Chapter 5

Discrete Random Variables

PEARSON

Definitions

A probability is a measure of the likelihood that an event in the future will happen.

- It can only assume a value between 0 and 1.
- A value near zero means the event is not likely to happen. A value near one means it is likely.

Definitions *continued*

- An experiment is the observation of some activity or the act of taking some measurement.
- An outcome is the particular result of an experiment.
- An event is the collection of one or more outcomes of an experiment.

Mutually Exclusive Events

Events are mutually exclusive if the occurrence of any one event means that none of the others can occur at the same time.

Events are independent if the occurrence of one event does not affect the occurrence of another.

Collectively Exhaustive Events

Events are collectively exhaustive if at least one of the events must occur when an experiment is conducted.

1. For example, faulty switches and working switches.

2. For example, $x < 50$, $50 \leq x < 100$, and $100 \leq x$.

Section 5.1

Discrete Random Variables and Probability Distributions

Definitions 5.1 & 5.2

Random Variable

A **random variable** is a quantitative variable whose value depends on chance.

Discrete Random Variable

A **discrete random variable** is a random variable whose possible values can be listed.

Discrete v.s. Continuous Distributions

- **Random Variable --** a variable which contains the outcomes of a chance experiment
- **Discrete Random Variable --** the set of all possible values is at most a finite or a countable infinite number of possible values
  - Number of new subscribers to a magazine
  - Number of bad checks received by a restaurant
  - Number of absent employees on a given day
- **Continuous Random Variable --** takes on values at every point over a given interval
  - Current Ratio of a motorcycle distributorship
  - Elapsed time between arrivals of bank customers
  - Percent of the labor force that is unemployed

Types of Probability Distributions

- A discrete probability distribution can assume only certain outcomes.

- A continuous probability distribution can assume an infinite number of values within a given range.

Types of Probability Distributions

Examples of a discrete distribution are:

- The number of students in a class.
- The number of children in a family.
- The number of cars entering a carwash in a hour.
- Number of transactions done in the Taiwan Stock market, per day.

Types of Probability Distributions

Examples of a continuous distribution include:

- The distance students travel to class.
- The daily petrol consumption of a car.
- The length of an afternoon nap.
- The length of time of a particular phone call.

Some Special Distributions

- **Discrete**
  - binomial
  - Poisson
- **Continuous**
  - normal
  - uniform
  - exponential
  - t
  - chi-square
  - F

Definition 5.3

Probability Distribution and Probability Histogram

Probability distribution: A listing of the possible values and corresponding probabilities of a discrete random variable, or a formula for the probabilities.

Probability histogram: A graph of the probability distribution that displays the possible values of a discrete random variable on the horizontal axis and the probabilities of those values on the vertical axis. The probability of each value is represented by a vertical bar whose height equals the probability.

EXAMPLE 5.2 Weekly TV-Viewing Times

Number of Siblings: Professor Weiss asked his introductory statistics students to state how many siblings they have. Table 5.1 presents frequency and relative frequency distributions for that information. The table shows, for instance, that 11 of the 40 students, or 27.5%, have two siblings. Let X denote the number of siblings of a randomly selected student.

a. Determine the probability distribution of the random variable X.

b. Construct a probability histogram for the random variable X.

EXAMPLE 5.2

Solution

We want to determine the probability of each of the possible values of the random variable *X*. To obtain, for instance, $P(X = 2)$, the probability that the student selected has two siblings, we apply the *f/N* rule.

$$P(X = 2) = \frac{f}{N} = \frac{11}{40} = 0.275.$$

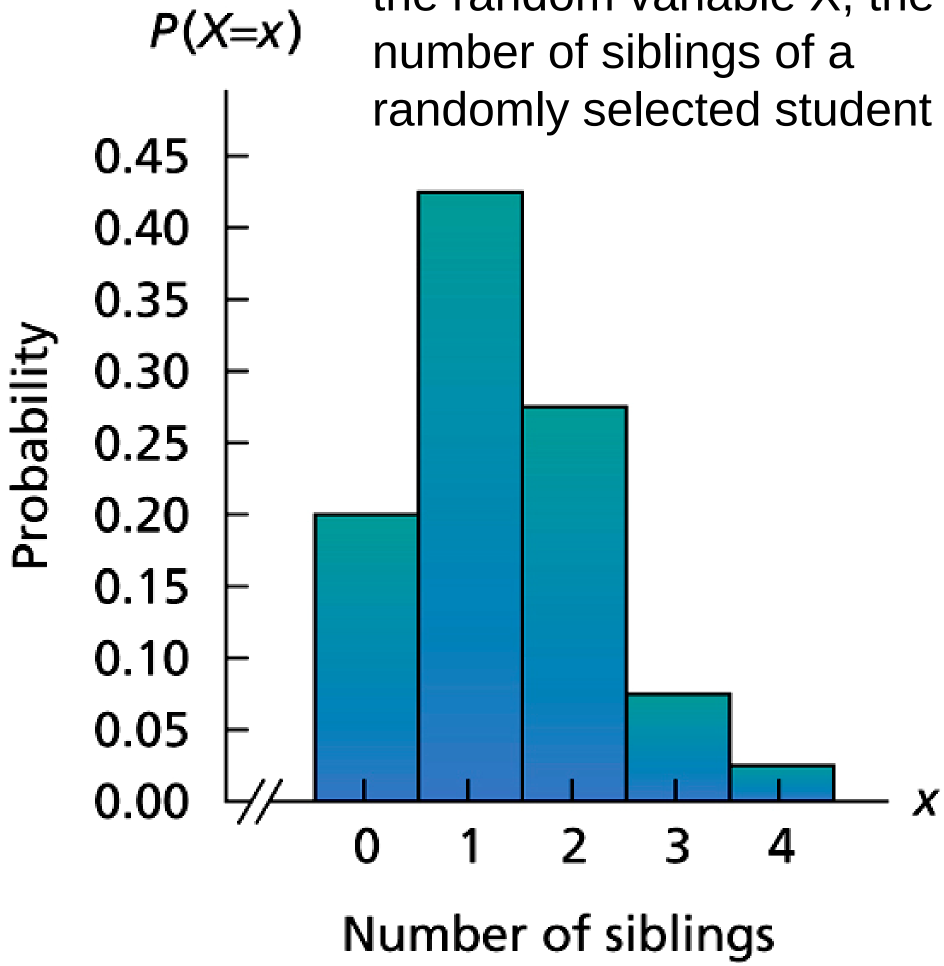
From Table 5.1, we find that Table 5.2 displays these probabilities and provides the probability distribution of the random variable X.

Table 5.2 & Figure 5.1

Probability distribution of the random variable X, the number of siblings of a randomly selected student

Siblings x	Probability $P(X = x)$
0	0.200
1	0.425
2	0.275
3	0.075
4	0.025
	1.000

Probability histogram for the random variable X, the number of siblings of a randomly selected student



KEY FACT 5.1

The sum of the probabilities of the possible values of a discrete random variable equals 1.

Sum of the Probabilities of a Discrete Random Variable

For any discrete random variable X, we have $\Sigma P(X = x) = 1.^{\dagger}$

EXAMPLE 5.5 Interpreting a Probability Distribution

Coin Tossing : Suppose we repeat the experiment of observing the number of heads, X, obtained in three tosses of a balanced dime a large number of times. Then the proportion of those times in which, say, no heads are obtained ($X = 0$) should approximately equal the probability of that event [$P(X = 0)$]. The same statement holds for the other three possible values of the random variable X. Use simulation to verify these facts.

