Ph.D. Preliminary Presentation

Optimal Mass Transport Theory and Its Applications

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Outline

Outline

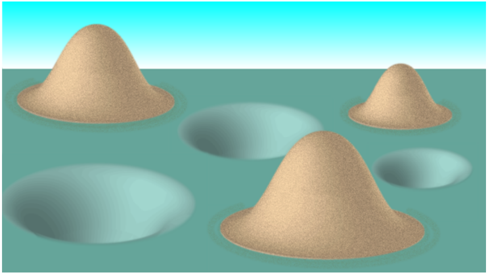


Figure 1: How do you best move given piles of sand to fill up given holes of the same total volume?

Monge Problem In the 18th century, Monge first introduced a problem minimizing the inter-domain transportation cost while preserving measure quantities.With modern notations, it can be stated as follows:

Problem 1

Given two probability measures $\mu \in \mathbb{P}(X)$ and $\nu \in \mathbb{P}(Y)$, and a cost function $c : X \times Y \to [0, +\infty]$, solve

$$\inf\{M(T) := \int c(x, T(x))d\mu(x) : T_{\#}\mu = \nu\}$$

where we recall that the measure denoted by $T_{\#}\mu$ is defined through $(T_{\#}\mu)(A) := \mu(T^{-1}(A))$ for every A, and is called image measure or push-forward of μ through T.

Kantorovich Problem Monge Problem makes it possible to merge mass but not to split mass. To overcome this difficulty, Kantorovich[2] proposed a relaxation of problem where mass can be both splitted and merged .

Problem 2

Given $\mu \in \mathbb{P}(X), \nu \in \mathbb{P}(Y)$ and $c : X \times Y \to [0, +\infty]$ we consider the problem

$$\inf\{K(\gamma) := \int_{X \times Y} c d\gamma : \gamma \in \Pi(\mu, \nu)\}$$

where $\Pi(\mu, \nu)$ is the set of the so-called transport plans,

$$\Pi(\mu, \nu) = \{\gamma \in \mathbb{P}(X \times Y) : (\pi_x)_{\#}\gamma = \mu, (\pi_y)_{\#}\gamma = \nu\}$$

where π_x and π_y are the two projections of $X \times Y$ onto X and Y.

Figure[2] shows the difference between Transport Map and Transport Plan.

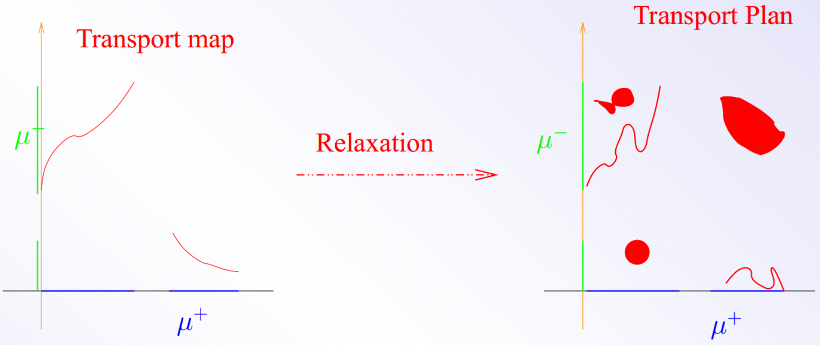


Figure 2: Transport Map vs Transport Plan

Optimal Mass Transport

The existence of such transport plans is guaranteed by the following theorems [3]

Theorem 1

Let X and Y be compact metric spaces, $\mu \in P(X)$, $\nu \in P(Y)$ and $c : X \times Y \to \mathbb{R}$ be a continuous function. Then Problem 2 has a solution.

Theorem 2

Let X and Y be compact metric spaces, $\mu \in P(X)$, $\nu \in P(Y)$ and $c : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous and bounded from below. Then Problem 2 has a solution.

Outline

Framework of Theory Concepts

Figure[3] shows the whole framework relationship of the theory concepts enrolled in my presentation.

