Current Approach to Muon Convolution

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1 Framework

1.1. Average Field

We want the average scalar (vertical component) B field experienced by the muons

$$\langle B \rangle = B(\mathbf{x},t) \otimes M(\mathbf{x},t) \tag{1}$$

This involves convolving the spacial and time structure of the muon beam and the B field. The spacial structures can be expanded into moments as $B(\mathbf{x},t) = \Sigma_n b_n(t)$ and $M(\mathbf{x},t) = \Sigma_n m_n(t)$ where $b_n(t)$ and $m_n(t)$ are the nth field and muon moment. However, there will be different uncertainty on the measurement of each moment and may want to weight the n contributions to final result accordingly, i.e. take the weighted sum

$$B(\mathbf{x},t) = \Sigma_n b_n(t) w_n^{\mathrm{b}} \tag{2}$$

and

$$M(\mathbf{x},t) = \Sigma_n m_n(t) w_n^{\mathrm{m}} \tag{3}$$

where w_n^{x} implicitly obeys some normalization condition, such as

$$\Sigma_{n=1}^{N} w_n^{\mathrm{x}} = N \tag{4}$$

Also note that w_n^x has no time structure. The average experienced B field can now be expressed as the weighted sum of products of the moments

$$\langle B \rangle = \Sigma_n b_n(t) w^{\mathrm{b}_n} \otimes \Sigma_n m_n(t) w^{\mathrm{m}_n} = \tag{5}$$

or

$$\langle B \rangle = \Sigma_n w_n^{\mathrm{b}} w_n^{\mathrm{m}} (b_n(t) \otimes m_n(t)) \tag{6}$$

or, letting $w_n = w_n^{\mathrm{b}} w_n^{\mathrm{m}}$,

$$\langle B \rangle = \Sigma_n w_n (b_n(t) \otimes m_n(t)) \tag{7}$$

We can express what is left of the convolution as a weighted average

$$b_n(t) \otimes m_n(t) = \frac{\int b_n(t) m_n(t) c_n(t) dt}{\int c_n(t) dt} \tag{8}$$

where $c_n(t)$ is, for now, a general weighting factor that can absorb various effects, such as the changing number of muons, field uncertainties, etc. Identifying $c_n(t)$ exactly is reserved for a later section. Substituting the above expression in for $\langle B \rangle$, we get

$$\langle B \rangle = \Sigma_n w_n \frac{\int b_n(t) m_n(t) c_n(t) dt}{\int c_n(t) dt} \tag{9}$$

Provided time is discretized finely enough so that, within any single interval, the spacial structures are effectively constant, we can write

$$\langle B \rangle = \Sigma_n w_n \frac{\Sigma_t b_{nt} m_{nt} c_{nt}}{\Sigma_t c_{nt}} \tag{10}$$

Letting $\Sigma_{nt} c_{nt} = \Sigma_n C_n$ gives

$$\boxed{\langle B \rangle = \Sigma_n [\frac{w_n}{C_n}(\Sigma_t b_{nt} m_{nt} c_{nt})].} \tag{11}$$

Note that we could let C_n absorb w_n, but to avoid confusion, we will not do so. Finally, we can write the contribution from a single (nth) moment as

$$\boxed{\langle B \rangle_n = [\frac{w_n}{C_n}(\Sigma_t b_{nt} m_{nt} c_{nt})]} \tag{12}$$

and that contribution in a single time-slice at τ as

$$\boxed{\langle B \rangle_{n\tau} = \frac{w_n b_{n\tau} m_{n\tau} c_{n\tau}}{C_n}} \tag{13}$$

1.2. Error on Average Field

Assuming that the muon and field structures (m_{nt} and b_{nt}) are uncorrelated. The variance in the nth moment at time t, $\sigma_{nt}^2 \equiv (\Delta \langle B \rangle_{nt})^2$, is

$$\sigma_{nt}^2 = (\frac{d \langle B \rangle_{nt}}{db_{nt}} \Delta b_{nt})^2 + (\frac{d \langle B \rangle_{nt}}{dm_{nt}} \Delta m_{nt})^2 + (\frac{d \langle B \rangle_{nt}}{dc_{nt}} \Delta c_{nt})^2 \tag{14}$$

