References

[1] Economic Capital Modeling, *EC Framework for Concentration Risk*, Economic Capital Program, 2011.

[2] Vasicek, O. *Loan portfolio value*, RISK, December 2002, 160 - 162.

[3] Gordy, M., Lutkebohmert, E., *Granularity adjustment for Basel II*, 2007.

[4] Economic Capital Modeling, *Credit EC - Corporate Correlation Methodology*, Internal Credit EC Correlation document, 2011.

[5] Economic Capital Modeling, *Credit EC - Retail Correlation Methodology*, Internal Credit EC Correlation document, 2011.

[6] Economic Capital Modeling, *Economic Capital Methodology - Loss Simulation*, Economic Capital Program, 2011.

[7] Economic Capital Program Workstream 4, *EC Data Sourcing, Limitations and Data Quality Issue Remediation*, Economic Capital Program, 29 July 2011.

[8] Economic Capital Modeling, *Economic Capital Methodology - Credit EC simulation engine data interface*, Economic Capital Program, 2011.

[9] Wilde, T. *Probing granularity*, Risk, pp. 103-106, 2001.

[10] Pykhtin, M. and Dev, A., *Analytical Approach to Credit Risk Modeling*, Risk, pp. S26-S32, 2002.

[11] Gordy, M. , *A Risk-Factor Model Foundation for Ratings-Based Bank Capital Rules*, Board of Governors of the Federal Reserve System, 2003.

[12] Gourieroux C. , Laurent J.P., Scaillet O., *Sensitivity analysis of Values at Risk.*, Journal of Empirical Finance 7, pp. 225-245, 2000.

[13] Martin, R. and Wilde, T. *Unsystematic credit risk*, Risk, pp. 123-128, 2002.

[14] Emmer, S., Tasche, D., *Calculating credit risk capital charges with the one-factor model*, 2005

[15] Hommels, J., Tchistiakov, V. *Name concentration correction*, Risk December 2010.

[16] Pykhtin, M., *Multi-factor adjustment*, Risk, pp 85-90, 2004.

[17] Xu, C., *Concentration Risk of the Fund's Credit Outstanding*, 2008.

[18] Basel Committee on Banking Supervision, *An Explanatory Note on the Basel II IRB Risk Weight Functions*, 2005.

[19] Economic Capital Modeling, *Phase 3 Credit Correlation Framework*, Internal Credit EC Correlation document, 2012.

C Appendix: Analyses of Gordy's Granularity Adjustment methodology

The model-based granularity adjustment approach was proposed by Wilde (2001) [9] and improved by Pykhtin and Dev (2002) [10] and Gordy (2003) [11]. Building on the work of Gourieroux et al. (2000) [12], Martin and Wilde (2002) [13] developed a general analytical form of granularity adjustment, which is derived from a second-order Taylor series expansion of loss quantile around its asymptotic value. Emmer and Tasche (2005) [14] extended this approach and induced a closed form of granularity adjustment through a single factor default-mode CreditMetrics model. Gordy and Lutkebohmert (2007)[3] provided another simplified form of granularity adjustment based on a single factor CreditRisk+ model.

The granularity adjustment is an extension of the ASFR model which forms the theoretical basis of the Internal Ratings-Based (IRB) approaches. Let X denote the systematic risk factor with the probability density function $h(X)$. Consider a portfolio consisting of N obligors indexed by $i = 1, 2, ..., N$. Denote the exposure at default of obligor i by EAD_i and let $s_i = \mathrm{EAD}_i / \sum_{i=1}^{N} \mathrm{EAD}_i$ be its share of the total portfolio exposure. The portfolio loss can be expressed as

$$L_N = \sum_{i=1}^{N} s_i D_i \mathrm{LGD}_i,$$

where D_i is a indicator variable which equals to 1 if obligor i defaults and 0 otherwise, and LGD_i is the loss given default of obligor i. Denote the qth percentile of the distribution of some random variable, say, Y by $\alpha_q(Y)$. The IRB formula estimates the qth quantile of the conditional expected loss $\alpha_q(\mathbb{E}[L_N|X])$. The difference

$$\alpha_q(L_N) - \alpha_q(\mathbb{E}[L_N|X]) \tag{10}$$

is the granularity adjustment for the effect of undiversified idiosyncratic risk in the portfolio. This interpretation is justified by the fact that $\alpha_q(\mathbb{E}[L_N|X])$ converges to $\alpha_q(L_N)$ as the portfolio becomes more and more fine-grained.

No analytical form can be strictly obtained for this difference. Based on the work of Gourieroux et al. (2000), Martin and Wilde (2002) developed a general analytical form for granularity adjustment (GA) through a second-order Taylor expansion of $\alpha_q(L_N)$

$$GA \approx -\frac{1}{2h(\alpha_q(X))} \frac{\partial}{\partial \alpha_q(X)} \left(\frac{\mathbb{V}[L_N|\alpha_q(X)] h(\alpha_q(X))}{\frac{\partial}{\partial \alpha_q(X)} \mathbb{E}[L_N|X]} \right) \tag{11}$$

Derivations of Eq.(11) is provided in Emmer and Tasche (2005)[14].

