Chapter 12

Analysis of Variance (ANOVA)

Is Seeing Really Believing?

Many adults look on the eyewitness testimony of children with skepticism. They believe that young witnesses' testimony is less accurate than the testimony of adults in court cases. Several statistical studies have been done on this subject.

In a preliminary study, three researchers selected fourteen 8-year-olds, fourteen 12-year-olds, and fourteen adults. The researchers showed each group the same video of a crime being committed. The next day, each witness responded to direct and crossexamination questioning. Then the researchers, using statistical methods explained in this chapter, were able to determine if there were differences in the accuracy of the testimony of the three groups on direct examination and on cross-examination. The statistical methods used here differ from the ones explained in Chapter 9 because there are three groups rather than two. See Statistics Today—Revisited.

前 言

兩種不同的車款每公升汽油可行駛的公里數是否相同？

1. 三種不同的車款每公升汽油可行駛的公里數是否相同？
2. 四種機器進行飲料填充，飲料容量是否會因機器不同而有差異？
3. 男性與女性工人及三種不同的機器對產量是否有影響？

◆上述三個研究之目的是想瞭解<u>3個或3個以上</u>的母體平均數是否相同的問題，如何比較？
- 兩兩比較？
- 若有五個母體，該如何？

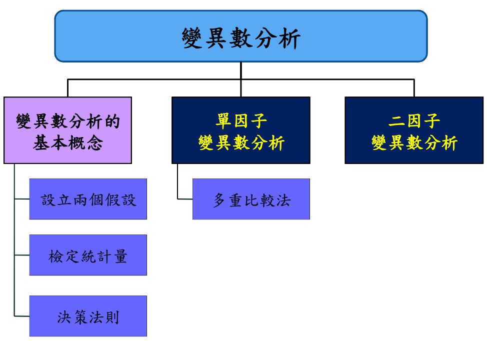
Chapter 11 Overview

Introduction

◆12-1 One-Way Analysis of Variance

◆12-2 The Scheffé Test and the Tukey Test

◆12-3 Two-Way Analysis of Variance



ANOVA範例(一)

◆消費者很想知道哪種車最省油，比較A, B, C<u>三種車款</u>每公升可以行駛的公里數如下：

	A	B	C
樣本資料	18.2	19.8	21.2
	19.4	21.0	21.8
	19.6	20.0	22.4
	19.0	20.8	22.0
	18.8	20.4	21.6

ANOVA範例(二)

◆某知名飲料公司欲了解其所生產的**飲料容量是否因機器的種類而不同**，分別在4台機器獨立隨機抽取5罐飲料，如下表所示：

	機器一	機器二	機器三	機器四
樣本資料	7	5	3	5
	6	8	4	3
	4	7	6	4
	4	6	3	7
	6	4	6	5

ANOVA範例(三)

◆探討性別與A, B, C三種不同機器對產量是否有影響?

機 器 類 型

性別		A	B	C
	男性	4, 9, 8, 9, 6, 8	1, 3, 4 5, 3, 3	3, 9, 6 5, 9, 8
	女性	3, 8, 5 6, 3, 7	7, 3, 4 2, 5, 3	11, 8, 10 12, 9, 13

變異數分析的基本觀念

◆在比較多組母體的平均值時，我們通常不採用兩兩比較的方式，原因：這種做法太浪費時間，因為比較多個母體可能產生很多的比較組

- 例如比較五個母體的平均值差異，如果以兩兩比較的方式，我們必須進行C^5_2=10次的比較。
- 採用變異數分析

◆所謂變異數分析是指，檢定三個或三個以上的**母體平均數是否相等**的檢定方法，或檢定因子對依變數是否有影響的統計方法

自變數與依變數

◆自變數、獨立變數、實驗變數或因子 (Factor)

- 引起資料發生差異的原因稱為因子，亦稱獨立變數、自變數、實驗變數等
- 因子水準：每因子內之處理方式稱為「水準」(levels)
  - 範例一，車款因子，ABC三種車款為三個衡量水準(level)
  - 範例二，生產機器為因子，衡量水準為4(考慮四種不同機器)
  - 範例三，考慮兩種不同的因子：性別與機器

◆依變數 (dependent variable)

- 研究者欲觀察的反應變數
  - 範例一，汽油每公升可以行駛的公里數
  - 範例二，飲料容量
  - 範例三，生產量

ANOVA分類(依因子數目)

