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Lagrangian manifolds and asymptotically optimal stabilizing feedback control ☆

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Abstract

Approximations to nonlinear optimal control based on solving a Riccati equation which varies with the state have been put forward in the literature. It is known that such algorithms are asymptotically optimal given large scale asymptotic stability. This paper presents an analysis for estimating the size of the region on which large scale asymptotic stability holds. This analysis is based on a geometrical construction of a viscosity-type Lyapunov function from a stable Lagrangian manifold. This produces a less conservative estimate than existing approaches in the literature by considering regions of state space over which the stable manifold is multi-sheeted rather than just single sheeted.

Keywords: Nonlinear optimal control; Riccati equation; Viscosity solution; Lagrangian manifold

1. Introduction

A number of authors have considered an approximation to nonlinear optimal control based on solving a Riccati equation which varies with the state—see for instance [1,9]. The first of these references applies the feedback from a finite time Riccati equation depending on the current state. This is shown to be stable under certain bounds on the derivatives of the functions defining the nonlinear dynamics. This reference also studies the feedback obtained from an infinite time algebraic Riccati equation, although as pointed out in [6] the stability arguments presented only apply to the scalar case. The second reference ([9]) applies the feedback from an infinite time algebraic Riccati equation depending on the state. It is shown that the feedback is locally asymptotically stable and, in the scalar case, is optimal. It is also shown that the Pontryagin necessary conditions for optimality are satisfied asymptotically by the algorithm under the assumption of large scale asymptotic stability. In this reference the algorithm is given the name ‘state dependent Riccati equation’ or SDRE control. Other versions of the algorithm are presented in the literature—see for instance [10].

In this paper, we present an analysis for determining the size of the region on which large scale asymptotic stability holds for the SDRE algorithm. This is motivated by our observation that, for numerous examples, the SDRE feedback gives stabilizing control from most, but not necessarily all, starting conditions. Our analysis is based on using the value function which solves the optimal control problem as a Lyapunov function. The context is that of infinite time optimal control with a control term which appears linearly in

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the dynamics and quadratically in the cost. The resulting test involves evaluating an inequality along trajectories of a Hamiltonian dynamical system—it is not necessary to find the value function. The form of the inequality in question is already known and has been used before to show that feedback controls sufficiently close to the optimal globally stabilizing feedback are themselves asymptotically stabilizing in the same domain. However, the key ingredient in such results is proof of existence of the Lyapunov function corresponding to the optimal feedback. In the literature to date, such proofs of existence have been based on an assumption that the Lyapunov function is smooth. These are the stable Lagrangian manifold arguments of [2,7,11,12] and others. This assumption of smoothness limits the domain within which the Lyapunov function is known to exist to the largest state space neighbourhood of the equilibrium point onto which the stable manifold has a well defined projection, i.e. is single-sheeted.

What is new in our analysis is to enlarge the region within which the Lyapunov function is known to exist to regions of state space over which the stable manifold has multiple sheets. This is done by applying a geometrical construction from [4] along with various convexity arguments to prove the existence of a non-smooth Lyapunov function at a point x in state space from the fact that x is covered by one or more sheets of the stable manifold. The resulting function solves the associated Bellman equation in a viscosity sense. The inequality referred to above can then be used on this much larger region to test the stability of the SDRE feedback. The resulting estimate of the domain of attraction for the SDRE feedback is thus likely to be far closer to the true domain of attraction than the conservative estimates arising from the smoothness assumptions of the existing literature.

To start with, the precise type of optimal control problem considered here is the following infinite time optimal regulator problem:

$$V(\xi) = \inf_{u(.)\in L_2(0,\infty)} \int_0^\infty \frac{1}{2}(x(t)^{\mathrm{T}} q(x(t))x(t) + u(t)^{\mathrm{T}} r(x(t))u(t))\,\mathrm{d}t, \quad (1)$$

subject to $\dot{x} = f(x) + g(x)u$, $x(0) = \xi$, $\lim_{t\to\infty} x(t) = 0$ where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and f, g, q and r are analytic functions of the appropriate dimensions. We assume there is an equilibrium at the origin, i.e. $f(0) = 0$, and that $q(x)$ and $r(x)$ are positive definite for all x. One would look for this value function $V(x)$ as a solution to the associated stationary Hamilton–Jacobi–Bellman (HJB) equation:

$$\max_u \left\{ -\frac{\partial V}{\partial x}(f(x) + g(x)u) - \frac{1}{2}x^{\mathrm{T}} q(x)x - \frac{1}{2}u^{\mathrm{T}} r(x)u \right\} = 0. \quad (2)$$

