\mathbf{P}(t)\mathbf{H}^T(\mathbf{0})\mathbf{V}^{-1}\mathbf{H}(\mathbf{0})\mathbf{P}(t)\Big\}dt. \end{aligned} \quad (18)$$

By the hypotheses on $(\mathbf{F}(\mathbf{0}), \mathbf{G}(\mathbf{0}), \mathbf{H}(\mathbf{0}))$, the linear filter is asymptotically stable (Kalman, 1963, Theorem 13.18). That is, the homogeneous part of the linear system (17),

$$d\hat{\mathbf{x}}(t) = \left[\mathbf{F}(\mathbf{0}) - \mathbf{P}(t)\mathbf{H}^T(\mathbf{0})\mathbf{V}^{-1}\mathbf{H}(\mathbf{0})\right]\hat{\mathbf{x}}(t)dt$$

is asymptotically stable with Lyapunov function $S(\hat{\mathbf{x}}, t) = \frac{1}{2}\hat{\mathbf{x}}^T\mathbf{P}^{-1}(t)\hat{\mathbf{x}}$. In addition, by Theorem 13.33 of Kalman (1963), $\mathbf{P}(t)$ tends to the *unique* positive-definite solution of

$$\begin{aligned} & \mathbf{G}(\mathbf{0})\mathbf{W}\mathbf{G}^T(\mathbf{0}) + \mathbf{F}(\mathbf{0})\mathbf{P} + \mathbf{P}\mathbf{F}^T(\mathbf{0}) \\ & \qquad -\mathbf{P}\mathbf{H}^T(\mathbf{0})\mathbf{V}^{-1}\mathbf{H}(\mathbf{0})\mathbf{P} = \mathbf{0}. \end{aligned} \quad (19)$$

In the steady state case, stability follows since $S(\hat{\mathbf{x}}, t) = \frac{1}{2}\hat{\mathbf{x}}^T\mathbf{P}^{-1}\hat{\mathbf{x}}$ is clearly greater than zero for $\hat{\mathbf{x}} \neq \mathbf{0}$, and

$$\begin{aligned} \frac{dS}{dt} &= \hat{\mathbf{x}}^T\mathbf{P}^{-1}\dot{\hat{\mathbf{x}}} + \dot{\hat{\mathbf{x}}}^T\mathbf{P}^{-1}\hat{\mathbf{x}} \\ &= \hat{\mathbf{x}}^T\mathbf{P}^{-1}\left[\mathbf{F}(\mathbf{0}) - \mathbf{P}\mathbf{H}^T(\mathbf{0})\mathbf{V}^{-1}\mathbf{H}(\mathbf{0})\right]\hat{\mathbf{x}} \\ &\quad + \hat{\mathbf{x}}^T\left[\mathbf{F}^T(\mathbf{0}) - \mathbf{H}^T(\mathbf{0})\mathbf{V}^{-1}\mathbf{H}(\mathbf{0})\mathbf{P}\right]\mathbf{P}^{-1}\hat{\mathbf{x}} \\ &= \hat{\mathbf{x}}^T\left[\mathbf{P}^{-1}\mathbf{F}(\mathbf{0}) + \mathbf{F}^T(\mathbf{0})\mathbf{P}^{-1} - 2\mathbf{H}^T(\mathbf{0})\mathbf{V}^{-1}\mathbf{H}(\mathbf{0})\right]\hat{\mathbf{x}}. \end{aligned}$$

From (19), however,

$$\begin{aligned} \mathbf{P}^{-1}\mathbf{F}(\mathbf{0}) + \mathbf{F}^T(\mathbf{0})\mathbf{P}^{-1} &= \mathbf{H}^T(\mathbf{0})\mathbf{V}^{-1}\mathbf{H}(\mathbf{0}) \\ &\quad -\mathbf{P}^{-1}\mathbf{G}(\mathbf{0})\mathbf{W}\mathbf{G}^T(\mathbf{0})\mathbf{P}^{-1}. \end{aligned}$$

