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The Poisson Voronoi Tessellation
I. A Basic Identity

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Abstract. This paper gives basic relations between the stationary Poisson point process and the point process of vertices of the corresponding Voronoi tessellation in $\mathbf{R}^d$ and of planar sections through it. The results are based on a study of the Palm distribution of the point process of vertices. An identity is given connecting the distribution of a Poisson point process and the Palm distribution with respect to the vertices of the corresponding Voronoi tessellation. Distributional properties for the edges are discussed. Finally, identities are given for characteristics of the "typical" edge and an edge chosen at random emanating from the "typical" vertex.

1. Introduction

Voronoi and Delaunay tessellations generated by random point processes are important models of stochastic geometry, which have been studied intensively for a long time. Many results have been found in the case when the generating point process is Poisson (MILES [2], MØLLER [3], and other authors). Much of these results are summarized in the book by OKABE, BOOTS and SUGIHARA [7], p. 273–334. But in spite of the simple structure of both models analytical expressions for the distribution of many characteristics are unknown until now.

The present paper gives identities which turn out to be fundamental for the derivation of new distributional characteristics of the d-dimensional Poisson Voronoi tessellation. The approach is based on special properties of the Palm distribution of the point process of vertices of the Poisson Voronoi tessellation, or, in other words, of the neighbourhood of the "typical" point of this point process. The word "typical" is used in the sense of STOYAN, KENDALL and MECKE [8], p. 110. A vertex is a point which has exactly $d + 1$ nearest neighbours of the points of the Poisson point process within the same distance. In other words, $d + 1$ Poisson points (centres) define a d-dimensional ball (d-ball). The midpoint of this d-ball is a vertex if and only if the interior of the d-ball does not contain any other centre. First, the distribution of the point process of that points (centres) outside of the d-ball circumscribing the "typical" vertex is investigated. It will be shown that its distributions is identical with that of the Poisson process outside of this b-ball.

These identities established for the Poisson Voronoi tessellation in $\mathbb{R}^d$ are extended to the case of planar sections through it.

Furthermore, distributional properties of edges are considered. A relation between distributional characteristics of the "typical" edge and an edge emanating from the "typical" vertex chosen at random is given.

2. Preliminaries

In Section 2 and 3, simple point processes are considered, i.e., point processes without multiple points. A simple point process Φ with distribution $P = \mathbb{P} \circ \Phi^{-1}$ is defined to be an $(\mathfrak{A}, \mathfrak{N})$-measurable mapping from a probability space $[\Omega, \mathfrak{A}, \mathbb{P}]$ into the measurable space $[N, \mathfrak{N}]$ of realizations. Here N is the system of subsets φ of $\mathbb{R}^d$ not having accumulation points (system of locally finite subsets of $\mathbb{R}^d$). The σ-algebra over N is generated by the sets $\{\varphi \in N : \text{card}\,(\varphi \cap B) = n\}$ for bounded Borel B and $n = 0, 1, 2, \ldots$

Let Φ be a stationary Poisson point process in $\mathbb{R}^d$, $2 \le d < \infty$, with distribution P and intensity λ. The Voronoi tessellation corresponding to Φ is denoted by $\mathfrak{V}$ and the Delaunay tessellation by $\mathfrak{D}$ (for definitions see e.g. [8], p. 259–261]. Let $\Psi \sim Q$ be the point process of vertices of $\mathfrak{V}$. It has the itensity

$$\lambda_\Psi = \frac{2^{d+1}\pi^{(d-1)/2}\Gamma\left(\frac{d^2+1}{2}\right)}{(d+1)\,d^2\Gamma\left(\frac{d^2}{2}\right)} \left\{\frac{\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(\frac{d+1}{2}\right)}\right\}^d \lambda\,,$$

(cf. [3], p. 67). Then to P-almost every $\varphi \in N$ there corresponds a realization ψ of Ψ with the following relations.

On one hand, for every $y \in \psi$ there exist $d + 1$ points $z_1, z_2, \ldots, z_{d+1} \in \varphi$ with the same distance δ_{d+1} to y,

$$\|y - z_1\| = \|y - z_2\| = \ldots = \|y - z_{d+1}\| = \delta_{d+1}$$

and

$$\varphi(b^{\text{int}}(y, \delta_{d+1})) = 0 \quad \text{and} \quad \varphi(b(y, \delta_{d+1})) = d + 1\,.$$

Here $\varphi(B)$ denotes the number of points of φ in the set B, and $b^{\text{int}}(x, r)$ is the interior of the closed d-ball $b(x, r)$ with radius r and midpoint x.

On the other hand, the midpoint of a d-ball defined by $d + 1$ pairwise different points of φ belongs to ψ if and only if the interior of this d-ball does not contain any other point of φ.