It is shown in [7,11,12] how to construct a smooth V geometrically in a neighbourhood of the origin under the assumption that the linearized system $(\partial f/\partial x(0), g(0), q^{1/2}(0))$ is stabilizable and detectable. This assumption implies the existence of a Lagrangian manifold L through the origin which locally has a well-defined projection onto state space. L is a stable manifold for the Hamiltonian dynamics arising from the maximum principle. These dynamics have a hyperbolic equilibrium at the origin. $V(x)$ is the generating function for L, i.e. L is the set of points $\{x, y = -\partial V/\partial x\}$ in phase space and V satisfies $\mathrm{d}V = -y\,\mathrm{d}x$ on L.

It is shown in [4] how to construct from L a stationary viscosity solution $V(x)$ to (2) beyond points at which optimal trajectories start to cross (going backwards in time) and smoothness breaks down. The construction works on simply connected regions of L. Any region of L in which all the points can be connected via Hamiltonian trajectories to a small neighbourhood of the origin on L will be simply connected. In such a region, the equation $\mathrm{d}S = -y\,\mathrm{d}x$ still has a well defined solution $S(x, y)$ on L, even if the solution is no longer well defined over x. A viscosity solution $V(x)$ to Eq. (2) is obtained from the formula:

$$V(x) = \min\{S(x, y) \colon y \text{ such that } (x, y) \in L\}, \quad (3)$$

i.e. by taking the minimum value of $S(x, y)$ over all points $(x, y) \in L$ which project onto x. In addition to the assumptions of local stabilizability and detectability at the origin, the arguments of [4] also require that the function $V(x)$ defined by Eq. (3) be Lipschitz in order for it to be a viscosity solution to (2)—see Theorem 3 of [4]. It is shown in [8] that the Lipschitz condition follows from the topological properties of L in the case where L is of dimension $\leqslant 5$. The proof for higher dimensional cases relies on a conjecture made in [8] regarding the nature of certain types of singularities in the projection of L onto state space. This conjecture is known to be true up to dimension 5. In this paper, we assume the Lipschitz property for higher dimensional cases and note that, at the very worst, it will hold on the region of the origin on which $V(x)$ is smooth and, in general, on a larger region. This viscosity solution $V(x)$ to (2) will be our Lyapunov function.

We now briefly outline the state dependent Riccati equation algorithm put forward in [1,9] that provides the approximate optimal feedback for the problem defined in Eq. (1). This involves factoring $f(x)$ in the form $A(x)x$ for some matrix valued function $A(x)$ such that $A(x) \to \partial f/\partial x(0)$ as $x \to 0$. In the region where $V(x)$ is a smooth solution to (2), the maximum is achieved by $u = -r^{-1}(x)g^{\mathrm{T}}(x)\partial V/\partial x$, and V is thus the solution to

$$-\frac{\partial V}{\partial x} f(x) + \frac{1}{2}\frac{\partial V^{\mathrm{T}}}{\partial x} g(x) r^{-1}(x) g^{\mathrm{T}}(x)\frac{\partial V}{\partial x} - \frac{1}{2}x^{\mathrm{T}} q(x) x = 0. \tag{4}$$

It is shown in [11] that $\partial V/\partial x(0) = 0$ and so we can write $\partial V/\partial x = P(x)x$ for some matrix valued function $P(x)$ and (4) becomes:

$$x^{\mathrm{T}}(-P^{\mathrm{T}}(x)A(x) - A^{\mathrm{T}}(x)P(x) + P^{\mathrm{T}}(x)g(x)r^{-1}(x) g^{\mathrm{T}}(x)P(x) - q(x))x = 0.$$

The approximation to the solution involves ignoring the requirement that $P(x)x$ be the gradient of some function and assuming instead that $P(x)$ is symmetric. Then, at any given x, the algorithm consists of finding the positive semi-definite solution $P(x)$ to the algebraic Riccati equation:

$$P(x)A(x) + A^{\mathrm{T}}(x)P(x) - P(x)g(x)r^{-1}(x)g^{\mathrm{T}}(x)P(x) + q(x) = 0 \tag{5}$$

and applying, at that x, the control $u = -r^{-1}(x)g^{\mathrm{T}}(x) P(x)x$.

