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Optimal control of distributed bilinear systems

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Abstract: The optimal control problem for a bilinear distributed parameter system subject to a quadratic cost functional is solved. It is shown that the optimal control is given by a convergent power series in the state with tensor coefficients.

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1. Introduction

In a recent paper, Banks and Yew [3] have obtained a class of suboptimal controls for a bilinear system

$$\dot{x} = Ax + uBx$$

where u is a scalar, x belongs to a Hilbert space and A and B are bounded operators, subject to the quadratic cost

$$J = \langle x, Gx \rangle + \int_0^{t_f} \{ \langle x, Mx \rangle + ru^2 \} \, \mathrm{d}t.$$

The feedback turns out to be a power series in x with tensor coefficients. The convergence of this series was not proved, however, and so we proposed truncated versions of the series as suboptimal controls. In this paper we wish to provide the missing convergence proof and also generalise the results to the case where A is an unbounded operator, which generates a semigroup.

In Section 2, we shall present the essential tensor theory for the development of the optimal control and then in Section 3 we shall prove the convergence of the series derived in Banks and Yew [3]. Finally, in Section 4, we shall give an example of a system in which A is self-adjoint and has compact resolvent which will enable us to obtain representations of the tensor coefficients of the feedback control series in terms of the spectrum of A.

We shall see that, in contrast to the linear-quadratic regulator problem, the optimal feedback control is only defined for states satisfying a certain bound, which depends on the horizon time. In other words, we must make the horizon time dependent on the initial states.

2. Tensor theory in Hilbert space

We shall first briefly review the theory of tensors defined in a Hilbert space (Greub [5]). If E and F are vector spaces and G is any vector space, then the tensor product of E and F is the pair $(E \otimes F, \otimes)$ where $E \otimes F$ is a vector space and $\otimes$ is a bilinear mapping with the universal property: if ϕ is any bilinear mapping then there exists a unique linear mapping $f: E \otimes$ G such that the diagram

$$\begin{array}{ll} E \times F \xrightarrow{\phi} G & \\ \otimes \downarrow \quad \nearrow f & \\ E \otimes F & \end{array}$$

commutes. If H is a Hilbert space then we define, by induction, the vector space $H_i = H \otimes \cdots \otimes H$ (i copies of H). Then we can make H_i into a Hilbert space by defining

$$\langle x_1 \otimes \cdots \otimes x_i, y_1 \otimes \cdots \otimes y_i \rangle_{H_i} = \prod_{j=1}^{i} \langle x_j, y_j \rangle_H.$$

Let $\mathscr{H}$ be the graded Hilbert space $\otimes_{i=1}^{\infty}(\otimes_i H) = \otimes_{i=1}^{\infty} H_i$ consisting of sequences $h = \{h_1, h_2, \dots\}$ ($h_i \in H_i$) such that

$$\| h \| \triangleq \left\{ \sum_{i=1}^{\infty} \| h_i \|_H^2 \right\}^{1/2} < \infty.$$

Next, denote by $\mathscr{L}(H)$ the space of bounded linear operators on the Hilbert space H. Then if $P \in \mathscr{L}(H_i)$, $C \in \mathscr{L}(H)$ we define the operator $PC \in \mathscr{L}(H_i)$ by

$$(PC)(x_1 \otimes \cdots \otimes x_i) = P(Cx_1 \otimes x_2 \otimes \cdots \otimes x_i) + P(x_1 \otimes Cx_2 \otimes \cdots \otimes x_i) + \cdots + P(x_1 \otimes x_2 \otimes \dots \otimes Cx_i)$$

(and by extension by linearity to all of H_i). Moreover, we define the adjoint operator P^* of P in the usual way:

$$\langle P^*(x_1 \otimes \cdots \otimes x_i), (y_1 \otimes \cdots \otimes y_i) \rangle = \langle (x_1 \otimes \cdots \otimes x_i), P(y_1 \otimes \cdots \otimes y_i) \rangle$$

and if $P \in \mathscr{L}(H_i)$, $Q \in \mathscr{L}(H_j)$ we define $P \otimes Q$ by

$$(P \otimes Q)(\xi \otimes \eta) = P\xi \otimes Q\eta, \quad \xi \in H_i, \ \eta \in H_j.$$

Then we have the following elementary results (see Banks and Yew [3]).

