Applied Mathematics and Computation 216 (2010) 655–661

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# On the calculation of the Kauffman bracket polynomial

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ARTICLE INFO

*Keywords:*
Knot invariant
Knot polynomial
Kauffman bracket polynomial
Computation
Cyclic permutation

ABSTRACT

In this paper, we present a new algorithm to evaluate the Kauffman bracket polynomial. The algorithm uses cyclic permutations to count the number of states obtained by the application of '*A*' and '*B*' type smoothings to the each crossing of the knot. We show that our algorithm can be implemented easily by computer programming.



## 1. Introduction

The theory of knots and links has a long and distinguished history dating back hundreds of years. Even the modern mathematical theory is nearly one hundred years old, starting essentially with the work of Alexander and Reidemeister [1,2]. It has gained more attention since it was associated with the study of dynamical systems, more specifically chaotic attractors as unstable periodic orbits of chaotic attractors which can be viewed as knots in three dimensional space [17–20]. More recently, a large number of isotopy invariants of knots and links have been developed many of which are generalization of the Alexander polynomial. These include the Jones and HOMFLY polynomials [3,4] and the Kauffman bracket polynomial [5]. There are also the finite type invariants of Vassiliev [6,7]. The main problem with any topological invariant of a knot is the explosion in computation with the number of crossings. There is also the problem that many invariants are 'pictorial' and require some human insight into their application. For example, the Kauffman bracket polynomial $\langle K \rangle$, for a knot or link $K$ requires a count of the number of simple closed curves (or states) which occur when a choice of an '*A*' or '*B*' smoothing is made at each crossing (see Section 2 for more details). Recognizing these closed curves is quite complex [8]. One way of determining these closed curves is to perform a pictorial state diagram which requires manual decoration each crossing of the knot diagram. However we shall show here that there is a simple computational interpretation of these curves and that they can be identified by finding the cycles in a certain permutation. This leads to a simple algorithm for computing the Kauffman bracket polynomial. In recent years there has been a boom in developing computer implementable algorithms to calculate knot invariants. Gouesbet et al. [9,10] developed a computer program written in C programming language to compute HOMFLY and the two variable Kauffman polynomials by interpreting the skein relations into algebraic symbols by using Gauss codes. Simsek et al. [11] wrote a computer program in Delphi programming language to compute the Alexander polynomial by using free derivatives that is obtained from the braid representation of a given knot. One can find other computational interpretations of the knot polynomials in the literature [12–15]. Each of these algorithms has some advantages depending on the specific knot polynomial. The algorithm we propose here has the advantage of using only simple enumeration of crossings which saves us a significant amount of pre-computational work for the evaluation of Kauffman bracket polynomial, which is a one variable polynomial. The application of this technique is possible for the two variable polynomials with some modifications of the algorithm which may increase the amount of computation.

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0096-3003/$ - see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2010.01.102

The outline of the paper is the following. In Section 2 we give a brief introduction to the Kauffman bracket polynomial, together with an example of its computation. In Section 3 we introduce the basic method of permutations and cyclic representation and give examples to illustrate the method. Finally we draw the conclusions.

## 2. The Kauffman bracket polynomial

Suppose we are given a standard knot (or link) diagram $K$ in the plane which has a finite number of double point crossings. To each double point crossing we associate two 'smoothings' as shown in Fig. 1.

We shall think of these smoothings as the removal of a small disk of the plane with centre the crossing point and the intersection of two curve segment as in Fig. 2.

