Current Approach to Muon Convolution

Jason Bono

1 Framework

1.1. Average Field

We want the average scalar (vertical component) B field experienced by the muons

$$\langle B \rangle = B(\mathbf{x}, t) \otimes M(\mathbf{x}, t) \tag{1}$$

This involves convolving the spacial and time structure of the muon beam and the B field. The spacial structures can be expanded into moments as $B(\mathbf{x}, t) = \Sigma_n b_n(t)$ and $M(\mathbf{x}, t) = \Sigma_n m_n(t)$ where $b_n(t)$ and $m_n(t)$ are the nth field and muon moment. However, there will be different uncertainty on the measurement of each moment and may want to weight the n contributions to final result accordingly, i.e. take the weighted sum

$$B(\mathbf{x}, t) = \Sigma_n b_n(t) w_n^{\mathrm{b}} \tag{2}$$

and

$$M(\mathbf{x}, t) = \Sigma_n m_n(t) w_n^{\mathrm{m}} \tag{3}$$

where w_n^{x} implicitly obeys some normalization condition, such as

$$\Sigma_{n=1}^{N} w_n^{\mathrm{x}} = N \tag{4}$$

Also note that w_n^x has no time structure. The average experienced B field can now be expressed as the weighted sum of products of the moments

$$\langle B \rangle = \Sigma_n b_n(t) w^{\mathrm{b}_n} \otimes \Sigma_n m_n(t) w^{\mathrm{m}_n} = \tag{5}$$

or

$$\langle B \rangle = \Sigma_n w_n^{\mathrm{b}} w_n^{\mathrm{m}} (b_n(t) \otimes m_n(t)) \tag{6}$$

or, letting $w_n = w_n^{\mathrm{b}} w_n^{\mathrm{m}}$,

$$\langle B \rangle = \Sigma_n w_n (b_n(t) \otimes m_n(t)) \tag{7}$$

We can express what is left of the convolution as a weighted average

$$b_n(t) \otimes m_n(t) = \frac{\int b_n(t) m_n(t) c_n(t) dt}{\int c_n(t) dt} \tag{8}$$

where $c_n(t)$ is, for now, a general weighting factor that can absorb various effects, such as the changing number of muons, field uncertainties, etc. Identifying $c_n(t)$ exactly is reserved for a later section. Substituting the above expression in for $\langle B \rangle$, we get

$$\langle B \rangle = \Sigma_n w_n \frac{\int b_n(t) m_n(t) c_n(t) dt}{\int c_n(t) dt} \tag{9}$$

Provided time is discretized finely enough so that, within any single interval, the spacial structures are effectively constant, we can write

$$\langle B \rangle = \Sigma_n w_n \frac{\Sigma_t b_{nt} m_{nt} c_{nt}}{\Sigma_t c_{nt}} \tag{10}$$

Letting $\Sigma_t c_{nt} = C_n$ gives

$$\boxed{\langle B \rangle = \Sigma_n [\frac{w_n}{C_n} (\Sigma_t b_{nt} m_{nt} c_{nt})].} \tag{11}$$

Note that we could let C_n absorb w_n, but to avoid confusion, we will not do so. Finally, we can write the contribution from a single (nth) moment as

$$\boxed{\langle B \rangle_n = [\frac{w_n}{C_n} (\Sigma_t b_{nt} m_{nt} c_{nt})]} \tag{12}$$

and that contribution in a single time-slice at τ as

$$\boxed{\langle B \rangle_{n\tau} = \frac{w_n b_{n\tau} m_{n\tau} c_{n\tau}}{C_n}} \tag{13}$$

