Stochastic Processes and their Applications 83 (1999) 273–288

stochastic processes and their applications

www.elsevier.com/locate/spa

Asymptotics for Voronoi tessellations on random samples

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Received 15 September 1998; received in revised form 23 March 1999; accepted 26 March 1999

Abstract

Let $V(X_1,\dots,X_n)$ denote the total edge length of the Voronoi tessellation on random variables $X_1,\dots,X_n$. If $X_1,X_2,\dots$ are independent and have a common continuous density $f(x)$ on the unit square which is bounded away from 0 and ∞ then it is shown that

$$\lim_{n\to\infty}\frac{V(X_1,\dots,X_n)}{n^{1/2}} = 2\int_{[0,1]^2}(f(x))^{1/2}\,\mathrm{d}x \quad \text{c.c.},$$

where c.c. denotes complete convergence. © 1999 Elsevier Science B.V. All rights reserved.

MSC: 60D05; 60F15; 05C80

Keywords: Voronoi tessellation; Subadditive and superadditive Euclidean functionals; Complete convergence; Delaunay triangulation

1. Introduction

The purpose of this paper is to describe the a.s. stochastic behavior of the total edge length of planar tessellations of random point sets. Although we restrict attention to Voronoi tessellations of random sets in $\mathbb{R}^2$, the approach appears suitable for describing the a.s. behavior of the Delaunay triangulation and related Euclidean graphs in $\mathbb{R}^d$, $d\geqslant 2$.

We recall the basic definition of the planar Voronoi tessellation. Given $x_1,\dots,x_n\in[0,1]^2$, consider the locus of points closer to x_i, $1\leqslant i\leqslant n$, than to any other point. This set of points is a cell and is denoted by $\mathscr{C}(i):=\mathscr{C}(x_i)$. $\mathscr{C}(i)$ is the intersection of $n-1$ half-planes and is a convex polygonal region with at most $n-1$ sides. The cells $\mathscr{C}(i)$, $1\leqslant i\leqslant n$, partition the square into a convex net which is variously called the Voronoi tessellation, Voronoi diagram, or Dirichlet tessellation of $[0,1]^2$. We let $\mathscr{V}(x_1,\dots,x_n)$ designate the graph of the Voronoi tessellation of $[0,1]^2$ associated with the points $x_1,\dots,x_n$, which are called the Voronoi generators of $\mathscr{V}(x_1,\dots,x_n)$. We

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[1] Research supported in part by NSA grant MDA904-97-1-0053.

0304-4149/99/$ - see front matter
PII: S0304-4149(99)00035-6

are interested in the total edge length $V(x_1,\dots,x_n)$ of the graph $\mathscr{V}(x_1,\dots,x_n)$. The Delaunay triangulation is a closely related graph: it puts an edge between generators x_i and x_j if the cells $\mathscr{C}(i)$ and $\mathscr{C}(j)$ are adjacent.

Voronoi tessellations are of widespread interest and have numerous applications. They have often been used as a model for natural phenomena in agriculture, astrophysics, cell biology, communication theory, crystallography, geology, metallography, and zoology. See Aurenhammer (1991) and Okabe et al. (1992). They can be naturally interpreted as a result of a growth process; we refer to Stoyan et al. (1995) and the encyclopedic work of Okabe et al. (1992) for details and a thorough treatment of the applications. In mathematics, the Voronoi tessellation forms one of the fundamental constructions of computational geometry (Preparata and Shamos, 1985).

This paper studies the behavior of Voronoi diagrams on *random* point sets $X_1,\dots,X_n$, where X_i, $i\geqslant 1$, are i.i.d. random variables with values in $[0,1]^2$. Voronoi diagrams on random point sets are of general interest. Voronoi diagrams on Poisson point sets are discussed in Moller (1994); they are used in quantum field theory (Christ et al., 1982, Drouffe and Itzykson, 1984) and in random networks (Jerauld et al., 1984). They also occur naturally in percolation models. Percolation is usually considered with respect to the edges on a fixed lattice, but this is somewhat restrictive. A more general approach involves percolation on the edges of a Voronoi diagram defined by random generators.

We are interested in the large n behavior of the total edge length $V(X_1,\dots,X_n)$ of the Voronoi tessellation. There has been little work in this area, save for the notable work of Miles (1970) and Avram and Bertsimas (1993) who investigate Voronoi tessellations and Delaunay triangulations on a Poisson number of uniformly distributed random variables on $[0,1]^2$. More precisely, Miles (1970) considers the functional $V(U_1,\dots,U_{N(n)})$ where U_i, $i\geqslant 1$, are i.i.d. with the uniform distribution on $[0,1]^2$ and where $N(n)$ is an independent Poisson random variable with parameter n. Using ergodic theoretic methods Miles (1970, p. 115) obtains precise asymptotics for the mean total edge length:

Theorem 1.1 (Miles, 1970). *With the above notation we have*

$$\lim_{n\to\infty} \frac{\mathbb{E}V(U_1,\dots,U_{N(n)})}{n^{1/2}} = 2. \tag{1.1}$$

Avram and Bertsimas (1993) show that the functional $V(n) := V(U_1,\dots,U_{N(n)})$ is the sum of approximately independent random variables and in this way they show a central limit theorem of the form

$$\frac{V(n)-\mathbb{E}V(n)}{\sqrt{\operatorname{Var} V(n)}} \xrightarrow{\text{d}} N(0,1).$$

Talagrand (1995) uses isoperimetry to show that $V(n)$ does not differ significantly from its mean and in this way he establishes that (1.1) holds in the sense of *complete convergence* (*c.c.*). Recall that random variables Y_i, $i\geqslant 1$, converge completely to a constant C if and only if for all $\varepsilon > 0$ we have $\sum_{i=1}^{\infty} P(|Y_i - C| > \varepsilon) < \infty$.

In this paper we use the theory of subadditive and superadditive Euclidean functionals to extend Miles' asymptotics (1.1) to non-uniform random variables on $[0,1]^2$. Additionally, the asymptotics hold in the sense of complete convergence.

Theorem 1.2. *Let Y_i, $i \geqslant 1$, be i.i.d. random variables on $[0,1]^2$ with a continuous density f_Y which is bounded away from* 0 *and* ∞. *Then*

$$\lim_{n\to\infty} \frac{V(Y_1,\ldots,Y_n)}{n^{\frac{1}{2}}} = 2\int_{[0,1]^2} (f_Y(x))^{1/2}\,\mathrm{d}x \quad \text{c.c.} \tag{1.2}$$

Remarks. (1) Theorem 1.2 bears a striking resemblance to the landmark result of Beardwood et al. (1959) describing the total edge length $T(X_1,\ldots,X_n)$ of the shortest tour on i.i.d. random variables $X_1, X_2, \ldots$ with a common density $f(x)$, $x \in [0,1]^2$. They showed that

$$\lim_{n\to\infty} \frac{T(X_1,\ldots,X_n)}{n^{(d-1)/d}} = \beta\int_{[0,1]^d} (f(x))^{(d-1)/d}\,\mathrm{d}x \quad \text{a.s.}, \tag{1.3}$$

where β is a positive constant depending only on the dimension d. Other functionals in Euclidean optimization, including the total edge length of the minimal spanning tree, minimal triangulation, and minimal matching, satisfy a limit similar to (1.3); see the monographs of Steele (1997) and Yukich (1998) for details and references. *Theorem* 1.2 *thus shows that the asymptotics of the Voronoi length functional resemble those of the classic problems of Euclidean combinatorial optimization.*

