Chapter 12

Analysis of Variance

https://www.youtube.com/watch?v=-yQb_ZJnFXw
https://www.youtube.com/watch?v=lTf4vHhyGpc

Chapter 12 Overview

Introduction

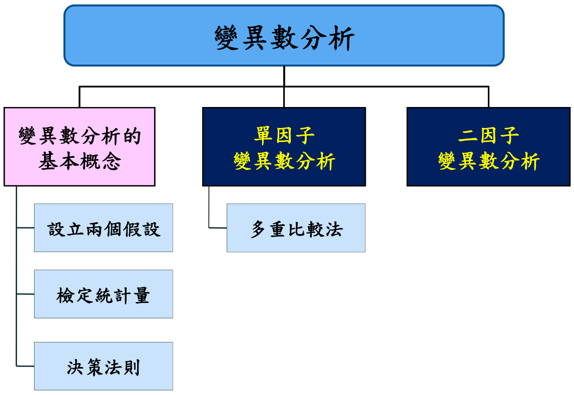
◆12-1 One-Way Analysis of Variance

◆12-2 The Scheffé Test and the Tukey Test

◆12-3 Two-Way Analysis of Variance

Chapter 12 Objectives

1. Use the one-way ANOVA technique to determine if there is a significant difference among three or more means.
2. Determine which means differ, using the Scheffé or Tukey test if the null hypothesis is rejected in the ANOVA.
3. Use the two-way ANOVA technique to determine if there is a significant difference in the main effects or interaction.



Section 12-2

The Scheffé Test and the Tukey Test

12-2 The Scheffé Test and the Tukey Test

◆When the null hypothesis is rejected using the *F* test, the researcher may want to know where the difference among the means is.

◆The **Scheffé test** and the **Tukey test** are procedures to determine where the significant differences in the means lie after the ANOVA procedure has been performed.

The Scheffé Test

- In order to conduct the **Scheffé test**, one must compare the two means at a time, using all possible combinations of means.
- For example, if there are three means, the following comparisons must be done:

$\bar{X}_1$ versus $\bar{X}_2$ $\quad$ $\bar{X}_1$ versus $\bar{X}_3$ $\quad$ $\bar{X}_2$ versus $\bar{X}_3$

Formula for the Scheffé Test

$$F_S = \frac{\left(\bar{X}_i - \bar{X}_j\right)^2}{\underbrace{s_W^2}_{MSW}\left[\left(1/n_i\right)+\left(1/n_j\right)\right]}$$

where $\bar{X}_i$ and $\bar{X}_j$ are the means of the samples being compared, n_i and n_j are the respective sample sizes, and the within-group variance is s_W^2 or MSW.

F Value for the Scheffé Test

- To find the **critical value F' for the Scheffé test**, multiply the critical value for the F test by $k - 1$:

$$F' = (k-1)(\text{C.V.})$$

- There is a significant difference between the two means being compared when F_s is greater than F'.

Example: Lowering Blood Pressure

A researcher wishes to try three different techniques to lower the blood pressure of individuals diagnosed with high blood pressure. The subjects are randomly assigned to three groups; the first group takes medication, the second group exercises, and the third group follows a special diet. After four weeks, the reduction in each person's blood pressure is recorded. At α = 0.05, test the claim that there is no difference among the means.

Medication	Exercise	Diet
10	6	5
12	8	9
9	3	12
15	0	8
13	2	4
$\bar{X}_1$ = 11.8	$\bar{X}_2$ = 3.8	$\bar{X}_3$ = 7.6
s_1^2 = 5.7	s_2^2 = 10.2	s_3^2 = 10.3

Example: Lowering Blood Pressure

Medication	Exercise	Diet
10	6	5
12	8	9
9	3	12
15	0	8
13	2	4
$\bar{X}_1$ = 11.8	$\bar{X}_2$ = 3.8	$\bar{X}_3$ = 7.6
s_1^2 = 5.7	s_2^2 = 10.2	s_3^2 = 10.3

Step 1: State the hypotheses and identify the claim.

