Binomial Heaps

Ch. 9.3

Leftist vs. Binomial heaps

- Time complexity for different operations

	Leftist trees	Binomial heaps	
		Actual	Amortized
Insert	O(log n)	O(1)	O(1)
Delete min (or max)	O(log n)	O(n)	O(log n)
Meld	O(log n)	O(1)	O(1)

Toy example of amortization

- Performing a sequence of insert and delete-min operations:

I1, I2, D1, I3, I4, I5, I6, D2, I7

Actual cost of each insertion: 1

Actual cost of D1: 8

Actual cost of D2: 10

Total cost: 25

- Cost transferring (amortization) scheme:

Transfer one unit of cost of a delete-min to the prior insertions.

	I1	I2	D1	I3	I4	I5	I6	D2	I7
Actual	1	1	8	1	1	1	1	10	1
Amortized	1 **+1**	1 **+1**	8 **-2**	1 **+1**	1 **+1**	1 **+1**	1 **+1**	10 **-4**	1

Amortization

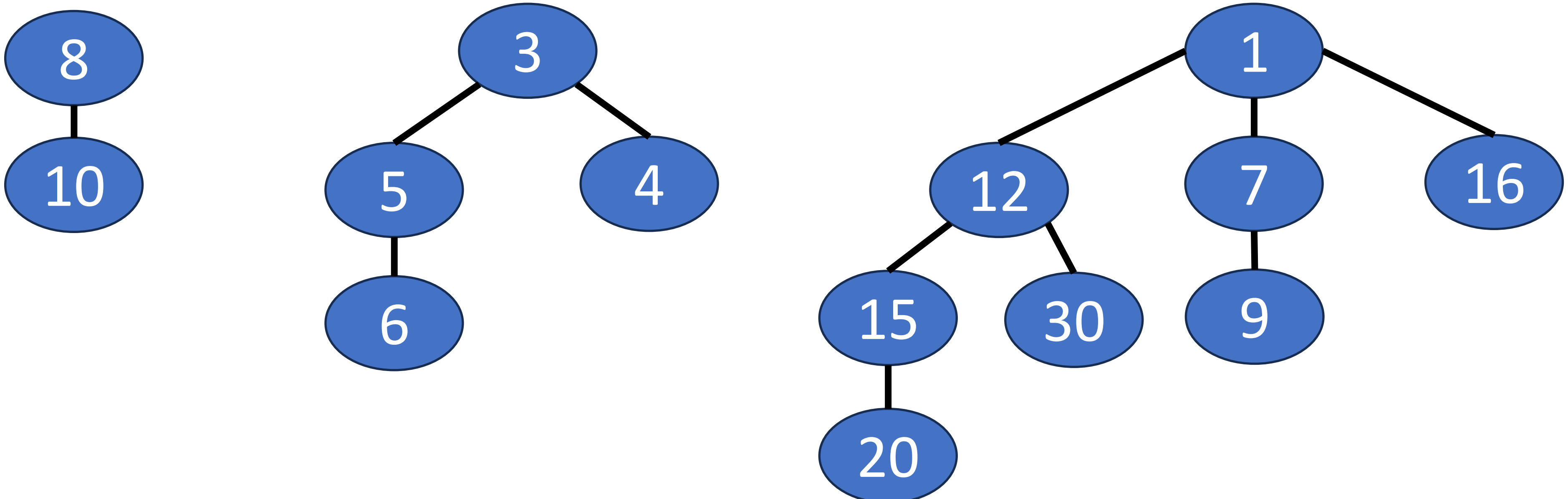
- Sometimes, we care more about the overall time to perform a sequence of operations.
    - The time to perform a single operation is less important.
    - For example: sorting.

Binomial heaps (B-heap)

- Max binomial heap
  - A collection of max trees
- Min binomial heap
  - A collection of min trees

(In this lecture, we will consider min binomial heaps only.)

Example: A B-heap with 3 min trees.

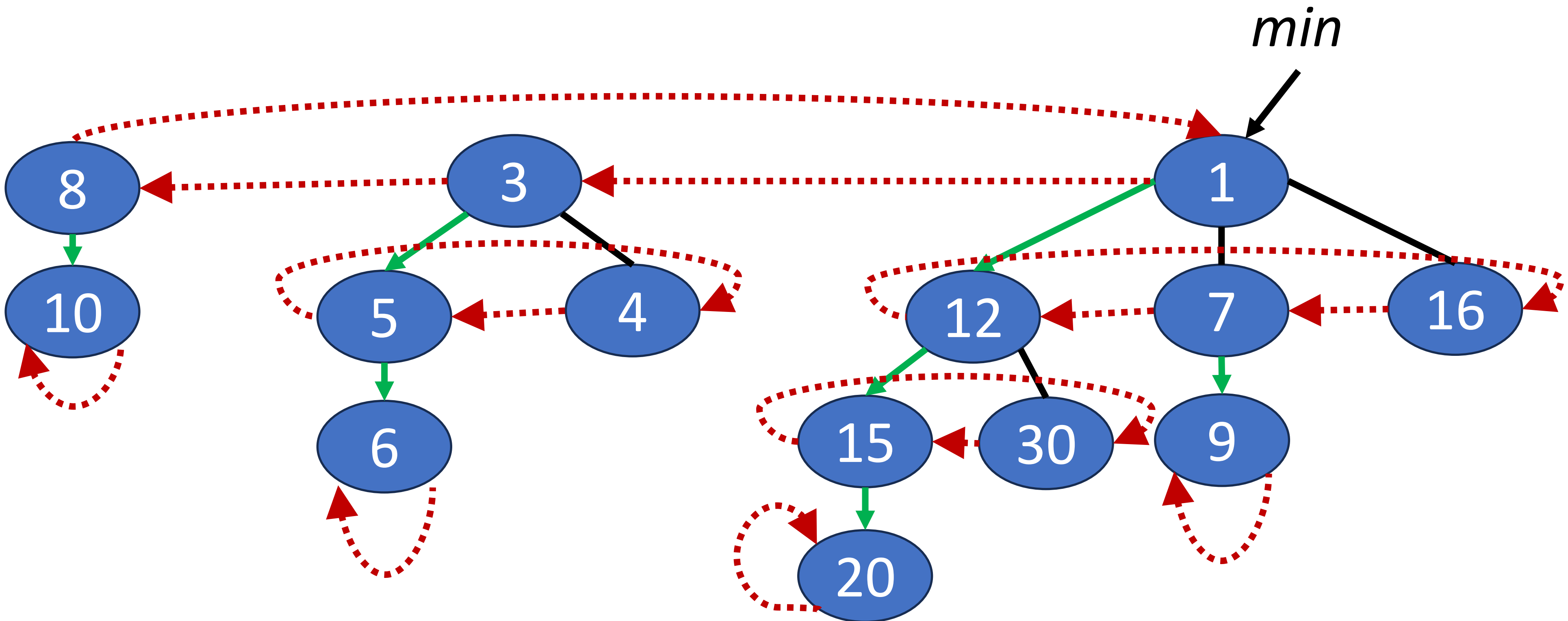


Binomial heap representation

- All roots of min trees form a circular linked list.
- All the children of a node form a circular linked list.

{10}, {6}, {5,4}, {20}, {15,30}, {9}, {12,7,16}, {8,3,1}

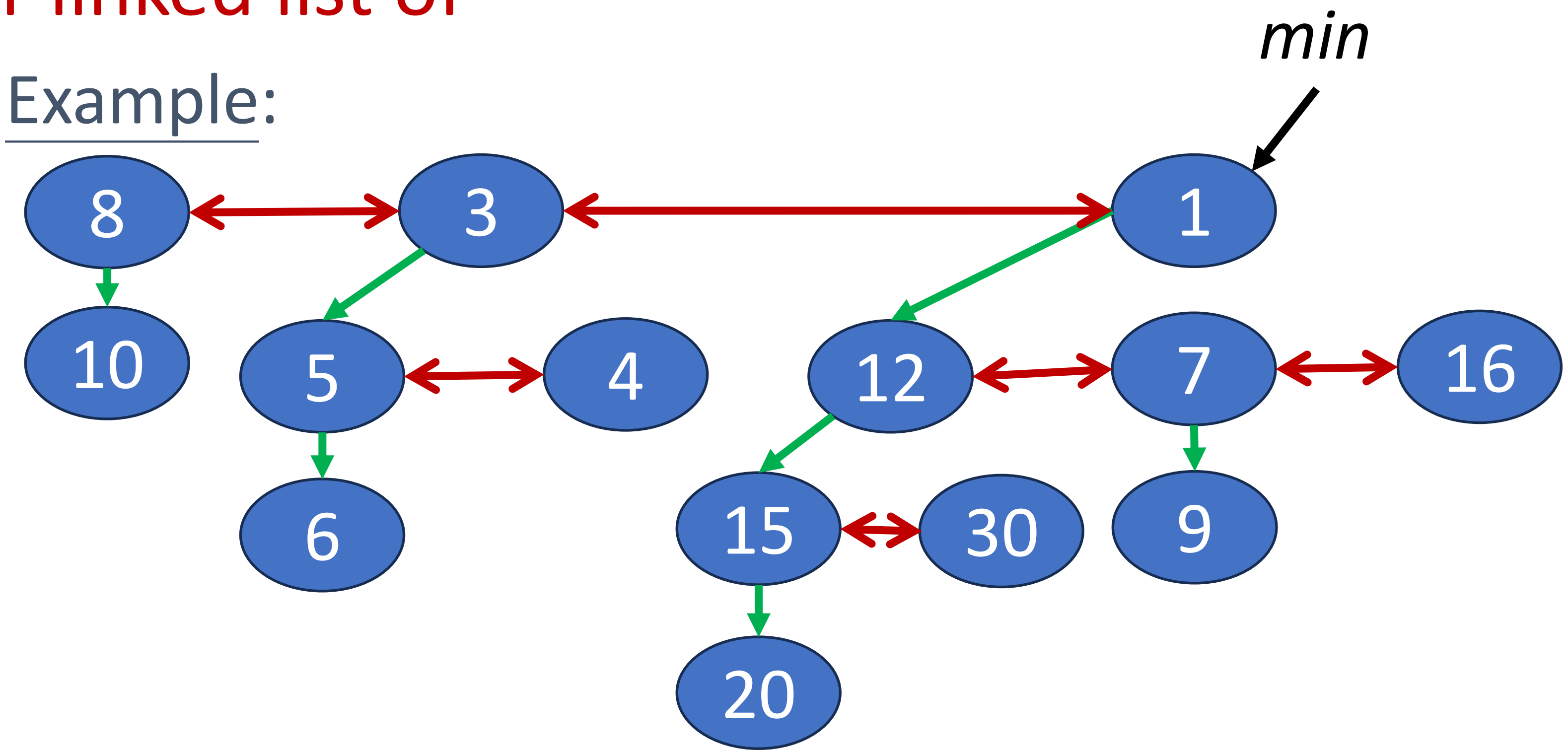
- *min*: pointer to the B-heap.