(2) The main attraction of complete convergence is not that it strengthens a.s. convergence, but that it provides convergence results for the two distinctly different ways to interpret the dependence of functionals $V(X_1,\ldots,X_n)$ and $V(X_1,\ldots,X_{n+1})$. Given the functional $V(X_1,\ldots,X_n)$, one can increment the number of existing sample points by one to get the new functional $V(X_1,\ldots,X_n,X_{n+1})$; this is the so-called *incrementing model of problem generation*. However, one can also consider the functional which is based on a completely new sample of points $\{X_1',\ldots,X_n',X_{n+1}'\}$ to get the new functional $V(X_1',\ldots,X_{n+1}')$. This second method is the *independent model of problem generation.* The difference between the limit theory for the two models is analogous to the difference between the limit theory of sequences and triangular arrays of random variables. A.s. limit results for the independent model imply a.s. limits for the incrementing model, but without extra assumptions, the converse is false in general. To prove a.s. limits for both models of problem generation we will show the complete convergence of $V(X_1,\ldots,X_n)/n^{1/2}$. Notice that the "hard" half of the Borel–Cantelli lemma shows that c.c. results are necessary if one is to obtain a.s. asymptotics in the context of the independent model. Weide (1978) was the first to recognize the need for complete convergence in the probabilistic analysis of algorithms.

(3) The limit (1.2) will in general fail without assumptions on the underlying distribution. For example, if the random variables Y_i, $i \geqslant 1$, have support on a linear subset of $[0,1]^2$ then $V(Y_1,\ldots,Y_n) = \Theta(n)$, violating the $n^{1/2}$ growth rate prescribed by (1.2).

(4) By Hölder's inequality the right-hand side of (1.2) is largest when f_Y is the uniform density on $[0,1]^2$.

As noted in Remark 1, the connection between (1.2) and (1.3) is hardly accidental and reflects similarities in the structure of Voronoi tessellations and graphs of problems in Euclidean combinatorial optimization. To make these ideas more precise, we first recall some of the properties enjoyed by the total edge length of a typical graph in combinatorial optimization. Here and elsewhere, $x_1,\ldots,x_n$ denotes points in $\mathbb{R}^d$, $d\geqslant 2$, α is a positive scalar, and L denotes a functional (such as the total edge length of a graph on $\{x_1,\ldots,x_n\}$) defined on pairs (F,R), where F is a finite set in $\mathbb{R}^d$ and R is a d-dimensional rectangle in $\mathbb{R}^d$. When $F\subset[0,1]^d$ we simply write $L(F)$ instead of $L(F,[0,1]^d)$.

$$\textit{Scaling}: L(\{\alpha x_1,\ldots,\alpha x_n\},\alpha R)=\alpha L(\{x_1,\ldots,x_n\},R) \text{ for all } \alpha>0, \tag{1.4}$$

$$\textit{Translation invariance}: L(\{x_1+y,\ldots,x_n+y\},R+y) = L(\{x_1,\ldots,x_n\},R) \text{ for all } y\in\mathbb{R}^d, \tag{1.5}$$

$$\textit{Subadditivity}: L(x_1,\ldots,x_n)\leqslant \sum\nolimits_{j=1}^{m^d} L(\{x_1,\ldots,x_n\}\cap Q_j,Q_j)+Cm^{d-1} \text{ where } \{Q_j\}_{j=1}^{m^d} \text{ is the partition of } [0,1]^d \text{ into subcubes of edge length } 1/m, \tag{1.6}$$

$$\textit{Superadditivity}: L(x_1,\ldots,x_n)\geqslant \sum\nolimits_{j=1}^{m^d} L(\{x_1,\ldots,x_n\}\cap Q_j,Q_j)-Cm^{d-1}, \text{ and} \tag{1.7}$$

$$\textit{Smoothness}: \text{ for all } n,k\in\mathbb{N}\ |L(x_1,\ldots,x_n)-L(x_1,\ldots,x_k)|\leqslant C|n-k|^{(d-1)/d}. \tag{1.8}$$

Here C is a positive constant depending only on L and d. L is a *Euclidean functional* if (1.4) and (1.5) are satisfied.

When functionals L satisfy conditions (1.4)–(1.8) then general theorems of Redmond and Yukich (1994,1996) describe the asymptotics of $L(X_1,\ldots,X_n)$, where X_i, $i\geqslant 1$, are i.i.d. random variables. The Voronoi length functional V satisfies (1.4) and (1.5) and is thus Euclidean. However, *V does not satisfy properties* (1.6)–(1.8) and therefore the general approach of Redmond and Yukich (1994,1996) may not be applied.

It turns out that V satisfies modified versions of (1.6)–(1.8); these versions are weaker but still strong enough to deliver the desired asymptotics (1.2). The following definitions play a key role; here X, X_i, $i\geqslant 1$, denote i.i.d. random variables with values in $[0,1]^d$ such that the law of X is nonatomic, and $N(n)$ is an independent Poisson random variable with parameter n.

Definition 1.3. A Euclidean functional L is *subadditive in mean* if

$$\mathbb{E}L(X_1,\ldots,X_{N(n)})\leqslant \sum_{j=1}^{m^d} \mathbb{E}L(\{X_1,\ldots,X_{N(n)}\}\cap Q_j,Q_j)+c_1(m)c_2(n) \tag{1.9}$$

and L is *superadditive in mean* if

$$\mathbb{E}L(X_1,\ldots,X_{N(n)})\geqslant \sum_{j=1}^{m^d} \mathbb{E}L(\{X_1,\ldots,X_{N(n)}\}\cap Q_j,Q_j)-c_1(m)c_2(n) \tag{1.10}$$

where $c_1(m)$ depends only on m and $c_2(n)=\mathrm{o}(n^{(d-1)/d})$.

The following notion of smoothness represents a natural weakening of the standard smoothness condition (1.8).

Definition 1.4. A Euclidean functional L is *smooth in mean* if there exists a constant C such that whenever X, X_i, $i \geqslant 1$, are i.i.d. random variables on $[0,1]^d$ and W, W_i, $i \geqslant 1$, are i.i.d. random variables on $[0,1]^d$ which are independent of X, X_i, then for all $1 \leqslant k \leqslant n/2$ we have

$$|\mathbb{E}L(X_1,\dots,X_n,W_1,\dots,W_k) - \mathbb{E}L(X_1,\dots,X_n)| \leqslant Ck^{(d-1)/d}. \tag{1.11}$$

Terminology: 1. If a Euclidean functional L satisfies conditions (1.9)–(1.11) then we omit the term "in mean" and simply say that L is subadditive, superadditive, and smooth, respectively.

2. Throughout we denote by C a positive constant whose value may change from line to line. C may occasionally depend on auxiliary constants α, β, γ, and m and sometimes the notation will reflect this.