or

$$\boxed{\sigma_{nt}^2 = \frac{w_n^2}{C_n^2}[(m_{nt}c_{nt}\Delta b_{nt})^2 + (b_{nt}c_{nt}\Delta m_{nt})^2 + (b_{nt}m_{nt}\Delta c_{nt})^2]} \tag{15}$$

substituting the above expression into

$$\sigma_n^2 = \Sigma_t \sigma_{nt}^2 \tag{16}$$

gives

$$\boxed{\sigma_n^2 = \frac{w_n^2}{C_n^2}\Sigma_t[(m_{nt}c_{nt}\Delta b_{nt})^2 + (b_{nt}c_{nt}\Delta m_{nt})^2 + (b_{nt}m_{nt}\Delta c_{nt})^2]} \tag{17}$$

Similarly, substituting the above expression into

$$\sigma^2 = \Sigma_n \sigma_n^2 \tag{18}$$

gives

$$\boxed{\sigma^2 = \Sigma_n[\frac{w_n^2}{C_n^2}\Sigma_t[(m_{nt}c_{nt}\Delta b_{nt})^2 + (b_{nt}c_{nt}\Delta m_{nt})^2 + (b_{nt}m_{nt}\Delta c_{nt})^2]]} \tag{19}$$

2 Application

2.1. A List

Applying the framework is just a matter of connecting the familiar observables to the variables introduced in this document. Below is a list of such connections. The list is only intended to be instructive for now, so many items are left out. Eventually, we will want to define everything exactly.

- b_{1t} The dipole field in time-slice t
- m_{1t} The muon beam's dipole moment in time-slice t (always equal to unity)
- b_{2t} The normal-quadrupole field in time-slice t
- m_{2t} The muon beam's mean horizontal position in time-slice t
- Δb_{2t} The error in the normal-quadrupole field in time-slice t
- Δb_{2t} The spread (RMS) in the muon beam's mean horizontal position in time-slice t
- c_{nt} Nominally identified with CTAGs in time-slice t
  - Then c_{nt} is reduced to c_t and C_n to C, where C is the total number of CTAGs integrated over the time interval in question
  - However, if we want to do an uncertainty weighted average, c_{nt} is the natural variable to do this with
  - Δc_{nt} Will probably be set to zero, but this needs to be thought about more
- w_n Nominally set to 1 or 0 for moments that are considered or not considered in the final sum over n
  - However, we may want to explicitly weight some moments more than others, and w_n is the natural variable to do this with
  - Δw_n has been assumed to be zero and will almost certainly remain this way

2.2. Simple Examples

2.2.1 Dipole

Here we calculate $\langle B \rangle_1$ and σ_1. We have

- $b_{1t} = \mathrm{D}$ (dipole)
- $\Delta b_{1t} = \mathrm{eD}$
- $c_{1t} = c_t = \mathrm{ctag}$ (number of muons)
- $\Delta c_{1t} = \Delta c_t = 0$
- $C_1 = C = \mathrm{ctag\_total}$ (cumulative sum of number of muons)
- $m_{1t} = 1$ (by definition)
- $\Delta m_{1t} = 0.$
- $w_1 = 1$ (by definition)

Subbing some of these values into the relevant equations, we get

$$\langle B \rangle_1 = \frac{\Sigma_t b_{1t} c_t}{C} = \frac{\Sigma_t D_t \mathrm{ctag}_t}{\mathrm{ctag\_total}} \tag{20}$$

and

$$\sigma_1 = \frac{1}{C}\sqrt{\Sigma_t[(c_t \Delta b_{1t})^2]} = \frac{\sqrt{\Sigma_t[(\mathrm{eD}_t \mathrm{ctag}_t)^2]}}{\mathrm{ctag\_total}} \tag{21}$$

2.2.2 Normal Quadrupole

Here we calculate $\langle B \rangle_2$ and σ_2. We have

- $b_{2t} = \mathrm{NQ}$ (normal quadrupole))
- $\Delta b_{2t} = \mathrm{eNQ}$
- $c_{2t} = c_t = \mathrm{ctag}$ (number of muons)
- $\Delta c_{2t} = \Delta c_t = 0$
- $C_2 = C = \mathrm{ctag\_total}$ (cumulative sum of number of muons)
- m_{2t} is the mean horizontal beam position
- Δm_{2t} is the horizontal RMS of the beam about its mean
- $w_2 = 1$ (by definition)

