An Approach to Muon Convolution

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1 Framework

1.1. Moments Method

1.1.1 Average Field

We want the average scalar (vertical component) B field experienced by the muons

$$\langle B\rangle = B(\mathbf{x},t)\otimes M(\mathbf{x},t) \tag{1}$$

This involves convolving the spacial and time structure of the muon beam and the B field. The spacial structures can be expanded into moments as $B(\mathbf{x},t) = \Sigma_n b_n(t)$ and $M(\mathbf{x},t) = \Sigma_n m_n(t)$ where $b_n(t)$ and $m_n(t)$ are the nth field and muon moment. However, there will be different uncertainty on the measurement of each moment and may want to weight the n contributions to final result accordingly, i.e. take the weighted sum

$$B(\mathbf{x},t) = \Sigma_n b_n(t) w_n^{\mathrm{b}} \tag{2}$$

and

$$M(\mathbf{x},t) = \Sigma_n m_n(t) w_n^{\mathrm{m}} \tag{3}$$

where w_n^{x} implicitly obeys some normalization condition, such as

$$\Sigma_{n=1}^{N} w_n^{\mathrm{x}} = N \tag{4}$$

Also note that w_n^x has no time structure. The average experienced B field can now be expressed as the weighted sum of products of the moments

$$\langle B\rangle = \Sigma_n b_n(t) w^{\mathrm{b}_n} \otimes \Sigma_n m_n(t) w^{\mathrm{m}_n} = \tag{5}$$

or

$$\langle B\rangle = \Sigma_n w_n^{\mathrm{b}} w_n^{\mathrm{m}} (b_n(t)\otimes m_n(t)) \tag{6}$$

or, letting $w_n = w_n^{\mathrm{b}} w_n^{\mathrm{m}}$,

$$\langle B\rangle = \Sigma_n w_n (b_n(t)\otimes m_n(t)) \tag{7}$$

We can express what is left of the convolution as a weighted average

$$b_n(t)\otimes m_n(t) = \frac{\int b_n(t) m_n(t) c_n(t) dt}{\int c_n(t) dt} \tag{8}$$

where $c_n(t)$ is, for now, a general weighting factor that can absorb various effects, such as the changing number of muons, field uncertainties, etc. Identifying $c_n(t)$ exactly is reserved for a later section. Substituting the above expression in for $\langle B\rangle$, we get

$$\langle B\rangle = \Sigma_n w_n \frac{\int b_n(t) m_n(t) c_n(t) dt}{\int c_n(t) dt} \tag{9}$$

Provided time is discretized finely enough so that, within any single interval, the spacial structures are effectively constant, we can write

$$\langle B\rangle = \Sigma_n w_n \frac{\Sigma_t b_{nt} m_{nt} c_{nt}}{\Sigma_t c_{nt}} \tag{10}$$

Letting $\Sigma_t c_{nt} = C_n$ gives

$$\boxed{\langle B\rangle = \Sigma_n [\frac{w_n}{C_n}(\Sigma_t b_{nt} m_{nt} c_{nt})].} \tag{11}$$

Note that we could let C_n absorb w_n, but to avoid confusion, we will not do so. Finally, we can write the contribution from a single (nth) moment as

$$\boxed{\langle B\rangle_n = [\frac{w_n}{C_n}(\Sigma_t b_{nt} m_{nt} c_{nt})]} \tag{12}$$

and that contribution in a single time-slice at τ as

$$\boxed{\langle B\rangle_{n\tau} = \frac{w_n b_{n\tau} m_{n\tau} c_{n\tau}}{C_n}} \tag{13}$$

1.1.2 Error in Average Field

Assume that the muon and field structures (m_{nt} and b_{nt}) are uncorrelated, and that the uncertainties on C_n and w_n are zero. The variance in the nth moment at time t, $\sigma_{nt}^2 \equiv (\Delta \langle B \rangle_{nt})^2$, is

$$\sigma_{nt}^2 = (\frac{d\langle B \rangle_{nt}}{db_{nt}}\Delta b_{nt})^2 + (\frac{d\langle B \rangle_{nt}}{dm_{nt}}\Delta m_{nt})^2 + (\frac{d\langle B \rangle_{nt}}{dc_{nt}}\Delta c_{nt})^2 \tag{14}$$

or

$$\boxed{\sigma_{nt}^2 = \frac{w_n^2}{C_n^2}[(m_{nt}c_{nt}\Delta b_{nt})^2 + (b_{nt}c_{nt}\Delta m_{nt})^2 + (b_{nt}m_{nt}\Delta c_{nt})^2]} \tag{15}$$

substituting the above expression into

$$\sigma_n^2 = \Sigma_t \sigma_{nt}^2 \tag{16}$$

gives

$$\boxed{\sigma_n^2 = \frac{w_n^2}{C_n^2}\Sigma_t[(m_{nt}c_{nt}\Delta b_{nt})^2 + (b_{nt}c_{nt}\Delta m_{nt})^2 + (b_{nt}m_{nt}\Delta c_{nt})^2]} \tag{17}$$