2. Auxiliary results

This section collects some results which will be useful in the sequel. Here and henceforth X_i, $i \geqslant 1$, and Y_i, $i \geqslant 1$, are i.i.d. random variables in $[0,1]^2$ with respective densities f_X and f_Y satisfying

$$\alpha \leqslant f_X \leqslant \gamma \quad \text{and} \quad \alpha \leqslant f_Y \leqslant \gamma, \tag{2.1}$$

where α and γ are positive, finite constants. Our main goal in this section is to show that when $d=2$, V satisfies subadditivity (1.9), superadditivity (1.10), and smoothness (1.11) *if we restrict attention to densities of the type* (2.1).

The first result provides some crude but handy upper bounds for $V(F)$, $F \subset [0,1]^2$.

Lemma 2.1. *For all finite* $F \subset [0,1]^2$, $V(F) \leqslant C \cdot \operatorname{card} F$.

Proof. We count the number of edges in $\mathscr{V}(F)$ which do not lie on the boundary of $[0,1]^2$, which we represent as a single vertex. The degree of every vertex in the resulting graph is at least three and since each edge has two vertices we obtain $3v \leqslant 2e$, where v and e denote the number of vertices and edges in the graph, respectively. Combining this bound with Euler's formula shows that there are at most 3card F edges in $\mathscr{V}(F)$. $\square$

The next result provides a "high probability" (i.e., with probability at least $1-n^{-\beta}$, β large) edge length bound for edges in the graph $\mathscr{V}(X_1,\dots,X_n)$.

Lemma 2.2. *For all* $\beta > 0$ *there is a constant* $C := C(\beta)$ *such that the length of the longest edge in the Voronoi diagram on* $\{X_1,\dots,X_n\}$ *is at most* $C\sqrt{\log n/\alpha n}$ *with probability at least* $1-n^{-\beta}$.

Proof. If the longest edge has a length $t > 0$, then there is a disk of radius Kt, $K > 0$ a constant, which does not contain any sample points. However for all $\beta > 0$ there is a constant C such that with probability at least $1-n^{-\beta}$, the largest hole in the sample $\{X_i\}_{i=1}^n$ has diameter at most $C\sqrt{\log n/\alpha n}$. $\square$

The next lemma captures the "high probability" local behavior of Voronoi diagrams. This local dependence was noted earlier by Avram and Bertsimas (1993).

Lemma 2.3. *For all $\beta > 0$ there is a constant $C := C(\beta)$ such that with probability at least $1 - n^{-\beta}$, $\{X_i\}_{i=1}^n$ has the property that for all $X \in \{X_i\}_{i=1}^n$ only points at a distance less than $C\sqrt{\log n/\alpha n}$ from X generate an edge in the Voronoi cell $\mathscr{C}(X)$ in $\mathscr{V}(X_1,\ldots,X_n)$.*

Proof. Suppose that X_1 generates an edge E belonging to the cell $\mathscr{C}(X)$ and that $||X_1 - X|| > C\sqrt{\log n/\alpha n}$. Consider the ball B centered around (X_1,X) of diameter $||X_1 - X||$. Let D denote that half of the ball B whose diameter coincides with the edge E and which contains X. Recall that with high probability $\{X_i\}_{i=1}^n$ does not have any holes with diameter larger than $C\sqrt{\log n/\alpha n}$. Thus the interior of D contains generators with high probability. Thus the distance between the points in $D \cap E$ and a generator in D is smaller than the distance between points in $D \cap E$ and X. This contradicts the definition of $\mathscr{C}(X)$. $\square$

The next result bounds the number of edges in the Voronoi cells in $\mathscr{V}(X_1,\ldots,X_n)$.

Corollary 2.4. *With high probability every Voronoi cell in the Voronoi diagram on $\{X_1,\ldots,X_n\}$ has at most $C(\alpha,\beta,\gamma)\log n$ edges.*

Proof. With high probability, by Lemma 2.3 only points within a distance of $C(\beta)$ $\sqrt{\log n/\alpha n}$ may generate an edge in a given Voronoi cell. Since f_X is bounded above by γ, it follows that with high probability there are at most $C(\alpha,\beta,\gamma)\log n$ such points. $\square$

We now have the tools to establish that V satisfies subadditivity (1.9) and superadditivity (1.10) for the i.i.d. random variables X_i, $i \geqslant 1$.

Lemma 2.5. *The Voronoi functional V is subadditive* (1.9) *and superadditive* (1.10).

Proof. We will only prove subadditivity; superadditivity is proved similarly. To prove subadditivity it suffices to show for fixed n

$$\mathbb{E}V(X_1,\ldots,X_n) \leqslant \sum_{j=1}^{m^2} \mathbb{E}V(\{X_1,\ldots,X_n\} \cap Q_j, Q_j) + C\log n. \tag{2.2}$$

Indeed, replacing n by the Poisson random variable $N(n)$ and taking expectations with respect to $N(n)$ gives subadditivity (1.9).

We show (2.2) as follows. For all $1 \leqslant j \leqslant m^2$, let $\mathscr{V}(j)$ be the Voronoi diagram on $\{X_1,\ldots,X_n\} \cap Q_j$. Let $\beta > 2$ and let $C := C(\beta)$ be as in Lemma 2.3. For all $1 \leqslant j \leqslant m^2$, construct a subsquare S_j of edge length $1/m - 4C\sqrt{\log n/\alpha n}$ at the center of Q_j. Let $G_j := Q_j \backslash S_j$ and $G := \bigcup_{j=1}^{m^2} G_j$ denote the "grating" of width $4C\sqrt{\log n/\alpha n}$. Lemma 2.3 implies that with high probability the restrictions of $\mathscr{V}(X_1,\ldots,X_n)$ and

$\mathscr{V}(j)$ to S_j coincide. Thus with high probability

$$\mathscr{V}(X_1,\dots,X_n)\subset\bigcup_{j=1}^{m^2}\mathscr{V}(j)\cup\mathscr{E},$$

where the inclusion is in the sense of edges and where $\mathscr{E}$ denotes $\mathscr{V}(X_1,\dots,X_n)\cap G$.

Therefore, on a high probability set E_n we have

$$V(X_1,\dots,X_n)\leqslant\sum_{j=1}^{m^2}V(\{X_1,\dots,X_n\}\cap Q_j,Q_j)+|\mathscr{E}|,$$

where $|\mathscr{E}|$ is the sum of the edge lengths in $\mathscr{E}$. The expected product of $V(X_1,\dots,X_n)$ and $1_{E_n^c}$ is, by Cauchy–Schwarz and Lemma 2.1, bounded above by a constant. Taking expectations in the above inequality yields

$$\mathbb{E}V(X_1,\dots,X_n)\leqslant\sum_{j=1}^{m^2}\mathbb{E}V(\{X_1,\dots,X_n\}\cap Q_j,Q_j)+\mathbb{E}|\mathscr{E}|+C. \tag{2.3}$$

We complete the proof by showing $\mathbb{E}|\mathscr{E}|\leqslant C(\alpha,\beta,\gamma,m)\log n$, where $C(\alpha,\beta,\gamma,m)$ is a constant depending on α,β,γ, and m. The number of points in G is with high probability bounded above by $C(\alpha,\beta,\gamma,m)(n\log n)^{1/2}$ and by Euler's formula the number of edges in $\mathscr{E}$ is bounded by $3C(\alpha,\beta,\gamma,m)(n\log n)^{1/2}$ with high probability. Lemma 2.2 shows that with high probability the length of each edge is at most $C(\beta)\sqrt{\log n/\alpha n}$ and thus $\mathbb{E}|\mathscr{E}|\leqslant C\log n$ as desired. $\square$

Subadditivity (1.9) and superadditivity (1.10) will play major roles in the proof of (1.2) as will smoothness (1.11). The following lemma establishes smoothness (1.11) for random variables having densities of the form (2.1); its involved proof is deferred to section four.

