Superstring Theory

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Abstract

We provide an introduction to superstring theory in the both the RNS and GS formalisms. The RNS formalism exhibits supersymmetry on the string worldsheet, and evidence for spacetime supersymmetry is seen after quantisation and the GSO projection. The GS formalism has manifest spacetime supersymmetry and is shown to resemble the RNS formalism after lightcone gauge quantisation. Though the bosonic theory is taken as given, comparisons are made frequently to provide motivation and to draw attention where similarities exist.

Contents

1 Introduction

It is well-known that the bosonic string theory suffers from the presence of a tachyon in its spectrum of states, and from the inability to describe fermions. This report gives an introduction to superstring theory, which not only solves both of these problems, but is ostensibly the best candidate for a coherent theory of quantum gravity.

First we discuss the Ramond-Neveu-Schwarz (RNS) formulation, wherein the bosonic string is generalised to include fermions on the string worldsheet. In this context we demonstrate global supersymmetry and develop the superspace formalism which makes worldsheet supersymmetry manifest. Boundary conditions are shown to give rise to two different sectors of string states, and mode expansions are given for each. The symmetries of the classical theory are addressed, along with the associated currents and the super-Virasoro constraints which impose important conditions on the spectrum of quantum states. We also review the locally supersymmetric theory and see the super-Virasoro constraints arise as gauge-fixing conditions. Quantisation is then considered in the covariant gauge and in the lightcone gauge, and it is shown that the two sectors allow the theory to describe both bosonic and fermionic states. The GSO projection is seen to ensure that the massless sector of the spectrum has an equal number of propagating bosonic and fermionic degrees of freedom, as necessary for spacetime supersymmetry. We also note the emergence of type IIA and type IIB theories.

Next we turn to the Green-Schwarz (GS) formalism, which makes spacetime supersymmetry manifest by embedding the worldsheet coordinates into superspace. The massive point particle is considered first, and both supersymmetry and kappa symmetry are demonstrated. The latter is a fermionic gauge symmetry necessary to ensure that there are again an equal number of bosonic and fermionic degrees of freedom. The string is then considered in both the classical and quantum settings. There is a phase-space constraint that makes covariant quantisation intractable, but lightcone gauge quantisation is still possible. We demonstrate that the quantum theory in this gauge looks very much like the RNS theory in the same gauge, and we again encounter the origins of the type I, type IIA, and type IIB theories.

2 The Classical RNS Superstring

We take as our starting point the action of the bosonic string in conformal gauge. This means that we have gauge-fixed the bosonic theory to make the worldsheet metric $h_{\alpha\beta}$ equal to the Minkowski spacetime metric $\eta_{\alpha\beta}$ up to a conformal factor,

$$h_{\alpha\beta} = e^{\phi}\eta_{\alpha\beta}. \tag{2.1}$$

The conformal factor drops out of the action, and we are left with[1]

$$S = -\frac{1}{2\pi}\int d^2\sigma\, \partial_\alpha X^\mu \partial^\alpha X_\mu \tag{2.2}$$

[1]We maintain the assumption $\alpha' = \frac{1}{2}$ throughout. Also, we will use letters from the beginning of the Greek alphabet $(\alpha, \beta, \ldots)$ to denote worldsheet indices, and letters from the end of the Greek alphabet $(\mu, \nu, \ldots)$ to denote spacetime indices.

The string worldsheet is a two-dimensional manifold described by $\sigma^\alpha = (\sigma^0, \sigma^1) = (\tau, \sigma)$, where τ is identified with "worldsheet time" and $\sigma \in [0, \pi]$ is a spatial parameter. The worldsheet scalar fields $X^\mu(\sigma^\alpha)$ are coordinate maps which embed these parameters into D-dimensional Minkowski spacetime. This action describes a free theory of massless scalar bosons on the two-dimensional worldsheet [4].

We can generalise the bosonic theory by including additional degrees of freedom propagating along the string. These internal degrees of freedom will be represented by spinor fields $\psi^\mu_A(\sigma^\alpha)$ obeying the classical anticommutation relations

$$\{\psi^\mu_A, \psi^\nu_B\} = 0. \tag{2.3}$$

More precisely, $\psi^\mu = \psi^\mu_A$ is a two-component worldsheet spinor (since $2^{D/2} = 2$ in two dimensions) composed of anticommuting Grassmann numbers, and A is the spinor index. We will frequently suppress spinor indices unless doing so obscures our meaning.

Though the fields X^μ are worldsheet scalars and ψ^μ are worldsheet spinors, both of these fields transform in the vector representation of the spacetime Lorentz group $SO(1, D-1)$. This distinction is subtle but it does not actually pose a problem; spacetime Lorentz symmetry is an internal symmetry from the worldsheet perspective, and as such it is not governed by the spin-statistics theorem [2]. This leads to unnatural results in the quantum theory, and we will see that the GSO projection is necessary to resolve these and other issues.

2.1 Superstring Action

We can make a simple generalisation to our bosonic action by adding D free massless fermions. Such fermions must satisfy the massless Dirac equation, $\rho^\alpha \partial_\alpha \psi_\mu = 0$, and therefore the *RNS superstring action* takes the form

$$S = -\frac{1}{2\pi} \int d^2\sigma \left(\partial_\alpha X^\mu \partial^\alpha X_\mu - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right) \tag{2.4}$$

where $\bar{\psi}^\mu = \psi^\dagger C = \psi^\dagger \rho^0$ is the Dirac conjugate, and ρ^α provides a two-dimensional representation of the Dirac algebra, whose defining relation is

$$\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}. \tag{2.5}$$

We can make our representation of the Dirac algebra explicit by choosing the basis (as in [1])

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.6}$$

Since this is a real basis, it makes sense to require the spinor components to be real as well. Then ψ^μ is a Majorana spinor, and the Dirac conjugate is given by $\bar{\psi}^\mu = (\psi^\mu)^\dagger \rho^0 = (\psi^\mu)^T \rho^0$.

The bosonic fields X^μ obey the two-dimensional wave equation $\partial^\alpha \partial_\alpha X^\mu = 0$, and as such they can be written in terms of left-moving and right-moving modes [4]

$$X^\mu(\tau, \sigma) = X^\mu_L(\sigma^-) + X^\mu_R(\sigma^+), \tag{2.7}$$

where we have introduced the worldsheet lightcone coordinates[2] $\sigma^\pm = \tau \pm \sigma$. The wave equation is given by $\partial^\alpha \partial_\alpha X^\mu = (\partial_\tau^2 - \partial_\sigma^2) X^\mu = 0$, which we can rewrite using $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$ as

$$\partial_+ \partial_- X^\mu = \partial_- \partial_+ X^\mu = 0. \tag{2.8}$$

The spinor components are given by

$$\psi^\mu = \begin{pmatrix} \psi_-^\mu \\ \psi_+^\mu \end{pmatrix}. \tag{2.9}$$

These fields satisfy the massless Dirac equation by construction, which we can rewrite in our basis as

$$\begin{aligned} \rho^\alpha \partial_\alpha \psi^\mu &= \begin{pmatrix} 0 & \partial_1 - \partial_0 \\ \partial_1 + \partial_0 & 0 \end{pmatrix} \begin{pmatrix} \psi_-^\mu \\ \psi_+^\mu \end{pmatrix} \\ &= 2 \begin{pmatrix} 0 & -\partial_- \\ \partial_+ & 0 \end{pmatrix} \begin{pmatrix} \psi_-^\mu \\ \psi_+^\mu \end{pmatrix} = 0. \end{aligned} \tag{2.10}$$

Therefore we have

$$\partial_+ \psi_-^\mu = \partial_- \psi_+^\mu = 0 \tag{2.11}$$

so again we can interpret $\psi_\pm^\mu$ as left-moving and right-moving modes.

The two-dimensional chirality operator $\rho^3 = \rho^0 \rho^1$ satisfies

$$\rho^3 \psi_\pm^\mu = \pm \psi_\pm^\mu \tag{2.12}$$

which tells us that the left-moving and right-moving modes are spinors of negative and positive chirality, respectively. Thus the Dirac equations coincide with Weyl conditions, and as a result $\psi_\pm^\mu$ are Majorana-Weyl spinors in the two-dimensional sense [2].

2.2 Global Worldsheet Supersymmetry

The RNS action is invariant under the infinitesimal supersymmetry transformations

$$\delta X^\mu = \bar{\epsilon} \psi^\mu \qquad \delta \psi_A^\mu = -i \rho^\alpha \partial_\alpha X^\mu \epsilon_A \tag{2.13}$$

where ϵ_A is a constant infinitesimal Majorana spinor, and A denotes the free spinor index [2]. These transformations mix the bosonic and fermionic fields, as expected. To see that these leave the action invariant, we calculate its supersymmetric variation

$$\begin{aligned} \delta S &= -\frac{1}{2\pi} \int d^2\sigma \left(2\partial_\alpha (\delta X^\mu) \partial^\alpha X_\mu - i (\delta \bar{\psi}^\mu) \rho^\alpha \partial_\alpha \psi_\mu - i \bar{\psi}^\mu \rho^\alpha \partial_\alpha (\delta \psi_\mu) \right) \\ &= -\frac{1}{2\pi} \int d^2\sigma \left(2\partial_\alpha (\delta X^\mu) \partial^\alpha X_\mu + i \partial_\alpha \bar{\psi}^\mu \rho^\alpha (\delta \psi_\mu) - i \bar{\psi}^\mu \rho^\alpha \partial_\alpha (\delta \psi_\mu) \right) \\ &= -\frac{1}{\pi} \int d^2\sigma \left(\partial_\alpha (\delta X^\mu) \partial^\alpha X_\mu - i \bar{\psi}^\mu \rho^\alpha \partial_\alpha (\delta \psi_\mu) \right) \end{aligned} \tag{2.14}$$

where we have anticommuted spinors and used partial integration to combine terms.

[2] These are not to be confused with the spacetime lightcone coordinates $X^\pm$ which appear in lightcone gauge quantisation.

Substituting the transformation rules for the fields gives,

$$
\begin{aligned}
\delta S &= -\frac{1}{\pi}\int d^2\sigma\,\Big(\partial_\alpha(\bar{\epsilon}\psi^\mu)\partial^\alpha X_\mu - i\bar{\psi}^\mu\rho^\alpha\partial_\alpha(-i\rho^\beta\partial_\beta X_\mu\epsilon)\Big)\\
&= -\frac{1}{\pi}\int d^2\sigma\,\Big(\bar{\epsilon}\partial_\alpha\psi^\mu\partial^\alpha X_\mu - \bar{\psi}^\mu\rho^\alpha\epsilon\partial_\alpha(\rho^\beta\partial_\beta X_\mu)\Big)\\
&= -\frac{1}{\pi}\int d^2\sigma\,\Big(\bar{\epsilon}\partial_\alpha\psi^\mu\partial^\alpha X_\mu + \bar{\epsilon}\rho^\alpha\rho^\beta\psi^\mu\partial_\alpha\partial_\beta X_\mu\Big)\\
&= -\frac{1}{\pi}\int d^2\sigma\,\Big(\bar{\epsilon}\partial_\alpha\psi^\mu\partial^\alpha X_\mu - \bar{\epsilon}\partial_\alpha\psi^\mu\partial^\alpha X_\mu\Big)\\
&= \;\; 0
\end{aligned}
\tag{2.15}
$$

where we have again anticommuted spinors, and used the Dirac algebra relation and partial integration. Thus the action is indeed supersymmetric.

A necessary feature of supersymmetry transformations is that their commutator closes into a spatial translation [5]. We must verify that this is true of our transformations, keeping in mind that a translation in our case will take place along the worldsheet. First we consider the commutator of the variations of the bosonic field,

$$
\begin{aligned}
[\delta_1,\delta_2]X^\mu &= \delta_1(\bar{\epsilon}_2\psi^\mu) - (1\leftrightarrow 2)\\
&= \bar{\epsilon}_2(-i\rho^\alpha\partial_\alpha X^\mu\epsilon_1) - (1\leftrightarrow 2)\\
&= -i(\bar{\epsilon}_2\rho^\alpha\epsilon_1 - \bar{\epsilon}_1\rho^\alpha\epsilon_2)\partial_\alpha X^\mu\\
&= 2i\bar{\epsilon}_1\rho^\alpha\epsilon_2\partial_\alpha X^\mu\\
&= a^\alpha\partial_\alpha X^\mu.
\end{aligned}
\tag{2.16}
$$

So we see that the commutator of infinitesimal supersymmetry transformations on X^μ gives a translation along the worldsheet as required, with magnitude $a^\alpha = 2i\bar{\epsilon}_1\rho^\alpha\epsilon_2$.

The commutator of the variations of the spinor gives,

$$
\begin{aligned}
[\delta_1,\delta_2]\psi^\mu &= \delta_1(-i\rho^\alpha\partial_\alpha X^\mu\epsilon_2) - (1\leftrightarrow 2)\\
&= -i\rho^\alpha\partial_\alpha(\bar{\epsilon}_1\psi^\mu)\epsilon_2^A - (1\leftrightarrow 2)\\
&= -i\rho^\alpha(\bar{\epsilon}_1\partial_\alpha\psi^\mu)\epsilon_2^A - (1\leftrightarrow 2)
\end{aligned}
\tag{2.17}
$$

where A denotes the free spinor index. Using the two-dimensional Fierz rearrangement with chirality operator ρ_3, we can write [5]

$$
(\bar{\epsilon}_1\partial_\alpha\psi^\mu)\epsilon_2 = -\tfrac{1}{2}(\bar{\epsilon}_1\epsilon_2)\partial_\alpha\psi^\mu - \tfrac{1}{2}(\bar{\epsilon}_1\rho_\beta\epsilon_2)\rho^\beta\partial_\alpha\psi^\mu - \tfrac{1}{2}(\bar{\epsilon}_1\rho_3\epsilon_2)\rho_3\partial_\alpha\psi^\mu \tag{2.18}
$$

and similarly for $(\bar{\epsilon}_2\partial_\alpha\psi^\mu)\epsilon_1$. Therefore, we have

$$
\begin{aligned}
[\delta_1,\delta_2]\psi^\mu &= \tfrac{i}{2}\rho^\alpha\Big((\bar{\epsilon}_1\epsilon_2-\bar{\epsilon}_2\epsilon_1)\partial_\alpha\psi^\mu + (\bar{\epsilon}_1\rho_\beta\epsilon_2-\bar{\epsilon}_2\rho_\beta\epsilon_1)\rho^\beta\partial_\alpha\psi^\mu + (\bar{\epsilon}_1\rho_3\epsilon_2-\bar{\epsilon}_2\rho_3\epsilon_1)\rho_3\partial_\alpha\psi^\mu\Big)\\
&= i\rho^\alpha(\bar{\epsilon}_1\rho_\beta\epsilon_2)\rho^\beta\partial_\alpha\psi^\mu
\end{aligned}
\tag{2.19}
$$

where we have anticommuted spinors to cancel terms. Applying the Dirac algebra relation to our result gives,

$$
\begin{aligned}
[\delta_1, \delta_2]\psi^\mu &= i\bar{\epsilon}_1(2\eta^\alpha{}_\beta - \rho_\beta\rho^\alpha)\epsilon_2\rho^\beta\partial_\alpha\psi^\mu \\
&= 2i\bar{\epsilon}_1\rho^\alpha\epsilon_2\partial_\alpha\psi^\mu - i\bar{\epsilon}_1\rho_\beta\rho^\alpha\epsilon_2\rho^\beta\partial_\alpha\psi^\mu \\
&= 2i\bar{\epsilon}_1\rho^\alpha\epsilon_2\partial_\alpha\psi^\mu - i\bar{\epsilon}_1\rho_\beta\epsilon_2(2\eta^{\beta\alpha} - \rho^\beta\rho^\alpha)\partial_\alpha\psi^\mu \\
&= 2i\bar{\epsilon}_1\rho^\alpha\epsilon_2\partial_\alpha\psi^\mu - 2i\bar{\epsilon}_1\epsilon_2\rho^\alpha\partial_\alpha\psi^\mu + i\bar{\epsilon}_1\rho_\beta\epsilon_2\rho^\beta\rho^\alpha\partial_\alpha\psi^\mu \\
&= a^\alpha\partial_\alpha\psi^\mu. \qquad (2.20)
\end{aligned}
$$

Thus we have a worldsheet translation with magnitude a^α defined as before. The other terms vanish by the Dirac equation $\rho^\alpha\partial_\alpha\psi^\mu = 0$, so the supersymmetry algebra is said to close *on-shell*. Consistency of the quantum theory requires that the same is true off-shell, and in the next section we will reformulate the theory to show this. [2]

Before moving on, we briefly acknowledge the assumptions that have brought us this far. First, our supersymmetric theory is founded upon a bosonic theory which is not as general as possible; in particular, the bosonic action we started with was already gauge-fixed. Therefore there is an implicit gauge choice built into our theory. Furthermore, we have taken the infinitesimal spinor parameter ϵ to be constant, and as a result the worldsheet supersymmetry of our theory is a *global symmetry*.

In light of this, it is not surprising that the RNS theory as described is in fact a gauge-fixed version of a more fundamental theory which exhibits *local supersymmetry*, meaning ϵ is allowed to depend on the worldsheet parameters; eg, $\epsilon = \epsilon(\sigma^\alpha)$. As we will see, the globally supersymmetric theory we have introduced arises from this underlying theory after gauge-fixing.

