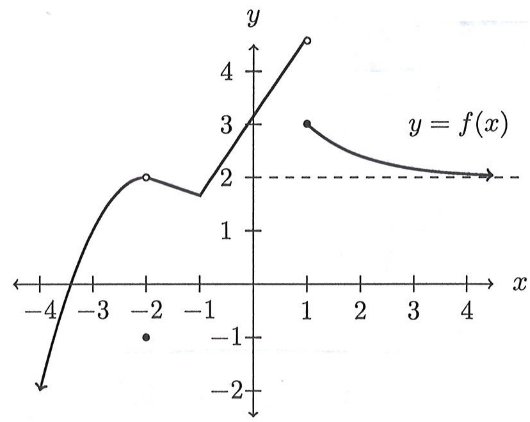
Answer Key

1. (25 points)

(a) (15 points) Use the figure below to answer the following questions. **If the limit or the derivative do not exist, write DNE.**



i. $\lim_{x\to -2^-} f(x)$ 2

ii. $\lim_{x\to -2} f(x)$ 2

iii. $\lim_{x\to 1^+} f(x)$ 3

iv. $\lim_{x\to 1} f(x)$ DNE

v. $\lim_{x\to \infty} f(x)$ 2

vi. Identify the $x-$values for all points where the function is discontinuous. $-2, 1$

vii. Identify the $x-$values for all points where the function is not differentiable. $-2, -1, 1$

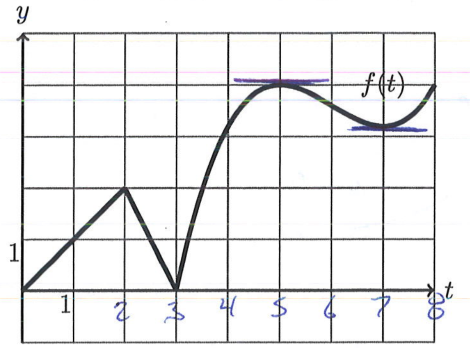
(b) (10 points) **Use the definition of the derivative** to compute $f'(x)$, where $f(x) = 2x^2 - 3x + 2$.

$$\lim_{h\to 0} \frac{2(x+h)^2 - 3(x+h) + 2 - (2x^2 - 3x + 2)}{h}$$

$$\lim_{h\to 0} \frac{2[\cancel{x^2} + 2xh + h^2] - \cancel{3x} - 3h \cancel{+2} - \cancel{2x^2} + \cancel{3x} \cancel{-2}}{h}$$

$$\lim_{h\to 0} \frac{4x\cancel{h} + h^{\cancel{2}} - 3\cancel{h}}{\cancel{h}} = \lim_{h\to 0} 4x + h - 3 = 4x-3$$

2. (25 points) A particle is moving vertically. The graph of the height (in meters) of a particle at time t, in seconds, is given by $f(t)$ below.



(a) At what t-value(s) is the velocity of the particle 0? Circle ALL correct answers.

↳ horizontal tangent on position graph

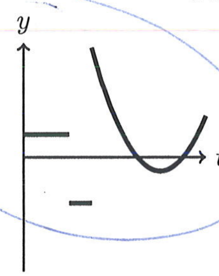
i. $t = 1$
ii. $t = 3$
iii. $t = 4$
iv. $t = 5$
v. $t = 7$

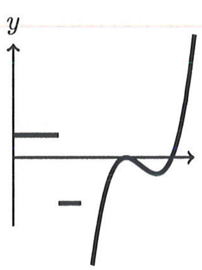
(b) Match each of the following mathematical notations with the appropriate statement. Write the correct letter in the blank beside each notation.

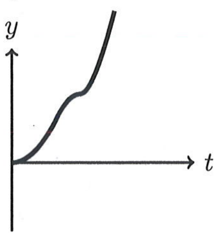
- $f'(4)$ B
- $\dfrac{f(4) - f(0)}{4 - 0}$ E
- $f''(4)$ C
- $f(4)$ A
- $\dfrac{1}{4-0}\displaystyle\int_0^4 f(x)dx$ D

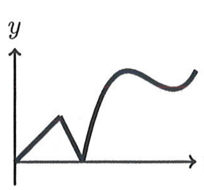
~~(A) The height of the particle after 4 seconds~~
~~(B) The velocity of the particle after 4 seconds~~
~~(C) The acceleration of the particle after 4 seconds~~
(D) The average height of the particle after 4 seconds
~~(E) The average rate of change of the particle after 4 seconds~~

(c) Which of the following graphs could represent a derivative of f? Circle ALL correct answers.