It can be seen easily that if at each crossing point of the knot diagram, we make a choice of $A$ or $B$ smoothing, then we obtain a member of simple closed (non-intersecting) curves. Then the bracket polynomial of the knot projection $K$ is defined as

$$\langle K\rangle(A,B,d)=\sum_{\substack{i_j\in\{0,1\}\\ 1\leqslant j\leqslant\gamma}} A^{i_1}B^{1-i_1}A^{i_2}B^{1-i_2}\cdots A^{i_\gamma}B^{1-i_\gamma}\times d^{L(i_1,i_2,\ldots,i_\gamma)-1}, \tag{1}$$

where we have assumed that $K$ contains $\gamma$ crossings, the sum is over all choices of $A$ or $B$ smoothings at each crossing ($2^\gamma$ choices in total) and $L^{(i_1,i_2,\ldots,i_\gamma)}$ is the number of closed loops which occurs for a particular choice of $A$ or $B$ crossings. $\langle K\rangle$ is a polynomial in the indeterminates $A$, $B$ and $d$, where $B=A^{-1}$ and $d=-A^2-A^{-2}$ and it is an invariant under the second and third Reidemeister moves. The normalized bracket polynomial $P(A)$ of $K$ is defined as

$$P(A)=(-A^3)^{-w(K)}\langle K\rangle(A,B,d)|_{B=A^{-1},d=-A^2-A^{-2}}, \tag{2}$$

which is now just a polynomial in $A$. Here, $w(K)$ is the writhe number of $K$ defined as follows:

$$w(K)=\sum_{i=1}^{\gamma}(\pm 1), \tag{3}$$

where the sum is over all the crossings of $K$ and we choose $\pm1$ according to the crossing, as in Fig. 3.

It can be shown that $P(A)$ is an invariant of ambient isotopy; in particular, it is invariant under the three Reidemeister moves [16].

**Example:.** In this example we show how to compute the bracket polynomial of the four knot given in Fig. 4.

In order to compute the bracket polynomial we first need to evaluate the states of the knot by sequentially applying $A$ and $B$ type smoothings to each crossing. The pictorial application of this process is very demanding and the probability of making mistakes is very high. We depict the states of the four knot in Fig. 5 in order to compare the manually obtained pictorial results with those obtained by our algorithm generated by the computer program which will be illustrated in Section 3. Fig. 5 contains information about the sequence of $A$ or $B$ type smoothing at each crossing. The order of the sequence follows the order given in Fig. 4. We underline that there is no rule for ordering and it can be done randomly as long as the smoothing sequence is applied consistently.

## 3. Knots and permutations

In this section we will present a new interpretation of knot diagrams in terms of cycles of permutations, which will provide a simple algorithm for the computation of knot polynomials. Consider any knot (or link) diagram with $N$ crossings (nodes) as shown in Fig. 6.

Around each crossing point we place a 'small' circle which does not contain any crossing. We number the points of intersection of the circles with the lines of the knot in the order as shown in Fig. 7 (i.e. we start numbering a node with the overcrossing). Since there are $N$ nodes there will be $2N$ arcs of the knot (obtained by removing the circles and their interiors) with a unique number from 1 to $4N$ at each end. This will position the set $\{1,2,\ldots,4N\}$ into $2N$ subsets $\{A_1,A_2,\ldots,A_{2N}\}$ where $A_i=\{i_1,i_2\}$ and $i_1,i_2$ are the number at the ends of an arc. To illustrate this, consider the four knot in example.