2.3 Superspace

Though the RNS theory is globally supersymmetric, this supersymmetry is not manifest. Furthermore, we have only been able to show that the supersymmetry algebra closes on-shell. We can gain greater insight into the supersymmetry of our theory by removing these qualifications. A minor but far-reaching generalisation allows us to reformulate the theory to exhibit manifest supersymmetry and off-shell closure of the supersymmetry algebra.

In particular, we can supplement the worldsheet parameters σ^α with two "fermionic" parameters $\theta_\pm$. These parameters are interpreted as the two (Grassmann) components of a Majorana spinor θ_A which satisfies the classical anticommutator relations $\{\theta_A, \theta_B\} = 0$. The parameters $(\sigma^\alpha, \theta_A)$ together describe a generalisation of the string worldsheet called *superspace*.

A function on superspace is called a *superfield*, and can be expressed in powers of θ as

$$Y(\sigma^\alpha, \theta_A) = X^\mu(\sigma^\alpha) + \bar{\theta}\psi^\mu(\sigma^\alpha) + \frac{1}{2}\bar{\theta}\theta B^\mu(\sigma^\alpha). \qquad (2.21)$$

The anticommutator relations ensure that any term of cubic order or higher in θ would vanish, so this power series expression is exact. Furthermore, since $\bar{\theta}\psi^\mu = \bar{\psi}^\mu\theta$ for Majorana spinors[3], a term linear in θ would be proportional to either $\bar{\theta}\psi^\mu$ or $\bar{\theta}\theta$, so this

[3]This is true because the charge conjugation matrix C is antisymmetric.

expression gives Y^μ in full generality [2]. We will soon show that B^μ is an auxiliary field which does not alter the physical content of the theory. Its purpose is to permit the formal improvements that we are after, namely manifest supersymmetry and off-shell closure of the supersymmetry algebra, without changing anything else.

The generators of supersymmetry transformations in the superspace formalism are proportional to the *supercharges* Q_A, given by

$$Q_A = (\partial_{\bar{\theta}})_A + i(\rho^\alpha \theta)_A \partial_\alpha. \tag{2.22}$$

The variation of the worldsheet parameter σ^α under a supersymmetry transformation is

$$\begin{aligned}
\delta\sigma^\alpha &= [\bar{\epsilon}Q, \sigma^\alpha] \\
&= \bar{\epsilon}(\partial_{\bar{\theta}} + i\rho^\beta\theta\partial_\beta)\sigma^\alpha - \sigma^\alpha\bar{\epsilon}(\partial_{\bar{\theta}} + i\rho^\beta\theta\partial_\beta) \\
&= i\bar{\epsilon}\rho^\beta\theta\partial_\beta(\sigma^\alpha) = i\bar{\epsilon}\rho^\beta\theta\delta^\alpha{}_\beta \\
&= i\bar{\epsilon}\rho^\alpha\theta
\end{aligned} \tag{2.23}$$

and the spinor parameter θ_A transforms as

$$\begin{aligned}
\delta\theta_A &= [\bar{\epsilon}Q, \theta_A] \\
&= \bar{\epsilon}(\partial_{\bar{\theta}} + i\rho^\alpha\theta\partial_\alpha)\theta_A - \theta_A\bar{\epsilon}(\partial_{\bar{\theta}} + i\rho^\alpha\theta\partial_\alpha) \\
&= \bar{\epsilon}\partial_{\bar{\theta}}(\theta) = \partial_{\bar{\theta}}(\bar{\theta})\epsilon \\
&= \epsilon_A.
\end{aligned} \tag{2.24}$$

Thus we see that $\bar{\epsilon}Q$, a superspace differential operator, generates supersymmetry transformations of the parameters. This allows us to interpret supersymmetry as a geometric property of superspace [2].

One of our goals in introducing the superspace formalism was to achieve a more general closure of the supersymmetry algebra. Since $\bar{\epsilon}Q$ generates all supersymmetry transformations in superspace, closure can be demonstrated by considering the commutator of this quantity with itself. The commutator is given by

$$\begin{aligned}
[\bar{\epsilon}_1 Q, \bar{\epsilon}_2 Q] &= (\bar{\epsilon}_1\partial_{\bar{\theta}} + i\bar{\epsilon}_1\rho^\alpha\theta\partial_\alpha)(\bar{\epsilon}_2\partial_{\bar{\theta}} + i\bar{\epsilon}_2\rho^\beta\theta\partial_\beta) - (1 \leftrightarrow 2) \\
&= \bar{\epsilon}_1\partial_{\bar{\theta}}(i\bar{\epsilon}_2\rho^\beta\theta)\partial_\beta - (1 \leftrightarrow 2) \\
&= i\bar{\epsilon}_1\bar{\epsilon}_2\rho^\beta(\partial_{\bar{\theta}}\theta)\partial_\beta - (1 \leftrightarrow 2) \\
&= -i\bar{\epsilon}_1(\partial_{\bar{\theta}}\bar{\theta})\rho^\beta\epsilon_2\partial_\beta - (1 \leftrightarrow 2) \\
&= -i(\bar{\epsilon}_1\rho^\beta\epsilon_2 - \bar{\epsilon}_2\rho^\beta\epsilon_1)\partial_\beta \\
&= -a^\alpha\partial_\alpha
\end{aligned} \tag{2.25}$$

where $a^\alpha = 2i\bar{\epsilon}_1\rho^\alpha\epsilon_2$ as before. We have not used the equations of motion to derive this result, so the supersymmetry algebra closes *off-shell*, as desired.

We can make contact with our original global supersymmetry transformations by considering the components of the supersymmetric variation of the superfield Y^μ. This variation is given by

$$
\begin{aligned}
\delta Y^\mu &= [\bar\epsilon Q, Y^\mu] = \bar\epsilon Q Y^\mu \\
&= \bar\epsilon(\partial_{\bar\theta} + i\rho^\alpha\theta\partial_\alpha)(X^\mu + \bar\theta\psi^\mu + \tfrac{1}{2}\bar\theta\theta B^\mu) \\
&= \bar\epsilon(\psi^\mu + \theta B^\mu) + i\bar\epsilon\rho^\alpha\theta\partial_\alpha(X^\mu + \bar\theta\psi^\mu) \\
&= \bar\epsilon\psi^\mu + \bar\theta\epsilon B^\mu - i\bar\theta\rho^\alpha\epsilon\partial_\alpha X^\mu + i\bar\epsilon\rho^\alpha\theta\bar\theta\partial_\alpha\psi^\mu \\
&= \bar\epsilon\psi^\mu + \bar\theta(-i\rho^\alpha\partial_\alpha X^\mu\epsilon + B^\mu\epsilon) + \tfrac{1}{2}\bar\theta\theta(-i\bar\epsilon\rho^\alpha\partial_\alpha\psi^\mu)
\end{aligned}
\tag{2.26}
$$

where in the last step we have used the two-dimensional identity [2],

$$\theta_A\bar\theta_B = -\frac{1}{2}\delta_{AB}\bar\theta_C\theta_C. \tag{2.27}$$

Peforming the same variation of the component fields of Y^μ gives

$$\delta Y^\mu = \delta X^\mu + \bar\theta\,\delta\psi^\mu + \tfrac{1}{2}\bar\theta\theta\,\delta B^\mu \tag{2.28}$$

and so by matching powers of θ, we see

$$\delta X^\mu = \bar\epsilon\psi^\mu \tag{2.29}$$

$$\delta\psi^\mu = -i\rho^\alpha\partial_\alpha X^\mu\epsilon + B^\mu\epsilon \tag{2.30}$$

$$\delta B^\mu = -i\bar\epsilon\rho^\alpha\partial_\alpha\psi^\mu. \tag{2.31}$$

Clearly these reduce to our original supersymmetry transformations when the auxiliary field B^μ is set to zero. After constructing the RNS action in the superspace formalism, we will see why this choice is legitimate.

2.3.1 Superspace Action

Now we would like to construct an action for the RNS superstring using the superspace formalism, but before we can do this we need to define how integration works in superspace. The obvious choice for a measure is to include all possible parameters, which gives the quantity [2]

$$\int d^2\sigma\, d^2\theta. \tag{2.32}$$

Integration over fermionic parameters is defined by the *Berezin rule*,

$$\int d^2\theta\,(a + b\theta_1 + c\theta_1\theta_2) = c, \tag{2.33}$$

which tells us that an integral with respect to $d^2\theta$ simply picks out the coefficient of the term that is quadratic in the spinors [2]. The Berezin rule allows us to integrate by parts in the usual way, since for an arbitrary superfield Y^μ we have

$$\int d^2\theta\,\frac{\partial Y^\mu}{\partial\bar\theta} = \int d^2\sigma\,(\psi^\mu + \tfrac{1}{2}\theta B^\mu) = 0 \tag{2.34}$$

where the integral vanishes because no quadratic terms are present.

We still need a suitable derivative operator in order to construct a Lagrangian capable of describing any kind of dynamics. In particular, we need a derivative operator that sends superfields to superfields. This resembles the situation in gauge theories or in general relativity where one defines a covariant derivative that sends tensors to tensors. In our case, we define the *supercovariant derivative*,

$$D_A = (\partial_{\bar{\theta}})_A - i(\rho^\alpha \theta)_A \partial_\alpha. \tag{2.35}$$

Both the supercovariant derivative and the supercharge carry spinor indices. In order to ensure that this derivative meets our requirements, consider their anticommutator

$$\begin{aligned}\{D_A, Q_B\} &= D_A Q_B \delta_{AB} + Q_B D_A \delta_{AB} \\ &= (\partial_{\bar{\theta}} - i\rho^\alpha \theta \partial_\alpha)(\partial_{\bar{\theta}} + i\rho^\beta \theta \partial_\beta) + (\partial_{\bar{\theta}} + i\rho^\beta \theta \partial_\beta)(\partial_{\bar{\theta}} - i\rho^\alpha \theta \partial_\alpha) \\ &= 0.\end{aligned} \tag{2.36}$$

We can use this result and the supervariation of Y^μ to write

$$D(\delta Y^\mu) = D(\bar{\epsilon} Q Y^\mu) = -\bar{\epsilon} Q(DY^\mu) = -\delta(DY^\mu). \tag{2.37}$$

Therefore if Y^μ transforms under supersymmetry as a superfield, then so does its supercovariant derivative DY^μ; in other words, the supercovariant derivative sends superfields to superfields as required.

Finally we note that the product of two superfields is again a superfield. To see this we can consider the supervariation of $Y_1 Y_2$,

$$\begin{aligned}\delta(Y_1 Y_2) &= (\delta Y_1) Y_2 + Y_1(\delta Y_2) = (\bar{\epsilon} Q Y_1) Y_2 + Y_1(\bar{\epsilon} Q Y_2) \\ &= \bar{\epsilon} Q(Y_1 Y_2)\end{aligned} \tag{2.38}$$

where we have used the fact that $\bar{\epsilon} Q$ satisfies the Leibniz rule, since it is a differential operator [2]. Thus $Y_1 Y_2$ transforms as a superfield. In fact, this suggests that an arbitrary product of superfields will transform as a superfield. This result is not surprising; a superfield is simply a function in superspace, and it makes sense that a product of such functions should give another function.

Now we can write down the action for a free superfield propagating in superspace,

$$S = \frac{1}{4\pi} \int d^2\sigma \, d^2\theta \, \overline{DY}^\mu DY_\mu \tag{2.39}$$

Since DY^μ is a superfield, the product $\overline{DY}^\mu DY_\mu$ is also a superfield. Thus the supersymmetric variation of our action is given by

$$\begin{aligned}\delta S &= \frac{1}{4\pi} \int d^2\sigma \, d^2\theta \, \bar{\epsilon} Q(\overline{DY}^\mu DY_\mu) \\ &= \frac{1}{4\pi} \int d^2\sigma \, d^2\theta \, \bar{\epsilon} \Big(\partial_{\bar{\theta}}(\overline{DY}^\mu DY_\mu) + i\rho^\alpha \theta \, \partial_\alpha(\overline{DY}^\mu DY_\mu) \Big) \\ &= 0.\end{aligned} \tag{2.40}$$

The variation vanishes because the integrand is a sum of total derivatives. This allows us to say that the action has *manifest supersymmetry.*

We can write this action more explicitly by expanding the Lagrangian and carrying out the fermionic integration. Our Lagrangian is proportional to

$$\begin{aligned} DY_\mu &= (\partial_{\bar\theta} - i\rho^\alpha\theta\partial_\alpha)(X_\mu + \bar\theta\psi_\mu + \tfrac{1}{2}\bar\theta\theta B_\mu) \\ &= \psi_\mu + \theta B_\mu - i\rho^\alpha\theta\partial_\alpha X_\mu - i\rho^\alpha\theta\bar\theta\partial_\alpha\psi_\mu \\ &= \psi_\mu + \theta B_\mu - i\rho^\alpha\theta\partial_\alpha X_\mu + \tfrac{i}{2}\bar\theta\theta\rho^\alpha\partial_\alpha\psi_\mu \end{aligned} \tag{2.41}$$

where we have used (2.27) to rearrange spinors, and

$$\begin{aligned} \overline{DY}^\mu &= (DY^\mu)^T\rho^0 \\ &= \bar\psi^\mu + B^\mu\bar\theta - i\partial_\alpha X^\mu\bar\theta\rho^\alpha - \tfrac{i}{2}\bar\theta\theta\partial_\alpha\bar\psi^\mu\rho^\alpha. \end{aligned} \tag{2.42}$$

Thus we see that the quadratic spinor terms in our Lagrangian are

$$\begin{aligned} &(B^\mu\bar\theta - i\partial_\alpha X^\mu\bar\theta\rho^\alpha)(\theta B_\mu - i\rho^\beta\theta\partial_\beta X_\mu) + \bar\psi^\mu(\tfrac{i}{2}\bar\theta\theta\rho^\alpha\partial_\alpha\psi_\mu) - \tfrac{i}{2}\bar\theta\theta\partial_\alpha\bar\psi^\mu\rho^\alpha\psi_\mu \\ &= B^\mu B_\mu\bar\theta\theta - \partial_\alpha X^\mu\bar\theta\rho^\alpha\rho^\beta\theta\partial_\beta X_\mu + \tfrac{i}{2}\bar\theta\theta(\bar\psi^\mu\rho^\alpha\partial_\alpha\psi_\mu - \partial_\alpha\bar\psi^\mu\rho^\alpha\psi_\mu) \\ &= -\bar\theta\theta(\partial_\alpha X^\mu\partial^\alpha X_\mu - i\bar\psi^\mu\rho^\alpha\partial_\alpha\psi_\mu - B^\mu B_\mu). \end{aligned} \tag{2.43}$$

The quantity $\bar\theta\theta$ is given in our basis by

$$\begin{aligned} \bar\theta\theta = \theta^T\rho^0\theta &= (\theta_- \;\; \theta_+)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \theta_- \\ \theta_+ \end{pmatrix} = (\theta_- \;\; \theta_+)\begin{pmatrix} -\theta_+ \\ \theta_- \end{pmatrix} \\ &= (-\theta_-\theta_+ + \theta_+\theta_-) = -2\theta_-\theta_+ \end{aligned} \tag{2.44}$$

and so for us, the fermionic integration gives

$$\int d^2\theta\,(\bar\theta\theta) = -2. \tag{2.45}$$

Therefore our action can be written

$$\begin{aligned} S &= \frac{1}{4\pi}\int d^2\sigma\, d^2\theta\,\overline{DY}^\mu DY_\mu \\ &= -\frac{1}{2\pi}\int d^2\sigma\,(\partial_\alpha X^\mu\partial^\alpha X_\mu - i\bar\psi^\mu\rho^\alpha\partial_\alpha\psi_\mu - B^\mu B_\mu). \end{aligned} \tag{2.46}$$

Clearly this coincides with our original superstring action when the field B^μ is omitted. The field equations give $B^\mu = 0$, which tells us that this field has no dynamics, and therefore that its removal would leave the physical content of our theory unchanged. This proves that B^μ is simply an auxiliary field, whose inclusion allows us to achieve manifest supersymmetry and off-shell closure of the supersymmetry algebra without altering the theory in a meaningful way.