Hence,

$$\begin{aligned} \tfrac{dS}{dt} &= \hat{\mathbf{x}}^T[\mathbf{H}^T(\mathbf{0})\mathbf{V}^{-1}\mathbf{H}(\mathbf{0}) - \mathbf{P}^{-1}\mathbf{G}(\mathbf{0})\mathbf{W}\mathbf{G}^T(\mathbf{0})\mathbf{P}^{-1} - 2\mathbf{H}^T(\mathbf{0})\mathbf{V}^{-1}\mathbf{H}(\mathbf{0})]\hat{\mathbf{x}} \\ &= -\hat{\mathbf{x}}^T\left[\mathbf{P}^{-1}\mathbf{G}(\mathbf{0})\mathbf{W}\mathbf{G}^T(\mathbf{0})\mathbf{P}^{-1} + \mathbf{H}^T(\mathbf{0})\mathbf{V}^{-1}\mathbf{H}(\mathbf{0})\right]\hat{\mathbf{x}} \\ &< 0. \end{aligned}$$

For the proof of stability when $\mathbf{P}$ depends on t, see Kalman (1963), Theorem 13.18.

Since $(\mathbf{F}(\mathbf{0}), \mathbf{G}(\mathbf{0}))$ is controllable, $\mathbf{P}$ is positive-definite and so $\mathbf{P}^{-1}$ exists and satisfies the algebraic Riccati equation

$$\begin{aligned} & \mathbf{P}^{-1}\mathbf{F}(\mathbf{0}) + \mathbf{F}^T(\mathbf{0})\mathbf{P}^{-1} \\ & \qquad + \mathbf{P}^{-1}\mathbf{G}(\mathbf{0})\mathbf{W}\mathbf{G}^T(\mathbf{0})\mathbf{P}^{-1} - \mathbf{H}^T(\mathbf{0})\mathbf{V}^{-1}\mathbf{H}(\mathbf{0}) = \mathbf{0}. \end{aligned} \quad (20)$$

This corresponds to the Hamilton-Jacobi equation for $S(\hat{\mathbf{x}})$, that is,

$$\begin{aligned} & \hat{\mathbf{x}}^T\Big[\mathbf{P}^{-1}\mathbf{A}(\mathbf{0}) + \mathbf{A}^T(\mathbf{0})\mathbf{P}^{-1} \\ & \qquad + \mathbf{P}^{-1}\mathbf{G}(\mathbf{0})\mathbf{Q}\mathbf{G}^T(\mathbf{0})\mathbf{P}^{-1} - \mathbf{H}^T(\mathbf{0})\mathbf{R}^{-1}\mathbf{H}(\mathbf{0})\Big]\hat{\mathbf{x}} = 0 \end{aligned}$$

since $\frac{\partial S}{\partial \hat{\mathbf{x}}} = \mathbf{P}^{-1}\hat{\mathbf{x}}$. Setting $\lambda \triangleq \frac{\partial S}{\partial \hat{\mathbf{x}}}$ gives

$$\begin{aligned} & \lambda^T\mathbf{F}(\mathbf{0})\hat{\mathbf{x}} + \tfrac{1}{2}\lambda^T\mathbf{G}(\mathbf{0})\mathbf{W}\mathbf{G}^T(\mathbf{0})\lambda \\ & \qquad -\tfrac{1}{2}\hat{\mathbf{x}}^T\mathbf{H}^T(\mathbf{0})\mathbf{V}^{-1}\mathbf{H}(\mathbf{0})\hat{\mathbf{x}} = 0, \end{aligned} \quad (21)$$

which implies that the Hamilton-Jacobi equation (21) has a solution $S(\hat{\mathbf{x}}) \geq 0$.

Let us now consider the following model for $\hat{\mathbf{x}}$ away from the equilibrium $\hat{\mathbf{x}} = \mathbf{0}$.

Hypotheses 2. $(\mathbf{F}(\hat{\mathbf{x}}), \mathbf{G}(\hat{\mathbf{x}}))$ and $(\mathbf{F}^T(\hat{\mathbf{x}}), \mathbf{H}^T(\hat{\mathbf{x}}))$ are stabilizable for all $\hat{\mathbf{x}}$.

Remark 1. Note that Hypotheses 2 holds at $\hat{\mathbf{x}} = \mathbf{0}$ by assumption on $(\mathbf{F}(\mathbf{0}), \mathbf{G}(\mathbf{0}), \mathbf{H}(\mathbf{0}))$ in Hypotheses 1.

