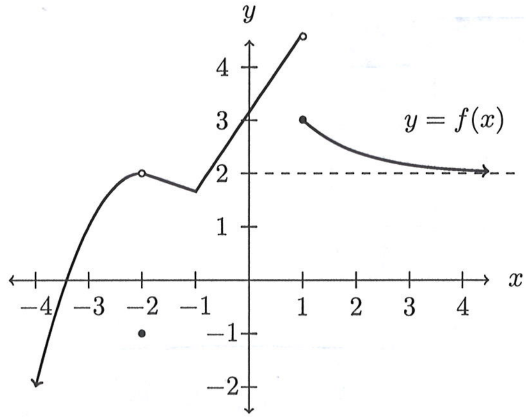
Answer Key

1. (25 points)

(a) (15 points) Use the figure below to answer the following questions. **If the limit or the derivative do not exist, write DNE.**



i. $\lim_{x \to -2^-} f(x)$ 2

ii. $\lim_{x \to -2} f(x)$ 2

iii. $\lim_{x \to 1+} f(x)$ 3

iv. $\lim_{x \to 1} f(x)$ DNE

v. $\lim_{x \to \infty} f(x)$ 2

vi. Identify the x−values for all points where the function is discontinuous. −2, 1

vii. Identify the x−values for all points where the function is not differentiable. −2, −1, 1

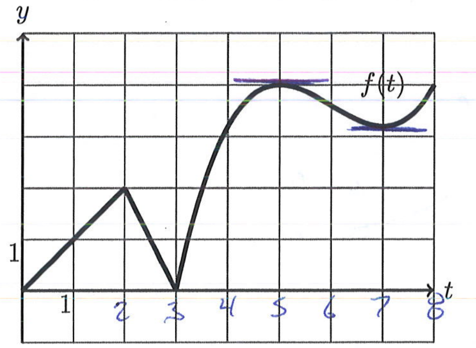
(b) (10 points) **Use the definition of the derivative** to compute $f'(x)$, where $f(x) = 2x^2 - 3x + 2$.

$$\lim_{h \to 0} \frac{2(x+h)^2 - 3(x+h) + 2 - (2x^2 - 3x + 2)}{h}$$

$$\lim_{h \to 0} \frac{2[x^2 + 2xh + h^2] - \cancel{3x} - 3h \cancel{+2} - 2x^2 \cancel{+3x} \cancel{-2}}{h}$$

$$\lim_{h \to 0} \frac{4x\cancel{h} + h^2 - 3\cancel{h}}{h} = \lim_{h \to 0} 4x + h - 3 = 4x - 3$$

2. (25 points) A particle is moving vertically. The graph of the height (in meters) of a particle at time t, in seconds, is given by $f(t)$ below.



(a) At what t-value(s) is the velocity of the particle 0? Circle ALL correct answers.

↳ horizontal tangent on position graph

i. $t = 1$

ii. $t = 3$

iii. $t = 4$

iv. $t = 5$

v. $t = 7$

(b) Match each of the following mathematical notations with the appropriate statement. Write the correct letter in the blank beside each notation.

- $f'(4)$ B
- $\dfrac{f(4) - f(0)}{4 - 0}$ E
- $f''(4)$ C
- $f(4)$ A
- $\dfrac{1}{4-0}\displaystyle\int_0^4 f(x)dx$ D

~~(A) The height of the particle after 4 seconds~~

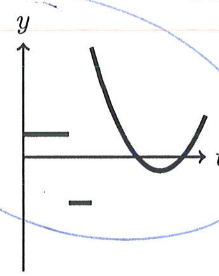
~~(B) The velocity of the particle after 4 seconds~~

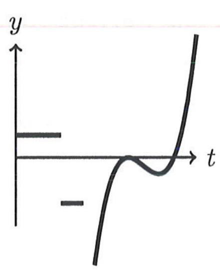
~~(C) The acceleration of the particle after 4 seconds~~

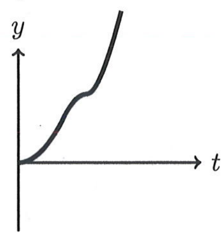
(D) The average height of the particle after 4 seconds

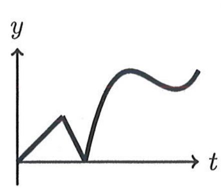
~~(E) The average rate of change of the particle after 4 seconds~~

(c) Which of the following graphs could represent a derivative of f? Circle ALL correct answers.

