Toroidal Networks

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Introduction

As long as networks have been studied, certain regular structures have held a fascination for engineers. In the world of electric circuits, ladder networks are one example. Likewise, in the areas of computer communication and computer intercommunication certain network structures have an appeal that is, in some sense, almost aesthetic. One such structure is the toroidal network.

A two dimensional toroidal network is simply a rectangular mesh where opposite nodes on the left and right boundaries are connected to each other and opposite nodes on the upper and lower boundaries are also connected to each other. Nodes correspond to processors or local access points and the edges to the communicating links.

Such toroidal structures have been proposed as a possible interconnection network for multiprocessor computers. This includes VLSI implementations. They have also been proposed as a possible architecture for metropolitan area networks. But why should one network structure be called upon to meet the demands of such a widely differing scale?

One reason is that addressing and routing is straightforward in a toroidal network. Moreover, the topology is isotropic; that is, every node has a similar set of connections to its neighbors. There are no edge effects, and the wrap around connections decrease path lengths. For mapping computation problems onto multiprocessors the toroidal topology can easily be associated with a Euclidean space. Similarly, for metropolitan area networks the topology easily covers a rectangular grid of streets and avenues.

In the following section the essential features of the toroidal network topology will be examined. This is followed by an exposition of some of the proposed applications of toroidal networks.

Performance Evaluation of the Toroidal Network Topology

Topology

The surface of a torus can be mapped onto a rectangular area where opposite sides are identified as being identical. This is a "plane model" of the surface of the torus. Actually the surface of the torus is an example of a special type of space that is of great interest to mathematicians. That space is a manifold.

A manifold is, essentially, a finite space without a boundary. This sounds contradictory until one realizes that because of the way in which they close upon themselves, both the surface of the torus and the surface of the sphere fit this description. Give one turn to the left and right boundaries of a rectangle before connecting them (connecting the upper and lower boundaries without a turn) and another manifold, the Klein bottle, results. There are actually four basic types of two dimensional manifolds, the last one being the projective plane [4,5,18].

When the graph of a network has been placed in a manifold, one can say that the network graph has been *embedded* into the space. This is illustrated for the toroidal network in two ways: in Fig. 1a by a plane model of a torus and in Fig. 1b by a three dimensional visualization of the surface of a torus.

Some embeddings have more symmetry and structure than others. The toroidal network is a planar type embedding. It also forms a complex; that is, each polygonal face has at least three links (edges) and each node lies on at least three links. Moreover, two distinct polygons meet only at a single node or only along a single link (see Firby and Gardiner [4] for an excellent introductory discussion).

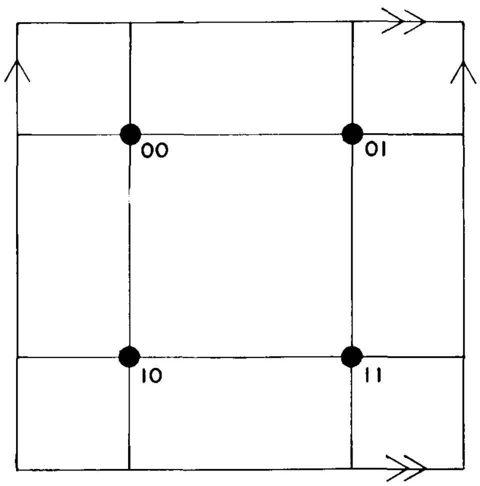

Fig. 1a. A toroidal network embedded in a plane model.

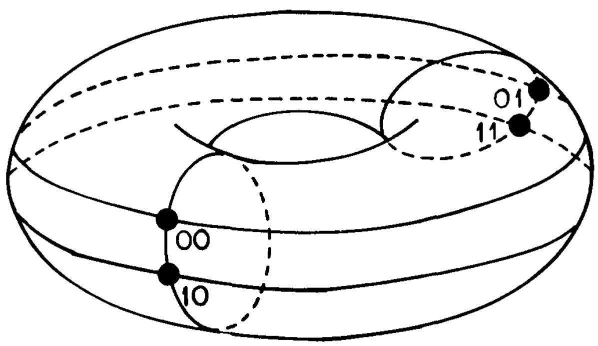

Fig. 1b. A toroidal network embedded in a torus.