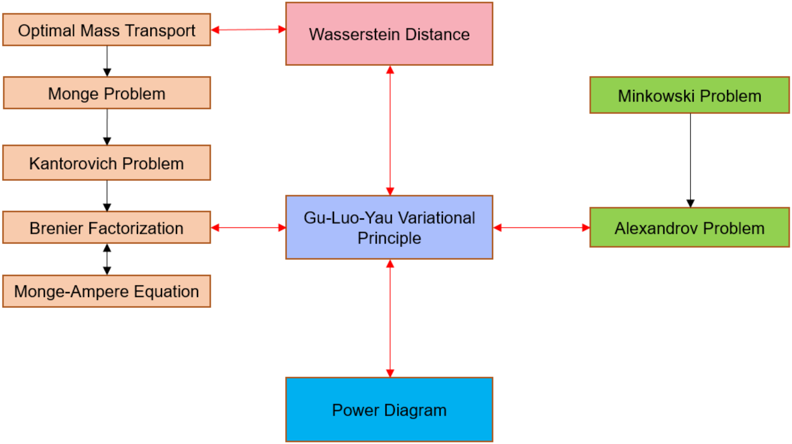


Figure 3: The framework of theory concepts

Problem 3

(**Minkowski problem for compact polytopes in $\mathbb{R}^n$.**) Suppose $n_1, n_2, \ldots, n_k$ are unit vectors which span $\mathbb{R}^n$ and $A_1, \ldots, A_k > 0$ so that $\sum_{i=1}^{k} A_i n_i = 0$. Find a compact convex polytope $P \subset \mathbb{R}^n$ with exactly k codimension-1 faces $F_1, \ldots, F_k$ so that n_i is the outward normal vector to F_i and the area of F_i is A_i.

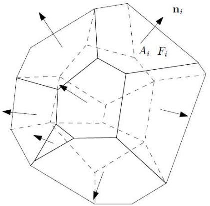


Figure 4: Minkowski problem

Theorem 3

Alexandrov Theorem *Suppose Ω is a compact convex polytope with non-empty interior in $\mathbb{R}^n$, $p_1, ..., p_k \subset \mathbb{R}^n$ are distinct k points and $A_1, ..., A_k > 0$ so that $\sum_{i=1}^{k} A_i = vol(\Omega)$. Then there exists a vector $h = (h_1, ..., h_k) \in \mathbb{R}^k$, unique up to adding the constant$(c, c, ..., c)$, so that the piecewise linear convex function*

$$u(x) = \max_{1 \leq i \leq k} \{x \cdot p_i + h_i\}$$

satisfies $vol(\{x \in \Omega | \nabla u(x) = p_i\}) = A_i$

Convex Geometry
Alexandrov Theorem

- The graph of the convex function u is an infinite convex polyhedron.
- The PL convex function produces a convex cell decomposition $\{W_i\}$

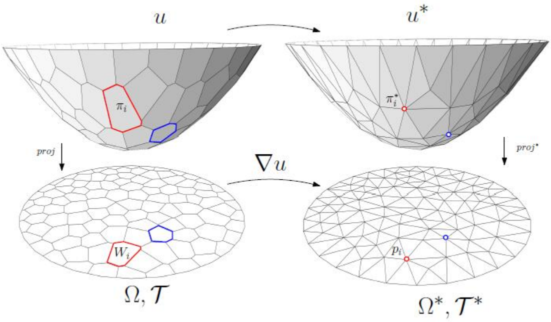


Figure 5: A PL convex function induces a cell decomposition of Ω. Each cell maps to a point

Brenier's Polar Factorization

Theorem 4

(Polar Factorization[4]). Let Ω_0 *and* Ω_1 *be two convex subdomains of* $\mathbb{R}^n$ *with smooth boundaries, each with a positive density function* μ_0, μ_1 *respectively, and of the same total mass* $\int_{\Omega_0} \mu_0 = \int_{\Omega_1} \mu_1$. *Let* $\phi : (\Omega_0, \mu_0) \to (\Omega_1, \mu_1)$ *be an diffeomorphic mapping, then* ϕ *has a unique decomposition of the form*

$$\phi = (\nabla u) \circ s$$

where $u : \Omega_0 \to \mathbb{R}$ *is a convex function,* $s : (\Omega_0, \mu_0) \to (\Omega_0, \mu_0)$ *is a measure-preserving mapping. This is called a polar factorization of* ϕ *with respect to* μ_0.

Brenier's Polar Factorization

A diffeomorphism $\phi : (\Omega_0, \mu_0) \to (\Omega_1, \mu_1)$, where $\mu_1 = \phi_{\#}\mu_0$, can be decomposed to the composition of a measure preserving map $s : (\Omega_0, \mu_0) \to (\Omega_0, \mu_0)$ and a L^2 optimal mass transport map [4] $\nabla u : (\Omega_0, \mu_0) \to (\Omega_1, \mu_1)$, and the composition is unique.
According to *Polar Decomposition*, $\nabla u^* = (\nabla u)^{-1} : (\Omega_1, \mu_1) \to (\Omega_0, \mu_0)$ is also an optimal mass transportation map. The measure-preserving map s can be computed directly by $s = (\nabla u)^{-1} \circ \phi$.

Definition 1

Alexandrov map The gradient map $\nabla u : x \mapsto \nabla u(x)$ is the Alexandrov map.

