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The k-ary n-cube network: modeling, topological properties and routing strategies

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Abstract

The class of k-ary n-cubes represents the most commonly used interconnection topology for parallel and distributed systems. The main shortcoming of this topology is exponential growth of its size, which is k^n, as n increases. In this paper, we describe a technique for modeling k-ary n-cube networks. The proposed technique is used to define and to analyze a variation of this topology, which we call incomplete k-ary n-cubes, or incomplete n:k cubes for short. We show that n:k cubes and incomplete n:k cubes are members of the class of linear recursive networks reported in [1] with generators $(k-1)^{\beta}k$ and $(k-1)^{\beta}$ respectively, where β is any positive integer. The significance of linearity of an incomplete n:k cube lies in the ability of having control over the growth (in size) of this type of networks. Thus, it is possible to design a communication network which satisfies most of the n:k cube properties but with significantly smaller size. In addition, an efficient routing algorithm for the class of incomplete n:k cube networks is presented. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: Linear interconnection topologies; Incomplete k-ary n-cube; Routing algorithms

1. Introduction

The performance and cost of a large multiprocessing system is greatly influenced by its interconnection topology. Thus, considering the trade-offs in network design becomes a critical requirement for building cost-effective, high performance systems.

Among many criteria for choosing an interconnection topology are small diameter, low

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degree, efficient routing and embedding algorithms, recursive structure, fault tolerance and scalability. In recent years many graphs have been studied as possible interconnection topologies, including hypercubes [2–4], star graphs [4], k-ary-cube [5,3,6], and generalized hypercubes [3].

Interconnection topologies can be classified as either *single-stage* or *multistage*. Multistage networks, such as the omega network [7], connect system resources through multiple intermediate stages of crossbar switching devices. The performance of multistage type networks has been extensively studied in the literature [8–12].

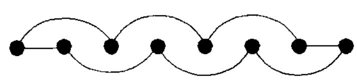
Single-stage networks incorporate the processing devices within the network itself, allowing the direct communication between processors. A single stage network has smaller average latency and is more fault tolerant in comparison with multistage networks of the same size. As a result, single stage networks are gaining in popularity and have been employed in many existing large scale computing systems, including the Connection Machine [13], Intel iPSC, Tera supercomputer [14], and CMU-Intel iWarp [15].

The most commonly used single stage networks are variants of the k-ary n-cube [12]. The k-ary n-cube has been used in several computers, such as the Cosmic cube [16] and the Connection Machine [13].

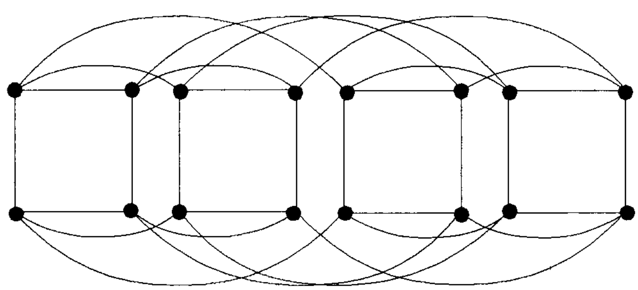
In this paper, we describe a methodology for the design and analysis of a class of networks which we call *incomplete k-ary n-cube* networks, or *incomplete n:k cubes* for short. In addition, we explore the topological properties and develop several routing algorithms for the class of incomplete n:k cube networks.

2. Preliminaries

The material of this section is informal and is intended to serve as motivation for the more formal material of the remainder of the paper.



(a)



(b)

Fig. 1. Example k-ary n-cube networks. (a) Folded 8-ary 1-cube; (b) 2-ary 4-cube.

The n:k cube $Q(n,k)$ consists of $N = k^n$ nodes. Each node X is labeled by a sequence of n digits, i.e., $X = x_{n-1}x_{n-2}\ldots x_0$, where $0 \le x_i < k-1$, for all $0 \le i < n$. Fig. 1 illustrates several different k-ary n-cubes. Each node is connected via a direct link to its nearest neighbors in each of n dimensions. Links can be bi- or unidirectional, and if bi-directional, the wrap-around links may be omitted. Examples of k-ary n-cubes include the ring ($n = 1$), 2–D mesh or torus ($n = 2$), 3–D mesh or torus ($n = 3$) and hypercube ($k = 2$). The generality and flexibility of the k-ary n-cube make it an excellent choice for exploring network design tradeoffs.

For a given system of size N, the primary design choice is the dimensionality, n, of the networks. Varying the dimensionality of the network affects the average communication latency, i.e., the number of links that messages must traverse and the average rate of traffic across the links.

