Finite Dimensional Inner Product Spaces

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Vector Spaces

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Linear Functions

CHAPTER 3

Linear Systems of Equations

We now apply fruits of our investigation into vector spaces and linearity to solve systems of linear equations.

DEFINITION 1. *A linear system of equations is a collection of m equations of the form:*

$$a_1x_1 + \cdots + a_nx_n = b$$

where $a_i, x_i, b \in \mathbb{F}$ for $1 \leq i \leq n$. Equivalently, we may say $Ax = b$ for an $m \times n$ matrix A, where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$. If $b = \mathbf{0}$, the linear system is said to be homogenous.

DEFINITION 2. *A solution to a linear system is a vector $s \in \mathbb{F}^n$ such that $As = b$*

THEOREM 1. *Let A be an $m \times n$ matrix over $\mathbb{F}$. If $m < n$, then the homogenous system $Ax = 0$ has a nontrivial solution.*

PROOF. Notice that, the solution set to the system $Ax = 0$ is $Ker(L_A)$, so by the dimension theorem, $dim(Ker(A)) = n - rank(L_A)$. Additionally, we know that $rank(A)$ is nothing but the number of linearly independent vectors defined by its rows which certainly cannot exceed m. Therefore $rank(A) \leq m < n$, in which case $n - rank(A) = dim(Ker(A)) > 0$, and so $ker(A) \neq \{0\}$. □

THEOREM 2. *For any solution s to the linear system $Ax = b$,*

$$S = \{s + s_0 : As_0 = \mathbf{0}\}$$

is its solution set.

PROOF. Suppose that $As = b$ and $As' = b$. Then $A(s' - s) = As' - As = b - b = 0$. It follows that $s + (s' - s) \in S$. Conversely, if $y \in S$, then $y = s + s'$, in which case $Ay = A(s + s') = As + As' = b + 0 = b$. That is, $Ay = b$. □

CHAPTER 4

Eigenspaces

CHAPTER 5

Orthogonality

Hello

DEFINITION 3. *There exists*