- u and $\nabla u(x)$ in the theorem is called the *Alexandrov potential* and *Alexandrov map*
- *Alexandrov map* is the unique OMT map
- The gradient map minimizes the energy

$$\int_{\Omega} ||x - T(x)||^2 dx$$

Variational Principle

Theorem 5

Let Ω be a compact convex domain in $\mathbb{R}^n$ and $\{p_1, ..., p_k\}$ a set of distinct points in $\mathbb{R}^n$ and $\sigma : \Omega \to \mathbb{R}$ be a positive continuous function. Then for any $A_1, ..., A_k > 0$ with $\sum_{i=1}^k A_i = \int_\Omega \sigma(x)dx$, there exists $b = (b_1, ..., b_k) \in \mathbb{R}^k$, unique up to adding a constant $(c, ..., c)$, so that $\int_{W_i(b)\cap\Omega} \sigma(x)dx = A_i$ for all i. The vectors b are exactly minimum points of the convex function

$$E(h) = \int_a^h \sum_{i=1}^k \int_{W_i(h)\cap\Omega} \sigma(x)dxdh_i - \sum_{i=1}^k h_i A_i$$

on the open convex set $H = \{h \in \mathbb{R}^k | vol(W_i(h) \cap \Omega) > 0 \forall i\}$.

Variational Principle

Theorem 5

(continued) In fact, $E(h)$ restricted to $H_0 = H \cap \{h | \sum_{i=1}^{k} h_i = 0\}$ is strictly convex. Furthermore, ∇u_b minimizes the quadratic cost $\int_\Omega |x - T(x)|^2 \sigma dx$ among all transport maps $T : (\Omega, \sigma dx) \to (\mathbb{R}^n, \sum_{i=1}^{k} A_i \delta_{p_i})$. where u_b is defined as the PL convex function

$$u_b(x) = \max_i \{x \cdot p_i + b_i\}$$

and the closed convex polytope is denoted as

$$W_i(h) = \{x \in \mathbb{R}^n | \nabla u(x) = p_i\} = \{x | x \cdot p_i + b_i \geq x \cdot p_j + b_j \; \textit{for all } j\}$$

- Voronoi Diagram $\forall$ p, the convex region $R(p) = \{x \in E^d | d(x,p) \leq d(x,q), \forall q \in M - \{p\}\}$.
- Power Voronoi Diagram use the power distance.

$$POW(x, p_i) = \frac{1}{2}||x - p_i||^2 - \frac{1}{2}h_i.$$

- Power Diagram is a partition of Euclidean plane,

$$W_i = \{x | Pow(x, p_i) \leq Pow(x, p_j), \forall j\}.$$

- Computing the Power Diagram is equivalent to computing the Alexandrov map.

Computational Geometry

Power Diagram

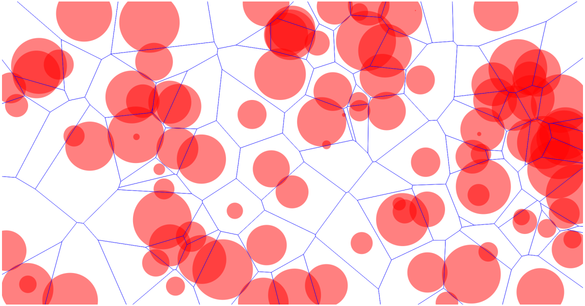


Figure 6: Power diagram.

Monge-Ampere Equation[5]

Theorem 6

Let μ and ν be two compactly supported probability measure on $\mathbb{R}^n$. If μ is absolutely continuous with respect to the Lebesgue measure, then

i. *there exists a unique solution T to the optimal mass transport problem with cost $c(x, y) = |x - y|^2/2$;*

ii. *there exists a convex function $u : \mathbb{R}^n \to \mathbb{R}$ such that the optimal map T is given by $T(x) = \nabla u(x)$ for $\mu - a.e.\ x$*

Monge-Ampere Equation

Theorem 6

(continued) Furthermore, if $\mu(dx) = f(x)dx$ and $\nu(dy) = g(y)dy$, then T is differentiable $\mu - a.e.$ and

$$|det(\nabla T(X))| = \frac{f(x)}{g(T(x))} \quad for\ \mu - a.e.\ x \in \mathbb{R}^n.$$

Since $T = \nabla u$,[4], the formula becomes:

$$det(D^2 u(x)) = \frac{f(x)}{g(\nabla(u))}$$

which is a non-linear elliptic PDE.