An n:k cube can be decomposed into k $(n-1)$:k cubes or k^2 $(n-2)$:k cubes or in general k^β:$(n-\beta)$-cubes for all $\beta \le n$. The relation between $Q(n,k)$ and its components can be described by the following recurrence relations: $Q(n,k) = kQ(n-1,k)$, or $Q(n,k) = k^2\, Q(n-2,k)$, or in general $Q(n,k) = k^\beta Q(n-\beta,k)$. We refer to these equations as *characteristic equations* of n:k cubes. Note that the recurrence equation $Q(n,k) = kQ(n-1,k)$ can be expanded to $Q(n,k) = (k-1)\; Q(n-1,k) + \; kQ(n-2,k) = (k-1)\; Q(n-1,k) + (k-1)$ $Q(n-2,k) + kQ(n-3,k) = (k-1)\; Q(n-1,k) + (k-1)\; Q(n-2,k) + \ldots + (k-1)\; Q(n-\beta,k) +$ $k\; Q(n-\beta-1,k)$.

Let the operator D be defined as $D^\beta(Q(n,k)) = Q(n-\beta,k)$. In this notation, the characteristic equation of the cube can be written as:

$$Q(n,k) = [(k-1)D + (k-1)D^2 + (k-1)D^3 + \ldots + (k-1)D^\beta + kD^{\beta+1}]Q(n,k). \tag{1}$$

A degree-β polynomial $a_1D + a_2D^2 + \ldots + a_{\beta-1}D^{\beta-1} + a_\beta D^\beta$ can be described by the string $A = a_1a_2\ldots a_\beta$ which is referred to as the *generator string (or sequence)* of the polynomial. Thus, the recurrence relation Eq. (1) can be constructed from the sequence $A = (k-1)(k-1)\ldots(k-1)k = (k-1)^\beta k$, where $(k-1)^\beta$ denotes concatenation of $(k-1)$ done β times.

Each choice of the generator sequence represents a different topology $X(n)$. These topologies satisfy the linear recurrence equation: $X(n) = (a_1D + a_2D^2 + \ldots + a_{\beta-1}D^{\beta-1} + a_\beta\; D^\beta)X(n)$. Throughout the paper, we refer to these topologies as *linear recursive graphs* (LRG).

2.1. Contributions of the paper

This paper summarizes research and the development of a class of interconnection topologies for distributed computing systems. The contributions of this paper are as follows:

- It applies an extension of a methodology reported in [1] to the modeling of n:k cubes by linear recurrence relations and presents the new class of *incomplete n:k cubes*. The complexity (size) of an incomplete n:k cube can be controlled by altering its generator sequence.
- It explores toplogical and routing properties of incomplete n:k cubes and relates these properties to the generators of this type of graphs.
- It presents an efficient routing algorithm for incomplete n:k cubes.

In the next section, we introduce some notations, necessary background, and definitions.

3. Notations and definitions

Following is the list of definitions and notations that are used throughout the paper.

- For a string of integers $A = a_1 \ldots a_i \ldots a_\beta$, the length and the weight of A are defined respectively as: $L(A) = \beta$ and $W(A) = a_1 + a_2 + \ldots + a_\beta$.
- $A \circ B$ denotes the concatenation of two sequences A and B. The empty string is denoted by e and satisfies the condition: $\forall A,\ A \circ e = e \circ A = A$. Clearly, $W(e) = 0$.
- A linear recurrence relation $X(n) = a_1 X(n-1) + \ldots + a_i X(n-i) + \ldots + a_\beta X(n-\beta)$ is characterized by the string $A = a_1 \ldots a_i \ldots a_\beta$.
- If the structural growth of a graph as well as its topology is described by a linear recurrence equation, then the graph is classified as a *linear graph*. Thus, a string $A = a_1 \ldots a_i \ldots a_\beta$ of digits characterizes a linear graph. The string A is referred to as the generator string, or *generator*, of the graph. The base of a linear graph, r, is taken to be $r = \max_i(a_i) + 1$. We restrict attention here to those A for which $W(A) > 1$. This notion is defined more precisely in Section 4.
- The number of occurrences of non-zero elements of the sequence A is referred to as the *depth of the graph.*

For example, a graph $G_n(A)$ of depth 3 is composed of a number of modules that are copies of $G_{n-1}(A)$, $G_{n-2}(A)$, *and* $G_{n-3}(A)$. Assuming $A = a_1 a_2 a_3$ and a_1, a_2, and a_3 are positive integers. Note that the generator $A = a_1 a_2 00 a_3$ is also characterizes a graph of depth three.

- The *Hamming sequence* between two strings $A = a_1 \ldots a_i \ldots a_\beta$, and $B = b_1 \ldots b_i \ldots b_\beta$, $H(A,B)$, is defined to be the string $h_1 \ldots h_i \ldots h_\beta$ such that, for all i, $h_i = 0$ if $a_i = b_i$, otherwise is $h_i = 1$.
- If S is a set of sequences, then $A \circ S$ denotes the set of concatenations of the string A with every string in S.
- For a generator string $A = a_1 \ldots a_i \ldots a_\beta$, A_i is the set of all strings $a_1 a_2 \ldots a_{i-1} a_i^*$ where, $a_i^* \in \{0,1,\ldots,a_i - 1\}$ and a_i greater than zero. The set A_i is called a *seed set* of A and the sequence $a_1 a_2 \ldots a_{i-1}$ is called the *prefix* of A_i.

