Finite Dimensional Inner Product Spaces

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Vector Spaces

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Linear Systems of Equations

Form

We now apply the fruits of our investigation into vector spaces and linearity to solve systems of linear equations.

Definition 1. *A linear system of equations is a collection of m equations of the form:*

$$a_1x_1 + \cdots + a_nx_n = b$$

where $a_i, x_i, b \in \mathbb{F}$ for $1 \leq i \leq n$. Equivalently, we may say $Ax = b$ for an $m \times n$ matrix A, where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$. If $b = \mathbf{0}$, the linear system is said to be homogenous.

Definition 2. *A solution to a linear system is a vector $s \in \mathbb{F}^n$ such that $As = b$*

Theorem 1. *Let A be an $m \times n$ matrix over $\mathbb{F}$. If $m < n$, then the homogenous system $Ax = 0$ has a nontrivial solution.*

Proof. Notice that, the solution set to the system $Ax = 0$ is $\ker(L_A)$, so by the dimension theorem, $dim(\ker(A)) = n - rank(L_A)$. Additionally, we know that $rank(A)$ is nothing but the number of linearly independent vectors defined by its rows which certainly cannot exceed m. Therefore $rank(A) \leq m < n$, in which case $n - rank(A) = dim(\ker(A)) > 0$, and so $\ker(A) \neq \{0\}$. □

Theorem 2. *For any solution s to the linear system $Ax = b$,*

$$S = \{s + s_0 : As_0 = \mathbf{0}\}$$

is its solution set.

Proof. Suppose that $As = b$ and $As' = b$. Then $A(s' - s) = As' - As = b - b = 0$. It follows that $s + (s' - s) \in S$. Conversely, if $y \in S$, then $y = s + s'$, in which case $Ay = A(s + s') = As + As' = b + 0 = b$. That is, $Ay = b$. □

Theorem 3. *Let $Ax = b$ for an $n \times n$ matrix A. If A is invertible, then the system has a single solution $A^{-1}b$. If the system has a single solution, then A is invertible.*

Proof. Suppose A is invertible. Then $A(A^{-1}b) = AA^{-1}(b) = b$. Furthermore, if $As = b$ for some $s \in \mathbb{F}^n$, then $A^{-1}(As) = A^{-1}b$ and so $s = A^{-1}b$. Next, suppose that the system has a unique solution s. Then by theorem 2, we know that the solution set $S = \{s + s_0 : As_0 = 0\}$. But this is only the case if $\ker(A) = \{0\}$, lest s not be unique. And so, by the dimension theorem, A is invertible. □

THEOREM 4. *The linear system $Ax = b$ has a nonempty solution set if and only if $rank(A) = rank(A|b)$.*

PROOF. If the system has a solution, then $b \in im(L_A)$. Additionally, $im(L_A) = L_A(F^n)$ and $L_A(e_i) = Ae_i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix}$. Therefore, since $L_A(\mathbb{F}^n) = span\{Ae_1, \dots Ae_n\}$, $im(L_A) = span\{A_1, \dots A_n\}$, where A_i is the i^{th} column of A. Certainly, $b \in span\{A_1, \dots A_n\}$ if and only if $span\{A_1, \dots A_n\} = span\{A_1, \dots A_n, b\}$, which is to say $dim(span\{A_1, \dots A_n\}) = dim(span\{A_1, \dots A_n, b\})$, or, $rank(A) = rank(A|b)$. $\square$

Solution

DEFINITION 3. *A matrix of the form $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$. Is said to be in reduced echelon form if $a_{ii} \neq 0$ implies that $a_{ij} = 1$ and all other entries $a_{lj} = 0$. Additionally, if $a_{ij} = 0$ for all $1 \leq j \leq n$, then $i < r$ for all nonzero rows $a_{r1}, \dots a_{rm}$.*

THEOREM 5. *Any matrix can be converted into reduced echelon form via a finite number of elementary row operations.*

The proof should be clear from the definitions of the three elementary. This form is of particular interest because reducing an augmented matrix is equivalent to solving a linear system of equations. We now have a procedure for solving arbitrary systems of linear equations. For example, we may now demonstrate that a set of vectors is linearly dependent by finding a nontrivial solution to a linear system of equations; similarly we may apply theorem 4 to demonstrate that a set of vectors is linearly dependent. In the following chapter, we will also see that computing the elements of an eigenspace is made possible by reducing a matrix.

CHAPTER 4

Eigenspaces

CHAPTER 5

Orthogonality

Hello

DEFINITION 4. *There exists*