Wasserstein Distance

Definition 2

Shape Distance Given two Riemannian surfaces, which are topological disks, $(S_1, \mathbf{g}_1)$ and $(S_2, \mathbf{g}_2)$, the Riemann mappings are ϕ_k, $k = 1, 2$ respectively. Let $\eta \in Mob(\mathbb{D})$ be a Mobius transformation, where $Mob(\mathbb{D})$ is the Mobius transformation group of the unit planar disk, then $\eta_k \circ \phi_k$ is still Riemann mapping. Each Riemann mapping $\eta_k \circ \phi_k$ determines a unique optimal mass transportation map $\tau_k(\phi_k, \eta_k)$. Then the distance between two surfaces is given by

$$d(S_1, S_2) := \min_{\eta_1, \eta_2 \in Mob(\mathbb{D})} \int_{\mathbb{D}} |\tau_1(\phi_1, \eta_1) - \tau_2(\phi_2, \eta_2)|^2 dxdy$$

Wasserstein Distance

Wasserstein Metric Space$(M, \mathbf{g})$ is a Riemannian manifold with a Riemannian metric $\mathbf{g}$, considering the set:

$$P_p(M) := \{\mu \in P(M) : \int |x|^p d\mu < +\infty\}.$$

For $\mu, \nu \in P_p(M)$, we define

$$W_p(\mu, \nu) := \inf_{T_{\#}\mu = \nu} (\int_M d(x, T(x))^p d\mu(x))^{\frac{1}{p}}.$$

- $W_p(\mu, \nu) \geq 0$.
- $W_p(\mu, \nu) = 0$ implies $\mu = \nu$.
- Satisfies the triangle inequality.

The quantity W_p is the *Wasserstein Distance* over $P_p(M)$.

Outline

Definition 3

(Discrete Metric). A discrete metric on a triangular mesh (S, T) is a function defined on the edges $d : E \to \mathbb{R}^+$, which satisfies the triangle inequality, on a face $[v_i, v_j, v_k]$,

$$d_{ij} + d_{jk} > d_{ki}; d_{ki} + d_{ij} > d_{jk}; d_{ik} + d_{kj} > d_{ji}.$$

Definition 4

(Delaunay Triangulation). A triangulation such that no point is inside the circumcircle of any triangle.

Definition 5

(Discrete Gauss Curvature). The discrete Gauss curvature function on a mesh is defined on vertices, $K : V \to \mathbb{R}$, such that

$$K(v) = \begin{cases} 2\pi - \sum_i \theta_i, & v \notin \partial S \\ \pi - \sum_i \theta_i, & v \in \partial S \end{cases}$$

where θ_i's are corner angles adjacent to the vertex v, and ∂S represents the boundary of the mesh.
Gauss-Bonnet:

$$\sum_i K(v_i) = 2\pi\chi(S)$$

where $\chi(S)$ is the Euler characteristic of S.

Definition 6

(Discrete Yamabe Flow). Given a surface (S, V) with a discrete metric d, given a target curvature function $\bar{K} : V \to \mathbb{R}$, $\bar{K}(v_i) \in (-\infty, 2\pi)$, and the total target curvature satisfies Gauss-Bonnet formula, the discrete Yamabe flow is defined as

$$\frac{du(v_i)}{dt} = \bar{K}(v_i) - K(v_i),$$

under the constraint $\sum_{v_i \in V} u(v_i) = 0$. As the Yamabe flow updates at each iteration, the triangulation on (S, V) always keep the property of the Delaunay triangulation.

The existence of the solution to the Yamabe flow is guarranteed by the following theorem.

Theorem 7

Suppose (S, V) is a closed connected surface and d is any discrete metric on (S, V). Then for any $\bar{K} : V \to (-\infty, 2\pi)$ satisfying Gauss-Bonnet formula, there exists a discrete metric $\bar{d}$, unique up to a scaling on (S, V), so that $\bar{d}$ is discrete conformal to d and the discrete curvature of $\bar{d}$ is $\bar{K}$. Furthermore, the $\bar{d}$ can be obtained by discrete Yamabe flow.