2.4 Boundary Conditions and Mode Expansions

The worldsheet fields can be expressed as general solutions to their respective equations of motion by expanding them into Fourier series, subject to the appropriate boundary conditions. For the worldsheet scalars X^μ, this procedure is unchanged from the bosonic theory. The fields obey the two-dimensional wave equation, and their boundary conditions depend upon their topology. For closed strings, periodicity is the only supplemental condition necessary. For open strings, we can require either Dirichlet or von Neumann boundary conditions. The general solution to the wave equation can be written as a sum of right-moving and left-moving modes, $X^\mu(\sigma^\alpha) = X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+)$, with

$$X_L^\mu(\sigma^+) = \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \tilde{a}_n^\mu e^{-in\sigma^+} \tag{2.47}$$

$$X_R^\mu(\sigma^-) = \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} a_n^\mu e^{-in\sigma^-} \tag{2.48}$$

where x^μ is the center of mass of the string, p^μ is the momentum, and α' is a constant proportional to the string tension. The fields X^μ must be Hermitian, so the oscillator coordinates satisfy $(a_n^\mu)^\dagger = a_{-n}^\mu$. [4]

In order to find the necessary boundary conditions for ψ^μ, we must see how our action varies under a general transformation on the spinors. We can expand the fermionic part of the action into spinor components as

$$\begin{aligned}
S_f &= \frac{i}{2\pi} \int d^2\sigma\, \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \\
&= \frac{i}{2\pi} \int d^2\sigma\, \psi^T \rho^0 \begin{pmatrix} 0 & \partial_1 - \partial_0 \\ \partial_1 + \partial_0 & 0 \end{pmatrix} \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \\
&= \frac{i}{\pi} \int d^2\sigma\, \psi^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\partial_-\psi_+ \\ \partial_+\partial_- \end{pmatrix} \\
&= -\frac{i}{\pi} \int d^2\sigma\, \begin{pmatrix} \psi_- & \psi_+ \end{pmatrix} \begin{pmatrix} \partial_+\psi_- \\ \partial_-\psi_+ \end{pmatrix} \\
&= -\frac{i}{\pi} \int d^2\sigma\, (\psi_-\partial_+\psi_- + \psi_+\partial_-\psi_+)
\end{aligned} \tag{2.49}$$

where we have used our Dirac basis and the worldsheet lightcone coordinates $\sigma^\pm$. This notation makes the massless Dirac equations $\partial_+\psi_- = \partial_-\psi_+ = 0$ explicit.

The variation of the fermionic action under a general transformation on the spinors is given by

$$\begin{aligned}
\delta S_f &= -\frac{i}{\pi} \int d^2\sigma \Big((\delta\psi_-)\partial_+\psi_- + \psi_-\partial_+(\delta\psi_-) + (\delta\psi_+)\partial_-\psi_+ + \psi_+\partial_-(\delta\psi_+) \Big) \\
&= -\frac{i}{\pi} \int d^2\sigma \Big(\partial_+(\psi_-\delta\psi_-) + \partial_-(\psi_+\delta\psi_+) \Big)
\end{aligned} \tag{2.50}$$

where we have used the equations of motion and partial integration to eliminate terms which vanish on-shell. Expanding partial derivatives with $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$ gives

$$\begin{aligned}
\delta S_f &= -\frac{i}{2\pi} \int d^2\sigma \Big(\partial_0(\psi_-\delta\psi_- + \psi_+\delta\psi_+) + \partial_1(\psi_-\delta\psi_- - \psi_+\delta\psi_+) \Big) \\
&= -\frac{i}{2\pi} \int d\tau\, (\psi_-\delta\psi_- - \psi_+\delta\psi_+)\big|_{\sigma=0}^{\sigma=\pi}
\end{aligned} \tag{2.51}$$

where the ∂_0 term is eliminated because the variation $\delta\psi$ vanishes at $\tau = \pm\infty$. [8]

The vanishing of the remaining spatial surface term requires the condition $\psi_-\delta\psi_- = \psi_+\delta\psi_+$ to hold at $\sigma = 0$ and $\sigma = \pi$, corresponding to the two endpoints of the open string. We can achieve this by setting $\psi_- = \pm\psi_+$ (and therefore $\delta\psi_- = \pm\delta\psi_+$) for each of these values. The relative sign at the first endpoint is a matter of convention, so without loss of generality we can set $\psi_-^\mu(\tau, 0) = \psi_+^\mu(\tau, 0)$. Then the relative sign at $\sigma = \pi$ becomes important, and the two possibilities corresepond to two possible sets of boundary conditions for the spinor ψ^μ [2]. We must consider their effects on strings of different topologies separately.

2.4.1 Open Strings

In the case of open strings, our first possible choice gives the *Ramond boundary conditions*

$$\psi_-^\mu(\tau, \pi) = \psi_+^\mu(\tau, \pi). \tag{2.52}$$

The mode expansions in the corresponding *Ramond sector* are

$$\psi_-^\mu(\tau, \sigma) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau-\sigma)} \tag{2.53}$$

$$\psi_+^\mu(\tau, \sigma) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau+\sigma)}. \tag{2.54}$$

The Fourier coefficients d_n^μ are interpreted as fermionic oscillators, which are analogous to the bosonic oscillators that appear in the mode expansions of X^μ. Because the index variable n is an integer, the fields $\psi_\pm^\mu$ are said to be *integer-moded*. The fields ψ^μ are two-dimensional Majorana spinors, and so the components $\psi_\pm^\mu$ must be real. Thus the fermionic oscillators satisfy the condition $(d_n^\mu)^\dagger = d_{-n}^\mu$. [1]

Alternatively, we can select the *Neveu-Schwarz boundary conditions*,

$$\psi_-^\mu(\tau, \pi) = -\psi_+^\mu(\tau, \pi). \tag{2.55}$$

The mode expansions in the *Neveu-Schwarz sector* are given by

$$\psi_-^\mu(\tau, \sigma) = \frac{1}{\sqrt{2}} \sum_{r\in\frac{1}{2}+\mathbb{Z}} b_r^\mu e^{-ir(\tau-\sigma)} \tag{2.56}$$

$$\psi_+^\mu(\tau, \sigma) = \frac{1}{\sqrt{2}} \sum_{r\in\frac{1}{2}+\mathbb{Z}} b_r^\mu e^{-ir(\tau+\sigma)}. \tag{2.57}$$

These fields are *half integer-moded* because the index variable takes values in the set $\frac{1}{2}+\mathbb{Z} = \{\pm\frac{1}{2}, \pm\frac{3}{2}, \ldots\}$. Following convention, we will use m, n to label integers and r, s to label half-integers. The Majorana condition again requires that the NS-sector oscillators satisfy $(b_r^\mu)^\dagger = b_{-r}^\mu$ [1].

It is clear from the preceding argument that we can only select one set of open string boundary conditions at a time. When we consider the quantum theory, we will see that fermionic excitations in the open string spectrum can therefore be created either from NS-sector oscillators or from R-sector oscillators, but not from both. This suggests that the spectra of open string states described by fermionic oscillators in different sectors does not overlap. In fact, states which arise from different sectors are represented in entirely different Fock spaces, and in particular the sectors will have different ground states. This fact provides a first step towards the interpretation of NS-sector excitations as *spacetime bosons*, and of R-sector excitations as *spacetime fermions*, thus achieving one of the principal goals of superstring theory. We will develop this argument in detail after further discussion of the classical superstring.

2.4.2 Closed Strings

Having examined the boundary conditions and mode expansions for the open string, we now turn our attention to closed strings. The left-moving and right-moving modes of the closed bosonic string are independent, and subsequently the left-moving and right-moving modes of X^μ depend upon two independent sets of bosonic oscillators denoted $\tilde{a}_n^\mu$ and a_n^μ, respectively [4]. The same is true of the bosonic fields X^μ in the RNS theory.

Because closed strings do not have endpoints, our boundary conditions will impose either periodicity or anti-periodicity on closed string modes. The periodic Ramond conditions are

$$\psi_-^\mu(\tau,\sigma) = \psi_+^\mu(\tau,\sigma+\pi) \tag{2.58}$$

and the anti-periodic Neveu-Schwarz conditions are

$$\psi_-^\mu(\tau,\sigma) = -\psi_+^\mu(\tau,\sigma+\pi). \tag{2.59}$$

The left-moving and right-moving modes of the worldsheet spinors are also independent for closed superstrings, so we may assign boundary conditions to the different modes of ψ^μ independently. Therefore the right-moving modes of the closed string must satisfy one of

$$\psi_-^\mu(\tau,\sigma) = \sum_{n\in\mathbb{Z}} d_n^\mu\, e^{-2in(\tau-\sigma)} \tag{2.60}$$

$$\text{or} \qquad \psi_-^\mu(\tau,\sigma) = \sum_{r\in\frac{1}{2}+\mathbb{Z}} b_r^\mu\, e^{-2ir(\tau-\sigma)} \tag{2.61}$$

and left-moving modes must satisfy one of

$$\psi_+^\mu(\tau,\sigma) = \sum_{n\in\mathbb{Z}} \tilde{d}_n^\mu e^{-2in(\tau-\sigma)} \tag{2.62}$$

$$\text{or} \qquad \psi_+^\mu(\tau,\sigma) = \sum_{r\in\frac{1}{2}+\mathbb{Z}} \tilde{b}_r^\mu e^{-2ir(\tau-\sigma)}. \tag{2.63}$$

Quantum states in the spectrum of the closed string are created by taking tensor products of left-moving and right-moving modes. Our freedom to select boundary conditions for these modes independently leads to four possible sectors of states for the closed superstring, denoted

$$NS\otimes NS \qquad NS\otimes R \tag{2.64}$$

$$R\otimes R \qquad R\otimes NS. \tag{2.65}$$

We have noted that NS-sector states represent spacetime bosons, and R-sector states represent spacetime fermions. Since bosonic states have integer spin and fermionic states have half-integer spin, and since spin is additive when tensor products are taken, we can therefore see that the NS-NS and R-R sectors of the closed string will describe bosonic states, while the mixed NS-R and R-NS sectors will describe fermionic states.

2.5 Symmetries and Conservation Laws

The RNS theory is invariant under global supersymmetry transformations and also under constant worldsheet translations $\delta\sigma^\alpha = \epsilon n^\alpha$. Both of these are continuous symmetries, and therefore are associated with conserved currents according to Noether's theorem. We can deduce these currents by varying our action under gauged versions of the symmetries via the *Noether method* [2].

In particular, the conserved current associated with a global symmetry given by $\phi \to \phi + \epsilon\delta\phi$, where $\phi = \phi(\sigma^\alpha)$ is some field in an arbitrary worldsheet theory, can be found by considering the gauged symmetry transformation

$$\phi \to \phi + \epsilon(\sigma^\alpha)\delta\phi. \tag{2.66}$$

The global symmetry need not hold locally, so this gauged transformation does not necessarily give us another symmetry. Instead, the variation of the action is proportional to the derivative of the gauged parameter [2]

$$\delta S = \int d^2\sigma(\partial_\alpha\epsilon)j^\alpha. \tag{2.67}$$

In an on-shell configuration any variation of the action is zero, which allows us to write

$$\delta S = \int d^2\sigma\,(\partial_\alpha\epsilon)j^\alpha = -\int d^2\sigma\,\epsilon(\partial_\alpha j^\alpha) = 0 \tag{2.68}$$

by partial integration. In order for this to be true for an arbitrary value of the parameter ϵ, the quantity j^α must be conserved on-shell.

2.5.1 The Energy-Momentum Tensor

The fields transform under a constant worldsheet translation as

$$\delta X^\mu = \epsilon n^\alpha\partial_\alpha X^\mu \qquad \delta\psi^\mu = \epsilon n^\alpha\partial_\alpha\psi^\mu \tag{2.69}$$

where ϵ is an infinitesimal parameter, and n^α is a constant vector. To verify that these transformations leave the action invariant, consider the variation

$$\begin{aligned}
\delta S &= -\frac{1}{2\pi}\int d^2\sigma\left(\partial_\alpha(\delta X^\mu)\partial^\alpha X_\mu + \partial_\alpha X^\mu\partial^\alpha(\delta X_\mu) - i(\delta\bar\psi^\mu)\rho^\alpha\partial_\alpha\psi_\mu - i\bar\psi^\mu\rho^\alpha\partial_\alpha(\delta\psi_\mu)\right)\\
&= -\frac{\epsilon n^\beta}{2\pi}\int d^2\sigma\left(\partial_\alpha(\partial_\beta X^\mu)\partial^\alpha X_\mu + \partial_\alpha X^\mu\partial^\alpha(\partial_\beta X_\mu) - i(\partial_\beta\bar\psi^\mu)\rho^\alpha\partial_\alpha\psi_\mu - i\bar\psi^\mu\rho^\alpha\partial_\alpha(\partial_\beta\psi_\mu)\right)\\
&= -\frac{\epsilon n^\beta}{2\pi}\int d^2\sigma\left(\partial_\beta(\partial_\alpha X^\mu\partial^\alpha X_\mu) - i\partial_\beta\bar\psi^\mu\rho^\alpha\partial_\alpha\psi_\mu + i\partial_\beta\bar\psi^\mu\rho^\alpha\partial_\alpha\psi_\mu\right)\\
&= -\frac{\epsilon n^\beta}{2\pi}\int d^2\sigma\,\partial_\beta(\partial_\alpha X^\mu\partial^\alpha X_\mu)\\
&= 0
\end{aligned} \tag{2.70}$$

where partial integration has been used to cancel spinor terms, and the variation vanishes because the remaining term in the integrand is a total derivative. Therefore constant worldsheet translations are a symmetry of the RNS theory, as expected.

The energy-momentum tensor is defined by varying a gauge-invariant action with respect to the worldsheet metric $h_{\alpha\beta}$ [4]. Due to our gauge-fixing choices this field is not present in our action, so we cannot derive the energy-momentum tensor directly. However we can use the Noether method by applying the steps described above to our translational symmetry.

The variation of the action under a gauged worldsheet translation is

$$\begin{aligned}\delta S &= -\frac{1}{2\pi}\int d^2\sigma\,\Big(2\partial_\alpha(\delta X^\mu)\partial^\alpha X_\mu - i(\delta\bar{\psi}^\mu)\rho^\alpha\partial_\alpha\psi_\mu - i\bar{\psi}^\mu\rho^\alpha\partial_\alpha(\delta\psi_\mu)\Big)\\ &= -\frac{n^\beta}{2\pi}\int d^2\sigma\,\Big(2(\partial_\alpha\epsilon)\partial_\beta X^\mu\partial^\alpha X_\mu + 2\epsilon\partial_\alpha\partial_\beta X^\mu\partial^\alpha X_\mu - i\epsilon\partial_\beta\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu\\ &\qquad - i\bar{\psi}^\mu\rho^\alpha(\partial_\alpha\epsilon)\partial_\beta\psi_\mu - i\bar{\psi}^\mu\rho^\alpha\epsilon\partial_\alpha\partial_\beta\psi_\mu\Big).\end{aligned} \tag{2.71}$$

This procedure gives us a current which is conserved on-shell, so we can ignore terms which are proportional to the Dirac equation $\rho^\alpha\partial_\alpha\psi_\mu$. We are left with

$$\delta S = -\frac{n^\beta}{2\pi}\int d^2\sigma\,\Big(2(\partial_\alpha\epsilon)\partial_\beta X^\mu\partial^\alpha X_\mu + 2\epsilon\partial_\alpha\partial_\beta X^\mu\partial^\alpha X_\mu - i\bar{\psi}^\mu\rho^\alpha(\partial_\alpha\epsilon)\partial_\beta\psi_\mu\Big) \tag{2.72}$$

Partially integrating the second term gives

$$\begin{aligned}\int d^2\sigma\,\epsilon\partial_\alpha\partial_\beta X^\mu\partial^\alpha X_\mu &= -\int d^2\sigma\,(\partial_\beta\epsilon)\partial_\alpha X^\mu\partial^\alpha X_\mu - \int d^2\sigma\,\epsilon\partial_\alpha X^\mu\partial_\beta\partial^\alpha X_\mu\\ &= -\int d^2\sigma\,(\partial_\beta\epsilon)\partial_\alpha X^\mu\partial^\alpha X_\mu - \int d^2\sigma\,\epsilon\partial_\alpha\partial_\beta X^\mu\partial^\alpha X_\mu\\ &= -\frac{1}{2}\int d^2\sigma\,(\partial_\beta\epsilon)\partial_\alpha X^\mu\partial^\alpha X_\mu\end{aligned} \tag{2.73}$$

which comes from moving the second term on the right to the other side of the equals sign, and dividing by two.

Therefore the variation can be written

$$\begin{aligned}\delta S &= -\frac{n^\beta}{2\pi}\int d^2\sigma\,(\partial^\alpha\epsilon)(\partial_\alpha X^\mu\partial_\beta X_\mu - i\bar{\psi}^\mu\rho_\alpha\partial_\beta\psi_\mu)\\ &= -\frac{n^\beta}{2\pi}\int d^2\sigma\,(\partial^\alpha\epsilon)T'_{\alpha\beta}.\end{aligned} \tag{2.74}$$

The term $T'_{\alpha\beta}$ should be conserved on-shell according to the reasoning above, and indeed we have

$$\begin{aligned}\partial^\alpha T'_{\alpha\beta} &= \partial^\alpha\partial_\alpha X^\mu\partial_\beta X_\mu + \partial_\alpha X^\mu\partial^\alpha\partial_\beta X_\mu - i\partial^\alpha\bar{\psi}^\mu\rho_\alpha\partial_\beta\psi_\mu - i\bar{\psi}^\mu\rho_\alpha\partial^\alpha\partial_\beta\psi_\mu\\ &= 2\partial^\alpha\partial_\alpha X^\mu\partial_\beta X_\mu + i\partial_\beta\bar{\psi}^\mu\rho_\alpha\partial^\alpha\psi_\mu - i\bar{\psi}^\mu\rho_\alpha\partial^\alpha\partial_\beta\psi_\mu\\ &= 0\end{aligned} \tag{2.75}$$

but we cannot yet interpret it as the energy-momentum tensor. The Noether method only determines a conserved current up to terms proportional to $\partial_\mu A_{\mu\nu}$, where $A_{\mu\nu}$ is an antisymmetric tensor. The energy-momentum tensor must be symmetric a priori, and traceless as a consequence of Weyl invariance. After imposing these conditions on our conserved quantity (via so-called “improvement terms” [6]) we arrive at the *energy-momentum tensor* in its correct form,[4]

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{i}{4}\bar{\psi}^\mu \rho_\alpha \partial_\beta \psi_\mu - \frac{i}{4}\bar{\psi}^\mu \rho_\beta \partial_\alpha \psi_\mu - (\text{trace}) \tag{2.76}$$

which is the conserved current associated with constant worldsheet translations. Since $T'_{\alpha\beta}$ is conserved on-shell, it is reasonable to imagine that making it symmetric and traceless will not alter this fact. We can verify this by considering

$$\begin{aligned}
\partial^\alpha T_{\alpha\beta} &= \partial^\alpha \partial_\alpha X^\mu \partial_\beta X_\mu + \partial_\alpha X^\mu \partial^\alpha \partial_\beta X_\mu - \frac{i}{4}\Big(\partial^\alpha \bar{\psi}^\mu \rho_\alpha \partial_\beta \psi_\mu + \bar{\psi}^\mu \rho_\alpha \partial^\alpha \partial_\beta \psi_\mu\Big) \\
&\quad - \frac{i}{4}\Big(\partial^\alpha \bar{\psi}^\mu \rho_\beta \partial_\alpha \psi_\mu + \bar{\psi}^\mu \rho_\beta \partial^\alpha \partial_\alpha \psi_\mu\Big) \\
&= 2\partial^\alpha \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{i}{2}\Big(\partial_\beta \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu + \bar{\psi}^\mu \partial_\beta(\rho^\alpha \partial_\alpha \psi_\mu)\Big) \\
&= 0
\end{aligned} \tag{2.77}$$

therefore our manipulations leave the physical content of the theory unchanged.