Lemma 2.6. *The Voronoi length functional is smooth in mean for random variables X satisfying the density condition* (2.1). *Thus, if $W_1,W_2,\dots$ are i.i.d. random variables on $[0,1]^2$ and independent of $X_1,X_2,\dots$ then for all $1\leqslant k\leqslant n/2$*

$$|\mathbb{E}V(X_1,\dots,X_n)-\mathbb{E}V(X_1,\dots,X_n,W_1,\dots,W_k)|\leqslant C(\alpha,\gamma)k^{1/2}. \tag{2.4}$$

Using Lemma 2.6 we can prove the following smoothness estimate relating the Voronoi functional on random variables $X_1,\dots,X_{N(n)}$ to the Voronoi functional on random variables $Y_1,\dots,Y_{N(n)}$ when the respective densities satisfy $pf_X\leqslant f_Y$ a.s., where $\frac{3}{4}<p<1$.

Lemma 2.7. *If the density of X satisfies condition* (2.1) *and the laws of X and Y are related by*

$$\mathscr{L}(Y)=p\mathscr{L}(X)+(1-p)\mu,$$

where μ is a probability measure and $\frac{3}{4}<p<1$, then there exists a positive constant $C(\alpha,\gamma)$ depending only on α and γ such that

$$|\mathbb{E}V(X_1,\dots,X_{N(n)})-\mathbb{E}V(Y_1,\dots,Y_{N(n)})|\leqslant C(\alpha,\gamma)(n(1-p))^{1/2}. \tag{2.5}$$

Proof. If W_i, $i \geqslant 1$, are i.i.d. with law μ, then the superposition principle for Poisson point processes tells us that

$$\{Y_1,\dots,Y_{N(n)}\} \stackrel{\mathrm{d}}{=} \{X_1,\dots,X_{N(pn)},W_1,\dots,W_{N(qn)}\},$$

where $N(pn)$ and $N(qn)$ are *independent* Poisson random variables with parameters pn and qn, respectively, with $q := 1-p$. Now

$$\begin{aligned}
&|\mathbb{E}V(\{Y_i\}_{i=1}^{N(n)}) - \mathbb{E}V(\{X_i\}_{i=1}^{N(n)})| \\
&\quad = |\mathbb{E}V(X_1,\dots,X_{N(pn)},W_1,\dots,W_{N(qn)}) - \mathbb{E}V(X_1,\dots,X_{N(pn)+N(qn)})| \\
&\quad \leqslant |\mathbb{E}V(X_1,\dots,X_{N(pn)},W_1,\dots,W_{N(qn)}) - \mathbb{E}V(X_1,\dots,X_{N(pn)})| \\
&\qquad + |\mathbb{E}V(X_1,\dots,X_{N(pn)+N(qn)}) - \mathbb{E}V(X_1,\dots,X_{N(pn)})|.
\end{aligned}$$

Each of the above terms may be bounded by $C(qn)^{1/2}$. To see this, on the set $\{N(qn) \leqslant N(pn)/2\}$ we may use (2.4) and Jensen's inequality and on the set $\{N(qn) \geqslant N(pn)/2\}$ we may use the bounds $V(X_1,\dots,X_{N(n)}) \leqslant CN(n)$, $\frac{3}{4} < p < 1$, and the exponential decay of $N(qn)$. □

The following smoothness result is a consequence of smoothness (2.4) and the exponential tails of a Poisson random variable.

Lemma 2.8. *Let $N(n)$ be an independent Poisson random variable with mean n. Then*

$$|\mathbb{E}V(\{X_i\}_{i=1}^{N(n)}) - \mathbb{E}V(\{X_i\}_{i=1}^{n})| \leqslant Cn^{1/4}.$$

Proof. Let $\mathbb{E}_{N,X}$ denote expectation with respect to (N,X) and let $\mathbb{E}_N$ and $\mathbb{E}_X$ be defined similarly. Consider the following four events: $A_1 := \{2n/3 \leqslant N(n) \leqslant n\}$, $A_2 := \{n \leqslant N(n) \leqslant 3n/2\}$, $A_3 := \{N(n) < 2n/3\}$, and $A_4 := \{N(n) > 3n/2\}$.

By independence and Fubini's Theorem,

$$\begin{aligned}
&|\mathbb{E}_{N,X}V(\{X_i\}_{i=1}^{N(n)}) - \mathbb{E}_{N,X}V(\{X_i\}_{i=1}^{n})| \\
&\quad \leqslant \mathbb{E}_N \,|\, \mathbb{E}_X V(\{X_i\}_{i=1}^{N(n)}) - \mathbb{E}_X V(\{X_i\}_{i=1}^{n})| \\
&\quad \leqslant \sum_{j=1}^{4} \mathbb{E}_N \,|\, (\mathbb{E}_X V(\{X_i\}_{i=1}^{N(n)}) - \mathbb{E}_X V(\{X_i\}_{i=1}^{n})) \cdot 1_{A_j}|.
\end{aligned}$$

Using smoothness (2.4), the definition of variance for a Poisson random variable, and Jensen's inequality, the first two summands may be bounded by $Cn^{1/4}$. By Lemma 2.1 and the exponential decay of $N(n)$, the last two summands are also bounded by $Cn^{1/4}$. □

3. Proof of main results

Equipped with the lemmas of section two we now prove our main result. The first step is to prove that (1.2) holds in expectation, that is,

$$\lim_{n\to\infty} \frac{\mathbb{E}V(Y_1,\dots,Y_n)}{n^{1/2}} = 2\int_{[0,1]^2} (f_Y(x))^{1/2}\,\mathrm{d}x. \tag{3.1}$$

We then use a modification of Azuma's inequality (Azuma, 1967) to establish complete convergence.

Proof of (3.1). The proof of (3.1) has two parts. We show that (3.1) holds when the density f_Y is (i) a step density bounded away from 0 and ∞, and (ii) continuous and bounded away from 0 and ∞ .

