The Receding Horizon Principle for Distributed Systems

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ABSTRACT: *The receding horizon principle is generalized to distributed parameter systems of the wave and diffusion types. This leads to a nonlinear control law which improves on the classical linear quadratic solution.*

I. Introduction

In multivariable linear systems theory, the use of nonlinear controllers is well-known (**1, 2**). Such a controller, if correctly chosen, can be made to respond quickly to large errors, while only responding slowly to small errors which may be due to noise disturbances in the system. A general method for deriving an appropriate nonlinear control law has been called the *receding horizon principle* (**3, 4**), and in this paper we seek to generalize the principle to certain distributed systems—in particular to systems defined by wave- or diffusion-type equations.

The basic idea in the finite-dimensional case is to produce a linear controller whose effective "time constant" is a function of some variable time t_1, corresponding to the length of the time interval over which the current minimization of a quadratic-type cost is being calculated. By allowing t_1 to depend on the current state error vector, a nonlinear control is obtained which has the desired properties.

In the next section we shall discuss the notation and basic semigroup theory for the convenience of the reader, and in Section III the theory of receding horizon control will be presented in the infinite-dimensional case. It will be convenient, for technical reasons, to consider the group case first and then to indicate the generalizations necessary for the semigroup formulation. Two examples will then be given in Section IV to illustrate the theory; these will be the wave and heat equations, which are representative of hyperbolic and parabolic systems.

II. Notation and Terminology

We shall make use in this paper of the standard theory of Hilbert spaces and semigroups as presented, for example, by Yosida (**5**). It will be useful to give a brief summary of the basic semigroup theory which we need. Suppose, therefore, that H is a Hilbert space; then a strongly continuous semigroup is a map $T(t)$ from R^+ to $\mathscr{L}(H)$ (the space of bounded linear operators on H) with the properties

(i) $T(t+s) = T(t)T(s), \quad s, t \geqslant 0,$

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(ii) $T(0) = I$,

(iii) $\|T(t)h - h\| \to 0$ as $t \to 0^+$, $\quad \forall h \in H$.

(In general, when denoting the norm of a vector in H we shall use the notation $\|.\|$ when the space H is clear from the context, or by $\|.\|_H$ when we wish to indicate explicitly that the norm is with respect to H.) Note that we shall sometimes denote a semigroup by T_t.

The generator A of the semigroup $T(t)$ is the operator defined by

$$Ah = \lim_{t \to 0^+} \frac{1}{t}(T(t) - I)h,$$

with domain $D(A)$ equal to the set of all elements $h \in H$ for which the limit exists.

Specific examples of semigroups with which we shall be dealing are those related to the wave equation and the heat equation. The former is generated by the operator

$$A = \begin{pmatrix} 0 & I \\ A_1 & 0 \end{pmatrix}$$

where A_1 is the (unbounded) operator defined on $L_2[0, 1]$ by

$$(A_1 f)(x) = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x), \quad f \in D(A_1), f(0) = f(1) = 0.$$

Then A is an operator on the space $\mathscr{H} = D(A_1^{1/2}) \times L_2[0, 1]$ with the inner product

$$\langle w, \bar{w} \rangle_{\mathscr{H}} = \langle A_1^{1/2} z, A_1^{1/2} \bar{z} \rangle_{L_2[0,1]} + \langle y, \bar{y} \rangle_{L_2[0,1]}$$

and domain $D(A) = D(A_1) \times D(A_1^{1/2})$. The semigroup generated by A is given by

$$T(t)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \sum\{\langle w_1, \phi_n \rangle_H \cos n\pi t + (n\pi)^{-1} \langle w_2, \phi_n \rangle_H \sin n\pi t\}\phi_n \\ \sum\{-n\pi \langle w_1, \phi_n \rangle_H \sin n\pi t + \langle w_2, \phi_n \rangle_H \cos n\pi t\}\phi_n \end{pmatrix},$$

for $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathscr{H}$, where $\phi_n(x) = \sqrt{2} \sin n\pi x$.

The semigroup related to the heat equation

$$z_t = z_{xx}, \quad z(0, t) = z(1, t) = 0$$

is

$$(T(t)z)(x) = \sum_{n=1}^{\infty} 2 \exp(-n^2\pi^2 t) \sin n\pi x \int_0^1 \sin(n\pi\rho) z(\rho)\, \mathrm{d}\rho$$

for $z \in L_2[0, 1]$, with generator

$$Az = z_{xx}, \quad z \in D(A) = H^2[0, 1] \bigcap H_0^1[0, 1].$$

Note finally that if $T(t)$ is a strongly continuous semigroup on a Hilbert space then the dual operators $T^*(t)$ have the properties:

(i) $T_0^* = I^* = I$,

(ii) $T_{t+s}^* = T_t^* T_s^*$, $\quad t, s \geqslant 0$,

(iii) $\lim_{t \to 0^+} T_t^* h = h$ in the weak topology,

(iv) $\lim_{t \to 0^+} \left(\frac{T_t^* - I}{t} \right) h = A^* h$, for $h \in D(A^*)$, again in the weak topology.

III. Receding Horizon Control

3.1. *The group case*

We shall consider first the standard linear quadratic problem for the system defined by the input–output relationship

$$x(t) = T(t-t_0)x_0 + \int_{t_0}^{t} T(t-s)Bu(s)\,\mathrm{d}s, \quad 0 \leqslant t_0 \leqslant t \leqslant t_1 < \infty, \tag{3.1}$$

where the output $x(t)$ belongs to a separable Hilbert space H (for each t) and the input $u(t)$ belongs to a Hilbert space U. The group $T(t)$ is generated by the operator A with domain $D(A)$ dense in H. The cost functional for this system is taken to be

$$J(u; t_0, x_0) = \langle x(t_0+t_1), Gx(t_0+t_1) \rangle + \int_{t_0}^{t_0+t_1} \{ \langle x(s), Mx(s) \rangle_H + \langle u(s), Ru(s) \rangle_U \}\,\mathrm{d}s \tag{3.2}$$

where $M, G \in \mathscr{L}(H)$ and $R \in \mathscr{L}(U)$ are self adjoint, non-negative and

$$\langle Ru, u \rangle \geqslant \alpha' \|u\|^2, \quad \forall u \in U \text{ and some } \alpha' > 0.$$