For example, if $A = 35$, then $A_1 = \{0,1,2\}$ and $A_2 = 3 \circ \{0,1,2,3,4\} = \{30,31,32,33,34\}$. The prefixes of A_1 and A_3 are the null sequence and 3 respectively.

- $Q(n,k)$ and $Q_{n,k}$ shall denote the n:k cube and an incomplete n:k cube (yet to be defined) respectively.

The primary motivation behind the introduction of incomplete n:k cubes is to reduce the complexity (size or number of nodes) of k-ary n-cubes. The size of the n:k cube grows exponentially, which is its main shortcoming. The incomplete n:k cube is a special subcube of an $n{:}k$ cube with an arbitrary depth β; specifically, the incomplete $n{:}k$ cube is characterized by the generator string $A = (k-1)^{\beta}$. The size of incomplete $n{:}k$ cube is less than that of k-ary n-cube, which is k^n. The notions of $n{:}k$ cube and incomplete $n{:}k$ cube will be made more precise in the following section.

4. Incomplete *n*:*k* cubes

In this section, we first describe the class of linear recursive graphs (LRGs) and then we show that $n{:}k$ cubes are members of the class of LRGs. Finally, we propose a new class of linear topologies, which we refer to as incomplete $n{:}k$ cubes.

As we pointed out in Section 2, the $n{:}k$ cube $Q(n,k)$ can be decomposed into k copies of $Q(n-1,k)$. The relation between $Q(n,k)$ and its components can be described by the following recurrence relations:

$$Q(n,k) = kQ(n-1,k) = (k-1)Q(n-1,k) + kQ(n-2,k),$$

or in general

$$Q(n,k) = (k-1)Q(n-1,k) + (k-1)Q(n-2,k) + \ldots + (k-1)Q(n-\beta+1,k) + kQ(n-\beta,k). \tag{2}$$

We refer to Eq. (2) as the *characteristic equation* of $n{:}k$ cubes. Let the operator D be defined as $D^{\beta}(Q(n,k)) = Q(n-\beta,k)$. In this notation, the characteristic equation of the cube can be written as:

$$Q(n,k) = [(k-1)D + (k-1)D^2 + (k-1)D^3 + \ldots + kD^{\beta+1}]Q(n,k). \tag{3}$$

A degree-$(\beta+1)$ polynomial $a_1D + a_2D^2 + \ldots + a_{\beta}D^{\beta} + a_{\beta+1}D^{\beta+1}$ can be described by the string $A = a_1a_2 \ldots a_{\beta+1}$. We refer to A as the *generator string (or sequence)* of the polynomial. Thus, the recurrence relation Eq. (3) can be constructed from the sequence $A = (k-1)(k-1)\ldots k = (k-1)^{\beta}k$, where $(k-1)^{\beta}$ denotes the string $(k-1)(k-1)\ldots(k-1)$ of length β.

Definition 4.1. Let $X(n)$ denote the size (number of nodes) of a graph G_n in which each node is labeled with n digit number. Furthermore, assume that $X(n)$ satisfies a linear recurrence relation of the form $X(n) = AX(n)$, where $A = a_1a_2\ldots a_{\beta}$ is the generator sequence of the relation, i.e., $X(n) = a_1X(n-1) + a_2X(n-2) + \ldots + a_{\beta}X(n-\beta)$. Then, G_n is said to be a *linear graph*.

Throughout the remainder of this paper, a linear graph generated by the string A is denoted by $G_n(A)$.

Lemma 4.1. *The following graphs are linear with specified generator sequences*:

1. *A binary n-cube with* $A = 1^{\beta}2$ *for any non-negative integer* β.
2. *An incomplete hypercube with* $A = a_1 \ldots a_i \ldots a_{\beta}$, *where* $a_i \in \{0,1\}$ *for all i.*
3. *A Fibonacci cube with* $A = 11$.
4. *A k-ary n-cube with* $A = (k-1)^{\beta}k$.

Proof.