Note that $\mathbb{E}[L_N|\alpha_q(X)]$, $\mathbb{V}[L_N|\alpha_q(X)]$, and $h(\alpha_q(X))$ are model-dependent quantities. Different model settings or assumptions would deliver different expressions for granularity adjustment. Pykhtin & Dev (2002) and Emmer & Tasche (2005) developed an analytical closed-form of granularity adjustment through on a one-factor default-mode CreditMetrics model. Gordy and Lutkebohmert (2007) provided another simplified form of granularity adjustment based on a one-factor CreditRisk+ model. A comparison of two approached is conducted by Xu (2008). The author points out that the conditional probability of default in Gordy and Lutkebohmert's model has a positive relationship with the latent systematic risk factor X, whereas the conditional probability of default is decreasing with X in the ASRF model.

Implementation

In this section we briefly review Gordy and Lutkebohmert's approach for determining GA used by RBS for SNC add-on calculations.

In analogy to Gordy and Lutkebohmert (2007), we re-parameterize the GA formula (11). Conditional on $X = x$, define the conditional expectation and conditional variance of obligor i by

$$\mu_i(x) = \mathbb{E}[D_i \mathrm{LGD}_i | x] \tag{12}$$

$$\sigma_i^2(x) = \mathbb{V}[D_i \mathrm{LGD}_i | x] \tag{13}$$

and on portfolio level

$$\mu_N(x) = \mathbb{E}[L_N | x] = \sum_{i=1}^{N} s_i \mu_i(x) \tag{14}$$

$$\sigma_N^2(x) = \mathbb{V}[L_N | x] = \sum_{i=1}^{N} s_i^2 \sigma_i^2(x). \tag{15}$$

The GA can be rewritten as

$$GA \approx -\frac{1}{2h(\alpha_q(x))} \frac{\partial}{\partial \alpha_q(x)} \left(\frac{\sigma_N^2(\alpha_q(x)) h(\alpha_q(x))}{\mu_N'(\alpha_q(x))} \right). \tag{16}$$

Following the CreditRisk+ methodology, we assume LGD_i is a random variable and independent of D_i with expectation of ELGD_i and volatility (here standard deviation) of VLGD_i. Conditional on $X = x$, the conditional default probability of obligor i is

$$\mathbb{E}[D_i | x] = \mathrm{PD}_i(x) = \mathrm{PD}_i(1 - \beta_i + \beta_i x) \tag{17}$$

where PD_i is unconditional default probability, and β_i is a factor loading specifying the extent to which obligor i depends on the systematic factor X. Thus, we can rewrite (12) as $\mu_i(x) = \mathrm{ELGD}_i \mathrm{PD}_i(x)$ and thus

$$\mathrm{PD}_i(x) = \mu_i(x)/\mathrm{ELGD}_i. \tag{18}$$

For every obligor i let $\mathcal{R}_i$ be the expected loss (EL) reserve requirement and $\mathcal{K}_i$ the IRB capital charge as share of EAD_i. In the default-mode setting of CreditRisk+ these quantities can be expressed as

$$\mathcal{R}_i = \mathbb{E}[D_i \mathrm{LGD}_i] = \mathrm{ELGD}_i \mathrm{PD}_i \tag{19}$$

$$\mathcal{K}_i = \mathbb{E}[D_i \mathrm{LGD}_i | \alpha_q(x)] - \mathbb{E}[D_i \mathrm{LGD}_i] = \mathrm{ELGD}_i \mathrm{PD}_i \beta_i (\alpha_q(x) - 1) \tag{20}$$

Moreover, let $\mathcal{K}^* = \sum_{i=1}^{N} s_i \mathcal{K}_i$ denote the required capital per unit exposure for the portfolio as a whole. Recall (12) and (17), we obtain

$$\mu_i\left(\alpha_q(x)\right) = \mathcal{R}_i + \mathcal{K}_i \tag{21}$$

$$\mu_i'\left(\alpha_q(x)\right) = \frac{\partial}{\partial \alpha_q(x)} (\mathcal{R}_i + \mathcal{K}_i) = 0 + \mathrm{ELGD}_i \mathrm{PD}_i \beta_i = \mathcal{K}_i / \left(\alpha_q(x) - 1\right) \tag{22}$$

$$\mu_i''\left(\alpha_q(x)\right) = \frac{\partial}{\partial \alpha_q(x)} (\mathrm{ELGD}_i \mathrm{PD}_i \beta_i) = 0 \tag{23}$$