Extra information on the example

Recap the random experiment in which a coin is tossed three times. Let x be the number of heads. Let H represent the outcome of a head and T the outcome of a tail.

Extra information on the example

- The possible outcomes for such an experiment will be:

  TTT, TTH, THT, THH,

  HTT, HTH, HHT, HHH.

- Thus the possible values of x (number of heads) are 0,1,2,3.

Extra information on the example

- The outcome of zero heads occurred once.
- The outcome of one head occurred three times.
- The outcome of two heads occurred three times.
- The outcome of three heads occurred once.
- From the definition of a random variable, x as defined in this experiment, is a *random variable*.

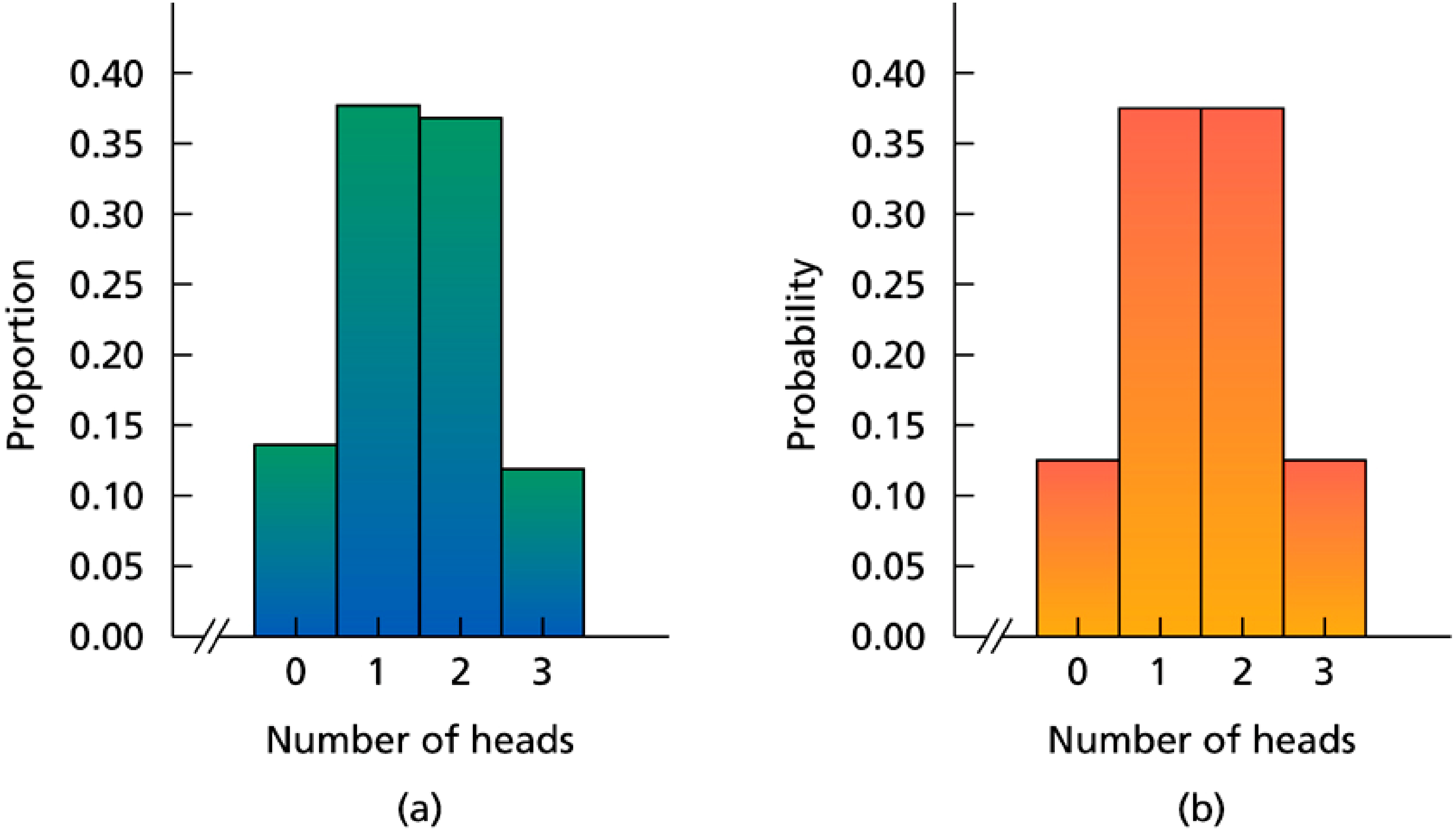
Table 5.7

TABLE 5.7
Frequencies and proportions for the numbers of heads obtained in three tosses of a balanced dime for 1000 observations

No. of heads x	Frequency f	Proportion $f/1000$
0	136	0.136
1	377	0.377
2	368	0.368
3	119	0.119
	1000	1.000

Figure 5.2

(a) Histogram of proportions for the numbers of heads obtained in three tosses of a balanced dime for 1000 observations; (b) probability histogram for the number of heads obtained in three tosses of a balanced dime



Requirements for a Discrete Probability Function

- Probabilities are between 0 and 1, inclusively

$$\boxed{0 \le P(X) \le 1 \quad \text{for all X}}$$

- Total of all probabilities equals 1

$$\boxed{\sum_{\text{over all x}} P(X) = 1}$$

KEY FACT 5.2

Interpretation of a Probability Distribution

In a large number of independent observations of a random variable X, the proportion of times each possible value occurs will approximate the probability distribution of X; or, equivalently, the proportion histogram will approximate the probability histogram for X.

Discrete Distribution -- Example

Distribution of Daily Crises	
Number of Crises	**Probability**
0	0.37
1	0.31
2	0.18
3	0.09
4	0.04
5	0.01

Requirements for a Discrete Probability Function -- Examples

X	P(X)
-1	.1
0	.2
1	.4
2	.2
3	.1
	1.0

X	P(X)
-1	-.1
0	.3
1	.4
2	.3
3	.1
	1.0

X	P(X)
-1	.1
0	.3
1	.4
2	.3
3	.1
	1.2

Section 5.2

The Mean and Standard Deviation of a Discrete Random Variable

PEARSON

Definition 5.4

Mean of a Discrete Random Variable

The **mean of a discrete random variable** $\boldsymbol{X}$ is denoted $\boldsymbol{\mu_X}$ or, when no confusion will arise, simply $\boldsymbol{\mu}$. It is defined by

$$\mu = \Sigma x P(X = x).$$

The terms **expected value** and **expectation** are commonly used in place of the term *mean*.†

The Mean of a Discrete Probability Distribution

The mean is computed by the formula:

$$\mu = \Sigma[xP(x)]$$

- where μ represents the mean and $P(x)$ is the probability of the various outcomes x.

Mean of a Discrete Distribution

$$\mu = E(X) = \sum X \cdot P(X)$$

X	P(X)	$X \cdot P(X)$
-1	.1	-.1
0	.2	.0
1	.4	.4
2	.2	.4
3	.1	.3
		1.0

EXAMPLE 5.7 The Mean of a Discrete Random Variable

Busy Tellers Prescott National Bank has six tellers available to serve customers. The number of tellers busy with customers at, say, 1:00 P.M. varies from day to day and depends on chance; hence it is a random variable, say, X. Past records indicate that the probability distribution of X is as shown in the first two columns of Table 5.10. Find the mean of the random variable X.