◆單因子變異數分析

- 係指探討**單一**分類性的解釋變數對依變數之間的關係
  - 例如：範例一、範例二及以下兩種問題
    - 大學中各年級的同學智商是否有別？
    - 三種不同的教學方法對於學生的成績是否有影響？

◆多因子變異數分析

- 係指探討兩個以上分類性的解釋變數對依變數之間的關係
  - 例如:(二因子)
    - 分析不同工人與不同廠牌機器對產量的影響(範例三)

單因子變異數分析—範例

◆消費者很想知道哪種車最省油，比較A, B, C三種車款每加崙可以行駛的里數如下：

	A	B	C
樣本資料	18.2	19.8	21.2
	19.4	21	21.8
	19.6	20	22.4
	19	20.8	22
	18.8	20.4	21.6

因子：車種
因子水準:3 (A,B,C)
依變數：每加崙里數

多因子變異數分析

◆係指探討兩個以上分類性的解釋變數對依變數之間的關係，最常見者為二因子變異數分析

◆二因子變異數分析係探討兩個分類性的解釋變數對依變數之間的關係

◆資料格式(二因子)

Factor A

Factor B

因子	A_1	A_2		A_c	平均
B_1	x_{11}	x_{12}	…	x_{1c}	$\bar{B}_1$
B_2	x_{21}	x_{22}	…	x_{2c}	$\bar{B}_2$
⋮	⋮	⋮	…	⋮	⋮
B_r	x_{r1}	x_{r2}	…	x_{rc}	$\bar{B}_r$
平均	$\bar{A}_1$	$\bar{A}_2$	…	$\bar{A}_c$	$\bar{\bar{x}}$

二因子變異數分析—範例

◆探討性別和X, Y, Z三種不同機器對產量是否有影響?

Factor A:機 器 類 型

		X	Y	Z
Factor B 性別	男性	4, 9, 8, 9, 6, 8	1, 3, 4 5, 3, 3	3, 9, 6 5, 9, 8
	女性	3, 8, 5 6, 3, 7	7, 3, 4 2, 5, 3	11, 8, 10 12, 9, 13

A因子水準: 3 (X, Y, Z)
B因子水準: 2 (男, 女)

Introduction

◆The *F* test, used to compare two variances, can also be used to compare three of more means.

◆This technique is called **analysis of variance** or **ANOVA**.

◆For three groups, the *F* test can only show whether or not a difference exists among the three means, not where the difference lies.

◆Other statistical tests, **Scheffé test** and the **Tukey test**, are used to find where the difference exists.

Section 12-1

One-Way Analysis of Variance

12-1 One-Way Analysis of Variance

◆When an *F* test is used to test a hypothesis concerning the means of three or more populations, the technique is called **analysis of variance** (commonly abbreviated as **ANOVA**).

◆Although the *t* test is commonly used to compare two means, it should not be used to compare three or more.

Assumptions for the *F* Test

The following assumptions apply when using the *F* test to compare three or more means.

1. The populations from which the samples were obtained must be normally or approximately normally distributed.
2. The samples must be independent of each other.
3. The variances of the populations must be equal.

The *F* Test

- In the *F* test, two different estimates of the population variance are made.
- The first estimate is called the **between-group variance**, and it involves finding the variance of the means.
- The second estimate, the **within-group variance**, is made by computing the variance using all the data and is not affected by differences in the means.

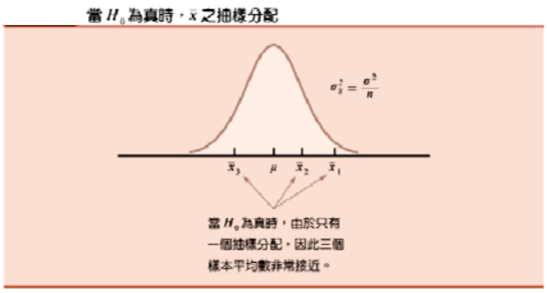
變異數分析介紹

H_0: $\mu_1 = \mu_2 = \mu_3 = \cdots = \mu_k$

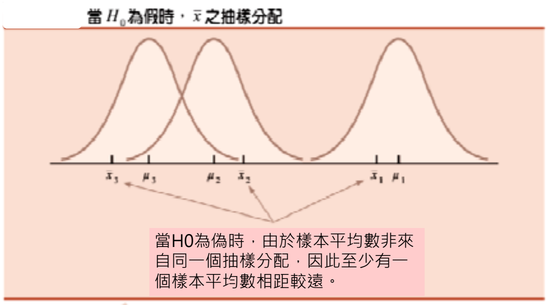
H_a: ______________________

- 如果拒絕H_0，我們不能下結論說所有的母體平均數都不相等。
- 拒絕H_0意指至少有兩個母體平均數不相等。

變異數分析介紹



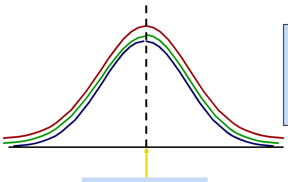
變異數分析介紹



One way ANOVA (單因子ANOVA)