Let us write

$$\mathbf{f}(\hat{\mathbf{x}}) = \mathbf{F}(\hat{\mathbf{x}})\hat{\mathbf{x}} \quad \text{and} \quad \mathbf{h}(\hat{\mathbf{x}}) = \mathbf{H}(\hat{\mathbf{x}})\hat{\mathbf{x}},$$

where $\mathbf{F}(\hat{\mathbf{x}}) \to \mathbf{F}(\mathbf{0})$ and $\mathbf{H}(\hat{\mathbf{x}}) \to \mathbf{H}(\mathbf{0})$ as $\hat{\mathbf{x}} \to \mathbf{0}$. Then $\hat{\mathbf{x}}$ is the solution of the proposed nonlinear (SDRE) filter (12) with initial condition $\hat{\mathbf{x}}(t_0) = E\{\mathbf{x}(t_0)\}$, where $\mathbf{P}(\hat{\mathbf{x}})$ satisfies at $\hat{\mathbf{x}}$ the SDRE (14).

Remark 2. Note that the assumptions in Hypotheses 2 on $(\mathbf{F}(\hat{\mathbf{x}}), \mathbf{G}(\hat{\mathbf{x}}))$ and $(\mathbf{F}^T(\hat{\mathbf{x}}), \mathbf{H}^T(\hat{\mathbf{x}}))$ ensure a positive-semidefinite solution to (14) exists for all $\hat{\mathbf{x}}$.

The proposed model is simpler to implement than the EKF and appears to be insensitive to errors in the initial condition, unlike the EKF. The subsequent discussion will attempt to answer the following three questions concerning this model:

1) Firstly, what meaning can be attached to $\hat{\mathbf{x}}$ as derived from this model?
2) Secondly, under what conditions is the model stable and thus insensitive to the initial condition?
3) Thirdly, how does this model relate to the EKF?

Clearly, (12) and (14) tend to the same limit as the EKF if $\hat{\mathbf{x}} \to \mathbf{0}$, namely, the steady-state form of the linear filter (17) and (19), which has been assumed stable. Let us now give conditions under which (12) and (14) will be stable on some neighborhood of $\mathbf{0}$. To do this, consider the following Hamilton-Jacobi equation for the function $\bar{S}(\hat{\mathbf{x}})$,

$$\bar{\lambda}^T \mathbf{f}(\hat{\mathbf{x}}) + \tfrac{1}{2}\bar{\lambda}^T \mathbf{G}(\hat{\mathbf{x}})\mathbf{W}\mathbf{G}^T(\hat{\mathbf{x}})\bar{\lambda} - \tfrac{1}{2}\hat{\mathbf{x}}^T \mathbf{H}^T(\hat{\mathbf{x}})\mathbf{V}^{-1}\mathbf{H}(\hat{\mathbf{x}})\hat{\mathbf{x}} = 0 \quad (22)$$

where $\bar{\lambda} \triangleq \frac{\partial \bar{S}}{\partial \hat{\mathbf{x}}}$. This equation is the nonlinear extension of (21).

Remark 3. The Hamiltonian flow corresponding to (22) has an equilibrium at $\hat{\mathbf{x}} = \bar{\lambda} = \mathbf{0}$ in phase space. Eq. (21) is the Hamilton-Jacobi equation corresponding to the linearization of this flow at $\hat{\mathbf{x}} = \bar{\lambda} = \mathbf{0}$. The Hamiltonian matrix describing this linearized flow is

$$\mathbf{M} = \begin{bmatrix} \mathbf{F}(\mathbf{0}) & \mathbf{G}(\mathbf{0})\mathbf{W}\mathbf{G}^T(\mathbf{0}) \\ \mathbf{H}^T(\mathbf{0})\mathbf{V}^{-1}\mathbf{H}(\mathbf{0}) & -\mathbf{F}^T(\mathbf{0}) \end{bmatrix}. \quad (23)$$

By the assumption on $(\mathbf{F}(\mathbf{0}), \mathbf{G}(\mathbf{0}), \mathbf{H}(\mathbf{0}))$ (Hypotheses 1), it follows that $(\mathbf{F}(\mathbf{0}), \mathbf{G}(\mathbf{0}))$ is stabilizable and $(\mathbf{F}(\mathbf{0}), \mathbf{H}(\mathbf{0}))$ is observable. Also, $\mathbf{W}$ is clearly positive-definite. So by Lemma 3 of Doyle et al. (1989) (or Theorem 12.2 of Wonham, 1985), $\mathbf{M}$ has no purely imaginary eigenvalues and the unstable eigenspace of $\mathbf{M}$ is spanned by the columns of the $2n \times n$ matrix

$$[\mathbf{I} \quad \mathbf{P}^{-1}]^T \quad (24)$$

where $\mathbf{P}^{-1}$ is the positive-definite solution of (20), thus showing again that (21) has a positive solution $\frac{1}{2}\hat{\mathbf{x}}^T \mathbf{P}^{-1}\hat{\mathbf{x}}$.