Then, it is well-known (**6**) that the optimal control u^*, which minimizes J subject to the dynamics (3.1), is given by

$$u^*(t) = -R^{-1}B^*Q(t)x(t)$$

where $Q(t)$ satisfies the inner product Riccati equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Q(t)h, k \rangle + \langle Q(t)h, Ak \rangle + \langle Ah, Q(t)k \rangle + \langle Mh, k \rangle = \langle Q(t)BR^{-1}B^*Q(t)h, k \rangle, \quad \text{on } [t_0, t_0+t_1], \quad Q(t_0+t_1) = G; \tag{3.3}$$

for $h, k \in D(A)$, and that the optimal cost is given by

$$J^* = \langle x_0, Q(t_0)x_0 \rangle, \quad x_0 = x(t_0).$$

We can write (3.3) in the form (taking $M = 0$),

$$\dot{Q} + A^*Q + QA = QBR^{-1}B^*Q \tag{3.4}$$

provided the derivative $\dot{Q}$ is interpreted in the weak sense of (3.3). Let us write the optimal control in the open-loop form; this can be done by noting that if the optimal control above is applied to the system, then the state satisfies the equation

$$\dot{x} = Ax - BR^{-1}B^*Qx \quad (\text{if } x_0 \in D(A))$$

and so

$$Q\dot{x} = QAx - QBR^{-1}B^*Qx$$
$$= -\dot{Q}x - A^*Qx$$

where again the derivative is interpreted in the weak sense. Hence,

$$\dot{\overline{Qx}} = -A^*Qx$$

and so

$$Qx(t) = T^*(-(t-t_0))Q(t_0)x_0$$

where T^* is the dual group of T. It follows that the optimal control can be written

$$u^* = -R^{-1}B^*T^*(t_0-t)Q(t_0)x_0. \tag{3.5}$$

In order to derive the receding horizon control, it is convenient to put $G = \alpha I$ in the initial condition of the Riccati equation above.

Consider also the equation

$$\dot{W}(t) = AW + WA^* - BR^{-1}B^*, \quad W(t_0+t_1) = \frac{I}{\alpha} \tag{3.6}$$

where (3.6) is again interpreted in the inner product sense of (3.3). This equation has the unique solution

$$W(t) = T(-(t_1-t+t_0))W(t_0+t_1)T^*(-(t_1-t+t_0))$$
$$+\int_0^{t_1-t+t_0} T(-\tau)BR^{-1}B^*T^*(-\tau)\,\mathrm{d}\tau. \tag{3.7}$$

We therefore have the following result:

Proposition 3.1.

The solution Q of the Riccati equation (3.4) is an invertible operator on $[t_0, t_0+t_1]$ for $\alpha > 0$.

Proof. Let $\tau \in [t_0, t_0+t_1]$. Then

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\langle Q(t)W(t)h, k\rangle\right)_{t=\tau} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\langle W(t)h, Q(t)k\rangle\right)_{t=\tau}$$
$$= \left(\frac{\mathrm{d}}{\mathrm{d}t}\langle W(t)h, Q(\tau)k\rangle\right)_{t=\tau} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\langle W(\tau)h, Q(t)k\rangle\right)_{t=\tau}$$
$$= \langle AW(\tau)h, Q(\tau)k\rangle + \langle W(\tau)A^*h, Q(\tau)k\rangle - \langle BR^{-1}B^*h, Q(\tau)k\rangle - \langle W(\tau)h, Q(\tau)Ak\rangle$$
$$-\langle W(\tau)A^*h, A^*Q(\tau)k\rangle + \langle W(\tau)h, Q(\tau)BR^{*-1}B^*Q(\tau)k\rangle$$
$$= \langle Q(\tau)W(\tau)A^*h, k\rangle - \langle A^*Q(\tau)W(\tau)h, k\rangle - \langle QBR^{-1}B^*h, k\rangle + \langle QBR^{-1}B^*QWh, k\rangle$$

for $h, k \in D(A)$. For a fixed solution $Q(t)$ of equation (3.4) this is a linear inner product equation in $Q(\tau)W(\tau)$ with final condition $(QW)(t_0+t_1) = \alpha I(I/\alpha) = I$. Hence it has the unique solution $Q(\tau)W(\tau) = I$, $\tau \in [t_0, t_0+t_1]$. Similarly, $W(\tau)Q(\tau) = I$ and the result is proved. □

We would now like to let $\alpha \to \infty$, thus forcing the final state $x(t_0+t_1)$ to zero. In order to do this we must first define exact controllability.

Definition 3.2.

The system (3.1) is exactly controllable on $[t_0, t_0+t_1]$ if, given any points $x_0, x_1 \in H$, there is a control $u \in L^p(t_0, t_0+t_1; U)$ (for any $1 < p < \infty$) such that

$$x(t_0) = x_0, x(t_0+t_1) = x_1$$

where $x(t)$ is the controlled state.

Then the following theorem can be shown (**6**):

Theorem 3.3

The system (3.1) is exactly controllable if and only if there exists a $\gamma > 0$ for each $t_1 > 0$ such that

$$\gamma \|B^*T^*(.)h\|_{L^q(0,t_1;U)} > \|h\|_H, \tag{3.8}$$

where $\frac{1}{p}+\frac{1}{q} = 1$ and we have identified H^* with H and U^* with U. Hence we have

Corollary 3.4.

If (3.1) is exactly controllable on $[t_0, t_0+t_1]$, then the solution $W(t;\alpha)$ of (3.7) (denoting the dependence on α explicitly) converges to a well-defined operator function $W(t;\infty)$ as $\alpha \to \infty$ which is invertible except at $t = t_0+t_1$.