Let $\Psi_0 \sim Q_0$ be the Palm version of the point process Ψ of vertices. Because of the close geometrical connection between vertices and centres, the point process of the generating points of the Voronoi tessellation corresponding to Ψ_0 has $d + 1$ points on the boundary of a d-ball $b(o, \Delta_{d+1})$ with a random radius Δ_{d+1} and no points in its interior. Let denote this point process by $\Phi_0^\Psi \sim P_0^\Psi$.

Similar statements can be made concerning the point process of vertices of a planar section through $\mathfrak{V}$. Let p be the (x_1, x_2)-plane in $\mathbb{R}^d$ and let $\Xi \sim R$ the point process of vertices of $\mathfrak{V} \cap p$. Its intensity is

$$\lambda_{\Xi} = \frac{2\pi\Gamma\left(\frac{3d-1}{2}\right)\left\{\Gamma\left(\frac{d}{2}+1\right)\right\}^{3-2/d}\Gamma\left(3-\frac{2}{d}\right)}{3d\Gamma\left(\frac{3d}{2}-1\right)\left\{\Gamma\left(\frac{d+1}{2}\right)\right\}^{3}}\lambda^{2/d}$$

(cf. [3]. p. 68). Then to P-almost every $\varphi \in N$ there corresponds a realization ξ of Ξ with the following relations.

Since a point of ξ is an intersection point of p and a $(d-2)$-dimensional hyperplane which lies in the boundaries of three cells, for P-almost every $\eta \in \xi$ there are exactly three nearest neighbours $\zeta_1, \zeta_2, \zeta_3 \in \varphi$ with

$$\|\eta - \zeta_1\| = \|\eta - \zeta_2\| = \|\eta - \zeta_3\| = \delta_3\,,$$
$$\varphi(b^{\mathrm{int}}(\eta, \delta_3)) = 0 \quad \text{and} \quad \varphi(b(\eta, \delta_3)) = 3\,.$$

Conversely, three pairwise different points generate exactly one d-ball having the midpoint in p. This midpoint belongs to ξ if and only if the d-ball contains no other points of φ.

Analogously to Ψ_0 and Φ_0^{Ψ}, the point processes $\Xi_0 \sim R_0$ and $\Phi_0^{\Xi} \sim P_0^{\Xi}$ will be considered here. The planar point process Ξ_0 is the Palm version of the point process of vertices in the section plane. Further, Φ_0^{Ξ} is the point process of generating points of the d-dimensional Voronoi tessellation corresponding to Ξ_0 and has three points on the boundary of the d-ball $b(o, \Delta_3)$ with a random Δ_3 and no points in its interior.

The point processes Φ_0^{Ψ} and Φ_0^{Ξ} are in close connection to the Poisson process Φ, because its points can be thought to result from a random shift of Φ. Their distributions P_0^{Ψ} and P_0^{Ξ} will be considered in the following.

3. Identities for Φ_0^{Ψ} and Φ_0^{Ξ}

Let χ be any point sequence of $\mathbb{R}^d$. Then $C(\chi)$ denotes the largest closed b-ball centred at the origin which does not contain a point of χ in its interior. Let X_1 and X_2 be two independent point processes in $\mathbb{R}^d$ and

$$\widetilde{X} = [X_1 \cap C(X_1)] \cup [X_2 \cap C^c(X_1)]\,, \tag{3.1}$$

where $B^c = \mathbb{R}^d \setminus B$. In the following the distribution of $\widetilde{X}$ will be investigated for the case $X_1 \sim P_0^{\Psi}$, $X_2 \sim P$. The measurable space corresponding to X_1 is denoted by $[N_0, \mathfrak{N}_0]$. Clearly, $C(X_1)$ is identical with $b(o, \Delta_{d+1})$ and $\partial b(o, \Delta_{d+1})$ contains $d+1$ points of X_1 with probability 1.

A reason for the fact that $\widetilde{X}$ has the distribution P_0^{Ψ} will be given, because it is an important fundamental for the determination of distributional properties of the Poisson Voronoi tessellation. An equivalent wording, which is more suitable for a proof, is the following measure-theoretical description of the distribution P_0^{Ψ}.

Lemma 1. *For any measurable function $u: N_0 \to [0, \infty)$ the equality*

$$\int u(\varphi)\, P_0^{\Psi}(\mathrm{d}\varphi) = \int\int u([\chi_1 \cap C(\chi_1)] \cup [\chi_2 \cap C^c(\chi_1)])\, P_0^{\Psi}(\mathrm{d}\chi_1)\, P(\mathrm{d}\chi_2) \tag{3.2}$$

holds.