Node structure

- **Child**
  - Pointer to one of the node's children.
  - Null iff node has no child.
- **Link**
  - Used for singly circular linked list of siblings.
- **Degree**
  - Number of children.
- **Data**

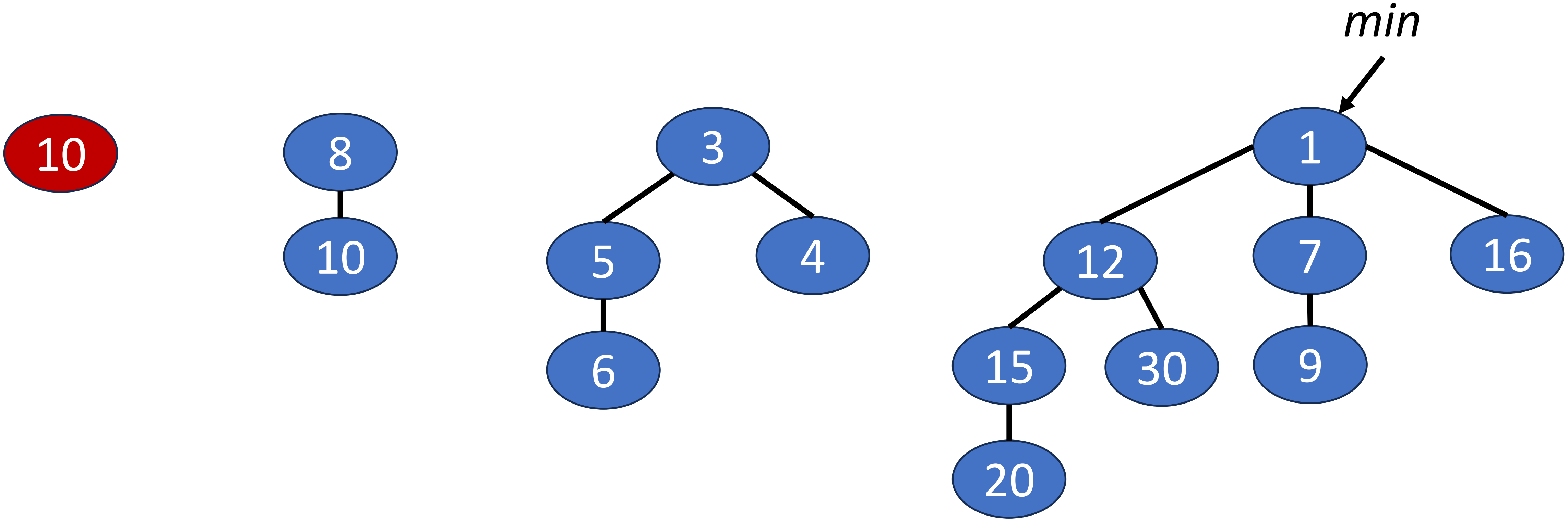
Example:



Operation: Insertion

- Add a new single-node min tree to the collection.
- Update min-element pointer if necessary.

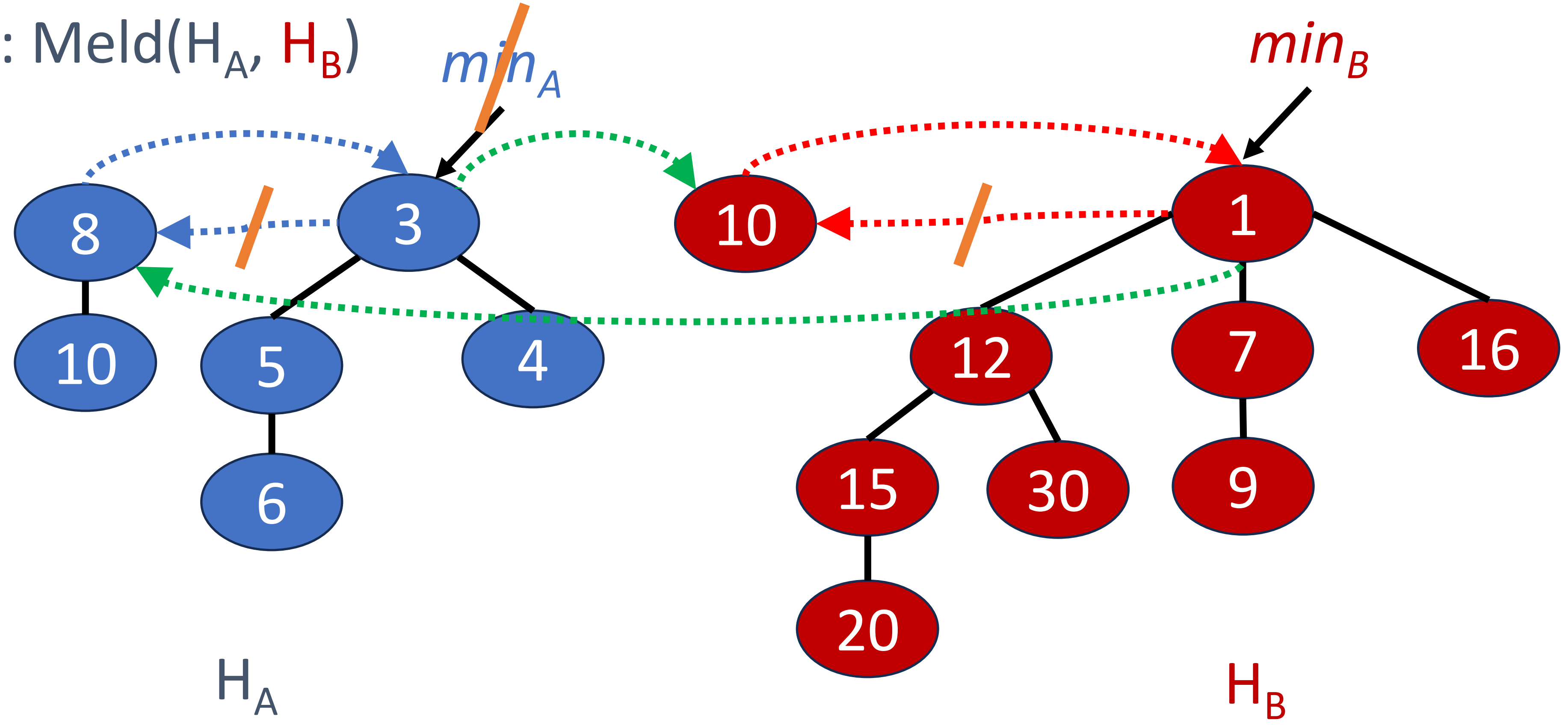
Example: Insert 10



Operation: Meld

- Meld the top circular lists of two B-heaps into one circular list
- Keep only the min pointer having the smaller key