A)


B)


C)


D)


3. (25 points)

(a) (10 points) Find the slope of the tangent line to the curve $x^3y + 2y^3 = 5x$ at the point $(2, 1)$.

product rule

i. $\frac{1}{2}$

ii. $-\frac{1}{2}$

iii. $\frac{17}{14}$

iv. $\frac{13}{8}$

v. $-\frac{13}{8}$

$$x^3y' + 3x^2y + 6y^2y' = 5$$

$$x^3y' + 6y^2y' = 5 - 3x^2y$$

$$y'[x^3 + 6y^2] = 5 - 3x^2y$$

$$y' = \frac{5 - 3x^2y}{x^3 + 6y^2}$$

$$@\ (2,1) \rightarrow m = \frac{5 - 3(2)^2\cdot(1)}{(2)^3 + 6(1)^2} = \frac{-7}{14} = -\frac{1}{2}$$

(b) (10 points) Find the linear approximation to $f(x) = x^{1/2}$ at $x = 9$. Is it an underestimate or overestimate? **Motivate your answer.**

$$L(x) = f(a) + f'(a)\cdot(x - a) \quad \text{where } a = 9$$

$$= 9^{1/2} + \tfrac{1}{2}(9)^{-1/2}\cdot(x - 9)$$

$$= 3 + \tfrac{1}{6}(x - 9)$$

$$f'(x) = \tfrac{1}{2}x^{-1/2}$$

$$f''(x) = -\tfrac{1}{4}x^{-3/2}$$

$$f''(9) < 0 \Rightarrow \text{concave down}$$

over estimate

(c) (5 points) Use the approximation of part (b) to estimate $f(12)$ with two correct decimals. **Circle the correct answer.**

i. 10.50

ii. 3.46

iii. 3.48

iv. 3.43

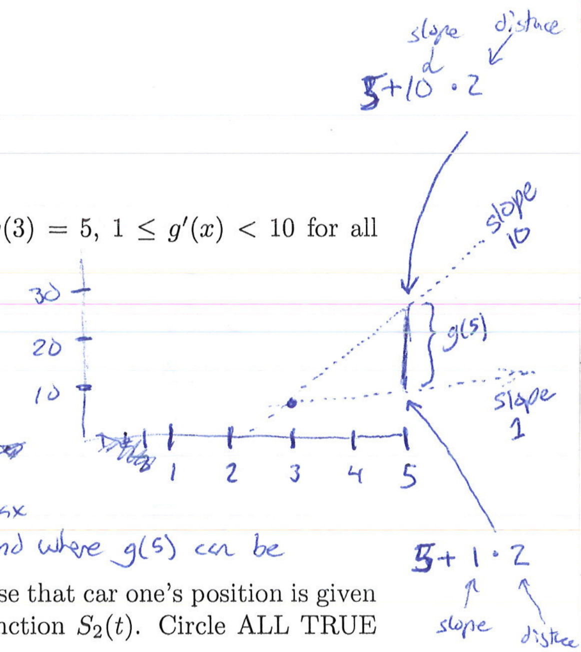
v. 3.50

$$L(12) = 3 + \tfrac{1}{6}(12 - 9) = 3 + \tfrac{1}{6}\cdot(3) = 3 + \tfrac{1}{2} = 3.50$$

Note: $f(12) = 12^{1/2} = \sqrt{12} \approx 3.4641$

4. (25 points)

(a) (5 points) Let $g(x)$ be a differentiable function, and suppose $g(3) = 5$, $1 \leq g'(x) < 10$ for all values of x. Circle ALL TRUE statements below:



i. $g(x)$ is a continuous function.