We get

$$\langle B \rangle_2 = \frac{\Sigma_t m_{2t} b_{2t} c_t}{C} = \frac{\Sigma_t m_{2t} NQ_t \mathrm{ctag}_t}{\mathrm{ctag\_total}} \tag{22}$$

$$\sigma_2 = \frac{1}{C}\sqrt{\Sigma_t[(m_{2t} c_t \Delta b_{2t})^2 + (b_{2t} c_t \Delta m_{2t})^2]} = \frac{\sqrt{\Sigma_t[(\mathrm{eNQ}_t \mathrm{ctag}_t m_{2t})^2 + (\mathrm{NQ}_t \mathrm{ctag}_t \Delta m_{2t})^2]}}{\mathrm{ctag\_total}} \tag{23}$$

2.2.3 Skew Quadrupole

Here we calculate $\langle B \rangle_3$ and σ_3. We have

- $b_{3t} = \text{SQ}$ (skew quadrupole)
- $\Delta b_{3t} = \text{eSQ}$
- $c_{3t} = c_t = \text{ctag}$ (number of muons)
- $\Delta c_{3t} = \Delta c_t = 0$
- $C_3 = C = \text{ctag\_total}$ (cumulative sum of number of muons)
- m_{3t} is the mean vertical beam position
- Δm_{3t} is the vertical RMS of the beam about its mean
- $w_3 = 1$ (by definition)

We get

$$\langle B \rangle_3 = \frac{\Sigma_t m_{3t} b_{3t} c_t}{C} = \frac{\Sigma_t m_{3t} SQ_t \text{ctag}_t}{\text{ctag\_total}} \tag{24}$$

$$\sigma_3 = \frac{1}{C}\sqrt{\Sigma_t [(m_{3t} c_t \Delta b_{3t})^2 + (b_{3t} c_t \Delta m_{3t})^2]} = \frac{\sqrt{\Sigma_t [(\text{eSQ}_t \text{ctag}_t m_{3t})^2 + (\text{SQ}_t \text{ctag}_t \Delta m_{3t})^2]}}{\text{ctag\_total}} \tag{25}$$

2.2.4 Adding contributions from the first three moments

In the above examples we found the contributions from moments 1,2, and 3. To get the total contribution, we take the simple sum

$$\langle B \rangle = \Sigma_{n=1}^{3} \langle B \rangle_n = \frac{\Sigma_t c_t [(b_{1t}) + (m_{2t} b_{2t}) + (m_{3t} b_{3t})]}{C} \tag{26}$$

For the total variance, we also take the simple sum,

$$\sigma^2 = \Sigma_{n=1}^{3} \sigma_n^2 = \frac{1}{C^2} \Sigma_t c_t^2 [(\Delta b_{1t})^2 + (m_{2t} \Delta b_{2t})^2 + (b_{2t} \Delta m_{2t})^2 + (m_{3t} \Delta b_{3t})^2 + (b_{3t} \Delta m_{3t})^2] \tag{27}$$

and so the error is

$$\sigma = \frac{\sqrt{\Sigma_t c_t^2 [(\Delta b_{1t})^2 + (m_{2t} \Delta b_{2t})^2 + (b_{2t} \Delta m_{2t})^2 + (m_{3t} \Delta b_{3t})^2 + (b_{3t} \Delta m_{3t})^2]}}{C} \tag{28}$$

Note that, in general, the values $w_{1,2,3}$ could have been made unequal in in order to weight the contributions from some moments over others, possibly based on sources of uncertainty. Similarly, identifying c_{nt} with ctags alone allowed for its contraction into c_t. This may not always be the case. For example, we may allow the c_{nt} weighting factor to incorporate uncertainty from the field determination, which may have distinct effects on the various moments.

2.3. Further Questions

Below are some questions that need to be answered:

- What are the problems with the approach to average and error on average?
- Are non-binary weights for w_n needed?
  - If so, what normalization condition should we use?
- Should anything other than ctags be put into c_{tn}? Eg the field uncertainty?
- Does the accumulation of error over time, as it's formulated make sense?
  - Does our error calculation have built-in invariance to sample-rate?
- How should we use the muon distribution to get the higher order moments (ie above quadrupole)