Algorithms

Algorithm 1 Discrete Surface Yamabe Flow

Require: : A triangular mesh Σ, A target curvature $\bar{K}$

Ensure: : A discrete metric

```
Initialize the discrete conformal factor u as 0 and conformal structure
coefficient η, such that η(e) equals to the initial edge length of e.
while max_i |K̄_i − K_i| > ε do
    compute the edge length from γ and η
    (Update the triangulation to be Delaunay using diagonal edge swap
    for each pair of adjacent faces)
    Compute the corner angle θ_i^{jk} from the edge length using cosine law
    Compute the vertex curvature K
    Compute the Hessian matrix H
    Solve linear system H δu = K̄ − K
    Update conformal factor u ← u − δu
end while
Output the result metric
```

The Hessian matrix H is defined explicitly:

$$h_{ij} = \begin{cases} -w_{ij} & v_i \sim v_j \ i \neq j \\ 0 & v_i \nsim v_j \ i \neq j \\ \sum_k w_{ik} & i = j \end{cases}$$

where w_{ij} is the cotangent edge weight defined as

$$w_{ij} := \begin{cases} cot\theta_k^{ij} + cot\theta_l^{ji} & [v_i, v_j] \notin \partial S \\ cot\theta_k^{ij} & [v_i, v_j] \in \partial S \end{cases}$$

Outline

Algorithm 2 Polar Factorization of Mapping

Require: Convex domains Ω_0 and Ω_1 in $\mathbb{R}^d$. A diffeomorphic mapping $\phi : (\Omega_0, \mu_0) \to (\Omega_1, \mu_1)$, satisfying $\mu_1 = \phi_\# \mu_0$.

Ensure: The polar factorization $\phi = \nabla u \circ s$, where s is measure-preserving and u is convex.

Compute the unique optimal mass transport map $\nabla v : (\Omega_1, \mu_1) \to (\Omega_0, \mu_0)$ using Alg.3. The convex function u is the Legendre dual of v, $u = v^*$

Compute the composition $s = \nabla v \circ \phi$

Outline

Theorem 8

Let Ω be a compact convex domain in $\mathbb{R}^n$, $\{p_1, ..., p_k\}$ be a set of distinct points in $\mathbb{R}^n$ and $\sigma : \Omega \to \mathbb{R}$ be a positive continuous function. Then for any $A_1, ..., A_k > 0$ with $\sum_{i=1}^k A_i = \int_\Omega \sigma(x)dx$, there exists $b = (h_1, ..., h_k) \in \mathbb{R}^k$, unique up to adding a constant $(c, c, ..., c)$, so that $\int_{W_i(b)\cap\Omega} \sigma(x)dx = A_i$ for all i. The vectors b are exactly minimum points of the convex function

$$E(h) = \int_a^h \sum_{i=1}^k \int_{W_i(h)\cap\Omega} \sigma(x)dxdh_i - \sum_{i=1}^k h_i A_i$$

on the open convex set $H = \{h \in \mathbb{R}^k | vol(W(h) \cap \Omega) > 0 \text{ for all } i\}$. Furthermore, ∇u_h minimizes the quadratic cost $\int_\Omega |x - T(x)|^2 \sigma(x)dx$ among all transport maps $T : (\Omega, \sigma dx) \to (\mathbb{R}^n, \sum_{i=1}^k A_i \delta_{p_i})$

In practice, the energy can be optimized using Newton's method, with the help of the computation of the energy gradient

$$\nabla E(\mathbf{h}) = (w_1(\mathbf{h}) - \nu_1, ..., w_k(\mathbf{h}) - \nu_k)^T$$

. The Hessian of $E(\mathbf{h})$ is given as following:

$$\frac{\partial^2 E(\mathbf{h})}{\partial h_i \partial h_j} = \begin{cases} \frac{\int_{e_{ij}} \mu(x)dx}{|y_j - y_i|} & W_i(\mathbf{h}) \cap W_j(\mathbf{h}) \cap \Omega \neq \emptyset \\ 0 & otherwise \end{cases}$$

Algorithm 3 Optimal Mass Transport Map

Require: The Input: (Ω,μ),(P,ν), $\nu_i > 0$, $\int_\Omega u(x)dx = \sum_{i=1}^{k} \nu_i$

The Output: The unique discrete OMT-Map $f:(\Omega,\mu)\rightarrow(P,\nu)$

Scale and translate P, such that $P \subset \Omega$

$\mathbf{h} \leftarrow (0,0,...,0)$

Compute the power diagram $D(\mathbf{h})$,dual power Delaunay triangulation $T(\mathbf{h})$,the cell areas $\mathbf{w}(\mathbf{h}) = (w_1(\mathbf{h}),...,w_k(\mathbf{h}))$

while $||\nabla E|| < \epsilon$ **do**

    Compute ∇E and Hessian matrix

    $\lambda \leftarrow 1$

    $\mathbf{h} \leftarrow \mathbf{h} - \lambda H^{-1} \nabla E(\mathbf{h})$

    Compute $D(\mathbf{h})$, $T(\mathbf{h})$, and $\mathbf{w}(\mathbf{h})$

    while $\exists w_i(\mathbf{h}) == 0$ **do**

        Update $\mathbf{h} \leftarrow \mathbf{h} + \lambda H^{-1}\nabla E(\mathbf{h})$, $\lambda \leftarrow \frac{1}{2}\lambda$,$\mathbf{h} \leftarrow \mathbf{h} - \lambda H^{-1}\nabla E(\mathbf{h})$