Lemma 2.1. *Let $P \in \mathscr{L}(H_i)$, $Q \in \mathscr{L}(H_j)$, with P self-adjoint (i.e. $P^* = P$). Then*

$$[\mathscr{F}_x \langle \otimes_i x, P \otimes_i x \rangle] x = 2i \langle \otimes_i x, P \otimes_i x \rangle, \tag{2.1}$$

where $\mathscr{F}_x$ is the Frechet derivative with respect to x and $\otimes_i x = x \otimes \cdots \otimes x$ (i factors), and, more generally, if $C \in \mathscr{L}(H)$,

$$[\mathscr{F}_x \langle \otimes_i x, P \otimes_i x \rangle] Cx = 2 \langle \otimes_i x, (PC) \otimes_i x \rangle. \tag{2.2}$$

Moreover, we have

$$\langle \otimes_i x, P \otimes_i x \rangle \langle \otimes_j x, Q \otimes_j x \rangle = \langle \otimes_{i+j} x, (P \otimes Q) \otimes_{i+j} x \rangle \tag{2.3}$$

and

$$\| P \otimes Q \|_{\mathscr{L}(H_{i+j})} \leqslant \| P \|_{\mathscr{L}(H_i)} \| Q \|_{\mathscr{L}(H_j)}. \quad \square \tag{2.4}$$

It follows easily from the definition of PC, where $P \in \mathscr{L}(H_i)$, $C \in \mathscr{L}(H)$, that

$$\| PC \|_{\mathscr{L}(H_i)} \leqslant i \| P \|_{\mathscr{L}(H_i)} \| C \|_{\mathscr{L}(H)}.$$

Let CP be defined as $(P^* C^*)^*$. We can express a tensor operator in terms of its components with respect to a basis in the following way. Let $\{e_k\}_{k \geqslant 1}$ be an orthonormal basis of H. Then $\{e_{k_1} \otimes \cdots \otimes e_{k_j}\}$ ($1 \leqslant k_j < \infty$, $1 \leqslant j \leqslant i$) is an orthonormal basis of H_i. If $P \in \mathscr{L}(H_i)$ we shall write its matrix representation in terms of such a basis as $P_{k_1 \cdots k_i}^{l_1 \cdots l_i}$,

$$P(e_{k_1} \otimes \cdots \otimes e_{k_i}) = \sum_{l_1=1}^{\infty} \cdots \sum_{l_i=1}^{\infty} P_{k_1 \cdots k_i}^{l_1 \cdots l_i} (e_{l_1} \otimes \cdots \otimes e_{l_i}).$$

Then P is self-adjoint if $P_{k_1 \cdots k_i}^{l_1 \cdots l_i} = P_{l_1 \cdots l_i}^{k_1 \cdots k_i}$.

3. Optimal control of bilinear systems

Consider now the bilinear system

$$\dot{x} = Ax + uBx \tag{3.1}$$

where $x \in H$, $u \in \mathbb{R}$ is a scalar control, A is a closed, densely-defined operator which generates a semigroup $T(t)$, and B is a bounded operator. We shall consider the minimisation of the cost functional

$$J = \langle x, Gx \rangle + \int_0^{t_f} \{ \langle x, Mx \rangle + ru^2 \} \, dt \tag{3.2}$$

subject to the dynamics (3.1). It can be shown (Banks and Yew [3]) that the optimal control is given by the series

$$u(t) = -r^{-1} \sum_{i=1}^{\infty} \langle \otimes_i x, (P_i B) \otimes_i x \rangle \tag{3.3}$$

where the tensor operators P_i are self-adjoint, $P_i \in \mathscr{L}(H_i)$ and

$$P_1(t) = T_1(t_f - t) G T_1^*(t_f - t) + \int_0^{t_f - t} T_1(t_f - t - s) M T_1^*(t_f - t - s) \, ds, \tag{3.4}$$

$$P_m(t) = -r^{-1} \sum_{\substack{i+j=m \\ i,j \geqslant 1}} \int_0^{t_f - t} T_m(t_f - t - s) P_i(t_f - s) B \otimes P_j(t_f - s) B T_m^*(t_f - t - s) \, ds, \tag{3.5}$$

provided the formal series in (3.3) converges. Here, $T_i(t)$ is the semigroup generated by the tensor operator $\mathscr{A}_i$ defined by

$$\mathscr{A}_i P_i = P_i A, \quad P_i \in \mathscr{L}(H_i) \cap D(\mathscr{A}_i), \quad i \geqslant 1.$$