ii. $\dfrac{g(6) - g(3)}{6 - 3} = g'(x)$ for **every** x in the interval $[3, 6]$

mean value Theorem → iii. $\dfrac{g(6) - g(3)}{6 - 3} = g'(x)$ for **some** x in the interval $[3, 6]$

iv. g is decreasing on $[3,6]$

v. $7 \leq g(5) < 35$ $7 \leq g(5) \leq 25$ — min & max slope bound where g(5) can be

(b) (5 points) Two cars leave from Lincoln NE to Omaha NE. Suppose that car one's position is given by the function $S_1(t)$ and car two's position is given by the function $S_2(t)$. Circle ALL TRUE statements below:

i. If $S_1'(t) < S_2'(t)$ for all t, and both cars arrive at Omaha at the same time then car two must have left Lincoln fist.

ii. If $S_1'(t) < S_2'(t)$ for all t, and both cars arrive at Omaha at the same time then car one must have left Lincoln fist.

iii. If $S_1'(t) > S_2'(t)$ for all t, and both cars leave Lincoln at the same time then car one gets to Omaha first.

iv. If $S_1'(t) > S_2'(t)$ for all t, and both cars leave Lincoln at the same time then car two gets to Omaha first.

v. Without knowing $S_1(t)$ or $S_2(t)$ explicitly there is no way to determine which car arrived first. (this is a poorly worded question. They are saying it is possible to decide who arrives first even without an equation as we did in parts ii, iii)

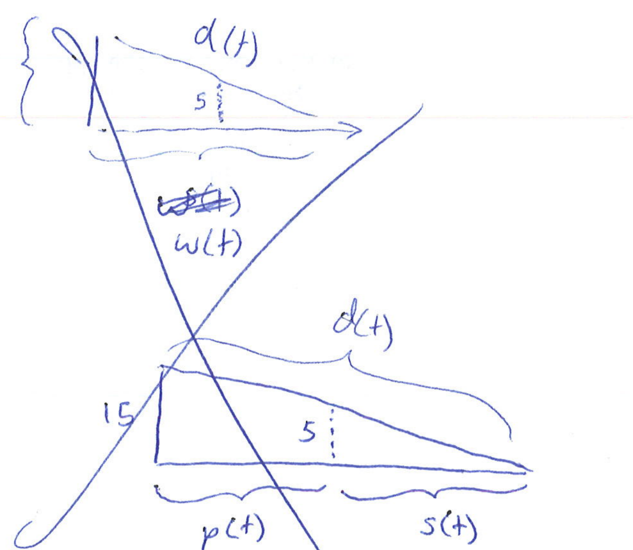
(c) (15 points) A street light is at the top of a 15 foot tall pole. A 5 foot tall woman walks away the pole with a speed of 2 ft/sec along a straight path. How fast is the tip of her shadow moving (away from the woman) when she is 15 feet from the base of the pole?

~~$15^2 + [w(t)]^2 = d(t)^2$~~

~~$2w(t)\, w'(t) = 2d(t) \cdot d'(t)$~~

~~$15^2 + (p(t) + s(t))^2 = (d(t))^2$~~

~~$2(p(t) + s(t)) \cdot (p'(t) + s'(t)) = 2d(t) \cdot d'(t)$~~



~~d(t) =~~

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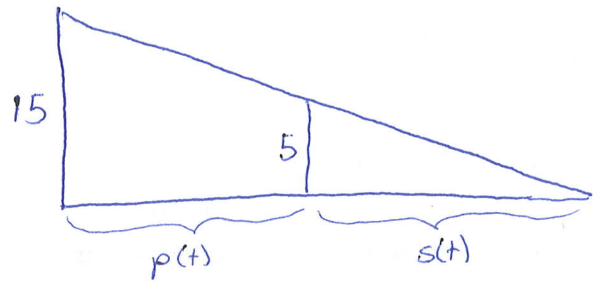