H_0: $\mu_1 = \mu_2 = \mu_3$ (claim)

H_1: At least one mean is different from the others.

Example: Lowering Blood Pressure

Medication	Exercise	Diet
10	6	5
12	8	9
9	3	12
15	0	8
13	2	4
$\bar{X}_1$ = 11.8	$\bar{X}_2$ = 3.8	$\bar{X}_3$ = 7.6
s_1^2 = 5.7	s_2^2 = 10.2	s_3^2 = 10.3

Step 2: Find the critical value.

Since k = 3, n = 15, and α = 0.05,

d.f.N. = k – 1 = 3 – 1 = 2

d.f.D. = n – k = 15 – 3 = 12

The critical value is 3.89, obtained from Table H.

Example: Lowering Blood Pressure

Step 3: Compute the test value.

a. Find the mean and variance of each sample (these were provided with the data).

b. Find the **grand mean**, the mean of all values in the samples.

$$\bar{X}_{GM} = \frac{\sum X}{N} = \frac{10+12+9+\dots+4}{15} = \frac{116}{15} = 7.73$$

c. Find the **between-group variance**, MSB or s_B^2 .

$$MSB = S_B^2 = \frac{SSB}{k-1} = \frac{\sum n_i\left(\bar{X}_i - \bar{X}_{GM}\right)^2}{k-1}$$

Example: Lowering Blood Pressure

Step 3: Compute the test value. (continued)

c. Find the **between-group variance**, MSB or s_B^2.

$$MSB = s_B^2 = \frac{5(11.8-7.73)^2+5(3.8-7.73)^2+5(7.6-7.73)^2}{3-1}$$

$$= \frac{160.13}{2} = 80.07$$

d. Find the **within-group variance**, MSW or s_W^2 .

$$MSW = s_W^2 = \frac{\sum (n_i-1)s_i^2}{\sum (n_i-1)}$$

$$= \frac{4(5.7)+4(10.2)+4(10.3)}{4+4+4} = \frac{104.80}{12} = 8.73$$

Example: Lowering Blood Pressure

Step 3: Compute the test value. (continued)

e. Compute the F value.

$$F = \frac{MSB}{MSW} = \frac{S_B^2}{S_W^2} = \frac{80.07}{8.73} = 9.17$$

Step 4: Make the decision.

Reject the null hypothesis, since 9.17 > 3.89.

Step 5: Summarize the results.

There is enough evidence to reject the claim and conclude that at least one mean is different from the others.

Example: Lowering Blood Pressure

◆The ANOVA table:

Source	Sum of Squares	d.f.	Mean Squares	*F*
Between	160.13	2	80.07	9.17
Within (error)	104.80	12	8.73	
Total	264.93	14		

CV: F(2,12) = 3.89

Example: Lowering Blood Pressure

Using the Scheffé test, test each pair of means in last example (Lowering Blood Pressure) to see whether a specific difference exists, at α = 0.05.

a. For $\bar{X}_1$ versus $\bar{X}_2$,

$$F_S = \frac{(\bar{X}_1-\bar{X}_2)^2}{s_W^2\left[(1/n_1)+(1/n_2)\right]} = \frac{(11.8-3.8)^2}{8.73\left[(1/5)+(1/5)\right]} = 18.33$$

b. For $\bar{X}_2$ versus $\bar{X}_3$,

$$F_S = \frac{(\bar{X}_2-\bar{X}_3)^2}{s_W^2\left[(1/n_2)+(1/n_3)\right]} = \frac{(3.8-7.6)^2}{8.73\left[(1/5)+(1/5)\right]} = 4.14$$

Example: Lowering Blood Pressure

Using the Scheffé test, test each pair of means in last example (Lowering Blood Pressure) to see whether a specific difference exists, at α = 0.05.

c. For $\bar{X}_1$ versus $\bar{X}_3$,

$$F_S = \frac{(\bar{X}_1-\bar{X}_3)^2}{s_W^2\left[(1/n_1)+(1/n_3)\right]} = \frac{(11.8-7.6)^2}{8.73\left[(1/5)+(1/5)\right]} = 5.05$$

Example: Lowering Blood Pressure

- The critical value for the ANOVA for the example: $\alpha = 0.05$, d.f.N. = 2, and d.f.D. = 12

$$F = 3.89$$

- In this case, the critical value is multiplied by $k - 1$:

$$F' = (k-1)(C.V.) = 2 \times 3.89 = 7.78$$

- Since only the F test value for part a ($\overline{X}_1$ versus $\overline{X}_2$) is greater than the critical value, 7.78, the only significant difference is between $\overline{X}_1$ and $\overline{X}_2$, that is, between medication and exercise.

