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On the optimal control of nonlinear systems

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Abstract: The Lie algebra of tensors on a Hilbert space is used to obtain optimal controls for a class of nonlinear systems.

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1. Introduction

In this paper we shall consider the optimal control of a nonlinear analytic system by writing the system in the form of an infinite-dimensional bilinear system, using Carleman linearization; see Carleman [6], Brockett [5], Krener [8], Takata [10], Banks [1], Banks and Ashtiani [3]. However we shall use the theory of tensors (cf. Greub [5]) and consider the Lie algebra of tensors generated by the operators in the associated bilinear system. This simplifies considerably the approach taken by Banks and Yew [4], where a 'power series' in the tensor operators is obtained for the optimal control. We show here that particularly simple results hold if the Lie algebra is nilpotent.

In Section 2 we shall introduce some simple tensor notation and in Section 3 the bilinear realisation of nonlinear systems will be considered. The application of simple Lie algebra theory to the bilinear representation will be discussed in Section 4, and finally, in Section 5, a very simple example will be considered. The example is chosen mainly for its illustrative clarity rather than its applicability.

2. Notation and terminology

We shall use the elementary theory of tensors on a Hilbert space, usually written in component form. Thus, a tensor $\Phi \in \otimes_{k=1}^{n} \ell^2$, for some n, will be written $\phi_{i_1 \cdots i_n}$ where $0 \leqslant i_j < \infty$ for $1 \leq j \leq n$. If A is a tensor operator in $\mathscr{L}(\otimes_{k=1}^{n} \ell^2)$, i.e. a bounded operator from $\otimes_{k=1}^{n} \ell^2$ to itself, we shall denote its componentwise operation, in an expression of the form $\Psi = A\Phi$, by

$$\psi_{i_1 \cdots i_n} = \sum_{j_1=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} A_{i_1 \cdots i_n}^{j_1 \cdots j_n} \phi_{j_1 \cdots j_n}.$$

(Matrix multiplication then becomes $\psi_i = \sum_{j=1}^{n} A_i^j \phi_i$.) Finally the transpose A' of A is the tensor defined by $(A')_{i_1 \cdots i_n}^{j_1 \cdots j_n} = A_{j_1 \cdots j_n}^{i_1 \cdots i_n}$.

3. Bilinear realisation of nonlinear systems

In this section we shall consider a general (analytic) nonlinear system

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \tag{3.1}$$

where u is assumed, for simplicity, to be a scalar control. As in Banks [2], we shall associate with this system the 'augmented' system

$$\dot{x} = f(x, u), \qquad \dot{u} = v \tag{3.2}$$

where we restrict the controls to be differentiable. Now introduce the functions

$$\phi_{i_1 \cdots i_n} = x_1^{i_1} \cdots x_n^{i_n} u^j. \tag{3.3}$$

We have

$$\frac{\mathrm{d}\phi_{i_1 \cdots i_n j}}{\mathrm{d}t} = = \sum_{k=1}^{n} i_k x_1^{i_1} \cdots x_k^{i_k - 1} \cdots x_n^{i_n} u^j \dot{x}_k + j x_1^{i_1} \cdots x_n^{i_n} u^{j-1} \dot{u}.$$

Since f is analytic it has a Taylor series of the form

$$f_k(x, u) = \sum_{\alpha_1 = 0}^{\infty} \cdots \sum_{\alpha_n = 0}^{\infty} \sum_{\beta = 0}^{\infty} C^k_{\alpha_1 \cdots \alpha_n \beta} x_1^{\alpha_1} \cdots x_n^{\alpha_n} u^\beta$$

for some constants $C^k_{\alpha_1 \cdots \alpha_n \beta}$. Hence

$$\begin{aligned} \frac{\mathrm{d}\phi_{i_1 \cdots i_n j}}{\mathrm{d}t} &= \sum_{k=1}^{n} \sum_{\alpha_1 = 0}^{\infty} \cdots \sum_{\alpha_n = 0}^{\infty} \sum_{\beta = 0}^{\infty} i_k C^k_{\alpha_1 \cdots \alpha_n \beta} x_1^{\alpha_1 + i_1} \cdots x_k^{i_k + \alpha_k - 1} \cdots x_n^{\alpha_n + i_n} u^{j+\beta} + j x_1^{i_1} \cdots x_n^{i_n} u^{j-1} \\ &= \sum_{k=1}^{n} \sum_{\alpha_1 = 0}^{\infty} \cdots \sum_{\alpha_n = 0}^{\infty} \sum_{\beta = 0}^{\infty} i_k C^k_{\alpha_1 \cdots \alpha_n \beta} \phi_{\alpha_1 + i_1, \ldots, \alpha_k + i_k - 1, \ldots, \alpha_n + i_n, j+\beta} + j \phi_{i_1 \cdots i_n j-1} v. \end{aligned} \tag{3.4}$$