A) 

B) 

C) 

D) 

3. (25 points)

(a) (10 points) Find the slope of the tangent line to the curve $x^3y + 2y^3 = 5x$ at the point $(2, 1)$.

product rule

i. $\frac{1}{2}$

ii. $-\frac{1}{2}$

iii. $\frac{17}{14}$

iv. $\frac{13}{8}$

v. $-\frac{13}{8}$

$$x^3y' + 3x^2y + 6y^2y' = 5$$

$$x^3y' + 6y^2y' = 5 - 3x^2y$$

$$y'[x^3 + 6y^2] = 5 - 3x^2y$$

$$y' = \frac{5 - 3x^2y}{x^3 + 6y^2}$$

$$@ (2,1) \to m = \frac{5 - 3(2)^2 \cdot (1)}{(2)^3 + 6(1)^2} = \frac{-7}{14} = -\frac{1}{2}$$

(b) (10 points) Find the linear approximation to $f(x) = x^{1/2}$ at $x = 9$. Is it an underestimate or overestimate? **Motivate your answer.**

$$L(x) = f(a) + f'(a) \cdot (x - a)$$

where $a = 9$

$$= 9^{1/2} + \frac{1}{2}(9)^{-1/2} \cdot (x - 9)$$

$$= 3 + \frac{1}{6}(x - 9)$$

$$f'(x) = \frac{1}{2}x^{-1/2}$$

$$f''(x) = -\frac{1}{4}x^{-3/2}$$

$f''(9) < 0 \Rightarrow$ concave down

over estimate

(c) (5 points) Use the approximation of part (b) to estimate $f(12)$ with two correct decimals. **Circle the correct answer.**

i. 10.50

ii. 3.46

iii. 3.48

iv. 3.43

v. 3.50

$$L(12) = 3 + \frac{1}{6}(12 - 9) = 3 + \frac{1}{6} \cdot (3) = 3 + \frac{1}{2} = 3.50$$

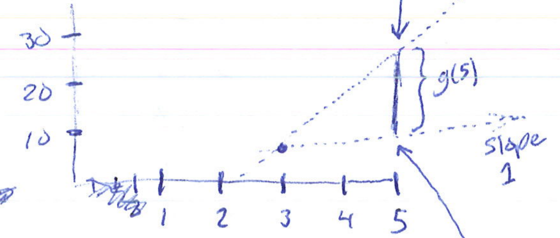
Note: $f(12) = 12^{1/2} = \sqrt{12} \approx \underline{3.4641}$

slope distance
$5+10\cdot 2$

4. (25 points)

(a) (5 points) Let $g(x)$ be a differentiable function, and suppose $g(3) = 5$, $1 \le g'(x) < 10$ for all values of x. Circle ALL TRUE statements below:

slope 10



i. $g(x)$ is a continuous function.

~~ii.~~ $\dfrac{g(6)-g(3)}{6-3} = g'(x)$ for **every** x in the interval $[3, 6]$

mean value Theorem → iii. $\dfrac{g(6)-g(3)}{6-3} = g'(x)$ for **some** x in the interval $[3, 6]$

~~iv.~~ g is decreasing on [3,6]

v. $7 \le g(5) < 35$ $\quad 7 \le g(5) \le 25$ — min & max slope bound where $g(5)$ can be

$5+1\cdot 2$
slope distance

(b) (5 points) Two cars leave from Lincoln NE to Omaha NE. Suppose that car one's position is given by the function $S_1(t)$ and car two's position is given by the function $S_2(t)$. Circle ALL TRUE statements below:

~~i.~~ If $S_1'(t) < S_2'(t)$ for all t, and both cars arrive at Omaha at the same time then car two must have left Lincoln fist.

ii. If $S_1'(t) < S_2'(t)$ for all t, and both cars arrive at Omaha at the same time then car one must have left Lincoln fist.

iii. If $S_1'(t) > S_2'(t)$ for all t, and both cars leave Lincoln at the same time then car one gets to Omaha first.

~~iv.~~ If $S_1'(t) > S_2'(t)$ for all t, and both cars leave Lincoln at the same time then car two gets to Omaha first.