Proof. That $W(t;\alpha) \to W(t;\infty)$ (uniformly) as $\alpha \to 0$ follows directly from (3.7). Also,

$$W(t;\infty) = \int_0^{t_1-t+t_0} T(-\tau)BR^{-1}B^*T^*(-\tau)\,\mathrm{d}\tau.$$

Hence, for each $h \in H$,

$$\begin{aligned}\langle W(t;\infty)h, h\rangle_H &= \int_0^{t_1-t+t_0} \langle T(-\tau)BR^{-1}B^*T^*(-\tau)h, h\rangle_H\,\mathrm{d}\tau \\ &\geqslant r\int_0^{t_1-t+t_0} \|B^*T^*(-\tau)h\|_H^2\,\mathrm{d}\tau \\ &= r\|B^*T^*(-\tau)h\|^2_{L^2[0,t_1-t+t_0-U^*]} \\ &\geqslant \frac{r}{\gamma}\|h\|_H^2,\end{aligned}$$

by Theorem 3.3, for some $r > 0$ and $\gamma > 0$ (the latter depending on t), since R is positive definite. (Note that Theorem 3.3 applies here since $T(t)$ is a group and controllability in forward time is equivalent to controllability in reverse time.) $\square$

However, we have seen above that $Q(t;\alpha)\cdot W(t;\alpha) = I$ for all $\alpha > 0$ (again denoting explicitly the dependence of Q and W on α), and so

$$Q(t;\alpha) = W^{-1}(t;\alpha).$$

By Corollary 3.4 and the continuity of the inverse, it follows that $Q(t;\infty) = W^{-1}(t;\infty)$ exists for all $t \in [t_0, t_0+t_1)$. The optimal control which drives the state to zero in time t_1 is therefore

$$u^* = -R^{-1}B^*T^*(t_0-t)Q(t_0;\infty)x_0. \tag{3.9}$$

The basic receding horizon philosophy is now to argue that, at each time t, we should apply the open loop control (3.9) as if we were beginning a new control interval of t_1 seconds. This amounts to replacing t_0 by t and x_0 by $x(t)$ in (3.9). Hence, the feedback control now becomes

$$\begin{aligned} u^* &= -R^{-1}B^*Q(t_0;\infty)x(t), \\ &= -R^{-1}B^*W^{-1}(t_0;\infty)x(t) \end{aligned} \tag{3.10}$$

where

$$W(t_0;\infty) = \int_0^{t_1} T(-\tau)BR^{-1}B^*T^*(-\tau)\,\mathrm{d}\tau. \tag{3.11}$$

3.2. *The semigroup case*

We have considered above the receding horizon principle for systems which are defined by a group of operators $\{T(t)\}$. The fact that the system generates a group was necessary in order to define expressions such as (3.7). We shall now show that the theory can be extended, in certain circumstances, to the semigroup case. Suppose therefore that $\{T(t)\}$ is now a semigroup of operators defined on the Hilbert space H. Then, for each $t \geqslant 0$, we may write

$$\begin{aligned} T(t)h &= T(t)\left\{\sum_{i=0}^{\infty} \langle h, e_i\rangle_H e_i\right\} \\ &= \sum_{i=0}^{\infty}\sum_{n=0}^{\infty} \langle h, e_i\rangle_H f(n,i,t)e_n \end{aligned}$$

where $\{e_i\}$ is an orthonormal basis of H and $f(n,i,t) = \langle T(t)e_i, e_n\rangle_H$. By Parseval's theorem,

$$\sum_{n=0}^{\infty}\left|\sum_{i=0}^{\infty} \langle h, e_i\rangle_H f(n,i,t)\right|^2 < \infty.$$

However, we may consider formally the functions $f(n,i,t)$ for $t \in [-t_1, 0)$. Let $H_1(t)$ (for $t \in [-t_1, 0)$) be the subspace of H consisting of all elements h_1 such that

$$\sum_{n=0}^{\infty}\left|\sum_{i=0}^{\infty} \langle h_1, e_i\rangle_H f(n,i,t)\right|^2 < \infty \tag{3.12}$$

and define the inner product on $H_1(t)$ by

$$\langle h_1, h_2\rangle_{H_1(t)} = \sum_{n=0}^{\infty}\left\{\sum_{i=0}^{\infty} \langle h_1, e_i\rangle_H f(n,i,t)\right\}\left\{\sum_{i=1}^{\infty} \langle h_2, e_i\rangle_H f(n,i,t)\right\}. \tag{3.13}$$

Then we assume that $H_1(t)$ is a Hilbert space, i.e. $H_1(t)$ is complete and we define the operator $T(t)$ on $[-t_1, 0)$ (using the same notation as for the semigroup) by

$$T(t)h_1 = \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} \langle h_1, e_i \rangle_H f(n, i, t) e_n, \quad t \in [-t_1, 0), \quad h_1 \in H_1(t).$$

It follows that $T(t)$ is a bounded operator from $H_1(t)$ to H, for each $t \in [-t_1, 0)$. Hence, the dual operators $T^*(t)$ are also defined and,

$$T^*(t): H^* \to H_1^*(t), \quad t \in [-t_1, 0).$$

In the following discussion we shall identify H^* with H under the usual isomorphism. However, since $H_1(t)$ is a subspace of H we must be careful not to identify $H_1(t)$ with $H_1^*(t)$. Then, we have the sequence of spaces

$$H_1(t) \subset H = H^* \subset H_1^*(t).$$

Of course, $H_1^*(t)$ can be identified with $H_1(t)$ in the canonical way, and so the operator $W(t)$ defined in formally the same way as in (3.7) is a bounded operator from $H_1(t)$ to $H_1^*(t)$. It will be convenient to introduce the spaces $H_2(t), t \in [-t, 0)$ such that $T(t)$ maps $H_1^*(t)$ into $H_2^*(t)$. Then the sequence (3.14) is expanded to the sequence

$$H_2(t) \subset H_1(t) \subset H = H^* \subset H_1^*(t) \subset H_2^*(t), \tag{3.15}$$

