

Q1. Let $x' = (1, -2)$ and $y' = (2, -1)$.

- (a) Graph the two vectors x and y .
- (b) Calculate and graph a vector $z = x - 2y$.
- (c) Calculate the angle between x and y .
- (d) Find the projection of z on y . What is the length of this projection? Provide geometric interpretation of this projection and its length.
- (e) Find the projection of x on y . What is the length of this projection? Provide geometric interpretation of this projection and its length.

Q2. Consider three vectors, $x'_1 = (3, 0, 0)$, $x'_2 = (4, 1, 0)$, and $x'_3 = (5, -6, 2)$.

- (a) Are these vectors linearly independent?
- (b) Construct perpendicular unit length vectors with the same linear span with these three vectors.

Q3. Let

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 4 & -2 & 3 \\ 1 & 0 & -1 \end{pmatrix}.$$

- (a) Find a vector \mathbf{x} such that $A\mathbf{x} = 0$.
- (b) What are the rank of A ?
- (c) Determine whether A is singular or nonsingular.

Q4. Let $A = \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix}$.

- (a) Find eigenvalues and eigenvectors of A .
- (b) Discuss whether or not eigenvectors of A are perpendicular.
- (c) Determine the spectral decomposition of A .
- (d) Find the eigenvalues and eigenvectors of A^{-3} .
- (e) Find the square root matrix of A , $A^{1/2}$.

Q5. For a matrix $A = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$ and a vector $d = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$, find the maximum value of $\frac{(x'd)^2}{x'Ax}$ for any non-zero vector $x' = (x_1, x_2)$.

Q6. Find the maximum and minimum values of the quadratic form $4x_1^2 + 4x_2^2 + 6x_1x_2$ for all points $x' = (x_1, x_2)$ such that $x'x = 1$.

Q7. Let $B_{k \times k}$ be a positive definite matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 0$ and associated normalized eigenvectors x_1, x_2, \dots, x_k . Then prove that

$$\min_{z \neq 0} \frac{z'Bz}{z'z} = \lambda_k \quad (\text{attained when } z = x_k).$$