We shall denote the infinite-dimensional rank $(n+1)$ tensor $\phi_{i_1 \cdots i_n j}$ by ϕ; i.e.

$$(\Phi)_{i_1 \cdots i_n j} = \phi_{i_1 \cdots i_n j}.$$

Equation (3.4) can then be written in the form

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = A\Phi + vB\Phi \tag{3.5}$$

where A is the tensor operator defined by

$$(A\Phi)_{i_1 \cdots i_n j} = \sum_{k=1}^{n} \sum_{\alpha_1 = 0}^{\infty} \cdots \sum_{\alpha_n = 0}^{\infty} \sum_{\beta = 0}^{\infty} i_k C^k_{\alpha_1 \cdots \alpha_n \beta} \phi_{\alpha_1 + i_1, \ldots, \alpha_k + i_k - 1, \ldots, \alpha_n + i_n, j+\beta}$$

and B is the tensor operator defined by

$$(B\Phi)_{i_1 \cdots i_n j} = j \phi_{i_1 \cdots i_n j-1}.$$

Remark 3.1. We must remember that (3.5) is a tensor differential equation and that A and B are operators on a space of tensors.

Remark 3.2. If $u \in \mathbb{R}^m$, $m > 1$, then in an exactly similar way to that given above, we can show that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = A\Phi + \sum_{i=1}^{m} v_i B_i \Phi \tag{3.6}$$

where $\dot{u}_i = v_i$ and the B_i operators are defined in an obvious way. (In this case $\Phi = (\phi_{i_1 \cdots i_n j_1 \cdots j_m})$ where $\phi_{i_1 \cdots i_n j_1 \cdots j_m} = x_1^{i_1} \cdots x_n^{i_n} u_1^{j_1} \cdots u_m^{j_m}$.)

It has been shown by Banks and Ashtiani [3] that with suitable scaling, if the solutions of (3.1) are bounded then we can regard equation (3.5) (or (3.6)) as being defined on the space $\otimes_{k=1}^{n+1} \ell^2$, i.e. the tensor product of $n+1$ copies of ℓ^2.

Remark 3.3. The tensor operators A and B can be written in the forms

$$(A\Phi)_{i_1\cdots i_n j} = \sum_{\alpha_1=0}^{\infty} \cdots \sum_{\alpha_n=0}^{\infty} \sum_{\beta=0}^{\infty} a_{i_1\cdots i_n j}^{\alpha_1\cdots\alpha_n\beta} \phi_{\alpha_1\cdots\alpha_n\beta} \tag{3.7}$$

and

$$(B\Phi)_{i_1\cdots i_n j} = \sum_{\alpha_1=0}^{\infty} \cdots \sum_{\alpha_n=0}^{\infty} \sum_{\beta=0}^{\infty} b_{i_1\cdots i_n j}^{\alpha_1\cdots\alpha_n\beta} \phi_{\alpha_1\cdots\alpha_n\beta} \tag{3.8}$$

where

$$a_{i_1\cdots i_n j}^{\alpha_1\cdots\alpha_n\beta} = \sum_{k=1}^{n} i_k C^k_{\alpha_1-i_1,\ldots,\alpha_k-i_k+1,\ldots,\alpha_n-i_n,\beta-j} \quad \text{and} \quad b_{i_1\cdots i_n j}^{\alpha_1\cdots\alpha_n\beta} = j\delta_{i_1}^{\alpha_1}\delta_{i_2}^{\alpha_2}\cdots\delta_{i_n}^{\alpha_n}\delta_{j-1}^{\beta}.$$

(The coefficients $C^k_{i_1 i_2\cdots i_n j}$ are interpreted as 0 if any of the indices $i_1,\ldots,i_n$, j are zero.)

4. Optimal control of nonlinear systems

Consider now the nonlinear control system

$$\dot{x} = f(x, u) \tag{4.1}$$

and suppose that we wish to make $\| x(T) \|$ as small as possible while maintaining $\| u(\cdot) \|_{L^2(0,T)}$ bounded. Then we can consider the cost function

$$J_x(u) = \frac{\pi^2}{T^2}\int_0^T u^2 \,\mathrm{d}t + \eta x'(T)Fx(T) \tag{4.2}$$

where η is chosen to be large, and the purpose of the factor π^2/T^2, which is introduced for convenience, will become clear shortly.