1. A binary n-cube $Q(n)$ has $N = 2^n$ nodes and can be decomposed into two binary $(n-1)$-cubes. Hence, $Q(n) = 2Q(n-1)$ or equivalently, $Q(n) = Q(n-1) + Q(n-2) + \ldots + Q(n-\beta) + 2Q(n-\beta-1)$. Thus, $Q(n)$ is a linear graph with the generator $A = 1^{\beta}2$.
2. An incomplete hypercube can be characterized by the sub-sequence 1^{β} of the generator A in part 1. To reduce the complexity of an incomplete hypercube, some of the bits in the string 1^{β} can be complemented to 0. Thus, the generator of an incomplete hypercube is $A = a_1 \ldots a_i \ldots a_{\beta}$, where $a_i \in \{0,1\}$ for all i.
3. A Fibonacci graph [17,18] of size n, $F(n)$, satisfies the recurrence relation $F(n) = F(n-1) + F(n-2)$. Thus, $F(n)$ is linear with the generator $A = 11$.
4. The proof is similar to part 1 but with $N = k^n$.□

Definition 4.2 An incomplete *n:k* cube $Q_{n,k}(A)$ is a linear graph with the generator $A = (k-1)^{\beta}$.

Definition 4.2 implies that an incomplete n:k cube contains $(k-1)$ copies of each of the subgraphs $(n-1)$:k cube through $(n-\beta)$:k cube. Thus, the size of an incomplete n:k cube is reduced by removing k copies of the subgraph $(n-\beta-1)$:k cube from the n:k cube.

It should be noted that there are numerous other choices for the generator of an incomplete n:k cube. For instance, some of the digits in the generator $(k-1)^{\beta}$ may be replaced by integers less than $k-1$. For example, for $k = 5$ and $\beta = 3$, some possible choices for generator A are: $A = 444$ or 432 or 404.

The choice of $(k-1)^{\beta}$ as the generator string simplifies the development of routing strategies for incomplete n:k cubes.

Next, we describe a methodology for generating recursively the node set $V_{n,k}(A)$ and the edge set $E_{n,k}(A)$ of the $Q_{n,k}(A)$ graph.

Definition 4.3. By an incomplete *n:k* cube we mean a linear graph $Q_{n,k}(A)$, where $A = (k-1)^{\beta}$ for some $\beta \geq 1$, with vertex (or node) set $V_{n,k}(A)$ defined recursively as follows: $V_{0,k}(A) = \{e\}$, $\forall n > 0$, $V_{n,k}(A) = A_1 \circ V_{n-1,k}(A) \cup \ldots \cup A_i \circ V_{n-i,k}(A) \cup \ldots \cup A_{\beta} \circ V_{n-\beta,,k}(A)$, where the complex expression $A_i V_{n-i,k}(A)$ denotes $\{e\}$ for the case with $i > n$.

Two nodes X and Y of $Q_{n,k}(A)$ are connected if and only if $W(H(X,Y)) = 1$, i.e., X and Y are directly connected if and only if their labels differ in exactly one position.

For example, $V_{n,3}(22) = \{0,1\} \circ V_{n-1,3}(22) \cup 2 \circ \{0,1\} \circ V_{n-2,3}(22)) = \{0,1\} \circ V_{n-1,3}(22) \cup \{20,21\} \circ V_{n-2,3}(22)$. Fig. 2 shows $Q_{n,3}(22)$ for $n = 1,2,3$ respectively.

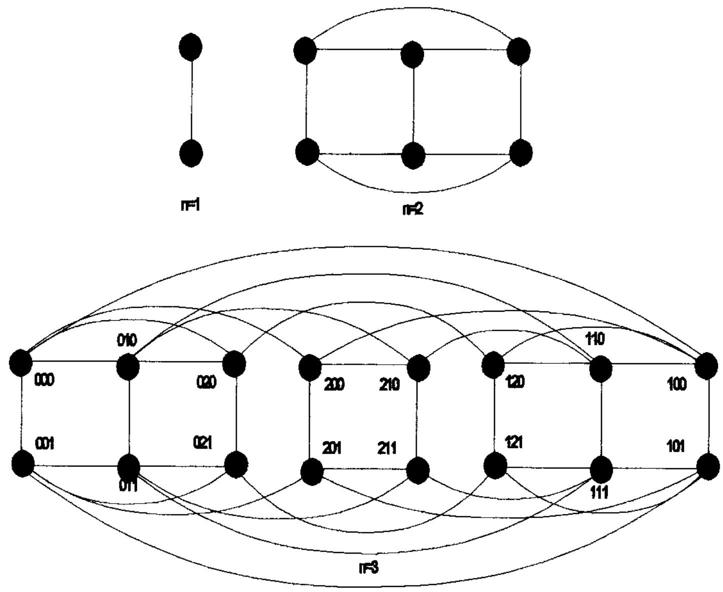


Fig. 2. Graph of $Q_{n,3}(22)$.

It should be noted that conditions (i) and (ii) of Definition 4.3 specifies the initial conditions for the characteristic equation of an incomplete n:k cube. By changing these conditions, other topologies may result. For instance, one may choose to define condition (ii) as $A_i V_{n-i,k}(A) = (k-1)^n$, for all $n<i$.

5. Topological properties of incomplete n:k cubes

In this section, we explore several useful topological properties of incomplete n:k cubes. We begin by defining a relation on the generator sequences.