~~v.~~ Without knowing $S_1(t)$ or $S_2(t)$ explicitly there is no way to determine which car arrived first. (this is a poorly worded question. They are saying it is possible to decide who arrives first even without an equation as we did in parts ii, iii)

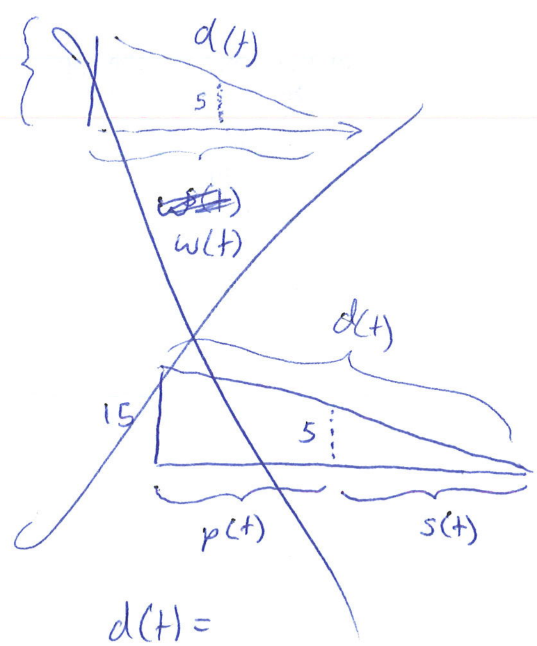
(c) (15 points) A street light is at the top of a 15 foot tall pole. A 5 foot tall woman walks away the pole with a speed of 2 ft/sec along a straight path. How fast is the tip of her shadow moving (away from the woman) when she is 15 feet from the base of the pole?

$$\cancel{15^2 + (w(t))^2 = d(t)^2}$$

$$\cancel{2w(t)\,w'(t) = 2d(t)\cdot d'(t)}$$

$$\cancel{15^2 + (p(t)+s(t))^2 = (d(t))^2}$$

$$\cancel{2(p(t)+s(t))\cdot(p'(t)+s'(t)) = 2d(t)\cdot d'(t)}$$



$d(t) =$

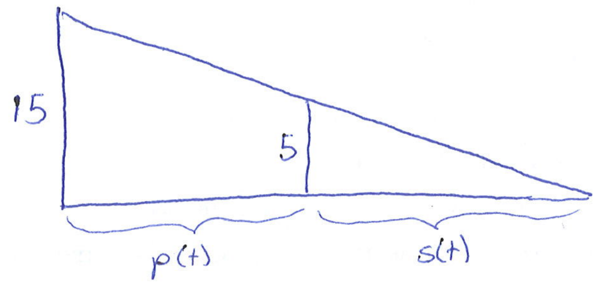
SEE NEXT PAGE

$p(t)$ = distance of woman from pole

$s(t)$ = lenth of her shadow

we know $p'(t) = 2\ \frac{ft}{sec}$

we want to find $s'(t)$



The small triangle and large triangle are similar, so the ratios of side lengths are the same

use the ratio of vertical side to horizontal side

$$\frac{15}{p(t)+s(t)} = \frac{5}{s(t)} \rightarrow 15\,s(t) = 5(p(t)+s(t))$$

$\rightarrow 15\,s(t) = 5p(t) + 5\,s(t)$

$\rightarrow 15\,s(t) - 5\,s(t) = 5\,p(t)$

$\rightarrow 10\,s(t) = 5\,p(t)$

$\rightarrow 2\,s(t) = p(t)$ ← This is the geometric equation we want. Take derivative of both sides

$\rightarrow 2\,s'(t) = p'(t)$

plug in $p'(t) = 2\ \frac{ft}{sec}$

$2\,s'(t) = 2$ ft/sec

$\rightarrow s'(t) = 1$ ft/sec

the tip of her shadow moves away from her at $1\ \frac{ft}{sec}$

5. (25 points)

(a) (7 points) Compute the limits below

(i) $\lim_{x\to 0} \frac{1-\cos x}{3x^2}$ $\quad \frac{1-1}{0}$ $\xrightarrow{L'H} \lim_{x\to 0} \frac{0+\sin(x)}{6x} = \frac{0}{0}$

$\xrightarrow{L'H} \lim_{x\to 0} \frac{\cos(x)}{6} = \frac{1}{6}$

(ii) (7 points) $\lim_{x\to 0^+} x\ln(x)$

$\lim_{x\to 0^+} x\cdot \ln(x) = 0\cdot(-\infty)$ which is not well defined, so we rewrite the original

$x\cdot \ln(x) = \frac{1}{x^{-1}}\cdot \ln(x)$

so $\lim_{x\to 0^+} \frac{1}{x^{-1}}\cdot \ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1}} = \frac{-\infty}{\infty} \xrightarrow{L'H} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-x^{-2}} = \lim_{x\to 0^+} \frac{-x^2}{x} = \lim_{x\to 0^+} -x$

$= 0$

(b) (5 points) The position of a particle at time t is given by the parametric curve

$$x(t) = 6t^2 - t^3 \text{ and } y(t) = 2t^2 - 4t.$$

Find all points (x, y) where the curve has a vertical tangent line.