2.5.2 The Supercurrent

Our action is invariant under global supersymmetry transformations, as we have discussed above. Therefore we can use the Noether method to derive the associated conserved current. The variation of our action a general transformation of the fields is

$$\begin{aligned}
\delta S &= -\frac{1}{2\pi}\int d^2\sigma \,\Big(2\partial_\alpha(\delta X^\mu)\partial^\alpha X_\mu - i(\delta\bar{\psi}^\mu)\rho^\alpha \partial_\alpha \psi_\mu - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha(\delta\psi_\mu)\Big) \\
&= -\frac{1}{\pi}\int d^2\sigma \,\Big(\partial_\alpha(\delta X^\mu)\partial^\alpha X_\mu + i\partial_\alpha \bar{\psi}^\mu \rho^\alpha (\delta\psi_\mu)\Big).
\end{aligned} \tag{2.78}$$

If we substitute the gauged versions of our global supersymmetry transformations, this becomes

$$\begin{aligned}
\delta S &= -\frac{1}{\pi}\int d^2\sigma \,\Big(\partial_\alpha(\bar{\epsilon}\psi^\mu)\partial^\alpha X_\mu + i\partial_\alpha \bar{\psi}^\mu \rho^\alpha(-i\rho^\beta \partial_\beta X^\mu \epsilon)\Big) \\
&= -\frac{1}{\pi}\int d^2\sigma \,\Big((\partial_\alpha \bar{\epsilon})\psi^\mu \partial^\alpha X_\mu + \bar{\epsilon}\partial_\alpha \psi^\mu \partial^\alpha X_\mu + \partial_\alpha \bar{\psi}^\mu \partial^\alpha X^\mu \epsilon\Big) \\
&= -\frac{1}{\pi}\int d^2\sigma \,(\partial_\alpha \bar{\epsilon})\psi^\mu \partial^\alpha X_\mu
\end{aligned} \tag{2.79}$$

where we have used the Dirac relation and anticommuted spinors to cancel terms. We can rewrite this as

$$\begin{aligned}
\delta S &= -\frac{1}{\pi}\int d^2\sigma \,(\partial_\alpha \bar{\epsilon})\psi^\mu_A \eta^{\alpha\beta}\partial_\beta X_\mu \\
&\equiv -\frac{2}{\pi}\int d^2\sigma \,(\partial_\alpha \bar{\epsilon}) J^\alpha_A.
\end{aligned} \tag{2.80}$$

[4]There appears to be a sign error in [2], as confirmed in [8].

where J_A^α is conserved on-shell. This quantity, called the *supercurrent*, is defined as

$$J_A^\alpha = \frac{1}{2}\rho^\beta \rho^\alpha \psi_A^\mu \partial_\beta X_\mu \tag{2.81}$$

which is equivalent to the integrand above via the defining relation of the Dirac algebra.

Evaluating the conservation law directly gives

$$\begin{aligned}\partial_\alpha J^\alpha &= \tfrac{1}{2}\rho^\beta \rho^\alpha \partial_\alpha \psi_A^\mu \partial_\beta X_\mu + \tfrac{1}{2}\rho^\beta \rho^\alpha \psi_A^\mu \partial_\alpha \partial_\beta X_\mu \\ &= \tfrac{1}{2}\rho^\beta \rho^\alpha \partial_\alpha \psi_A^\mu \partial_\beta X_\mu + \tfrac{1}{2}\psi_A^\mu \partial^\alpha \partial_\alpha X_\mu \\ &= 0\end{aligned} \tag{2.82}$$

where the first term is proportional to the Dirac equation, and we have used the Dirac algebra relation to make the second term proportional to the wave equation. This confirms that $\partial_\alpha J^\alpha = 0$ on-shell.

2.5.3 The Classical Super-Virasoro Constraints

The bosonic theory retains a conformal symmetry after gauge-fixing which requires the energy-momentum tensor to vanish,[5]

$$T_{\alpha\beta} = 0. \tag{2.83}$$

We can reformulate this classical constraint by expanding $T_{\alpha\beta}$ in terms of its Fourier modes and requiring all of these to vanish. The Fourier modes are called the Virasoro generators, denoted L_m, and they are defined for the open string by[6]

$$\begin{aligned}L_n &= \frac{1}{\pi}\int_0^\pi d\sigma\,(e^{in\sigma}T_{++} + e^{-in\sigma}T_{--}) \\ &= \frac{1}{\pi}\int_{-\pi}^\pi d\sigma\, e^{in\sigma}T_{++}\end{aligned} \tag{2.84}$$

where T_{++} and T_{--} are the non-vanishing lightcone components of $T_{\alpha\beta}$ (the off-diagonal components vanish due to tracelessness [2]), given by

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu + \frac{i}{2}\psi_+^\mu \partial_+ \psi_+^\nu \eta_{\mu\nu} \tag{2.85}$$

$$T_{--} = \partial_- X^\mu \partial_- X_\mu + \frac{i}{2}\psi_-^\mu \partial_- \psi_-^\nu \eta_{\mu\nu}. \tag{2.86}$$

The classical Virasoro constraints can then be written

$$L_n = 0 \quad \forall n \tag{2.87}$$

or equivalently,

$$T_{++} = T_{--} = 0. \tag{2.88}$$

[5]This follows from the Euler-Lagrange equations of the worldsheet metric $h_{\alpha\beta}$, which appears in the bosonic string action before the conformal gauge is fixed. [4]

[6]This definition relies upon a technical modification to the definitions of X_L and X_R, owing to the fact that the functions $e^{in\sigma}$ are not orthogonal in the interval $0 \leq \sigma \leq \pi$. [2]

These constraints define important restrictions on the spectrum of states in the quantum theory. In particular, they help to remove unphysical states from the spectrum, such as negative-norm states which can appear as excitations of timelike oscillators. We will see how these arise, and how they decouple from the spectrum, when we consider quantisation in detail.

The Virasoro constraints (along with other important quantum conditions, also to be considered later) provide strong enough restrictions on the spectrum of the bosonic string to allow us to construct a sensible physical theory. However the RNS theory is a generalisation of the bosonic theory, and therefore we can imagine that stronger constraints will be needed to perform the same function for the superstring spectrum. Evidence for this can be seen in the fact that the quantum RNS theory has additional fermionic oscillators which can create negative-norm states, and which the Virasoro constraints do not address.

Therefore we want something more general than the Virasoro constraints, and subsequently something more general than conformal symmetry. The RNS symmetries that we have considered include translational symmetry, whose conserved current is the energy-momentum tensor, and supersymmetry, which gives us the supercurrent J_A^α.

The lightcone components of the supercurrent are $(J_-)_A$ and $(J_+)_A$, and both of these are two-component spinors of fixed chirality. However we have not yet considered the identity

$$\rho_\alpha J_A^\alpha = 0, \tag{2.89}$$

which is a consequence of the fact that $\rho_\alpha \rho^\beta \rho^\alpha = 0$ in two dimensions. This identity can be interpreted as the analog of the tracelessness of $T_{\alpha\beta}$, and as a result there are only two nonzero lightcone components of J_A^α, which are given by

$$J_+ = \psi_+^\mu \partial_+ X^\mu \tag{2.90}$$
$$J_- = \psi_-^\mu \partial_- X^\mu. \tag{2.91}$$

These components can be shown to satisfy the classical relations [2]

$$\{J_+(\sigma), J_+(\sigma')\} = \pi\delta(\sigma - \sigma')T_{++}(\sigma) \tag{2.92}$$
$$\{J_-(\sigma), J_-(\sigma')\} = \pi\delta(\sigma - \sigma')T_{--}(\sigma) \tag{2.93}$$
$$\{J_+(\sigma), J_-(\sigma')\} = 0 \tag{2.94}$$

If we consider the Virasoro constraints $T_{++} = T_{--} = 0$ again, it is apparent from the relations above that we cannot impose these conditions without also requiring $J_\pm$ to vanish. Therefore we have a candidate for a set of generalised Virasoro constraints which takes supersymmetry into account. These can be written

$$J_+ = J_- = T_{++} = T_{--} = 0. \tag{2.95}$$

Though for the moment we can only postulate that these relations will be useful, our reasoning is in fact correct: these are the *super-Virasoro constraints*. Whereas the Virasoro constraints are the requirements of conformal symmetry, the super-Virasoro constraints enforce *superconformal symmetry* on the theory, and we will see that they define important restrictions on the quantum states. In the next section we will show that these constraints arise from the equations of motion for the locally supersymmetric theory.

We can express the super-Virasoro constraints in terms of Fourier modes, in analogy with the bosonic theory. The Fourier modes of $T_{\alpha\beta}$ are the same as the bosonic Virasoro generators L_n. Their definition appears at the beginning of this section.

Since the supercurrent is a spinor it will have different expansions depending on the sector, and its Fourier modes will give the fermionic super-Virasoro generators. In the case of the open string, these are defined in the NS-sector by

$$G_r = \frac{\sqrt{2}}{\pi} \int_0^{\pi} d\sigma \left(e^{ir\sigma} J_+ + e^{-ir\sigma} J_- \right) = \frac{\sqrt{2}}{\pi} \int_{-\pi}^{\pi} d\sigma \, e^{ir\sigma} J_+ \tag{2.96}$$

and in the R-sector by

$$F_m = \frac{\sqrt{2}}{\pi} \int_0^{\pi} d\sigma \left(e^{im\sigma} J_+ + e^{-im\sigma} J_- \right) = \frac{\sqrt{2}}{\pi} \int_{-\pi}^{\pi} d\sigma \, e^{im\sigma} J_+ . \tag{2.97}$$

The classical super-Virasoro constraints can then be written, depending on the sector, as

$$\begin{array}{lll} \text{(NS)} & L_n = G_r = 0 & \forall m, r \\ \text{(R)} & L_n = F_m = 0 & \forall n, m \end{array} \tag{2.98}$$

where again, r is a half-integer and m is an integer. [2]

Since the right-moving and left-moving modes of closed strings are independent, closed superstrings have two sets of super-Virasoro generators. The right-moving set is as defined here, and the left-moving set depends on the left-movers T_{--} and J_-. The left-moving super-Virasoro conditions are then defined as above, but in terms of $\tilde{L}_n, \tilde{G}_r$, and $\tilde{F}_m$.

2.6 Local Worldsheet Supersymmetry

Thus far we have concerned ourselves exclusively with global supersymmetry on the string worldsheet; we began with a gauge-fixed bosonic action which we then generalised to include free fermions propagating along the string, and we have seen that the resulting action is invariant under supersymmetry transformations generated by a constant parameter ϵ. This framework has allowed us to explore many features of the classical superstring in detail. However we have seen that the super-Virasoro constraints, which play a central role in the quantum theory, can only be postulated in this framework. To derive them properly requires a more general approach than we have taken; in particular, we must relax our assumption that the string worldsheet is conformally flat.[7]

In the bosonic theory, the Virasoro constraints are derived by gauge-fixing an action which is invariant under general worldsheet reparameterizations, or diffeomorphisms [4]. Therefore we can expect the super-Virasoro constraints to arise from gauge-fixing a supersymmetric action that is also diffeomorphism-invariant. Such an action will depend on the worldsheet metric $h_{\alpha\beta}$ as well as two fields which we have not yet encountered.

[7] In fixing the conformal gauge we made the assumption $h_{\alpha\beta} = e^{\phi}\eta_{\alpha\beta}$, which means the string worldsheet has the same geometry as Minkowski spacetime, up to a conformal transformation.

The first new field we must introduce is the *zweibein* e^a_α, which can be heuristically regarded as the "square root" of the worldsheet metric, since it satisfies the relation

$$h_{\alpha\beta} = e^a_\alpha e^b_\beta \eta_{ab} \tag{2.99}$$

at each point p on the worldsheet, where $h_{\alpha\beta}$ is the worldsheet metric and η_{ab} is the metric of the tangent space to the worldsheet at p. The Roman indices a, b label the dimensions of the tangent space, and the zweibein provides an orthonormal basis for this tangent space. There is another new field which will appear, called a *Rarita-Schwinger field* and denoted $\chi_{A\alpha}(\tau, \sigma)$, which carries both a spinor index and a vector index. [2]

We can now write the gauge-invariant superstring action,

$$S = -\frac{1}{\pi}\int d^2\sigma\, e^a_\lambda \Big(\tfrac{1}{2}h^{\alpha\beta}(\partial_\alpha X^\mu \partial_\beta X_\mu - i\bar{\psi}^\mu \rho_\alpha \partial_\beta \psi_\mu) + \bar{\chi}_\alpha \rho^\beta \rho^\alpha \psi^\mu \partial_\beta X_\mu + \tfrac{1}{4}\bar{\psi}^\mu \psi_\mu \bar{\chi}_\alpha \rho^\beta \rho^\alpha \chi_\beta \Big). \tag{2.100}$$

Besides being diffeomorphism-invariant and invariant under local Lorentz transformations, this action also has a local Weyl symmetry, a fermionic gauge symmetry, and it is invariant under the local supersymmetry transformations

$$\delta X^\mu = \bar{\epsilon}\psi^\mu \qquad \delta\psi^\mu = -i\rho^\alpha \epsilon(\partial_\alpha X^\mu - \bar{\psi}^\mu \chi_\alpha) \tag{2.101}$$

$$\delta e^a_\lambda = -2i\bar{\epsilon}\rho^a \chi_\lambda \qquad \delta\chi_\alpha = \partial_\alpha \epsilon \tag{2.102}$$

where $\epsilon = \epsilon(\sigma^\alpha)$. [2]

Together, these symmetries can be used to gauge the four components of the zweibein into the form $e^a_\alpha = \delta^a_\alpha$ and the four components of χ_α to zero. These gauge choices define the *superconformal gauge*, and they bring the action into the globally supersymmetric form that we are familiar with. [2]

The new fields do not appear in the action after gauge-fixing, so their equations of motion must be evaluated before gauge-fixing and subsequently imposed as constraints. The equation of motion for $\bar{\chi}$ is

$$\frac{\delta S}{\delta \bar{\chi}} = -\frac{e}{\pi}\Big(\rho^\beta \rho^\alpha \psi^\mu_A \partial_\beta X_\mu + \frac{1}{2}\bar{\psi}^\mu \psi_\mu \rho^\beta \rho^\alpha \chi_{\beta A}\Big) = 0 \tag{2.103}$$

where A is the free spinor index. Our gauge choices give $\chi = 0$, and with the appropriate rescaling this becomes

$$J^\alpha_A = -\frac{\pi}{2e}\frac{\delta S}{\delta \bar{\chi}} = \frac{1}{2}\rho^\beta \rho^\alpha \psi^\mu_A \partial_\beta X_\mu = 0. \tag{2.104}$$

So we see the vanishing of the supercurrent J^α_A arising from gauge-fixing the equations of motion for χ_α.

The equations of motion for the zweibein e^a_α give, for $\chi = 0$,

$$\frac{\delta S}{\delta e} = -\frac{1}{2\pi}(\partial_\alpha X^\mu \partial_\beta X_\mu - i\bar{\psi}^\mu \rho_\alpha \partial_\beta \psi_\mu) = 0. \tag{2.105}$$

The expression in parentheses is the same as $T'_{\alpha\beta}$, which we derived from translational invariance via the Noether method, and which gives the energy-momentum tensor $T_{\alpha\beta}$ after symmetrization and removal of the trace. Thus the vanishing of $T_{\alpha\beta}$ is imposed by the e equations of motion in the superconformal gauge.

3 The Quantum RNS Superstring

Now that we have explored the classical theory in detail we are ready to quantise. We will focus first on canonical (old covariant) quantisation, in which superconformal symmetry must be imposed on the spectrum of states by the super-Virasoro constraints. Subsequently we will discuss quantisation in lightcone gauge, which is not manifestly covariant but which results in a manifestly physical spectrum.