Definition 5.1. For any two generator sequences A and A', A is a substring of A', denoted by $A \subseteq A'$, if and only if there exists a string B such that $A' = A \circ B$.

Lemma 5.1. *If $A \subseteq A'$, then $Q_{n,k}(A)$ is a subgraph of $Q_{n,k}(A')$.*

Proof. Follows from Definition 4.3 and by noticing that the seed sets of $Q_{n,k}(A)$ are subsets of the seed sets of $Q_{n,k}(A')$.□

Lemma 5.2. *Let $Q_{n,k}(A)$ be an incomplete n:k cube with the generator $A=(k-1)^{\beta}$. Then, for all $n \geq \beta$, $Q_{n,k}$ can be partitioned into β blocks of subgraphs such that: Block i contains $k-1$ subgraphs of $Q_{n,k}$ which are isomorphic images of $Q_{n-i,k}$ for $1 \leq i \leq \beta$.*

Proof. The prefix of the node set $V_{n-i,k}$, $1 \leq i \leq \beta$, (Definition 4.3) is A_i. It is fairly simple to observe that seed sets of a generator string are mutually disjoint. Thus, the node set of the graph $Q_{n,k}(A)$ can be decomposed into β mutually exclusive classes. The proof is then completed by noting that the cardinality of each seed set A_i, $1 \leq i \leq \beta$, is $k-1$. Thus, each block contains $k-1$ graphs that are isomorphic images of each other.□

For example, let $A = (k-1)^{\beta} = 2^2 = 22$. The graph of $Q_{n,3}(A)$ for $n=3$ is shown in Fig. 3. As shown in Fig. 3, $Q_{n,3}(A)$ can be decomposed into two types of mutually exclusive subgraphs, namely $Q_{n-1,3}(A)$ and $Q_{n-2,3}(A)$. The number of subgraphs of each type is $k-1$, which in this case is two.

Since the nodes and the edges of an incomplete n:k cube are based on the generator A, one should expect that the structural properties as well as the implementation cost of the network are also related to A. The dependency of $Q_{n,k}$ networks to generator sequences are examined next.

Theorem 5.1. *Let $Q_{n,k}(A)$ be an incomplete n:k cube with generator the $A=(k-1)^{\beta}$. Then:*

1. *$Q_{n,k}(A)$ is isomorphic to $Q(n,k)$ for all $n \leq \beta$,*
2. *$Q_{n,k}(A)$ is a subgraph of $Q(n,k)$ for all n,*
3. *$Q_{n,k}(A)$ is a subgraph of $Q_{n,k}(A')$ for all A' such that $A \subseteq A'$*
4. *$Q_{n,k}(A)$ can be decomposed into $(k-1)$ node-disjoint subgraphs which are isomorphic to $Q_{n-i,k}(A)$ for all $i \leq \beta$.*

Proof.

1. In constructing $Q_{n,k}(A)$, only the first n digits of generator $A = (k-1)^{\beta}$ are examined (Definition 4.3). However, since for all $n \leq \beta$ these digits are identical to the first n digits of the generator of the n:k cube, the structures are isomorphic images of each other.

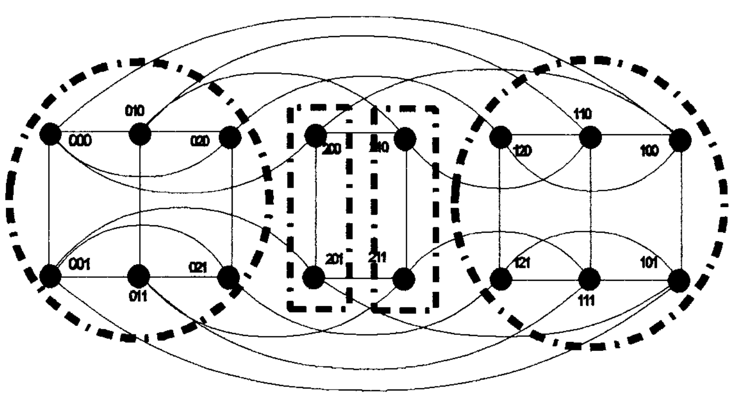


Fig. 3. Decomposition of $Q_{n,3}(A)$.

2. Since the generator of $Q(n,k)$ is $(k-1)^{\beta}k$ and $A \subseteq (k-1)^{\beta}k$, it then follows from Lemma 5.1 that $Q_{n,k}(A)$ is a subgraph of $Q(n,k)$ for all n.
3. Follows from Lemma 5.1.
4. Follows from Def. 4.3 and by noticing that the prefix of seed sequences are distinct, thus the subgraphs are node-disjoint.□

By Theorem 5.1(2), the size (number of nodes) of a $Q_{n,k}(A)$ cannot exceed k^n. The formal statement of this property is given by the next theorem.