H_0: $\mu_1 = \mu_2 = \mu_3 = \cdots = \mu_k$

H_1 : Not all μ_j are the same



All means are the same:
The null hypothesis is true
(no treatment effect)

$\mu_1 = \mu_2 = \mu_3$

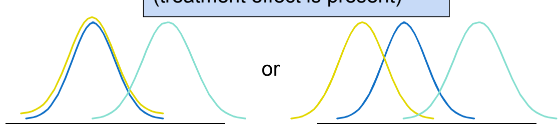
One way ANOVA (單因子ANOVA)

(continued)

H_0: $\mu_1 = \mu_2 = \mu_3 = \cdots = \mu_k$

H_1 : Not all μ_j are the same

At least one mean is different:
The null hypothesis is NOT true
(treatment effect is present)



$\mu_1 = \mu_2 \neq \mu_3$ or $\mu_1 \neq \mu_2 \neq \mu_3$

分割總變異數

◆總變異(Total variation)可分割成二部份:

SST = Total Sum of Squares
(Total variation)

SSB (SSTr) = Sum of Squares Between Groups (Treatments)
(Between-group variation)

SSW (SSE) = Sum of Squares Within Groups
(Within-group variation)

Total Variation

總變異數 (SST)
d.f. = n – 1

= 由於因子所形成平方和 (SSB) + 由於隨機抽樣所形成平方和 (SSW)

d.f. = k – 1 | d.f. = n – k

Commonly referred to as:
- **Sum of Squares Between Groups**
- **Sum of Squares Among Groups**
- **Sum of Squares Explained**
- **Among Groups Variation**

Commonly referred to as:
- Sum of Squares Within Groups
- Sum of Squares Error
- Sum of Squares Unexplained
- Within-Group Variation

Total Variation

$\boxed{SST}$ = SSB + SSW

Where:

SST : Total sum of squares

k : number of groups (levels or treatments)

n_j : number of observations in the j^{th} group

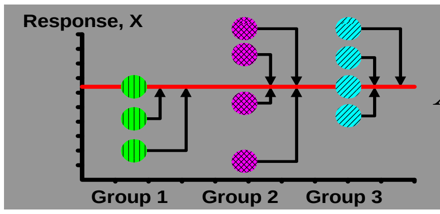
X_{ij} = the i^{th} observation in the j^{th} group

$\overline{\overline{X}}$ = the aggregate mean (the grand mean, $\overline{X}_{GM}$)

Total Variation

(continued)

$$SST = (X_{11} - \overline{\overline{X}})^2 + (X_{12} - \overline{\overline{X}})^2 + \ldots + (X_{k\,n_k} - \overline{\overline{X}})^2$$



$\overline{\overline{X}}$

Sum of Squares Between Groups

$$SST = \boxed{SSB} + SSW$$

$$SSB = \sum_{j=1}^{k} n_j (\overline{X}_j - \overline{\overline{X}})^2$$

Where:

SSB : Sum of Squares Between Groups

k : number of groups (levels or treatments)

n_j : number of observations in the j^{th} group

$\overline{X}_j$: Mean value of the j^{th} treatment

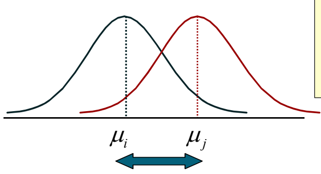
$\overline{\overline{X}}$: Aggregate mean

Sum of Squares Between Groups

(continued)

$$SSB = \sum_{j=1}^{k} n_j (\overline{X}_j - \overline{\overline{X}})^2$$

Variation Due to Differences Between Groups



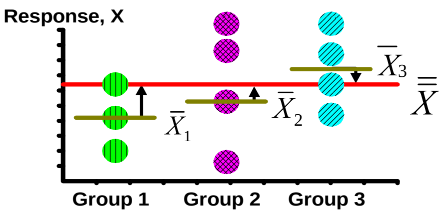
$$MSB = \frac{SSB}{k-1}$$

因子間之平均變異
(Mean Square Between Groups)
=因子間之總變異數 / 自由度

Sum of Squares Between Groups

(continued)

$$SSB = n_1(\bar{x}_1 - \bar{\bar{x}})^2 + n_2(\bar{x}_2 - \bar{\bar{x}})^2 + \ldots + n_k(\bar{x}_k - \bar{\bar{x}})^2$$