In order that (5) have a positive semi-definite solution for all x, it is sufficient that the 'frozen' linear systems $(A(x), g(x), q^{1/2}(x))$ be stabilizable and detectable for all x. The algorithm then gives a smooth feedback. Even if this is not true for all x, the algorithm still works, although the feedback then becomes discontinuous and there are values of x at which the algorithm blows up. This is illustrated in the example presented at the end of this paper. Note also that, while the above heuristic presentation of the algorithm made reference to the smoothness of V, this is not necessary for its application.

2. A stability test

Assume, as outlined in the Section 1, that the linearized system $(\partial f/\partial x(0), g(0), q^{1/2}(0))$ is stabilizable and detectable and that the function $V(x)$ defined by Eq. (3) is locally Lipschitz in a region Ω of the origin. Note that Ω is covered by L, i.e. for all $x \in \Omega$ there exists y such that $(x, y) \in L$. Note also that it follows from the positive definiteness of $q(x)$, for all $x \neq 0$, that $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$.

We start by noting that the positive semi-definite Riccati matrix solving the linearized problem at the origin is $\partial^2 V/\partial x^2(0)$, see [11]. Further, since the factorization $f(x) = A(x)x$ satisfies $A(0) = \partial f/\partial x(0)$, the solution $P(0)$ to (5) at $x=0$ satisfies $P(0) = \partial^2 V/\partial x^2(0)$. Note also that points $(x, y) \in L$ can be generated by following trajectories of the Hamiltonian dynamics corresponding to (2) backwards in time from final conditions $(x_{\mathrm{f}}, y_{\mathrm{f}}) \in L$ lying close to the origin. The dynamics are given by

$$\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}, \tag{6}$$

where

$$\begin{aligned} H &= \max_u \{ y^{\mathrm{T}} f(x) + y^{\mathrm{T}} g(x) u - \tfrac{1}{2} x^{\mathrm{T}} q(x) x - \tfrac{1}{2} u^{\mathrm{T}} r(x) u \} \\ &= y^{\mathrm{T}} f(x) + \tfrac{1}{2} y^{\mathrm{T}} g(x) r^{-1}(x) g^{\mathrm{T}}(x) y - \tfrac{1}{2} x^{\mathrm{T}} q(x) x. \end{aligned}$$

For points x_{f} lying in a small ball $B_\varepsilon \backslash \{0\}$, the final condition $(x_{\mathrm{f}}, y_{\mathrm{f}}) \in L$ can be approximated arbitrarily closely by taking $(x_{\mathrm{f}}, y_{\mathrm{f}})$ to lie on the tangent plane to L at the origin. This is given by

$$y_{\mathrm{f}} = -\frac{\partial^2 V}{\partial x^2}(0) x_{\mathrm{f}} = -P(0) x_{\mathrm{f}}.$$

In the following let Ω_t, for $t > 0$, be the set of all $x \in \mathbb{R}^n$ which are projections of points $(x, y) \in L$ which can be reached in time t or less along reverse trajectories of (6) starting from some $(x_{\mathrm{f}}, y_{\mathrm{f}}) \in L$, $x_{\mathrm{f}} \in B_\varepsilon \backslash \{0\}$. We can now state the stability test, which essentially involves checking that the error between the true feedback and the approximation is small in some sense.

Proposition 1. *For any $t > 0$ such that $\Omega_t \subset \Omega$, $V(x)$ is strictly decreasing along trajectories of the feedback algorithm $\dot{x} = f(x) - g(x)r^{-1}(x)g^{\mathrm{T}}(x) P(x)x$ for all $x \in \Omega_t \backslash \{0\}$ provided*

$$\tfrac{1}{2}(y + P(x)x)^{\mathrm{T}} g(x) r^{-1}(x) g^{\mathrm{T}}(x)(y + P(x)x) - \tfrac{1}{2} x^{\mathrm{T}} P(x) g(x) r^{-1}(x) g^{\mathrm{T}}(x) P(x) x \leqslant 0 \tag{7}$$

for all $(x, y) \in L$ such that $x \in \Omega_t \backslash \{0\}$.