The toroidal network is also a regular complex in that each polygonal face has the same number of links and each node is connected to the same number of links. Are there other regular complexes which could be embedded on the surface of a torus? An important characteristic of a manifold is its Euler characteristic, $\chi(M)$. This is zero for the torus. Now [4]:

$$\chi(M) = N - L + F$$

where there are N vertices, L links and F faces. Each link belongs to two polygons so if a is the number of links per face:

$$2L = aF$$

Similarly, each link belongs to two nodes so if b is the number of links connected to a node:

$$2L = bN$$

Substituting, one obtains:

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{2}$$

This can be solved as:

$$\begin{array}{ll} a=4 & b=4 \\ a=6 & b=3 \\ a=3 & b=6 \end{array}$$

The first case is just the rectangular toroidal network of Fig. 1. The latter two cases consist of embeddings of hexagons and triangles, respectively. In the remainder of this work only the rectangular toroidal network will be considered.

It is possible to generalize the two dimensional surface of a torus into higher dimensions. For instance, consider a three dimensional cubic volume of space where the opposite sides of the cube are identified as being equal. This forms a three dimensional toroidal manifold or hypertorus. A three dimensional cubic graph can be very naturally embedded into this space. While such three dimensional topologies have been considered for multiprocessor interconnection, almost all work to date has involved two dimensional structures.

Characteristics of the Toroidal Network

A D dimensional toroidal network has:

$$\begin{array}{rcl} N \text{ nodes} & = & W^D \\ L \text{ links} & = & DW^D \\ C \text{ connections} & = & 2DW^D \end{array}$$

where W is the width of the network [13]. The longest path between nodes (network diameter) for the first three dimensonalities is:

$$\begin{array}{lll} D=1 & \frac{W}{2} & \frac{N}{2} \\ D=2 & W & N^{1/2} \\ D=3 & \frac{3W}{2} & \frac{3N^{1/3}}{2} \end{array}$$

As pointed out in [6], for 1,024 nodes the ratio of the longest path for one, two, and three dimensions is 512:32:15. The improvement from one to two dimensions is greater than that from two to three dimensions.

To calculate the average distance between nodes for uniform traffic, one can make use of the fact that each node is connected to D orthogonal rings of circumference W. This is actually a useful conceptualization for addressing purposes. A node address is simply a vector of D numbers of size W.

Now in one dimension the average distance, $\overline{d}$, is [14,19]:

$$\overline{d}_{1D} = \frac{\sum_{i=0}^{W-1} \min(i, W-i)}{W} = \frac{W}{4} - \frac{1}{4W} \quad W \text{ odd}$$

$$= \frac{W}{4} \quad W \text{ even}$$

Since dimensions are independent and scaling to account for a node not sending messages to itself:

$$\overline{d}_D = D \left(\frac{W^D}{W^D - 1}\right) \overline{d}_{1D} \approx \frac{DW}{4}$$

One advantage of the toroidal topology is the large multiplicity of potential paths. Suppose that two nodes are i rows and j columns apart in a two dimensional network. The number of minimum distance paths between the nodes, for W odd, is [12]:

$$\binom{i+j}{j} \quad \text{for} \quad 0 < i + j < W-1, \quad 0 \leq j < \frac{W-1}{2}$$

$$4\binom{i+j}{j} \quad \text{for} \quad i + j = W-1, \quad i = j = \frac{W-1}{2}$$

A useful quantity is the number of nodes or the fraction of nodes k hops away from a source. For odd W this fraction is:

$$f_k = \begin{cases} \dfrac{4k}{W^D - 1} & 1 \leq k \leq \dfrac{W-1}{2} \\[2ex] \dfrac{4(W-k)}{W^D - 1} & \dfrac{W+1}{2} \leq k \leq W-1 \end{cases}$$

As an example of its utility, consider a situation described by Kamoun [7]. In a two dimensional toroidal packet network, P_s is the probability that a packet reaches its destination successfully. The probability of a packet being blocked on a link is P_B. Then:

$$P_s = \sum_{k \geq 1} (1-P_B)^k f_k$$

Now the sequence f_k, $k \geq 1$ is a numerical sequence like any other. It is thus possible to take the z-transform of the sequence. This "topology z-transform" is:

$$H(z) = \sum_k f_k z^k$$

so:

$$P_s = H(1-P_B)$$

where $H(z)$ is given by [7]:

$$H(z) = \frac{4z\,(1-z^{(W-1)/2})\,(1-z^{(W+1)/2})}{(W^D-1)\,(1-z)^2}$$

Other performance measures can be determined based on this z-transform [13,14].