i. $(0, 0)$
ii. $(1, -2)$
iii. $(5, -2)$
iv. $(32, 16)$
v. $(0, 1)$

vertical tangent means $\frac{dx}{dy} = 0$ so $\frac{dx}{dy} = \frac{dx/dt}{dy/dt} = 0$ (so dx/dt must be 0)

$\frac{dx}{dt} = 12t - 3t^2 = t(12-3t)$

if $\frac{dx}{dt} = 0$ then $t=0$ or $t=4$

plug in $(x(0), y(0)) = (0,0)$

and $(x(4), y(4)) = (32, 16)$

(c) (6 points) Given the same parametric curve as problem (b), $x(t) = 6t^2 - t^3$ and $y(t) = 2t^2 - 4t$, find the equation of a line tangent to the curve at the point $t = 2$. Put the tangent in slope intercept form, that is $y = mx + b$.

$$y = \boxed{\tfrac{1}{3}}\, x + \boxed{\tfrac{-16}{3}}$$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t-4}{12t-3t^2}$

when $t=2$, $m = \frac{4(2)-4}{12(2)-3(2)^2} = \frac{4}{12} = \frac{1}{3}$ ← slope

next find the point at $t=2$

$(x(2), y(2)) = (6(2)^2-(2)^3,\ 2(2)^2-4(2))$

$= (16, 0)$

now with the slope and point,

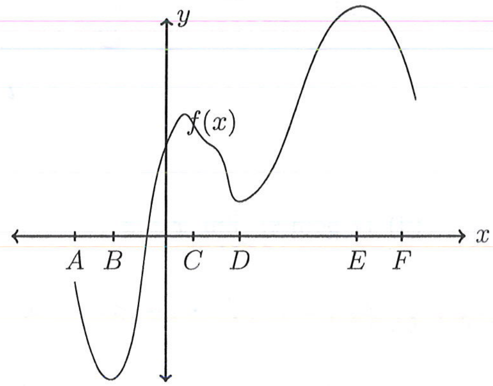
$y - 0 = \frac{1}{3}(x-16) \Rightarrow y = \frac{1}{3}x - \frac{16}{3}$

should check that $\frac{dy}{dt} \neq 0$ @ $t=0, t=4$

$\frac{dy}{dt} = 4t-4 \rightarrow y'(0) = -4 \neq 0$ and $y'(4) = 16-4 \neq 0$ ✓

6. (25 points)

(a) (5 points) In the graph of the function f given to the right, identify all points that are



i. local maxima on the interval $[A, F]$

C, E

ii. local minima on the interval $[A, F]$

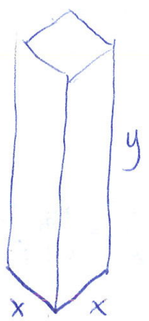
B, D

iii. global maximum on the interval $[A, F]$

E

iv. global minimum on the interval $[A, F]$

B



(b) (20 points) A closed box has height y inches and a **square base** of edge length x inches. Let V be the volume of the box and S the surface area. In this problem you will find the values of x and y that minimize the surface area S of the box, given that its volume V is 1000 inches^3.

a. Give S and V in terms of x and y.

$$S = 2x^2 + 4xy$$

$$V = x^2 y$$

b. Use the fact that $V = 1000$ in^3 to write S as a function of x only.

$$1000 = V = x^2 y \rightarrow y = \frac{1000}{x^2}$$

$$\rightarrow S = 2x^2 + 4x\left(\frac{1000}{x^2}\right) = 2x^2 + \frac{4000}{x}$$

c. Find the value of x that minimizes the value of $S(x)$ on the interval $0 < x < \infty$.

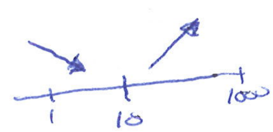
$$S'(x) = 4x - \frac{4000}{x^2}$$

set $S'(x) = 0 = 4x - \frac{4000}{x^2} \rightarrow \frac{4000}{x^2} = 4x \rightarrow x^3 = 1000$

$\rightarrow x = 10$

check that $x = 10$ is local min by checks slope (S') on both sides of 10.