Remark 4. Note, to apply Lemma 3 of Doyle et al. (1989), the phase-space transformation $\bar{\lambda} \to -\bar{\lambda}$ must be applied to get $\mathbf{M}$ in the correct form, and so their result, which concerns the stable eigenspace, becomes a result about the unstable eigenspace.

Remark 5. It can easily be checked that $[\mathbf{I} \quad \mathbf{P}]$ spans the stable eigenspace for the Hamiltonian matrix corresponding to (19) and so asymptotic stability for the Kalman filter (17), (18) corresponds to asymptotic instability for the $\hat{\mathbf{x}}$-dynamics of (21) with the feedback $\lambda = \mathbf{P}^{-1}\hat{\mathbf{x}}$.

So, from Remark 4, the equilibrium at $\hat{\mathbf{x}} = \bar{\lambda} = \mathbf{0}$ for the flow given by (22) is hyperbolic. By the stable manifold theorem there therefore exist n-dimensional stable and unstable manifolds for this equilibrium. Furthermore, the columns of (24) span the tangent space to the unstable manifold at the origin and so the unstable manifold can be parameterized by the $\hat{\mathbf{x}}$ coordinates in a neighborhood U_1 of the origin. This parameterization has the form $\bar{\lambda} = \partial\bar{S}/\partial\hat{\mathbf{x}}$, where $\bar{S}$ is the solution to (22) in U_1 satisfying $\bar{S}(\mathbf{0}) = 0$, $\partial\bar{S}(\mathbf{0})/\partial\hat{\mathbf{x}} = \mathbf{0}$ and $\partial^2\bar{S}(\mathbf{0})/\partial\hat{\mathbf{x}}^2 = \mathbf{P}^{-1}$. This argument comes from Van der Schaft (1991).

Since $\mathbf{P}^{-1}$ is positive-definite and, to second order locally at $\hat{\mathbf{x}} = \mathbf{0}$,

$$\bar{S} = \tfrac{1}{2}\hat{\mathbf{x}}^T \tfrac{\partial^2\bar{S}(\mathbf{0})}{\partial\hat{\mathbf{x}}^2}\hat{\mathbf{x}} = \tfrac{1}{2}\hat{\mathbf{x}}^T \mathbf{P}^{-1}\hat{\mathbf{x}},$$

it follows that $\bar{S}$ is positive-definite in some neighborhood $U_2 \subset U_1$ of $\hat{\mathbf{x}} = \mathbf{0}$. $\bar{S}$ is then a Lyapunov function for the homogeneous part of (12) provided a certain inequality holds on U_2. Along the trajectories of the homogeneous part of (12),

$$\frac{d\bar{S}}{dt} = \frac{\partial\bar{S}}{\partial\hat{\mathbf{x}}}\frac{d\hat{\mathbf{x}}}{dt} = \bar{\lambda}^T\left[\mathbf{f}(\hat{\mathbf{x}}) - \mathbf{P}(\hat{\mathbf{x}})\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{V}^{-1}\mathbf{H}(\hat{\mathbf{x}})\hat{\mathbf{x}}\right].$$

Substituting (22) for $\bar{\lambda}^T\mathbf{f}(\hat{\mathbf{x}})$, therefore, gives

$$\begin{aligned}\frac{d\bar{S}}{dt} &= -\tfrac{1}{2}\bar{\lambda}^T\mathbf{G}(\hat{\mathbf{x}})\mathbf{W}\mathbf{G}^T(\hat{\mathbf{x}})\bar{\lambda} + \tfrac{1}{2}\hat{\mathbf{x}}^T\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{V}^{-1}\mathbf{H}(\hat{\mathbf{x}})\hat{\mathbf{x}} \\ &\quad -\bar{\lambda}^T\mathbf{P}(\hat{\mathbf{x}})\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{V}^{-1}\mathbf{H}(\hat{\mathbf{x}})\hat{\mathbf{x}},\end{aligned} \quad (25)$$