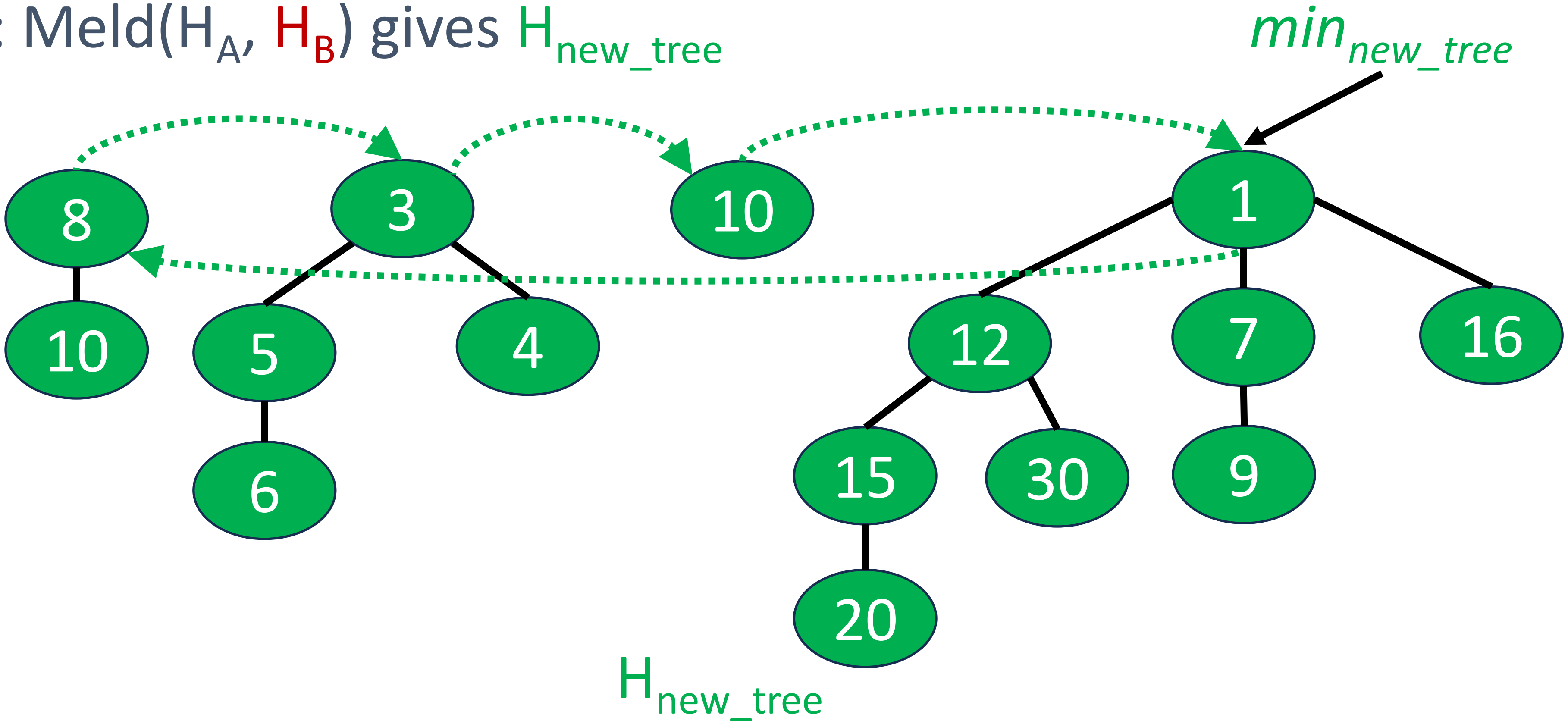
Example: Meld(H_A, H_B)



Operation: Meld

- Meld the top circular lists of two B-heaps into one circular list
- Keep only the min pointer having the smaller key

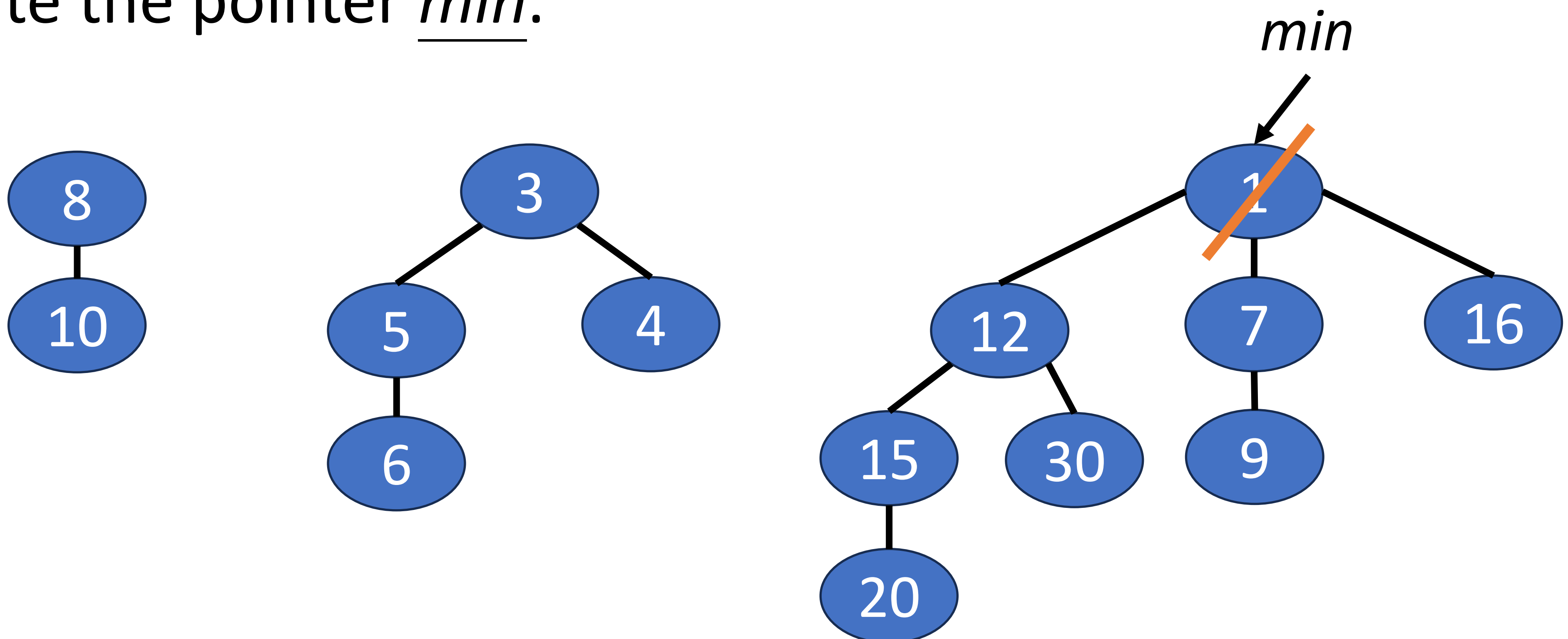
Example: Meld(H_A, H_B) gives H_{new_tree}



Operation: Delete-min

- If *min* is 0, the B-heap is empty. ➔ Cannot perform deletion.
- If *min* points to a node, the B-heap is not empty.
  1) Remove the node pointed by *min*.
  2) Reinsert subtrees of the removed node.
  3) Update the pointer *min*.

Example:

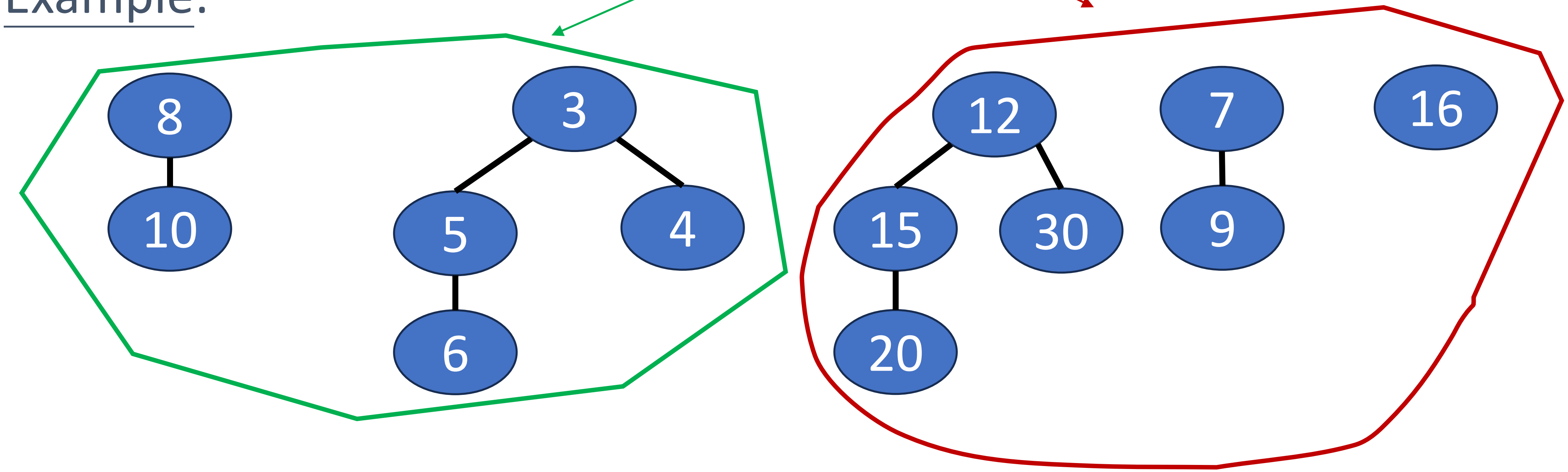


Operation: Delete-min

1) Remove the node pointed by *min*.

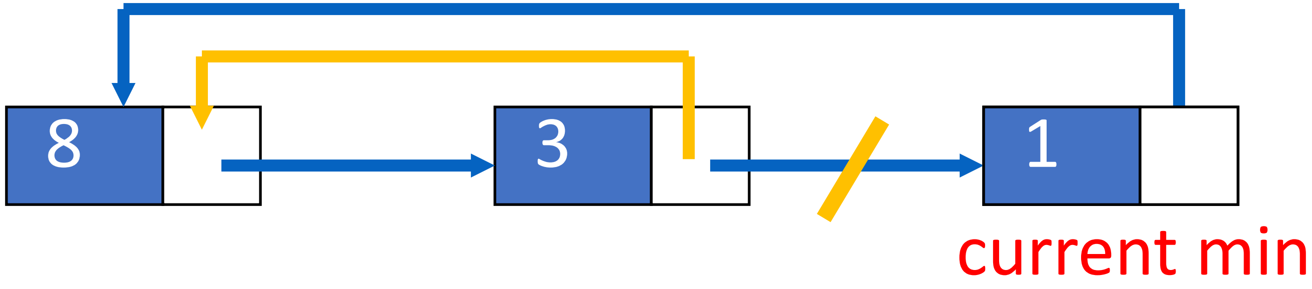
➔ New B-heap consists of the remaining min trees and the sub-min tree of the deleted node.

Example:



Remove min node from B-heap

- Delete the min node from its circular list.
- Same as removing a node from a circular list.



- If no next node ➔ The list is empty after remove min node.