Theorem 5.2. *For an incomplete n:k cube with the generator $A=(k-1)^{\beta}$ and vertex set $V_{n,k}(A)$:*

1. $\|V_{n,k}(A)\| = O[\delta^n(\beta)]$, *where* $k-1 \le \delta(\beta) < k$ *for all* β.
2. *The node degree of $Q_{n,k}(A)$ is upper bounded by n.*

Proof.

1. The characteristic equation of the recurrence relation Eq. (1) is $f(X) = X^{\beta} - (k-1)X^{\beta-1} - (k-1)X^{\beta-2} - \ldots - (k-1)X - (k-1) = 0$. Clearly, $f(0)<0$, and $f(k) > 0$. Thus, there exists a positive root $\delta<k$. The positive root δ of the characteristic equation can be calculated for certain values of β. For instance, for $\beta=1$, the corresponding δ is $k-1$. Since $\beta \ge 1$, a tighter bound on δ would be $k-1 \le \delta < k$.
2. Immediately follows from Theorem 5.1(2).□

The next theorem enables us to decompose the $Q_{n,k}(A)$ and counts the number of subgraphs $Q_{n-j,k}(A)$ embedded in it.

Theorem 5.3. *For $\beta>0$ and $k>1$, let $A=(k-1)^{\beta}$ be the generator of an incomplete n:k cube. Then, the number of subgraphs, B_i, isomorphic to $Q_{n-i,k}(A)$, is given recursively as*:

$$B_1 = k-1$$

$$B_{i+1} = kB_i \ \forall i \ge 1.$$

Proof. (1) By induction.

Basic step: For simplicity, we drop mention of A in the following. Since $Q_{n,k}$ satisfies the linear recurrence relation $Q_{n,k} = (k-1)Q_{n-1,k} + \ldots + (k-1)Q_{n-I,k} + \ldots + Q_{n-\beta,k}$, the number of subgraphs isomorphic to $Q_{n-1,k}$ is $k-1$.

Induction step: In order to find the number of subgraphs isomorphic to $Q_{n-i,k}$, the recurrence relation $Q_{n,k} = (k-1)Q_{n-1,k} + \ldots + (k-1)Q_{n-I,k} + \ldots + (k-1)Q_{n-\beta,k}$ should be transformed into the form $Q_{n,k} = B_i\, Q_{n-i,k} + B_iQ_{n-i-1,k} + \ldots$. But since $Q_{n-i,k}$ must satisfy the recurrence relation $Q_{n-i,k} = (k-1)Q_{n-i-1,k} + (k-1)Q_{n-I-2,k} + \ldots + (k-1)Q_{n-i-\beta,k}$ one may conclude that: $Q_{n,k} = B_i[(k-1)Q_{n-i-1,k} + \ldots + (k-1)Q_{n-I,k} + \ldots + (k-1)Q_{n-\beta,k}] + B_iQ_{n-i-1,k} + \ldots = kB_iQ_{n-i-1,k} + \ldots$. Thus, the number of subgraphs isomorphic to $Q_{n-i-1,k}$, $B_{i+1} = kB_i$.□

For example, let $A = (k-1)^3 = 333$, then $B_1 = k - 1 = 3$, $B_2 = kB_1 = 12$, *and* $B_3 = kB_2 = 48$. Thus, there are 48 copies of $Q_{n-3,4}(A)$ embedded into $Q_{n,4}(A)$.

Several relevant, but easily verifiable, properties of incomplete n:k cubes are listed below.

Lemma 5.3.

1. $\forall n \geq 0$, $0^n \in V_{n,k}(A)$.
2. *For every generator string A, $Q_{n,k}(A)$ is a connected graph.*
3. *The diameter of $Q_{n,k}(A)$ is bounded above by n.*

6. Routing in incomplete *n*:*k* cubes

In this section, we will present a one-to-one routing algorithm on incomplete n:k cubes. Given a source S and destination node D, the algorithm constructs a shortest path P connecting S to D.

The concept of seed sets of a generator string is crucial to our discussion and is repeated here for easy reference.

For a generator string $A = (k-1)^{\beta}$, A_i is called a seed set of A and is defined recursively as: $A_1 = \{0,1,\ldots,k-2\}$ and $A_i = (k-1) \circ A_{i-1}$, $1 < i \leq \beta$, or equivalently, $A_i = (k-1)^{i-1}A_1$.

For example, for the generator $A = 3^4$, the seed sets of A are:

$A_1 = \{0,1,2\}$,
$A_2 = 3\circ\{0,1,2\} = \{30,31,32\}$,
$A_3 = 3\circ A_2 = 3\circ\{30,31,32\} = \{330,331,332\} = 3^2A_1$, and
$A_4 = 3\circ A_3 = 3\circ\{330,331,332\} = \{3330,3331,3332\} = 3^3A_1$.