For the conditional variance, we have

$$\begin{aligned}
\sigma_i^2(x) &= \mathbb{E}[D_i^2 \mathrm{LGD}_i^2 | x] - \left(\mathbb{E}[D_i \mathrm{LGD}_i | x]\right)^2 \\
&= \mathbb{E}[\mathrm{LGD}_i^2 | x] \mathbb{E}[D_i^2 | x] - \left(\mathrm{PD}_i(x)\right)^2 \mathrm{ELGD}_i^2 \\
&= \mathbb{E}[\mathrm{LGD}_i^2 | x] \mathbb{E}[D_i^2 | x] - \left(\mu_i(x)\right)^2
\end{aligned} \tag{24}$$

due to the assumption of LGD_i. Under the CreditRisk+ framework, D_i given X is assumed to be Poisson distributed. Therefore, we have $\mathbb{E}[D_i|X] = \mathbb{V}[D_i|X] = \mathrm{PD}_i(X)$, which implies

$$\mathbb{E}[D_i^2|X] = \mathbb{V}[D_i|X] - (\mathbb{E}[D_i|X])^2 = \mathrm{PD}_i(X) + (\mathrm{PD}_i(X))^2. \tag{25}$$

For the term $\mathbb{E}[\mathrm{LGD}_i^2]$ in (24), we have

$$\mathbb{E}[\mathrm{LGD}_i^2] = \mathbb{V}[\mathrm{LGD}_i] + (\mathbb{E}[\mathrm{LGD}_i])^2 = \mathrm{VLGD}_i^2 + \mathrm{ELGD}_i^2$$

Hence, we can rewrite (24) as

$$\begin{aligned}\sigma_i^2(x) &\overset{(25)}{=} \left(\mathrm{VLGD}_i^2 + \mathrm{ELGD}_i^2\right)\left(\mathrm{PD}_i(X) + (\mathrm{PD}_i(X))^2\right) - (\mu_i(x))^2 \\ &\overset{(18)}{=} \mathcal{C}_i + (\mu_i(x))^2 \frac{\mathrm{VLGD}_i^2}{\mathrm{ELGD}_i^2} \end{aligned} \tag{26}$$

where $\mathcal{C}_i$ is defined as

$$\mathcal{C}_i = \frac{\mathrm{ELGD}_i^2 + \mathrm{VLGD}_i^2}{\mathrm{ELGD}_i^2}. \tag{27}$$

Thus, we have

$$\frac{\partial}{\partial \alpha_q(x)} \sigma_i^2(\alpha_q(x)) = \mathcal{C}_i \mu_i'(\alpha_q(x)) + 2\mu_i'(\alpha_q(x))\mu_i(\alpha_q(x)) \frac{\mathrm{VLGD}_i^2}{\mathrm{ELGD}_i^2} \tag{28}$$

Note that by (22) and (23), we have

$$\begin{aligned} \mu_N'(x) &= \sum_{i=1}^{N} s_i \mathcal{K}_i / (\alpha_q(x) - 1) = \mathcal{K}^* / (\alpha_q(x) - 1) \\ \mu_N''(x) &= 0. \end{aligned} \tag{29}$$

Therefore, (16) can be reformulated as

$$GA = \frac{1}{2\mathcal{K}^*} \left(\delta \sigma_N^2(\alpha_q(x)) - (\alpha_q(x) - 1) \frac{\partial}{\partial \alpha_q(x)} \sigma_N^2(\alpha_q(x)) \right) \tag{30}$$

where

$$\delta = -(\alpha_q(x) - 1) \frac{h'(\alpha_q(x))}{h(\alpha_q(x))} \tag{31}$$

Substituting $\mu_N(\alpha_q(x))$ and $\sigma_N^2(\alpha_q(x))$ and their derivatives obtaining from (29) and (28), we can conclude that

$$GA = \frac{1}{2\mathcal{K}^*} \sum_{i=1}^{N} s_i^2 \left[\delta(\mathcal{K}_i + \mathcal{R}_i) \left(\mathcal{C}_i + (\mathcal{K}_i + \mathcal{R}_i) \frac{\mathrm{VLGD}_i^2}{\mathrm{ELGD}_i^2} \right) - \mathcal{K}_i \left(\mathcal{C}_i + 2(\mathcal{K}_i + \mathcal{R}_i) \frac{\mathrm{VLGD}_i^2}{\mathrm{ELGD}_i^2} \right) \right]. \tag{32}$$

In conclusion, we derived the analytical solution of GA (32) proposed by Gordy and Lütkebohmert (2007).

As pointed out by Gordy and Lutkenbohmert, Gordy's granularity adjustment faces a "model mismatch" problem due to its inconsistency with the ASRF model underpinning the IRB function. This "model mismatch" might result in potential misleading calculations. The default-mode CreditMetrics model used by Emmer & Tasche (2005) and Pykhtin & Dev (2002) is more consistent with the Basel ASRF model.

Gordy's Granularity Adjustment: Sensitivity analyses

Sensitivity of Gordy's GA to portfolio granularity

Lets examine sensitivity of the analytic GA to the cardinality (number of obligors) of homogeneous portfolios. Consider the following sets of homogeneous portfolio,