which can be written as

$$\begin{aligned}\frac{d\bar{S}}{dt} &= -\tfrac{1}{2}\bar{\lambda}^T\mathbf{G}(\hat{\mathbf{x}})\mathbf{W}\mathbf{G}^T(\hat{\mathbf{x}})\bar{\lambda} + \tfrac{1}{2}\left[\mathbf{P}^{-1}(\hat{\mathbf{x}})\hat{\mathbf{x}} - \bar{\lambda}\right]^T \\ &\quad \times\mathbf{P}(\hat{\mathbf{x}})\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{V}^{-1}\mathbf{H}(\hat{\mathbf{x}})\mathbf{P}(\hat{\mathbf{x}})\left[\mathbf{P}^{-1}(\hat{\mathbf{x}})\hat{\mathbf{x}} - \bar{\lambda}\right] \\ &\quad -\tfrac{1}{2}\bar{\lambda}^T\mathbf{P}(\hat{\mathbf{x}})\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{V}^{-1}\mathbf{H}(\hat{\mathbf{x}})\mathbf{P}(\hat{\mathbf{x}})\bar{\lambda},\end{aligned}$$

where $\mathbf{P}(\hat{\mathbf{x}})$ is the solution to (14). This is less than zero provided

$$\begin{aligned}&\left[\mathbf{P}^{-1}(\hat{\mathbf{x}})\hat{\mathbf{x}} - \bar{\lambda}\right]^T\mathbf{P}(\hat{\mathbf{x}})\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{V}^{-1}\mathbf{H}(\hat{\mathbf{x}})\mathbf{P}(\hat{\mathbf{x}})\left[\mathbf{P}^{-1}(\hat{\mathbf{x}})\hat{\mathbf{x}} - \bar{\lambda}\right] \\ &< \bar{\lambda}^T\mathbf{P}(\hat{\mathbf{x}})\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{V}^{-1}\mathbf{H}(\hat{\mathbf{x}})\mathbf{P}(\hat{\mathbf{x}})\bar{\lambda}\end{aligned} \quad (26)$$

This inequality essentially says that the difference between the true feedback and the approximation given by (14) is less than the true feedback as measured with respect to the norm given by $\mathbf{P}(\hat{\mathbf{x}})\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{V}^{-1}\mathbf{H}(\hat{\mathbf{x}})\mathbf{P}(\hat{\mathbf{x}})$. Since $\mathbf{P}(\hat{\mathbf{x}}) \to \mathbf{P}$ (that is, the solution to (19)) as $\hat{\mathbf{x}} \to \mathbf{0}$, one would expect this inequality to hold on a neighborhood of the origin.

Remark 6. Note from (25) that it is enough to test if

$$\tfrac{1}{2}\hat{\mathbf{x}}^T\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{V}^{-1}\mathbf{H}(\hat{\mathbf{x}})\hat{\mathbf{x}} < \bar{\lambda}^T\mathbf{P}(\hat{\mathbf{x}})\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{V}^{-1}\mathbf{H}(\hat{\mathbf{x}})\hat{\mathbf{x}},$$

which avoids one having to consider whether $\mathbf{P}^{-1}(\hat{\mathbf{x}})$ exists. Note also that the value of $\bar{\lambda} = \partial\bar{S}/\partial\hat{\mathbf{x}}$ can be obtained by following characteristics of (22) along the unstable manifold starting near the origin without having to solve (22) for $\bar{S}$.

So suppose inequality (26) holds on some neighborhood U_3 of the origin contained within U_2. Then, the homogeneous part of (12) will be asymptotically stable on the largest sublevel set of $\bar{S}$ contained within U_3, that is, for the largest $\gamma > 0$ such that $\{\hat{\mathbf{x}} : \bar{S}(\hat{\mathbf{x}}) \le \gamma\} \subset U_3$. So to summarize, the following result has been obtained, which answers the second and third questions previously posed above.

Proposition 1. *Suppose* $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ *and* $\mathbf{h}(\mathbf{0}) = \mathbf{0}$. *Suppose also that the linear system* $(\mathbf{F}(\mathbf{0}), \mathbf{G}(\mathbf{0}), \mathbf{H}(\mathbf{0}))$ *is completely controllable and completely observable. Then Eqs. (12) and (14) give a model for* $\hat{\mathbf{x}}$ *which agrees asymptotically as* $\hat{\mathbf{x}} \to \mathbf{0}$ *with the optimal (minimum variance) estimate given by the steady-state limit of the EKF (should the EKF attain this limit). If* U *is the largest neighborhood of the origin on which both a positive solution* $\bar{S}$ *to the Hamilton-Jacobi equation (22) exists and the inequality (26) holds, then the algorithm will be asymptotically stable on the largest sublevel of* $\bar{S}$ *contained within* U.