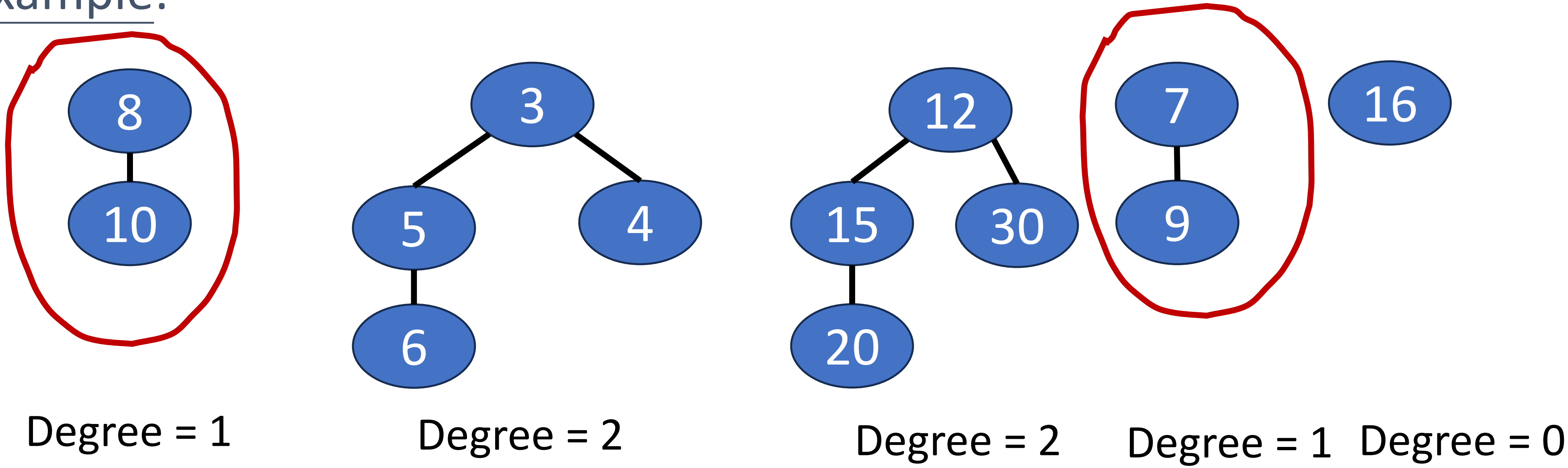
Operation: Delete-min

2) Min-tree joining.

- Pairs of min trees having the same degree must be joined.

The degree of a min tree is the degree of its root.

Example:



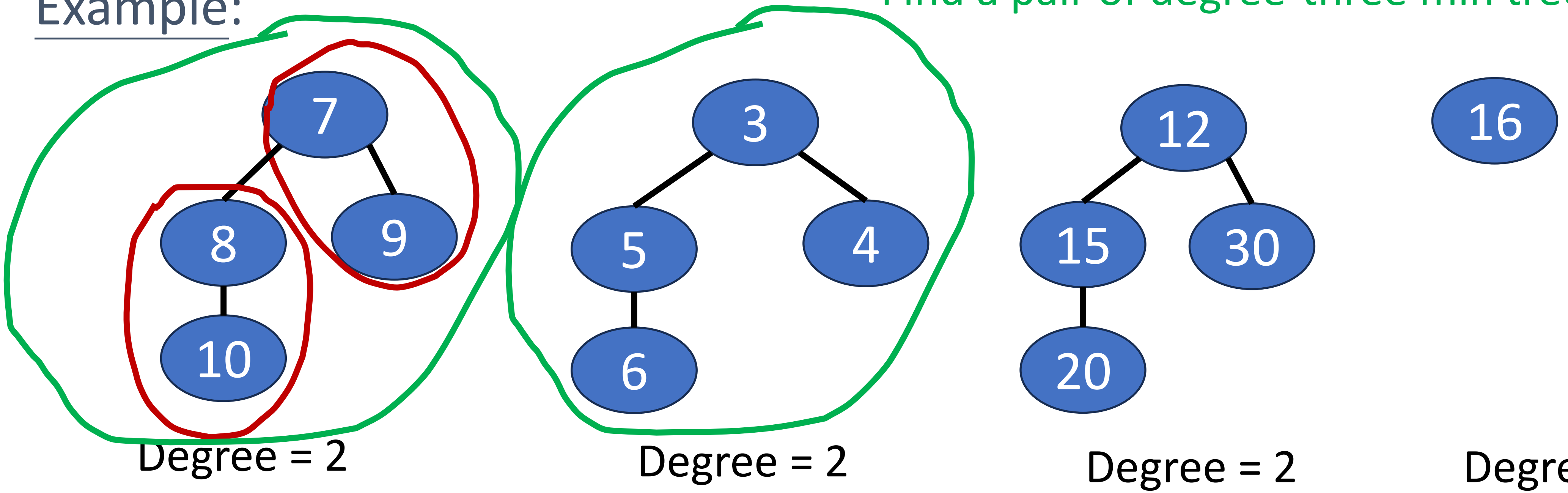
Operation: Delete-min

2) Min-tree joining.

- Pairs of min trees having the same degree must be joined.
- The min tree(root has a larger key) becomes a subtree of the min tree(root has a smaller key) . **To maintain property of min tree.**

Example:

Find a pair of degree-three min trees



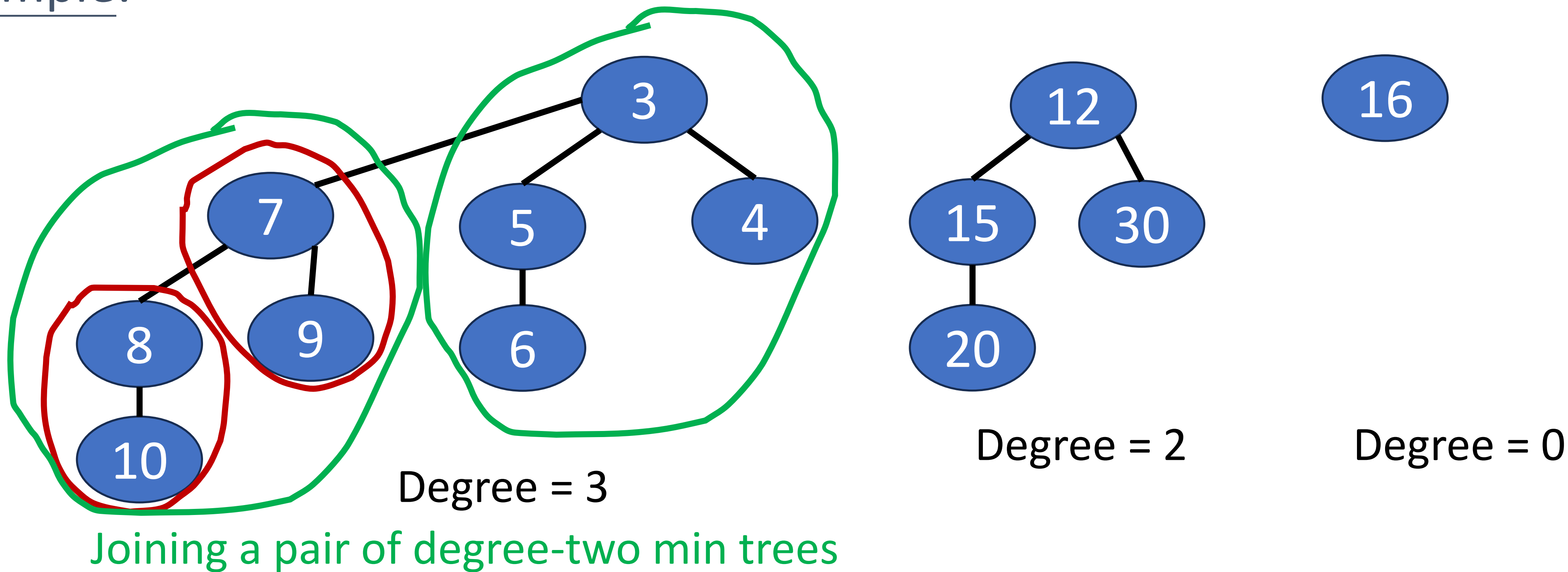
Joining a pair of degree-one min trees

Operation: Delete-min

2) Min-tree joining.

- Pairs of min trees having the same degree must be joined.
- The min tree(root has a larger key) becomes a subtree of the min tree(root has a smaller key) .

Example:



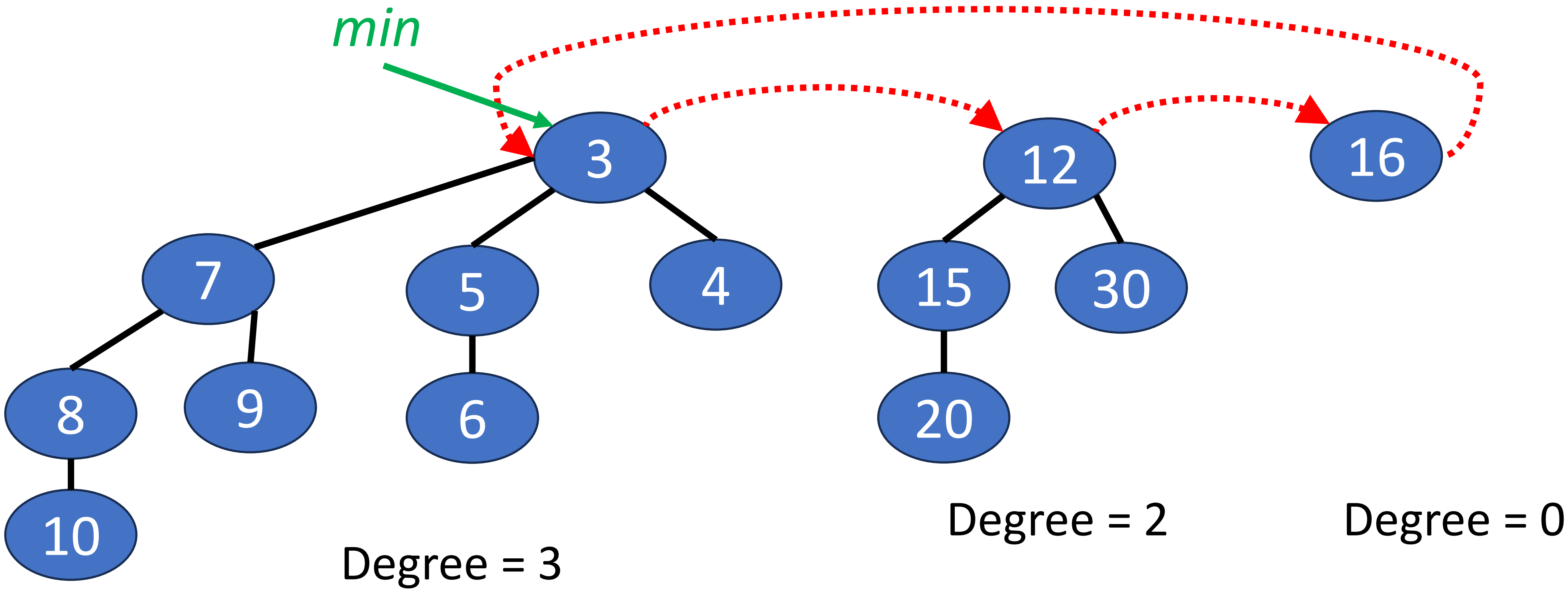
Operation: Delete-min

See programs 9.3 and 9.4 in textbook for more details.

3) Form min-tree root list and update *min* pointer.

- Link roots of the remaining min-trees together using linked circular list.
- Set min to point to the root with minimum key.

Example:



Importance of circular lists in B-heaps

- Reinsert sub-trees due to removal of min
  - Deleting *min* node results in several sub-min trees.
  - With the circular list, those sub-min trees can be visited.

- Find *min* node
  - Searching the top-level circular list.

- Meld two binomial heaps
  - Melding the top-level circular lists of the two B-heaps.

Complexity of Delete-min

1) Remove the node pointed by *min*.
   - O(1)

2) Min-tree joining.
   - O(*s*), where *s* is the number of min trees.
   - s = O(*n*), since there may be several one-node subtrees.

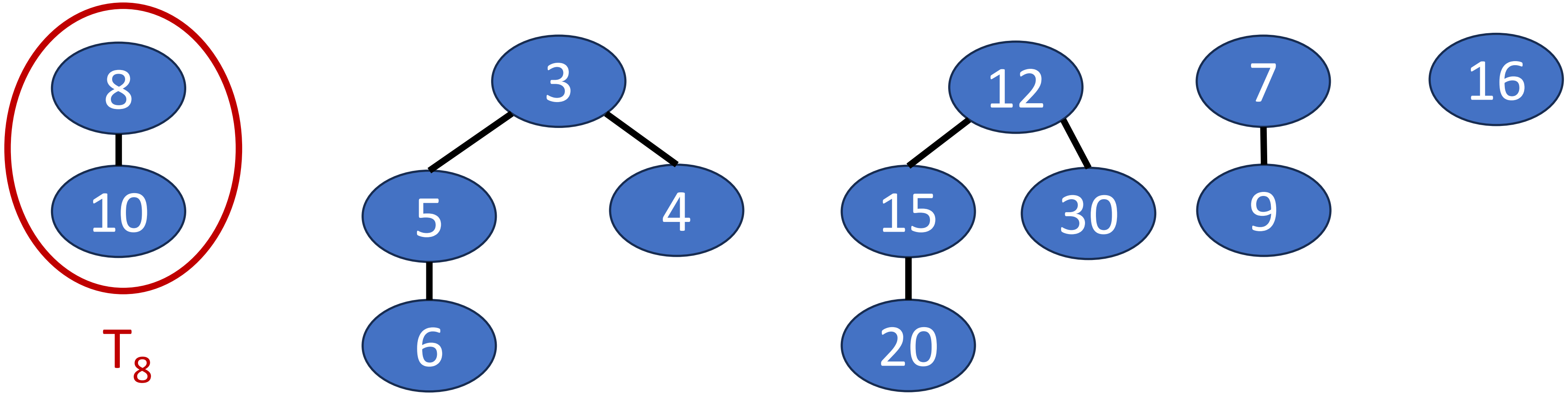
3) Form min-tree root list and update *min* pointer.
   - The final number of min trees ≤ *s*.

Overall complexity of delete-min is O(n).

Pairwise combine

- Can be implemented using an array *tree* to keep track of trees by degree.

Example:



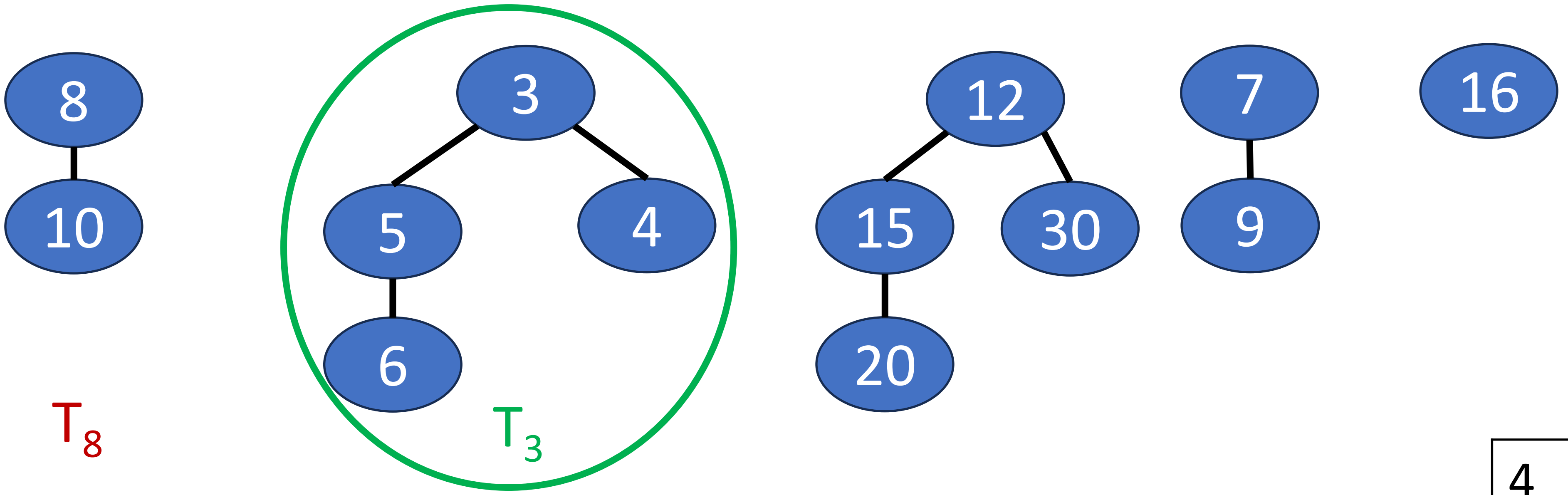
tree

4	
3	
2	
1	T_8
0	

Visiting the min trees following the top-level circular list.
Recording the degree of visited min trees.

Pairwise combine

Example:



tree

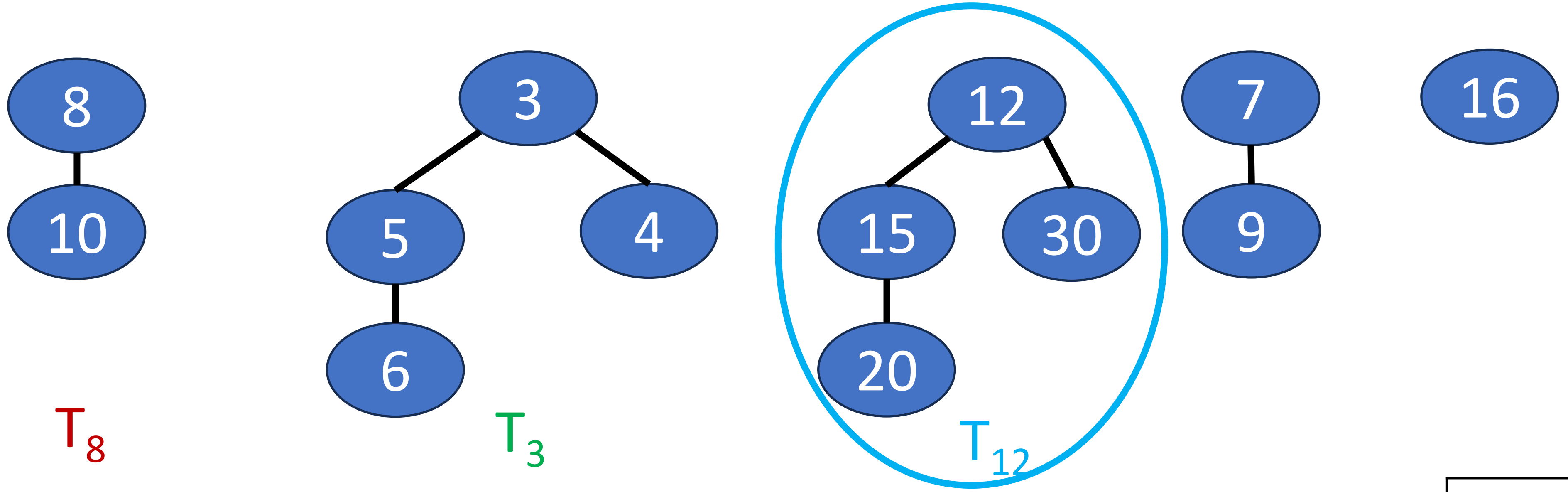
4	
3	
2	T_3
1	T_8
0	

Visiting the next min tree.