1.2. Error on Average Field

Assume that the muon and field structures (m_{nt} and b_{nt}) are uncorrelated, and that the uncertainties on C_n and w_n are zero. The variance in the nth moment at time t, $\sigma_{nt}^2 \equiv (\Delta \langle B \rangle_{nt})^2$, is

$$\sigma_{nt}^2 = (\frac{d \langle B \rangle_{nt}}{db_{nt}} \Delta b_{nt})^2 + (\frac{d \langle B \rangle_{nt}}{dm_{nt}} \Delta m_{nt})^2 + (\frac{d \langle B \rangle_{nt}}{dc_{nt}} \Delta c_{nt})^2 \tag{14}$$

or

$$\boxed{\sigma_{nt}^2 = \frac{w_n^2}{C_n^2} [(m_{nt} c_{nt} \Delta b_{nt})^2 + (b_{nt} c_{nt} \Delta m_{nt})^2 + (b_{nt} m_{nt} \Delta c_{nt})^2]} \tag{15}$$

substituting the above expression into

$$\sigma_n^2 = \Sigma_t \sigma_{nt}^2 \tag{16}$$

gives

$$\boxed{\sigma_n^2 = \frac{w_n^2}{C_n^2} \Sigma_t [(m_{nt} c_{nt} \Delta b_{nt})^2 + (b_{nt} c_{nt} \Delta m_{nt})^2 + (b_{nt} m_{nt} \Delta c_{nt})^2]} \tag{17}$$

Similarly, substituting the above expression into

$$\sigma^2 = \Sigma_n \sigma_n^2 \tag{18}$$

gives

$$\boxed{\sigma^2 = \Sigma_n [\frac{w_n^2}{C_n^2} \Sigma_t [(m_{nt} c_{nt} \Delta b_{nt})^2 + (b_{nt} c_{nt} \Delta m_{nt})^2 + (b_{nt} m_{nt} \Delta c_{nt})^2]]} \tag{19}$$

2 Application

2.1. A List

Applying the framework is just a matter of connecting the familiar observables to the variables introduced in this document. Below is a list of such connections. The list is only intended to be instructive for now, so many items are left out. Eventually, we will want to define everything exactly.

- b_{1t} The dipole field in time-slice t
- m_{1t} The muon beam's dipole moment in time-slice t (always equal to unity)
- b_{2t} The normal-quadrupole field in time-slice t
- m_{2t} The muon beam's mean horizontal position in time-slice t
- Δb_{2t} The error in the normal-quadrupole field in time-slice t
- Δb_{2t} The spread (RMS) in the muon beam's mean horizontal position in time-slice t
- c_{nt} Nominally identified with CTAGs in time-slice t
  - Then c_{nt} is reduced to c_t and C_n to C, where C is the total number of CTAGs integrated over the time interval in question
  - However, if we want to do an uncertainty weighted average, c_{nt} is the natural variable to do this with
  - Δc_{nt} Will probably be set to zero, but this needs to be thought about more
- w_n Nominally set to 1 or 0 for moments that are considered or not considered in the final sum over n
  - However, we may want to explicitly weight some moments more than others, and w_n is the natural variable to do this with
  - Δw_n has been assumed to be zero and will almost certainly remain this way

2.2. Simple Examples

2.2.1 Dipole

Here we calculate $\langle B \rangle_1$ and σ_1. We have

- b_{1t} = D (dipole)
- Δb_{1t} = eD
- $c_{1t} = c_t$ = ctag (number of muons)
- $\Delta c_{1t} = \Delta c_t = 0$
- $C_1 = C$ = ctag_total (cumulative sum of number of muons)
- $m_{1t} = 1$ (by definition)
- $\Delta m_{1t} = 0$.
- $w_1 = 1$ (by definition)

Subbing some of these values into the relevant equations, we get

$$\langle B \rangle_1 = \frac{\Sigma_t b_{1t} c_t}{C} = \frac{\Sigma_t D_t \mathrm{ctag}_t}{\mathrm{ctag\_total}} \tag{20}$$

and

$$\sigma_1 = \frac{1}{C}\sqrt{\Sigma_t[(c_t \Delta b_{1t})^2]} = \frac{\sqrt{\Sigma_t[(\mathrm{eD}_t \mathrm{ctag}_t)^2]}}{\mathrm{ctag\_total}} \tag{21}$$