One-Way ANOVA: Medication, Exercise, Diet

```
Source  DF      SS     MS     F      P
Factor   2  160.13  80.07  9.17  0.004
Error   12  104.80   8.73
Total   14  264.93

                                 Individual 95% CIs For Mean
                                 Based on Pooled StDev
Level        N    Mean  StDev  --------+---------+---------+---------+--
Medication   5  11.800  2.387                        (--------*--------)
Exercise     5   3.800  3.194  (--------*--------)
Diet         5   7.600  3.209              (--------*--------)
                               --------+---------+---------+---------+--
                                     3.5       7.0      10.5      14.0
Pooled StDev = 2.955
```

Reject the null hypothesis. There is enough evidence to conclude that there is a difference between the treatments.

SPSS報表

ANOVA

Reduction

	平方和	自由度	平均平方和	F	顯著性
組間	160.133	2	80.067	9.168	.004
組內	104.800	12	8.733		
總和	264.933	14			

Scheffe 法[a]

Method	個數	alpha = 0.05 的子集 1	2
Exercise	5	3.80	
Diet	5	7.60	7.60
Medication	5		11.80
顯著性		.169	.122

顯示的是同質子集中組別的平均數。

a. 使用調和平均數樣本大小 = 5.000。

An Additional Note

- On occasion, when the *F* test value is greater than the critical value, the Scheffé test may not show any significant differences in the pairs of means. This result occurs because the difference may actually lie in the average of two or more means when compared with the other mean. The Scheffé test can be used to make these types of comparisons, but the technique is beyond the scope of this book.

The Tukey Test

- The **Tukey test** can also be used after the analysis of variance has been completed to make pairwise comparisons between means when the groups have **the same sample size**.
- The symbol for the test value in the Tukey test is *q*.

Formula for the Tukey Test

$$q = \frac{\overline{X}_i - \overline{X}_j}{\sqrt{s_W^2 / n}}$$

Where $\overline{X}_i$ and $\overline{X}_j$ are the means of the samples being compared, *n* is the size of the sample, and the within-group variance is s_W^2 or *MSW*.

Example: Lowering Blood Pressure

Using the Tukey test, test each pair of means in the example of Lowering Blood Pressure to see whether a specific difference exists, at $\alpha = 0.05$.

a. For $\overline{X}_1$ versus $\overline{X}_2$,

$$q = \frac{\overline{X}_1 - \overline{X}_2}{\sqrt{s_W^2/n}} = \frac{11.8 - 3.8}{\sqrt{8.73/5}} = 6.06$$

b. For $\overline{X}_1$ versus $\overline{X}_3$,

$$q = \frac{\overline{X}_1 - \overline{X}_3}{\sqrt{s_W^2/n}} = \frac{11.8 - 7.6}{\sqrt{8.73/5}} = 3.18$$

Example: Lowering Blood Pressure

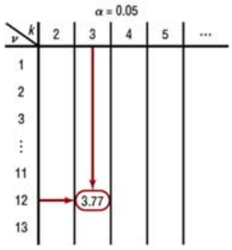
Using the Tukey test, test each pair of means in the example of Lowering Blood Pressure to see whether a specific difference exists, at $\alpha = 0.05$.

c. For $\overline{X}_2$ versus $\overline{X}_3$,

$$q = \frac{\overline{X}_2 - \overline{X}_3}{\sqrt{s_W^2/n}} = \frac{3.8 - 7.6}{\sqrt{8.73/5}} = -2.88$$

Example: Lowering Blood Pressure

- To find the critical value for the Tukey test, use **Table N**.
- The number of means *k* is found in the row at the top, and the degrees of freedom for are found in the left column (denoted by *v).*
- Since *k* = 3, d.f. = 12, and α = 0.05, the C.V. is 3.77.