EXAMPLE 5.2

Solution

TABLE 5.10
Table for computing the mean of the random variable X, the number of tellers busy with customers

x	$P(X=x)$	$xP(X=x)$
0	0.029	0.000
1	0.049	0.049
2	0.078	0.156
3	0.155	0.465
4	0.212	0.848
5	0.262	1.310
6	0.215	1.290
		4.118

The third column of Table 5.10 provides the products of x with P(X = x), which, in view of Definition 5.4, are required to determine the mean of X. Summing that column gives

$$\mu = \Sigma x P(X = x) = 4.118.$$

Interpretation: The mean number of tellers busy with customers is 4.118.

Key Fact 5.3

Interpretation of the Mean of a Random Variable

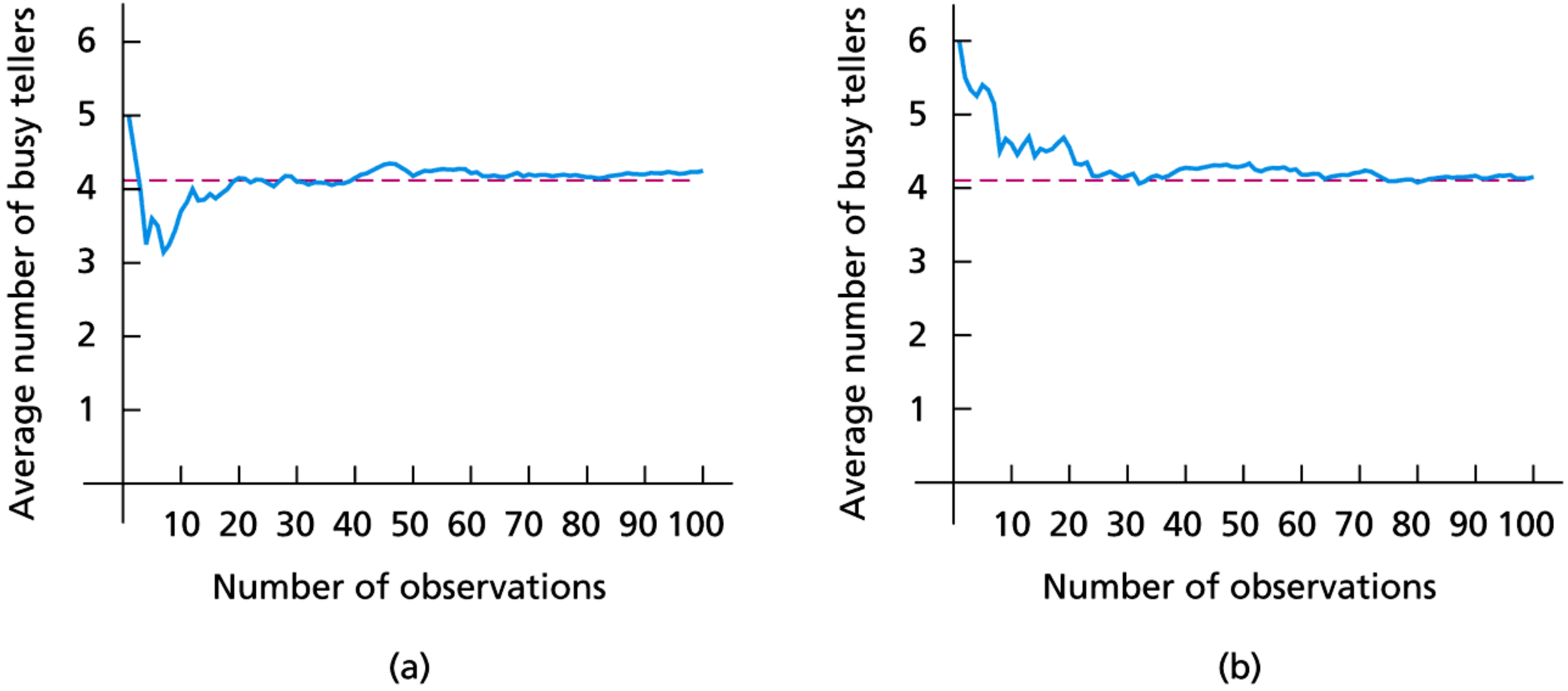
In a large number of independent observations of a random variable X, the average value of those observations will approximately equal the mean, μ, of X. The larger the number of observations, the closer the average tends to be to μ.

The Mean of a Discrete Probability Distribution

- The mean:
  - reports the central location of the data.
  - is the long-run average value of the random variable.
  - is also referred to as its expected value, $E(X)$, in a probability distribution.
  - is a weighted average.

Figure 5.3

Graphs showing the average number of busy tellers versus the number of observations for two simulations of 100 observations each



The Variance of a Discrete Probability Distribution

- The variance measures the amount of spread (variation) of a distribution.
- The variance of a discrete distribution is denoted by the Greek letter σ^2 (sigma squared).
- The standard deviation is σ.

The Variance of a Discrete Probability Distribution

- The variance of a discrete probability distribution is computed from the formula:

$$\sigma^2 = \Sigma[(x-\mu)^2 P(x)]$$

Definition 5.5

Standard Deviation of a Discrete Random Variable

The **standard deviation of a discrete random variable X** is denoted σ_X or, when no confusion will arise, simply σ. It is defined as

$$\sigma = \sqrt{\Sigma(x-\mu)^2 P(X=x)}.$$

The standard deviation of a discrete random variable can also be obtained from the computing formula

$$\sigma = \sqrt{\Sigma x^2 P(X=x) - \mu^2}.$$

EXAMPLE

Dan Desch, owner of College Painters, studied his records for the past 20 weeks and reports the following number of houses painted per week:

# of Houses Painted	Weeks
10	5
11	6
12	7
13	2

EXAMPLE *continued*

Probability Distribution:

Number of houses painted, x	Probability, $P(x)$
10	.25
11	.30
12	.35
13	.10
Total	1.00

EXAMPLE *continued*

Compute the mean number of houses painted per week:

$$\mu = E(x) = \Sigma[xP(x)]$$
$$= (10)(.25) + (11)(.30) + (12)(.35) + (13)(.10)$$
$$= 11.3$$

EXAMPLE *continued*

Compute the variance of the number of houses painted per week:

$$
\begin{aligned}
\sigma^2 &= \Sigma[(x-\mu)^2 P(x)] \\
&= (10-11.3)^2(.25) + \ldots + (13-11.3)^2(.10) \\
&= 0.4225 + 0.0270 + 0.1715 + 0.2890 \\
&= 0.91
\end{aligned}
$$

Variance and Standard Deviation of a Discrete Distribution

$$\sigma^2 = \sum (X - \mu)^2 \cdot P(X) = 1.2$$

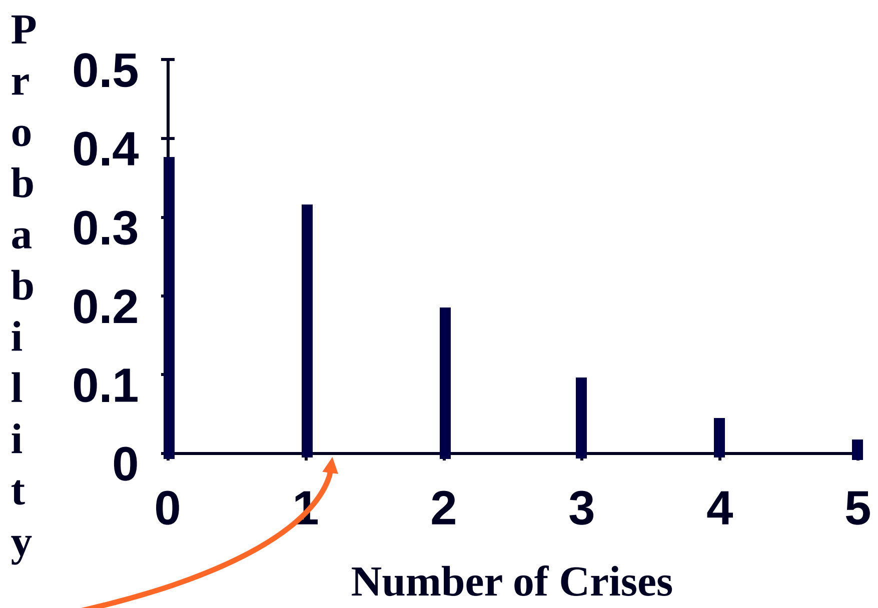
$$\sigma = \sqrt{\sigma^2} = \sqrt{1.2} \cong 1.10$$