The RNS theory exhibits supersymmetry on the worldsheet, and though it is not obvious, it is possible to show that its spectrum admits spacetime supersymmetry as well [2]. We will not undertake a full proof of spacetime supersymmetry in the RNS framework, since it is manifest in the GS formalism which we discuss later. Nonetheless, our goal in developing the quantum theory will be to uncover evidence which suggests that spacetime supersymmetry is present.

We have seen that the bosonic fields X^μ satisfy two-dimensional wave equations, and the fermionic fields ψ^μ satisfy massless Dirac equations. These are the equations of free field theory, and so canonical quantisation can proceed easily. Quantisation of X^μ is carried out by imposing the canonical commutator relations

$$[X^\mu(\tau,\sigma), \Pi_X^\nu(\tau,\sigma')] = i\eta^{\mu\nu}\delta(\sigma-\sigma') \tag{3.1}$$

where Π_X^μ is the conjugate momentum of X^μ. Denoting the Lagrangian by $\mathcal{L}$, we have $\Pi_X^\mu = \delta\mathcal{L}/\delta\dot{X}_\mu = -\frac{1}{\pi}\dot{X}^\mu$, so we can rewrite the commutator as

$$[X^\mu(\tau,\sigma), \dot{X}^\nu(\tau,\sigma')] = -i\pi\eta^{\mu\nu}\delta(\sigma-\sigma'). \tag{3.2}$$

This defines relations satisfied by the bosonic oscillators [4],

$$[a_m^\mu, a_n^\nu] = m\eta^{\mu\nu}\delta_{m+n}. \tag{3.3}$$

The spinors are quantised by the canonical anticommutator relations [2]

$$\{\psi_A^\mu(\tau,\sigma),\ \psi_B^\nu(\tau,\sigma')\} = \pi\eta^{\mu\nu}\delta_{AB}\,\delta(\sigma-\sigma') \tag{3.4}$$

where $A = \pm$. This defines the anticommutator relations obeyed by the fermionic oscillators; for example, the anticommutator of the right-moving spinor fields in the R-sector can be expanded to give

$$\begin{aligned}\{\psi_+^\mu(\tau,\sigma), \psi_+^\nu(\tau,\sigma')\} &= \tfrac{1}{2}\sum_m\sum_n e^{-im(\tau+\sigma)}e^{-in(\tau+\sigma')}\{d_m^\mu, d_n^\nu\} \\ &= \tfrac{1}{2}\sum_m\sum_n e^{-i(m+n)\tau}e^{-i(m\sigma+n\sigma')}\{d_m^\mu, d_n^\nu\}.\end{aligned} \tag{3.5}$$

From this we can see that all terms in the sum for which $n \neq -m$ must vanish in order to cancel the τ dependence. This leaves us with

$$\{\psi_+^\mu(\tau,\sigma), \psi_+^\nu(\tau,\sigma')\} = \tfrac{1}{2}\sum_m e^{-im(\sigma-\sigma')}\{d_m^\mu, d_{-m}^\nu\}. \tag{3.6}$$

Now using the identity

$$\delta(\sigma-\sigma') = \tfrac{1}{2\pi}\sum_n e^{-in(\sigma-\sigma')} \tag{3.7}$$

we see that agreement with (3.4) requires that the oscillators obey

$$\{d^\mu_m, d^\nu_n\} = \eta^{\mu\nu}\delta_{m+n} \tag{3.8}$$

where $\delta_{m+n} = 1$ iff $m + n = 0$. A similar calculation shows that the NS-sector fermionic oscillators satisfy

$$\{b^\mu_r, b^\nu_s\} = \eta^{\mu\nu}\delta_{r+s}. \tag{3.9}$$

These relations hold for the oscillators of the open superstring, or for the right-moving oscillators of the closed superstring. The left-moving oscillators $\tilde{a}^\mu_n$, $\tilde{b}^\mu_r$, $\tilde{d}^\mu_m$ of the closed superstring satisfy analogous relations which differ only in that they require tildes. We will therefore omit the left-moving oscillators in our exposition for the sake of clarity, though it remains implicit throughout our discussion that the left-movers are present for the closed string, and that they obey similar relations.

The oscillator algebras are proportional to the algebra of the ladder operators of the quantum harmonic oscillator, $[a, a^\dagger] = 1$, which allows us to interpret the oscillators as ladder operators as well. Since $(a^\mu_n)^\dagger = a^\mu_{-n}$ and similarly for the other oscillators, we regard positive-frequency (or "positive-moded") oscillators as annihilation operators, and negative-frequency oscillators as creation operators whose action increases the energy of a state.

Every state is an eigenstate of the number operator N, and the energy level of a state can be determined by its N-eigenvalue. In the NS-sector, the number operator is defined by [2]

$$N_{NS} = \sum_{n=1}^{\infty} a^\mu_{-n} a^\nu_n \,\eta_{\mu\nu} + \sum_{r=1/2}^{\infty} r b^\mu_{-r} b^\nu_r \,\eta_{\mu\nu} \tag{3.10}$$

and in the R-sector by

$$N_R = \sum_{n=1}^{\infty} a^\mu_{-n} a^\nu_n \,\eta_{\mu\nu} + \sum_{m=1}^{\infty} m d^\mu_{-m} d^\nu_m \,\eta_{\mu\nu}. \tag{3.11}$$

3.1 Ground States & Spacetime Spin-Statistics

Our next step towards evidence of spacetime supersymmetry is to construct the spectrum of quantum states. This proceeds along the same lines as in the bosonic theory; we posit a Fock space of multiparticle states wherein the energy vacuum is represented by the ground state, which we assume to have unit norm. Positive-moded oscillators annihilate the ground state, and negative-moded oscillators acting on the ground state create excitations whose energies are proportional to their frequencies. [4]

However we soon encounter significant departures from the bosonic theory. We noted before that the different sectors which arise from the two possible sets of fermionic boundary conditions correspond to quantum spectra which occupy separate and disjoint Fock spaces. Furthermore we suggested that one of these sectors produces states which are spacetime bosons, and that the other admits the spacetime fermionic states that provide much of the motivation for superstring theory.

In order to prove this we must first consider the ground state carefully. Its defining feature is that it is annihilated by positive-frequency oscillators. We have seen that the sectors have different oscillators, and so it follows that we must have different ground states for each sector. These are defined by the relations (for $n, m, r > 0$)

$$a_n^\mu \,|\, 0 \,\rangle_{NS} = b_r^\mu \,|\, 0 \,\rangle_{NS} = 0 \tag{3.12}$$

$$a_n^\mu \,|\, 0 \,\rangle_R = d_m^\mu \,|\, 0 \,\rangle_R = 0 \tag{3.13}$$

where $|0\rangle_{NS}$ denotes the NS-sector ground state, and $|0\rangle_R$ denotes the R-sector ground state.

The ground states are eigenstates of the momentum operator, which we denote $\hat{p}^\mu$ to distinguish it from its eigenvalue p^μ. Therefore the ground state is most accurately written as $|0; p\rangle$ with

$$\hat{p}^\mu \,|\, 0; p \,\rangle = p^\mu \,|\, 0; p \,\rangle \tag{3.14}$$

in either sector.

The action of an oscillator on the ground state can result in one of three possible outcomes. As we know, positive-frequency oscillators annihilate the ground state. A negative-frequency oscillator acting on the ground state creates a state of higher energy (or equivalently, higher mass-squared). An oscillator with zero frequency can act on the ground state without affecting its energy; these zero modes, if present, introduce degeneracy in the definition of the ground state.

The NS-sector is half integer-moded, and as a result zero modes do not occur here (0 does not belong to the indexing set $\frac{1}{2} + \mathbb{Z}$). This eliminates the third possibility mentioned above, so all NS-sector oscillators either create excited states or annihilate the ground state. Thus the NS-sector ground state is unique, and we can therefore identify it as a non-degenerate state of spin zero, a spacetime scalar. Since all oscillators (in both sectors) are spacetime vectors, NS-sector excitations have integer spin and are interpreted as *spacetime bosons* [2].

The situation is more complicated in the R-sector. Here the oscillators are integer-moded, and so zero modes occur which can act on the ground state without affecting its energy. We can show this explicitly by considering the energy of the state $d_0^\mu|0\rangle_R$,

$$\begin{aligned}
N_R(d_0^\mu \,|\, 0 \,\rangle_R) &= \eta_{\lambda\rho} \sum_{n=1}^{\infty} a_{-n}^\lambda a_n^\rho d_0^\mu \,|\, 0 \,\rangle_R + \eta_{\lambda\rho} \sum_{m=1}^{\infty} m d_{-m}^\lambda d_m^\rho d_0^\mu \,|\, 0 \,\rangle_R \\
&= d_0^\mu \eta_{\lambda\rho} \sum a_{-n}^\lambda a_n^\rho \,|\, 0 \,\rangle_R + \eta_{\lambda\rho} \sum m d_{-m}^\lambda (\{d_m^\rho, d_0^\mu\} - d_0^\mu d_m^\rho) \,|\, 0 \,\rangle_R \\
&= \eta_{\lambda\rho} \sum m d_{-m}^\lambda (\eta^{\rho\mu} \delta_m - 0) \,|\, 0 \,\rangle_R \\
&= 0
\end{aligned} \tag{3.15}$$

where we have used the fact that bosonic and fermionic oscillators commute, and the anticommutator vanishes because m is never equal to zero in these sums. This shows that $d_0^\mu|0\rangle_R$ has the same energy as the ground state, but this is possible only if $d_0^\mu|0\rangle_R$ is a ground state itself. This is what is meant by *ground state degeneracy*; the zero modes map one ground state to another.

From the anticommutator relations defined earlier, we see that the zero modes satisfy the relation

$$\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu} \tag{3.16}$$

which is proportional to relation that defines the Dirac algebra. Thus we can interpret the d_0^μ as scaled Dirac matrices [2]. So we have zero modes which are proportional to Dirac matrices, and which map ground states to ground states. These facts taken together indicate that the ground states must furnish an irreducible representation of the Dirac algebra, and therefore we can regard the R-sector ground state as a spacetime spinor. Furthermore, since oscillators are spacetime vectors, all R-sector excitations inherit half-integer spin from the spinor ground state. Thus we see that R-sector states are interpreted as *spacetime fermions*.

Therefore we have established that superstring theory is capable of describing spacetime bosons and spacetime fermions. Obviously this is a necessary feature of any realistic physical theory, and it is also a step towards demonstrating that our spectrum of states can exhibit spacetime supersymmetry.

3.2 The Super-Virasoro Algebra

The classical RNS theory relies on supplemental constraints, the super-Virasoro constraints, to preserve the superconformal symmetry that is present in the gauge-invariant (locally supersymmetric) theory, but which is not manifest after gauge-fixing.

Quantum states in the spectrum of any theory must furnish representations of the symmetry groups whose transformations leave the action invariant. From the above, we can expect that the quantum RNS theory will require quantum super-Virasoro constraints to restrict the naive spectrum of states to those which belong to the superconformal symmetry multiplet. These are called the *physical states* of the theory.

We have seen that the quantised oscillators obey (anti-) commutator relations which are proportional to the Minkowski metric. This metric is not positive-definite; in particular we have $\eta^{00} = -1$. As a result, excitations constructed from timelike oscillators can create states of negative norm. For example, taking $n > 0$, the norm of the excited state $a^0_{-n}|0\rangle$ is given by

$$\begin{aligned} \langle 0; p \,|\, a^0_n a^0_{-n} \,|\, 0; p' \rangle &= \langle 0; p \,|\, ([a^0_n, a^0_{-n}] - a^0_{-n} a^0_n) \,|\, 0; p' \rangle \\ &= n\eta^{00} \langle 0; p \,|\, 0; p' \rangle \\ &= -n\delta(p - p'). \end{aligned} \tag{3.17}$$

These negative-norm states cannot exist in a sensible theory, and so they must decouple from the physical spectrum. This is achieved through the use of the super-Virasoro constraints and the determination of the critical values of the theory, which we discuss below.

The super-Virasoro generators can be expanded in terms of their Fourier modes to give expressions which depend upon the oscillators, and which subsequently vary by sector. In the NS sector, the generators are given by [2]

$$L_m = \frac{1}{2} \sum_n : a_{-n} \cdot a_{m+n} : + \frac{1}{2} \sum_r \left(r + \tfrac{m}{2}\right) : b_{-r} \cdot b_{m+r} : \tag{3.18}$$

$$G_r = \sum_n a_{-n} \cdot b_{r+n} \tag{3.19}$$

where $n \in \mathbb{Z}$, $r \in \frac{1}{2}+\mathbb{Z}$, and the dot implies a contraction of spacetime indices. Expressions such as $: a_{-n} \cdot a_{m+n} :$ are interpreted as being normal-ordered, meaning annihilation operators appear to the right of creation operators.

The mode expansions of the super-Virasoro generators in the R-sector are

$$L_m = \frac{1}{2} \sum_n : a_{-n} \cdot a_{m+n} : + \frac{1}{2} \sum_n (n + \tfrac{m}{2}) : d_{-n} \cdot d_{m+n} : \tag{3.20}$$

$$F_m = \sum_n a_{-n} \cdot d_{m+n} \tag{3.21}$$

where again the oscillators are normal-ordered. Normal-ordering is relevant only for the zero-mode L_0, since oscillators commute for other values of m (there is no ambiguity in the definition of F_0, since a_n^μ and d_m^ν commute). Therefore, as in the bosonic theory, we must include an arbitrary constant in expressions involving L_0. [2]

The classical super-Virasoro constraints require all modes of the generators to vanish. In the quantum theory, we make the weaker demand that positive-moded generators annihilate physical states. Since the generators inherit the reality condition $(L_{-n})^\dagger = L_n$ from the oscillators, this is sufficient to ensure that all generators vanish in expectation value, since for two physical states $|\phi\rangle$ and $|\phi'\rangle$ we have

$$\langle\, \phi \,|\, L_n - a\delta_n \,|\, \phi' \,\rangle = 0 \quad \forall n \tag{3.22}$$

where $\delta_n = 1$ for $n = 0$, and a is the arbitrary (real) constant which arises from the normal-ordering ambiguity of L_0.

The *quantum super-Virasoro constraints* are therefore given in the NS-sector by

$$G_r \,|\, \phi \,\rangle = 0 \quad r > 0 \tag{3.23}$$

$$L_n \,|\, \phi \,\rangle = 0 \quad n > 0 \tag{3.24}$$

$$(L_0 - a_{NS}) \,|\, \phi \,\rangle = 0 \tag{3.25}$$

where $|\phi\rangle$ is a physical bosonic state, and in the R-sector by

$$F_n \,|\, \psi \,\rangle = 0 \quad n \geq 0 \tag{3.26}$$

$$L_n \,|\, \psi \,\rangle = 0 \quad n > 0 \tag{3.27}$$

$$(L_0 - a_R) \,|\, \psi \,\rangle = 0 \tag{3.28}$$

for a physical fermionic state $|\psi\rangle$. [2]

In the NS-sector, the generator L_0 can be written

$$\begin{aligned} L_0 &= \tfrac{1}{2} \sum_n : a_{-n} \cdot a_n : + \tfrac{1}{2} \sum_r r : b_{-r} \cdot b_r : \\ &= \tfrac{1}{2} a_0^2 + \tfrac{1}{2} \sum_1^\infty : a_{-n} \cdot a_n : + \tfrac{1}{2} \sum_r r : b_{-r} \cdot b_r : \\ &= \alpha' p^2 + N_{NS} = -\alpha' M^2 + N_{NS} \end{aligned} \tag{3.29}$$

where we have used the relation $a_0^\mu = \sqrt{2\alpha'} p^\mu$, which comes from the mode expansion of the bosonic field, and the relativistic identity $p^2 = -M^2$. Then the constraint (3.25) implies

$$\alpha' M^2 = N_{NS} - a_{NS}. \tag{3.30}$$

This defines the *mass-shell relation* for the superstring in the NS-sector, and similar calculation gives the same result for the R-sector.

Since the R-sector is integer-moded, we have an extra constraint coming from the zero-moded fermionic generator. We can rewrite the generator F_0 as

$$\begin{aligned} F_0 &= \sum_n a_{-n} \cdot d_n \\ &= a_0 \cdot d_0 + \sum_1^\infty (a_{-n} \cdot d_n + d_{-n} \cdot a_n) \\ &= \frac{\alpha'}{2} p \cdot \Gamma + \sum_1^\infty (a_{-n} \cdot d_n + d_{-n} \cdot a_n) \end{aligned} \tag{3.31}$$

where we have used $a_0^\mu = \frac{\alpha'}{\sqrt{2}} p^\mu$, which comes from the mode expansion of X^μ, and $d_0^\mu = \frac{1}{\sqrt{2}} \Gamma^\mu$, where Γ^μ is a D-dimensional Dirac matrix.

