ST495/590 Assignment 4 - Solutions

(1) $Y|\lambda \sim \textit{Poisson}(\lambda)$, *prior* $p(\lambda) = 1$ *for* $\lambda > 0$. Since

$$\int_0^\infty p(\lambda)\, d\lambda = \infty \neq 1,$$

the prior is improper.

Posterior:

$$p(\lambda|Y) \propto p(Y|\lambda)p(\lambda) \propto \frac{e^{-\lambda}\lambda^Y}{Y!} \times 1 \propto e^{-\lambda}\lambda^Y,$$

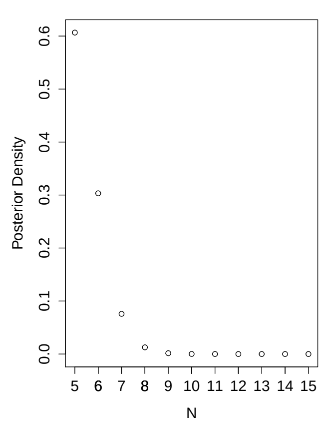
so $\lambda|Y \sim \text{Gamma}(Y+1, 1)$, which is proper.

(2) $Y|N \sim \textit{Binomial}(N, 1/2)$, *prior* $N \sim \textit{Poisson}(1)$.

Posterior:

$$\begin{aligned} P(N = N|Y = 5) &= \frac{P(Y|N = N)P(N = N)}{\sum_{n=0}^{\infty}(Y|N = n)P(N = n)} = \frac{\binom{N}{Y}\left(\frac{1}{2}\right)^N \frac{e^{-1}}{N!}}{\sum_{n=0}^{\infty}(Y|N = n)P(N = n)} = \frac{\frac{(\frac{1}{2})^{N-5}}{(N-5)!}}{\sum_{n=0}^{\infty}\frac{(\frac{1}{2})^{n-5}}{(n-5)!}} \\ &= \frac{\frac{(\frac{1}{2})^{N-5}}{(N-5)!}}{\sum_{n=5}^{\infty}\frac{(\frac{1}{2})^{n-5}}{(n-5)!}} = \frac{\left(\frac{1}{2}\right)^{N-5}}{(N-5)!}e^{-1/2} \end{aligned}$$

Note that $P(Y = 5|N = n) = 0$ if $n = 0, 1, 2, 3, 4$. Thus $N - 5|Y \sim \text{Poisson}(1/2)$.



(3) $y|\lambda \sim \textit{Exponential}(\lambda)$, $\lambda \sim \textit{Gamma}(a, b)$.

Posterior:

$$p(\lambda|y) \propto p(y|\lambda)p(\lambda) \propto \lambda \exp(-\lambda y)\lambda^{a-1}\exp(-b\lambda) \propto \lambda^{a+1-1}\exp(-(b+y)\lambda)$$

$\lambda|y \sim \text{Gamma}(a+1, b+y)$.

(4) *Compare the effectiveness of three drugs.*

Suppose the success probability for each drug is θ_i for $i = 1, 2, 3$. Now $Y|\theta_i \sim \text{Binomial}(100, \theta_i)$ and $\theta \sim \text{Uniform}(0, 1)$. The posterior:

$$p(\theta|Y) \propto p(Y|\theta)p(\theta) \propto \theta^Y(1-\theta)^{100-Y} \times 1,$$

so $\theta_i|Y_i \sim \text{Beta}(Y_i + 1, 101 - Y_i)$. So for drug 1, $\theta_1|Y_1 \sim \text{Beta}(13, 89)$; for drug 2, $\theta_2|Y_2 \sim \text{Beta}(19, 83)$; for drug 3, $\theta_3|Y_3 \sim \text{Beta}(11, 91)$.

The posterior mean for a $\text{Beta}(a, b)$ distribution is $a/(a+b)$, and the posterior standard deviation is $\sqrt{\frac{ab}{(a+b)^2(a+b+1)}}$. So the posterior mean for the three drugs are $0.127, 0.186, 0.108$ respectively, and the standard deviation is $0.033, 0.038, 0.031$. Drug 2 has the highest success probability.