Recording the degree of visited min trees.

Pairwise combine

Example:



tree

4	
3	
2	T_3
1	T_8
0	

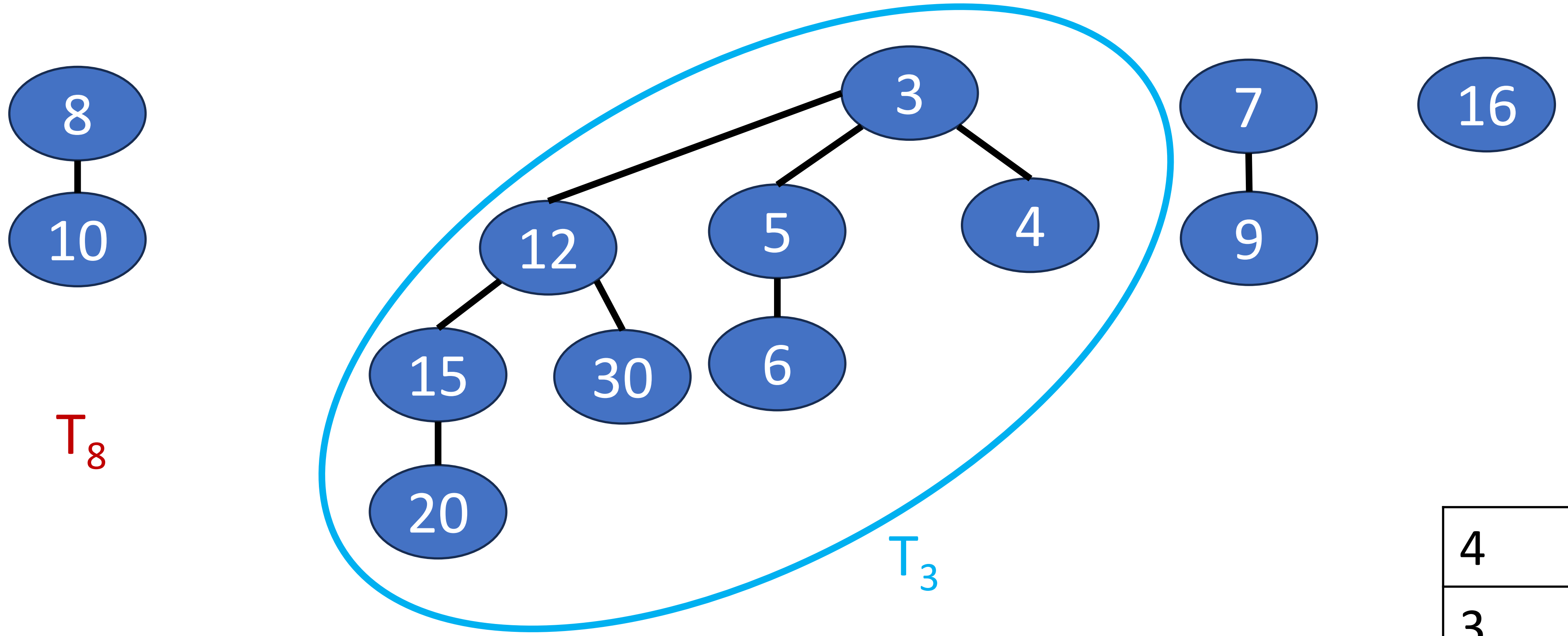
Visiting the next min tree.

Recording the degree of visited min trees.

Finding another min tree T_3 having same degree. ➔ Meld them.

Pairwise combine

Example:

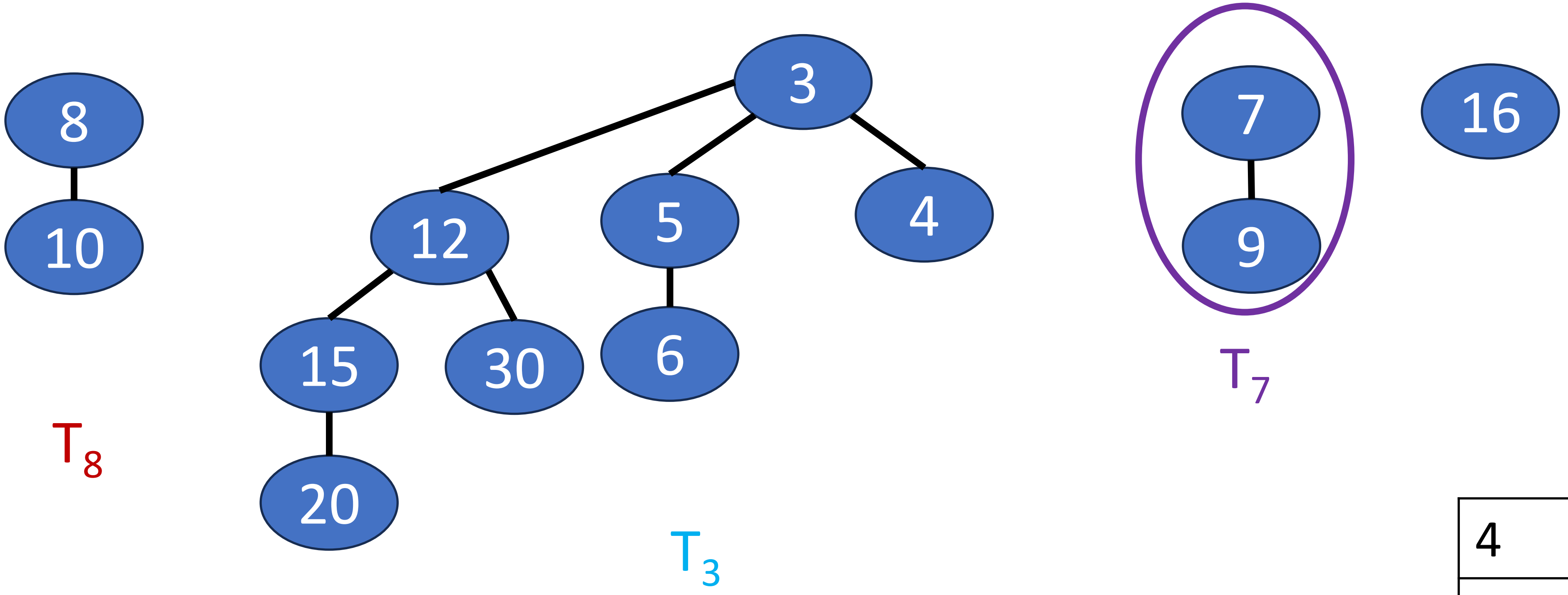


	tree
4	
3	T_3
2	
1	T_8
0	

Recording the degree of new min-tree.

Pairwise combine

Example:



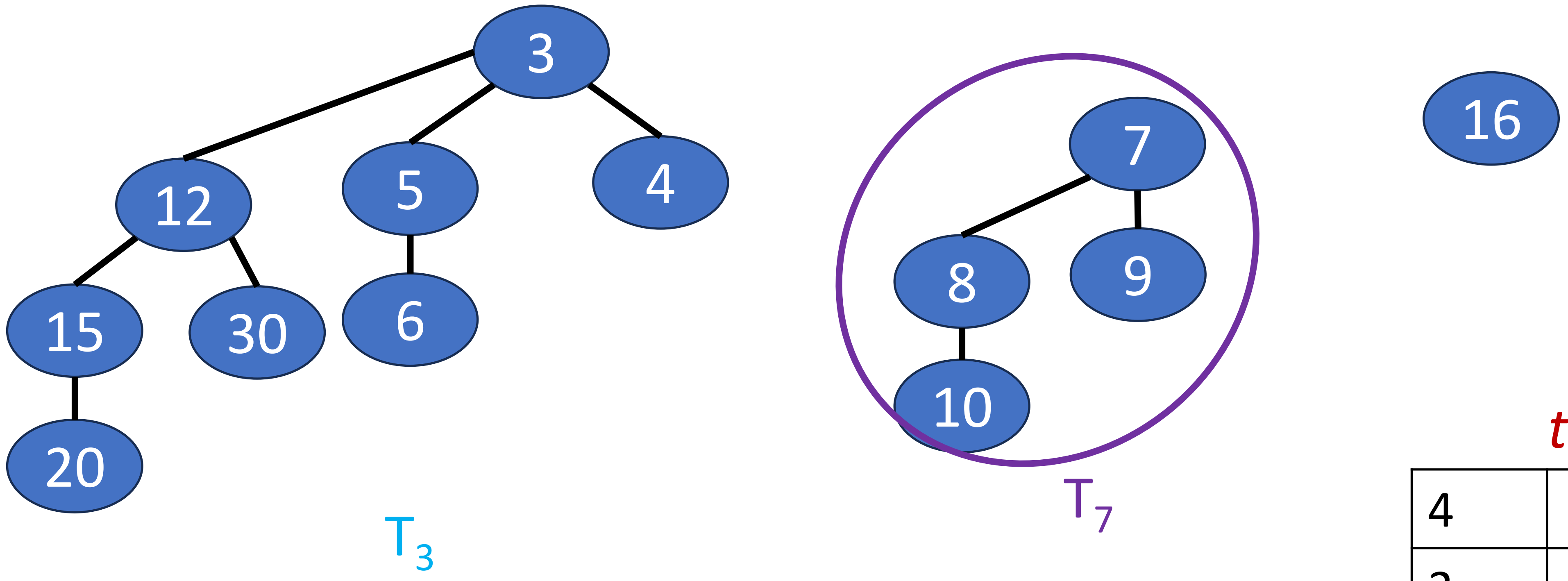
tree

4	
3	T_3
2	
1	T_8
0	

Visiting the next min tree.
Recording the degree of visited min trees.
Finding another min tree T_8 having same degree. ➔ Meld them.