and again, we have

$$W(t): H_2(t') \to H \quad \text{and} \quad W(t): H \to H_2^*(t'),$$

where $t' = -t_1 + t - t_0$. In a similar way, we can restrict the operator $A: D(A) \to H$ to an operator $A: D(A) \bigcap H_2(t) \to H_2(t)$ or $A: D(A) \bigcap H_1(t) \to H_1(t)$, or extend A in an obvious way to an operator on $H_2^*(t)$ with domain $D_2(A; t)$, so as to retain the standard properties

$$\frac{\mathrm{d}T(t)h}{\mathrm{d}t} = AT(t)h = T(t)Ah, \quad t \in [-t_1, 0)$$

for $h \in D(A) \bigcap H_1(t)$, for example. It is now easy to see that $W(t)$ satisfies (3.6) as before, provided that the inner product is replaced by the appropriate duality; for example, $W(t)$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle W(t)h, k\rangle = \langle AWh, k\rangle + \langle WA^*h, k\rangle - \langle BR^{-1}B^*h, k\rangle$$

in the $H_2^*(t')$, $H_2(t')$ duality, for $h \in D(A)$, $k \in D(A) \bigcap H_2(t')$. Now

$$T(t): H \to H_1(-t), \quad t > 0.$$

Hence, from the integrated form of (3.4) it is easy to see that $Q(t)$ is also smoothing and, in fact,

$$Q(t): H \to H_2(t'), \quad t_0 \leqslant t < t_1 + t_0,$$

where again $t' = -t_1 + t - t_0$.

We can now conclude that Proposition 3.1 is still true and the proof is as before, provided again that the inner products are interpreted in the appropriate duality pairings. An analogue of Theorem 3.3 is also necessary, but we shall not assume exact controllability in the semigroup case. In fact, we shall assume that the system is approximately controllable in the following sense:

Definition 3.5.

We say that (3.1) is approximately controllable on $[0, t_1]$ if

$$\overline{\text{Range}\,(G)} = H,$$

where

$$Gu = \int_0^{t_1} T(t_1 - s)Bu(s)\,\mathrm{d}s,$$

for $u \in L^p[0, t_1; -U]$. Then we have

Theorem 3.6.

If the system (3.1) is approximately controllable on $[0, t_1]$, then

$$\overline{\text{Range}\,(G')} = H_1^*(-t_1)$$

where

$$G'u = \int_{-t_1}^{0} T(-t_1 - s)Bu(s)\,\mathrm{d}s : L^p[-t_1, 0; U] \to H_1^*(-t_1),$$

and the closure is taken in the topology of $H_1^*(-t_1)$.

Proof. Suppose that Range (G') is not dense in $H_1^*(-t_1)$. Then there is a point x and a ball $B_{\varepsilon,x}$ of radius ε and centre x in $H_1^*(-t_1)$ such that

$$\overline{\text{Range}\,(G')} \bigcap B_\varepsilon = \phi.$$

Let $x_0 \varepsilon$ Range (G) be chosen so that $\|x_0\|_H$ is sufficiently small in order that $\|T(-t_1)x_0\|_{H_1^*(-t_1)} < \dfrac{\varepsilon}{2}$. This is possible since $T(-t_1): H \to H_1^*(-t_1)$ is continuous. Since the system is approximately controllable, there is a control u which takes $x' \in H$ to x_0, where $x' \in B_{\varepsilon/2,x}$. (Note that $H_1^*(-t_1)$ is dense in H.) Hence, reversing time,

$$T(-t_1)x_0 - \int_{-t_1}^{0} T(-t_1 - s)Bu(s)\,\mathrm{d}s = x'.$$

Now,

$$\left\| x - \int_{-t_1}^{0} T(-t_1 - s)B(-u(s))\,\mathrm{d}s \right\|_{H_1^*(-t)} \leqslant \|x - x'\|_{H_1^*(-t_1)} + \|T(-t_1)x_0\|_{H_1^*(-t_1)} < \varepsilon.$$

Hence, $G'(-u) \in B_\varepsilon$ and we have a contradiction. □

However, it is well-known (**6**) that

$$\overline{\text{Range}\,(G')} = H_1^*(-t_1)$$

is equivalent to

$$\ker(G'^*) = \{0\},$$

and since

$$G'^* z^* = B^* T^*(-t_1 - s) z^*$$

it follows as in Corollary 3.4 that $W(t;\infty)$ is invertible, for

$$\langle W(t;\infty)h_1, h_1\rangle_{H_1^*(t'),H_1(t')} \geqslant r\|B^*T^*(-\tau)h_1\|^2_{L^2[0,t_1-t+t_0;u^*]}$$

for $h_1 \in H_1(-t_1)$ and the right hand side is zero only if $h_1 = 0$. We cannot conclude that $W(t;\infty)$ has a bounded inverse, however. But if $W(t;\infty)$ has closed range $R_t(W)$ then $W(t;\infty)$ has a bounded inverse for $t \in [t_0, t_0 + t_1)$ and so $Q(t;\alpha)$ converges to $Q(t;\infty)$ on $R_t(W)$. We can therefore state that the feedback control in the semigroup case is as in (3.10).

3.3. *Stability*

Having obtained the receding horizon feedback control we would now like to prove that the system (3.1) with the control (3.10) is stable, thus justifying the use of this control. In fact, we have the following result, which will be proved first in the case of a group, to avoid the technical difficulties initially:

Theorem 3.7.

The system (3.1) with the control (3.10) where $T(t)$ is a group is stable and the function

$$V(x) = \langle x(t), W^{-1}(t_0)x(t)\rangle$$

is a Lyapunov function. (Note that we are now denoting $W(t;\infty)$ by $W(t)$, for simplicity.)