The vertex set $V_{n,k}(A)$ of $Q_{n,k}(A)$ is defined (Def. 4.3) recursively as:

$$V_{n,k}(A) = A_1 \circ V_{n-1,k}(A) \cup A_2 \circ V_{n-2,k}(A) \cup A_3 \circ V_{n-3,k}(A) \cup A_4 \circ V_{n,k}(A).$$

Before describing the routing algorithm we need to make an observation and also define a relation on the vertex set of a $Q_{n,k}$ graph. First observe that, since a $Q_{n,k}$ graph is generated by the string $A = (k-1)^{\beta}$, any node of the graph is composed of a sequence of $p_i \in A_i$. (Note that there is no restriction on the number of appearances of a specific string p_i in a node label.) As a result, any sequence of length n consisting of p_i, $1 \leq i \leq \beta$, is a legitimate node label.

Next, we describe a relation between seed sequences. Based on this relation we develop a routing algorithm for incomplete n:k cubes.

Lemma 6.1. *For any node label X of a $Q_{n,k}$ graph, if $p_i = (k-1)^{i-1}p_1$ appears in the label of X, by replacing digit $(k-1)$ in position j, $1 \leq j \leq i-1$, by p_1, i.e., replacing $p_i = (k-1)^{i-1}p_1 = (k-1)^{j-1}(k-1)(k-1)^{i-j-1}p_1$ by $(k-1)^{j-1}p_1(k-1)^{i-j-1}p_1 = [(k-1)^{j-1}p_1][(k-1)^{i-j-1}p_1] = p_jp_{i-j}$, the new sequence Y is a legitimate node label and is adjacent to X, i.e., $H(X,Y) = 1$.*

Proof. Clearly $H(p_i,p_j\ p_{i-j}) = H((k-1)^{i-1}\ p_1,(k-1)^{j-1}p_1(k-1)^{i-j-1}p_1) = 1$. Since the rest of sequences in X and Y are the same, $H(X,Y) = 1$ and X and Y are directly connected.□

Based on Lemma 6.1, one may conclude the following:

Lemma 6.2. *For all A, $Q_{n,k}(A)$ is a connected graph.*

Proof. Let $S = s_{n-1}\dots s_i\dots s_0$ and $D = d_{n-1}\dots d_i\dots d_0$ be two nodes of $Q_{n,k}(A)$. According to Lemma 6.1, by substituting each non-zero digit of S by zero, a path P of length $L(P) = L((H(S,0^n))$ can be established between S and the node 0^n. Similarly, there exists a path P' of length $L(P') = L(H(D,0^n)$ between 0^n and D. Thus, every pair of nodes of $Q_{n,k}(A)$ are connected by a path of length $L(H(S,0^n) + L(H(D,0^n)$, which implies that $Q_{n,k}(A)$ is connected.□

Next, we present a *one-to-one routing* algorithm on incomplete *n*:*k* cubes. A one-to-one routing algorithm constructs a single path in the network between any pair of nodes.

6.1. One-to-one routing on $Q_{n,k}$ graphs

We first give an intuitive description of our algorithms. For a given source-destination pair of nodes (S,D), we scan S and D from left to right till they disagree in position i. Without loss of generality, assume that the bit i in S is p_1 and is $k-1$ in D. Obviously, i cannot be the last bit of the sequence, otherwise D is not a legitimate sequence. We then replace bit i of D with p_1 to obtain a new node D'. By Lemma 6.1 node S is adjacent to D'. We then continue with scanning the rest of the sequence and perform a similar substitution, if required, until all digits of source and destination nodes agree.

The routing algorithm, called ONE-TO-ONE ROUTING, and which is presented formally below generates a path of length $H(S,D)$ between nodes S and D of a $G_{n,k}(A)$ network.

Algorithm:. ONE-TO-ONE ROUTING ON INCOMPLETE *n*:*k* CUBES INPUT: a pair of source-destination nodes $S = a_1\dots a_i\dots a_n$ and $D = d_1\dots d_i\dots d_n$. OUTPUT: a sequence of node labels forming a path between S and D.

```
Initialize vectors S(I) and D(J) to S(1) = S and D(1) = D
  For k=1 to n do
    if a_k = d_k continue with the next k;
    if a_k > d_k set a_k to d_k and assign S to S(I). Increment I if S(I) = D(J) end the search,
    otherwise continue with the next k.
    if d_k > a_k set d_k to a_k and assign D to D(J). Increment J
    if S(I) = D(J) end the search, otherwise continue with the next k.
  endif
Form concatenation of components of vectors S and D.
```

For example, for A = 333, S = 313302 and D = 332310 algorithm-one-to-one routing generates the following sequence of node labels: (313302)(312302)(312300)(312310)(332310).