Proof. $V(x)$ is a viscosity solution of

$$\max_u \left\{ -\frac{\partial V}{\partial x}(f(x)+g(x)u) - \frac{1}{2}x^{\mathrm{T}}q(x)x - \frac{1}{2}u^{\mathrm{T}}r(x)u \right\} = 0 \tag{8}$$

by Theorem 3 of [4]. So, in particular, it is a supersolution, i.e. for all $p \in D^- V$, the subdifferential of V, we have

$$-p^{\mathrm{T}}f + \tfrac{1}{2}p^{\mathrm{T}}gr^{-1}g^{\mathrm{T}}p - \tfrac{1}{2}x^{\mathrm{T}}qx \geqslant 0,$$

where, for a given $p \in D^- V$, the maximum in (8) is achieved by $u = -r^{-1}g^{\mathrm{T}}p$. Then, denoting $P(x)x$ by $\hat{p}$, the 'subderivative' of V along trajectories of the feedback algorithm is

$$-p(f - gr^{-1}g^{\mathrm{T}}\hat{p}) \geqslant -\tfrac{1}{2}pgr^{-1}g^{\mathrm{T}}p + \tfrac{1}{2}x^{\mathrm{T}}qx + pgr^{-1}g^{\mathrm{T}}\hat{p}. \tag{9}$$

By Theorem I.14 of [3], V is strictly decreasing along trajectories of $\dot{x} = f - gr^{-1}g^{\mathrm{T}}\hat{p}$ provided the right-hand side of (9) is strictly positive for all $p \in D^- V$. Since $q(x)$ is positive definite for $x \neq 0$, this will follow provided:

$$\tfrac{1}{2}pgr^{-1}g^{\mathrm{T}}p - pgr^{-1}g^{\mathrm{T}}\hat{p} = \tfrac{1}{2}(p-\hat{p})^{\mathrm{T}}gr^{-1}g^{\mathrm{T}}(p-\hat{p}) - \tfrac{1}{2}\hat{p}gr^{-1}g^{\mathrm{T}}\hat{p} \leqslant 0 \tag{10}$$

for all $p \in D^- V$. Since V is Lipschitz, $D^- V \subset \partial V$, the generalized gradient (see [5]). Further, it is shown in the proof of Theorem 3 of [4] that for $V(x)$ given by (3), $\partial V \subset \mathrm{co}\{-y\colon (x,y) \in L\}$ where co denotes the convex hull. Since the expression for p in (10) is convex, (10) will follow if

$$\tfrac{1}{2}(-y-\hat{p})^{\mathrm{T}}gr^{-1}g^{\mathrm{T}}(-y-\hat{p}) - \tfrac{1}{2}\hat{p}gr^{-1}g^{\mathrm{T}}\hat{p} = \tfrac{1}{2}(y+\hat{p})^{\mathrm{T}}gr^{-1}g^{\mathrm{T}}(y+\hat{p}) - \tfrac{1}{2}\hat{p}gr^{-1}g^{\mathrm{T}}\hat{p} \leqslant 0$$

as required. □

As stated above, $P(0) = \partial^2 V/\partial x^2(0)$. Hence $P(x)x \to \partial^2 V/\partial x^2(0)x$ as $x \to 0$. It is also true that $-y \to \partial^2 V/\partial x^2(0)x$ as $x \to 0$. Thus, (7) will hold in a sufficiently small ball B_ε centred on the origin. The stability test then involves following trajectories of (6) backwards in time from points $x_{\mathrm{f}} \in \partial B_\varepsilon$, $y_{\mathrm{f}} = -\partial^2 V/\partial x^2(0)x_{\mathrm{f}}$ and estimating the largest t for which (7) holds in Ω_t. The feedback algorithm will then be asymptotically stable in the sublevel set $\{x \in \mathbb{R}^n\colon V(x) \leqslant c\}$ where $c = \min\{V(x)\colon x \in \partial\Omega_t\}$. The above analysis could be done with weaker conditions than $q(x)$ positive definite for $x \neq 0$; for example some form of zero-state detectability (see [12] and the references therein).

Note that the use of inequalities of a similar form to (7) to prove the asymptotic stability of feedbacks which approximate the optimal control sufficiently closely is not in itself new. However, the deduction of stability for an approximating feedback from the fact that it satisfies (7) on a given domain is only valid if the Lyapunov function associated with the optimal control can be shown to exist in that domain. As mentioned in Section 1, current approaches in the literature to proving the existence of this Lyapunov function from that of the stable manifold result in conservative estimates of the domain of stability. This is due to the need to restrict attention to the region of state space over which the stable manifold is single sheeted. The boundary of this region is thus often reached (going backwards in time from the origin) well before (7) fails to hold true.