X	P(X)	$X - \mu$	$(X - \mu)^2$	$(X - \mu)^2 \cdot P(X)$
-1	.1	-2	4	.4
0	.2	-1	1	.2
1	.4	0	0	.0
2	.2	1	1	.2
3	.1	2	4	.4
				1.2

Mean of the Crises Data Example

$$\mu = E(X) = \sum X \cdot P(X) = 1.15$$

X	P(X)	X•P(X)
0	.37	.00
1	.31	.31
2	.18	.36
3	.09	.27
4	.04	.16
5	.01	.05
		1.15



Variance and Standard Deviation of Crises Data Example

$$\sigma^2 = \sum (X - \mu)^2 \cdot P(X) = 1.41$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{1.41} = 1.19$$

X	P(X)	(X- μ)	(X- μ)²	(X- μ)²•P(X)
0	.37	-1.15	1.32	.49
1	.31	-0.15	0.02	.01
2	.18	0.85	0.72	.13
3	.09	1.85	3.42	.31
4	.04	2.85	8.12	.32
5	.01	3.85	14.82	.15
				1.41

EXAMPLE 5.8 The Standard Deviation of a Discrete Random Variable

Busy Tellers Recall Example 5.7, where *X* denotes the number of tellers busy with customers at 1:00 P.M. Find the standard deviation of *X*.

TABLE 5.12
Table for computing the standard deviation of the random variable *X*, the number of tellers busy with customers

x	$P(X=x)$	x^2	$x^2P(X=x)$
0	0.029	0	0.000
1	0.049	1	0.049
2	0.078	4	0.312
3	0.155	9	1.395
4	0.212	16	3.392
5	0.262	25	6.550
6	0.215	36	7.740
			19.438

EXAMPLE 5.8

Solution

We apply the computing formula given in Definition 5.5. To use that formula, we need the mean of X, which we found in Example 5.7 to be 4.118, and columns for x^2 and $x^2 P(X = x)$, which are presented in the last two columns of Table 5.12. From the final column of Table 5.12, $x^2 P(X = x) = 19.438$. Thus

$$\sigma = \sqrt{\Sigma x^2 P(X = x) - \mu^2} = \sqrt{19.438 - (4.118)^2} = 1.6.$$

Interpretation: Roughly speaking, on average, the number of busy tellers is 1.6 from the mean of 4.118 busy tellers.

Section 5.3

The Binomial Distribution

Example of Bernoulli Trials

1. Testing the effectiveness of a drug: Several patients take the drug (the trials), and for each patient, the drug is either effective or not effective (the two possible outcomes).

2. Weekly sales of a car salesperson: The salesperson has several customers during the week (the trials), and for each customer, the salesperson either makes a sale or does not make a sale (the two possible outcomes).

3. Taste tests for colas: A number of people taste two different colas (the trials), and for each person, the preference is either for the first cola or for the second cola (the two possible outcomes).

Definition 5.8

Bernoulli Trials

Repeated trials of an experiment are called **Bernoulli trials** if the following three conditions are satisfied:

1. The experiment (each trial) has two possible outcomes, denoted generically ***s***, for **success,** and ***f***, for **failure.**
2. The trials are independent.
3. The probability of a success, called the **success probability** and denoted ***p***, remains the same from trial to trial.

EXAMPLE 5.11 Introducing the Binomial Distribution

Mortality: Mortality tables enable actuaries to obtain the probability that a person at any particular age will live a specified number of years. Insurance companies and others use such probabilities to determine life-insurance premiums, retirement pensions, and annuity payments. According to tables provided by the National Center for Health Statistics in *Vital Statistics of the United States*, a person of age 20 years has about an 80% chance of being alive at age 65 years. Suppose three people of age 20 years are selected at random.

a. Formulate the process of observing which people are alive at age 65 as a sequence of three Bernoulli trials.

Table 5.13 and 5.14

Outcomes and probabilities for observing whether each of three people is alive at age 65

