Lecture 2 - Propositional Logic

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1 Propositional Logic

1.1 Propositions

Definition 1.1. *A* ***proposition*** *is a statement that is either true or false, but not both.*

1.2 Logical Operations

A nice way to represent operations on propositions is by using truth tables which lists all the possible values of arguments and states, for each, the resulting value.

Definition 1.2. *The* ***negation*** *of a proposition p is denoted by $\neg p$ and is true if p is false, and false if p is true.*

Definition 1.3. *The* ***conjunction*** *(and) of propositions p and q is denoted by $p \wedge q$ and is true if both p and q are true, and false otherwise. This is often called*

Definition 1.4. *The* ***disjunction*** *(inclusive OR) of propositions p and q is denoted by $p \vee q$ and is true if at least one of p and q is true, and false otherwise.*

1.3 De Morgan's Laws

Conjunction and disjunction are related by De Morgan's laws.

Theorem 1.5. ***De Morgan's Laws****:*

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

There is also an intuitive way to think about it. A fun example: "I will not eat my vegetables and I will not eat my fruits" is equivalent to "I will not eat my vegetables or fruits".

You can also prove De Morgan's laws using truth tables.

1.4 Implication

Definition 1.6. *The* ***implication*** *of propositions p and q is denoted by $p \rightarrow q$ and is true if p is false or q is true, and false otherwise.*

An example: "If it is raining, then I will take an umbrella". If I do take an umbrella even if it doesn't rain, the statement is still true.

Implication is not neccessarily causation. For example, "If I am in Istanbul, then I am in Turkey" is true even if I am not in Istanbul.

Equivalent words: "if...then", "only if", "implies", "is sufficient for", "is necessary for".

1.5 Biconditional / bi-implication

Definition 1.7. *The* ***biconditional*** *of propositions p and q is denoted by $p \leftrightarrow q$ and is true if p and q have the same truth value, and false otherwise.*

Equivalent words: "if and only if", "iff", "is equivalent to". An example of this is that a right-angle triangle must satisfy $a^2 + b^2 = c^2$

1.6 Tautologies and Logical Equivalence

Definition 1.8. *A **tautology** is a proposition that is always true, regardless of the truth values of its variables.*

In other words, the right hand column of the truth table is all true.

Definition 1.9. *Two statements are **logically equivalent** if their truth tables are identical.*

If P and Q are logically equivalent, $P \leftrightarrow Q$ is a tautology.

1.7 Laws of Boolean Algebra

A full listing of laws of Boolean algebra:

$$\begin{aligned} \neg\text{True} &= \text{False} & \neg\neg P &= P \\ P \wedge Q &= Q \wedge P & P \vee Q &= Q \vee P \\ (P \wedge Q) \wedge R &= P \wedge (Q \wedge R) & (P \vee Q) \vee R &= P \vee (Q \vee R) \\ P \wedge P &= P & P \vee P &= P \\ P \wedge \neg P &= \text{False} & P \vee \neg P &= \text{True} \\ P \wedge \text{True} &= P & P \vee \text{False} &= P \\ P \wedge \text{False} &= \text{False} & P \vee \text{True} &= \text{True} \end{aligned}$$

Distributive Laws

$$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$$
$$P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$$

De Morgan's Laws

$$\neg(P \vee Q) = \neg P \wedge \neg Q$$
$$\neg(P \wedge Q) = \neg P \vee \neg Q$$

1.8 Disjunctive Normal Form

Definition 1.10. *A boolean expression is in **disjunctive normal form** if it is a disjunction of one or more conjunctions of literals. This is like the 'expanded' form of a boolean expression.*

* A **literal** is a variable or its negation.

Problem with DNF is that, in real life, logical rules are not usually specified in a form that is amenable to DNF. They are typically described by listing conditions that must be satisfied together.

1.9 Conjunctive Normal Form

When DNF is a disjunction of literals, CNF is simply a conjunction of literals.

Definition 1.11. *A boolean expression is in **conjunctive normal form** if it is a conjunction of one or more disjunctions of literals.*

Each disjunction of literals is called a **clause**.

1.10 Representing Logical Statements