Example: Lowering Blood Pressure

- Hence, the only *q* value that is greater in **absolute value** than the critical value is the one for the difference between $\overline{X}_1$ and $\overline{X}_2$. The conclusion, then, is that there is a significant difference in means for medication and exercise.
- These results agree with the Scheffé analysis.

One-Way ANOVA: Medication, Exercise, Diet

```
Source   DF      SS     MS     F      P
Factor    2  160.13  80.07  9.17  0.004
Error    12  104.80   8.73
Total    14  264.93

                               Individual 95% CIs For Mean
                               Based on Pooled StDev
Level        N    Mean  StDev  -------+---------+---------+---------+--
Medication   5  11.800  2.387                      (--------*-------)
Exercise     5   3.800  3.194  (-------*-------)
Diet         5   7.600  3.209            (-------*-------)
                               -------+---------+---------+---------+--
                                    3.5       7.0      10.5      14.0
Pooled StDev = 2.955
```

Reject the null hypothesis. There is enough evidence to conclude that there is a difference between the treatments.

A researcher wishes to see whether there is any difference in the weight gains of athletes following one of three special diets. Athletes are randomly assigned to three groups and placed on the diet for 6 weeks. The weight gains (in pounds) are shown here. At a 0.05, can the researcher conclude that there is a difference in the diets?
If the difference is significant, test each pair of means to see whether a specific difference exists, at α = 0.05.

Diet A	Diet B	Diet C
3	10	8
6	12	3
7	11	2
4	14	5
	8	
	6	

H_0: $\mu_1 = \mu_2 = \mu_3$

H_1: At least one mean is different from the others. (claim)

$$\bar{X}_{GM} = \frac{\sum X}{n} = \frac{99}{14} = 7.07$$

$$s_B^2 = \frac{101.095}{2} = 50.548$$

$$s_W^2 = \frac{71.833}{11} = 6.530$$

$$F = \frac{s_B^2}{s_W^2} = \frac{50.548}{6.530} = 7.74$$

P-value = 0.00797

Reject since P-value < 0.05. There is enough evidence to support the claim that at least one mean is different from the others.

Section 12-3

Two-Way Analysis of Variance

12-3 Two-Way Analysis of Variance

- ◆二因子變異數分析
  - 兩個自變項的變異數分析
- ◆探討兩個分類性的解釋變數對依變數之間的關係
  - 分析促銷工具在不同時段，對銷貨收入之影響。
  - 分析不同性別之青年、中年、老年三個年齡層的族群，每日水分攝取量是否有差異。
  - 探討教室氣氛和教學方法對學生學習成就的影響
  - 分析不同工人與不同廠牌機器對產量的影響

Example: Two-Way NOVA

A research wishes to test the effects of two different types of plant food and two different types of soil on the growth of certain plants.

Two independent variables:

→ the type of plant food

→ the type of soil

One dependent variable:

→ The plant growth

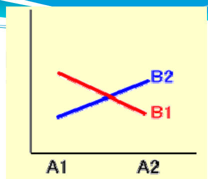
Plant food	Soil type I	Soil type II
A1	Group 1	Group 2
A2	Group 3	Group 4

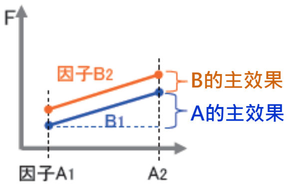
Example: Two-Way NOVA

Questions:

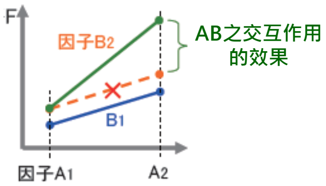
1. Whether there is a difference in means of heights of plants growing **using different soil**?
2. Whether there is a difference in means of heights of plants growing **using different foods**?
3. Whether there is an **interaction effect** between type of plant food used and type of soil used on plant growth?