        Compute $D(\mathbf{h})$, $T(\mathbf{h})$, and $\mathbf{w}(\mathbf{h})$

    end while

end while

Output the result mapping $f:\Omega \rightarrow P, W_i(\mathbf{h}) \rightarrow p_i,\ i = 1,2,...,k.$

Outline

Suppose S is a topological disk, with Riemannian metric g. Scale the surface so that the area is π. According to Riemann Mapping theorem, there is a conformal mapping $\phi : (S, \mathbf{g}) \to (\mathbb{D}, dzd\bar{z})$, such that $\mathbf{g} = e^{2\lambda(z)}dzd\bar{z}$. Then we can find the OMT map $\tau : (\mathbb{D}, dzd\bar{z}) \to (\mathbb{D}, e^{2\lambda}dzd\bar{z})$, and the composition $\tau^{-1} \circ \phi : (S, \mathbf{g}) \to (\mathbb{D}, dzd\bar{z})$ gives the area-preserving mapping.

Algorithm 4 Topological Disk Area-preserving Parameterization

Require: The inputs: a triangular mesh M, which is a topological disk; three vertices $\{v_0, v_1, v_2\} \subset \partial M$
The output: The area-preserving parameterization $f : M \to \mathbb{D}$, which maps $\{v_0, v_1, v_2\}$ to $\{1, i, -1\}$ respectively.

1: Scale M such that the total area is π
2: Compute the conformal parameterization $\phi : M \to \mathbb{D}$, such that the images of $\{v_0, v_1, v_2\}$ are $\{1, i, -1\}$
3: For each vertex $v_i \in M$, define $p_i = \phi(v_i)$, ν_i to be $\frac{1}{3}$ of the total area of the faces adjacent to v_i. Set $P = \{p_i\}, \nu = \{\nu_i\}$
4: Compute the *Discrete Optimal Mass Transport Map* with Algorithm 3
5: Construct the mapping $\tau^{-1} \circ \phi : M \to \mathbb{D}$, which maps each vertex $v_i \in M$ to the centroid of $W_i(\mathbf{h}) \subset \mathbb{D}$

Outline

After we have the OMT map between two surfaces M_1, M_2 with topological disk algorithm, we will have the map: $f : \Omega \to P, W_i(\mathbf{h}) \to p_i$. Therefore, the Wasserstein distance between M_1 and M_2 can be defined as

$$d_W(\mu, \nu) = \sum_{i=1}^{n} \int_{W_i} (x - p_i)^2 \mu(x) dx$$

Algorithm 5 Computing Wasserstein Distance for Two Surfaces

Require: **Input:** Two topological disk surfaces: $(M_1, g_1), (M_2, g_2)$.
Output: The Wasserstein distance between M_1 and M_2

1: Scale and normalize M_1 and M_2 such that the total area of each is π.
2: Compute the conformal maps $\phi_1 : M_1 \to \mathbb{D}_1$, and $\phi_2 : M_2 \to \mathbb{D}_2$ defined above.
3: Construct the convex planar domain (Ω, μ) from $\mathbb{D}_1$, $\mu \leftarrow e^{2\lambda_1} d_A$
4: Discretize $\mathbb{D}_2$ into a planar point set with measure (P, ν)
5: With (Ω, μ) and (P, ν) as inputs, compute the Optimal Mass Transport map f with Algorithm 3
6: Output the Wasserstein distance $d_W(\mu, \nu)$.

Outline

Utilizing the Wasserstein distance to classify the patients' and normal person's hippocampus. Table[7] and Fig.[8] demonstrate our method has better classification performance than other methods.

Wasserstein Distance for Shape Analysis[6]

Surface Matching

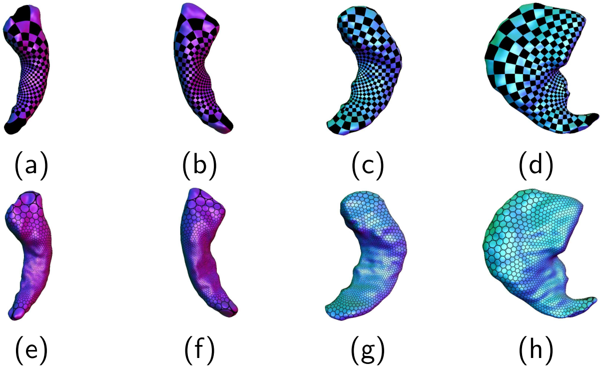


Figure 7: Conformal texture-mapping results. (a)-(d) conformal checkerboard-texture mapping results of left and right hippocampus surfaces in normal control and epilepsy data groups, respectively; (e)-(h)conformal circle-texture mapping results of left and right hippocampus surfaces in normal control and epilepsy data groups, respectively.