Part (i): We show that

$$\lim_{n\to\infty} \frac{\mathbb{E}V(Y_1,\ldots,Y_{N(n)})}{n^{1/2}} = 2\int_{[0,1]^2} (\varphi_Y(x))^{1/2}\,\mathrm{d}x \tag{3.2}$$

whenever $\{Y_i\}_{i=1}^{N(n)}$ are i.i.d. random variables with a step density

$$\varphi_Y := \sum_{j=1}^{m^2} \alpha_j 1_{Q_j}, \quad \alpha \leqslant \alpha_j \leqslant \gamma.$$

Notice that $\mathrm{card}\{i \leqslant N(n): Y_i \in Q_j\}$, $1 \leqslant j \leqslant m^2$, is a Poisson random variable $N(n\alpha_j m^{-2})$. Let U_i, $i \geqslant 1$, be i.i.d. uniform random variables on $[0,1]^2$. By subadditivity (1.9) and scaling (1.4)

$$\mathbb{E}V(Y_1,\ldots,Y_{N(n)}) \leqslant \frac{1}{m}\sum_{j=1}^{m^2} \mathbb{E}V(\{U_i\}_{i=1}^{N(n\alpha_j m^{-2})}) + c_1(m)c_2(n).$$

Therefore by (1.1)

$$\overline{\lim_{n\to\infty}} \frac{\mathbb{E}V(Y_1,\ldots,Y_{N(n)})}{n^{1/2}} \leqslant \sum_{j=1}^{m^2} \overline{\lim_{n\to\infty}} \frac{\mathbb{E}V(\{U_i\}_{i=1}^{N(n\alpha_j m^{-2})})}{n^{1/2}\alpha_j^{1/2} m^{-1}} \alpha_j^{1/2} m^{-2}$$

$$= 2\int_{[0,1]^2} (\varphi_Y(x))^{1/2}\,\mathrm{d}x.$$

Similarly by superadditivity (1.10) and scaling (1.4) we see that

$$\underline{\lim_{n\to\infty}} \frac{\mathbb{E}V(Y_1,\ldots,Y_{N(n)})}{n^{1/2}} \geqslant 2\int_{[0,1]^2} (\varphi_Y(x))^{1/2}\,\mathrm{d}x.$$

Therefore

$$\lim_{n\to\infty} \frac{\mathbb{E}V(Y_1,\ldots,Y_{N(n)})}{n^{1/2}} = 2\int_{[0,1]^2} (\varphi_Y(x))^{1/2}\,\mathrm{d}x. \tag{3.3}$$

Part (ii): Assume that f_Y is a continuous density, $0 < \alpha \leqslant f_Y \leqslant \gamma$. In this case, for all $0 < \varepsilon < \frac{1}{4}$ we know by the continuity of f_Y that there is a random variable $X := X(\varepsilon)$ having a step density of the form

$$\varphi_X := \sum_{j=1}^{m^2} \alpha_j 1_{Q_j}, \quad \alpha \leqslant \alpha_j \leqslant \gamma,$$

and whose law is related to that of Y by

$$\mathscr{L}(Y) = p\mathscr{L}(X) + (1-p)\mu, \tag{3.4}$$

where $1 - p \leqslant \varepsilon$ and where μ is a probability measure. By (2.5) we obtain

$$|\mathbb{E}V(X_1,\dots,X_{N(n)}) - \mathbb{E}V(Y_1,\dots,Y_{N(n)})| \leqslant C(\alpha,\gamma)(\varepsilon n)^{1/2}. \tag{3.5}$$

By the triangle inequality

$$\begin{aligned}
&\left| \frac{\mathbb{E}V(Y_1,\dots,Y_{N(n)})}{n^{1/2}} - 2\int_{[0,1]^2} (\varphi_X(x))^{1/2}\,\mathrm{d}x \right| \\
&\quad \leqslant \left| \frac{\mathbb{E}V(Y_1,\dots,Y_{N(n)})}{n^{1/2}} - \frac{\mathbb{E}V(X_1,\dots,X_{N(n)})}{n^{1/2}} \right| \\
&\qquad + \left| \frac{\mathbb{E}V(X_1,\dots,X_{N(n)})}{n^{1/2}} - 2\int_{[0,1]^2} (\varphi_X(x))^{1/2}\,\mathrm{d}x \right|.
\end{aligned} \tag{3.6}$$

Letting $n \to \infty$ in (3.6) gives by (3.2) and (3.5)

$$\varlimsup_{n\to\infty} \left| \frac{\mathbb{E}V(Y_1,\dots,Y_{N(n)})}{n^{1/2}} - 2\int_{[0,1]^2} (\varphi_X(x))^{1/2}\,\mathrm{d}x \right| \leqslant C(\alpha,\gamma)\varepsilon^{1/2}. \tag{3.7}$$

Since $|a^{1/2} - b^{1/2}| \leqslant |a-b|^{1/2}$ for all a, $b > 0$ it follows that

$$\int_{[0,1]^2} |(f_Y(x))^{1/2} - (\varphi_X(x))^{1/2}|\,\mathrm{d}x \leqslant \int_{[0,1]^2} |f_Y(x) - \varphi_X(x)|^{1/2}\,\mathrm{d}x.$$

Moreover, by (3.4) it follows that

$$\int_{[0,1]^2} |f_Y(x) - \varphi_X(x)|\,\mathrm{d}x \leqslant 2(1-p) \leqslant 2\varepsilon.$$

Combining the last two estimates with Jensen's inequality shows that

$$\left| \int_{[0,1]^2} \varphi_X(x)^{1/2}\,\mathrm{d}x - \int_{[0,1]^2} f_Y(x)^{1/2}\,\mathrm{d}x \right| \leqslant \int_{[0,1]^2} |f_Y(x) - \varphi_X(x)|^{1/2}\,\mathrm{d}x \leqslant (2\varepsilon)^{1/2}. \tag{3.8}$$

By (3.7), (3.8), and the arbitrariness of ε we obtain

$$\lim_{n\to\infty} \frac{\mathbb{E}V(Y_1,\dots,Y_{N(n)})}{n^{1/2}} = 2\int_{[0,1]^2} (f_Y(x))^{1/2}\,\mathrm{d}x.$$

This last limit may be de-Poissonized via Lemma 2.8. This finishes the proof of Part (ii) and completes the proof of (3.1). □

Next we show that (1.2) holds in the sense of complete convergence. By the definition of complete convergence it will suffice to show that for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\{|V(Y_1,\dots,Y_n) - \mathbb{E}V(Y_1,\dots,Y_n)| > \varepsilon n^{1/2}\} < \infty. \tag{3.9}$$

Consider

$$|V(Y_1,\dots,Y_i,\dots,Y_n) - V(Y_1,\dots,\hat{Y}_i,\dots,Y_n)|,$$

where the hat symbol signals a missing variable. By Corollary 2.4 the deletion of Y_i affects at most $C\log n$ edges, which by Lemma 2.2 each have length at most $C\sqrt{\log n/n}$ with high probability. Thus the displayed difference is with high probability bounded by $C(\log n)^{3/2}/n^{1/2}$. The same estimates hold if Y_i is replaced by Y_i', where Y_i' is an independent copy of Y_i. The triangle inequality shows therefore that

$$|V(Y_1,\dots,Y_i,\dots,Y_n)-V(Y_1,\dots,Y_i',\dots,Y_n)|\leqslant C\frac{(\log n)^{3/2}}{n^{1/2}},\quad 1\leqslant i\leqslant n, \tag{3.10}$$

on a high probability set $A_{n,i}$.