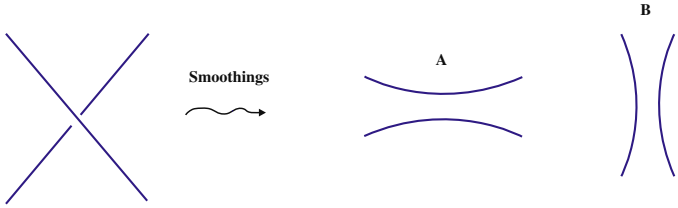



**Fig. 1.** $A$ and $B$ type smoothings.

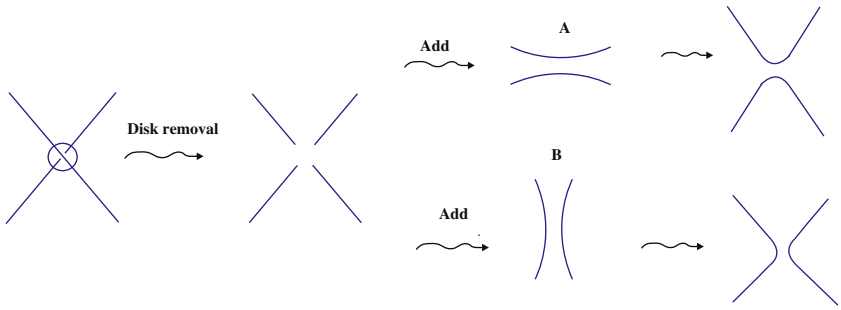



**Fig. 2.** *A* and *B* type smoothings by 'surgery'.

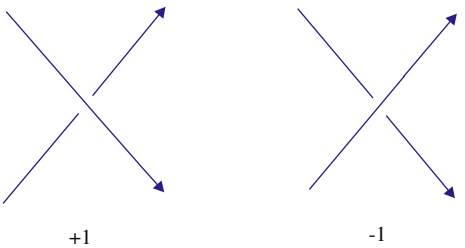



**Fig. 3.** +1 and −1 type crossings.

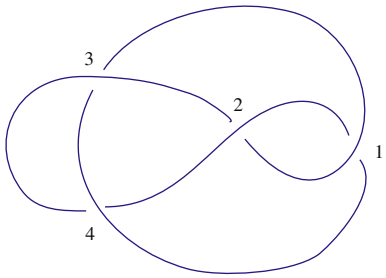



**Fig. 4.** The four knot.

We write this information in the form

$$P_1 = \begin{pmatrix} 1_1 & 1_2 \\ 2_1 & 2_2 \\ \vdots & \vdots \\ (2N)_1 & (2N)_2 \end{pmatrix}. \tag{4}$$

Note that each number $i_j$ in this array belongs to $1, 2, \ldots, 4N$ and each element of this set appears exactly once in this array (Fig. 8(b)).

We form a second array corresponding to counting around the nodes in clockwise order (Fig. 8(c)).

$$P_2 = \begin{pmatrix} 1 & 2 \\ 4 & 3 \\ 5 & 6 \\ 8 & 7 \\ \vdots & \vdots \\ 4N-3 & 4N-2 \\ 4N & 4N-1 \end{pmatrix}. \tag{5}$$

