Survey of Space Complexity and Connectivity

Jason Vander Woude

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University of Nebraska–Lincoln

Foundations

Terminology

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Graph We mean a directed graph unless stated otherwise

Connectivity/Reachability

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Theorem 2.

STCONN *is* **NL**-*complete*

Savitch's Theorem

Theorem 3 [Savitch, 1970].

Let $s : \mathbb{N} \to \mathbb{N}$ be a space constructible function with $s(n) \geq \log n$. Then $\mathbf{NSPACE}(s(n)) \subseteq \mathbf{DSPACE}(s^2(n))$.

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It suffices to prove that $\mathbf{NL} \subseteq \mathbf{DSPACE}(\log^2(n))$ because for any $s(n)$ we can then show that reachability on the configuration graph ($O(c^{s(n)})$ nodes) can be solved in $O(\log^2(c^{s(n)})) = O(s^2(n))$ space.

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Corollary 4.

$\mathbf{PSPACE} = \mathbf{NPSPACE}$

Savitch's Algorithm

Algorithm 1 Find a directed path in a graph of length at most k

procedure $\textsc{FindDirectedPath}((V, E), s, t, k)$
  if $k == 0$ **then return** $s == t$
  end if
  if $k == 1$ **then return** $s == t$ OR $(s, t) \in E$
  end if
  for $v \in V$ **do**
    if $\textsc{FindDirectedPath}((V, E), s, v, \lfloor \frac{k}{2} \rfloor)$ AND $\textsc{FindDirectedPath}((V, E), v, t, \lceil \frac{k}{2} \rceil)$ **then**
      return TRUE
    end if
  end for
  return FALSE
end procedure

Immerman-Szelepcsènyi Theorem

Theorem 5 [Immerman, 1988, Szelepcsnyi, 1988].
Let $s : \mathbb{N} \to \mathbb{N}$ *be a function such that* $s(n) \geq \log n$. *Then* $\mathbf{NSPACE}(s(n)) = \mathbf{coNSPACE}(s(n))$ *(even if* $s(n)$ *is not space constructible).*

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Corollary 6.

$\mathbf{NL} = \mathbf{coNL}$

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- This is a major open question.
- One attempt at answering in the affirmative is efficient algorithms
- One attempt at answering in the negative is to show $\mathrm{USTCONN} \notin \mathbf{L}$
  - In a breakthrough result, Reingold showed this is a fruitless attempt

Reingold's Results

Theorem 7 [Reingold, 2008].

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A directed graph is said to be Eulerian if the indegree and outdegree of each vertex is the same. That is, there are the same number of arcs into and out of every vertex.

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Theorem 9 [Reingold et al., 2006].

There exists a logspace algorithm for deciding reachability in Eulerian directed graphs

Restricted Graphs

Planar Graphs

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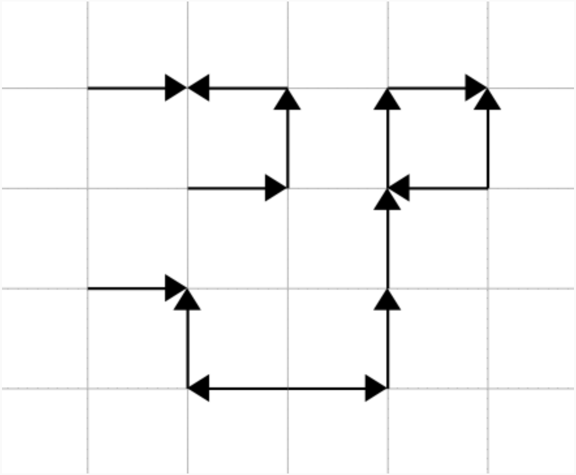
Theorem 10 [Allender and Mahajan, 2000, Reingold, 2008].
Given an adjacency matrix of a graph, the problem of finding a combinatorial embedding is **L***-complete*

Grid Graphs



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Theorem 13 [Allender et al., 2005].

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Further Restricted Grid Graphs

- Two local restrictions

  1GGR outdegree at most 1

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Theorem 14 [Allender et al.,].

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- There is weak evidence that GGRB is not **NL**-complete [Allender et al.,]
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- Is LGGR easier than GGR?
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    - Next result shows that no more is known about LGGR than GGR

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Theorem 16 [Tewari, 2007].

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Corollary 17 [Allender et al., 2005, Tewari, 2007].

$\mathrm{PLANARREACH} \leq_l \mathrm{GGR} \in$ **UL**, *so*
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If G, H are graphs and G does not contain H as a minor, then G is called H-free

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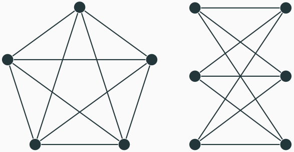
Theorem 21 [Wagner, 1937].

A graph is planar if and only if it is both K_5-free and $K_{3,3}$ free



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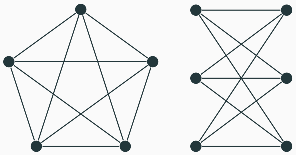

Theorem 22 [Thierauf and Wagner, 2009].

Reachability for K_5-free graphs and for $K_{3,3}$-free graphs logspace reduces to $\mathrm{PLANARREACH}$

Space and Time

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Polynomial time, sublinear space algorithms also exist

Theorem 25 [Barnes et al.,].

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- When considering time and space bounds together, planar graphs and grid graphs are not known to be equivalent because the known reductions from planar graphs to grid graphs result in a polynomial blowup in the size of the graph

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The former result also holds for graphs that are almost planar

Theorem 28 [Chakraborty et al., 2014].

Reachability for a directed graph which is either K_5*-free or* $K_{3,3}$*-free can be decided in* $\mathbf{TISP}(n^{O(1)}, n^{1/2+\epsilon})$ *for any* $\epsilon > 0$

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Given a graph G and natural number k, it is **NP***-complete to determine if the genus of G is less than or equal to k*

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Theorem 30 [Chakraborty et al., 2014].

Given a combinatorial embedding of a graph on a genus g surface, reachability can be decided in $\mathbf{TISP}(n^{O(1)}, n^{2/3}\log^{O(1)}(n)\cdot g^{1/3}\log^{O(1)}(g))$

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Theorem 33 [Jain and Tewari, 2019].

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Conclusion

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    - $\mathbf{ReachFewL} = \mathbf{ReachUL}$

Acknowledgements

I'd like to thank my advisor Vinod Variyam for helping me find a starting point for this project and also meeting weekly to answer questions and help me understand the material more deeply.

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