Proof. Suppose first that $x_0 \in D(A)$. Then, the solution $x(t) \in D(A)$, and

$$\begin{aligned}
\dot V(x) &= \langle \dot x(t), W^{-1}(t_0)x(t)\rangle + \langle x(t), W^{-1}(t_0)\dot x(t)\rangle \\
&= \langle Ax(t) - BR^{-1}B^*W^{-1}(t_0)x(t), W^{-1}(t_0)x(t)\rangle \\
&\quad + \langle x(t), W^{-1}(t_0)(A - BR^{-1}B^*W^{-1}(t_0))x(t)\rangle \\
&= \langle Ax(t), W^{-1}(t_0)x(t)\rangle + \langle x(t), W^{-1}(t_0)Ax(t)\rangle \\
&\quad - 2\langle W^{-1}(t_0)x(t), BR^{-1}B^*W^{-1}(t_0)x(t)\rangle \\
&= \langle AW(t_0)W^{-1}(t_0)x(t), W^{-1}(t_0)x(t)\rangle \\
&\quad + \langle W^{-1}(t_0)x(t), AW(t_0)W^{-1}(t_0)x(t)\rangle \\
&\quad - 2\langle W^{-1}(t_0)x(t), BR^{-1}B^*W^{-1}(t_0)x(t)\rangle \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\langle W(t)W^{-1}(t_0)x(t), W^{-1}(t_0)x(t)\rangle|_{t=t_0} \\
&\quad - \langle W^{-1}(t_0)x(t), BR^{-1}B^*W^{-1}(t_0)x(t)\rangle
\end{aligned}$$

by the inner product form of (3.6). However,

$$\frac{\mathrm{d}}{\mathrm{d}t} W(t) = -T(-t)BR^{-1}B^*T^*(-t)$$

is a nonpositive bounded operator, and so

$$\dot{V}(x) = \langle W^{-1}(t_0)x, \left(\frac{\mathrm{d}W}{\mathrm{d}t}(t_0) - BR^{-1}B^*\right)W^{-1}(t_0)x\rangle \tag{3.16}$$

for all $x \in H$. Moreover,

$$\dot{V}(x) \leqslant 0, \quad x \in H,$$

and the result is proved. □

Suppose that $\dot{V}(x) = 0$, then by the non-negativity of $-\mathrm{d}W(t)/\mathrm{d}t$ and $BR^{-1}B^*$, we have

$$0 = -\left\langle \frac{\mathrm{d}W(t)}{\mathrm{d}t} y, y \right\rangle \bigg|_{t=t_0} = \langle T(-t_1)BR^{-1}B^*T^*(-t_1)y, y\rangle$$
$$\geqslant r\|B^*T^*(-t_1)y\|^2$$

for all $y \in H$, where r is as before. Hence, if the system is controllable on any interval $[t_0, t_0+t_1]$, by Theorem 3.3, $y = 0$, i.e. $y = W^{-1}(t_0)x = 0$ and so $x = 0$.

We have therefore shown that

$$\ker(\mathrm{d}W(t_0)/\mathrm{d}t - BR^{-1}B^*) = \{0\}.$$

Let $S = \dfrac{-\mathrm{d}W(t_0)}{\mathrm{d}t} + BR^{-1}B^* = T(-t_1)BR^{-1}B^*T^*(-t_1) + BR^{-1}B^*$ and define $S_1 = S^{1/2}$. Then $\ker S^{1/2} = \{0\}$ and

$$\dot{V}(x) \leqslant -\|S^{1/2}W^{-1}(t_0)x\|^2.$$

We cannot show in general that the system is asymptotically stable, although we can say that since $\ker S^{1/2} = \{0\}$,

$$|||x||| = \|S^{1/2}x\|$$

is a norm on H and

$$V(x) \geqslant C_1\|x\|^2 \geqslant \frac{C_1}{\|S^{1/2}\|^2}|||x|||^2$$

for some constant C_1, and

$$\dot{V}(x) \leqslant -C_2|||x|||^2$$

for some constant C_2. Hence the system is asymptotically stable in the norm $|||.|||$. However, if $S^{1/2}$ has closed range (in particular if B is invertible), then $S^{-1/2}$ exists and is bounded (on $RaS^{1/2}$) and so

$$\dot{V}(x) \leqslant -C_3\|x\|^2$$

for some new constant C_3, and the system with receding horizon control is stable in the original norm.

Consider now the semigroup case. We define the function $V(x)$ by

$$V(x) = \langle x(t), W^{-1}(t_0)x(t)\rangle_H = \langle x(t), Q(t_0)x(t)\rangle_H.$$

The formal computations in Theorem 3.7 again go through and we obtain (3.16). The only remaining problem is to interpret the inner product in (3.16) in the correct duality. However, if we assume that our semigroup is sufficiently smoothing so that $H_1(-t) \subseteq D(A)$ for all $t > 0$ (e.g. in the case of analytic semigroups), then it follows easily from (3.7) that the operator

$$S(\alpha) = W^{-1}(t_0;\alpha)\left(\frac{\mathrm{d}W}{\mathrm{d}t}(t_0;\alpha) - BR^{-1}B^*\right)W^{-1}(t_0;\alpha)$$

is a bounded operator on H, where we have again introduced the dependence of W on α. Also, we have $S(\alpha)x$ converges for each $x \in H$ as $\alpha \to \infty$ and so by the uniform boundedness principle $S(\infty)$ exists and is a bounded linear operator. Of course, $S(\infty)$ is also non-negative and so $S^{1/2}(\infty)$ exists and the same comments made above apply here with S_1 replaced by $S^{1/2}(\infty)$. Similarly, if $S^{1/2}(\infty)$ has closed range then the system is again asymptotically stable.

In the above discussion the length of the control period t_1 has been fixed throughout. We now wish to vary t_1 so that the control law responds quickly to large errors but much more slowly to small errors. In order to bring out the dependence of W on t_1 explicitly, put

$$\bar{W}(t) = W(t-(t_1-t_0)).$$

Then,

$$\bar{W}(t_1) = W(t_0).$$

An obvious choice for t_1 is $\|x\|_H^{-p}$ for some $p \geqslant 1$. The norm in H may be replaced by another norm if we require the control to respond quickly to, say, large gradients of the state. We therefore have, finally, a nonlinear control with the desired properties. The theory will now be applied to two examples in the next section.