Using this, the condition $F_0 |\psi\rangle = 0$ can be rewritten as

$$\left(p \cdot \Gamma + \frac{2}{\alpha'} \sum_1^\infty (a_{-n} \cdot d_n + d_{-n} \cdot a_n) \right) |\, \psi \,\rangle = 0. \tag{3.32}$$

Because this equation involves a momentum operator (which is a derivative in position space) contracting with a Dirac matrix, we can interpret the condition $F_0 |\psi\rangle = 0$ as a string-theoretic generalisation of the Dirac equation, called the *Dirac-Ramond equation.*

The generators defined here are the conserved currents associated with superconformal symmetry, and the associated symmetry algebra is defined by the relations these generators satisfy. This symmetry algebra is the *super-Virasoro algebra*, whose NS-sector representation is

$$[L_m, L_n] = (m - n) L_{m+n} + A_{NS}(m) \delta_{m+n} \tag{3.33}$$

$$[L_m, G_r] = (\tfrac{1}{2} m - r) G_{m+r} \tag{3.34}$$

$$\{G_r, G_s\} = 2 L_{r+s} + B_{NS}(r) \delta_{r+s}. \tag{3.35}$$

The terms $A(m)$ and $B(r)$ are quantum anomalies which are proportional to the identity operator. They can be determined by evaluating vacuum expectation values to get [2]

$$A_{NS}(m) = \tfrac{1}{8} D (m^3 - m) \tag{3.36}$$

$$B_{NS}(r) = \tfrac{1}{2} D (r^2 - \tfrac{1}{4}) \tag{3.37}$$

The R-sector representation of the super-Virasoro algebra is given by the relations

$$[L_m, L_n] = (m - n) L_{m+n} + A_R(m) \delta_{m+n} \tag{3.38}$$

$$[L_m, F_n] = (\tfrac{1}{2} m - n) F_{m+n} \tag{3.39}$$

$$\{F_m, F_n\} = 2 L_{m+n} + B_R(m) \delta_{m+n} \tag{3.40}$$

and here the anomaly terms are given by [2]

$$A_R(m) = \tfrac{1}{8}Dm^3 \tag{3.41}$$

$$B_R(m) = \tfrac{1}{2}Dm^2 \tag{3.42}$$

As with the bosonic string, the dimension of spacetime and the values of the normal-ordering constants a_{NS}, a_R must be deduced from within the theory. In the covariant gauge, these parameters have ranges of values for which negative-norm states appear, and ranges for which such states are absent. At the boundary of these ranges are the *critical values*, for which large numbers of zero-norm states appear in the spectrum. Such states form the foundation for gauge symmetry in string theory, so these critical values coincide with an enlarged gauge symmetry which makes the theory particularly attractive. [2]

The anticommutator of the F_n generators tell us that the fermionic zero-modes satisfy $F_0^2 = L_0$. Taken together with the constraints $F_0|\psi\rangle = (L_0 - a_R)|\psi\rangle = 0$, this immediately shows that $a_R = 0$. The critical dimension $D = 10$ is forced upon us by cancellation of the anomaly which appears in the super-Virasoro algebra in the covariant gauge, or by the cancellation of the Lorentz anomaly which occurs in lightcone gauge. In either case, the upshot is that superconformal symmetry and Lorentz covariance are compatible only in ten dimensions. Both the critical spacetime dimension and the value $a_{NS} = \frac{1}{2}$ can be shown to give rise to large families of zero-norm states, but rather than enter into these suggestive discussions we simply take these values as given. [2]

3.3 Lightcone Gauge

All of our analysis thus far has relied upon an assumption that we made at the very beginning. In particular, we assumed that worldsheet metric $h_{\alpha\beta}$ was gauge-fixed to coincide with the Minkowski metric. This gauge choice puts us in a manifestly covariant gauge with additional gauge freedom remaining. Quantisation in the covariant gauge requires supplemental constraints (the super-Virasoro constraints) to ensure that superconformal symmetry is preserved.

We will describe a different approach here, lightcone gauge quantisation, in which an additional gauge-fixing choice is made that allow us to solve the super-Virasoro constraints explicitly and incorporate superconformal symmetry into the quantum theory a priori. Though it sacrifices manifest Lorentz covariance, the benefit of this approach is that the resulting spectrum consists only of physical states, so we need not worry about unphysical excitations such as negative-norm states.

First we introduce the *spacetime lightcone coordinates*,

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}),\ X^i \qquad (i = 1, \ldots, D-2) \tag{3.43}$$

which are somewhat analogous to the worldsheet lightcone coordinates used before, but in this case we have broken the Lorentz symmetry by treating two of the dimensions of spacetime differently from the rest. As a result Lorentz covariance need not be present in general, and it can be shown that covariance is preserved only in the critical spacetime dimension $D = 10$. [2]

Our remaining gauge invariance allows us to make transformations to the worldsheet lightcone coordinates such as

$$\sigma^+ \to \tilde{\sigma}^+(\sigma^+), \qquad \sigma^- \to \tilde{\sigma}^-(\sigma^-) \tag{3.44}$$

where $\tilde{\sigma}^{\pm} = \frac{1}{2}(\tilde{\tau} \pm \tilde{\sigma})$, and also local supersymmetry transformations that do not disrupt the earlier gauge choices. [2]

The open-string boundary conditions restrict $\tilde{\tau}$ to a solution of the two-dimensional wave equation. Our spacetime coordinates X^μ solve the same equation, since they are free bosonic fields on the worldsheet, so we can choose $\tilde{\tau}$ to coincide with one of these coordinates. In particular we can choose $\tilde{\tau} = X^+/p^+ + x^+$, or as it is more typically written

$$X^+ = x^+ + p^+\tau \tag{3.45}$$

where x^+ is a constant, and p^+ denotes the momentum of X^+. [4]

Furthermore we can use the residual local fermionic symmetry to set

$$\psi^+ = 0 \tag{3.46}$$

assuming we keep the zero-mode in the R-sector, which is necessary to preserve its interpretation as the sector of fermionic states. [1]

Checking the global supersymmetric variation of X^+ shows that this causes no inconsistency, since

$$\delta X^+ = \bar{\epsilon}\psi^+ = 0. \tag{3.47}$$

These gauge choices put us into *lightcone gauge*.

The super-Virasoro constraints can be written as

$$\psi \cdot \partial_+ X = 0 \tag{3.48}$$

$$(\partial_+ X)^2 + \frac{i}{2}\psi \cdot \partial_+ \psi = 0. \tag{3.49}$$

Using our gauge choices and the fact that the inner product of two vectors is given in lightcone coordinates by [2]

$$V \cdot W = V^i W_i - V^+ W^- - V^- W^+ \tag{3.50}$$

we can use the super-Virasoro constraints to solve explicitly for $\partial_+ X^-$ and ψ^-. The first constraint gives

$$\begin{aligned} 0 &= \psi^i \partial_+ X_i - \psi^+ \partial_+ X^- - \psi^- \partial_+ X^+ \\ &= \psi^i \partial_+ X_i - \psi^- \partial_+ X^+ \\ &= \psi^i \partial_+ X_i - \tfrac{1}{2}\psi^- p^+ \\ \Rightarrow \quad \psi^- &= \tfrac{2}{p^+}\psi^i \partial_+ X^i \end{aligned} \tag{3.51}$$

where we have used $\partial_+ X^+ = \frac{1}{2}(\partial_0 - \partial_1)X^+ = \frac{1}{2}p^+$. The second constraint gives

$$\begin{aligned}
0 &= \partial_+ X^i \partial_+ X_i - \partial_+ X^+ \partial_+ X^- - \partial_+ X^- \partial_+ X^+ + \tfrac{i}{2}\psi \cdot \partial_+ \psi \\
&= \partial_+ X^i \partial_+ X_i - \partial_+ X^- p^+ + \tfrac{i}{2}(\psi^i \partial_+ \psi_i - \psi^+ \partial_+ \psi^- - \psi_- \partial_+ \psi^+) \\
&= \partial_+ X^i \partial_+ X_i - \partial_+ X^- p^+ + \tfrac{i}{2}\psi^i \partial_+ \psi_i \\
\Rightarrow \quad \partial_+ X^- &= \tfrac{1}{p^+}(\partial_+ X^i \partial_+ X_i + \tfrac{i}{2}\psi^i \partial_+ \psi_i).
\end{aligned} \tag{3.52}$$

So X^+ and ψ^+ are fixed by our gauge choices, and $\partial_+ X^-$ and ψ^- are both seen to be uniquely determined by the super-Virasoro constraints. Therefore we have only $D-2$ physical degrees of freedom in lightcone gauge. By substituting mode expansions into the expressions for $\partial_+ X^-$ and ψ^- and evaluating zero-modes, one can derive mass-shell relations which depend only upon the transverse oscillators a^i_n, b^i_r, d^i_n. It follows that excitations in lightcone gauge are constructed from transverse oscillators only.

3.4 The GSO Projection

The mass-shell condition is derived from the zero-mode Virasoro generator L_0 as in the bosonic theory. In the NS-sector, it takes the form [2]

$$\begin{aligned}
\alpha' M^2 &= N_{NS} - a_{NS} \\
&= \sum_{n=1}^{\infty} a^i_{-n} \cdot a^i_n + \sum_{r=1/2}^{\infty} r b^i_{-r} \cdot b^i_r - \frac{1}{2}
\end{aligned} \tag{3.53}$$

where $a_{NS} = \frac{1}{2}$, as we have seen, and only transverse oscillators appear because we are in lightcone gauge. Since the ground state $|0\rangle_{NS}$ has zero energy, the mass-shell condition tells us that

$$\alpha' M^2 \,|\, 0 \,\rangle_{NS} = -\frac{1}{2} \,|\, 0 \,\rangle_{NS} \tag{3.54}$$

and therefore the ground state is a *tachyon*, as in the bosonic theory. This means that we are expanding around a local maximum of the energy potential rather than a local minimum, and so our vacuum is unstable. [9]

Furthermore, we have seen that the fields ψ^μ are worldsheet spinors but spacetime vectors, and while this does not directly contradict the spin-statistics theorem, it is not ideal. To illustrate the source of this tension, consider the action of ψ^μ on the bosonic state $|\phi\rangle$. The resulting state $\psi^\mu|\phi\rangle$ is a boson, since ψ^μ is a spacetime vector, so we have anticommuting operators which map bosons to bosons. More generally, the state

$$\psi^{\mu_1} \cdots \psi^{\mu_n} \,|\, \phi \,\rangle \tag{3.55}$$

is a boson for any value of n. This is normal for even n, since the product of two anticommuting operators is a commuting operator, but the fact that the same is true for odd n does not seem quite right. [2]

Therefore we would like to truncate the spectrum in a way that removes the tachyon, and that also clears up the spin-statistics discrepancies which arise from viewing worldsheet spinors in the context of spacetime. It turns out that this can be done easily using the concept of G-parity.

We will first need to introduce the *worldsheet fermion number* operator, whose eigenvalue for a given state is equal to the number of fermionic excitations present. This is defined according to sector by [1]

$$F_{NS} = \sum_{r=1/2}^{\infty} b_{-r} \cdot b_r \tag{3.56}$$

$$F_R = \sum_{n=1}^{\infty} d_{-n} \cdot d_n. \tag{3.57}$$

Next we define the *G-parity operator* which inherits its name from the original hadronic interpretation of string theory. In the NS-sector, this is given by

$$G_{NS} = (-1)^{1+F_{NS}} \tag{3.58}$$

and in the R-sector by

$$G_R = \Gamma_{11}(-1)^{F_R} \tag{3.59}$$

where $\Gamma_{11} = \Gamma_0 \cdots \Gamma_9$ denotes the ten-dimensional chirality operator. [1]

We can refer to NS-sector states with an even number of fermionic excitations as having negative G-parity, since for example the tachyonic ground state, which has no fermionic excitations, satisfies

$$G_{NS}\,|\,0\,\rangle_{NS} = (-1)^{1+0}\,|\,0\,\rangle_{NS} = -1\,|\,0\,\rangle_{NS}. \tag{3.60}$$

Since the states we want to be rid of are bosons arising from an odd number of anticommuting operators, these states also have negative G-parity. Therefore in the NS-sector, we can achieve both of the goals described above by projecting out states with negative G-parity, and keeping those which have positive G-parity. This is called the *GSO projection.*

The R-sector ground state has no fermionic excitations present, and so G-parity is the same as chirality for this state, since

$$G_R\,|\,0\,\rangle_R = \Gamma_{11}(-1)^{F_R}\,|\,0\,\rangle_R = \Gamma_{11}\,|\,0\,\rangle_R. \tag{3.61}$$

This will give ± 1, and therefore since there is no tachyon to worry about, we can remove states in the R-sector with either positive or negative G-parity without loss of generalisation. [1]

3.5 The Spectrum of States

3.5.1 The Massless Sector of the Open String

Only states with an odd number of fermionic excitations remain in the NS-sector after the GSO projection. The tachyonic ground state is among those which are discarded, but the first excited state

$$b^i_{-1/2}\,|\,0\,\rangle_{NS} \tag{3.62}$$

survives. This is the lowest-energy state in the spectrum after the tachyon is removed, so we can regard it as the new NS-sector ground state.

The mass of the new ground state is given by

$$\alpha' M^2 (b^i_{-1/2} \,|\, 0 \,\rangle_{NS}) = (N_{NS} - \tfrac{1}{2}) b^i_{-1/2} \,|\, 0 \,\rangle_{NS} = 0. \tag{3.63}$$

Furthermore, since $b^i_{-1/2}$ is a transverse vector in ten dimensions, it has eight polarization states, and thus $b^i_{-1/2} \,|\, 0 \,\rangle_{NS}$ has eight propagating modes.[8] Therefore it is a massless vector with eight physical on-shell degrees of freedom.

The R-sector ground state is a spinor in ten-dimensional spacetime which has $2^{10/2} = 32$ complex components, in general. In ten dimensions we can apply both Majorana and Weyl conditions to the spinors [5]. The Majorana condition restricts the spinors to 32 real components, and the Weyl condition projects the spinors onto a single chirality, removing half the components in the process. Furthermore, the ground state must satisfy the Dirac-Ramond equation

$$F_0 \,|\, 0 \,\rangle_R = 0 \tag{3.64}$$

which reduces the number of independent degrees of freedom by half. Therefore the combined effect of these conditions is to give us a spinor ground state with eight independent degrees of freedom, and one of two possible chiralities.

The R-sector mass-shell condition is given in lightcone gauge by

$$\begin{aligned} \alpha' M^2 &= N_R - a_R \\ &= \sum_{n=1}^{\infty} a^i_{-n} \cdot a^i_n + \sum_{m=1}^{\infty} m d^i_{-m} \cdot d^i_m \end{aligned} \tag{3.65}$$

with $a_R = 0$, and therefore the R-sector ground state is massless, since

$$\alpha' M^2 \,|\, 0 \,\rangle_R = N_R \,|\, 0 \,\rangle_R = 0. \tag{3.66}$$

The result of this analysis is that the ground states of the RNS theory (after the GSO projection) include a massless vector boson and a massless spinor, and both of these have eight physical on-shell degrees of freedom. Spacetime supersymmetry requires an equal number of bosons and fermions at a given mass level, so this result satisfies a necessary, though not sufficient, condition to form a supersymmetry multiplet.

3.5.2 The Massless Sector of the Closed String

We have seen that closed string states are made from tensor products of left-moving and right-moving modes, and that the four resulting sectors of closed string states are denoted NS-NS, R-R, NS-R, and R-NS. The GSO projection proceeds as above, but since left-moving and right-moving modes are independent, they must be treated separately.

We project onto positive G-parity states in the NS-sector in order to remove the tachyon. In the R-sector, the projection can remove either positive or negative G-parity states,

[8] In fact, this is sufficient to characterise the state as a massless vector in a ten-dimensional covariant theory, so this provides one way to establish that $a_{NS} = 1/2$. [1]

depending on the chirality of the ground state. Since the closed string spectrum involves two separate R-sectors, we can choose the chiralities of their ground states (and subsequently, the G-parity of the states which remain) to be either the same or different. This choice results in two distinct types of theories.

Selecting the same chirality (chosen to be positive, without loss of generality) for both ground states makes the G-parity of our left-moving and right-moving R-sectors the same. The massless sector can be written

$$\tilde{b}^i_{-1/2}|0\rangle_{NS} \otimes b^j_{-1/2}|0\rangle_{NS} \tag{3.67}$$

$$|+\rangle_R \otimes |+\rangle_R \tag{3.68}$$

$$\tilde{b}^i_{-1/2}|0\rangle_{NS} \otimes |+\rangle_R \tag{3.69}$$

$$|+\rangle_R \otimes b^i_{-1/2}|0\rangle_{NS} \tag{3.70}$$

where $|+\rangle_R$ denotes the ground state of positive chirality. This choice means that fermionic states (those in the NS-R and R-NS sectors) have the same chirality, and this defines the *type IIB theory*.