Proof. A representation of P_0^Ψ. The Palm distribution P_0^Ψ of the point process Ψ of vertices of $\mathfrak{V}$ is defined by

$$\lambda_\Psi \int u(\varphi)\, P_0^\Psi(\mathrm{d}\varphi) = \int \sum_{y \in \psi} h(y)\, u(\varphi - y)\, P(\mathrm{d}\varphi)$$

for any measurable function $u: N_0 \to [0, \infty)$, where $\varphi - y = \{z - y : z \in \varphi\}$ and $h: \mathbb{R}^d \to [0, \infty)$ is a measurable function with $\int_{\mathbb{R}^d} h(y)\, \mathrm{d}y = 1$. Replacing the summation over all points (vertices) of ψ by a summation (denoted by $\sum^*$) over all $(d + 1)$-tuples of pairwise distinct points (centres) $z_1, z_2, \ldots, z_{d+1}$ of φ, yields

$$(3.3) \qquad \lambda_\Psi \int u(\varphi)\, P_0^\Psi(\mathrm{d}\varphi) = \frac{1}{(d+1)!} \int \sum^*_{z_1, z_2, \ldots, z_{d+1}} h(c^{(1,\ldots,d+1)})\, I^{(1,\ldots,d+1)}(\varphi)\, u(\varphi - c^{(1,\ldots,d+1)})\, P(\mathrm{d}\varphi)\,.$$

Here $c^{(1,\ldots,d+1)}$ denotes the centre of the d-ball spanned by $z_1, z_2, \ldots, z_{d+1}$ and

$$I^{(1,\ldots,d+1)}(\varphi) = \begin{cases} 1\,, & \text{if there is no point of } \varphi \text{ in the interior of the } d\text{-ball} \\ & \text{corresponding to } z_1, z_2, \ldots, z_{d+1}\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Properties of the Poisson process Φ. A fundamental property of the stationary Poisson process is given by

$$(3.4) \qquad \int \sum_{z \in \varphi} g(z, \varphi)\, P(\mathrm{d}\varphi) = \lambda \int_{\mathbb{R}^d} \int g(z, \varphi \cup \{z\})\, P(\mathrm{d}\varphi)\, \mathrm{d}z$$

for any measurable function $g: \mathbb{R}^d \times N \to [0, \infty)$, cf. [1]. Furthermore,

$$(3.5) \qquad \int g_1(\varphi \cap B)\, g_2(\varphi \cap B^c)\, P(\mathrm{d}\varphi) = \int\int g_1(\varphi_1 \cap B)\, g_2(\varphi_2 \cap B^c)\, P(\mathrm{d}\varphi_1)\, P(\mathrm{d}\varphi_2)$$

with measurable $g_1, g_2: N \to [0, \infty)$ for any Borel $B \subset \mathbb{R}^d$ holds. (Property of independence for the Poisson process).

A further representation of P_0^Ψ. Now the fact that P is the distribution of a Poisson process will be used. Repeated use of (3.4) to the right-hand side of (3.3) for a measurable $f: N_0 \to [0, \infty)$ (which takes the place of u) leads to

$$(d+1)!\, \lambda_\Psi \int f(\varphi)\, P_0^\Psi(\mathrm{d}\varphi) = \int \underbrace{\sum_{z_1 \in \varphi} \sum_{z_2 \in \varphi \setminus \{z_1\}} \cdots \sum_{z_d \in \varphi \setminus \{z_1\} \setminus \ldots \setminus \{z_{d-1}\}} \sum_{z_{d+1} \in \varphi \setminus \{z_1\} \setminus \ldots \setminus \{z_d\}}}_{d\,+\,1 \text{ summations}} \times h(c^{(1,\ldots,d+1)})\, I^{(1,\ldots,d+1)}(\varphi)\, f(\varphi - c^{(1,\ldots,d+1)})\, P(\mathrm{d}\varphi)$$

$$= \lambda \int_{\mathbf{R}^d} \int \underbrace{\sum_{z_2 \in \varphi} \sum_{z_3 \in \varphi \setminus \{z_2\}} \cdots \sum_{z_d \in \varphi \setminus \{z_2\} \setminus \ldots \setminus \{z_{d-1}\}} \sum_{z_{d+1} \in \varphi \setminus \{z_2\} \setminus \ldots \setminus \{z_d\}}}_{d \text{ summations}}$$

$$\times h(c^{(1,\ldots,d+1)})\, I^{(1,\ldots,d+1)}(\varphi \cup \{z_1\})\, f([\varphi \cup \{z_1\}] - c^{(1,\ldots,d+1)})\, P(\mathrm{d}\varphi)\, \mathrm{d}z_1$$