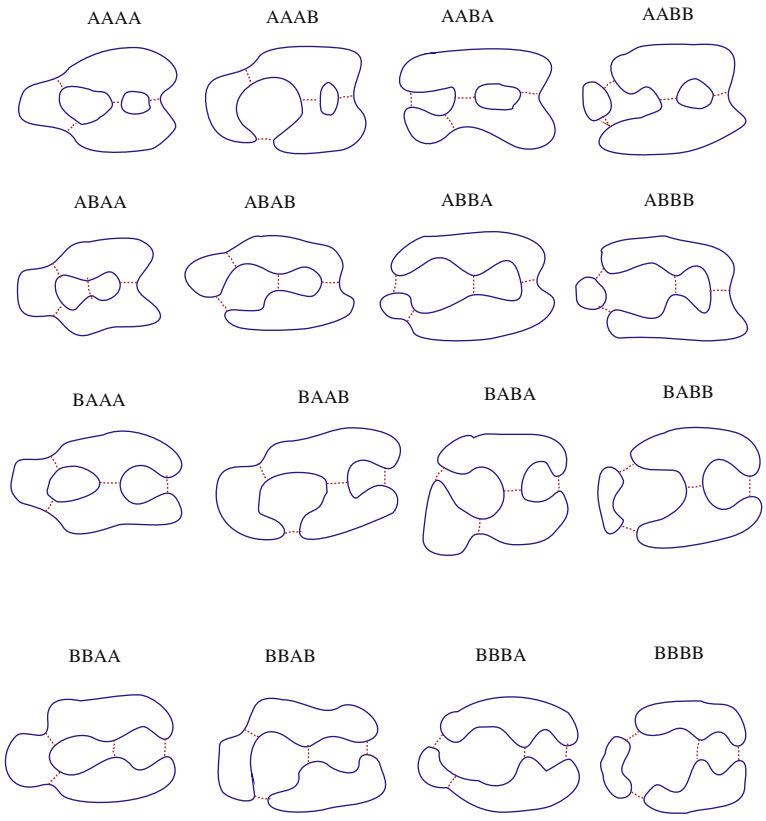



**Fig. 5.** Closed curves of four knot.

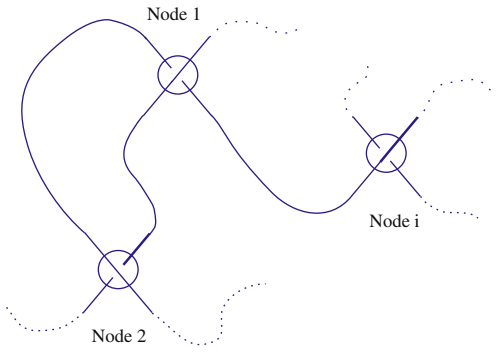



**Fig. 6.** A portion of a knot diagram.

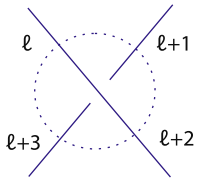
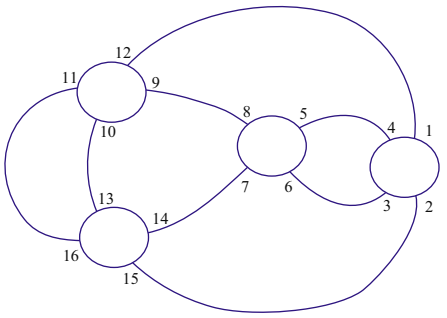



**Fig. 7.** Numbering the knot arcs.




(a)

$$\begin{pmatrix} 1 & 12 \\ 2 & 15 \\ 3 & 6 \\ 4 & 5 \\ 7 & 14 \\ 8 & 9 \\ 10 & 13 \\ 11 & 16 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \\ 5 & 6 \\ 8 & 7 \\ 9 & 10 \\ 12 & 11 \\ 13 & 14 \\ 16 & 15 \end{pmatrix}$$

(c)

**Fig. 8.** Numbered arcs from example.

In the array $P_2$ we think of 1 being connected to 2 and 3 being connected to 4, etc. In this way, $P_2$ contains the information at the nodes where we perform '$A$' type smoothings at every node. If we choose a '$B$' type smoothing at node 1, then $P_2$ is replaced by

$$P_2' = \begin{pmatrix} 4 & 1 \\ 3 & 2 \\ 5 & 6 \\ 8 & 7 \\ \vdots & \vdots \end{pmatrix}. \tag{6}$$

In a similar way, we introduce a '$B$' type smoothing at the $i^{th}$ node by rotating the numbers $(4i-3, 4i-2, 4i-1, 4i)$ in $P_2$ once clockwise. Note that, throughout the algorithm, the array $P_1$ is fixed. Next we form a permutation of the set $\{1, 2, \ldots, 4N\}$ as follows. Start with $P_1$ and choose $1_1$, i.e. the first number in $P_1$, and write $\sigma(1_1) = 1_2$. Then in $P_2$ find the number $1_2$ and write $\sigma(1_2) = \ell_2$, where $\ell_2$ is the other number in the row of $P_2$ containing $1_2$. Then return to $P_1$ and find $\ell_2$ in $P_1$ and write $\sigma(\ell_2) = k_2$, where $(\ell_2, k_2)$ or $(k_2, \ell_2)$ is a row in $P_1$. Proceeding in this way we obtain a permutation $\sigma$. We can do the same for any choice of '$A$' or '$B$' smoothings at the nodes by considering the appropriate modification of $P_2$ (such as $P_2'$).