$p(t)$ = distance of woman from pole

$s(t)$ = lenth of her shadow

we know $p'(t) = 2 \frac{ft}{sec}$

we want to find $s'(t)$

The small triangle and large triangle are similar, so the ratios of side lengths are the same

use the ratio of vertical side to horizontal side

$$\frac{15}{p(t)+s(t)} = \frac{5}{s(t)} \rightarrow 15\,s(t) = 5\left(p(t)+s(t)\right)$$

$$\rightarrow 15\,s(t) = 5p(t) + 5\,s(t)$$

$$\rightarrow 15\,s(t) - 5\,s(t) = 5\,p(t)$$

$$\rightarrow 10\,s(t) = 5\,p(t)$$

$$\rightarrow 2\,s(t) = p(t)$$

This is the geometric equation we want. Take derivative of both sides

$$\rightarrow 2\,s'(t) = p'(t)$$

plug in $p'(t) = 2 \frac{ft}{sec}$

$$2\,s'(t) = 2 \text{ ft/sec}$$

$$\rightarrow s'(t) = 1 \text{ ft/sec}$$

the tip of her shadow moves away from her at $1 \frac{ft}{sec}$

Math 106: Quiz #7

Names: ______________________ ______________________ ______________________

Directions: You may work with everyone in your group, but you must turn in team quizzes in groups of two or three. You may not work in groups smaller than two or in groups larger than 3 (otherwise you will earn a zero). You may not use your notes, worksheets, or other resources on this team quiz. **Be sure to show work and/or explain your reasoning.**

(4 points)

1. The function f has values given in the table below. Use these values to estimate $\int_0^4 f(x)dx$.

x	0	1	2	3	4	5	6
$f(x)$	4	2	3	4	5	6	2

(6 points)

2. (~~2 points~~) Use the fundamental theorem to evaluate the following integrals.

$\int_0^3 2x dx$ $\int_0^{\pi/2} \cos(\theta)d\theta$ $\int_2^4 (3x^2 + 2x)\, dx$

5. (25 points)

(a) (7 points) Compute the limits below

(i) $\lim_{x\to 0} \frac{1-\cos x}{3x^2}$

$\frac{1-1}{0} \xrightarrow{L'H} \lim_{x\to 0} \frac{0+\sin(x)}{6x} = \frac{0}{0}$

$\xrightarrow{L'H} \lim_{x\to 0} \frac{\cos(x)}{6} = \frac{1}{6}$

(ii) (7 points) $\lim_{x\to 0^+} x\ln(x)$

$\lim_{x\to 0^+} x \cdot \ln(x) = 0 \cdot (-\infty)$ which is not well defined, so we rewrite the original

$x \cdot \ln(x) = \frac{1}{x^{-1}} \cdot \ln(x)$

so $\lim_{x\to 0^+} \frac{1}{x^{-1}} \cdot \ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1}} = \frac{-\infty}{\infty} \xrightarrow{L'H} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-x^{-2}} = \lim_{x\to 0^+} \frac{-x^2}{x} = \lim_{x\to 0^+} -x$

$= 0$

(b) (5 points) The position of a particle at time t is given by the parametric curve

$$x(t) = 6t^2 - t^3 \text{ and } y(t) = 2t^2 - 4t.$$

Find all points (x, y) where the curve has a vertical tangent line.

i. $(0, 0)$
ii. $(1, -2)$
iii. $(5, -2)$
iv. $(32, 16)$
v. $(0, 1)$

vertical tangent means $\frac{\partial x}{\partial y} = 0$ so $\frac{\partial x}{\partial y} = \frac{\partial x/\partial t}{\partial y/\partial t} = 0$ (so $\partial x/\partial t$ must be 0)

$\frac{\partial x}{\partial t} = 12t - 3t^2 = t(12-3t)$

if $\frac{\partial x}{\partial t} = 0$ then $t=0$ or $t=4$

plug in $(x(0), y(0)) = (0,0)$

and $(x(4), y(4)) = (32, 16)$

(c) (6 points) Given the same parametric curve as problem (b), $x(t) = 6t^2 - t^3$ and $y(t) = 2t^2 - 4t$, find the equation of a line tangent to the curve at the point $t = 2$. Put the tangent in slope intercept form, that is $y = mx + b$.