$$= \ldots$$

$$= \lambda^d \underbrace{\int_{\mathbf{R}^d} \cdots \int_{\mathbf{R}^d}}_{d \text{ integrations}} \int \sum_{z_{d+1} \in \varphi} h(c^{(1,\ldots,d+1)})\, I^{(1,\ldots,d+1)}\left(\varphi \cup \bigcup_{i=1}^{d} \{z_i\}\right)$$

$$\times f\left(\left[\varphi \cup \bigcup_{i=1}^{d} \{z_i\}\right] - c^{(1,\ldots,d+1)}\right) P(\mathrm{d}\varphi)\, \mathrm{d}z_d\, \mathrm{d}z_{d-1} \ldots \mathrm{d}z_2\, \mathrm{d}z_1$$

$$= \lambda^{d+1} \underbrace{\int_{\mathbf{R}^d} \cdots \int_{\mathbf{R}^d}}_{d+1 \text{ integrations}} \int h(c^{(1,\ldots,d+1)})\, I^{(1,\ldots,d+1)}\left(\varphi \cup \bigcup_{i=1}^{d+1} \{z_i\}\right)$$

$$\times f\left(\left[\varphi \cup \bigcup_{i=1}^{d+1} \{z_i\}\right] - c^{(1,\ldots,d+1)}\right) P(\mathrm{d}\varphi)\, \mathrm{d}z_{d+1}\, \mathrm{d}z_d \ldots \mathrm{d}z_2\, \mathrm{d}z_1 .$$

The identity

$$I^{(1,\ldots,d+1)}\left(\varphi \cup \bigcup_{i=1}^{d+1} \{z_i\}\right) = I^{(1,\ldots,d+1)}(\varphi)$$

yields

$$(3.6) \qquad (d+1)!\, \lambda_\Psi \int f(\varphi)\, P_0^\Psi(\mathrm{d}\varphi) = \lambda^{d+1} \int_{\mathbf{R}^d} \cdots \int_{\mathbf{R}^d} h(c^{(1,\ldots,d+1)}) \int I^{(1,\ldots,d+1)}(\varphi)$$

$$\times f\left(\left[\varphi \cup \bigcup_{i=1}^{d+1} \{z_i\}\right] - c^{(1,\ldots,d+1)}\right) P(\mathrm{d}\varphi)\, \mathrm{d}z_{d+1}\, \mathrm{d}z_d \ldots \mathrm{d}z_2\, \mathrm{d}z_1 .$$

Proof of (3.2). Let $\widetilde{C}(z_1, z_2, \ldots, z_{d+1})$ be the closed d-ball generated by the boundary points $z_1, z_2, \ldots, z_{d+1}$. Setting $f = u$ in (3.6) and using (3.5) yield

$$(d+1)!\, \lambda_\Psi \int u(\varphi)\, P_0^\Psi(\mathrm{d}\varphi)$$

$$= \lambda^{d+1} \int_{\mathbf{R}^d} \cdots \int_{\mathbf{R}^d} h(c^{(1,\ldots,d+1)}) \int I^{(1,\ldots,d+1)}(\varphi \cap \widetilde{C}(z_1, z_2, \ldots, z_{d+1}))$$

$$\times u\left(\left[(\varphi \cap \widetilde{C}^c(z_1, z_2, \ldots, z_{d+1})) \cup \bigcup_{i=1}^{d+1} \{z_i\}\right] - c^{(1,\ldots,d+1)}\right) P(\mathrm{d}\varphi)$$

$$\times \mathrm{d}z_{d+1}\, \mathrm{d}z_d \ldots \mathrm{d}z_2\, \mathrm{d}z_1$$

$$= \lambda^{d+1} \int_{\mathbf{R}^d} \dots \int_{\mathbf{R}^d} h(c^{(1,\dots,d+1)}) \int I^{(1,\dots,d+1)}(\varphi \cap \widetilde{C}(z_1, z_2, \dots, z_{d+1}))\, P(\mathrm{d}\varphi)$$

$$\times \int u\left(\left[(\chi \cap \widetilde{C}^c(z_1, z_2, \dots, z_{d+1})) \cup \bigcup_{i=1}^{d+1} \{z_i\}\right] - c^{(1,\dots,d+1)}\right) P(\mathrm{d}\chi)$$

$$\times\, \mathrm{d}z_{d+1}\, \mathrm{d}z_d \dots \mathrm{d}z_2\, \mathrm{d}z_1$$

$$= \lambda^{d+1} \int_{\mathbf{R}^d} \dots \int_{\mathbf{R}^d} h\left(c^{(1,\dots,d+1)}\right) \int I^{(1,\dots,d+1)}(\varphi)\, P(\mathrm{d}\varphi)$$

$$\times \int u(s)\, P(\mathrm{d}\chi)\, \mathrm{d}z_{d+1}\, \mathrm{d}z_d \dots \mathrm{d}z_2\, \mathrm{d}z_1\,,$$

with

$$s = \bigcup_{i=1}^{d+1} \{z_i - c^{(1,\dots,d+1)}\}$$

$$\times \cup \left[(\chi - c^{(1,\dots,d+1)}) \cap \widetilde{C}^c(z_1 - c^{(1,\dots,d+1)}, \dots, z_{d+1} - c^{(1,\dots,d+1)})\right].$$