It can be seen that $\mathscr{A}_i$ is densely defined and does indeed generate a semigroup $T_i(t)$ which satisfies

$$\| T_i(t) \| < N \, e^{i\omega t} \tag{3.6}$$

where N and ω are positive numbers such that

$$\| T(t) \| < N \, e^{\omega t}, \tag{3.7}$$

where $T(t)$ is the semigroup generated by A. The main point remaining is to establish the convergence of the formal series (3.3). To do this we first estimate $P_1(t)$ from (3.4), using (3.7), to obtain

$$\| P_1(t) \| \leqslant N^2 \, e^{2\omega(t_f - t)} \| G \| + N^2 (t_f - t) \| M \| \left(\sup_{s \in [0, t_f]} e^{2\omega(t_f - s)} \right) e^{-2\omega t} \leqslant \alpha \, e^{-2\omega t} \tag{3.8}$$

where

$$\alpha = N^2 \, e^{2\omega t_f} \| G \| + N^2 t_f \| M \| \sup_{s \in [0, t_f]} e^{2\omega(t_f - s)}.$$

Similarly, from (3.5) and (3.6) we have

$$\| P_m(t) \| \leqslant r^{-1} \| B \|^2 N^2 \sum_{i+j=m} \int_0^{t_f - t} e^{2\omega m(t_f - t - s)} ij \, \| P_i(t_f - s) \| \cdot \| P_j(t_f - s) \| \, ds.$$

Define

$$p_m(t) = M \, e^{2m\omega t} \alpha^{-m} \left(r^{-1} \| B \|^2 N^2 \right)^{-m+1} \| P_m(t) \|, \quad m \geqslant 1. \tag{3.9}$$

Then, it follows that

$$p_m(t) \leqslant m \sum_{\substack{i+j=m \\ i,j\geqslant 1}} \int_0^{t_f-t} p_i(t_f-s) p_j(t_f-s)\, ds. \tag{3.10}$$

with $p_1(t) \leqslant 1$.

Lemma 3.1. *Let p_i' be given by*

$$p_1'(t) = 1, \quad t \in [0, t_f], \tag{3.11}$$

$$p_m'(t) = m \sum_{i+j=m} \int_0^{t_f-t} p_i'(t_f-s) p_j'(t_f-s)\, ds, \quad m \geqslant 2. \tag{3.12}$$

Then $p_i(t) \leqslant p_i'(t)$, $t \in [0, t_f]$. (This follows easily by induction.) □

Lemma 3.2. *If p_m satisfies* (3.11) *and* (3.12), *then it is of the form*

$$p_m = q_m (t_f - t)^{m-1}, \quad m \geqslant 2,$$

where q_m is constant on $[0, t_f]$ *and satisfies the difference equation*

$$q_m = \frac{m}{m-1} \sum_{i+j=m} q_i q_j \quad (m \geqslant 2), \quad q_1 = 1. \tag{3.13}$$

Proof. Again this follows by induction if we note that

$$q_m (t_f - t)^{m-1} = m \sum_{i+j=m} \int_0^{t_f-t} q_i q_j s^{i-1} s^{j-1}\, ds = \frac{m}{m-1} \sum_{i+j=m} (t_f - t)^{m-1} q_i q_j. \quad \square$$

Lemma 3.3. *Suppose that q_m satisfies* (3.13) *and that r_m satisfies the difference equation*

$$r_m = \sum_{i+j=m} r_i r_j, \quad r_1 = 2; \tag{3.14}$$

then $q_m \leqslant r_m$, $m \geqslant 1$.

