Lecture 2 - Propositional Logic

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1 Propositional Logic

1.1 Propositions

Definition 1.1. A **proposition** is a statement that is either true or false, but not both.

1.2 Logical Operations

A nice way to represent operations on propositions is by using truth tables which lists all the possible values of arguments and states, for each, the resulting value.

Definition 1.2. The **negation** of a proposition p is denoted by $\neg p$ and is true if p is false, and false if p is true.

Definition 1.3. The **conjunction** (and) of propositions p and q is denoted by $p \wedge q$ and is true if both p and q are true, and false otherwise. This is often called

Definition 1.4. The **disjunction** (inclusive OR) of propositions p and q is denoted by $p \vee q$ and is true if at least one of p and q is true, and false otherwise.

1.3 De Morgan's Laws

Conjunction and disjunction are related by De Morgan's laws.

Theorem 1.5. ***De Morgan's Laws:***

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

There is also an intuitive way to think about it. A fun example: "I will not eat my vegetables and I will not eat my fruits" is equivalent to "I will not eat my vegetables or fruits".

You can also prove De Morgan's laws using truth tables.

1.4 Implication

Definition 1.6. The **implication** of propositions p and q is denoted by $p \to q$ and is true if p is false or q is true, and false otherwise.

An example: "If it is raining, then I will take an umbrella". If I do take an umbrella even if it doesn't rain, the statement is still true.

Implication is not neccessarily causation. For example, "If I am in Istanbul, then I am in Turkey" is true even if I am not in Istanbul.

Equivalent words: "if...then", "only if", "implies", "is sufficient for", "is necessary for".

1.5 Biconditional / bi-implication

Definition 1.7. The **biconditional** of propositions p and q is denoted by $p \leftrightarrow q$ and is true if p and q have the same truth value, and false otherwise.

Equivalent words: "if and only if", "iff", "is equivalent to". An example of this is that a right-angle triangle must satisfy $a^2 + b^2 = c^2$

1.6 Tautologies and Logical Equivalence

Definition 1.8. A **tautology** is a proposition that is always true, regardless of the truth values of its variables.

In other words, the right hand column of the truth table is all true.

Definition 1.9. Two statements are **logically equivalent** if their truth tables are identical.

If P and Q are logically equivalent, $P \leftrightarrow Q$ is a tautology.

1.7 Laws of Boolean Algebra

A full listing of laws of Boolean algebra:

$$\begin{aligned}
\neg\text{True} &= \text{False} & \neg\neg P &= P \\
P \wedge Q &= Q \wedge P & P \vee Q &= Q \vee P \\
(P \wedge Q) \wedge R &= P \wedge (Q \wedge R) & (P \vee Q) \vee R &= P \vee (Q \vee R) \\
P \wedge P &= P & P \vee P &= P \\
P \wedge \neg P &= \text{False} & P \vee \neg P &= \text{True} \\
P \wedge \text{True} &= P & P \vee \text{False} &= P \\
P \wedge \text{False} &= \text{False} & P \vee \text{True} &= \text{True}
\end{aligned}$$

Distributive Laws

$$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$$
$$P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$$

De Morgan's Laws

$$\neg(P \vee Q) = \neg P \wedge \neg Q$$
$$\neg(P \wedge Q) = \neg P \vee \neg Q$$

1.8 Disjunctive Normal Form

Definition 1.10. A boolean expression is in **disjunctive normal form** if it is a disjunction of one or more conjunctions of literals. This is like the 'expanded' form of a boolean expression.

Remark 1.11. A **literal** is a variable or its negation.

Problem with DNF is that, in real life, logical rules are not usually specified in a form that is amenable to DNF. They are typically described by listing conditions that must be satisfied together.

1.9 Conjunctive Normal Form

When DNF is a disjunction of literals, CNF is simply a conjunction of literals.

Definition 1.12. A boolean expression is in **conjunctive normal form** if it is a conjunction of one or more disjunctions of literals.

Each disjunction of literals is called a **clause**.

1.10 Representing Logical Statements