Finally, because of the stationarity of P, we have

$$(d+1)!\, \lambda_\Psi \int u(\varphi)\, P_0^\Psi(\mathrm{d}\varphi) = \lambda^{d+1} \int_{\mathbf{R}^d} \dots \int_{\mathbf{R}^d} h(c^{(1,\dots,d+1)}) \int I^{(1,\dots,d+1)}(\varphi)\, P(\mathrm{d}\varphi) \tag{3.7}$$

$$\times \int u\left(\bigcup_{i=1}^{d+1} \{z_i - c^{(1,\dots,d+1)}\} \cup \left[\chi \cap \widetilde{C}^c\left(z_1 - c^{(1,\dots,d+1)}, \dots, z_{d+1} - c^{(1,\dots,d+1)}\right)\right]\right)$$

$$\times P(\mathrm{d}\chi)\, \mathrm{d}z_{d+1}\, \mathrm{d}z_d \dots \mathrm{d}z_2\, \mathrm{d}z_1\,.$$

Inserting

$$f(\varphi) = \int u([\varphi \cap C(\varphi)] \cup [\chi \cap C^c(\varphi)])\, P(\mathrm{d}\chi)$$

in (3.6) yields

$$(d+1)!\, \lambda_\Psi \int\int u([\varphi \cap C(\varphi)] \cup [\chi \cap C^c(\varphi)])\, P(\mathrm{d}\chi)\, P_0^\Psi(\mathrm{d}\varphi)$$

$$= \lambda^{d+1} \int_{\mathbf{R}^d} \dots \int_{\mathbf{R}^d} h(c^{(1,\dots,d+1)}) \int I^{(1,\dots,d+1)}(\varphi)$$

$$\times \int u([\widetilde{\varphi} \cap C(\widetilde{\varphi})] \cup [\chi \cap C^c(\widetilde{\varphi})])\, P(\mathrm{d}\varphi)\, P(\mathrm{d}\chi)\, \mathrm{d}z_{d+1}\, \mathrm{d}z_d \dots \mathrm{d}z_2\, \mathrm{d}z_1\,,$$

where $\widetilde{\varphi} = \left(\varphi \cup \bigcup_{i=1}^{d+1} \{z_i\}\right) - c^{(1,\dots,d+1)} = (\varphi - c^{(1,\dots,d+1)}) \cup \bigcup_{i=1}^{d+1} \{z_i - c^{(1,\dots,d+1)}\}$. If $I^{(1,\dots,d+1)}(\varphi) = 1$, then

$$C(\widetilde{\varphi}) = \widetilde{C}(z_1 - c^{(1,\dots,d+1)}, \dots, z_{d+1} - c^{(1,\dots,d+1)})$$

and the intersection $\widetilde{\varphi} \cap C(\widetilde{\varphi})$ is just $\bigcup_{i=1}^{d+1} \{z_i - c^{(1,\dots,d+1)}\}$. Thus

(3.8)
$$(d+1)!\,\lambda_\Psi \int\int u([\varphi \cap C(\varphi)] \cup [\chi \cap C^c(\varphi)])\, P(d\chi)\, P_0^\Psi(d\varphi)$$
$$= \lambda^{d+1} \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} h\left(c^{(1,\dots,d+1)}\right) \int I^{(1,\dots,d+1)}(\varphi)\, P(d\varphi)$$
$$\times \int u\left(\bigcup_{i=1}^{d+1} \{z_i - c^{(1,\dots,d+1)}\} \cup \left[\chi \cap \widetilde{C}^c\left(z_1 - c^{(1,\dots,d+1)}, \dots, z_{d+1} - c^{(1,\dots,d+1)}\right)\right]\right)$$
$$\times P(d\chi)\, dz_{d+1}\, dz_d \dots dz_2\, dz_1\,.$$

Comparison of (3.7) and (3.8) yields (3.2). □

Now consider a planar section through the Poisson Voronoi tessellation $\mathfrak{V}$ in $\mathbb{R}^d$ again. The distribution of $\widetilde{X}$ in (3.1) will be considered for $X_1 \sim P_0^\Xi$ and $X_2 \sim P$. In that case is $C(X_1) = b(o, \Delta_3)$. It will be shown that $\widetilde{X} \sim P_0^\Xi$.