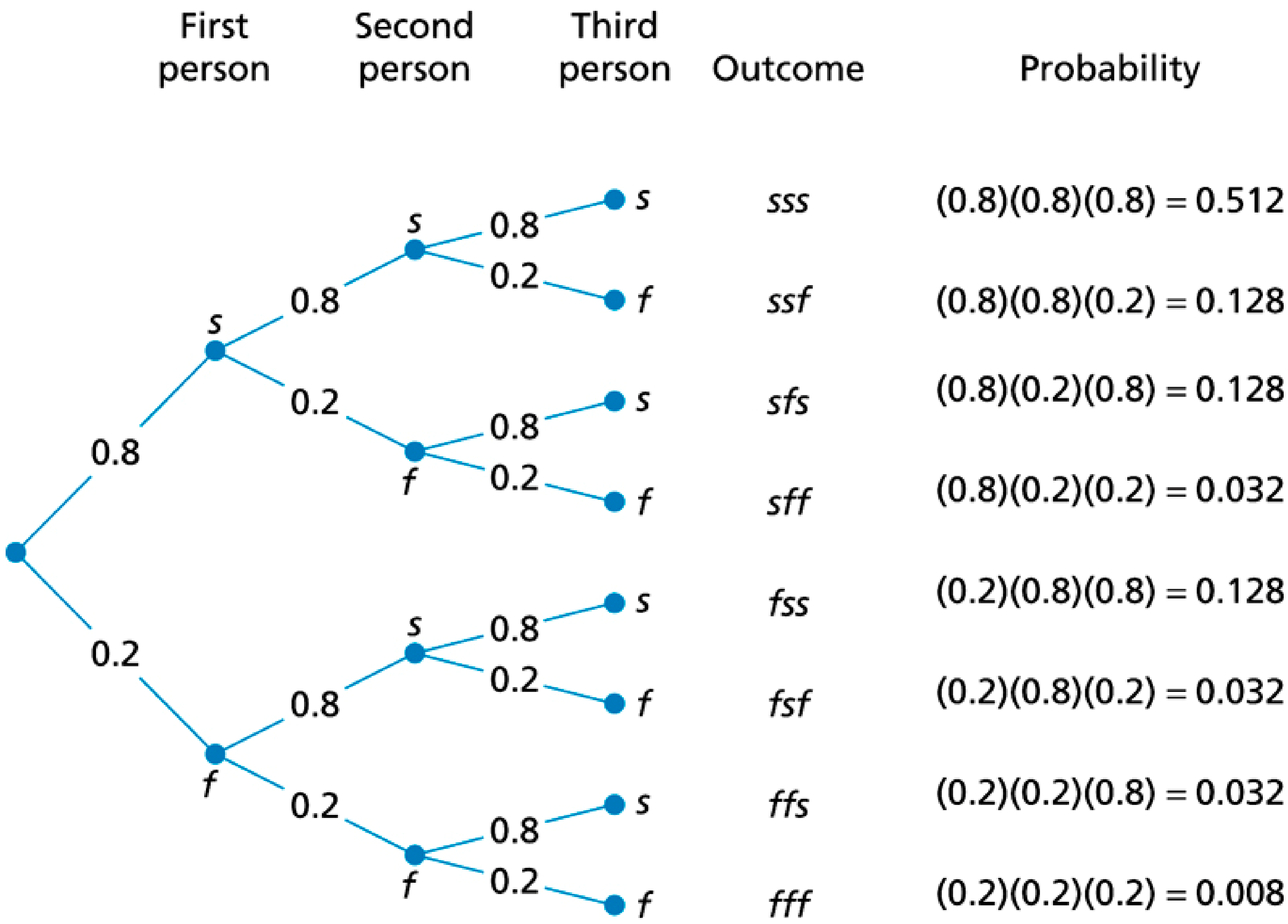
TABLE 5.13
Possible outcomes

sss	*ssf*	*sfs*	*sff*
fss	*fsf*	*ffs*	*fff*

Outcome	Probability
sss	$(0.8)(0.8)(0.8) = 0.512$
ssf	$(0.8)(0.8)(0.2) = 0.128$
sfs	$(0.8)(0.2)(0.8) = 0.128$
sff	$(0.8)(0.2)(0.2) = 0.032$
fss	$(0.2)(0.8)(0.8) = 0.128$
fsf	$(0.2)(0.8)(0.2) = 0.032$
ffs	$(0.2)(0.2)(0.8) = 0.032$
fff	$(0.2)(0.2)(0.2) = 0.008$

Figure 5.4

Tree diagram corresponding to Table 5.14



KEY FACT 5.4

Number of Outcomes Containing a Specified Number of Successes

In n Bernoulli trials, the number of outcomes that contain exactly x successes equals the binomial coefficient $\binom{n}{x}$.

Procedure 5.1

To Find a Binomial Probability Formula

Assumptions

1. n trials are to be performed.
2. Two outcomes, success or failure, are possible for each trial.
3. The trials are independent.
4. The success probability, p, remains the same from trial to trial.

Step 1 **Identify a success.**

Step 2 **Determine p, the success probability.**

Step 3 **Determine n, the number of trials.**

Step 4 **The binomial probability formula for the number of successes, X, is**

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}.$$

EXAMPLE 5.12 Obtaining Binomial Probabilities

Mortality According to tables provided by the National Center for Health Statistics in *Vital Statistics of the United States*, there is roughly an 80% chance that a person of age 20 years will be alive at age 65 years. Suppose that three people of age 20 years are selected at random. Find the probability that the number alive at age 65 years will be

a. exactly two. **b.** at most one. **c.** at least one.

d. Determine the probability distribution of the number alive at age 65.

EXAMPLE 5.12

Solution

Let X denote the number of people of the three who are alive at age 65. To solve parts (a)–(d), we first apply Procedure 5.1.

Step 1 Identify a success.

A success is that a person currently of age 20 will be alive at age 65.

Step 2 Determine p, the success probability.

The probability that a person currently of age 20 will be alive at age 65 is 80%, so $p = 0.8$.

Step 3 Determine n, the number of trials.

The number of trials is the number of people in the study, which is three, so $n = 3$.

EXAMPLE 5.12

Solution

Step 4 **The binomial probability formula for the number of successes, X, is**

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}.$$

Because $n = 3$ and $p = 0.8$, the formula becomes

$$P(X = x) = \binom{3}{x} (0.8)^x (0.2)^{3-x}.$$

We see that X is a binomial random variable and has the binomial distribution with parameters $n = 3$ and $p = 0.8$. Now we can solve parts (a)–(d) relatively easily.

EXAMPLE 5.12

Solution

a. Applying the binomial probability formula with $x = 2$ yields

$$P(X = 2) = \binom{3}{2}(0.8)^2(0.2)^{3-2} = \frac{3!}{2!\,(3-2)!}(0.8)^2(0.2)^1 = 0.384.$$

Interpretation Chances are 38.4% that exactly two of the three people will be alive at age 65.

b. The probability that at most one person will be alive at age 65 is

$$\begin{aligned} P(X \le 1) &= P(X = 0) + P(X = 1) \\ &= \binom{3}{0}(0.8)^0(0.2)^{3-0} + \binom{3}{1}(0.8)^1(0.2)^{3-1} \\ &= 0.008 + 0.096 = 0.104. \end{aligned}$$

EXAMPLE 5.12

Solution

c. The probability that at least one person will be alive at age 65 is $P(X \geq 1)$, which we can obtain by first using the fact that

$$P(X \geq 1) = P(X = 1) + P(X = 2) + P(X = 3)$$

and then applying the binomial probability formula to calculate each of the three individual probabilities. However, using the complementation rule is easier:

$$\begin{aligned} P(X \geq 1) &= 1 - P(X < 1) = 1 - P(X = 0) \\ &= 1 - \binom{3}{0}(0.8)^0(0.2)^{3-0} = 1 - 0.008 = 0.992. \end{aligned}$$

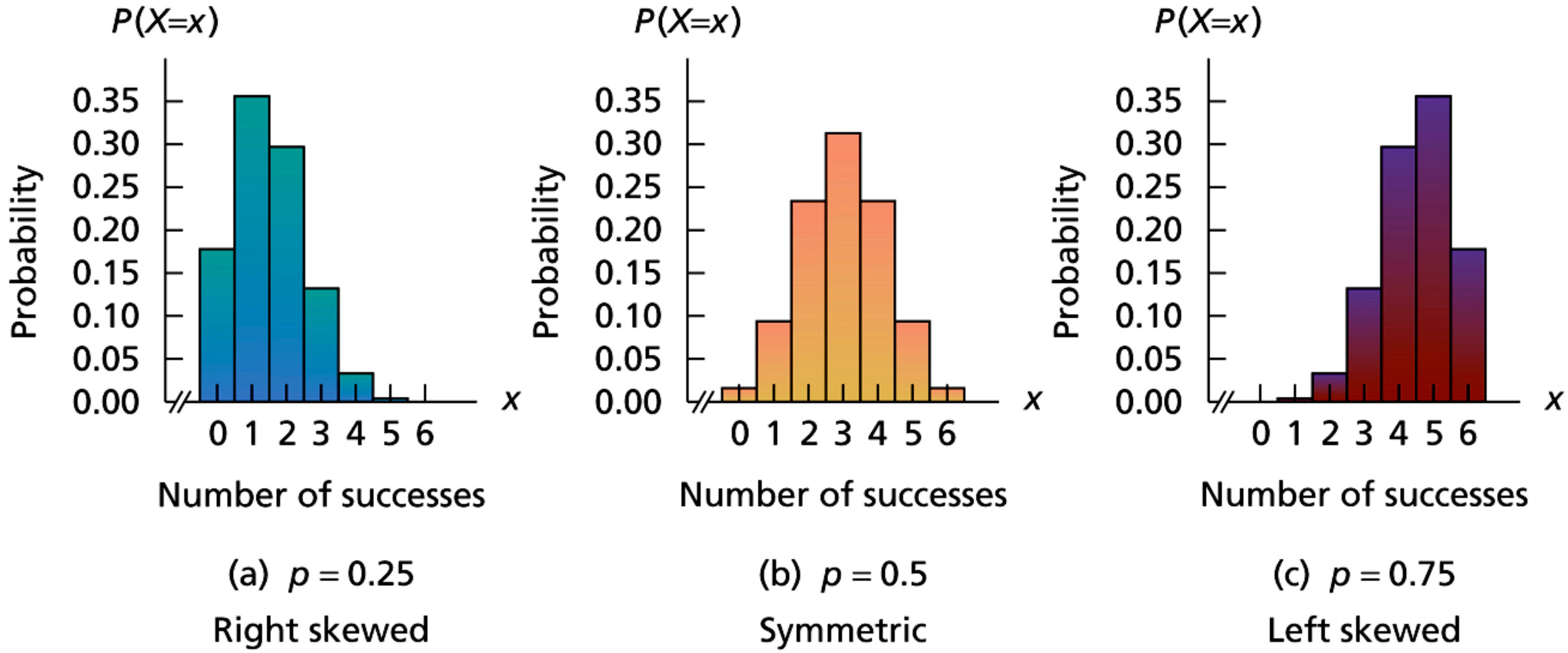
Interpretation Chances are 99.2% that one or more of the three people will be alive at age 65.