Sum of Squares Within Groups

$$SST = SSB + \boxed{SSW}$$

$$SSW = \sum_{j=1}^{k} \sum_{i=1}^{n_j} (X_{ij} - \bar{X}_j)^2$$

Where:

SSW : Sum of Squares Within Groups

k : number of groups (levels or treatments)

n_j : number of observations in the j^{th} group

$\bar{X}_j$: Mean value of the j^{th} treatment

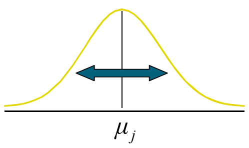
X_{ij} : the i^{th} observation in the j^{th} group

Sum of Squares Within Groups

(continued)

$$SSW = \sum_{j=1}^{k} \sum_{i=1}^{n_j} (X_{ij} - \bar{X}_j)^2$$

Summing the variation within each group and then adding over all groups



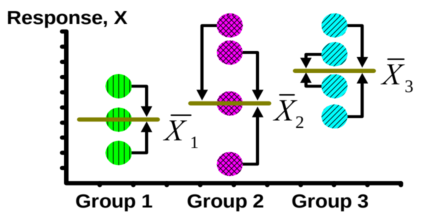
$$MSW = \frac{SSW}{n-k}$$

因子內之平均變異數 (Mean Square Within) =因子內之變異數 /自由度

Sum of Squares Within Groups

(continued)

$$SSW = (x_{11} - \bar{X}_1)^2 + (X_{12} - \bar{X}_2)^2 + \ldots + (X_{k\,n_k} - \bar{X}_k)^2$$



平均變異數

$$MSB = \frac{SSB}{k-1}$$

$$MSW = \frac{SSW}{n-k}$$

One-Way ANOVA Table

Source of Variation	SS	df	MS (Variance)	F ratio
Among Groups				
Within Groups				
Total				

c = number of groups
n = sum of the sample sizes from all groups
df = degrees of freedom

One-Way ANOVA
F Test Statistic

H_0: $\mu_1 = \mu_2 = \ldots = \mu_k$
H_1: At least one mean is different from the others

- test value

$$F = \frac{MSB}{MSW}$$

MSB is the **Mean Square Between Groups**
MSW is the **Mean Square Within Groups**

- Degrees of freedom
  - $df_1 = k - 1$ (k = number of groups)
  - $df_2 = n - k$ (n = sum of sample sizes from all populations)

The procedure of ANOVA

- **Step 1 State the hypotheses and identify the claim.**

H_0: $\mu_1 = \mu_2 = \ldots = \mu_k$
H_1: At least one mean is different from the others

- **Step 2 Find the critical value.**
  - **F-test,**
    - **df.N = *k*-1, df.D = *n* - *k***
- **Step 3 Compute the test value**
- **Step 4 Make the decision.**
- **Step 5 Summarize the results.**

The procedure of ANOVA

- **Step 3 Compute the test value**
  - Find the mean and variance of each sample
  - Find the **Mean Square Between Groups** or **between-group variance**, MSB or s_B^2
    - Sum of Squares Between Groups (SSB) ➔
  - Find the **Mean Square Within Groups** or **within-group variance**, MSW or s_W^2
    - Sum of Squares Within Groups (SSW) ➔
  - **Test value:**
- **Step 4 Make the decision.**
- **Step 5 Summarize the results.**

The procedure of ANOVA

- **Step 1 State the hypotheses and identify the claim.**
- **Step 2 Find the critical value.**
- **Step 3 Compute the test value**
- **Step 4 Make the decision.**
- **Step 5 Summarize the results.**

The *F* Test

- If there is no difference in the means, the between-group variance will be approximately equal to the within-group variance, and the *F* test value will be close to 1—*do not reject null hypothesis*.
- However, when the means differ significantly, the between-group variance will be much larger than the within-group variance; the *F* test will be significantly greater than 1—*reject null hypothesis*. (組間變異大、組內變異小)

Example 12-1: Miles per Gallon

A researcher wishes to see if there is a difference in the fuel economy for city driving for three different types of automobiles: small automobiles, sedans, and luxury automobiles. He randomly samples four small automobiles, five sedans, and three luxury automobiles. The miles per gallon for each is shown. At $\alpha = 0.05$, test the claim that there is no difference among the means.