$$y = \boxed{\frac{1}{3}}\, x + \boxed{\frac{-16}{3}}$$

$\frac{\partial y}{\partial x} = \frac{\partial y/\partial t}{\partial x/\partial t} = \frac{4t-4}{12t-3t^2}$

when $t=2$, $m = \frac{4(2)-4}{12(2)-3(2)^2} = \frac{4}{12} = \frac{1}{3}$ ← slope

next find the point at $t=2$

$(x(2), y(2)) = (6(2)^2 - (2)^3, 2(2)^2 - 4(2))$

$= (16, 0)$

now with the slope and point,

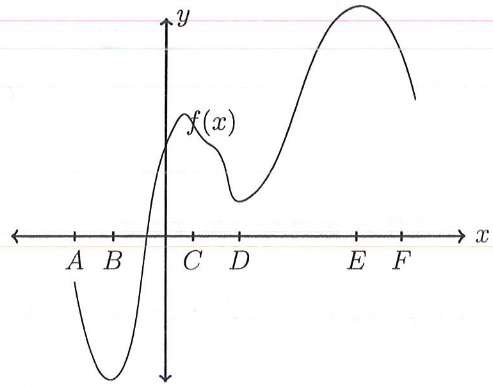
$y - 0 = \frac{1}{3}(x-16) \Rightarrow y = \frac{1}{3}x - \frac{16}{3}$

should check that $\frac{dy}{dt} \neq 0$ @ $t=0, t=4$

$\frac{dy}{dt} = 4t-4 \rightarrow y'(0) = -4 \neq 0$ and $y'(4) = 16-4 \neq 0$ ✓

6. (25 points)

(a) (5 points) In the graph of the function f given to the right, identify all points that are



i. local maxima on the interval $[A, F]$

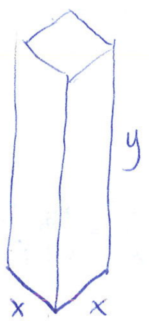
C, E

ii. local minima on the interval $[A, F]$

B, D

iii. global maximum on the interval $[A, F]$

E



iv. global minimum on the interval $[A, F]$

B

(b) (20 points) A closed box has height y inches and a **square base** of edge length x inches. Let V be the volume of the box and S the surface area. In this problem you will find the values of x and y that minimize the surface area S of the box, given that its volume V is 1000 inches^3.

a. Give S and V in terms of x and y.

$$S = 2x^2 + 4xy$$

$$V = x^2 y$$

b. Use the fact that $V = 1000$ in^3 to write S as a function of x only.

$$1000 = V = x^2 y \rightarrow y = \frac{1000}{x^2}$$

$$\rightarrow S = 2x^2 + 4x\left(\frac{1000}{x^2}\right) = 2x^2 + \frac{4000}{x}$$

c. Find the value of x that minimizes the value of $S(x)$ on the interval $0 < x < \infty$.

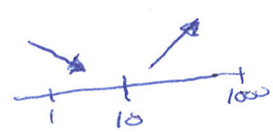
$$S'(x) = 4x - \frac{4000}{x^2}$$

$$\text{set } S'(x) = 0 = 4x - \frac{4000}{x^2} \rightarrow \frac{4000}{x^2} = 4x \rightarrow x^3 = 1000.$$

$$\rightarrow x = 10$$

check that $x=10$ is local min by checking slope (s') on both sides of 10.

$S'(1) = 4 - 4000 < 0$ and $S'(1000) = 4000 - \frac{4000}{1000^2} > 0$

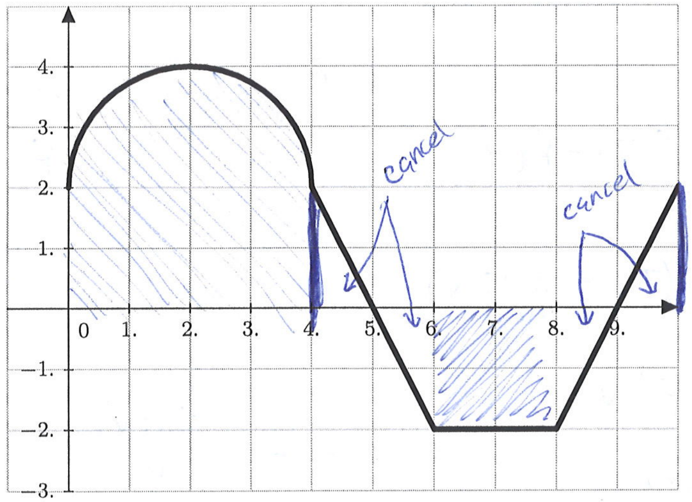


d. Use the result of part (c) to find the height y of the box of minimum surface area.

$$y = \frac{1000}{x^2} = \frac{1000}{10^2} = \frac{1000}{100} = 10$$

$$\boxed{x = y = 10 \text{ in}}$$

7. (25 points) The graph of a function $f(x)$ is given below.