d. To obtain the probability distribution of the random variable X, we need to use the binomial probability formula to compute $P(X = x)$ for $x = 0, 1, 2,$ and 3. We have already done so for $x = 0, 1,$ and 2 in parts (a) and (b). For $x = 3$, we have

$$P(X = 3) = \binom{3}{3}(0.8)^3(0.2)^{3-3} = (0.8)^3 = 0.512.$$

Figure 5.6

Probability histograms for three different binomial distributions with parameter n = 6

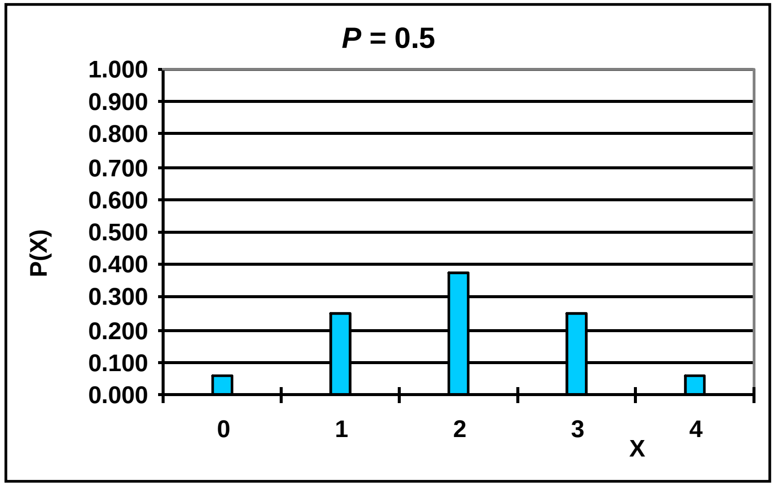


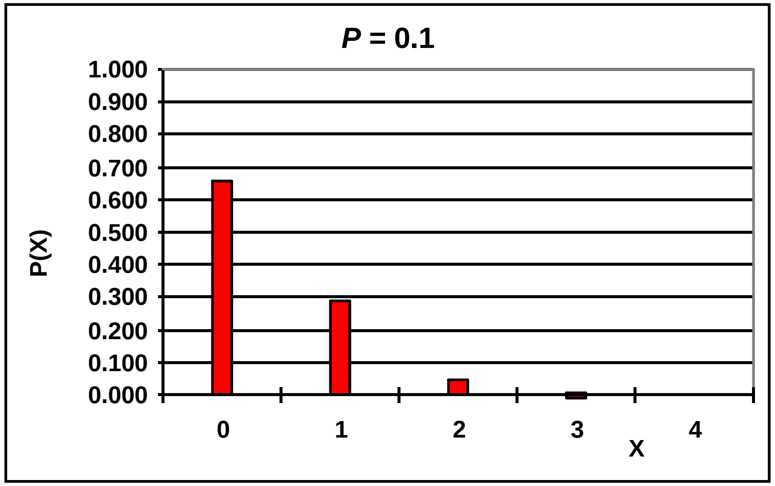
Generally, a binomial distribution is right skewed if p <0.5, is symmetric if p =0.5, and is left skewed if p >0.5.

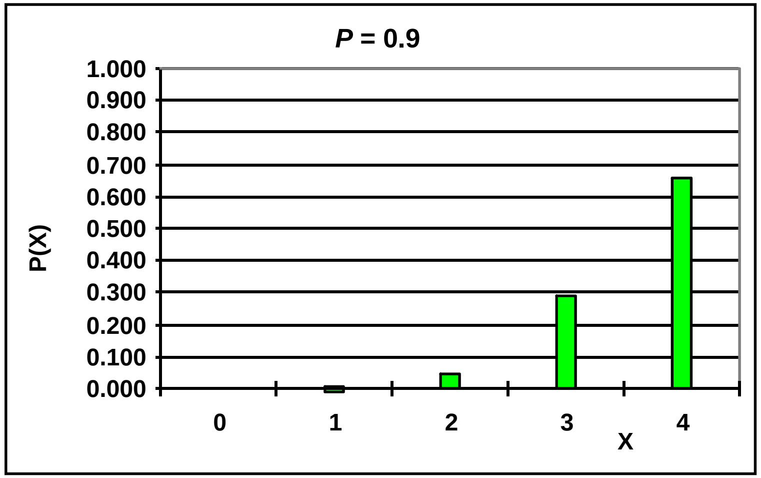
Graphs of Selected Binomial Distributions

n = 4 PROBABILITY

X	0.1	0.5	0.9
0	0.656	0.063	0.000
1	0.292	0.250	0.004
2	0.049	0.375	0.049
3	0.004	0.250	0.292
4	0.000	0.063	0.656







Formula 5.2

Mean and Standard Deviation of a Binomial Random Variable

The mean and standard deviation of a binomial random variable with parameters n and p are

$$\mu = np \quad \text{and} \quad \sigma = \sqrt{np(1-p)},$$

respectively.

EXAMPLE 5.13 Mean and Standard Deviation of a Binomial Random Variable

Mortality: For three randomly selected 20-year-olds, let X denote the number who are still alive at age 65. Find the mean and standard deviation of X.

EXAMPLE 5.13

Solution

As we stated in the previous example, X is a binomial random variable with parameters n = 3 and p = 0.8. Applying Formula 5.2 gives

$$\mu = np = 3 \cdot 0.8 = 2.4$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{3 \cdot 0.8 \cdot 0.2} = 0.69.$$

Interpretation: On average, 2.4 of every three 20-year-olds will still be alive at age 65. And, roughly speaking, on average, the number out of three given 20-year-olds who will still be alive at age 65 will differ from the mean number of 2.4 by 0.69.

Binomial Distribution

- Experiment involves n identical trials
- Each trial has exactly two possible outcomes: success and failure
- Each trial is independent of the previous trials

p is the probability of a success on any one trial

$q = (1-p)$ is the probability of a failure on any one trial

p and q are constant throughout the experiment

X is the number of successes in the n trials

Binomial Distribution

- Probability function

$$\boxed{P(X) = \frac{n!}{X!(n-X)!} p^X \cdot q^{n-X} \quad \text{for} \quad 0 \le X \le n}$$

- Mean value

$$\boxed{\mu = n \cdot p}$$

- Variance and standard deviation

$$\boxed{\begin{aligned} \sigma^2 &= n \cdot p \cdot q \\ \sigma &= \sqrt{\sigma^2} = \sqrt{n \cdot p \cdot q} \end{aligned}}$$

Section 5.4

The Poisson Distribution

Poisson Distribution

- Describes discrete occurrences over a continuum or interval
- A discrete distribution
- Describes rare events
- Each occurrence is independent any other occurrences.
- The number of occurrences in each interval can vary from zero to infinity.
- The expected number of occurrences must hold constant throughout the experiment.