In this way we will obtain $2^N$ permutations $\sigma_k, 1 \leqslant k \leqslant 2^N$ of the set $\{1, 2, \ldots, 4N\}$, each one corresponding to one of the $2^N$ possible choices of '$A$' or '$B$' type smoothnigs at the nodes. The importance of these permutations is that the members of cycles of each permutation gives the number of states in the Kauffman diagram (i.e. the number of distinct closed curves which appear in the knot diagram when a given choice of '$A$' or '$B$' smoothings is made). This is clear, since any cycle of a permutation corresponds to one circuit of a set of arcs which form a simple closed curve.

**Example:.** Consider again the four knot in the previous example. We illustrate the above ideas by considering two '$A$' type smoothings and two '$B$' type smoothings. In this case the states appear as in Fig. 9(a).

The $P_1$ and $P_2$ arrays are

$$P_1 = \begin{pmatrix} 1 & 12 \\ 2 & 15 \\ 3 & 6 \\ 4 & 5 \\ 7 & 14 \\ 8 & 9 \\ 10 & 13 \\ 11 & 16 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 2 \\ 4 & 3 \\ 5 & 6 \\ 8 & 7 \\ 12 & 9 \\ 11 & 10 \\ 16 & 13 \\ 15 & 14 \end{pmatrix} \tag{7}$$

and the permutation associated with the diagram is

$$\begin{pmatrix} 1 & 12 & 9 & 8 & 7 & 14 & 15 & 2 & 3 & 6 & 5 & 4 & 10 & 13 & 16 & 11 \\ 12 & 9 & 8 & 7 & 14 & 15 & 2 & 1 & 6 & 5 & 4 & 3 & 13 & 16 & 11 & 10 \end{pmatrix}. \tag{8}$$

The cyclic decomposition of this permutation is

$$(1 \quad 12 \quad 9 \quad 8 \quad 7 \quad 14 \quad 15 \quad 2), \quad (3 \quad 6 \quad 5 \quad 4), \quad (10 \quad 13 \quad 16 \quad 11), \tag{9}$$

corresponding to the three closed curves in the state diagram. If instead we consider three '$A$' type and one '$B$' type smoothings as in Fig. 9(b), we get the permutation

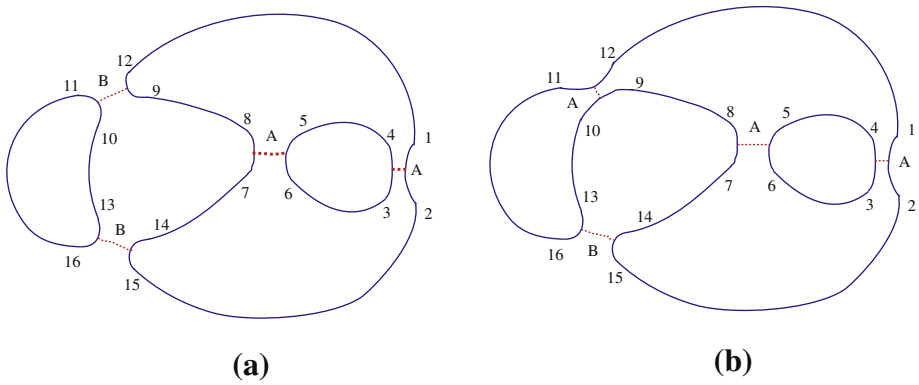



**Fig. 9.** States for AABB and AAAB type smoothings.

$$\begin{pmatrix} 1 & 12 & 11 & 16 & 13 & 10 & 9 & 8 & 7 & 14 & 15 & 2 & 3 & 6 & 5 & 4 \\ 12 & 11 & 16 & 13 & 10 & 9 & 8 & 7 & 14 & 15 & 2 & 1 & 6 & 5 & 4 & 3 \end{pmatrix}, \tag{10}$$

from which we see that there are two closed curves corresponding to the cycles

$$(1 \quad 12 \quad 11 \quad 16 \quad 13 \quad 10 \quad 9 \quad 8 \quad 7 \quad 14 \quad 15 \quad 2), \quad (3 \quad 6 \quad 5 \quad 4). \tag{11}$$

We have given the full list of decomposition of the cyclic permutations for the four knot in Table 1, which is generated by a computer program written in MATLAB environment. The computation time for the example is obtained as 0.026 s. on a desktop PC with an Intel Core2Duo 2.0 processor, 1 GB RAM. However less computation time can be obtained depending on the performance of the computer, programming language and ability of the programmer. One can see that the number of cycles obtained from cyclic permutations method by our computer program given in Table 1 is equal to the number of closed curves obtained by manual drawing given in Fig. 5.