Some Examples of Toroidal Networks

Metropolitan Area Network Communication

Metropolitan area networks are intended to interconnect high speed local area networks over metropolitan sized regions. Local area networks operate at high data rates with simple protocols whereas wide area packet networks operate at much lower speeds with relatively complex protocols. There is thus a need for an intermediate network that can operate over distances larger than LAN distances at comparatively high data rates and with reasonable protocol complexity.

Metropolitan area network design is at an early stage, and several possible topologies have been proposed [3]. The Manhattan Street Network developed by N.F. Maxemchuk [9] is one such topology. It consists of a rectangular array of nodes connected in a toroidal fashion. This is illustrated in Fig. 2. Note that the links are unidirectional.

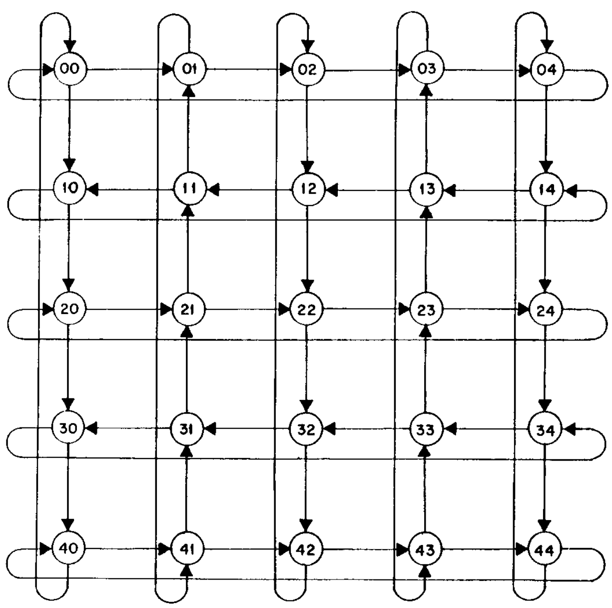

Fig. 2. A Manhattan street network.

The Manhattan Street Network is a slotted packet system. Packets arriving at a node are immediately retransmitted on outgoing links. Since each node has an equal number of incoming and outgoing links, packets are not buffered as in a store-and-forward network. A node only accepts a new packet if an empty slot is available. Thus, as the network becomes heavily loaded new traffic is prevented from entering the network.

This is not all that much different from what happens in a multiple access bus or a ring under similar conditions.

Why would one want to eliminate buffers? The motivation is the increasing speed of fiber optics transmission media relative to the electronics at communications nodes. The network algorithms implemented at nodes must be simple enough so that the nodal electronics does not become a performance bottleneck. The algorithms for store and forward packet networks are relatively complex because of the need for adaptive routing, flow control, deadlock avoidance, and packet resequencing [9]. If nodal processing is simple enough, it can be performed in hardware at transmission speeds. Input buffers may still be included but it is the buffering of transiting packets that is desirable to eliminate.

The toroidal topology makes routing very direct. At most two packets need to be routed in each slot. It can be demonstrated that 50 percent of the time it does not matter which outgoing link a packet takes (both belong to shortest route paths). Moreover, in the worst case, the increase in path length — remember that each packet must be routed on some outgoing link — is never more than four.

Other advantages of the toroidal topology include the same number of links per node, a symmetrical geometry, and a low maximal number of hops. It is also possible to add additional nodes and create hierarchies.

Another toroidal based metropolitan area network is the HR^4-NET proposed by Borgonovo and Cadorin [2]. The links here are full duplex. One advantage of this is that when a link fails, each node still has the same amount of ingoing and outgoing capacity.

In HR^4-NET, buffers are again not included in the network. Up to four packets may need to be routed outbound from a node simultaneously. There may be several equally good assignments of packets to outgoing links; therefore, HR^4-NET will randomly choose an assignment. No more than two packets will use an inefficient line.

Multiprocessor Interconnection

In designing computers consisting of multiple processors, the interconnection network for connecting the processors to each other is of critical importance. Many such interconnection network structures have been proposed [1,11,13-15,19]. One such structure, which has been implemented, is the toroidal network.