Poisson Probability Distribution

- The binomial distribution becomes more skewed to the right (positive) as the probability of success become smaller.

- The limiting form of the binomial distribution where the probability of success p is small and *n* is large is called the Poisson probability distribution.

Example of Bernoulli Trials

The Poisson distribution is often used to model the frequency with which a specified event occurs during a particular period of time. For instance, we might apply the Poisson distribution when analyzing

1. the number of patients who arrive at an emergency room between 6:00 P.M. and 7:00 P.M.

2. the number of telephone calls received per day at a switchboard, or

3. the number of alpha particles emitted per minute by a radioactive substance.

Poisson Distribution: Applications

- Arrivals at queuing systems
  - airports -- people, airplanes, automobiles, baggage
  - banks -- people, automobiles, loan applications
  - computer file servers -- read and write operations

- Defects in manufactured goods
  - number of defects per 1,000 feet of extruded copper wire
  - number of blemishes per square foot of painted surface
  - number of errors per typed page

Formula 5.3

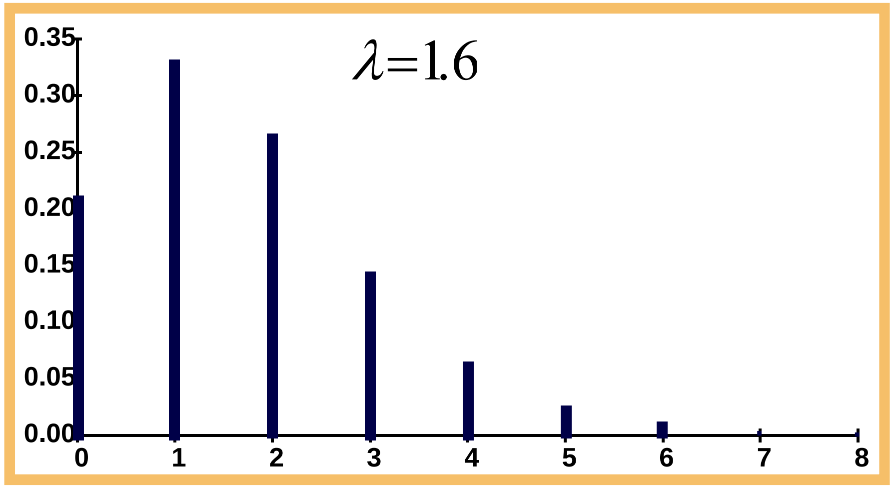
Poisson Probability Formula

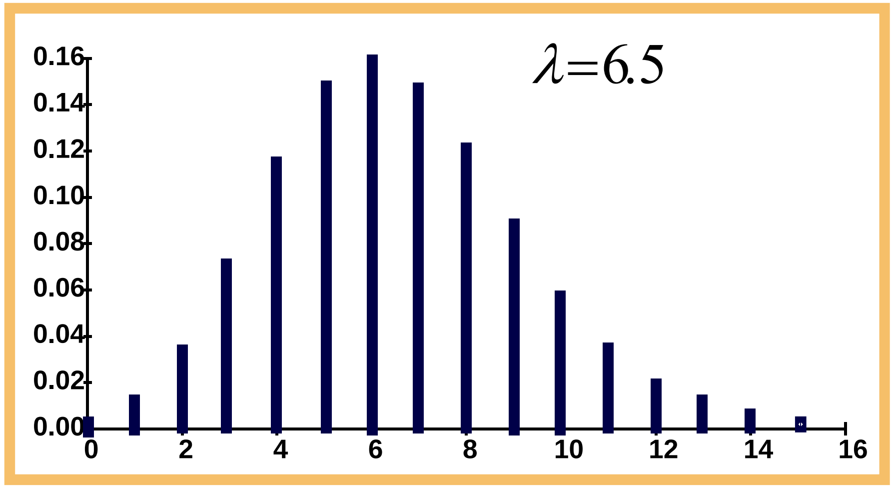
Probabilities for a random variable X that has a Poisson distribution are given by the formula

$$P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}, \qquad x = 0, 1, 2, \ldots,$$

where λ is a positive real number and $e \approx 2.718$. (Most calculators have an e key.) The random variable X is called a **Poisson random variable** and is said to have the **Poisson distribution** with parameter λ.

Poisson Distribution: Graphs





Poisson Distribution

- Probability function

$$P(X) = \frac{\lambda^{X} e^{-\lambda}}{X!} \quad \text{for} \quad X = 0, 1, 2, 3, \ldots$$

$where:$

$\lambda = long-run\ average$

$e = 2.718282\ldots \quad$ (the base of natural logarithms)

- **Mean value** $\boxed{\lambda}$
- **Variance** $\boxed{\lambda}$
- **Standard deviation** $\boxed{\sqrt{\lambda}}$

EXAMPLE 5.15 The Poisson Distribution

Emergency Room Traffic Desert Samaritan Hospital keeps records of emergency room (ER) traffic. Those records indicate that the number of patients arriving between 6:00 P.M. and 7:00 P.M. has a Poisson distribution with parameter $\lambda = 6.9$. Determine the probability that, on a given day, the number of patients who arrive at the emergency room between 6:00 P.M. and 7:00 P.M. will be:

EXAMPLE 5.15 The Poisson Distribution

a. exactly 4.

b. at most 2.

c. between 4 and 10, inclusive.

d. Obtain a table of probabilities for the random variable X, the number of patients arriving between 6:00 P.M. and 7:00 P.M. Stop when the probabilities become zero to three decimal places.

e. Use part (d) to construct a (partial) probability histogram for X.

f. Identify the shape of the probability distribution of X.

EXAMPLE 5.15

Solution

Let the random variable X be the number of patients arriving between 6:00 P.M. and 7:00 P.M, and it has a Poisson distribution with parameter $\lambda = 6.9$. Thus, by Formula 5.3, the probabilities for X are given by the Poisson probability formula,

$$P(X = x) = e^{-6.9} \frac{(6.9)^x}{x!}.$$

EXAMPLE 5.15

Solution

Using this formula, we can now solve parts (a)–(f).

a. Applying the Poisson probability formula with $x = 4$ gives

$$P(X = 4) = e^{-6.9}\frac{(6.9)^4}{4!} = e^{-6.9} \cdot \frac{2266.7121}{24} = 0.095.$$

Interpretation Chances are 9.5% that exactly 4 patients will arrive at the ER between 6:00 P.M. and 7:00 P.M.

EXAMPLE 5.15

Solution

b. The probability of at most 2 arrivals is

$$\begin{aligned} P(X \le 2) &= P(X = 0) + P(X = 1) + P(X = 2) \\ &= e^{-6.9}\frac{(6.9)^0}{0!} + e^{-6.9}\frac{(6.9)^1}{1!} + e^{-6.9}\frac{(6.9)^2}{2!} \\ &= e^{-6.9}\left(\frac{6.9^0}{0!} + \frac{6.9^1}{1!} + \frac{6.9^2}{2!}\right) \\ &= e^{-6.9}(1 + 6.9 + 23.805) = e^{-6.9} \cdot 31.705 = 0.032. \end{aligned}$$

Interpretation Chances are only 3.2% that 2 or fewer patients will arrive at the ER between 6:00 P.M. and 7:00 P.M.