Counting cycles is, of course, a very simple algorithm. Once we know the value of $L(i_1, i_2, \ldots, i_\gamma)$ in Eq. (1), the algorithm is then a simple matter of counting the terms in polynomial. For the four knot the value of $L(i_1, i_2, \ldots, i_\gamma)$ is given in Table 1. Hence, the bracket polynomial for the four knot is

$$\langle K \rangle = A^8 - A^4 - A^{-4} + A^{-8} + 1. \tag{12}$$

Since the writhe number of the four knot $w(K) = 0$, we obtain the normalized bracket polynomial $P(A)$ is same with the $\langle K \rangle$ given in Eq. (12). One can easily apply our algorithm to any knot by numbering the crossings in the knot diagram. The sequence of the numbers is not important as long as the construction of the $P_1$ and $P_2$ matrices are consistent with the numberings. The MATLAB code of our algorithm is available upon request.

**Table 1**
Computer generated cyclic permutations for the four knot.

| Smoothing Sequence | Number of cycles | Cyclic permutation decomposition |
|---|---|---|
| *AAAA* | 3 | (1 12 11 16 15 2)(3 6 5 4)(7 14 13 10 9 8) |
| *AAAB* | 2 | (1 12 11 16 13 10 9 8 7 14 15 2)(3 6 5 4) |
| *AABA* | 2 | (1 12 9 8 7 14 13 10 11 16 15 2)(3 6 5 4) |
| *AABB* | 3 | (1 12 9 8 7 14 15 2)(3 6 5 4)(10 13 16 11) |
| *ABAA* | 2 | (1 12 11 16 15 2)(3 6 7 14 13 10 9 8 5 4) |
| *ABAB* | 1 | (1 12 11 16 13 10 9 8 5 4 3 6 7 14 15 2) |
| *ABBA* | 1 | (1 12 9 8 5 4 3 6 7 14 13 10 11 16 15 2) |
| *ABBB* | 2 | (1 12 9 8 5 4 3 6 7 14 15 2)(10 13 16 11) |
| *BAAA* | 2 | (1 12 11 16 15 2 3 6 5 4)(7 14 13 10 9 8) |
| *BAAB* | 1 | (1 12 11 16 13 10 9 8 7 14 15 2 3 6 5 4) |
| *BABA* | 1 | (1 12 9 8 7 14 13 10 11 16 15 2 3 6 5 4) |
| *BABB* | 2 | (1 12 9 8 7 14 15 2 3 6 5 4)(10 13 16 11) |
| *BBAA* | 1 | (1 12 11 16 15 2 3 6 7 14 13 10 9 8 5 4) |
| *BBAB* | 2 | (1 12 11 16 13 10 9 8 5 4)(2 15 14 7 6 3) |
| *BBBA* | 2 | (1 12 9 8 5 4)(2 15 16 11 10 13 14 7 6 3) |
| *BBBB* | 3 | (1 12 9 8 5 4)(2 15 14 7 6 3)(10 13 16 11) |

## 4. Conclusion

We have presented a novel method to evaluate the number of states in the Kauffman bracket polynomial. The method uses cyclic permutations which is computer implementable and more reliable than manual methods. The method also saves a significant amount of pre-computational work since it only utilizes a simple numbering of the crossings. As a future work, we shall extend our algorithm to interpret the Reidemeister moves which may yield some simplification in a given knot. Moreover one can also implement the cyclic permutations method to the two variable knot polynomials such as HOMFLY and the Kauffman polynomials by possible interpretations of the skein-template algorithm.

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