Binghamton University
Department of Mathematical Sciences

Advanced Calculus of Several Variables by C.H Edwards: Selected Solutions

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Section 1

Question 1.2

Let V and W be subspaces of $\mathbb{R}^n$. Then $V \cap W$ is too.

Answer. Let $v_i \in V \cap W$ for $1 \leq i \leq n$ We know that each of these vectors must be in both V and W, whence any linear combination of vectors in V is in V, and similarly, any linear combination of vectors in W is in W. Thus, any linear combination involving v_i must necessarily be in $V \cap W$

Question 1.3

If V and W are subspaces of $\mathbb{R}^n$, then the set $V + W = \{v + w : v \in V \text{ and } w \in W\}$ is too.

Answer. If $v_1 + w_1 \in V + W$ then,since $VandW$ are closed under scalar multiplication, $av_1 + aw_1 \in V + W$ for any scalar $a \in \mathbb{R}$. Furthermore, supposing $v_1 + w_1 \in V + W$ and $v_2 + w_2 \in V + W$ for $v_1, v_2 \in V$ and $w_1, w_2 \in W$, we have:

$$(v_1 + w_1) + (v_2 + w_2) = (v_1 + v_2) + (w_1 + w_2) \in V + W$$

because both V and W are closed under vector addition.

Question 1.5

Let D_0 be the set of all real-valued differentiable functions on $[0, 1]$ where $f(0) = f(1) = 0$. Then D_0 is a vector space if we define scalar multiplication and vector addition in the natural way.

Answer. If $f, g \in D_0$ then $f + g$ must necessarily have the same domain as its component functions and be differentiable. Additionally, adding any two such functions will give you another, and multiplying by scalars does as well. Thus, it is clear that D_0 is a vector space.

However, if we, instead, have that $f(0) = 0$ and $f(1) = 1$ for any function $f \in D_0$ D_0 can not be a vector space, for if $f \in D_0$ and $a \in \mathbb{R}$, we have $af(1) = a \neq 1$. And so $af(1) \notin D_0$.

Section 2

Question 2.5

Let $V_1, \ldots V_k$ be a set of linearly independent vectors in a vector space V. If $k < n = \dim V$, then there exist vectors $V_{k+1}, \ldots V_n$ in V such that $V_1, \ldots, V_k, V_{k+1}, \ldots V_n$ is a basis for V.

Answer. By Zorn's Lemma, we know that any vector space has a basis. Furthermore, by definition of dimension, we know that any set of linearly independent vectors in V has at most n elements. Thus, since a basis is a linear independent generating set of vectors, and such a set must necessarily contain as many elements as the dimension of the space, it is clear that the set

$$\{V_1, \ldots, V_k\} \cup \{V_{k+1}, \ldots V_n\}$$

is a basis for V, where $\{V_{k+1}, \ldots V_n\}$ is the unique set of remaining vectors such that the union with our original set constitutes a basis.

Question 2.6

The following two statements are equivalent:

1. If $\dim V = n$, then each basis for V has exactly$n$ vectors.
2. If the system

$$\begin{aligned} a_{11}x_1 + \ldots a_{1n}x_n &= 0 \\ &\vdots \\ a_{n1}x_1 + \ldots a_{nn}x_n &= 0 \end{aligned}$$

   has only the trivial solution, then if the zero vector is replaced by $b = (b_1, \ldots b_n)$ above, the system has a unique solution.

Answer. Suppose that the first statement is true. Now, if our system above has only trivial solution, then that means the vectors

$$\begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{bmatrix} \cdots \begin{bmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nn} \end{bmatrix}$$

are linearly independent by definition. Furthermore, by assumption they must form a basis for V. Hence, we can uniquely express any vector $b = (b_1, \ldots b_n) \in V$ uniquely as a linear combination of the above linearly independent vectors, which is to say that there is a unique solution to the above system where the zero vector is replaced by b.

Conversely, assuming the second statement, it is clear that the vectors

$$\begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1k} \end{bmatrix} \cdots \begin{bmatrix} a_{k1} \\ a_{k2} \\ \vdots \\ a_{kk} \end{bmatrix}$$

form a basis for the space consisting of all vectors $b = (b_1, \dots b_k)$, by the same logic. Therefore if we suppose that $\dim V = n$ and $k < n$ then clearly there exist vectors in V that are not generated by our linearly independent vectors because the dimension of our space is strictly less than $\dim V$. Similarly, if $k > n$, our space will generate vectors that are not in V.

Question 2.7

Any two colinear vectors are linearly dependent. And any 3 coplanar vectors are linearly dependent.

Answer. Any vector space of dimension n can have at most n linearly independent vectors. Therefore, along any line (a one dimensional vector space), there can exist at most one linearly independent vector, and within a plane, two.

Section 3

Question 3.5

If $\cdot$ is the standard inner product on $\mathbb{R}^n$, then $X \cdot Y = \frac{1}{4}(|X+Y|^2 - |X-Y|^2)$ for any two vectors $X, Y \in \mathbb{R}^n$.

Answer. Applying the definition of the Euclidean norm, we have
$\frac{1}{4}(|X+Y|^2 - |X-Y|^2) = \frac{1}{4}((X+Y) \cdot (X-Y))$. And by definition of the dot product, we have
$\frac{1}{4}((X+Y) \cdot (X-Y)) = \frac{1}{4}((x_1+y_1)^2 + \cdots + (x_n+y_n)^2 - ((x_1-y_1)^2 + \dots (x_n-y_n)^2))$.
Multiplying our terms, it is clear that we are left with

$$\frac{1}{4}(4x_1y_1 + \dots 4x_ny_n) = X \cdot Y$$