To show (3.9) we will use a modification of Azuma's inequality. Consider the martingale difference representation

$$V(Y_1,\dots,Y_n)-\mathbb{E}(V(Y_1,\dots,Y_n))=\sum_{i=1}^n d_i,$$

where $d_i:=\mathbb{E}(V(Y_1,\dots,Y_n)\,|\,\mathscr{F}_i)-\mathbb{E}(V(Y_1,\dots,Y_n)\,|\,\mathscr{F}_{i-1})$, and where $\mathscr{F}_i$ denotes the σ-field generated by the random variables $Y_1,\dots,Y_i$.

Notice that

$$|d_i|=|\mathbb{E}(V(Y_1,\dots,Y_n)|\,\mathscr{F}_i)-\mathbb{E}(V(Y_1,\dots,Y_i',\dots,Y_n)|\,\mathscr{F}_i)|,$$

where Y_i' is an independent copy of Y_i. Thus by the conditional Jensen inequality and (3.10) we have for all $1\leqslant i\leqslant n$,

$$\begin{aligned}|d_i| &\leqslant \mathbb{E}(|V(Y_1,\dots,Y_n)-V(Y_1,\dots,Y_i',\dots,Y_n)|\,|\,\mathscr{F}_i)\\ &\leqslant C(\log n)^{3/2}n^{-1/2}\end{aligned} \tag{3.11}$$

on some high probability set. Now for any martingale difference sequence d_i, $i\geqslant 1$, and for all sequences w_i, $i\geqslant 1$, of positive numbers we have for all $t>0$,

$$\begin{aligned}P\left\{\left|\sum_{i=1}^n d_i\right|>t\right\} &\leqslant 2\exp\left(\frac{-t^2}{32\sum_{i=1}^n w_i^2}\right)\\ &\quad+\left(1+2t^{-1}\sup_{1\leqslant i\leqslant n}\|d_i\|_\infty\right)\sum_{i=1}^n P(|d_i|>w_i).\end{aligned} \tag{3.12}$$

See e.g. Lemma 1 of Chalker et al. (1999). Letting $w_i=C(\log n)^{3/2}n^{-1/2}$, $t=\varepsilon n^{1/2}$, using (3.11–3.12), and noting that $\|\alpha_i\|_\infty\leqslant C_n$, we obtain (3.9) as desired. The proof of Theorem 1.2 is complete. □

4. Regularity of Voronoi tessellations

This section verifies the smoothness (2.4) of the Voronoi length functional V. Our approach centers on the following two deterministic lemmas which describe the regularity properties of the Voronoi length functional when generators are added and deleted.

Let $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ be two collections of points on $[0,1]^2$. Lemma 4.1 bounds the length of the Voronoi tessellation on $k+l$ points by the length of the Voronoi tessellation on a "pruned" set of k points plus an error term.

Lemma 4.1. *For all $k,\ l \in \mathbb{N}$*

$$V(x_1,\dots,x_k,y_1,\dots,y_l) \leqslant V(x_1,\dots,x_k) + \sum_{j=1}^{l} E(y_j,\{x_i\}_{i=1}^{k}), \tag{4.1}$$

where $E(y_j,\{x_i\}_{i=1}^{k})$ denotes the sum of the edge lengths of the cell $\mathscr{C}(y_j)$ in $\mathscr{V}(x_1,\dots,x_k,y_j)$.

Proof. Consider the Voronoi diagram $\mathscr{V}(x_1,\dots,x_k)$ on $x_1,\dots,x_k$. Given $y_1 \in [0,1]^2$, construct $\mathscr{V}(x_1,\dots,x_k,y_1)$ by deleting portions of edges in $\mathscr{V}(x_1,\dots,x_k)$ and inserting the edges $\mathscr{E}(y_1,\{x_i\}_{i=1}^{k})$ of the cell $\mathscr{C}(y_1)$ in $\mathscr{V}(x_1,\dots,x_k,y_1)$. That is,

$$\mathscr{V}(x_1,\dots,x_k,y_1) \subset \mathscr{V}(x_1,\dots,x_k) \cup \mathscr{E}(y_1,\{x_i\}_{i=1}^{k}),$$

where the inclusion is in the sense of edges. Therefore, for all $x_1,\dots,x_k,y_1 \in [0,1]^2$

$$V(x_1,\dots,x_k,y_1) \leqslant V(x_1,\dots,x_k) + E(y_1,\{x_i\}_{i=1}^{k}), \tag{4.2}$$

where $E(y_1,\{x_i\}_{i=1}^{k})$ denotes the sum of the edge lengths of the cell $\mathscr{C}(y_1)$ in $\mathscr{V}(x_1,\dots,x_k,y_1)$.

Iterating (4.2) yields with the obvious notation

$$V(x_1,\dots,x_k,y_1,\dots,y_l) \leqslant V(x_1,\dots,x_k) + \sum_{j=1}^{l} E(y_j,\{x_1,\dots,x_k,y_1,\dots,y_{j-1}\}). \tag{4.3}$$

Since additional generators can only decrease the sum of the edge lengths of the cell around y_j, $1 \leqslant j \leqslant l$, we deduce from (4.3) that

$$V(x_1,\dots,x_k,y_1,\dots,y_l) \leqslant V(x_1,\dots,x_k) + \sum_{j=1}^{l} E(y_j,\{x_i\}_{i=1}^{k})$$

which completes the proof of Lemma 4.1. □

The following lemma is a companion to Lemma 4.1. It bounds the length of the Voronoi diagram on k points by the length of the Voronoi diagram on $k+l$ points plus an error term.

Lemma 4.2. *For every $k,\ l \in \mathbb{N}$*

$$V(x_1,\dots,x_k) \leqslant V(x_1,\dots,x_k,y_1,\dots,y_l) + \sum_{j=1}^{l} \hat{E}(y_j,\{x_i\}_{i=1}^{k}), \tag{4.4}$$

where $\hat{E}(y_j,\{x_i\}_{i=1}^{k})$ denotes the combined lengths of the edges in the intersection of the interior of the Voronoi cell in $\mathscr{V}(x_1,\dots,x_k,y_j)$ around y_j and the Voronoi graph $\mathscr{V}(x_1,\dots,x_k)$.