Proof. Note that, from (3.13), we have, for $m \geqslant 2$,

$$q_m \leqslant 2 \sum_{i+j=m} q_i q_j, \quad q_1 = 1.$$

Then it is easy to see that $q_m \leqslant 2^{m-1} A_m q_1^m = A_m 2^{m-1}$ and $r_m = A_m r_1^m = A_m 2^m$ for some A_m. Hence the result follows. □

Lemma 3.4. *The power series $\sum_{i=1}^{\infty} r_i Z^{2i}$ has radius of convergence $1/2\sqrt{2}$, where r_i is given by* (3.14).

Proof. Consider the formal power series $R(Z) = \sum_{i=1}^{\infty} r_i Z^{2i}$. It is easy to check that the coefficients of this formal series satisfy (3.14) if and only if

$$r_1 Z^2 + R^2(Z) = R(Z). \tag{3.15}$$

Hence the formal series is convergent to an analytic function in some region if and only if the equation (3.15) has an analytic solution $R(Z)$. However, (3.15) implies that

$$R(Z) = \frac{1 \pm \sqrt{1 - 8Z^2}}{2},$$

each branch of which is analytic inside the disc $\{Z: |Z| < 1/2\sqrt{2}\}$. □

From (3.9) and Lemma 3.1, we have

$$\| P_m(t) \| = \frac{1}{m} \, \mathrm{e}^{-2m\omega t}\alpha^m \left(r^{-1} \| B \|^2 N^2\right)^{m-1} p_m(t)$$
$$\leqslant \frac{1}{m} \, \mathrm{e}^{-2m\omega t}\alpha^m \left(r^{-1} \| B \|^2 N^2\right)^{m-1} (t_f - t)^{m-1} q_m$$

and so

$$\left(\| P_m(t) \| \right)^{1/2m} = \left(\frac{1}{m}\right)^{1/2m} \mathrm{e}^{-\omega t}\alpha^{1/2} \left(r^{-1} \| B \|^2 N^2\right)^{1/2-1/2m} (t_f - t)^{1/2-1/2m} (q_m)^{1/2m}$$
$$\leqslant \left(\frac{1}{m}\right)^{1/2m} \mathrm{e}^{-\omega t}\alpha^{1/2} r^{-1/2} \| B \| N (t_f - t)^{1/2} \cdot 2\sqrt{2}$$
$$\to 2\sqrt{2} \, \mathrm{e}^{-\omega t} \left(\frac{\alpha}{r}\right)^{1/2} \| B \| N (t_f - t)^{1/2}$$

as $m \to \infty$. Hence the optimal control (3.3) exists as a power series in the state provided

$$\| x \| < \frac{1}{2\sqrt{2}} \frac{\mathrm{e}^{\omega t}}{\| B \| N} \left(\frac{r}{\alpha(t_f - t)} \right)^{1/2}.$$

Since the optimal cost is

$$J(x_0) = \sum_{i=1}^{\infty} \langle \otimes_i x_0, \, P_i \otimes_i x_0 \rangle$$

(provided the series converges), we have:

Theorem 3.5. *The bilinear-quadratic regulator problem*

$$\dot{x} = Ax + uBx, \quad x(0) = x_0,$$
$$J = \langle x(t_f), \, Gx(t_f) \rangle + \int_0^{t_f} \{ \langle x, \, Mx \rangle + ru^2 \} \, \mathrm{d}t,$$

has the optimal solution

$$u(t) = -r^{-1} \sum_{i=1}^{\infty} \langle \otimes_i x, \, (P_i B) \otimes_i x \rangle,$$

where P_i satisfies (3.4) *and* (3.5), *provided*

$$\| x_0 \| < \frac{1}{2\sqrt{2}} \frac{1}{\| B \| N} \left(\frac{r}{\alpha t_f} \right)^{1/2} \tag{3.16}$$

and where $\| T(t) \| \leqslant N \, \mathrm{e}^{\omega t}$ and α is given by (3.8). □