2.2.2 Normal Quadrupole

Here we calculate $\langle B \rangle_2$ and σ_2. We have

- b_{2t} = NQ (normal quadrupole))
- Δb_{2t} = eNQ
- $c_{2t} = c_t$ = ctag (number of muons)
- $\Delta c_{2t} = \Delta c_t = 0$
- $C_2 = C$ = ctag_total (cumulative sum of number of muons)
- m_{2t} is the mean horizontal beam position
- Δm_{2t} is the horizontal RMS of the beam about its mean
- $w_2 = 1$ (by definition)

We get

$$\langle B \rangle_2 = \frac{\Sigma_t m_{2t} b_{2t} c_t}{C} = \frac{\Sigma_t m_{2t} NQ_t \mathrm{ctag}_t}{\mathrm{ctag\_total}} \tag{22}$$

$$\sigma_2 = \frac{1}{C}\sqrt{\Sigma_t[(m_{2t} c_t \Delta b_{2t})^2 + (b_{2t} c_t \Delta m_{2t})^2]} = \frac{\sqrt{\Sigma_t[(\mathrm{eNQ}_t \mathrm{ctag}_t m_{2t})^2 + (\mathrm{NQ}_t \mathrm{ctag}_t \Delta m_{2t})^2]}}{\mathrm{ctag\_total}} \tag{23}$$

2.2.3 Skew Quadrupole

Here we calculate $\langle B \rangle_3$ and σ_3. We have

- b_{3t} = SQ (skew quadrupole)
- Δb_{3t} = eSQ
- $c_{3t} = c_t$ = ctag (number of muons)
- $\Delta c_{3t} = \Delta c_t = 0$
- $C_3 = C$ = ctag_total (cumulative sum of number of muons)
- m_{3t} is the mean vertical beam position
- Δm_{3t} is the vertical RMS of the beam about its mean
- $w_3 = 1$ (by definition)

We get

$$\langle B \rangle_3 = \frac{\Sigma_t m_{3t} b_{3t} c_t}{C} = \frac{\Sigma_t m_{3t} SQ_t \mathrm{ctag}_t}{\mathrm{ctag\_total}} \tag{24}$$

$$\sigma_3 = \frac{1}{C}\sqrt{\Sigma_t[(m_{3t}c_t\Delta b_{3t})^2 + (b_{3t}c_t\Delta m_{3t})^2]} = \frac{\sqrt{\Sigma_t[(\mathrm{eSQ}_t\mathrm{ctag}_t m_{3t})^2 + (\mathrm{SQ}_t\mathrm{ctag}_t\Delta m_{3t})^2]}}{\mathrm{ctag\_total}} \tag{25}$$

2.2.4 Adding contributions from the first three moments

In the above examples we found the contributions from moments 1,2, and 3. To get the total contribution, we take the simple sum

$$\langle B \rangle = \Sigma_{n=1}^{3} \langle B \rangle_n = \frac{\Sigma_t c_t[(b_{1t}) + (m_{2t}b_{2t}) + (m_{3t}b_{3t})]}{C} \tag{26}$$

For the total variance, we also take the simple sum,

$$\sigma^2 = \Sigma_{n=1}^{3}\sigma_n^2 = \frac{1}{C^2}\Sigma_t c_t^2[(\Delta b_{1t})^2 + (m_{2t}\Delta b_{2t})^2 + (b_{2t}\Delta m_{2t})^2 + (m_{3t}\Delta b_{3t})^2 + (b_{3t}\Delta m_{3t})^2] \tag{27}$$

and so the error is

$$\sigma = \frac{\sqrt{\Sigma_t c_t^2[(\Delta b_{1t})^2 + (m_{2t}\Delta b_{2t})^2 + (b_{2t}\Delta m_{2t})^2 + (m_{3t}\Delta b_{3t})^2 + (b_{3t}\Delta m_{3t})^2]}}{C} \tag{28}$$

Note that, in general, the values $w_{1,2,3}$ could have been made unequal in in order to weight the contributions from some moments over others, possibly based on sources of uncertainty. Similarly, identifying c_{nt} with ctags alone allowed for its contraction into c_t. This may not always be the case. For example, we may allow the c_{nt} weighting factor to incorporate uncertainty from the field determination, which may have distinct effects on the various moments.