Pairwise combine

Example:

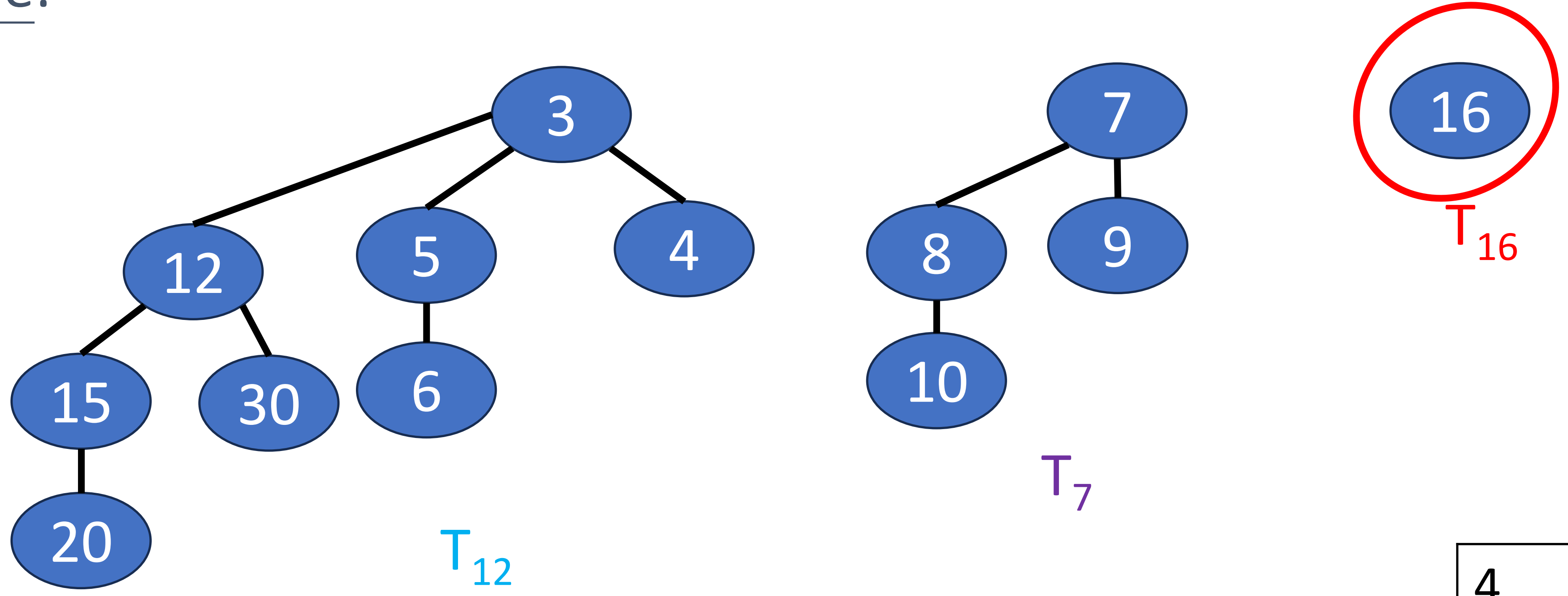


	tree
4	
3	T_3
2	T_7
1	
0	

Recording the degree of new min-tree.

Pairwise combine

Example:



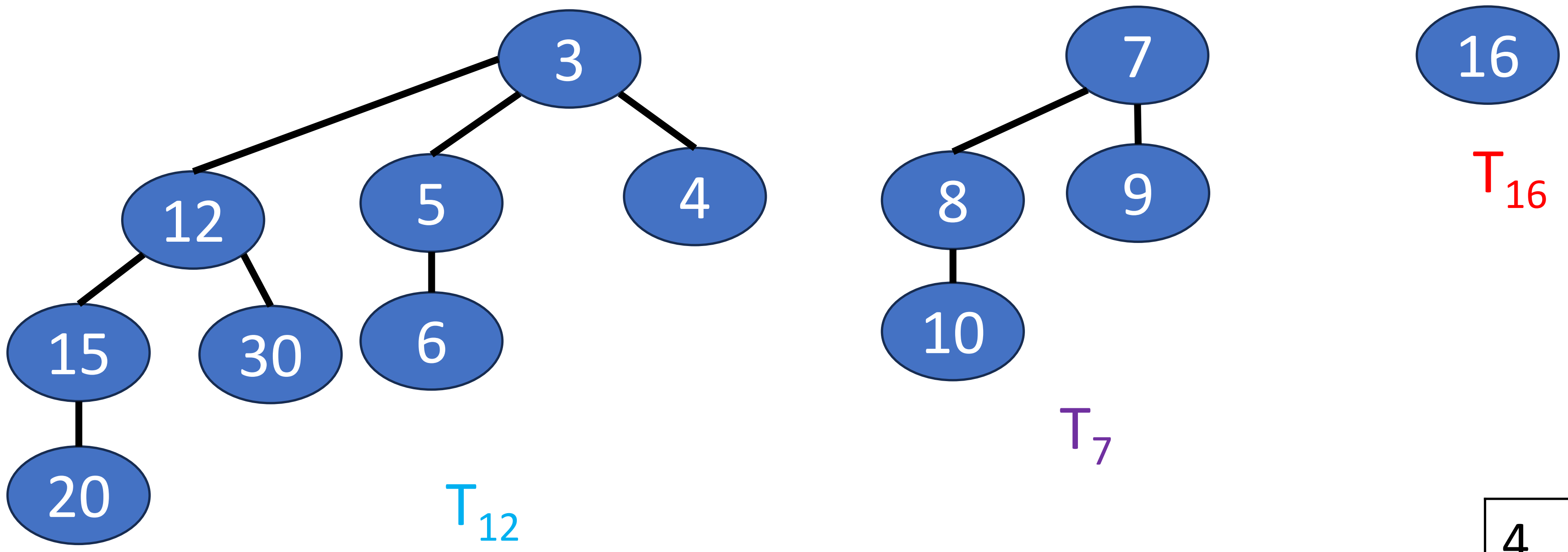
tree

4	
3	T_{12}
2	T_7
1	
0	T_{16}

Visiting the next min tree.
Recording the degree of visited min trees.

Pairwise combine

Example:



tree

4	
3	T_{12}
2	T_7
1	
0	T_{16}

Create circular list of remaining trees in the *tree* table.

Complexity of Delete-min

2) Min-tree joining.

- Create and initialize tree table.
  - O(MaxDegree).
  - Done once only.
- Examine s min trees and pairwise combine.
  - O(s), where s is the number of min trees.

Number of joins is at most s-1 as each join reduces number of min-trees by 1.

3) Form min-tree root list and update *min* pointer.

- Collect remaining trees from tree table, reset table entries to null, and set *min* pointer.
  - O(MaxDegree).

Overall complexity of delete-min is O(MaxDegree + s).

Exercise

- Q7: Into an empty B-heap, insert elements with priorities 20, 10, 5, 18, 6, 12, 14, 4, and 22 (in this order). Each insertion operation includes min-tree joining (pairwise combine). Please write the roots and degrees of min trees in the final B-heap.

- Q8: Delete the min element from the final B-heap of Q7. Please write the roots and degrees of min trees in the resulting B-heap.

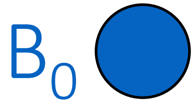
Please reply your answers of Q7-Q8 via the following link:

Group members: 1~3 people

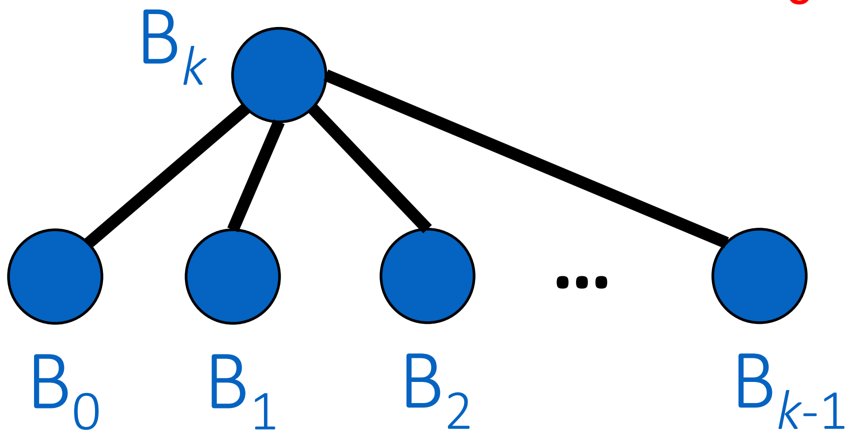
Binomial trees

- If performing inserts, melds, and delete-mins only, the min trees in B-heap are binomial trees.