The novel aspect of the above result is the evaluation of inequality (7) on a much larger region of state space on which the existence of a non-smooth viscosity-type Lyapunov function can be deduced from the fact that the region is covered by the stable manifold, even if this covering is multi-sheeted. In this situation, (7) is more likely to fail before the boundary is reached, resulting in a far less conservative estimate of the domain of attraction.

The approaches to estimating the domain of attraction compared in the above two paragraphs rely on the existence of a Lyapunov function for the optimal control and the backward integration of Hamiltonian trajectories in phase space. There is, of course, another approach which is simply to integrate the closed loop state space dynamics given by the SDRE feedback forwards from a variety of initial conditions and identify which trajectories converge to the equilibrium. This involves fewer equations in the integration (n instead of $2n$) but, on the other hand, it requires iteration through numerous sets of initial conditions in order to identify the boundary of the domain of attraction. As all the other terms in (7), apart from the n adjoint variables y, have to be evaluated anyway as part of the SDRE algorithm, it is not clear that there is any advantage, in terms of computational load, from forward state space as opposed to backward phase space simulation. The advantage of the latter approach, at least from a philosophical perspective, is that inequality (7) gives some measure of the distance between the SDRE feedback and the optimal control, i.e. a measure of the degree of sub-optimality. It also shows that sufficient conditions for optimality are asymptotically satisfied by the SDRE feedback.

Example. The algorithm described in Eq. (5) was applied to the simulated stabilization around the vertical upright position of a pendulum mounted on a cart. The cart moved on a linear track and was subject to a control force acting along the track. Taking the pendulum to be 1 m long with a 1 kg bob and 1 kg cart and gravitational acceleration to be 10 ms^{-2}, the equations of motion can be written in the factored form $A(x)x$ as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \dfrac{20\sin x_1}{(1+\sin^2 x_1)x_1} & \dfrac{-(\sin x_1 \cos x_1)x_2}{1+\sin^2 x_1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{-10\sin x_1 \cos x_1}{(1+\sin^2 x_1)x_1} & \dfrac{(\sin x_1)x_2}{1+\sin^2 x_1} & 0 & 0 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{-\cos x_1}{1+\sin^2 x_1} \\ 0 \\ \dfrac{1}{1+\sin^2 x_1} \end{pmatrix} u,$$

where x_1, x_2 are the angular position and velocity respectively of the pendulum measured from the upright vertical position and x_3, x_4 are the linear position and velocity respectively of the cart on the track. The integrand of the cost function is taken to be

$$\tfrac{1}{2}x_4^2 + \tfrac{1}{2}(x_4 + x_2\cos x_1)^2 + \tfrac{1}{2}x_2^2\sin^2 x_1 + 10(1-\cos x_1) + \tfrac{1}{2}u^2.$$

This is the mechanical Lagrangian of the unforced system, i.e. the kinetic minus the potential energy with the potential taken to be zero at the vertical upright position, plus a control cost term. The factorization $1/2x^{\mathrm{T}}q(x)x$ is obtained in the obvious way by writing $1-\cos x_1 = ((1-\cos x_1)/x_1^2)x_1^2$. For this example $q(x)$ is positive definite if one ignores x_3, the position on the track. This is justified as x_3 does not, effectively, enter the dynamics and we have not imposed any cost on x_3. Consequently, there is an equilibrium at $x_1=x_2=x_4=0$ for all x_3.

