1

Consider a forward SDE

$$dx_t = f(x_t, t)dt + g(x_t, t)dW_t,$$

show that the corresponding probability flow ODE is written as

$$dx_t = \left[f(x_t, t) - \frac{1}{2}\frac{\partial}{\partial x} g^2(x_t, t) - \frac{g^2(x_t, t)}{2}\frac{\partial}{\partial x}\log p(x_t, t) \right] dt.$$

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1. **SDE 的 Fokker-Planck 方程式 (機率密度 p 的演化):**

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}[fp] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[g^2 p]$$

2. **ODE ($dx_t = vdt$) 的連續性方程式:**

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}[vp]$$

3. **令兩者相等 ($\frac{\partial p}{\partial t} = \frac{\partial p}{\partial t}$):**

$$-\frac{\partial}{\partial x}[vp] = -\frac{\partial}{\partial x}[fp] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[g^2 p]$$

4. **對 x 積分 (消除 $\frac{\partial}{\partial x}$):**

$$-vp = -fp + \frac{1}{2}\frac{\partial}{\partial x}[g^2 p]$$

5. **求解 v:**

$$vp = fp - \frac{1}{2}\frac{\partial}{\partial x}[g^2 p]$$
$$v = f - \frac{1}{2p}\frac{\partial}{\partial x}[g^2 p]$$

6. **展開 $\frac{\partial}{\partial x}[g^2 p]$:**

$$v = f - \frac{1}{2p}\left[\left(\frac{\partial g^2}{\partial x}\right) p + g^2 \left(\frac{\partial p}{\partial x}\right) \right]$$

7. 化簡並代換 $\frac{1}{p}\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\log p$:

$$v = f - \frac{1}{2}\frac{\partial g^2}{\partial x} - \frac{g^2}{2}\left(\frac{1}{p}\frac{\partial p}{\partial x}\right)$$
$$v = f - \frac{1}{2}\frac{\partial g^2}{\partial x} - \frac{g^2}{2}\frac{\partial}{\partial x}\log p$$

8. 結論 ($dx_t = vdt$):

$$dx_t = \left[f(x_t, t) - \frac{1}{2}\frac{\partial}{\partial x}g^2(x_t, t) - \frac{g^2(x_t, t)}{2}\frac{\partial}{\partial x}\log p(x_t, t)\right]dt$$