Small	Sedans	Luxury
36	43	29
44	35	25
34	30	24
35	29	
	40	

Example 12-1: Miles per Gallon

Small	Sedans	Luxury
36	43	29
44	35	25
34	30	24
35	29	
	40	

Step 1: State the hypotheses and identify the claim.

Step 2: Find the critical value.

Example 12-1: Miles per Gallon

Step 3: Compute the test value.

a. Find the mean and variance of each sample.

Small	Sedans	Luxury
36	43	29
44	35	25
34	30	24
35	29	
	40	

b. Find the **grand mean.**

c. Find the **between-group variance.**

Example 12-1: Miles per Gallon

Step 3: Compute the test value. (continued)

c. Find the **SSB** and **MSB**.

d. Find the **SSW** and **MSW**.

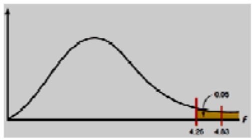
Example 12-1: Miles per Gallon

Step 3: Compute the test value. (continued)

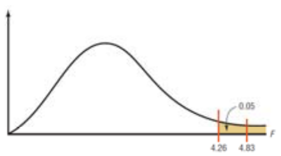
e. Compute the F value.

$$F = \frac{\text{variance between groups}}{\text{variance within groups}}$$

Step 4: Make the decision.



Example 12-1: Miles per Gallon



Step 5: Summarize the results.

There is enough evidence to reject the claim and conclude that at least one mean is different from the others.

Example: Lowering Blood Pressure

A researcher wishes to try three different techniques to lower the blood pressure of individuals diagnosed with high blood pressure. The subjects are randomly assigned to three groups; the first group takes medication, the second group exercises, and the third group follows a special diet. After four weeks, the reduction in each person's blood pressure is recorded. At $\alpha = 0.05$, test the claim that there is no difference among the means.

Medication	Exercise	Diet
10	6	5
12	8	9
9	3	12
15	0	8
13	2	4
$\bar{X}_1 = 11.8$	$\bar{X}_2 = 3.8$	$\bar{X}_3 = 7.6$
$s_1^2 = 5.7$	$s_2^2 = 10.2$	$s_3^2 = 10.3$

Example: Lowering Blood Pressure

Medication	Exercise	Diet
10	6	5
12	8	9
9	3	12
15	0	8
13	2	4
$\bar{X}_1 = 11.8$	$\bar{X}_2 = 3.8$	$\bar{X}_3 = 7.6$
$s_1^2 = 5.7$	$s_2^2 = 10.2$	$s_3^2 = 10.3$

Step 1: State the hypotheses and identify the claim.

Example: Lowering Blood Pressure

Medication	Exercise	Diet
10	6	5
12	8	9
9	3	12
15	0	8
13	2	4
$\bar{X}_1 = 11.8$	$\bar{X}_2 = 3.8$	$\bar{X}_3 = 7.6$
$s_1^2 = 5.7$	$s_2^2 = 10.2$	$s_3^2 = 10.3$

Step 2: Find the critical value.

Example: Lowering Blood Pressure

Step 3: Compute the test value.

a. Find the mean and variance of each sample (these were provided with the data).

b. Find the **grand mean**, the mean of all values in the samples.

c. Find the **between-group variance**, MSB or s_B^2 .

Example: Lowering Blood Pressure

Step 3: Compute the test value. (continued)

c. Find the **between-group variance**, MSA, MSB, s_B^2.

d. Find the **within-group variance**, MSW, s_W^2 .

W

Example: Lowering Blood Pressure

Step 3: Compute the test value. (continued)

e. Compute the F value.

Step 4: Make the decision.

Step 5: Summarize the results.

ANOVA

- The between-group variance is sometimes called the **mean square, *MSB***.
- The numerator of the formula to compute ***MSB*** is called the **sum of squares between groups, *SSB***.
- The within-group variance is sometimes called the **mean square, *MSW***.
- The numerator of the formula to compute ***MSW*** is called the **sum of squares within groups, *SSW***.