2.3. Sanity Checks

2.3.1 Invariance to Total CTAGs

The quantity c_{nt} will always incorporate the number of muons (though CTAGS), even if it incorporates other factors simultaneously. Intuitively, the average field experience by the muons should be invariant to the total number of muons. We can represent an error factor and offset, f, in our ability to count muons. We can then represent this as

$$c_{nt} \rightarrow f c_{nt} \tag{29}$$

which implies

$$C_n \rightarrow f C_n \tag{30}$$

which implies

$$\langle B \rangle = \Sigma_n [\frac{w_n}{C_n}(\Sigma_t b_{nt} m_{nt} c_{nt})] \rightarrow \Sigma_n [\frac{w_n}{f C_n}(\Sigma_t b_{nt} m_{nt} f c_{nt})] = \langle B \rangle \tag{31}$$

ie the calculated field is unchanged.

Furthermore, for the uncertainty, we get $\Delta c_{nt} \rightarrow f \Delta cnt$, which implies

$$\begin{gathered}\sigma^2 = \Sigma_n [\frac{w_n^2}{C_n^2} \Sigma_t [(m_{nt} c_{nt} \Delta b_{nt})^2 + (b_{nt} c_{nt} \Delta m_{nt})^2 + (b_{nt} m_{nt} \Delta c_{nt})^2]] \rightarrow \\ \Sigma_n [\frac{w_n^2}{f^2 C_n^2} \Sigma_t [(m_{nt} f c_{nt} \Delta b_{nt})^2 + (b_{nt} f c_{nt} \Delta m_{nt})^2 + (b_{nt} m_{nt} f \Delta c_{nt})^2]] = \sigma^2\end{gathered} \tag{32}$$

ie the calculated uncertainty in the field is unchanged.

2.3.2 Invariance to Sample Rate

Analytically, our calculations should also not be affected by the sample rate, provided it is fine enough. Note that b_{nt} always represents the time averaged magnetic field in time-slice t, so its value does not change when re sampled. m_{nt} always represents a geometric representation of the beam, and its value also does not change when re-sampled. c_{nt} on the other hand always incorporates the (sometimes fractional) number of muons in time-slice t. It is therefore a linearly cumulative variable (its value changes proportionally to the sample rate). c_{nt} may also incorporate other variables such as the error in b_{nt} and m_{nt}, but these are not cumulative. Provided these statements hold true, the product $b_{nt} m_{nt} c_{nt}$ is linearly cumulative variable in t. Therefore

$$\langle B \rangle_{nt} = \frac{w_n b_{nt} m_{nt} c_{nt}}{C_n} \tag{33}$$

is also a linearly cumulative variable and thus is invariant to sampling rate, provided the sampling is done over a fixed time interval.

By similar reasoning, Δb_{nt} and Δm_{nt} are non-cumulative variables, so if Δc_{nt} is linearly cumulative, zero, or negligible, then

$$\sigma_{nt} = \frac{w_n}{C_n} \sqrt{[(m_{nt} c_{nt} \Delta b_{nt})^2 + (b_{nt} c_{nt} \Delta m_{nt})^2 + (b_{nt} m_{nt} \Delta c_{nt})^2]} \tag{34}$$

is also linearly cumulative and therefore invariant. In most cases, though perhaps not all, the restriction on Δc_{nt} will hold.

2.4. Questions

Below are some questions that need to be answered:

- What are the problems with the approach to average and error on average?
- Are non-binary weights for w_n needed?
  - If so, what normalization condition should we use?
- Should anything other than ctags be put into c_{tn}? Eg the field uncertainty?
- Does the accumulation of error over time, as it's formulated make sense?
  - Does our error calculation have built-in invariance to sample-rate? (see sanity checks)
  - Does our field and error calculation have built-in invariance to total CTAGs? (see sanity checks)
- How should we use the muon distribution to get the higher order moments (ie above quadrupole)