(a) (3 points each) Use the graph to evaluate the following integrals.

(i) $\int_0^4 f(x)dx$

$8+\frac{\pi(2)^2}{2}$

$8+2\pi$

(ii) $\int_4^{10} f(x)dx$

-4

(iii) $\int_0^{10} f(x)dx$

add previous answers

~~$12+2\pi$~~

$4+2\pi$

(b) (8 points) Use your answer from part (a) to evaluate $\int_0^{10} 3f(x) - \frac{x}{2}\, dx$.

$$= \int_0^{10} 3f(x)dx - \int_0^{10} \frac{x}{2}dx$$

$$= 3\int_0^{10} f(x)dx - \left(\frac{x^2}{4}\Big|_0^{10}\right)$$

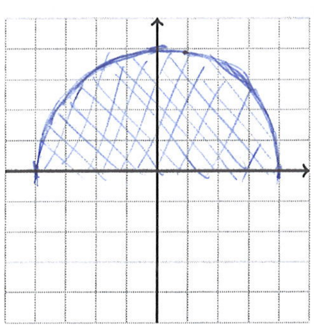
$$= 3(\cancel{12}\,4+2\pi) - \left(\left[\frac{10^2}{4}\right]-\left[\frac{0^2}{4}\right]\right)$$

$$= \cancel{36}\,12 + 6\pi - \frac{100}{4} + 0$$

$$= \cancel{36}\,12 + 6\pi - 25$$

$$\cancel{11+6\pi} = -13+6\pi$$

(c) (8 points) Draw an appropriate sketch for $\int_{-4}^{4} \sqrt{16-x^2}\, dx$ and then evaluate this integral.



top half of circle of radius $\sqrt{16}=4$ centered at the origin

Area of circle: πr^2

Area of half circle: $\frac{\pi r^2}{2}$

$$\int_{-4}^{4} \sqrt{16-x^2}\, dx = \frac{\pi(4)^2}{2} = 8\pi$$

8. (25 points)

(a) (5 points) Compute the following:

$$\frac{d}{dv}\int_v^7 \underbrace{\frac{\sin(t)}{t}}_{f(t)}\,dt$$

let F be a function so that $F' = f$

$$\frac{\partial}{\partial v}\int_v^7 \frac{\sin(t)}{t}\,dt = \frac{d}{dv}\left(F(7)-F(v)\right)$$

$$= \frac{d}{dv}F(7) - \frac{d}{dv}F(v)$$

$$= 0 - F'(v)$$

$$= -f(v) \longrightarrow = \frac{-\sin(v)}{v}$$

(b) Compute the indefinite integrals:

i. (10 points)

$$\int\left(\frac{2}{\sqrt{x}} - 3x^2 + \frac{1}{2}\sec^2(x) - \frac{2}{\sqrt{1-x^2}}\right)dx$$

$2x^{-1/2}$

$$= 4x^{1/2} - x^3 + \frac{1}{2}\tan(x) - 2\arcsin(x) + C$$

ii. (10 points)

$$\int \frac{3x}{x^2+3}\,dx$$

$u = x^2+3$

$du = 2x\,dx$

$\rightarrow dx = \frac{1}{2x}\,du$

$$= \int \frac{3x}{x^2+3}\left(\frac{1}{2x}\right)du$$

$$= \int \frac{3}{2}\,\frac{1}{x^2+3}\,du$$

$$= \int \frac{3}{2}\,\frac{1}{u}\,du$$

$$= \frac{3}{2}\int \frac{1}{u}\,du$$

$$= \frac{3}{2}\ln|u| + C$$

$$= \frac{3}{2}\ln|x^2+3| + C$$