What about the first question posed above on what meaning can be attached to the estimate $\hat{\mathbf{x}}$ produced by (12) and (14) away from the origin? We make no claim about this in the above result as it is not clear that away from $\hat{\mathbf{x}} = \mathbf{0}$ there is any relationship between $\hat{\mathbf{x}}$ and $\mathbf{x}$. Probably the most that can be said is that if the EKF attains a steady-state $d\mathbf{P}(t) = \mathbf{0}$ away from $\hat{\mathbf{x}} = \mathbf{0}$, then (12) and (14) provide an approximate solution of this steady state, the approximation being that $\mathbf{F}(\hat{\mathbf{x}})$ and $\mathbf{H}(\hat{\mathbf{x}})$ are used instead of $\partial\mathbf{f}(\hat{\mathbf{x}})/\partial\hat{\mathbf{x}}$ and $\partial\mathbf{h}(\hat{\mathbf{x}})/\partial\hat{\mathbf{x}}$. There are no known conditions, however, for determining whether the EKF will attain a steady state. Thus a meaning can only be attached to $\hat{\mathbf{x}}$ as $\hat{\mathbf{x}} \to \mathbf{0}$, namely $\hat{\mathbf{x}}$ in this limit gives the minimum variance estimate of $\mathbf{x}$. However, there are conditions which imply that the dynamics of $\hat{\mathbf{x}}$ are stable in a region around the origin and so one can expect this interpretation to hold eventually for trajectories which start in and remain in this region. These conditions also imply that errors in the initial conditions $\hat{\mathbf{x}}(t_0)$ eventually disappear, unlike in the case of the EKF where there are no guarantees that errors in $\hat{\mathbf{x}}(t_0)$ and $\mathbf{P}(t_0)$ are insignificant. In order to analyze the performance of the SDRE filter and illustrate the theory developed above, a simple example is studied next by Monte-Carlo simulation.

5. EXAMPLE: THE PENDULUM PROBLEM

In this section, the example from Mracek et al. (1996) is reconsidered, which represents a two-dimensional pendulum system. A comparative study is carried out between the SDRE filter, the LKF, and the EKF on this simple problem. The pendulum model has two state variables, $\mathbf{x} = [x_1(t) \quad x_2(t)]^T$, where $x_1(t)$ and $x_2(t)$ are the angular position and angular rate of the pendulum, respectively. The equations of motion are given by

$$\dot{x}_1(t) = x_2(t) + w_1(t), \tag{27}$$

$$\dot{x}_2(t) = -\tfrac{g}{L}\sin x_1(t) + w_2(t). \tag{28}$$

where $w_1(t)$ and $w_2(t)$ are small random forcing functions, which are assumed to be white. The measurement is obtained from an accelerometer attached to the bob of the pendulum. Thus, the measurement equation is

$$y(t) = -\tfrac{g}{L}\sin x_1(t) + v(t). \tag{29}$$

which is corrupted by additive white noise $v(t)$. The application of the SDRE filter requires representation of (27)-(29) in factored state-space form. An obvious parameterization for the SDRE filter is

$$\mathbf{F}(\hat{\mathbf{x}}) = \begin{bmatrix} 0 & 1 \\ -\tfrac{g}{L}\operatorname{sinc}\hat{x}_1 & 0 \end{bmatrix}, \quad \mathbf{H}(\hat{\mathbf{x}}) = \begin{bmatrix} -\tfrac{g}{L}\operatorname{sinc}\hat{x}_1 & 0 \end{bmatrix}, \tag{30}$$

where $\operatorname{sinc}\hat{x}_1 = 1$ if $\hat{x}_1 = 0$, and $\operatorname{sinc}\hat{x}_1 = \frac{\sin\hat{x}_1}{\hat{x}_1}$ otherwise, which is well behaved at the origin, and consequently $\mathbf{F}(\mathbf{0})$ and $\mathbf{H}(\mathbf{0})$ are well defined. Since this is a simple problem, a closed-form solution can be obtained by solving the state-dependent Riccati equation (14) analytically with

$$\mathbf{P} \triangleq \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix},$$