IV. Examples

We shall now present two examples to illustrate the theory. In order to compare the classical linear-quadratic solution with the receding horizon control, the examples will be the simple cases of the wave and heat equations, where analytical solutions can be obtained.

Example 4.1.

Consider first the controlled wave equation (**6**)

$$\begin{aligned} z_{tt} &= z_{\xi\xi} + u(t,\xi), \\ z(0,t) &= z(1,t) = 0, z(\xi,0) = z_0(\xi), z_t(\xi,0) = z_1(\xi), \end{aligned} \tag{4.1}$$

with the cost functional

$$J(u) = \tfrac{1}{2}\int_0^1 (z_\xi^2(\xi,t_1) + z_t^2(\xi,t_1))\,\mathrm{d}\xi + \int_0^{t_1}\int_0^1 (\tfrac{1}{4}z_t^2(\xi,t) + u^2(\xi,t))\,\mathrm{d}\xi\,\mathrm{d}t. \tag{4.2}$$

The system (4.1) is, of course, exactly controllable. On the Hilbert space $\mathscr{H} = D(A^{1/2}) \times L_2(0, 1)$ introduced in Section II, (4.1) become

$$w(t) = \begin{pmatrix} z(t) \\ z_t(t) \end{pmatrix} = T_t \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} + \int_0^t T_{t-s} \begin{pmatrix} 0 \\ I \end{pmatrix} u(s)\, ds$$

and (4.2) may be written

$$J(u) = \tfrac{1}{2}\langle w(t_1), w(t_1)\rangle_{\mathscr{H}} + \int_0^{t_1} (\tfrac{1}{2}\langle Mw(t), w(t)\rangle_{\mathscr{H}} + \langle u, u\rangle_{L_2(0,1)})\, dt$$

with $M = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$. The inner product on H is given by

$$\langle w, \bar{w}\rangle = \langle w^1, \bar{w}^1\rangle_H + \langle w^2, \bar{w}^2\rangle_H$$

for $w = \begin{pmatrix} w^1 \\ w^2 \end{pmatrix}$, $\bar{w} = \begin{pmatrix} \bar{w}^1 \\ \bar{w}^2 \end{pmatrix}$, where $H = L_2(0, 1)$. Then, if

$$Q(t) = \begin{pmatrix} Q_1(t) & Q_2(t) \\ Q_3(t) & Q_4(t) \end{pmatrix}$$

and

$$Q_l(t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha_{ij}^l \phi_i \langle \cdot \phi_j\rangle_H, \quad 1 \leqslant l \leqslant 4$$

where $\phi_i(\xi) = \sqrt{2}\sin(\pi i \xi)$, it follows easily that the Riccati equation has the unique solution given by

$$\alpha_{nm}^2 = \alpha_{nm}^3 = 0, \quad \alpha_{nm}' = \alpha_{nm}^4 = \tfrac{1}{2}\delta_{nm}, \quad n, m \geqslant 1.$$

Hence,

$$Q(t) = \tfrac{1}{2}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

and the optimal control is $u(t, \xi) = -\tfrac{1}{2}z_t(t, \xi)$.

Consider now the receding horizon control for the cost functional

$$J(u) = \int_0^{t_1} \int_0^1 u^2(\xi, t)\, d\xi\, dt.$$

Instead of solving the inner product Riccati equation (3.3) as was done above, it is easier to determine $W(t_0)$ from (3.7) (with $\alpha = 0$) and then invert this operator to find $Q(t_0) = W^{-1}(t_0)$. Now the group for the wave equation is

$$T_t \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \sum\{\langle w_1, \phi_n\rangle_H \cos n\pi t + \dfrac{1}{n\pi}\langle w_2, \phi_n\rangle_H \sin n\pi t\}\phi_n \\ \sum\{-n\pi\langle w_1, \phi_n\rangle_H \sin n\pi t + \langle w_2, \phi_n\rangle_H \cos n\pi t\}\phi_n \end{pmatrix}$$

for $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathscr{H}$. Recall that the inner product on $\mathscr{H}$ is given by

$$\langle w, \bar{w} \rangle_{\mathscr{H}} = \int_0^1 w_{1\xi}(\xi)\bar{w}_{1\xi}(\xi)\, d\xi + \int_0^1 w_2(\xi)\bar{w}_2(\xi)\, d\xi.$$

Using the facts that $B^* = [0, I]$, $R = I$ and $T_t^* = T_{-t}$, it follows easily that

$$T_{-\tau}BR^{-1}B^*T_\tau \begin{pmatrix} \phi_i \\ \phi_j \end{pmatrix} = \begin{pmatrix} \sin^2 i\pi\tau\phi_i - (j\pi)^{-1} \cos j\pi\tau \sin j\pi\tau\phi_j \\ -i\pi \sin i\pi\tau \cos i\pi\tau\phi_i + \cos^2 j\pi\tau\phi_j \end{pmatrix}.$$

Hence, if

$$W(t_0)(= \bar{W}(t_1)) = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix}$$

then each W_i has a diagonal matrix representation given by

$$(W_1)_{ii} = \frac{(i\pi)^2}{2}\left(t_1 - \frac{\sin 2i\pi t_1}{2i\pi}\right), \quad (W_4)_{ii} = \tfrac{1}{2}\left(t_1 + \frac{\sin 2i\pi t_1}{2\pi i}\right),$$

$$(W_2)_{ii} = (W_3)_{ii} = \tfrac{1}{4}(\cos 2i\pi t_1 - 1).$$

Since $W(t_0)$ is invertible, it follows that

$$\begin{aligned} Q(t_0) &= W^{-1}(t_0) \\ &= \begin{pmatrix} -W_3^{-1}W_4(-W_1W_3^{-1}W_4 + W_2)^{-1} & -W_1^{-1}W_2(-W_3W_1^{-1}W_2 + W_4)^{-1} \\ (-W_1W_3^{-1}W_4 + W_2)^{-1} & (-W_3W_1^{-1}W_2 + W_4)^{-1} \end{pmatrix} \end{aligned}$$