Plant food	Soil type I	Soil type II
A1	Group 1	Group 2
A2	Group 3	Group 4



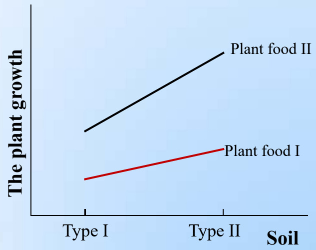


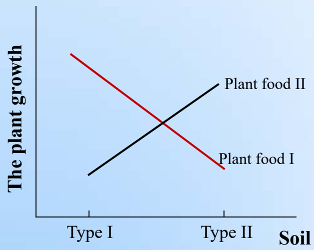
【不存在交互作用】



【存在交互作用】

Example: Two-Way NOVA





12-3 Two-Way Analysis of Variance

- The purposes of the two-way ANAVA
  - To test the effects of two independent variables or factors on one dependent variable.
  - To test the interaction effect of the two variables
- Variables
  - Two independent variables– categorical
  - One dependent variable– continuous

Two-Way Analysis of Variance

- Variables or factors are changed between two levels (i.e., two different treatments).
- The groups for a two-way ANOVA are sometimes called treatment groups.
- A two-way ANOVA has several null hypotheses. There is one for each independent variable and one for the interaction.

Assumptions for Two-Way ANOVA

1. The populations from which the samples were obtained must be normally or approximately normally distributed.
2. The samples must be independent.
3. The variances of the populations from which the samples were selected must be equal.
4. The groups must be equal in sample size.

Two-Way ANOVA Summary Table

Source	Sum of Squares	d.f.	Mean Squares	*F*
Factor A	SS_A	a-1	MS_A	F_A
Factor B	SS_B	b-1	MS_B	F_B
Interaction A X B	SS_{AxB}	$(a-1)(b-1)$	MS_{AxB}	F_{AxB}
Within (error)	SS_W	$ab(n-1)$	MS_W	
Total	SST	n-1		

Two-way ANOVA

- What are factors and factor levels?
- What are the hypothesizes?
- What is the critical values for the hypothesizes?
- Finish the ANOVA table and make a conclusion.

Source	SS	d.f.	MS	F
Factor A	11.61	$\boxed{}$	5.80	$\boxed{}$
Factor B	5.82	2	2.91	
Interaction A × B	$\boxed{}$	4	$\boxed{}$	
Within (error)	67.50	$\boxed{}$		
Total	110.55	53		

Example 12-5: Gasoline Consumption

A researcher wishes to see whether the type of gasoline used and the type of automobile driven have any effect on gasoline consumption. Two types of gasoline, regular and high-octane, will be used, and two types of automobiles, two-wheel- and four-wheel-drive, will be used in each group. There will be two automobiles in each group, for a total of eight automobiles used. Use a two-way analysis of variance at $\alpha = 0.05$.

	Type of automobile	
Gas	Two-wheel-drive	Four-wheel-drive
Regular	26.7 25.2	28.6 29.3
High-octane	32.3 32.8	26.1 24.2

Example 12-5: Gasoline Consumption

A researcher wishes to see whether the type of gasoline used and the type of automobile driven have any effect on gasoline consumption. Two types of gasoline, regular and high-octane, will be used, and two types of automobiles, two-wheel- and four-wheel-drive, will be used in each group.

Research question:
Is there difference in ***means of gasoline consumption*** for differing type of gasoline and type of automobile?

Two factors:
Factor A: **type of gasoline**—
two levels: regular and high-octane
Factor B: **type of automobile**—
two levels: two-wheel- and four-wheel-drive

	Type of automobile	
Gas	Two-wheel-drive	Four-wheel-drive
Regular	26.7 25.2	28.6 29.3
High-octane	32.3 32.8	26.1 24.2

Example 12-5: Gasoline Consumption

Step 1: State the hypotheses.
The hypotheses for the gasoline types are

H_0: There is no difference between the means of gasoline consumption for two types of gasoline.

H_1: There is a difference between the means of gasoline consumption for two types of gasoline.