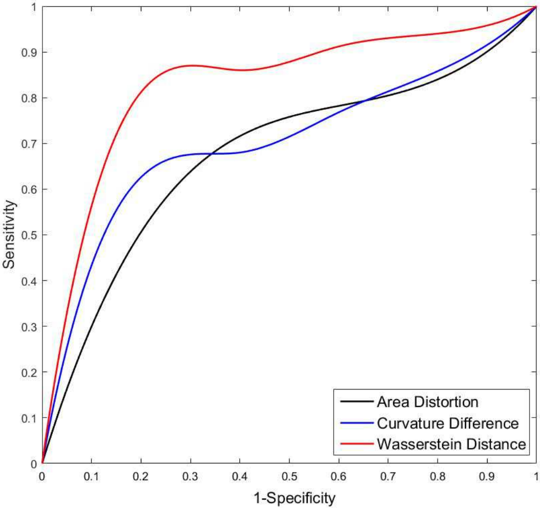


Figure 8: Comparison of average ROC curves for three methods.

Table 1: Average AUC value.

Methods	Average AUC value
Area Distortion	0.6948
Curvature Difference	0.7342
Curvature Difference + Area Distortion	0.7542
Wasserstein Distance	0.8834

Outline

Optimal Mass Transport For Visualization

Our work[7] allows users to fully control the texture area of region of interests, which will be helpful in medical image field. By adjusting the target measure, the user can zoom or shrink specific regions on the surface as shown in Fig.[9] and [10]. The top row demonstrates that the user can control the areas of the holes, the bottom row shows the user can enlarge/shrink the nose region with different scaling factors in Fig.[9]. The similar observation is also obtained from skull model shown in Fig.[10].

Optimal Mass Transport For Visualization

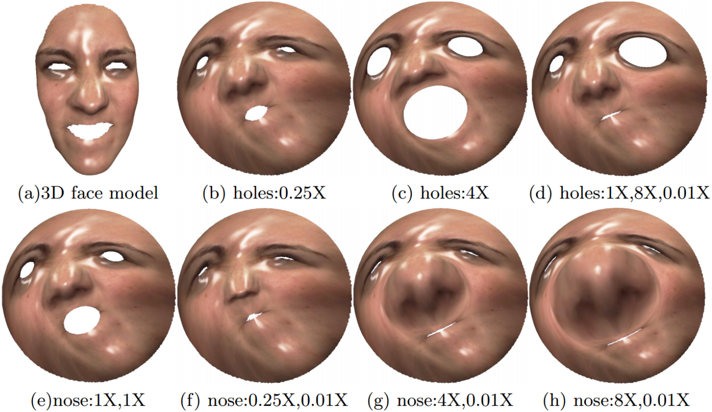
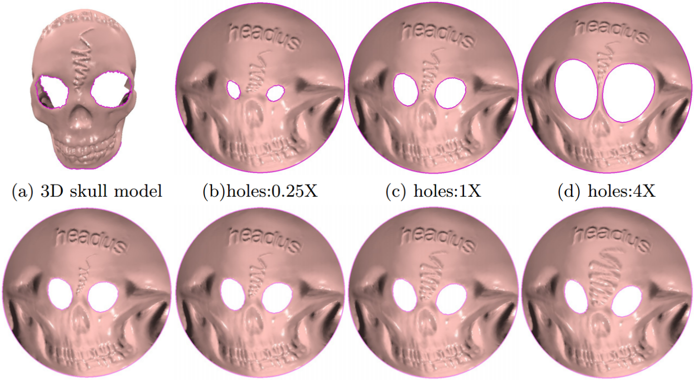


Figure 9: Importance driven surface parameterization for human face; (a) the 3d face model; (b)-(d) shows importance driven results of the holes with different scale factors; (e)-(h) shows the importance driven results of the nose and holes with different scale factors.

Optimal Mass Transport For Visualization



(a) 3D skull model (b)holes:0.25X (c) holes:1X (d) holes:4X

(e) curve:0.25X (f) curve:1X (g) curve:2X (h) curve:4X

Figure 10: Importance driven surface parameterization for skull; (a) the 3d skull model; (b)-(d) shows importance driven results of the holes with different scale factors; (e)-(h) shows the importance driven results of the curve with different scale factors.

Outline

Volume Preserving Morphing

In our paper[8], our OMT method Fig.[12] can produce more smooth morphing result than Fig.[11].