provided W_1 and $(W_3W_2 - W_1W_4)W_3^{-1}$ are invertible. It is an elementary exercise to show that, in the present case, these operators are invertible provided $t_1 > 0$ and it_1 is not a positive integer. In the latter case it follows that the receding horizon control is

$$\begin{aligned} u^* &= -R^{-1}B^*W^{-1}(t_0)w(t) \\ &= -\{(-W_1W_3^{-1}W_4 + W_2)^{-1}z_\xi(t) + (-W_3W_1^{-1}W_2 + W_4)^{-1}z_t(t)\} \\ &= -(-W_1W_4 + W_3W_2)^{-1}\{W_3z_\xi(t) - W_1z_t(t)\}, \end{aligned}$$

and so

$$\langle u^*, \phi_i \rangle_H = 2\left((i\pi)^2\left(t_1^2 - \frac{\sin^2 2i\pi t_1}{4(i\pi)^2}\right) - \tfrac{1}{4}(\cos 2i\pi t_1 - 1)^2\right)^{-1}$$
$$\times \left(\tfrac{1}{2}(\cos 2i\pi t_1 - 1)\langle z_\xi(t), \phi_i \rangle_H - (i\pi)^2\left({}^{Q}t_1 - \frac{\sin 2i\pi t_1}{2i\pi}\right)\langle z_t \phi_i \rangle_H\right).$$

If (it_1) is a positive integer, then

$$\langle u^*, \phi_i \rangle_H = -\frac{2}{t_1}\langle z_t, \phi_i \rangle_H,$$

and if $t_1 = 4$, then the control is $-\frac{1}{2}z_t(t, \xi)$ as in the linear quadratic solution. It can be seen therefore that the receding horizon control gives much more flexibility than the linear quadratic case, and by choosing, for example

$$t_1 = \|(z, z_t)\|_{\mathscr{H}}^{-1}$$

the control will respond quickly to large values of $\|(z, z_t)\|_{\mathscr{H}}^{-1}$ but more slowly when the latter is small.

Example 4.2.

Consider the controlled heat equation **(6)**

$$\begin{aligned} z_t &= z_{\xi\xi} + u(t, \xi), \\ z_\xi(0, t) = 0 &= z_\xi(1, t); \quad z(\xi, 0) = z_0(\xi) \end{aligned} \tag{4.2}$$

together with the cost functional

$$J(u) = \int_0^1 z^2(t_1, \xi)\, \mathrm{d}\xi + \int_0^{t_1} \left(\int_0^1 z^2(t, \xi) + u^2(t, \xi) \mathrm{d}\xi \right) \mathrm{d}t.$$

The system (4.2) is only approximately controllable. Let $\phi_i = \sqrt{2}\cos \pi i \xi$, $i = 1, 2, \ldots$; $\phi_0 = 1$ be a basis of $L^2[0, 1]$. Then the solution of this linear–quadratic problem is known to be given by the operator Q such that

$$Q(t)h = \sum_{i=0}^{\infty} q_{ii}(t)\phi_i \langle h, \phi_i \rangle, \quad \forall h \in H$$

where each q_{ii} satisfies

$$q_{ii}(t) = \frac{a_i(1-b_i) - b_i(1-a_i)\mathrm{e}^{-\alpha_i(t-t_1)}}{(1-b_i) - (1-a_i)\mathrm{e}^{-\alpha_i(t-t_1)}} \tag{4.3}$$

where

$$\begin{aligned} \alpha_i &= 2\sqrt{\pi^4 i^4 + 1}, \\ a_i &= -\pi^2 i^2 - \sqrt{\pi^4 i^4 + 1}, \\ b_i &= -\pi^2 i^2 + \sqrt{\pi^4 i^4 + 1}. \end{aligned}$$

The optimal control is

$$u(t) = \sum_{i=0}^{\infty} u_i(t)\phi_i = -\sum_{i=0}^{\infty} q_{ii}(t) z_i(t) \phi_i,$$

where the state $z(t)$ of the system is given by

$$z(t) = \sum_{i=0}^{\infty} z_i(t)\phi_i.$$

The controlled state therefore satisfies

$$\begin{aligned} \dot{z}_0(t) &= -q_{00}(t)z_0(t) = -z_0(t), \\ \dot{z}_i(t) &= -(\pi i)^2 z_i(t) - q_{ii}(t) z_i(t), \quad i \geqslant 1. \end{aligned} \tag{4.4}$$

Consider now the receding horizon solution and let

$$J(u) = \int_{t_0}^{t_1+t_0} u^2(t,\xi)\, \mathrm{d}\xi\, \mathrm{d}t$$

be the appropriate cost functional.

Then solving (3.3) as above, with final condition $Q(t_1) = \alpha$ and $M = 0$ we obtain the unique solution

$$q_{00}(t) = \left(t_1 - t + \frac{1}{\alpha}\right)^{-1},$$

$$q_{ii}(t) = \left(\left(\frac{1}{\alpha} + \frac{1}{2\pi^2 i^2}\right) \mathrm{e}^{2\pi^2 i^2 (t_1 - t)} - \frac{1}{2\pi^2 i^2}\right)^{-1}, \; i \geqslant 1,$$

$$q_{ij}(t) = 0, \quad i \neq j.$$

(Note that since the equations for W and Q are time-invariant, we can solve them over the interval $[0, t_1]$ rather than $[t_0, t_1 + t_0]$.) In order to identify the spaces $H_1(t), H_2(t)$ introduced in Section III, we first identify $H = L_2[0,1]$ with l_2 in the usual way, i.e.

$$x \in L_2 \leftrightarrow \{\langle x, e_i \rangle\}_{i \geqslant 0} \in l_2.$$