Lemma 1′. *For any measurable function $u: N_0 \to [0, \infty)$ the equality*

(3.9)
$$\int u(\varphi)\, P_0^\Xi(d\varphi) = \int\int u([\chi_1 \cap C(\chi_1)] \cup [\chi_2 \cap C^c(\chi_1)])\, P_0^\Xi(d\chi_1)\, P(d\chi_2)$$

holds.

Proof. This identity for P_0^Ξ can be shown in quite analogy to that considerations made in the proof of lemma 1: Using (3.4) the analog to (3.6) is

$$6\lambda_\Xi \int f(\varphi)\, P_0^\Xi(d\varphi) = \lambda^3 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h\left(c_p^{(1,2,3)}\right) \int I_p^{(1,2,3)}(\varphi)$$
$$\times f\left([\varphi \cup \{\zeta_1\} \cup \{\zeta_2\} \cup \{\zeta_3\}] - c_p^{(1,2,3)}\right) P(d\varphi)\, d\zeta_3\, d\zeta_2\, d\zeta_1\,,$$

where $c_p^{(1,2,3)} \in p$ is the midpoint of the d-ball with the points $\zeta_1, \zeta_2, \zeta_3$ on its boundary and $I_p^{(1,2,3)}$ is the indicator of the event that there is no other point of φ inside of this d-ball centered in $c_p^{(1,2,3)}$. Let $\widetilde{C}_p(\zeta_1, \zeta_2, \zeta_3)$ be the closed d-ball with $\zeta_1, \zeta_2, \zeta_3$ on the boundary centred in p. Put $f = u$ and $f(\varphi) = \int u([\varphi \cap C(\varphi)] \cup [\chi \cap C^c(\varphi)])\, P(d\chi)$ as above and the use of (3.5) leads to

$$6\lambda_\Xi \int u(\varphi)\, P_0^\Xi(d\varphi) = 6\lambda_\Xi \int\int u([\varphi \cap C(\varphi)] \cup [\chi \cap C^c(\varphi)])\, P(d\chi)\, P_0^\Xi(d\varphi)$$
$$= \lambda^3 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h(c_p^{(1,2,3)}) \int I_p^{(1,2,3)}(\varphi)$$
$$\times \int u\left(\bigcup_{i=1}^{3} \{\zeta_i - c_p^{(1,2,3)}\} \cup [\chi \cap \widetilde{C}_p^c(\zeta_1 - c_p^{(1,2,3)}, \zeta_2 - c_p^{(1,2,3)}, \zeta_3 - c_p^{(1,2,3)})]\right)$$
$$\times P(d\chi)\, d\zeta_3\, d\zeta_2\, d\zeta_1. \quad \square$$

4. Identities for Edges

Let $\mathfrak{T}$ be a random normal and stationary tessellation in $\mathbb{R}^d$. A tessellation is said to be normal, if every of its s-faces lies in the boundaries of $d - s + 1$ cells, $0 \le s \le d - 1$, (cf. [3], p. 43). Each cell of $\mathfrak{T}$ is the interior of a d-polytope and each vertex of $\mathfrak{T}$ has $d + 1$ emanating edges. Further, let each edge $E^{(i)}$ of $\mathfrak{T}$ be associated with an n-dimensional vector $\Theta^{(i)} = (\Theta_1^{(i)}, \Theta_2^{(i)}, \Theta_n^{(i)})$, ($n$ fixed), with non-negative components which is invariant under

Euclidean motions of $\mathfrak{T}$. For example, this vector can be thought to be the d-tuple of angles spanned between $E^{(i)}$ and its adjacent edges in one of its endpoints ($n = d$) or simply the length of $E^{(i)}$ ($n = 1$). Let $E_1^{\mathfrak{T}}$ be the "typical" edge and $E_0^{\mathfrak{T}}$ a randomly chosen edge emanating from the "typical" vertex of the random tessellation $\mathfrak{T}$. The corresponding distribution functions of their characteristics are $F_1^{\mathfrak{T}}(\vartheta_1, \vartheta_2, \ldots, \vartheta_n)$ and $F_0^{\mathfrak{T}}(\vartheta_1, \vartheta_2, \ldots, \vartheta_n)$, respectively.

Lemma 2. *The distribution functions $F_0^{\mathfrak{T}}$ and $F_1^{\mathfrak{T}}$ are identical.*

Proof. In the following, point processes are needed which may have multiple points. Such point processes can be described as random counting measures, see e.g. [8].