Example 12-5: Gasoline Consumption

Step 1: State the hypotheses.
The hypotheses for the types of automobile driven are

H_0: There is no difference between the means of gasoline consumption for two-wheel-drive and four-wheel-drive automobiles.

H_1: There is a difference between the means of gasoline consumption for two-wheel-drive and four-wheel-drive automobiles.

Example 12-5: Gasoline Consumption

Step 1: State the hypotheses.
The hypotheses for the interaction are these:

H_0: There is no interaction effect between type of gasoline used and type of automobile a person drives on gasoline consumption.

H_1: There is an interaction effect between type of gasoline used and type of automobile a person drives on gasoline consumption.

Example 12-5: Gasoline Consumption

Step 2: Find the critical value for each.

Since $\alpha = 0.05$, d.f.N. = 1, and d.f.D. = 4 for each of the factors, the critical values are the same, obtained from Table H as $\text{C.V.} = 7.71$

Step 3: Find the test values.

SSA, SSB, SSAB, SSW & SST.

Example 12-5: Gasoline Consumption

◆Step 3: Find the test values.

Gas	Type of automobile: Two-wheel-drive	Four-wheel-drive
Regular	26.7 25.2	28.6 29.3
High-octane	32.3 32.8	26.1 24.2

Two-Way ANOVA Summary Table

Source	Sum of Squares	d.f.	Mean Squares	*F*
Factor A	SS_A	a-1	MS_A	F_A
Factor B	SS_B	b-1	MS_B	F_B
Interaction A X B	SS_{AxB}	$(a-1)(b-1)$	MS_{AxB}	F_{AxB}
Within (error)	SS_W	$ab(n-1)$	MS_W	
Total	SST	n-1		

Example 12-5: Gasoline Consumption

Two-Way ANOVA Summary Table

Source	Sum of Squares	d.f.	Mean Squares	*F*
Gasoline A	3.920	1	3.920	4.752
Automobile B	9.680	1	9.680	11.733
Interaction A X B	54.080	1	54.080	65.552
Within (error)	3.300	4	0.825	
Total	70.890	7		

Example 12-5: Gasoline Consumption

Step 4: Make the decision.

Since $F_B = 11.733$ and $F_{AXB} = 65.552$ are greater than the critical value 7.71, the null hypotheses concerning the type of automobile driven and the interaction effect should be rejected.

Step 5: Summarize the results.

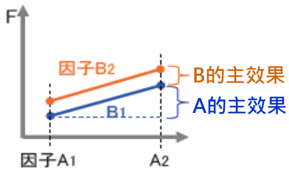
Since the null hypothesis for the interaction effect was rejected, it can be concluded that the combination of type of gasoline and type of automobile does affect gasoline consumption.

Main effect

In the preceding analysis, the effect of the type of gasoline used and the effect of the type of automobile driven are called the *main effects*. If there is no significant interaction effect, the main effects can be interpreted independently. However, if there is a significant interaction effect, the main effects must be interpreted cautiously, if at all.

To interpret the results of a two-way ANOVA, researchers suggest drawing a graph, plotting the means of each group, and interpreting the result.

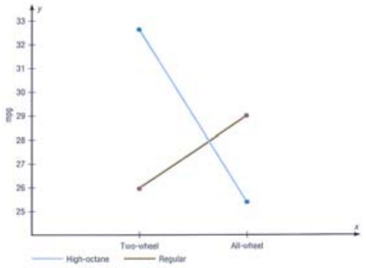
When there is no significant interaction effect, the lines in the graph will be parallel or approximately parallel. When this situation occurs, the main effects can be interpreted independently of each other because there is no interaction.



【不存在交互作用】

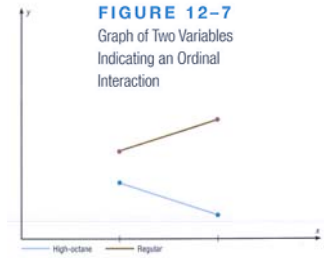
The graph of the means for each of the variables is shown in Figure 12–6. In this graph, the lines cross each other. When such an intersection occurs and the interaction is significant, the interaction is said to be a **disordinal interaction.** When there is a disordinal interaction, you should not interpret the main effects without considering the interaction effect.