which is symmetric. Eq. (13) leads to the SDRE gain

$$\mathbf{K}_{SDRE}(\hat{\mathbf{x}}) \triangleq [k_1 \quad k_2]^T = -\tfrac{1}{V}\tfrac{g}{L}\operatorname{sinc}\hat{x}_1[P_{11} \quad P_{12}]^T \tag{31}$$

where, for a unique positive-definite solution $\mathbf{P}$,

$$P_{11} = \tfrac{L}{g}\tfrac{1}{\operatorname{sinc}\hat{x}_1}\sqrt{V(2P_{12}+\Gamma_{11})}, \quad P_{12} = \tfrac{L}{g}\tfrac{1}{\operatorname{sinc}\hat{x}_1}V\left(1-\sqrt{1+\tfrac{\Gamma_{22}}{V}}\right), \tag{32}$$

$$\mathbf{GWG}^T \triangleq \begin{bmatrix} \Gamma_{11} & 0 \\ 0 & \Gamma_{22} \end{bmatrix}.$$

This yields

$$k_1(\hat{x}_1) = -\sqrt{\tfrac{L}{g}\tfrac{1}{\operatorname{sinc}\hat{x}_1}\left(-2+2\sqrt{1+\tfrac{\Gamma_{22}}{V}}\right)+\tfrac{\Gamma_{11}}{V}}, \quad k_2 = 1-\sqrt{1+\tfrac{\Gamma_{22}}{V}} \tag{33}$$

The SDRE filter equations are then obtained from (12), giving

$$\dot{\hat{x}}_1 = \hat{x}_2 + k_1(y-\hat{y}), \quad \dot{\hat{x}}_2 = -\tfrac{g}{L}\sin\hat{x}_1 + k_2(y-\hat{y}), \tag{34}$$

$$\hat{y} = -\tfrac{g}{L}\sin\hat{x}_1. \tag{35}$$

Note that there is a singularity in the SDRE filter gain for $\hat{x}_1 = \pi$ rad. This can possibly be overcome by parameterizing the state-dependent coefficient parameterization (30) in terms of in terms of a parameter α (based on a second distinct parameterization) and selecting α so that the singularity is avoided, allowing the SDRE filter to track the pendulum through 360° (Mracek et al., 1996). However, this is not investigated in the paper.

The continuous EKF is defined by the propagation equations (15) and (16) where, for the pendulum,

$$\frac{\partial\mathbf{f}(\hat{\mathbf{x}})}{\partial\hat{\mathbf{x}}} = \begin{bmatrix} 0 & 1 \\ -\tfrac{g}{L}\cos\hat{x}_1 & 0 \end{bmatrix}, \frac{\partial\mathbf{h}(\hat{\mathbf{x}})}{\partial\hat{\mathbf{x}}} = \begin{bmatrix} -\tfrac{g}{L}\cos\hat{x}_1 & 0 \end{bmatrix}.$$

The EKF equations thus become

$$\dot{\hat{x}}_1 = \hat{x}_2 - \tfrac{1}{V}\tfrac{g}{L}\cos\hat{x}_1 P_{11}(y-\hat{y}), \quad \dot{\hat{x}}_2 = -\tfrac{g}{L}\sin\hat{x}_1 - \tfrac{1}{V}\tfrac{g}{L}\cos\hat{x}_1 P_{12}(y-\hat{y})$$

where $\hat{y} = -\tfrac{g}{L}\sin\hat{x}_1$ and

$$\dot{P}_{11} = \Gamma_{11} + 2P_{12} - \tfrac{1}{V}\tfrac{g^2}{L^2}\cos^2\hat{x}_1 P_{11}{}^2$$

$$\dot{P}_{12} = -\tfrac{g}{L}\cos\hat{x}_1 P_{11} + P_{22} - \tfrac{1}{V}\tfrac{g^2}{L^2}\cos^2\hat{x}_1 P_{11}P_{12}$$

$$\dot{P}_{22} = \Gamma_{22} - 2\tfrac{g}{L}\cos\hat{x}_1 P_{12} - \tfrac{1}{V}\tfrac{g^2}{L^2}\cos^2\hat{x}_1 P_{12}{}^2$$

with the EKF gain $\mathbf{K}_{EKF} = -\tfrac{1}{V}\tfrac{g}{L}\cos\hat{x}_1[P_{11} \quad P_{12}]^T$. The LKF is based on the linearization of the pendulum equations about the nominal trajectory $\hat{x}_1(t) = \hat{x}_2(t) = 0$. The LKF equations are obtained from (17) and (19), such that the LKF equations give a simplified version of the SDRE filter equations (31) -(35), with $\operatorname{sinc}\hat{x}_1 \equiv 1$ and $\sin\hat{x}_1$ replaced by $\hat{x}_1$.