An advantage of toroidal topology involves the problem of mapping. In placing a problem onto a parallel processor, individual parts of the problem must be "mapped" onto individual processing elements in some reasonable manner. The toroidal topology is a natural choice for problems with the "proximity property" [6]. That is, the solution at each point in two or three dimensional space is a function of the solution at adjoining points of space. The topology provides Cartesian oriented communication paths between adjoining processors. This communication can take place in packet format or by shared variables in the memory of one of two adjoining processors.

What about the boundary connections? These also are useful for such operations as norm calculation, summation, maximum/minimum determination, and for periodic boundary problems in physics.

One example of such a toroidal based multiprocessor is the PAX series (Processor Array Experiment) from the Institute of Engineering Mechanics at the University of Tsukuba. Units have been constructed with up to 128 processors. A PAX system is illustrated in Fig 3. An excellent discussion of the PAX system by T. Hoshino appears in [6].

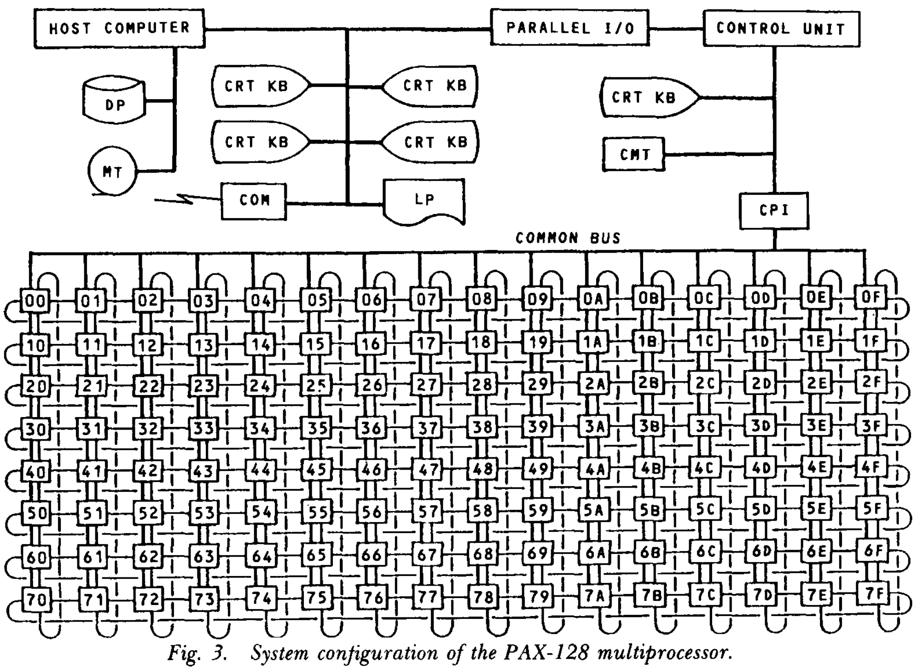

Fig. 3. System configuration of the PAX-128 multiprocessor.

Multiprocessors are becoming more accessible through the introduction of such off-the-shelf processing units as the INMOS Transputer. The Transputer, with its four connections per processor, is a natural choice for implementing a two dimensional toroidal multiprocessor. In fact, if two Transputers are combined per node, a three dimensional toroidal multiprocessor can be implemented.

VLSI Interconnection

Toroidal networks can also be used to interconnect multiprocessors implemented using VLSI technology. An interesting discussion of this by C. Sequin of the University of California at Berkeley appears in [16]. The problem involved is that of mapping a binary tree onto a toroidal network of processors.

Binary trees are a useful data structure for many situations, such as divide and conquer algorithms. In a binary tree each node has two children. Suppose that the root is mapped into the upper left hand corner processor of a toroidal network of processors, (X_0, Y_0). Its direct children are at (X_1, Y_0) and (X_0, Y_1). Its grandchildren are at (X_2, Y_0), (X_0, Y_2) and (X_1, Y_1). Note that two children are mapped onto the same processor (X_1, Y_1).

Each generation of the problem thus consists of processors located along a diagonal line running thru the array. Therefore only a small fraction of the total number of processors is busy at one time. How can one spread the problem over more processors?

