BINGHAMTON UNIVERSITY
DEPARTMENT OF MATHEMATICS

Selected Solutions to Probability Theory and Examples by Rick Durrett

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Question 1.1.1

Let $\Omega = \mathbb{R}$, $\mathcal{F}$ be the set of all subsets of $\mathbb{R}$ such that it or it's complement is countable. Let $P(A) = 0$ if A is countable and $P(A) = 1$ if A^c is countable. Then $(\Omega, \mathcal{F}, P)$ is a probability space.

Answer. We begin by showing that $\mathcal{F}$ is a sigma algebra. Suppose that $x \in \mathcal{F}$. Then either x countable or $\mathbb{R} \backslash x$ is countable. In the first case, $\mathbb{R} \backslash (\mathbb{R} \backslash x) = x \in \mathcal{F}$. In the latter, $\mathbb{R} \in (F)$. Now suppose that $x_1, x_2, \ldots \in \mathcal{F}$. If at least one of $\mathbb{R} \backslash x_i$ is countable, then so must $\bigcap_i \mathbb{R} \backslash x_i$. Otherwise, it must be that all x_i are uncountable, in which case $\bigcup_i x_i$ is too. Now, to demonstrate that our probability measure is well-defined, it is clear that by definition $P(x) \geq 0$ for any set $x \in \mathcal{F}$. Additionally, supposing $x_1, x_2, \ldots \in \mathcal{F}$ are disjoint sets we have either $P(\bigcup_i x_i) = 0$ or $P(\bigcup_i x_i) = 1$. In the first case, we know that the union is countable, which implies that each of x_i must be countable, lest our union be uncountable. This is to say $P(\bigcup_i x_i) = \sum_i 0 = \sum_i P(x_i)$. In the second case, we have that $P(\bigcup_i x_i) = 1$. And notice that if x_i are disjoint then only a single member of that collection can have an uncountable complement, for otherwise they could not be disjoint. This is to say that $P(\bigcup_i x_i) = 1 + 0 + 0 + \ldots = \sum_i P(x_i)$.