EXAMPLE 5.15

Solution

c. The probability of between 4 and 10 arrivals, inclusive, is

$$P(4 \leq X \leq 10) = P(X = 4) + P(X = 5) + \cdots + P(X = 10)$$

$$= e^{-6.9}\left(\frac{6.9^4}{4!} + \frac{6.9^5}{5!} + \cdots + \frac{6.9^{10}}{10!}\right) = 0.821.$$

Interpretation Chances are 82.1% that between 4 and 10 patients, inclusive, will arrive at the ER between 6:00 P.M. and 7:00 P.M.

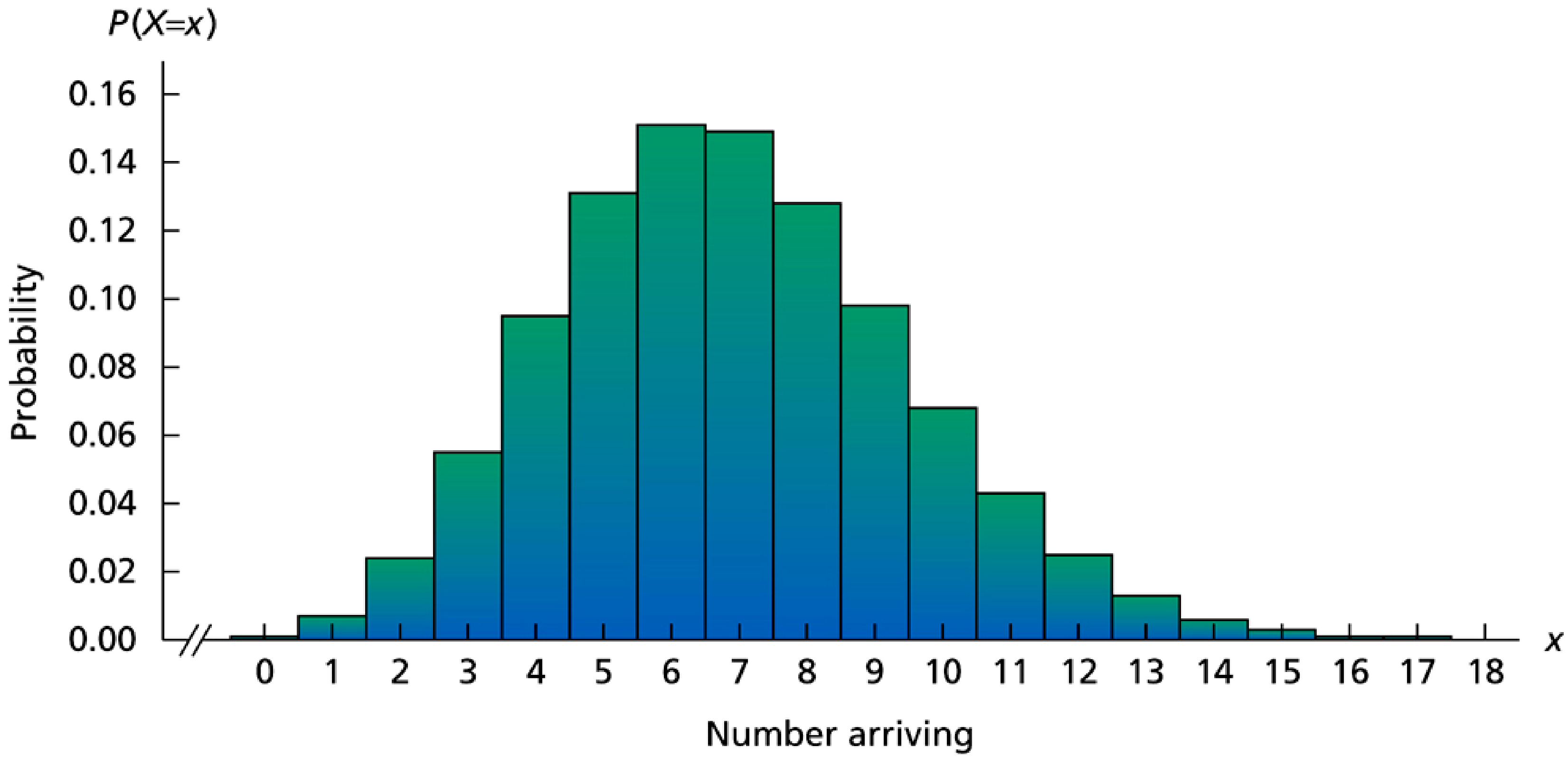
Table 5.16

Partial probability distribution of the random variable X, the number of patients arriving at the emergency room between 6:00 P.M. and 7:00 P.M.

Number arriving x	Probability $P(X = x)$	Number arriving x	Probability $P(X = x)$
0	0.001	10	0.068
1	0.007	11	0.043
2	0.024	12	0.025
3	0.055	13	0.013
4	0.095	14	0.006
5	0.131	15	0.003
6	0.151	16	0.001
7	0.149	17	0.001
8	0.128	18	0.000
9	0.098		

Figure 5.7

Partial probability histogram for the random variable X, the number of patients arriving at the emergency room between 6:00 P.M. and 7:00 P.M.



Procedure 5.2

To Approximate Binomial Probabilities by Using a Poisson Probability Formula

Step 1 **Find n, the number of trials, and p, the success probability.**

Step 2 **Continue only if $n \geq 100$ and $np \leq 10$.**

Step 3 **Approximate the binomial probabilities by using the Poisson probability formula**

$$P(X = x) = e^{-np}\frac{(np)^x}{x!}.$$

Poisson Approximation of the Binomial Distribution

- Binomial probabilities are difficult to calculate when n is large.
- Under certain conditions binomial probabilities may be approximated by Poisson probabilities.

$$\boxed{\text{If } n > 20 \text{ and } n \cdot p \le 7, \text{ the approximation is acceptable.}}$$

- Poisson approximation

$$\boxed{\text{Use } \lambda = n \cdot p.}$$

Poisson Approximation of the Binomial Distribution

X	Poisson $\lambda = 1.5$	Binomial $n = 50$ $p = .03$	Error
0	0.2231	0.2181	-0.0051
1	0.3347	0.3372	0.0025
2	0.2510	0.2555	0.0045
3	0.1255	0.1264	0.0009
4	0.0471	0.0459	-0.0011
5	0.0141	0.0131	-0.0010
6	0.0035	0.0030	-0.0005
7	0.0008	0.0006	-0.0002
8	0.0001	0.0001	0.0000
9	0.0000	0.0000	0.0000

X	Poisson $\lambda = 3.0$	Binomial $n = 10,000$ $p = .0003$	Error
0	0.0498	0.0498	0.0000
1	0.1494	0.1493	0.0000
2	0.2240	0.2241	0.0000
3	0.2240	0.2241	0.0000
4	0.1680	0.1681	0.0000
5	0.1008	0.1008	0.0000
6	0.0504	0.0504	0.0000
7	0.0216	0.0216	0.0000
8	0.0081	0.0081	0.0000
9	0.0027	0.0027	0.0000
10	0.0008	0.0008	0.0000
11	0.0002	0.0002	0.0000
12	0.0001	0.0001	0.0000
13	0.0000	0.0000	0.0000