Let $T_{\vartheta_1, \ldots, \vartheta_n}$ be the system of all edges of $\mathfrak{T}$ satisfying at least one of the relations $\Theta_1^{(i)} \geq \vartheta_1$, $\Theta_2^{(i)} \geq \vartheta_2$, ... or $\Theta_n^{(i)} \geq \vartheta_n$. Further, let Π_1 be the point process of midpoints of all edges belonging to $T_{\vartheta_1, \ldots, \vartheta_n}$ and Π_0 be the point process of all endpoints of these edges. Clearly, Π_1 and Π_0 are stationary. Their intensities are $\mu_1(\vartheta_1, \ldots, \vartheta_n)$ and $\mu_0(\vartheta_1, \ldots, \vartheta_n)$. While Π_1 is simple, Π_0 can also contain multiple points with a multiplicity $m \in \{1, 2, \ldots, d+1\}$. The distribution function for the "typical" edge is given by

$$F_1^{\mathfrak{T}}(\vartheta_1, \vartheta_2, \ldots, \vartheta_n) = 1 - \frac{\mu_1(\vartheta_1, \ldots, \vartheta_n)}{\mu_1(0, \ldots, 0)}, \qquad \vartheta_1, \ldots, \vartheta_n \geq 0 .$$

Since every edge has exactly one midpoint and two endpoints, the identity

$$\mu_1(\vartheta_1, \ldots, \vartheta_n) = \frac{1}{2}\, \mu_0(\vartheta_1, \ldots, \vartheta_n)$$

holds, which leads to

$$F_1^{\mathfrak{T}}(\vartheta_1, \vartheta_2, \ldots, \vartheta_n) = 1 - \frac{\mu_0(\vartheta_1, \ldots, \vartheta_n)}{\mu_0(0, \ldots, 0)}, \qquad \vartheta_1, \ldots, \vartheta_n \geq 0 . \tag{4.1}$$

Now Π_0 will be thinned as follows to obtain a simple point process $\tilde{\Pi}_0$.

In every vertex of $\mathfrak{T}$ exactly one endpoint of an adjacent edge will be signed by the mark 1 and the d other endpoints of adjacent edges by the mark 0. The event to receive mark 1 has equal probability for every edge endpoint and does not depend on other vertices. In $\tilde{\Pi}_0$ remain only those points of Π_0 having mark 1. In contrast to Π_0 the thinned point process $\tilde{\Pi}_0$ is simple. Let $\tilde{\mu}_0(\vartheta_1, \ldots, \vartheta_n)$ denote the intensity of $\tilde{\Pi}_0$. Then the distribution function $F_0^{\mathfrak{T}}$ is given by

$$F_0^{\mathfrak{T}}(\vartheta_1, \vartheta_2, \ldots, \vartheta_n) = 1 - \frac{\tilde{\mu}_0(\vartheta_1, \ldots, \vartheta_n)}{\tilde{\mu}_0(0, \ldots, 0)}, \qquad \vartheta_1, \ldots, \vartheta_n \geq 0 . \tag{4.2}$$

A point of $\tilde{\Pi}_0$ occurs with probability $m/(d+1)$ at the same location where a point of Π_0 lies with multiplicity m. Thus

$$\tilde{\mu}_0(\vartheta_1, \ldots, \vartheta_n) = \frac{1}{d+1}\, \mu_0(\vartheta_1, \ldots, \vartheta_n)$$

and comparison of (4.1) and (4.2) shows the identity

$$F_0^{\mathfrak{T}} = F_1^{\mathfrak{T}} . \qquad \square \tag{4.3}$$

Now a class of non-normal tessellations will be considered, for which similar properties can be shown.

Let $\mathfrak{U}$ be a stationary random tessellation in $\mathbb{R}^d$ consisting of non-degenerate space-filling d-dimensional simplices. In particular, each cell has a positive d-dimensional Lebesgue measure and its boundary consists of $d + 1$ $(d - 1)$-faces being $(d - 1)$-simplices spanned by d vertices. Let each $(d - 1)$-face $S^{(i)}$ be connected with an n-dimensional random vector $\Theta^{(i)} = (\Theta_1^{(i)}, \Theta_2^{(i)}, \ldots, \Theta_n^{(i)})$, in the same manner as above. Further, let $S_{d-1}^{\mathfrak{U}}$ denote the "typical" $(d - 1)$-face and let $S_d^{\mathfrak{U}}$ be a randomly chosen $(d - 1)$-face of the "typical" cell of $\mathfrak{U}$. The distribution functions of the corresponding n-dimensional vectors are denoted by $F_{d-1}^{\mathfrak{U}}$, and $F_d^{\mathfrak{U}}$, respectively.

