The St. Petersburg game

Notes on Behavioural Economics

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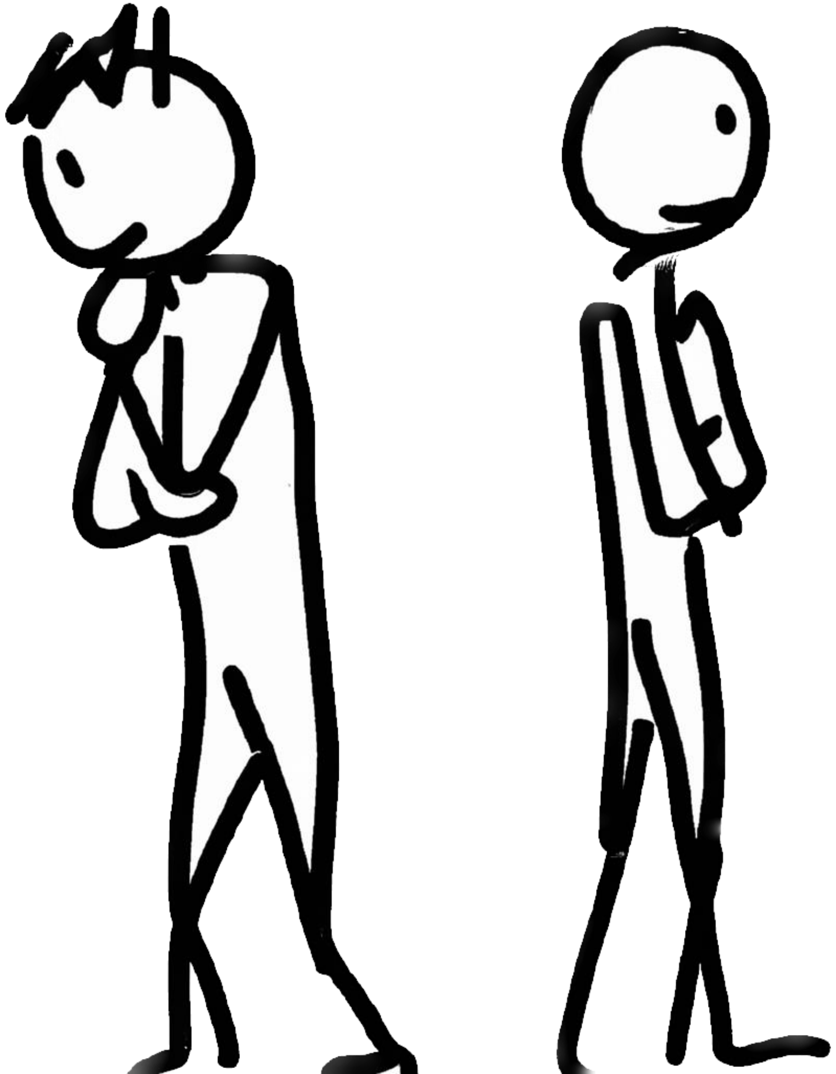

The St. Petersburg game

Tail on the 1st flip:	\$2
Tail on the 2nd flip:	\$4
Tail on the 3rd flip:	\$8
Tail on the 4th flip:	\$16
And so on.	

The St. Petersburg game

The expected value of this game X is equal to:

$$E[X] = \underbrace{\frac{1}{2} \times 2}_{\text{Tail first}} + \underbrace{\left(\frac{1}{2} \times \frac{1}{2}\right) \times 4}_{\text{Tail second}} + \underbrace{\left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) \times 8}_{\text{Tail third}} + \underbrace{\left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) \times 16}_{\text{Tail fourth}} + \cdots$$

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$$= 1 + 1 + 1 + 1 + \ldots$$

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$$= \infty$$

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$$= \sum_{k=1}^{k=\infty} \frac{1}{2^k} U\left(W + 2^k\right)$$

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What is the maximum $\$c$ a risk neutral player with $U(x) = x$ would be willing to pay to play the game?

They will be indifferent when:

$$U(W) = E[U(X - c)]$$

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$$c = \sum_{k=1}^{k=\infty} 1 = \infty$$

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What is the maximum $\$c$ a risk-averse player with $U(x) = ln(x)$ would be willing to pay to play the game? How does their wealth affect their willingness to pay?

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$$\ln(W) = \sum_{k=1}^{k=\infty} \frac{1}{2^k} \ln\left(W + \$2^k - c\right)$$

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Wealth	Willing to pay
$0.01	$2.01
$1000	$10.95
$1 million	$20.87

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$$= \sum_{k=1}^{k=\infty} \frac{k}{2^k} ln(2)$$

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$$= 2\ln(2)$$

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We can calculate what wealth is equivalent to this expected utility.

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$$W = e^{2\ln(2)} = 4$$