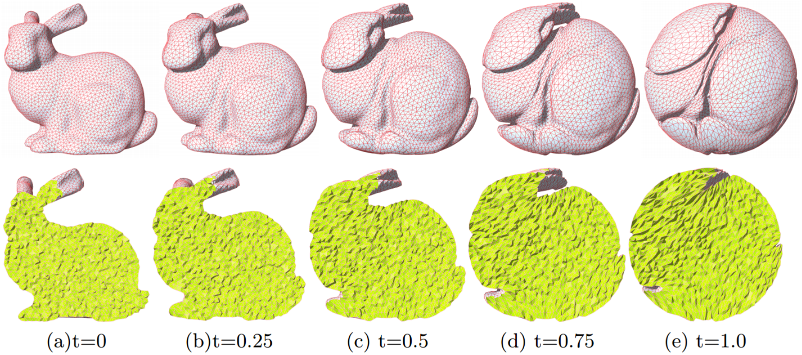


Figure 11: The volume morphing of Bunny(side view)using Bruno Levy method at typical time

Volume Preserving Parameterization

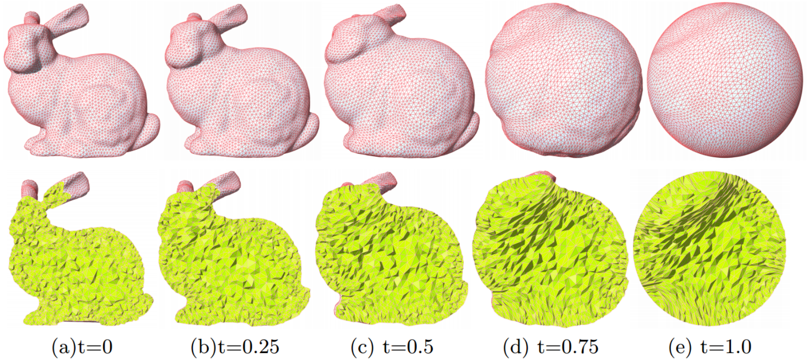


Figure 12: The volume morphing of Bunny(side view)using OMT at typical time

Conclusion

In this presentation, I introduce the history of Optimal Mass Transport(OMT) and our present research approach and result of OMT.

1. Brenier's polar factorization[4] and Yau-Luo-Gu's[9] variational principle method establish the bridge to link $Minkowski - Alexandrov$ problem and $Monge - Kantorovich$ problem.
2. Introduce the Wasserstein distance to describe the similarity of two Riemannian surfaces.
3. OMT and Wasserstein distance's applications in Shape Analysis, Visualization and Volume-Preserving morphing or Volume Parameterization.

Future Works

In future, there are three directions need more effort.

1. High dimension OMT algorithm
2. OMT Map in $3D$ human face tracking
3. Volume parametrization

Gaspard Monge.
Mémoire sur la théorie des déblais et des remblais.
De l'Imprimerie Royale, 1781.

Leonid V Kantorovich.
On the transfer of masses.
In *Dokl. Akad. Nauk. SSSR*, volume 37, pages 227–229, 1942.

Filippo Santambrogio.
Optimal transport for applied mathematicians.
Progress in Nonlinear Differential Equations and their applications, 87, 2015.

Yann Brenier.
Polar factorization and monotone rearrangement of vector-valued functions.
Communications on pure and applied mathematics, 44(4):375–417, 1991.

Guido De Philippis and Alessio Figalli.
The monge–ampère equation and its link to optimal transportation.

Bulletin of the American Mathematical Society, 51(4):527–580, 2014.

Zhengyu Su, Yalin Wang, Rui Shi, Wei Zeng, Jian Sun, Feng Luo, and Xianfeng Gu.
Optimal mass transport for shape matching and comparison.
IEEE transactions on pattern analysis and machine intelligence, 37(11):2246–2259, 2015.

Kehua Su, Li Cui, Kun Qian, Na Lei, Junwei Zhang, Min Zhang, and Xianfeng David Gu.
Area-preserving mesh parameterization for poly-annulus surfaces based on optimal mass transportation.
Computer Aided Geometric Design, 2016.

Kehua Su, Wei Chen, Na Lei, Junwei Zhang, Kun Qian, and Xianfeng Gu.
Volume preserving mesh parameterization based on optimal mass transportation.
Computer-Aided Design, 2016.

Xianfeng Gu, Feng Luo, Jian Sun, and S-T Yau.

Variational principles for minkowski type problems, discrete optimal transport, and discrete monge-ampere equations.
arXiv preprint arXiv:1302.5472, 2013.