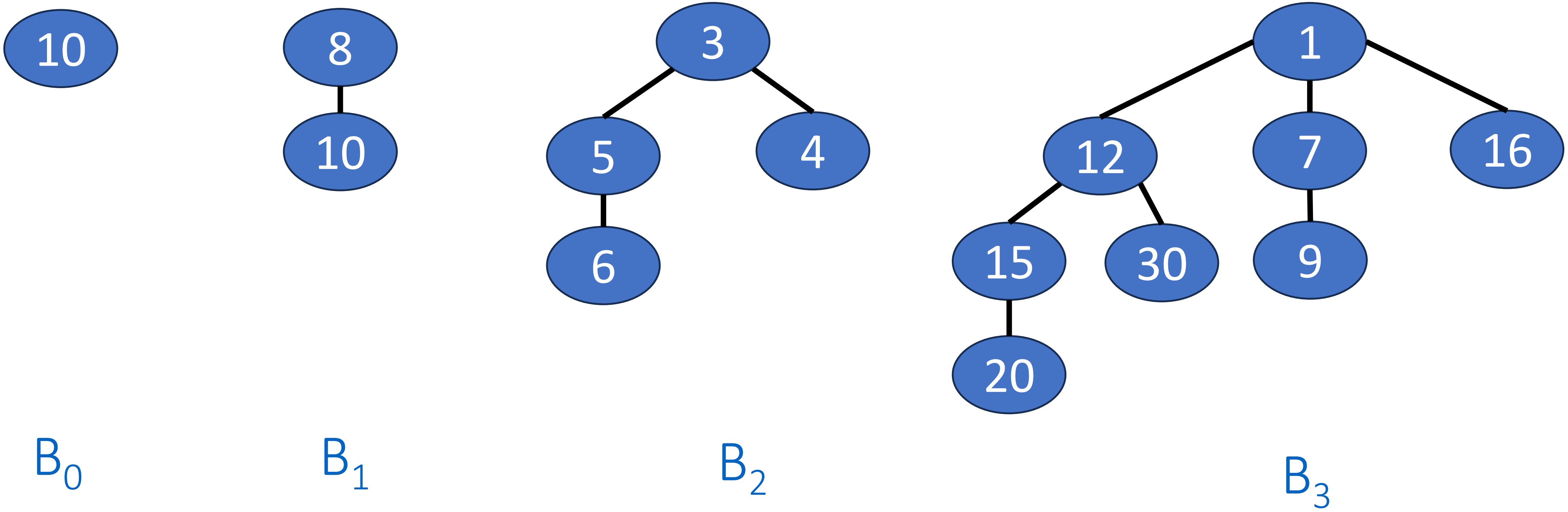
- B_k is degree-k binomial tree.
  - $k=0$, the tree has one node.



  - $k>0$, root with degree k and subtrees are B_0, B_1,…, B_{k-1}.

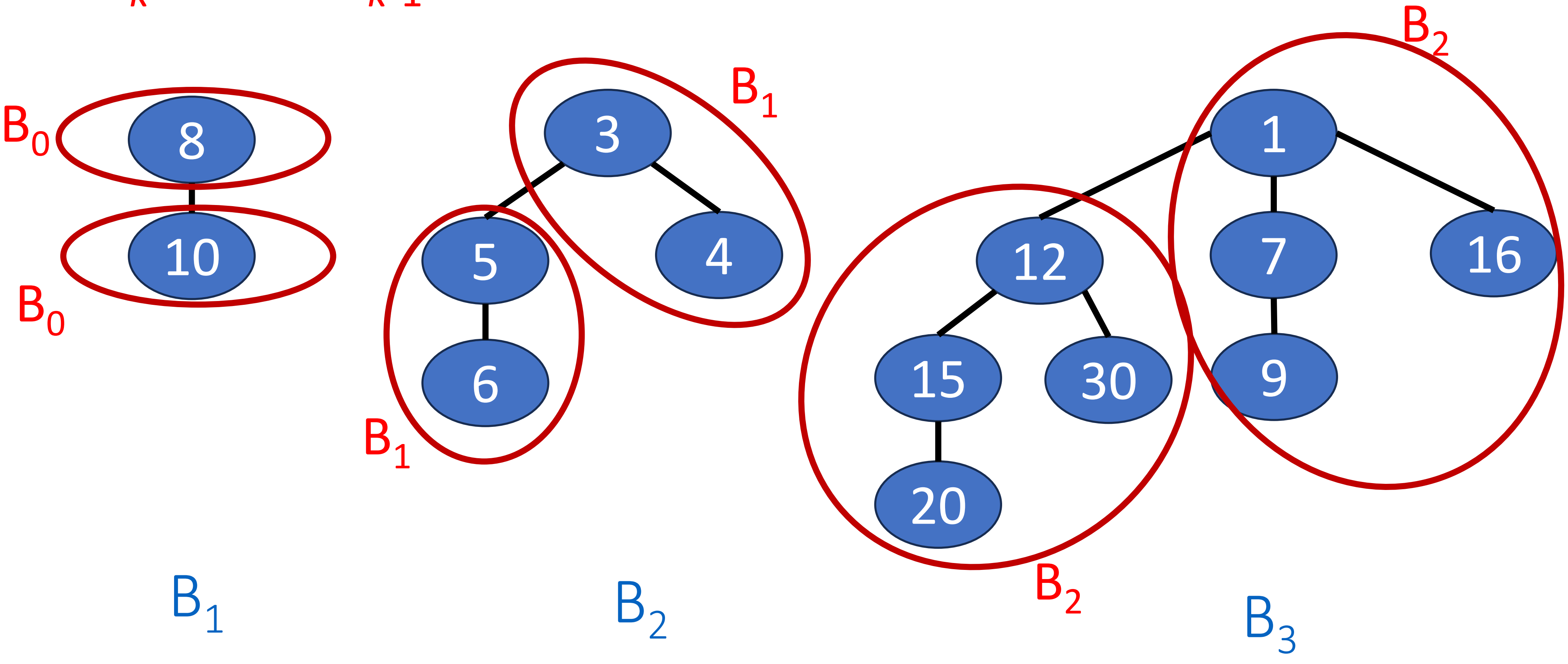


Example of binomial trees



Equivalent definition of binomial trees

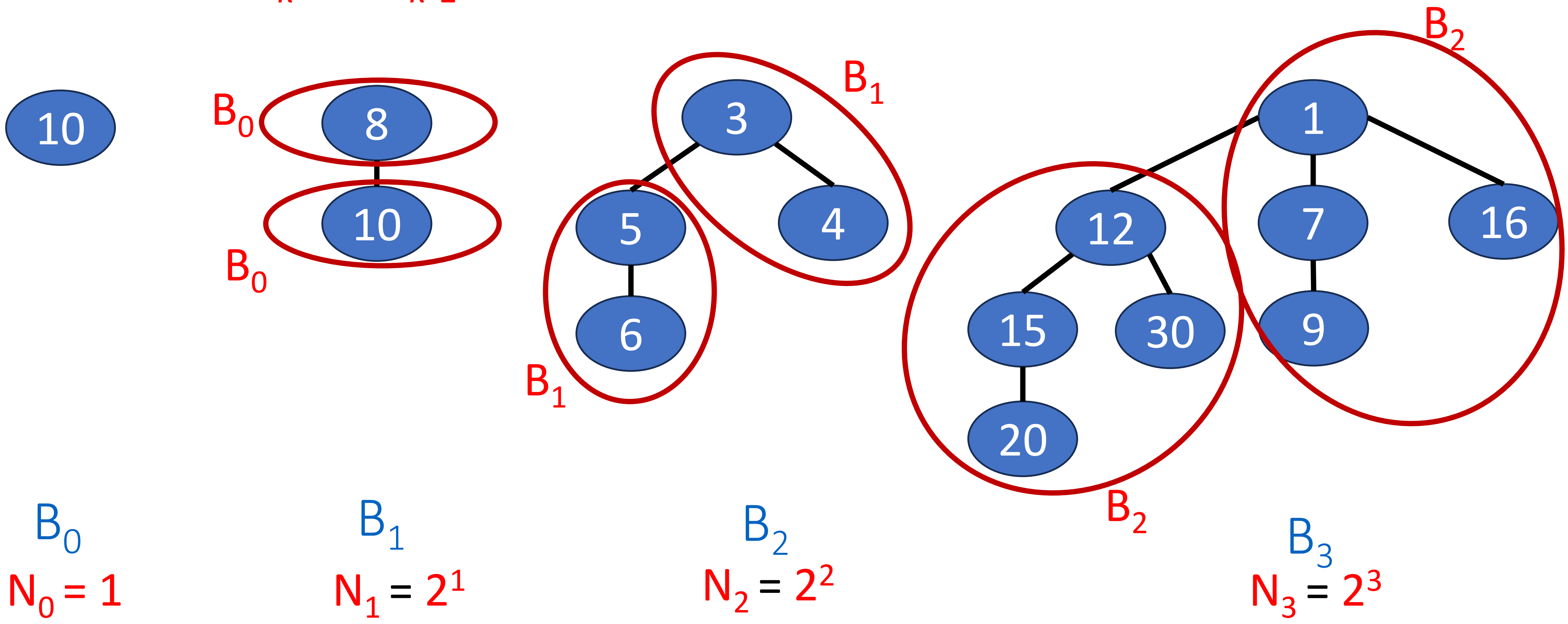
- B_k is degree-k binomial tree.
  - $K>0$, B_k is two B_{k-1}.



Number of nodes in binomial trees

- N_k is number of nodes in degree-k binomial tree.
  - $N_0 = 1$
  - $N_k = 2N_{k-1} = 2^k$

In B-heap, all trees are binomial trees.
Thus MaxDegree of a min tree is O(log n).



Performance analysis

See Theorem 9.1 in textbook for more details.

	Leftist trees	Binomial heaps	
		Actual	Amortized
Insert	O(log n)	O(1)	O(1)
Delete min (or max)	O(log n)	O(n)	O(log n)
Meld	O(log n)	O(1)	O(1)

- Actual cost of a delete-min: O(MaxDegree + *s*) = O(log n + *s*).
- If a *sequence* of insert, meld, and delete-min operations is performed on empty B-heaps, we can transfer the cost of delete-min *s* to prior insertions.
- Amortized cost of insertion becomes O(1+1) = O(1) and that of delete-min becomes O(MaxDegree) = O(log n).

Summary

- Binomial heaps
  - Operations: Insert, Meld, Delete-min, Delete-min using tree table.
- Binomial trees
  - Definition
  - Number of nodes
- Performance analysis
- Amortized cost