This system is stabilizable and detectable at the origin. It can also be clearly seen that the matrix $A(x)$ tends to the linearization of the system at the origin. The expression for the feedback $u = -r^{-1}(x)g^{\mathrm{T}}(x)P(x)x$ reduces, in this case, to $u = (\eta_2\cos x_1 - \eta_4)/(1+\sin^2 x_1)$ where $\eta = P(x)x$. The solution of the state dependent Riccati equation (5) was calculated analytically and reduced to the solution of a quartic polynomial. This was then implemented symbolically in MAPLE to obtain the simulated feedback at any given state point, with the MAPLE Runge–Kutta routine calculating the resulting trajectories through state space. Simulations were run starting from rest for different initial angles. The algorithm provides stabilization from all initial angles except $\pm\pi/2$. This is because the frozen linear system $(A(x), g(x))$ is unstabilizable at $x_1 = \pm\pi/2$. At these points the feedback algorithm blows up. This is dealt with by imposing bounds in the function for u in a neighbourhood of these points. Starting from rest from initial angles close to $\pm\pi/2$, the size of the initial control impulse produces a large overshoot which then requires a number of oscillations or complete revolutions to dissipate. However, even in these extreme cases the algorithm recovers and eventually stabilizes. The other consequence of unstabilizability is a discontinuity in the feedback produced by the algorithm at $\pm\pi/2$. This is because the positive definite root of (5) switches as the pendulum passes through $\pm\pi/2$ and the feedback goes from pushing to pulling or vice versa.

To estimate the region of stability, the left-hand side of (7) was evaluated backwards in time along trajectories of the dynamics (6) for this example. The final points for these trajectories lay on a small four-dimensional ball centred on the origin. This ball, in turn, lay on the four-plane tangent at the origin to the stable manifold L corresponding to the dynamics (6). The inequality (7) was found to hold out to about $x_1 = 1.5$ rad on a sample of trajectories. This is where the discontinuity occurs in the feedback algorithm.

There are numerical issues involved in generating a sufficiently wide spread of points (x, y) on L. Suppose α and ω are the largest and smallest (in absolute magnitude) eigenvalues for the linearized system

$$\dot{x} = \partial f/\partial x(0)x - g(0)r^{-1}(0)g^{\mathrm{T}}(0)\partial^2 V/\partial x^2(0)x.$$

Then trajectories on L, when projected onto state space, will on average approach the origin along the direction of the eigenvector corresponding to ω and leave (in reverse time) along the eigendirection corresponding to α. It is thus necessary to concentrate the final conditions $x_{\mathrm{f}} \in \partial B_\varepsilon$ along the eigendirection corresponding to ω in order to generate a spread of points on L in reverse time. □

To conclude, we have presented a test for estimating the extent of the region on which a state dependent Riccati feedback control will be asymptotically stable. Results in the literature indicate that asymptotic optimality of the feedback will then hold on the same region.

References

[1] S.P. Banks, K.J. Mhana, Optimal control and stabilization for nonlinear systems, IMA J. Math Control Inform. 9 (1992) 179–196.

[2] P. Brunovsky, On the optimal stabilization of nonlinear systems, Czechoslovak Math. J. 18 (1968) 278–293.

[3] M.G. Crandall, P.L. Lions, Viscosity solutions of Hamilton–Jacobi equations, Trans. Amer. Math. Soc. 277 (1) (1983) 1–42.

[4] M.V. Day, On Lagrange manifolds and viscosity solutions, J. Math. Systems, Estimation Control 8 (1998) http://www.math.vt.edu/people/day/research/LMVS.pdf.

[5] H. Frankowska, Hamilton–Jacobi equations: viscosity solutions and generalised gradients, J. Math. Anal. Appl. 141 (1989) 21–26.

[6] C. Gong, S. Thompson, A comment on stabilization and optimal control for nonlinear systems, IMA J. Math. Control Inform. 12 (1995) 395–398.

[7] D.L. Lukes, Optimal regulation of nonlinear dynamical systems, SIAM J. Control 7 (1) (1969) 75–100.

[8] D. McCaffrey, S.P. Banks, Lagrangian manifolds, viscosity solutions and Maslov index, preprint.

[9] C.P. Mracek, J.R. Cloutier, Control designs for the nonlinear bench-mark problem via the state-dependent Riccati equation method, Internat. J. Robust Nonlinear Control 8 (1998) 401–433.

[10] Z. Qu, J.R. Cloutier, C.P. Mracek, A new sub-optimal nonlinear control design technique—SDARE, Proceedings of the IFAC96 13th Triennial World Congress, San Francisco, CA, Vol. 2b-114, 1996, pp. 365–370.

[11] A.J. van der Schaft, On a state space approach to nonlinear H_∞ control, Systems Control Lett. 16 (1991) 1–8.

[12] A.J. van der Schaft, L_2 gain analysis of nonlinear systems and nonlinear state feedback H_∞ control, IEEE Trans. Automat. Control AC-37 (6) (1992) 770–784.