Now replace (4.1) by the augmented system (3.2), i.e.

$$\dot{x} = f(x, u), \qquad \dot{u} = v. \tag{4.3}$$

We have seen that this system is equivalent to a system of the form

$$\dot{\Phi} = A\Phi + vB\Phi \tag{4.4}$$

where $\Phi \in \otimes_{k=1}^{n+1}\ell^2$ and A, $B \in \mathscr{L}(\otimes_{k=1}^{n+1}\ell^2)$. In order to develop a cost function for (4.4) first note that since (4.3) now contains u as a state variable we require an initial condition on u. Without loss of generality we shall set $u(0) = 0$. Consider the function

$$w(t) = u(t) - \frac{t}{T}u(T), \quad 0 \le t \le T.$$

We have

$$w(0) = w(T) = 0, \qquad \dot{w}(t) = \dot{u}(t) - u(T)/T = v(t) - u(T)/T \tag{4.5}$$

and

$$u^2(t) = w^2(t) + \frac{2t}{T}u(t)u(T) + \frac{t^2}{T^2}u^2(T). \tag{4.6}$$

Now recall the well-known inequality

$$\pi^2 \| \phi \|_{L^2}^2 \le \| \dot{\phi} \|_{L^2}^2, \quad \phi \in H_0^1([0,1]),$$

where H_0 is the usual Sobolev space. Hence we have, using (4.5) and (4.6),

$$\|u\|_{L^2}^2 \le \|w(t)\|_{L^2}^2 + \frac{2}{T}|u(T)|\left(\tfrac{1}{3}T^3\right)^{1/2}\|u\|_{L^2} + Tu^2(T)$$

$$\le \frac{T^2}{\pi^2}\|v(t) - u(T)/T\|_{L^2}^2 + \frac{2}{T}|u(T)|\left(\tfrac{1}{3}T^3\right)^{1/2}\|u\|_{L^2} + Tu^2(T).$$

Hence in the limit as $u(T) \to 0$ with bounded $\|u\|_{L^2}$ we have

$$\|u\|_{L^2}^2 \le \frac{T^2}{\pi^2}\|v\|_{L^2}^2. \tag{4.7}$$

It follows that if we choose the cost function

$$J_\Phi(v) = \int_0^T v^2\,\mathrm{d}t + \Phi'(T)\Gamma\Phi(T) \tag{4.8}$$

where $\Gamma \in \mathscr{L}(\otimes_{k=1}^{n+1}\ell^2)$ is defined by

$$\Gamma_{i_1 \cdots i_n j}^{x_1 \cdots x_n \beta} = \eta \sum_{l=1}^{n}\sum_{k=1}^{n} F_l^k \delta_0^{\alpha_1} \cdots \delta_1^{\alpha_k} \cdots \delta_0^{\alpha_n}\delta_0^{\beta}\delta_{i_1}^0 \cdots \delta_{i_l}^1 \cdots \delta_{i_n}^0\delta_j^0 + \nu\delta_0^{\alpha_1} \cdots \delta_0^{\alpha_n}\delta_{i_1}^0 \cdots \delta_{i_n}^0\delta_1^{\beta}\delta_j^1 \tag{4.9}$$

then making η and ν large will make $\|x(T)\|^2$ and $\|u(T)\|^2$ small and $\|u\|_{L^2}^2$ will be bounded by (4.7) (which is approximately valid for small $|u(T)|$) and in the limit as $u(T) \to 0$,

$$J_x(u) \le J_\Phi(v).$$

We shall now consider the problem (4.4) with the cost function (4.8) to replace (4.1) and (4.2). Although the two problems are not equivalent, it seems reasonable to minimise the control u at the final time and the square of $\dot{u}$ over $[0,T]$.