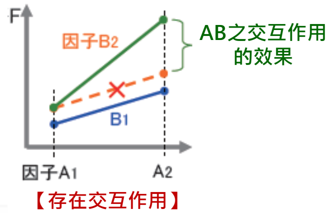
FIGURE 12–6
Graph of the Means of the Variables in Example 12–5



The other type of interaction that can occur is an *ordinal interaction.* Figure 12–7 shows a graph of means in which an ordinal interaction occurs between two variables. The lines do not cross each other, nor are they parallel. If the *F* test value for the interaction is significant and the lines do not cross each other, then the interaction is said to be an **ordinal interaction** and the main effects can be interpreted independently of each other.

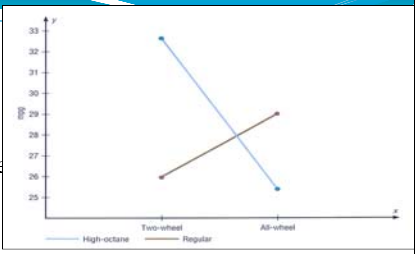
FIGURE 12–7
Graph of Two Variables Indicating an Ordinal Interaction





Main effect of

- Drawing means plot
  - Find the means for e in the chart.



Gas	Type of automobile	
	Two-wheel drive	four-wheel drive
Regular	$\overline{X}=\frac{26.7+25.2}{2}=25.95$	$\overline{X}=\frac{28.6+29.3}{2}=28.95$
High-	$\overline{X}=\frac{32.3+32.8}{2}=32.55$	$\overline{X}=\frac{26.1+24.2}{2}=25.15$

Exercise -1

- What are factors and factor levels?
- What are the hypothesizes?
- What is the critical values for the hypothesizes?
- Finish the ANOVA table and make a conclusion.

Source	SS	d.f.	MS	F
Factor A	5.64			
Factor B	11.28	2		
Interaction A x B		4		
Within (error)	35.10		1.3	
Total	60.45	35		

Exercise-2

- The manager of a grocery store wants to determine if there is a difference in the amount of money that the store brings in at different times of the day for different days of the week. The amount of money spent by five randomly selected customers is listed below.
- Analyze the data using a two-way ANOVA at $\alpha = 0.05$.
- Draw the main effect plot and interpret the result.

	Wednesday	Saturday
Morning	85	175
	120	112
	65	78
	95	130
	60	80
Afternoon	110	160
	150	150
	99	148
	120	152
	100	120
Evening	50	65
	53	85
	26	72
	48	83
	23	59

Exercise -1

- What are factors and factor levels?
- What are the hypothesizes?
- What is the critical values for the hypothesizes?
- Finish the ANOVA table and make a conclusion.

Source	SS	d.f.	MS	F
Factor A	5.64			
Factor B	11.28	2		
Interaction A x B		4		
Within (error)	35.10		1.3	
Total	60.45	35		

Source of Variation	SS	df	MS	F	P-value	F crit
Sample	28020.0667	2	14010.0333	26.3066283	8.9307E-07	3.40282611
Columns	7207.5	1	7207.5	13.5335169	0.00118127	4.25967721
Interaction	12.2	2	6.1	0.01145397	0.98861678	3.40282611
Within	12781.6	24	532.566667			
Total	48021.3667	29				

12. Home Building Times A contractor wishes to see whether there is a difference in the time (in days) it takes two subcontractors to build three different types of homes. At $\alpha = 0.05$, analyze the data shown here, using a two-way ANOVA. See below for raw data.

	Home type		
Subcontractor	**I**	**II**	**III**
A	25, 28, 26, 30, 31	30, 32, 35, 29, 31	43, 40, 42, 49, 48
B	15, 18, 22, 21, 17	21, 27, 18, 15, 19	23, 25, 24, 17, 13

ANOVA Summary Table for Exercise 12

Source	SS	d.f.	MS	F
Subcontractor	1672.553			
Home type	444.867			
Interaction	313.267			
Within	328.800			
Total	2759.487			