Of course,

$$\sum_{i=0}^{\infty} |\langle x, e_i \rangle|^2 < \infty.$$

Now, let $H_2(-t)$ be the space of sequences of real numbers $\{x_i\}$ such that

$$\sum_{i=0}^{\infty} a_i^2(t) x_i^2 < \infty$$

together with the inner product

$$\langle \{x_i\}, \{y_i\} \rangle = \sum_{i=0}^{\infty} a_i^2 x_i y_i,$$

where

$$a_i(t) = \exp\{2\pi^2 i^2 t\}.$$

Then, clearly $H_2(-t)$ is complete and $H_2(-t) \subseteq l_2$ since $a_i \to \infty$ as $i \to \infty$ for $t > 0$. Identifying $Q(t;\alpha)$ with an operator on $H_2^*(-t_1 + t)$ in the obvious way, then we have $Q(t;\alpha) : H_2^*(t - t_1) \to l_2$, for $t \in [0, t_1)$. Note that

$$W_{ii}(t;\alpha) = \left(\left(\frac{1}{\alpha} + \frac{1}{2\pi^2 i^2}\right) \mathrm{e}^{2\pi^2 i^2 (t_1 - t)} - \frac{1}{2\pi^2 i^2}\right), \quad i \geqslant 1,$$

$$W_{00}(t;\alpha) = t_1 - t + \frac{1}{\alpha}$$

and so $W(t): l_2 \rightarrow H_2^*(t-t_1)$ as we expected from the general theory. Also, we have for the semigroup $T(-t)$;

$$T(-t): l_2 \rightarrow H_1^*(-t), \quad t > 0$$

where $H_1(-t)$ is the space of sequences $\{x_i\}$ such that

$$\sum_{i=1}^{\infty} a_i(t) x_i^2 < \infty.$$

Now, as $\alpha \rightarrow \infty$, we have

$$W_{ii}(t) = \frac{1}{2\pi^2 i^2} e^{2\pi^2 i^2 (t_1 - t)},$$

$$W_{00}(t) = t_1 - t$$

and so $W(0): l_2 \rightarrow \text{Range}\,(W(0))$, where Range $(W(0))$ is the space of sequences $\{x_i\}$ such that

$$\sum_{i=0}^{\infty} \frac{x_i^2}{b_i^2} < \infty,$$

where

$$b_i = 2\pi^2 i^2 e^{-2\pi^2 i^2 t_1}.$$

Since Range $(W(0)) \subset H_2^*(-t_1)$, with continuous injection, $W(0;\alpha) \rightarrow W(0;\infty)$ $(= W(0))$ in $\mathscr{L}(l_2, H_2^*(-t_1))$. However, it is easy to see that $Q(0;\alpha) \rightarrow Q(0;\infty)$ in the space $\mathscr{L}(\text{Range}\,(W(0)), l_2)$.

The feedback control is now

$$u(t) = -\sum_{i=0}^{\infty} q_{ii}(0) z_i(t) \phi_i$$

and the controlled system equations are

$$\begin{aligned} \dot{z}_0(t) &= -\frac{1}{t_1} z_0(t), \\ \dot{z}_i(t) &= -(\pi i)^2 z_i(t) - \frac{2\pi^2 i^2}{\{\exp(2\pi^2 i^2 t_1) - 1\}} z(t), \end{aligned} \tag{4.5}$$

and all that remains is to choose t_1 as a function of the state. The differences in the control laws obtained from the normal linear–quadratic solution (4.4) and the receding horizon solution (4.5) are now readily seen. For example, the zeroth harmonic in the former case is controlled to zero with time constant unity, whereas in the latter case z_0 converges to zero with "time constant" t_1. Hence if, for example, $|z_0|$ is large initially and we choose $t_1 = \frac{1}{|z_0|}$, then the large initial values of z_0 will be reduced much more quickly. Consider the high harmonics as $i \rightarrow \infty$; then

$$\alpha_i \simeq 2\pi^2 i^2,\ a_i \simeq -2\pi^2 i^2,\ b_i \simeq 0,\ a_i b_i \simeq 1.$$

Hence the value of q_{ii} in (4.3) is approximately

$$\frac{a_i}{1-a_i e^{-\alpha_i(t-t_1)}} = \frac{-2\pi^2 i^2 - \exp(-2\pi^2 i(t-t_1))}{-2\pi^2 i^2 \exp(2\pi^2 i^2(t_1-t)+1} \simeq \frac{1}{2\pi^2 i^2} \simeq 0$$

if $t \neq t_1$ and i is large. Hence in the case of linear optimal feedback, the control has little effect on the high harmonics, at least for times t which are small compared with t_1. Hence the ith harmonic will converge to zero with time constant $(\pi i)^{-2}$, in accordance with the natural unforced dynamics. However, in the receding horizon case, the value of q_{ii} is $2\pi^2 i^2 \{\exp(2\pi^2 i^2 t_1)-1\}^{-1}$ and, if t_1 is chosen to be $(2\pi^2 i^2)^{-1}$, for example, then $q_{ii}(0) = 2\pi^2 i^2(\mathrm{e}-1)^{-1}$ and $|z_i|$ decreases much faster than before.

It is clear, therefore, that by choosing t_1 to be a suitable function of $\|z\|$ the nonlinear feedback law will respond quickly to large values of $\|z\|$ but more slowly to small values of $\|z\|$. If one required the control to reduce large spatial gradients of the state quickly, then t_1 could be taken as a suitable function of

$$\|z\|_{H^2(0,1)} = \{\|z\|^2_{L^2[0,1]} + \|z_\xi\|^2_{L^2[0,1]}\}^{1/2}.$$

V. Conclusions

In this paper we have generalized the receding horizon principle to certain distributed systems. If the operator of the system generated a group then the generalization is straightforward, but in the case of a system defined by a semigroup it is necessary to introduce the spaces $H_1(-t)$, $H_2(-t)$ as we saw in Section III. However, when the operator A is self-adjoint and has a discrete spectral resolution, the feedback control can be found explicitly as in the two examples above. The control has been shown to have the desired property of reacting more quickly than the linear quadratic solution to large error states but more slowly in the presence of small disturbances.

The operator B in this paper has been assumed to be bounded. In the case of boundary control B is unbounded and it would be interesting to see whether the theory can be generalized to cover such types of control. This will be examined in a future paper.

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