Proof. For all $1 \leqslant j \leqslant l$, let $\mathscr{C}(y_j)$ denote the Voronoi cell in $\mathscr{V}(x_1,\dots,x_k,y_1,\dots,y_l)$ around y_j. Notice that the intersection

$$\mathscr{V}(x_1,\dots,x_k) \cap \bigcup_{j=1}^{l} \mathscr{C}(y_j)$$

defines precisely those edges (or subsets thereof) in $\mathscr{V}(x_1,\dots,x_k)$ which are modified by the insertion of additional generators $\{y_1,\dots,y_l\}$ and thus

$$\mathscr{V}(x_1,\dots,x_k) \subset \mathscr{V}(x_1,\dots,x_k,y_1,\dots,y_l) \cup \bigcup_{j=1}^{l} (\mathscr{C}(y_j) \cap \mathscr{V}(x_1,\dots,x_k)).$$

Clearly

$$\mathscr{C}(y_j) \subset \mathscr{C}(y_j,\{x_i\}_{i=1}^k),$$

where $\mathscr{C}(y_j,\{x_i\}_{i=1}^k)$ denotes the Voronoi cell in $\mathscr{V}(x_1,\dots,x_k,y_j)$ around y_j. Consider the edge set

$$\hat{\mathscr{E}}(y_j,\{x_i\}_{i=1}^k) := \mathscr{C}(y_j,\{x_i\}_{i=1}^k) \cap \mathscr{V}(x_1,\dots,x_k). \tag{4.5}$$

We obtain the relation

$$\mathscr{V}(x_1,\dots,x_k) \subset \mathscr{V}(x_1,\dots,x_k,y_1,\dots,y_l) \cup \bigcup_{j=1}^{l} \hat{\mathscr{E}}(y_j,\{x_i\}_{i=1}^k)$$

which shows that

$$V(x_1,\dots,x_k) \leqslant V(x_1,\dots,x_k,y_1,\dots,y_l) + \sum_{j=1}^{l} \hat{E}(y_j,\{x_i\}_{i=1}^k).$$

This completes the proof of (4.4). □

We turn now to the

Proof of smoothness (2.4). We assume that X_i, $i \geqslant 1$, are i.i.d. random variables with densities satisfying (2.1) and that W_i, $i \geqslant 1$, are i.i.d. on $[0,1]^2$ independent of X_i, $i \geqslant 1$. We will show that

$$|\mathbb{E}V(X_1,\dots,X_n,W_1,\dots,W_k) - \mathbb{E}V(X_1,\dots,X_n)| \leqslant C(\alpha,\gamma)k^{1/2}, \tag{4.6}$$

for all $1 \leqslant k \leqslant n/2$. By Lemma 4.1 $V(X_1,\dots,X_n,W_1,\dots,W_k)$ is bounded by

$$V(X_1,\dots,X_n) + \sum_{j=1}^{k} E(W_j,\{X_i\}_{i=1}^n), \tag{4.7}$$

where $E(W_j,\{X_i\}_{i=1}^n)$, $1 \leqslant j \leqslant k$, denotes the combined lengths of the edges of the cell consisting of points closer to W_j than to $X_1,\dots,X_n$ and where $\sum_{j=1}^{0} E(\ ,\ ) := 0$.

Taking expectations in (4.7) gives

$$\begin{aligned}\mathbb{E}V(X_1,\dots,X_n,W_1,\dots,W_k) &\leqslant \mathbb{E}V(X_1,\dots,X_n) + \mathbb{E}\left(\sum_{j=1}^{k} E(W_j,\{X_i\}_{i=1}^n)\right)\\ &\leqslant \mathbb{E}V(X_1,\dots,X_n) + k\mathbb{E}E(W_1,\{X_i\}_{i=1}^n).\end{aligned} \tag{4.8}$$

Next we bound $\mathbb{E}V(X_1,\dots,X_n)$ in terms of $\mathbb{E}V(X_1,\dots,X_n,W_1,\dots,W_k)$. By Lemma 4.2

$$V(X_1,\dots,X_n) \leqslant V(X_1,\dots,X_n,W_1,\dots,W_k) + \sum_{j=1}^{k} \hat{E}(W_j,\{X_i\}_{i=1}^n), \tag{4.9}$$

where $\sum_{j=1}^{k} \hat{E}(W_j, \{X_i\}_{i=1}^n)$ represents the sum of the lengths of the edges in the intersection of the interior of the Voronoi cell in $\mathscr{V}(X_1,\dots,X_n,W_j)$ around W_j and the Voronoi diagram $\mathscr{V}(X_1,\dots,X_n)$.

Take expectations in (4.9) to obtain

$$\mathbb{E}V(X_1,\dots,X_n) \leqslant \mathbb{E}V(X_1,\dots,X_n,W_1,\dots,W_k) + k\mathbb{E}\hat{E}(W_1,\{X_i\}_{i=1}^n). \tag{4.10}$$

Considering (4.8) and (4.10) it is clear that to complete the proof of (4.6) we need to show

$$\mathbb{E}E(W_1,\{X_i\}_{i=1}^n) \leqslant C(\alpha,\gamma)n^{-1/2} \tag{4.11}$$

and

$$\mathbb{E}\hat{E}(W_1,\{X_i\}_{i=1}^n) \leqslant C(\alpha,\gamma)n^{-1/2}. \tag{4.12}$$

We now prove (4.11) and (4.12).

Define for all $x, y_1,\dots,y_l \in [0,1]^2$

$$D(x,\{y_i\}_{i=1}^l) := \text{diameter of the cell containing } x \text{ in the Voronoi diagram on } x, y_1,\dots,y_l \tag{4.13}$$

and

$$K(x,\{y_i\}_{i=1}^l) := \text{number of sides in the cell containing } x \text{ in the Voronoi diagram on } x, y_1,\dots,y_l. \tag{4.14}$$

$E(W_1,\{X_i\}_{i=1}^n)$ is bounded by the product of $D(W_1,\{X_i\}_{i=1}^n)$ and $K(W_1,\{X_i\}_{i=1}^n)$ and to show (4.11) it suffices to show

$$\mathbb{E}\{D(W_1,\{X_i\}_{i=1}^n)\cdot K(W_1,\{X_i\}_{i=1}^n)\} \leqslant C(\alpha,\gamma)n^{-1/2}. \tag{4.15}$$

Next, concerning $\hat{E}(W_1,\{X_i\}_{i=1}^n)$, we notice that it is bounded by the product of $D(W_1,\{X_i\}_{i=1}^n)$ and the number η of edges in the intersection of the Voronoi cell $\mathscr{C}(W_1)$ in $\mathscr{V}(X_1,\dots,X_n,W_1)$ and the Voronoi diagram $\mathscr{V}(X_1,\dots,X_n)$. Let $K := K(W_1,\{X_i\}_{i=1}^n)$ and let $g_1,\dots,g_K$ denote the generators of the cells adjacent to $\mathscr{C}(W_1)$. $\mathscr{C}(W_1)$ is contained in the union of the cells in $\mathscr{V}(g_1,\dots,g_K)$ and so η is bounded by the number of edges in $\mathscr{V}(g_1,\dots,g_K)$. Thus, by Euler's formula, η is bounded by $3\cdot K(W_1,\{X_i\}_{i=1}^n)$. Thus to show (4.12) it also suffices to show (4.15).