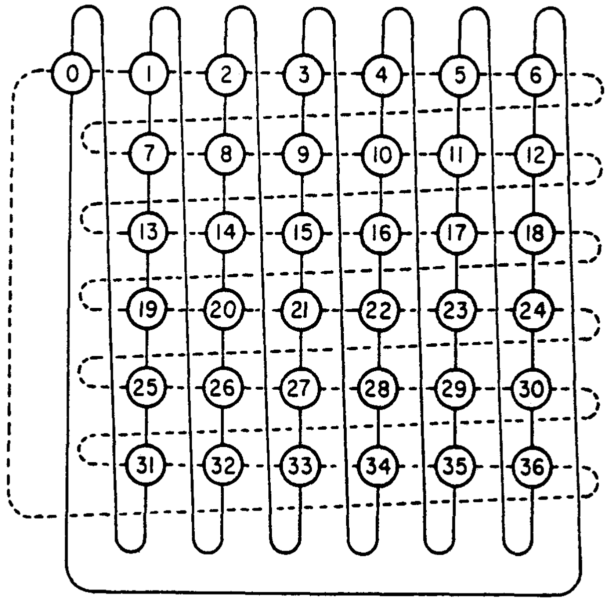

Fig. 4. Doubly twisted toroidal network.

One approach is to add a "twist" to the toroidal network. The processor at the end of each row is connected to the first processor in the row below it [8]. This will indeed spread each generation over more processors. If the tree is not symmetrical, one may attempt a double twist (vertically and horizontally). One must add the vertical twist in the opposite sense and insert an additional node, zero, in order to assure an isotropic structure [16] (Fig. 4).

In transferring such structures onto the planar surface of VLSI silicon, one must be careful as the boundary twists can lead to excessive crossings or an irregular interconnection pattern. A compromise arrangement with serpentine interconnections can be arrived at that would present a roughly homogeneous placement of processors (Fig. 5).

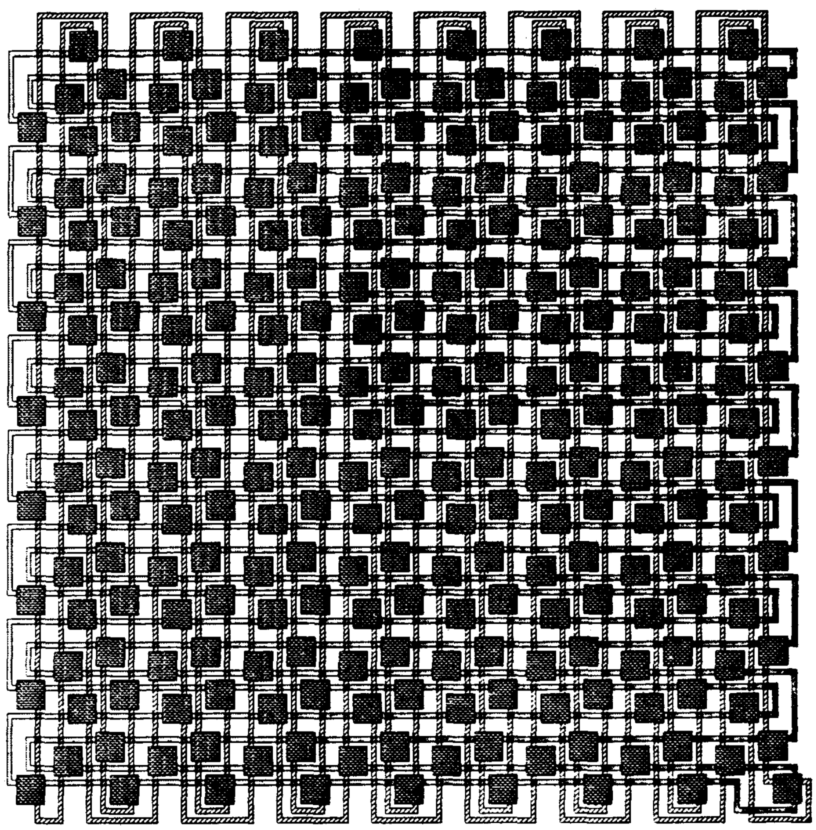

Fig. 5. Planar layout of doubly twisted toroidal network.

Conclusion

Toroidal networks are viable for a number of large and small scale applications. They are attractive because of their homogeneity, lack of boundaries, and Cartesian like geometry. Surprisingly, in an often planar world, torii can be quite useful.

Acknowledgment

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