The above Algorithm can be modified, as described below, to generate all shortest paths between two nodes $S = a_1 \ldots a_i \ldots a_n$ and $D = d_1 \ldots d_i \ldots d_n$ of $Q_{n,k}(A)$. First, we need to decompose the Hamming sequence $H(S,D)$ into two mutually exclusive sequences $H_S(S,D)$ and $H_D(S,D)$ such that bit i of $H_S(S,D)$ is one if and only if $s_i > d_i$, and zero otherwise. $H_D(S,D)$ is defined similarly, i.e., bit i of $H_D(S,D)$ is one if and only if $d_i > s_i$.

The number of nodes in vector $S(I)$ of the above algorithm is the same as the $W(H_S(S,D))$. Similarly, the number of nodes in $D(I)$ is the same as $W(H_D(S,D))$. The number of ways that components of $S(I)$ and $D(I)$ can be arranged are $W(H_S(S,D))$ and $W(H_D(S,D))$ respectively. Thus, the number of distinct shortest paths between S and D is $[W(H_S(S,D))!\ [(W(H_D(S,D))!]$.

For example, let $S = 32041$ and $D = 13242$ then, $H_S(S,D) = 10000$ and $H_D(S,D) = 01101$. The number of shortest paths between S and D is (1!)(3!) = 6. These paths are:

1. (32041)(12041)(12042)(12242)(13242)
2. (32041)(12041)(12241)(12242)(13242)
3. (32041)(12041)(12241)(13241)(13242)
4. (32041)(12041)(13041)(13241)(13242)
5. (32041)(12041)(13041)(13042)(13242)
6. (32041)(12041)(12042)(13042)(13242)

Based on the above observation, the necessary and sufficient conditions for the existence of parallel (or node disjoint) paths in $Q_{n,k}(A)$ can be stated as follows.

Lemma 6.3. *Let S and D be two nodes of a $Q_{n,k}(A)$ network. Then, there exist at least two parallel paths between S and D if and only if any one of the following condition holds*:

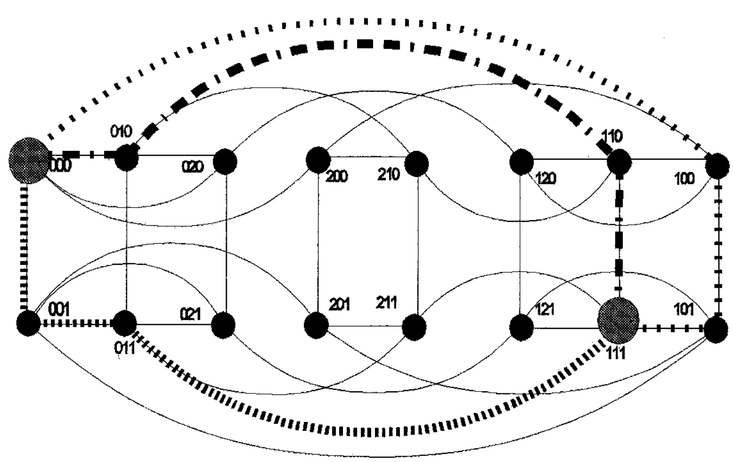


Fig. 4. Node disjoint paths between $S = 000$ and $D = 111$.

1. $W(H_S(S,D)) \geq 2$ and $W(H_D(S,D)) \geq 2$
2. $W(H_S(S,D)) = 0$ and $W(H_D(S,D)) \geq 2$
3. $W(H_S(S,D)) \geq 2$ and $W(H_D(S,D)) = 0$.

For example, the parallel paths between nodes $S=000$ and $D=111$ of $G_{3,3}(22)$ are (000)(100)(101)(111), (000)(010)(110)(111), and (000)(001)(011)(111) as shown in Fig. 4.

7. Concluding remarks and future research

This paper summarizes the results of the development of, and research into, a new interconnection topology, the incomplete *n*:*k* cube. It is shown that this topology can be recursively defined and characterized by a string of base-*k* digits called a generator string. The incomplete *n*:*k* cube, while more compact that of the *k*-ary *n*-cube topology have many of its useful properties. The impact of the generator string on the topological and routing properties of incomplete *n*:*k* cubes is studied. Finally, an efficient point-to-point routing algorithm for incomplete *n*:*k* cube networks is presented.

Following is a list of several open problems.

1. Developing efficient embedding algorithms for $Q_{n,k}$ graphs.
2. Investigating fault-tolerance of incomplete *n*:*k* cube networks.
3. Developing routers for faulty incomplete *n*:*k* cube networks.

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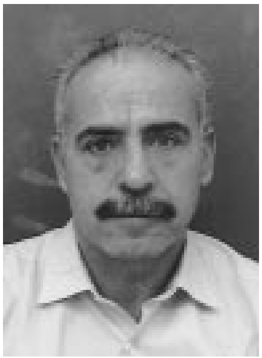

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