4. Example

As a simple example of the theory we shall consider the system

$$\dot{\phi} = A\phi + u\phi, \quad \phi \in L^2(0, 1),$$

where $B = I$ and A is a closed, self-adjoint, densely-defined operator with compact resolvent (see Dunford and Schwartz [4] or Banks [1]). Then the spectrum of A consists of eigenvalues λ_i with finite multiplicity such that $|\lambda_i| \to \infty$ as $i \to \infty$. Moreover, there is a complete orthonormal set of eigenvectors ϕ_i such that, for any $h \in L^2(0, 1)$,

$$h = \sum_{i=1}^{\infty} \langle h, \phi_i \rangle \phi_i, \qquad Ah = \sum_{i=1}^{\infty} \lambda_i \langle h, \phi_i \rangle \phi_i,$$

and

$$R(\lambda;\, A)h = \sum_{i=1}^{\infty} \frac{1}{\lambda - \lambda_i} \langle h, \phi_i \rangle \phi_i, \quad \lambda \in \rho(A),$$

where $R(\lambda;\, A)$ is the resolvent of A and $\rho(A)$ is the resolvent set. Here (λ_i), $1 \leqslant i < \infty$, is the sequence of eigenvalues counted according to multiplicity. Since $\{\phi_i\}$ is a basis of $L^2(0, 1)$ we can write

$$Q(\phi_{k_1} \otimes \cdots \otimes \phi_{k_i}) = \sum_{l_1=1}^{\infty} \cdots \sum_{l_i=1}^{\infty} Q_{k_1 \cdots k_i}^{l_1 \cdots l_i} (\phi_{l_1} \otimes \cdots \otimes \phi_{l_i})$$

for any tensor operator $Q \in \mathscr{L}(\otimes_i L^2(0, 1))$. Note also that, if $T(t)$ is the semigroup generated by A, then

$$T(t) = \sum_{i=1}^{\infty} \mathrm{e}^{\lambda_i t} \langle \cdot\, , \phi_i \rangle \phi_i .$$

In the cost functional (3.2) we shall assume, for simplicity, that $G = M = I$ and so from (3.4) we have $(P_1)_i^j = p_{1i} \delta_{ij}$, where

$$p_{1i}(t) = \mathrm{e}^{2\lambda_i(t_f - t)} + \frac{1}{2\lambda_i}(\mathrm{e}^{2\lambda_i(t_f - t)} - 1).$$

Note next that it is easy to see by induction that P_m is also 'diagonal' in the sense that

$$(P_m)_{i_1 \cdots i_m}^{j_1 \cdots j_m} = p_{m, i_1 \cdots i_m} \delta_{i_1}^{j_1} \cdots \delta_{i_m}^{j_m}$$

for some tensor $p_{1, i_1 \cdots i_m}$. Moreover, p_m is symmetric in all indices so that $p_{m, i_1 \cdots i_m} = p_{m, \sigma(i_1 \cdots i_m)}$ where σ is any permutation of the indices. It follows easily that if $\mathscr{A}_m P_m = P_m A$, then

$$(T_m(t) P_m)_{i_1 \cdots i_m}^{j_1 \cdots j_m} = \sum_{k=1}^{\infty} \delta_{i_1}^{k} \, \mathrm{e}^{m\lambda_k t} p_{m, k, i_2 \cdots i_m} \delta_{i_1}^{j_1} \cdots \delta_{i_m}^{j_m}.$$

Hence

$$p_{m, i_1 \ldots i_m}(t) = -r^{-1} \sum_{\substack{k+l=m \\ k,l>1}} kl \int_0^{t_f - t} \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \delta_{i_1}^{\alpha} \, \mathrm{e}^{k\lambda_\alpha (t_f - s)} p_{k, \alpha, i_2 \cdots i_k}(t_f - s)$$
$$\cdot \delta_{i_{k+1}}^{\beta} \, \mathrm{e}^{l\lambda_\beta (t_f - s)} p_{l, \beta, i_{k+2} \cdots i_m}(t_f - s) \, \mathrm{d}s.$$

5. Conclusions

This paper has been concerned with the optimal control of distributed bilinear systems. In an earlier paper (Banks and Yew [3]) we were unable to prove the convergence of the power series solution for the control. Convergence has now been established for certain initial states (unlike the linear-quadratic problem where the feedback rule is valid for all states). In particular, the inequality (3.16) shows that for large initial states we must choose a small horizon time t_f, whereas for a small initial state we could choose t_f to be somewhat larger. This resembles receding horizon control (Banks [2]).

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