Let us now illustrate the efficacy of each filter on the pendulum problem. Simulations are run in a MATLAB® environment, with $g = 9.81\ \mathrm{ms}^{-2}$ and $L = 0.3$ m. Numerical integration is carried out using a simple Euler routine with a step size of $dt = 1$ ms. Note that the EKF requires the integration of the covariance matrix $\mathbf{P}$. This is in contrast to the LKF or the SDRE filter, which only require the integration of the state equations. Numerous cases have been considered using the pendulum problem, investigating the mean (average) SDRE, LKF and EKF estimates for a Monte-Carlo run of 100 randomly initialized experiments. In the case of perfect knowledge of the initial states, where the initial variance $\mathbf{P}_0 = \mathbf{0}_{2\times 2}$ and in each run the initial state estimates are selected the same as the actual ones, the SDRE filter and the EKF provided almost identical performances, which were significantly better than the LKF. However, when the initial variance was increased, such that the filter initial conditions do not coincide with the actual ones, the performance of the EKF deteriorated, the mean estimates (of 100 Monte-Carlo simulation runs) eventually diverging from the actual states. The SDRE filter, on the other hand, still provided superior performance. Fig. 1 illustrates the dispersion of the estimation errors over 100 runs for a scenario when the actual initial positions are $x_1(0) = 1$ rad and $x_2(0) = 0$ rad/s, with initial variance set to $\mathbf{P}_0 = \mathbf{I}_{2\times 2}$. A measurement noise intensity of $V = 2$ has been selected for all filters, with the process noise intensity taken such that $\Gamma_{11} = \Gamma_{22} = 0.05$. Note how the EKF estimates often diverge from the true states. The SDRE filter, however, provides nonlinear estimates of the states whereas the LKF, because it is a linear filter, cannot capture the nonlinearities, thus producing a phase shift and amplitude scaling, and sometimes even filter instability.

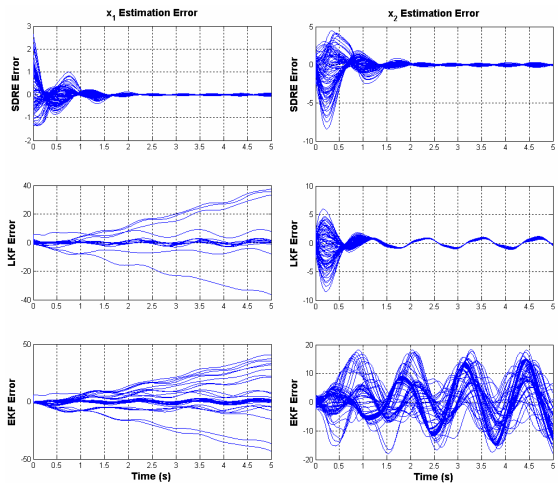

Fig. 1. Estimation-error distribution of Monte-Carlo simulation tests for 100 runs of the pendulum problem when $x_1(0) = 1$ rad, $x_2(0) = 0$ rad/s, $\mathbf{P}_0 = \mathbf{I}_{2\times 2}$, $V = 2$, and $\mathbf{W} = 0.05\mathbf{I}_{2\times 2}$.

6. CONCLUSIONS

A theoretical investigation of the SDRE filter has been carried out in this paper, which represents the dual of the SDRE control algorithm that has proved to be highly effective in solving nonlinear optimal control problems. A numerical example has been considered using a simple pendulum operating in the nonlinear regime. For this example, the SDRE filter is computationally simpler to implement than the EKF as it does not require propagation of the covariance matrix. The SDRE filter can also cope with highly misleading initial states, converging rapidly to the true states. The asymptotic behavior of the SDRE filter yields much improved performance and convergence properties compared with local approximations, such as the usual linearized and extended Kalman filters. As predicted from theoretical considerations in the paper, it is quite evident from simulation results that the SDRE filter becomes significant for increasing values of initial error and noise variances. The filter is capable of tracking the true values of the states, with little sensitivity to the selection of the statistics, or even to severe differences in the initial state estimates when compared with the EKF.

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