If we select ground states of opposite chiralities, we get opposite G-parity in the two R-sectors. In this case the massless sector is given by

$$\tilde{b}^i_{-1/2}|0\rangle_{NS} \otimes b^j_{-1/2}|0\rangle_{NS} \tag{3.71}$$

$$|-\rangle_R \otimes |+\rangle_R \tag{3.72}$$

$$\tilde{b}^i_{-1/2}|0\rangle_{NS} \otimes |+\rangle_R \tag{3.73}$$

$$|-\rangle_R \otimes b^i_{-1/2}|0\rangle_{NS} \tag{3.74}$$

where $|-\rangle_R$ denotes the ground state of negative chirality. With this choice we get the *type IIA theory*. Fermionic states come in two chiralities in this theory, but otherwise the states in the type IIA theory and the type IIB theory are similar. Each sector has 64 states in both theories, since the ground states all have eight physical degrees of freedom. [1]

A chiral theory, like the Standard Model, is one which is physically altered by a simultaneous change in the chiralities of all particles. From the above we can see that the type IIA theory is non-chiral, since it contains particles of both chiralities already, whereas type IIB is a chiral theory. [6]

4 The GS Formalism

We have seen evidence for spacetime supersymmetry in the RNS formalism by observing that there are an equal number of bosonic and fermionic degrees of freedom in the massless sector. This is a necessary condition for the states to form a supersymmetry multiplet, and though we have not shown as much, it is true at each mass level. [2]

However the RNS formalism is not without drawbacks. We encountered some mild confusion over spacetime spin-statistics early in our discussion, and our naive spectrum of states contained a tachyon. Luckily, the GSO projection rid us of these problems and also left behind a spectrum for which spacetime supersymmetry seemed like a strong possibility. Despite this fortuitous outcome, the origins of spacetime supersymmetry are quite obscure in the RNS formalism.

In the following sections we will describe the Green-Schwarz (GS) formalism, which makes spacetime supersymmetry manifest. This approach requires no supplemental constraints nor any truncation of the spectrum, and furthermore, it unites bosonic and fermionic states in a single Fock space. Though covariant quantisation extremely difficult, quantisation in lightcone gauge is still possible. Along the way we will encounter a new type of local fermionic symmetry which we must use to make the theory consistent.

4.1 The Classical Superparticle

We will begin by generalising a point particle to have manifest spacetime supersymmetry, and then do the same for the string. The worldline action of a massive point particle propagating in Minkowski spacetime can be written

$$S_{pp} = -\frac{1}{2}\int d\tau\,(e^{-1}\dot{x}^2 - em^2) \tag{4.1}$$

where e is an auxiliary coordinate which preserves reparametrization invariance, and can be regarded as a function of the worldline metric [4].

In our discussion of the RNS string, we saw that we could make supersymmetry manifest by generalising the string worldsheet to a superspace with anticommuting spinor coordinates. Using similar logic, we now suppose that the worldline parameter τ is embedded into a superspace which generalises Minkowski spacetime. The superspace coordinates are given by the spacetime vector $x^\mu(\tau)$ and the spacetime spinor $\theta^{Aa}(\tau)$, where $A = 1, \ldots, N$ gives the number of supersymmetries which will be present, and $a = 1, \ldots, 2^{D/2}$ labels the spinor components.

Supersymmetry is realised on the superspace coordinates by the transformations [2]

$$\delta\theta^A = \epsilon^A \qquad \delta x^\mu = i\bar{\epsilon}^A\Gamma^\mu\theta^A \tag{4.2}$$

$$\delta\bar{\theta}^A = \bar{\epsilon}^A \qquad \delta e = 0 \tag{4.3}$$

where Γ^μ gives a D-dimensional representation of the Dirac algebra. The spinor parameter ϵ^A does not depend on τ, so these are global supersymmetry transformations.

Now we can write the action for a supersymmetric point particle propagating in superspace,

$$S = \frac{1}{2}\int d\tau\, e^{-1}(\dot{x}^\mu - i\bar{\theta}^A\Gamma^\mu\dot{\theta}^A)^2. \tag{4.4}$$

The variation under a supersymmetry transformation is given by

$$\begin{aligned}
\delta S &= \int d\tau\, e^{-1}(\dot{x}^\mu - i\bar{\theta}\Gamma^\mu\dot{\theta})(\delta\dot{x}^\mu - i\,\delta\bar{\theta}\,\Gamma^\mu\dot{\theta} - i\bar{\theta}\Gamma^\mu\delta\dot{\theta}) \\
&= \int d\tau\, e^{-1}(\dot{x}^\mu - i\bar{\theta}\Gamma^\mu\dot{\theta})(i\bar{\epsilon}\Gamma^\mu\dot{\theta} - i\bar{\epsilon}\Gamma^\mu\dot{\theta} - 0) \\
&= 0
\end{aligned} \tag{4.5}$$

so the point particle action is supersymmetric.

We can rewrite our action as

$$S = \frac{1}{2} \int d\tau \, e^{-1} p^2 \tag{4.6}$$

where we define $p^\mu = \dot{x}^\mu - i\bar{\theta}^A \Gamma^\mu \dot{\theta}^A$. The equations of motion can quickly be seen to give (for e, x, and $\bar{\theta}$, respectively)

$$p^2 = \dot{p}^\mu = \Gamma_\mu p^\mu \dot{\theta} = 0. \tag{4.7}$$

The equations of motion give us an important identity,

$$(\Gamma^\mu p_\mu)^2 = \tfrac{1}{2} \{\Gamma^\mu, \Gamma^\nu\} \, p_\mu p_\nu = \eta^{\mu\nu} p_\mu p_\nu = p^2 = 0 \tag{4.8}$$

which means the matrix $\Gamma^\mu p_\mu$ has half the maximum possible rank. Since θ is always multiplied by $\Gamma^\mu p_\mu$, we can conclude that half of the components of θ are decoupled from the theory, or effectively gauged away. This is the result of a new type of symmetry.

4.1.1 Kappa Symmetry for the Superparticle

This new symmetry is a local fermionic symmetry which depends on a spinor parameter $\kappa^{Aa}(\tau)$, where again $A = (1, \ldots, N)$ and $a = (1, \ldots, 2^{D/2})$. The fields transform under a kappa transformation as [2]

$$\delta x^\mu = i\bar{\theta}^A \Gamma^\mu \delta\theta^A \qquad \delta\theta^A = i\Gamma^\mu p_\mu \kappa^A \qquad \delta e = -4e\dot{\bar{\theta}}^A \kappa^A. \tag{4.9}$$

The variation of the worldline action is therefore

$$\begin{aligned}
\delta S &= \frac{1}{2} \int d\tau \left(\delta(e^{-1}) p^2 + 2e^{-1} p_\mu (\delta\dot{x}^\mu - i\, \delta\bar{\theta}\Gamma^\mu \dot{\theta} - i\bar{\theta}\Gamma \delta\dot{\theta}) \right) \\
&= \frac{1}{2} \int d\tau \left(\delta(e^{-1}) p^2 + 2e^{-1} p_\mu (i\dot{\bar{\theta}}\Gamma^\mu \delta\theta + i\bar{\theta}\Gamma^\mu \delta\dot{\theta} - i\delta\bar{\theta}\Gamma^\mu \dot{\theta} - i\bar{\theta}\Gamma^\mu \delta\dot{\theta}) \right) \\
&= \frac{1}{2} \int d\tau \left(\delta(e^{-1}) p^2 + 2e^{-1} p_\mu (2i\dot{\bar{\theta}}\Gamma^\mu \delta\theta) \right) \\
&= \frac{1}{2} \int d\tau \left(\delta(e^{-1}) p^2 + 4ie^{-1} p_\mu \dot{\bar{\theta}}\Gamma^\mu (i\Gamma^\nu p_\nu \kappa) \right. \\
&= \frac{1}{2} \int d\tau \, (\delta(e^{-1}) p^2 - 4e^{-1} p^2 \dot{\bar{\theta}}\kappa)
\end{aligned} \tag{4.10}$$

where we have anticommuted spinors to combine terms. So the action is invariant if $\delta(e^{-1}) = 4e^{-1}\dot{\bar{\theta}}\kappa$, which is precisely the transformation law that our definition for δe gives[9]. Therefore the kappa transformations give a fermionic gauge symmetry of the massive superparticle action, which ensures that θ describes the correct number of degrees of freedom.

[9]Our definition of δe differs from [2] in sign because they take $\{\Gamma^\mu, \Gamma^\nu\} = -2\eta^{\mu\nu}$.

Though the kappa transformations mix bosonic and fermionic fields, they are not supersymmetry transformations. Their commutator on the spinor coordinate is given by

$$\begin{aligned}
[\delta_1, \delta_2]\, \theta^A &= \delta_1(i\Gamma^\mu p_\mu \kappa_2^A) - (1 \leftrightarrow 2) \\
&= i\Gamma^\mu \kappa_2^A\, \delta_1 p_\mu - (1 \leftrightarrow 2) \\
&= i\Gamma^\mu \kappa_2^A\, (2i\dot{\bar{\theta}}^B \Gamma_\mu \delta_1 \theta^B) - (1 \leftrightarrow 2) \\
&= i\Gamma^\mu \kappa_2^A\, (2i\dot{\bar{\theta}}^B \Gamma_\mu)\, (i\Gamma^\nu p_\nu \kappa_1^B) - (1 \leftrightarrow 2) \\
&= -2i\Gamma^\mu \kappa_2^A \dot{\bar{\theta}}^B \Gamma_\mu \Gamma^\nu p_\nu \kappa_1^B - (1 \leftrightarrow 2)
\end{aligned} \tag{4.11}$$

where we have used the fact that $\delta p_\mu = 2i\dot{\bar{\theta}}\Gamma_\mu \delta\theta$. Anticommuting gamma matrices then gives

$$\begin{aligned}
[\delta_1, \delta_2]\, \theta^A &= -2i\, \Gamma^\mu \kappa_2^A \dot{\bar{\theta}}^B\, (-\Gamma^\nu \Gamma_\mu + 2\eta^\nu{}_\mu)\, p_\nu \kappa_1^B - (1 \leftrightarrow 2) \\
&= 2i\, \Gamma^\mu \kappa_2^A \dot{\bar{\theta}}^B\, \Gamma^\nu p_\nu \Gamma_\mu \kappa_1^B - 4i\, \Gamma^\nu p_\nu \kappa_2^A \dot{\bar{\theta}}^B \kappa_1^B - (1 \leftrightarrow 2) \\
&= 2i\, \Gamma^\mu \kappa_2^A \bar{\kappa}_1^B \Gamma^\nu p_\nu \dot{\theta}^B \Gamma_\mu + i\Gamma^\nu p_\nu \tilde{\kappa}^A - (1 \leftrightarrow 2).
\end{aligned} \tag{4.12}$$

The first term is proportional to $\Gamma^\nu p_\nu \dot{\theta}$, and so it vanishes on-shell. The second term gives a kappa transformation with parameter $\tilde{\kappa}^A = -4\kappa_2^A \dot{\bar{\theta}}^B \kappa_1^B - (1 \leftrightarrow 2)$, so we see that the commutator of two kappa transformations on θ^A closes into another kappa transformation. This unusual result occurs because there is no conserved charge associated with kappa symmetry; the quantites which we would expect to be conserved on-shell vanish altogether [2].

We can see proof of this by trying to construct a conserved current using Noether's theorem. The general prescription for a conserved current J^α arising from a continuous symmetry of the Lagrangian $\mathcal{L}(\phi, \partial\phi)$ is given by

$$J^\alpha = \frac{\delta \mathcal{L}}{\delta(\partial\phi)} \delta\phi. \tag{4.13}$$

Attempting to use this to construct a conserved current for x gives,

$$\begin{aligned}
J^\alpha &= \frac{\delta \mathcal{L}}{\delta \dot{x}} \delta x \\
&= e^{-1} p_\mu\, \delta x^\mu = e^{-1} p_\mu (i\bar{\theta}\Gamma^\mu \delta\theta) \\
&= -e^{-1} (\Gamma^\mu p_\mu)^2 \bar{\theta}\kappa = 0.
\end{aligned} \tag{4.14}$$

where J^α vanishes by the equation of motion $p^2 = 0$, and a similar result holds for θ.

The conjugate momenta of the superspace coordinates are given by

$$\pi_x^\mu = \frac{\delta \mathcal{L}}{\delta \dot{x}_\mu} = e^{-1} p^\mu \qquad \pi_\theta^A = \frac{\delta \mathcal{L}}{\delta \dot{\theta}^A} = -ie^{-1} \bar{\theta}^A \Gamma_\mu p^\mu \tag{4.15}$$

and so we have the phase-space constraint [2]

$$\pi_\theta^A = -i\bar{\theta}^A \Gamma_\mu \pi_x^\mu. \tag{4.16}$$

This constraint makes covariant quantisation intractable, so quantisation of the string action must be carried out in lightcone gauge.

4.2 The Classical Superstring

The reparametrization-invariant action of the bosonic string is (for $\alpha' = \frac{1}{2}$)

$$S_{bos} = -\frac{1}{2\pi} \int d^2\sigma \, \sqrt{h} \, h^{\alpha\beta} \, \partial_\alpha X^\mu \partial_\beta X_\mu \tag{4.17}$$

so a reasonable guess for the reparametrization-invariant superstring action would be

$$S_1 = -\frac{1}{2\pi} \int d^2\sigma \, \sqrt{h} \, h^{\alpha\beta} \, \Pi^\mu_\alpha \Pi^\nu_\beta \, \eta_{\mu\nu} \tag{4.18}$$

where $\Pi^\mu_\alpha = \partial_\alpha X^\mu - i\bar{\theta}^A \Gamma^\mu \partial_\alpha \theta^A$ generalises the quantity p^μ from the point particle theory. This action has N global supersymmetries, but it is not invariant under kappa transformations, and so θ describes twice as many independent degrees of freedom as it should. We can remedy this by adding another term to our action which preserves the symmetries of S_1 and makes the sum $S = S_1 + S_2$ kappa-invariant; this term is

$$S_2 = \frac{1}{\pi} \int d^2\sigma \, \Big(- i\epsilon^{\alpha\beta} \partial_\alpha X^\mu (\bar{\theta}^1 \Gamma_\mu \partial_\beta \theta^1 - \bar{\theta}^2 \Gamma_\mu \partial_\beta \theta^2) + \epsilon^{\alpha\beta} \bar{\theta}^1 \Gamma^\mu \partial_\alpha \theta^1 \bar{\theta}^2 \Gamma_\mu \partial_\beta \theta^2 \Big) \tag{4.19}$$

where $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ is a tensor density, and so it replaces the factor $\sqrt{h}$. Kappa symmetry restricts the superstring theory to $N = 2$, so we can describe at most two supersymmetries. This explains our replacement of A with $1, 2$ in S_2. [2]

The global supersymmetry transformations are given by

$$\delta\theta^A = \epsilon^A \qquad \delta X^\mu = i\bar{\epsilon}^A \Gamma^\mu \theta^A \tag{4.20}$$

and the variation of S_2 is therefore

$$\begin{aligned} \delta S_2 = \frac{1}{\pi} \int d^2\sigma \, \epsilon^{\alpha\beta} \Big(& - i\partial_\alpha (\delta X^\mu)(\bar{\theta}^1 \Gamma_\mu \partial_\beta \theta^1 - \bar{\theta}^2 \Gamma_\mu \partial_\beta \theta^2) - i\partial_\alpha X^\mu (\bar{\epsilon}^1 \Gamma_\mu \partial_\beta \theta^1 - \bar{\epsilon}^2 \Gamma_\mu \partial_\beta \theta^2) \\ & + \bar{\epsilon}^1 \Gamma^\mu \partial_\alpha \theta^1 \cdot \bar{\theta}^2 \Gamma_\mu \partial_\beta \theta^2 + \bar{\theta}^1 \Gamma^\mu \partial_\alpha \theta^1 \cdot \bar{\epsilon}^2 \Gamma_\mu \partial_\beta \theta^2 \Big) \\ = \frac{1}{\pi} \int d^2\sigma \, \epsilon^{\alpha\beta} \Big(& \bar{\epsilon}^A \Gamma^\mu \partial_\alpha \theta^A (\bar{\theta}^1 \Gamma_\mu \partial_\beta \theta^1 - \bar{\theta}^2 \Gamma_\mu \partial_\beta \theta^2) \\ & + \bar{\epsilon}^1 \Gamma^\mu \partial_\alpha \theta^1 \cdot \bar{\theta}^2 \Gamma_\mu \partial_\beta \theta^2 + \bar{\theta}^1 \Gamma^\mu \partial_\alpha \theta^1 \cdot \bar{\epsilon}^2 \Gamma_\mu \partial_\beta \theta^2 \Big) \end{aligned} \tag{4.21}$$

where the second term in the first line vanishes because it is a total derivative [2]. We can continue by expanding the supersymmetry index A to get

$$\begin{aligned} \delta S_2 = \frac{1}{\pi} \int d^2\sigma \, \epsilon^{\alpha\beta} \Big(& (\bar{\epsilon}^1 \Gamma^\mu \partial_\alpha \theta^1 + \bar{\epsilon}^2 \Gamma^\mu \partial_\alpha \theta^2)(\bar{\theta}^1 \Gamma_\mu \partial_\beta \theta^1 - \bar{\theta}^2 \Gamma_\mu \partial_\beta \theta^2) \\ & + \bar{\epsilon}^1 \Gamma^\mu \partial_\alpha \theta^1 \cdot \bar{\theta}^2 \Gamma_\mu \partial_\beta \theta^2 + \bar{\theta}^1 \Gamma^\mu \partial_\alpha \theta^1 \cdot \bar{\epsilon}^2 \Gamma_\mu \partial_\beta \theta^2 \Big) \\ = \frac{1}{\pi} \int d^2\sigma \, \epsilon^{\alpha\beta} \Big(& \bar{\epsilon}^1 \Gamma^\mu \partial_\alpha \theta^1 \cdot \bar{\theta}^1 \Gamma_\mu \partial_\beta \theta^1 - \bar{\epsilon}^1 \Gamma^\mu \partial_\alpha \theta^1 \cdot \bar{\theta}^2 \Gamma_\mu \partial_\beta \theta^2 + \bar{\epsilon}^2 \Gamma^\mu \partial_\alpha \theta^2 \cdot \bar{\theta}^1 \Gamma_\mu \partial_\beta \theta^1 \\ & - \bar{\epsilon}^2 \Gamma^\mu \partial_\alpha \theta^2 \cdot \bar{\theta}^2 \Gamma_\mu \partial_\beta \theta^2 + \bar{\epsilon}^1 \Gamma^\mu \partial_\alpha \theta^1 \cdot \bar{\theta}^2 \Gamma_\mu \partial_\beta \theta^2 + \bar{\theta}^1 \Gamma^\mu \partial_\alpha \theta^1 \cdot \bar{\epsilon}^2 \Gamma_\mu \partial_\beta \theta^2 \Big). \end{aligned} \tag{4.22}$$