Lemma 2′. *The distribution functions $F_{d-1}^{\mathfrak{U}}$ and $F_d^{\mathfrak{U}}$ are identical.*

Proof. For each $(d - 1)$-face $S^{(i)}$, there exists a uniquely determined smallest $(d - 1)$-ball circumscribing $S^{(i)}$. The corresponding corcumcentres $c^{(i)}$ of these $(d - 1)$-balls generate a stationary point process. From this point process two further stationary point processes are derived. On the one hand, consider the simple point process of all those $c^{(i)}$ whose associated vector $\Theta^{(i)}$ satisfies at least one of the relations $\Theta_1^{(i)} \geq \vartheta_1$, $\Theta_2^{(i)} \geq \vartheta_2$, ... or $\Theta_n^{(i)} \geq \vartheta_n$. On the other hand, the $d + 1$ $(d - 1)$-faces of each cell of $\mathfrak{U}$ are labeled with marks 0 and 1, respectively, in the same manner as above. This generates a second point process consisting of those circumcentres having mark 1 and an associated vector $\Theta^{(i)}$ which satisfies at least one of the relations $\Theta_1^{(i)} \geq \vartheta_1$, $\Theta_2^{(i)} \geq \vartheta_2$, ... or $\Theta_n^{(i)} \geq \vartheta_n$. Since each $(d - 1)$-face belongs to exactly two cells, this point process may contain double points. The ratio of the intensities of both of these stationary point processes is equal to $(d + 1)/2$ and thus the assertion of Lemma 2′ is immediately seen. □

Examples. Setting $\mathfrak{T} = \mathfrak{V}$, $n = 1$ and $\Theta^{(i)} = L^{(i)}$ in (4.3), where $L^{(i)}$ denotes the random edge length, gives

$$F_{L_0}^{\mathfrak{V}} = F_{L_1}^{\mathfrak{V}}.$$

(The distribution function of the length of a randomly chosen edge emanating from the "typical" vertex is identical to the distribution function of the length of the "typical" edge of $\mathfrak{V}$).

Put $\mathfrak{U} = \mathfrak{D}$, $n = 1$ and $\Theta^{(i)} = A^{(i)}$, where $A^{(i)}$ is the angle between the normal of $S_{d-1}^{\mathfrak{D}}$ and the normal of one of its neighbouring $(d - 1)$-faces having a joint $(d - 2)$-face with $S_{d-1}^{\mathfrak{D}}$. Then

$$F_{A_{d-1}}^{\mathfrak{D}} = F_{A_d}^{\mathfrak{D}}.$$

(The distribution function of the angle between the normals of the "typical" $(d - 1)$-face $S_{d-1}^{\mathfrak{D}}$ and a randomly chosen $(d - 1)$-face $S_d^{\mathfrak{D}}$ adjacent to the same cell like $S_{d-1}^{\mathfrak{D}}$ is identical to the distribution function of the angle between the normals of a randomly chosen pair of distinct $(d - 1)$-faces of the "typical" cell of $\mathfrak{D}$).

The close connection between $\mathfrak{V}$ and $\mathfrak{D}$ (the parallelism of the considered normals and the edges of $\mathfrak{V}$) leads to

$$F_{A_0}^{\mathfrak{V}} = F_{A_1}^{\mathfrak{V}}.$$

(The distribution function of the angle between a randomly chosen pair of distinct edges emanating from the „typical" vertex of $\mathfrak{V}$ is identical to the distribution function of the angle of the "typical" edge $E_1^{\mathfrak{V}}$ of $\mathfrak{V}$ to a randomly chosen edge having a joint endpoint with $E_1^{\mathfrak{V}}$).

5. Discussion

The knowledge of the distributions of the point processes Φ_0^{Ψ} and Φ_0^{Ξ} simplifies investigations of geometrical tessellation characteristics in the neighbourhood of the "typical" vertex, because the well-known formulae for the Poisson point process are applicable to these point processes. This fact in connection with the identities given in Section 4 allows the determination of several new geometrical characteristics for the Poisson Voronoi tessellation $\mathfrak{V}$. The Formulae (3.2) and (4.3) have been used already for the special case $d = 2$, cf. [4]. There the behaviour of the point process Ψ is investigated and the distribution function of the length of the "typical" edge of $\mathfrak{V}$ in $\mathbb{R}^2$ is given. Basing on (3.2), (3.9) and (4.3) the method introduced in [4] will be extended in [5] to $\mathfrak{V}$ in $\mathbb{R}^d$ and to planar sections through it. Finally, in [6] by use of the identities given in Section 4 distribution functions of several angles adjacent to the "typical" edge of $\mathfrak{V}$ and other geometrical characteristics for $\mathfrak{V}$ and $\mathfrak{D}$ in $\mathbb{R}^3$ will be given.

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