The Hamiltonian for this problem is

$$H = v^2 + \Lambda'(A\Phi + vB\Phi)$$

and so we obtain the equations

$$\dot{\Lambda}' = -(\Lambda'A + v\Lambda'B), \qquad \Lambda(T) = \Gamma\Phi(T), \qquad \dot{\Phi} = A\Phi + vB\Phi, \qquad 2v = -\Lambda'B\Phi. \tag{4.10}$$

Since $\dot{v} = (-\mathrm{d}/\mathrm{d}t)(\Lambda'B\Phi) = -\Lambda'[B,A]\Phi$, we have that the optimal control is a constant if B commutes with A (as tensor operators). More generally we have (Krener [9]):

Lemma 4.1. *For any constant tensor operator X we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Lambda'X\Phi) = \Lambda'[X, A + vB]\Phi.$$

Proof. From (4.10) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Lambda'X\Phi) = \dot{\Lambda}'X\Phi + \Lambda'X\dot{\Phi} = -(\Lambda'A + v\Lambda'B)X\Phi + \Lambda'X(A + vB)\Phi = \Lambda'[X, A + v]\Phi. \qquad \square$$

Now consider the Lie algebra generated by the tensor operators A, B, i.e. the set of linear combinations of all brackets generated by A and B. Denote this Lie algebra by $T(A, B)$ and denote the Lie algebra of all tensor operators (of rank $2(n+1)$) by T. In order to consider finite-dimensional Lie algebras we shall truncate the bilinear representation (3.5) of the original nonlinear system (3.2) so that the tensors involved have indices ranging from 0 to, say, l. Since the system is analytic and we are controlling the system in some bounded region, as l increases we shall obtain an accurate representation of the nonlinear system. We will denote the corresponding Lie algebras by $T_l(A, B)$ and T_l. However, to avoid further confusing subscripts we shall not write A_l, B_l, etc. for the truncated tensors, but use the same notation A and B, leaving it to the context to indicate which is intended.

Since T_l is a finite-dimensional Lie algebra (of dimension $l^{2(n+1)}$) and $T_l(A, B)$ is a subalgebra of T_l it follows that $T_l(A, B)$ is a finite-dimensional Lie algebra of dimension $m_l \leq l^{2(n+1)}$. Let $X_1, \ldots, X_{m_l}$ be a basis of $T_l(A, B)$ and define

$$v_1 = -2v = \Lambda' B\Phi, \quad v_2 = \Lambda' X_1 \Phi, \quad \ldots, \quad v_{m_l+1} = \Lambda' X_{m_l}\Phi.$$

Thus,

$$\dot{v}_1 = \Lambda'[B, A]\Phi, \quad \dot{v}_2 = \Lambda'[X_1, A + vB]\Phi, \quad \ldots, \quad \dot{v}_{m_l+1} = \Lambda'\left[X_{m_l}, A + vB\right]\Phi. \tag{4.11}$$

Now, each term of the form $[X_i, A]$, $[X_i, B]$ belongs to $T_l(A, B)$ and so we may write

$$[X_i, A] = \sum_{j=1}^{m_l} \alpha_{ij} X_j, \qquad [X_i, B] = \sum_{j=1}^{m_l} \beta_{ij} X_j$$

for some constants α_{ij}, β_{ij}. Similarly, $[B, A] \in T_l(A, B)$ so

$$[B, A] = \sum_{j=1}^{m_l} b_j X_j,$$

for some constants b_j. Substituting into (4.11) we have

$$\begin{aligned} \dot{v}_1 &= \sum_{j=1}^{m_l} b_j v_{j+1}, \\ \dot{v}_i &= \sum_{j=1}^{m_l} \alpha_{ij} v_{j+1} + \tfrac{1}{2} v_1 \sum_{j=1}^{m_l} \beta_{ij} v_{j+1}, \quad 2 \leq i \leq m_l + 1. \end{aligned} \tag{4.12}$$

Write $\mu = (v_1, v_2, \ldots, v_{m_l+1})^{\mathrm{T}}$. Then (4.12) is of the form $\dot{\mu} = f(\mu)$ for some function f. Suppose μ_0 is a guess at $\mu(0)$. Then solving (4.12) (numerically) gives $\mu(t)$, $t \in [0, T]$, which can then be used to find $x(t)$ from

$$\dot{\Phi} = A\Phi + \mu_1 B\Phi, \quad \Phi(0) = \Phi_0$$

(again numerically). Then the cost functional becomes

$$J(\mu_0) = \int_0^T \mu_1^2(t;\, \mu_0)\, \mathrm{d}t + \Phi'(T;\, \mu_0)\Gamma\Phi(T;\, \mu_0)$$

which is a function of $m_l + 1$ variables and can be optimised numerically.