The second and fifth terms cancel trivially, while the third and sixth cancel due to the antisymmetry of $\epsilon^{\alpha\beta}$, leaving us with

$$\delta S_2 = \frac{1}{\pi}\int d^2\sigma\,\epsilon^{\alpha\beta}(\bar{\epsilon}^1\Gamma^\mu\partial_\alpha\theta^1\cdot\bar{\theta}^1\Gamma_\mu\partial_\beta\theta^1 - \bar{\epsilon}^2\Gamma^\mu\partial_\alpha\theta^2\cdot\bar{\theta}^2\Gamma_\mu\partial_\beta\theta^2). \tag{4.23}$$

The integrand can be rewritten as

$$A = \bar{\epsilon}\Gamma^\mu\dot{\theta}\bar{\theta}\Gamma_\mu\theta' - \bar{\epsilon}\Gamma^\mu\theta'\bar{\theta}\Gamma_\mu\dot{\theta} = A_1 + A_2 \tag{4.24}$$

where we have dropped superscripts and written out the worldsheet components [2]. Then we can expand A_1, A_2 such that

$$\begin{aligned} A_1 &= \tfrac{2}{3}\bar{\epsilon}\Gamma^\mu(\dot{\theta}\bar{\theta}\Gamma_\mu\theta' - \theta'\bar{\theta}\Gamma_\mu\dot{\theta} + \theta\bar{\theta}'\Gamma_\mu\dot{\theta}) \\ &= \tfrac{2}{3}\bar{\epsilon}\Gamma^\mu(\dot{\theta}\bar{\theta}\Gamma_\mu\theta' + \theta'\dot{\bar{\theta}}\Gamma_\mu\theta + \theta\bar{\theta}'\Gamma_\mu\dot{\theta}) \end{aligned} \tag{4.25}$$

$$\begin{aligned} A_2 &= \tfrac{1}{3}\bar{\epsilon}\Gamma^\mu(\dot{\theta}\bar{\theta}\Gamma_\mu\theta' + \theta'\dot{\bar{\theta}}\Gamma_\mu\theta - 2\theta\bar{\theta}'\Gamma_\mu\dot{\theta}) \\ &= \tfrac{1}{3}\bar{\epsilon}\Gamma^\mu(\dot{\theta}\bar{\theta}\Gamma_\mu\theta' - \theta'\bar{\theta}\Gamma_\mu\dot{\theta} - \theta\bar{\theta}'\Gamma_\mu\dot{\theta} + \theta\dot{\bar{\theta}}\Gamma_\mu\theta') \\ &= \tfrac{1}{3}\partial_0(\bar{\epsilon}\Gamma^\mu\theta\bar{\theta}\Gamma_\mu\theta') - \tfrac{1}{3}\partial_1(\bar{\epsilon}\Gamma^\mu\theta\bar{\theta}\Gamma_\mu\dot{\theta}) \end{aligned} \tag{4.26}$$

from which we see that A_2 is a total derivative, and can therefore be dropped. Next we rewrite A_1 as

$$\begin{aligned} A_1 &= \tfrac{2}{3}\bar{\epsilon}\Gamma^\mu(\theta\bar{\theta}'\Gamma_\mu\dot{\theta} + \theta'\dot{\bar{\theta}}\Gamma_\mu\theta + \dot{\theta}\bar{\theta}\Gamma_\mu\theta') \\ &= \tfrac{1}{3}\bar{\epsilon}\Gamma^\mu(\theta\bar{\theta}'\Gamma_\mu\dot{\theta} + \theta'\dot{\bar{\theta}}\Gamma_\mu\theta + \dot{\theta}\bar{\theta}\Gamma_\mu\theta' - \theta\dot{\bar{\theta}}\Gamma_\mu\theta' - \theta'\bar{\theta}\Gamma_\mu\dot{\theta} - \dot{\theta}\bar{\theta}'\Gamma_\mu\theta) \\ &= 2\bar{\epsilon}\Gamma_\mu\psi_{[1}\bar{\psi}_2\Gamma^\mu\psi_{3]} \end{aligned} \tag{4.27}$$

where the bracket on the subscripts antisymmetrizes the spinors $(\psi_1, \psi_2, \psi_3) = (\theta, \theta', \dot{\theta})$. This term has the same form as the cubic spinor term which occurs in super-Yang-Mills theories, and it vanishes only for $D = 3, 4, 6, 10$ with additional conditions on the spinors. These special cases give the circumstances under which "pure super-Yang-Mills" theories, or super-Yang-Mills theories with minimal field content, exist [5]. They are also the only cases for which the classical GS theory is supersymmetric. The case $D = 10$ is singled out in the quantum theory, and the spinors are required to satisfy Majorana-Weyl conditions. [2]

4.3 Kappa Symmetry for the Superstring

We still need to restrict the number of degrees of freedom that θ^A describes, and so we must show that the action $S = S_1 + S_2$ has kappa symmetry. For the point particle we had $\delta\theta^A = i\Gamma_\mu p^\mu\kappa^A$. In the string case, we replace p^μ by the quantity Π^μ_α, so the parameter κ^A must acquire a worldsheet vector index in order to preserve covariance. Furthermore using the projection tensors

$$P^{\alpha\beta}_{\pm} = \tfrac{1}{2}(h^{\alpha\beta} \pm \tfrac{1}{\sqrt{h}}\epsilon^{\alpha\beta}) \tag{4.28}$$

we restrict $\kappa^{A\alpha}$ to obey[10]

$$\kappa^{1\alpha} = P^{\alpha\beta}_{-}\kappa^1_\beta \qquad \kappa^{2\alpha} = P^{\alpha\beta}_{+}\kappa^2_\beta. \tag{4.29}$$

[10] A two-dimensional Lorentz vector can be decomposed into a sum of its self-dual and anti-self-dual parts; this condition requires κ^A to be anti-self-dual for $A = 1$ and self-dual for $A = 2$. [2]

The kappa transformations which leave the action invariant are

$$\delta X^\mu = i\bar{\theta}\Gamma^\mu \delta\theta \qquad\qquad \delta\theta = 2i\Gamma_\mu \Pi^\mu_\alpha \kappa^\alpha_A \tag{4.30}$$

$$\delta(\sqrt{h}h^{\alpha\beta}) = -16\sqrt{h}(P^{\alpha\lambda}_- \bar{\kappa}^{1\beta}\partial_\lambda\theta^1 + P^{\alpha\lambda}_+ \bar{\kappa}^{2\beta}\partial_\lambda\theta^2). \tag{4.31}$$

We omit the proof due to considerations of length, but the integrand in the variation reduces to a cubic spinor term like of the type we saw before; thus the local κ symmetry exists only for $D = 3, 4, 6, 10$ with accompanying conditions on the spinors.

Thus we have a theory with a fermionic gauge symmetry formulated in terms of spacetime spinor fields $\kappa^{A\alpha}$ that transform as worldsheet vectors. This is reminiscent of the spin-statistics ambiguity that appeared in the RNS theory, but because the spacetime spin-statistics are as they should be, there is no need for a GSO-type correction to the spectrum of quantum states. [2]

4.4 Lightcone Gauge Quantisation

The phase-space constraints which we encountered make the use of lightcone gauge necessary in quantisation. We assume that $D = 10$, $N = 2$, and θ^1, θ^2 are Majorana-Weyl spinors of either chirality. The consequences of their relative chirality are important, as we will discuss below.

The local symmetries of the GS superstring allow several gauge choices to be made. First we can use the reparametrization invariance and local Weyl symmetry to fix the conformal gauge, so that $h_{\alpha\beta} = \eta_{\alpha\beta}$ as before. We can also use the residual reparametrization invariance and local kappa symmetry to impose the conditions [2]

$$\Gamma^+\theta^A = 0 \tag{4.32}$$

where $A = 1, 2$ and $\Gamma^\pm$ are lightcone components of the $D = 10$ Dirac algebra, given by

$$\Gamma^\pm = \frac{1}{\sqrt{2}}(\Gamma^0 \pm \Gamma^9). \tag{4.33}$$

These matrices are nilpotent,

$$\begin{aligned}(\Gamma^\pm)^2 &= \tfrac{1}{2}\big(\Gamma^0\Gamma^0 \pm \{\Gamma^0, \Gamma^9\} + \Gamma^9\Gamma^9\big) \\ &= \tfrac{1}{4}(\{\Gamma^0, \Gamma^0\} \pm 2\{\Gamma^0, \Gamma^9\} + \{\Gamma^9, \Gamma^9\}) \\ &= \tfrac{1}{4}(\eta^{00} + \eta^{99}) = 0 \end{aligned} \tag{4.34}$$

but their sum is nonsingular,

$$\det(\Gamma^+ + \Gamma^-) = \frac{2}{\sqrt{2}}\det(\Gamma^0) \neq 0. \tag{4.35}$$

Therefore exactly half of their eigenvalues are zero, and so the gauge-fixing choice above eliminates half the components of θ^A [2]. A Majorana-Weyl spinor has 16 real components in ten dimensions, so this choice leaves us with eight components. Though we used the kappa symmetry to fix the gauge, this property of $\Gamma^\pm$ ensures that θ^A still describes the correct number of degrees of freedom.

The GS theory has a residual conformal symmetry with which we can impose the condition

$$X^+(\sigma^\alpha) = x^+ + p^+\tau \tag{4.36}$$

familiar from lightcone gauge quantisation of the bosonic string and the RNS superstring. This leaves eight propagating transverse bosonic modes, which we denote X^i as before. Thus we have an equal number of propagating bosonic and fermionic modes, as necessary for supersymmetry.

Using S^A to denote the eight components of θ^A which remain after gauge-fixing, we can make the substitutions

$$\sqrt{p^+}\theta^1 \quad \longrightarrow \quad S^{1a} \text{ or } S^{1\dot{a}} \tag{4.37}$$

$$\sqrt{p^+}\theta^2 \quad \longrightarrow \quad S^{2a} \text{ or } S^{2\dot{a}} \tag{4.38}$$

where a and $\dot{a}$ distinguish between spinors of different chiralities. We will use undotted indices in the equations that follow, but both chiralities are possible.

With this choice of gauge, the equations of motion of the GS action simplify to [2]

$$(\partial_1^2 - \partial_0^2)X^i = 0 \quad \Longrightarrow \quad \partial_+\partial_- X^i = \partial_-\partial_+ X^i = 0 \tag{4.39}$$

$$(\partial_0 + \partial_1)S^{1a} = 0 \quad \Longrightarrow \quad \partial_+ S^{1a} = 0 \tag{4.40}$$

$$(\partial_0 - \partial_1)S^{2a} = 0 \quad \Longrightarrow \quad \partial_- S^{2a} = 0. \tag{4.41}$$

If we interpret S^{1a} as the right-moving mode and S^{2a} as the left-moving mode of θ^A, then we see that these are the same equations of motion that arise in the RNS formalism. Therefore in this gauge, we can write the effective action (for $\alpha' = 1/2$)

$$S = -\frac{1}{2\pi}\int d^2\sigma\, (\partial_\alpha X^\mu \partial^\alpha X_\mu - i\bar{S}^a \rho^\alpha \partial_\alpha S^a). \tag{4.42}$$

with the important exception that the S^{Aa} are spacetime spinors, whereas the ψ^μ that appear in the RNS theory are spacetime vectors (and worldsheet spinors).

The quantisation, boundary conditions, and mode expansions of the bosonic coordinate X^i are unchanged from their analogues in the bosonic and RNS theories. For the spinors, we impose the canonical anticommutation relations

$$\{S^{Aa}(\tau,\sigma),\, S^{Bb}(\tau,\sigma)\} = \pi\delta^{ab}\delta^{AB}\delta(\sigma - \sigma'). \tag{4.43}$$

Open string boundary conditions require a relation between S^{1a} and S^{2a} at the string endpoints. In order to preserve at least one of the supersymmetries, we must make the same choice of sign at both endpoints; this means we do not have separate sectors as in the RNS theory, and therefore quantum states are united in a single Fock space. [2]

The open string boundary conditions are

$$S^{1a}(\tau, 0) = S^{2a}(\tau, 0) \tag{4.44}$$

$$S^{1a}(\tau, \pi) = S^{2a}(\tau, \pi) \tag{4.45}$$

and the mode expansions are given by

$$S^{1a} = \frac{1}{\sqrt{2}}\sum_n S_n^a e^{-in(\tau-\sigma)} \tag{4.46}$$

$$S^{2a} = \frac{1}{\sqrt{2}}\sum_n S_n^a e^{-in(\tau+\sigma)}. \tag{4.47}$$

For closed strings, the only boundary condition is periodicity,

$$S^{Aa}(\tau, \sigma) = S^{Aa}(\tau, \sigma + \pi) \tag{4.48}$$

and the mode expansions have doubled exponents as before,

$$S^{1a}(\tau, \sigma) = \sum_n S_n^a e^{-2in(\tau - \sigma)} \tag{4.49}$$

$$S^{2a}(\tau, \sigma) = \sum_n \tilde{S}_n^a e^{-2in(\tau + \sigma)}. \tag{4.50}$$

In either case, the expansion coefficients must satisfy the reality conditions $(S_n^a)^\dagger = S_{-n}^a$ and the anticommutator relations defined by the mode expansions,

$$\{S_m^a, S_n^b\} = \delta^{ab}\delta_{m+n}. \tag{4.51}$$

4.5 Type I & Type II Theories

As we have seen, the classical GS theory is supersymmetric only for certain values of the spacetime dimension D, and there are further restrictions in the quantum theory which specify $D = 10$, as in the RNS formulation. With this condition comes the additional requirement that the spinors satisfy Majorana-Weyl conditions, which means in particular that they are of definite chirality.

The chirality of θ^A is not restricted in general, and various possibilities arise from this. In the general case we have $N = 2$ supersymmetry and two possible chiralities, so there are four possible configurations, but only two of these are physically distinct: either the two chiralities are the same, or they are different. This is the same situtation we encountered in the RNS theory after the GSO projection, when we were faced with the a choice of chirality for the closed-string R-sector ground states.

Open string boundary conditions require us to equate S^{1a} and S^{2a} at the string endpoints, which means these spinors must be of the same chirality. These conditions also have the effect of equating the spinor parameters ϵ^1 and ϵ^2; therefore our two supersymmetries become one, and we are left with $N = 1$ supersymmetry only [2]. The possible superstring theories which exist in these conditions are are called *type I* theories. The classical theories contain only open strings, but quantum consistency demands that the endpoints of an open string are allowed to join to form a closed string, so both topologies are present in the quantum theory. [2]

A theory of only closed strings has in general $N = 2$ supersymmetry, since the periodicity boundary conditions place no restrictions on N. If θ^1 and θ^2 are of opposite chirality we get *type IIA* superstring theories, while if they have the same chirality then we get *type IIB* theories. This too resembles the situation we saw before, and we see the same implications for the types of theories. Since type IIA theories contain one spinor of each chirality they are symmetric under a simultaneous interchange of all chiralities, and therefore these are *nonchiral theories.* On the other hand, type IIB theories contain spinors of a single chirality, so these are antisymmetric under such an interchange, and therefore they are *chiral theories.*

5 Conclusion

This paper has described the key features of the RNS and GS formalism of superstring theory, and we have seen the origins of various types of string theories emerging from both formalisms. The chief reference for this work has been the first volume of the famous textbook by Green, Schwarz, and Witten. It is now more than twenty years since this book was written, and though its age is apparent in some places (eg, it makes no mention of Dirichlet boundary conditions), it is without doubt still a useful pedagogic tool.

The lecture notes of various instructors have also proved quite useful, most notably those of Dr. Neil Lambert, with which I began to learn string theory, and Dr. David Tong, whose writing style is a model of clarity. I also owe a great deal of thanks to Dr. Lambert for his patient and helpful guidance as instructor and advisor over the past several months.

As with any report of this type, there is inevitably much left unsaid due to the necessary restrictions on time and length. A different tack through the same material might focus on the tools of conformal field theory, including BRST quantisation. Also left undiscussed were the no-ghost theorem, which ensures the equivalence of old covariant quantisation and lightcone gauge quantisation for the RNS formalism, and the details of quantum anomalies and their role in determining the critical values of superstring theory. A more detailed study of local worldsheet supersymmetry and supergravity theories would also have been interesting, but would have taken us too far afield.

Further research in a similar vein could cover the details and applications of D-branes, and eventually M-theory. Though topics such as these are outside the present scope, they are fascinating to me. Therefore it is my hope that this report, while modest, contains a clear exposition of the introductory details of superstring theory that will motivate and enable my future studies in theoretical physics.

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