$S'(1) = 4 - 4000 < 0$ and $S'(1000) = 4000 - \frac{4000}{1000^2} > 0$

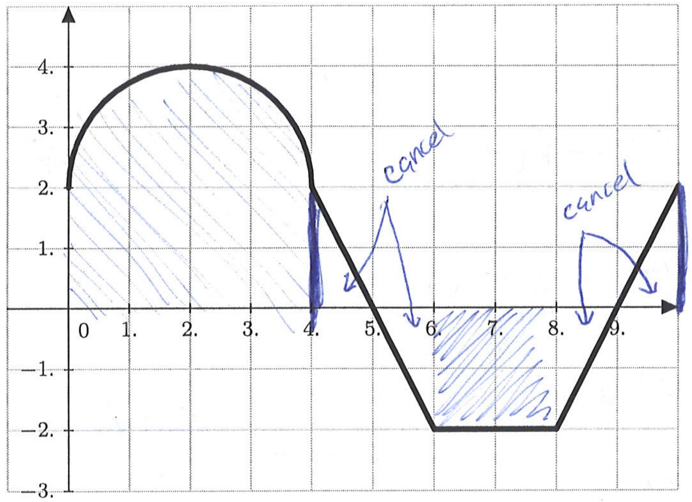


d. Use the result of part (c) to find the height y of the box of minimum surface area.

$$y = \frac{1000}{x^2} = \frac{1000}{10^2} = \frac{1000}{100} = 10$$

$$\boxed{x = y = 10 \text{ in}}$$

7. (25 points) The graph of a function $f(x)$ is given below.



(a) (3 points each) Use the graph to evaluate the following integrals.

(i) $\int_0^4 f(x)dx$

$8 + \frac{\pi(2)^2}{2}$

$8+2\pi$

(ii) $\int_4^{10} f(x)dx$

-4

(iii) $\int_0^{10} f(x)dx$

add previous answers

$\cancel{12+2\pi}$

$4+2\pi$

(b) (8 points) Use your answer from part (a) to evaluate $\int_0^{10} 3f(x) - \frac{x}{2}\,dx$.

$= \int_0^{10} 3f(x)dx - \int_0^{10} \frac{x}{2}dx$

$= 3\int_0^{10} f(x)dx - \left(\frac{x^2}{4}\Big|_0^{10}\right)$

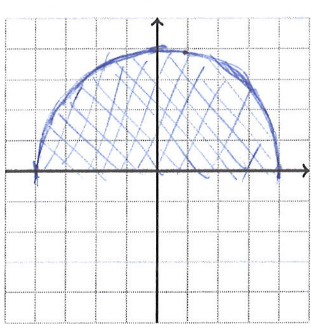
$= 3(\cancel{12}\,4+2\pi) - \left(\left[\frac{10^2}{4}\right] - \left[\frac{0^2}{4}\right]\right)$

$= \cancel{36}\,12 + 6\pi - \frac{100}{4} + 0$

$= \cancel{36}\,12 + 6\pi - 25$

$= \cancel{11+6\pi} = -13+6\pi$

(c) (8 points) Draw an appropriate sketch for $\int_{-4}^{4} \sqrt{16-x^2}\,dx$ and then evaluate this integral.



top half of circle of radius $\sqrt{16} = 4$ centered at the origin

Area of circle: πr^2

Area of half circle: $\frac{\pi r^2}{2}$

$\int_{-4}^{4} \sqrt{16-x^2}\,dx = \frac{\pi(4)^2}{2} = 8\pi$

8. (25 points)

(a) (5 points) Compute the following:

$$\frac{d}{dv}\int_v^7 \underbrace{\frac{\sin(t)}{t}}_{f(t)}\,dt$$

let F be a function so that $F' = f$

$$\frac{d}{dv}\int_v^7 \frac{\sin(t)}{t}\,dt = \frac{d}{dv}\left(F(7) - F(v)\right)$$

$$= \frac{d}{dv}F(7) - \frac{d}{dv}F(v)$$

$$= 0 - F'(v)$$

$$= -f(v) = -\frac{\sin(v)}{v}$$

(b) Compute the indefinite integrals:

i. (10 points)

$$\int \left(\frac{2}{\sqrt{x}} - 3x^2 + \frac{1}{2}\sec^2(x) - \frac{2}{\sqrt{1-x^2}}\right) dx$$

$2x^{-1/2}$

$$= 4x^{1/2} - x^3 + \frac{1}{2}\tan(x) - 2\arcsin(x) + C$$

ii. (10 points)

$$\int \frac{3x}{x^2+3}\,dx$$

$u = x^2 + 3$

$du = 2x\,dx$

$dx = \frac{1}{2x}\,du$

$$= \int \frac{3\cancel{x}}{x^2+3}\left(\frac{1}{2\cancel{x}}\right) du$$

$$= \int \frac{3}{2}\,\frac{1}{x^2+3}\,du$$

$$= \int \frac{3}{2}\,\frac{1}{u}\,du$$

$$= \frac{3}{2}\int \frac{1}{u}\,du$$

$$= \frac{3}{2}\ln|u| + C$$

$$= \boxed{\frac{3}{2}\ln|x^2+3| + C}$$