We show (4.15) as follows. We will use the fact that the density of X is bounded away from 0. Fix $x\in[0,1]^2$. Construct 12 disjoint congruent isosceles triangles $T_j(t) := T_j(x,t), 1\leqslant j\leqslant 12$, such that x is a vertex of each $T_j(t)$ and thus $T_j(t)$ has two edges of length t. The union of the $T_j(t)$ is a regular 12-sided polygonal region which may not lie entirely inside $[0,1]^2$. Let the random variable $T_0 := T_0(x)$ be the minimum t such that each triangle $T_j(t), 1\leqslant j\leqslant 12$, lying *wholly* in $[0,1]^2$ contains at least one point from $X_1,\dots,X_n$. There are some configurations for which T_0 does not exist and in this case we set $T_0 = 1$. Simple geometric considerations show that for all $x\in[0,1]^2$, including those near the boundary of $[0,1]^2$, we have

$$D := D(x,\{X_i\}_{i=1}^n) \leqslant 2T_0. \tag{4.16}$$

Let N be the total number of points in the region given by the union of the 12 triangles $T_j(3T_0), 1\leqslant j\leqslant 12$. Then $\mathbb{E}(N|T_0) \leqslant C(\gamma)T_0^2 n$. Since points farther than $3T_0$ away from x do not contribute edges to the cell around x, we have $K := K(x, \{X_i\}_{i=1}^n) \leqslant N$. Thus, for a fixed $x \in [0,1]^2$ we have by (4.16)

$$\mathbb{E}(DK) = \mathbb{E}(\mathbb{E}(DK|T_0)) \leqslant \mathbb{E}\mathbb{E}(2T_0 N|T_0) = \mathbb{E}(2T_0 \mathbb{E}(N|T_0)) \leqslant C\mathbb{E}(nT_0^3).$$

We now claim that $\mathbb{E}(T_0^3) \leqslant Cn^{-3/2}$. Indeed, since

$$\{T_0 > t\} \subset \bigcup_{j: T_j(t) \subset [0,1]^2} (\{X_i\}_{i=1}^n \cap T_j(t) = \emptyset)$$

we have

$$P(T_0 > t) \leqslant \sum_{j: T_j(t) \subset [0,1]^2} P(\{X_i\}_{i=1}^n \cap T_j(t) = \emptyset) \leqslant \max(12(1 - C\alpha t^2)^n, 0), \qquad (4.17)$$

since $\varphi_X \geqslant \alpha > 0$, $T_j(t) \subset [0,1]^2$, and the area of $T_j(t)$ is Ct^2. Therefore, by (4.17)

$$\mathbb{E}T_0^3 \leqslant 3 \int_0^\infty t^2 P(T_0 > t)\,\mathrm{d}t \leqslant 36 \int_0^{(\alpha C)^{-1/2}} t^2 (1 - C\alpha t^2)^n\,\mathrm{d}t.$$

Letting $v = C\alpha t^2$, and noting that the function $v \to v^{1/2}(1-v)^{n/2}$ is decreasing on $[1/n, 1]$, the above integral equals

$$= C \int_0^{1/n} v^{1/2}(1-v)^n\,\mathrm{d}v + C \int_{1/n}^1 v^{1/2}(1-v)^{n/2}(1-v)^{n/2}\,\mathrm{d}v$$

$$\leqslant C(1/n)^{3/2} + C \int_{1/n}^1 (1/n)^{1/2}(1-v)^{n/2}\,\mathrm{d}v$$

$$\leqslant C(1/n)^{3/2}.$$

Thus we have for the fixed x

$$\mathbb{E}(DK) \leqslant Cn^{-1/2}$$

as desired. Taking $x = W_1$ and integrating over W_1 gives (4.15). This completes the proof of Lemma 2.6. □

5. Concluding remarks

As indicated earlier, the methods of this paper have the potential for describing the total surface area of Voronoi tessellations of point sets in the unit cube. Further likely extensions and modifications include:

1. Delaunay triangulations. Much of this paper represents a simplification and generalization of McGivney (1997). We anticipate that further modifications show that the total edge length of the Delaunay triangulation satisfies subadditivity (1.9), superadditivity (1.10), and smoothness (1.11). In this way we would obtain asymptotics for the total edge length of a Delaunay triangulation on $[0,1]^2$-valued random variables $X_1, \ldots, X_n$.

2. Rates of convergence. We also anticipate that the error term in (2.2) can be improved to $C_1(m) = \mathrm{O}(m)$. This would lead to rates of convergence for $\mathbb{E}V(U_1, \ldots, U_n)$ of the form
$$|\mathbb{E}V(U_1, \ldots, U_n) - 2n^{1/2}| \leqslant C.$$
3. General densities. It is unclear whether Theorem 1.2 holds without boundedness assumptions on the underlying density f_Y.

Acknowledgements

It is a pleasure to thank Mathew Penrose for numerous, detailed comments which corrected and substantially improved upon the original version. In particular the comments led to a much shortened proof of smoothness (2.4).

References

Aurenhammer, F., 1991. Voronoi diagrams – a survey of a fundamental geometric data structure. ACM Comput. Surv. 23 (3), 345–405.

Avram, F., Bertsimas, D., 1993. On central limit theorems in geometrical probability. Ann. Appl. Probab. 3 (4), 1033–1046.

Azuma, K., 1967. Weighted sums of certain dependent random variables. Tohuku Math. J. 19, 357–367.

Beardwood, J., Halton, J.H., Hammersley, J.M., 1959. The shortest path through many points. Proc. Cambridge Philos. Soc. 55, 299–327.

Chalker, T.K., Godbole, A.P., Hitczenko, P., Radcliff, J., Ruehr, O.G., 1999. On the size of a random sphere of influence graph. Adv. Appl. Probab., in press.

Christ, N.H., Friedberg, R., Lee, T.D., 1982. Random lattice field theory: general formulation. Nucl. Phys. B 202, 89–125.

Drouffe, J.M., Itzykson, C., 1984. Random geometry and statistics of two dimensional cells. Nucl. Phys. B 235, 45–53.

Jerauld, G.R., Hatfield, J.C., Scriven, L.E., Davis, H.T., 1984. Percolation and conduction on Voronoi and triangular networks: a case study in topological disorder. J. Phys. C 17, 1519–1529.

McGivney, K., 1997. Probabilistic limit theorems for combinatorial optimization problems. Ph.D. Thesis, Lehigh University.

Miles, R.E., 1970. On the homogeneous planar point process. Math. Biosci. 6, 85–127.

Moller, J., 1994. Lectures on random Voronoi tessellations. Lecture Notes in Statistics, Vol. 87. Springer, Berlin.

Okabe, A., Boots, B., Sugihara, K., 1992. Spatial Tessellations – Concepts and Applications of Voronoi Diagrams. Wiley, England.

Preparata, F.P., Shamos, M.I., 1985. Computational Geometry, An Introduction. Springer, Berlin.

Redmond, C., Yukich, J.E., 1994. Limit theorems and rates of convergence for subadditive Euclidean functionals. Ann. Appl. Probab. 4 (4), 1057–1073.

Redmond, C., Yukich, J.E., 1996. Asymptotics for Euclidean functionals with power weighted edges. Stochast. Process. Appl. 61, 289–304.

Steele, J.M., 1997. Probability theory and combinatorial optimization, Vol. 69. CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, Philadelphia.

Stoyan, D., Kendall, W.S., Mecke, J., 1995. Stochastic Geometry and Its Applications, 2nd Edition. Wiley, New York.

Talagrand, M., 1995. Concentration of measure and isoperimetric inequalities in product spaces. I.H.E.S. 81, 73–205.

Weide, B., 1978. Statistical methods in algorithm design and analysis. Ph.D. Thesis, Computer Science Department, Carnegie Mellon University.

Yukich, J.E., 1998. Probability Theory of Classical Euclidean Optimization Problems. Lecture Notes in Mathematics, Vol. 1675. Springer, Berlin.