變異數分析的基本概念

- 總差異

總差異=因子引起的差異+隨機差異

$SST = SSB + SSW$

- 因子引起的差異(組間差異) SSB

$$SSB = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(\overline{X}_i - \overline{X}_{GM})^2 = \sum_{i=1}^{k} n_i(\overline{X}_i - \overline{X}_{GM})^2$$

- 隨機差異(組內差異) SSE or SSW

$$SSW = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \overline{X}_i)^2 = \sum_{i=1}^{k}(n-1)S_i^2$$

變異數分析的基本概念

- 因子引起的變異數(組間變異)

$$MSB = \frac{SSB}{k-1}$$

- 隨機變異數(組內變異)

$$MSW = \frac{SSW}{n-k}$$

變異數分析的基本概念

- F檢定統計量

$$F = \frac{MSB}{MSW}$$

- 決策法則

①若 $F > F_{k-1,\sum n-k,\alpha}$，則拒絕 H_0。

②若 $F \le F_{k-1,\sum n-k,\alpha}$，則接受 H_0。

ANOVA Summary Table

Source	Sum of Squares	d.f.	Mean Squares	*F*
Between	***SSB***	$k-1$	$MSB = \frac{SSB}{k-1}$	$\frac{MSB}{MSW}$
Within (error)	***SSW***	$n-k$	$MSW = \frac{SSW}{n-k}$	
Total	SST	$n-1$		

ANOVA Summary Table for Example 12-1

Source	Sum of Squares	d.f.	Mean Squares	*F*
Between				
Within (error)				
Total				

ANOVA Table
Example of Lowering Blood Pressure

Source	Sum of Squares	d.f.	Mean Squares	*F*
Between				
Within (error)				
Total				

Report from MINITAB

One-Way ANOVA: Medication, Exercise, Diet

```
Source  DF      SS      MS     F      P
Factor   2  160.13   80.07  9.17  0.004
Error   12  104.80    8.73
Total   14  264.93

                              Individual 95% CIs For Mean
                              Based on Pooled StDev
Level       N    Mean  StDev  --------+---------+---------+---------+--
Medication  5  11.800  2.387                       (--------*-------)
Exercise    5   3.800  3.194  (--------*-------)
Diet        5   7.600  3.209            (-------*--------)
                              --------+---------+---------+---------+--
                                    3.5       7.0      10.5      14.0
Pooled StDev = 2.955
```

Reject the null hypothesis. There is enough evidence to conclude that there is a difference between the treatments.

Example 12-2: Toll Road Employees

A state employee wishes to see if there is a significant difference in the number of employees at the interchanges of three state toll roads. The data are shown. At $\alpha = 0.05$, can it be concluded that there is a significant difference in the average number of employees at each interchange?

Pennsylvania Turnpike	Greensburg Bypass/ Mon-Fayette Expressway	Beaver Valley Expressway
7	10	1
14	1	12
32	1	1
19	0	9
10	11	1
11	1	11
$\bar{X}_1 = 15.5$	$\bar{X}_2 = 4.0$	$\bar{X}_3 = 5.8$
$s_1^2 = 81.9$	$s_2^2 = 25.6$	$s_3^2 = 29.0$

Example 12-2: Toll Road Employees

Pennsylvania Turnpike	Greensburg Bypass/ Mon-Fayette Expressway	Beaver Valley Expressway
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Step 1: State the hypotheses and identify the claim.

Example 12-2: Toll Road Employees

Pennsylvania Turnpike	Greensburg Bypass/ Mon-Fayette Expressway	Beaver Valley Expressway
7	10	1
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19	0	9
10	11	1
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$\bar{X}_1 = 15.5$	$\bar{X}_2 = 4.0$	$\bar{X}_3 = 5.8$
$s_1^2 = 81.9$	$s_2^2 = 25.6$	$s_3^2 = 29.0$

Step 2: Find the critical value.

Example 12-2: Toll Road Employees

Step 3: Compute the test value.

a. Find the mean and variance of each sample (these were provided with the data).

b. Find the **grand mean**, the mean of all values in the samples.

c. Find the **between-group variance**.

Example 12-2: Toll Road Employees

Step 3: Compute the test value. (continued)

c. Find the **between-group variance**, s_B^2.

d. Find the **within-group variance**, s_W^2.

Example 12-2: Toll Road Employees

Step 3: Compute the test value. (continued)

e. Compute the F value.

Step 4: Make the decision.

Step 5: Summarize the results.

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ANOVA Summary Table for Example 12-2

Source	Sum of Squares	d.f.	Mean Squares	*F*
Between				
Within (error)				
Total				

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