However, the computation involved in this method is likely to be considerable and the main purpose of the present paper is to show that we can simplify the application of the method a great deal when the algebra $T_l(A, B)$ is nilpotent. This will then generalize the above remark that the optimal control is constant when $[A, B] = 0$. Recall that a Lie algebra L is nilpotent if $(\mathrm{Ad}\ L)^k = 0$ for some $k > 0$, where $(\mathrm{Ad}\ L)X = [L, X]$, $X \in L$.

Lemma 4.2. *If $X \in (\mathrm{Ad}\ T_l(A, B))^l B$ then $(\mathrm{d}/\mathrm{d}t)(\Lambda' X\Phi) = \Lambda' Y\Phi + v\Lambda' Z\Phi$ where $Y, Z \in (\mathrm{Ad}\ T_l(A, B))^{l+1}B$. Hence if $T_l(A, B)$ is nilpotent and $(\mathrm{Ad}\ T_l(A, B))^k = 0$ then $\Lambda' X\Phi$ is constant for any $X \in (\mathrm{Ad}\ T_l(A, B))^{k-1}$.*

Proof. This follows from Lemma 4.1, since

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Lambda' X\Phi) = \Lambda'[X, A + vB]\Phi = -\Lambda'(\mathrm{Ad} A)X\Phi - v\Lambda'(\mathrm{Ad} B)X\Phi. \qquad \square$$

It is then easy to see that equations (4.12) can be written in the form

$$\dot{v}_1 = b'\mu \qquad \dot{\mu} = \Gamma\mu + \tfrac{1}{2}v_1\Delta\mu,$$

where $b = (b_1, \ldots, b_m)'$ and Γ, Δ are nilpotent matrices. For example, if $(\operatorname{Ad} T_t(A, B))^2 = 0$, then

$$2\dot{v} = \Lambda'[B, A]\Phi \quad \text{and} \quad 2\ddot{v} = \Lambda'[[B, A], A + uB]\,\Phi,$$

so $\ddot{v} = 0$ and the optimal control is of the form $v^* = c_1 + c_2 t$.

5. Example

To illustrate the theory developed above we consider the simple system

$$\dot{x} = x^2 u, \qquad J(u) = \int_0^T u^2\,\mathrm{d}t + x^2(T). \tag{5.1}$$

Then we have

$$\dot{x} = x^2 u, \qquad \dot{u} = v,$$

and putting $\phi_{ij} = x^i u^j$ it follows that

$$\dot{\phi}_{ij} = ix^{i-1}u^j\dot{x} + jx^i u^{j-1}\dot{u} = ix^{i+1}u^{j+1} + jx^i u^{j-1}v = i\phi_{i+1j} + j\phi_{ij-1}v.$$

Hence, $A_{ij}^{kl} = i\delta_{j+1}^k\delta_{j+1}^l$, $B_{ij}^{kl} = j\delta_i^k\delta_{j-1}^l$, and so

$$A_{ij}^{kl}B_{kl}^{mn} = i(j+1)\delta_{i+1}^m\delta_j^n, \qquad B_{ij}^{kl}A_{kl}^{mn} = ij\delta_{i+1}^m\delta_j^n.$$

Thus,

$$C_{ij}^{kl} \triangleq [A, B]_{ij}^{kl} = i\delta_{i+1}^k\delta_j^l.$$

However, $[C, A] = [C, B] = 0$ and so the optimal control u^* is of the form $\dot{u}^* = c_1 t + c_2$, i.e.

$$u^* = \tfrac{1}{2}c_1 t^2 + c_2 t + c_3$$

for some constants c_1, c_2, c_3. Now choose c_3 so that $u^*(T) = 0$, so that the inequality (4.7) is valid. Hence we may write

$$u^* = c_1(t - T)^2 + c_2(t - T) \tag{5.2}$$

for new constants c_1, c_2, and so

$$\frac{1}{x(T)} - \frac{1}{x(0)} = -\int_0^T u^*(t)\,\mathrm{d}t = -\tfrac{1}{3}c_1T^3 + \tfrac{1}{2}c_2T^2. \tag{5.3}$$

Substituting for $x(T)$ and u^* from (5.2) and (5.3) into (5.1) gives $J(u^*)$ as a function of c_1 and c_2 which can easily be minimized.

6. Conclusions

In this paper we have derived a general method for obtaining a dynamical model of the optimal control for certain nonlinear systems. The method has been shown to be effective mainly when the Lie algebra generated by the system matrices A and B is nilpotent. The application of the method to the study of controllability, stabilizability, etc. of nonlinear systems would also be desirable and this will be considered in future papers.

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