Similarly, substituting the above expression into

$$\sigma^2 = \Sigma_n \sigma_n^2 \tag{18}$$

gives

$$\boxed{\sigma^2 = \Sigma_n[\frac{w_n^2}{C_n^2}\Sigma_t[(m_{nt}c_{nt}\Delta b_{nt})^2 + (b_{nt}c_{nt}\Delta m_{nt})^2 + (b_{nt}m_{nt}\Delta c_{nt})^2]]} \tag{19}$$

1.1.3 Beam Moments

The multipole expansion of some field, F, is

$$F(r,\theta) = c_0 + \sum_{n=1} r^n(c_n \cos(n\theta) + s_n \sin(n\theta)) \tag{20}$$

Note that integrating gives

$$\int_{r=0}^{R}\int_{\theta=0}^{2\pi} F(r,\theta) r d\theta dr = c_0 \int_{r=0}^{R}\int_{\theta=0}^{2\pi} r d\theta dr = c_0 \pi R^2 \tag{21}$$

or

$$\boxed{c_0 = \frac{\int_{r=0}^{R}\int_{\theta=0}^{2\pi} F(r,\theta) r d\theta dr}{A}} \tag{22}$$

where the area $A = \pi R^2$. In other words, c_0 is the average value of the field over the area in question.

The higher order moments can be found by

$$\int_{r=0}^{R}\int_{\theta=0}^{2\pi} F(r,\theta) r^n \cos(n\theta) r d\theta dr = \int_{r=0}^{R}\int_{\theta=0}^{2\pi} r^{2n+1} c_n \cos^2(n\theta) d\theta dr = c_n \int_{r=0}^{R} r^{2n+1} dr = c_n \frac{R^{2n+2}}{2n+2} \tag{23}$$

so

$$\boxed{c_n = \frac{2n+2}{R^{2n+2}} \int_{r=0}^{R}\int_{\theta=0}^{2\pi} F(r,\theta) r^{n+1} \cos(n\theta) d\theta dr} \tag{24}$$

Similarly, we have

$$\boxed{s_n = \frac{2n+2}{R^{2n+2}} \int_{r=0}^{R}\int_{\theta=0}^{2\pi} F(r,\theta) r^{n+1} \sin(n\theta) d\theta dr} \tag{25}$$

As a test, we can compute c_n and s_n for the general field. We get

$$c_0 = \frac{\int_{r=0}^{R}\int_{\theta=0}^{2\pi} F(r,\theta) r d\theta dr}{A} = \frac{\int_{r=0}^{R}\int_{\theta=0}^{2\pi} c_0 r d\theta dr}{A} = c_0 \tag{26}$$

and

$$c_n = \frac{(2n+2)\int_{r=0}^{R}\int_{\theta=0}^{2\pi}(c_0 + \sum_{n=1} r^n (c_n\cos(n\theta) + s_n\sin(n\theta)))r^{n+1}\cos(n\theta)d\theta dr}{R^{2n+2}} \tag{27}$$

or

$$c_n = \frac{(2n+2)}{R^{2n+2}}\int_{r=0}^{R}\int_{\theta=0}^{2\pi} r^n c_n \cos^2(n\theta) r^{n+1} d\theta dr = c_n \tag{28}$$

Similarly,

$$s_n = \frac{(2n+2)\int_{r=0}^{R}\int_{\theta=0}^{2\pi}(c_0 + \sum_{n=1} r^n (c_n\cos(n\theta) + s_n\sin(n\theta)))r^{n+1}\sin(n\theta)d\theta dr}{R^{2n+2}} = s_n \tag{29}$$

which is what we set out to show.

We have a B field and an (in general un-normalized) muon distribution, which we can call F_B and F_M. What we want is the average value of the multiplication of the two fields,

$$\bar{B} = \frac{\int F_B F_M dA}{\int F_M dA} \tag{30}$$

where we will call the denominator

$$\boxed{I_0 \equiv \int F_M dA} \tag{31}$$

We express the two fields generally by

$$F_B(r,\theta) = c_0 + \sum_{n=1} r^n (c_n\cos(n\theta) + s_n\sin(n\theta)) \tag{32}$$

and

$$F_M(r,\theta) = C_0 + \sum_{n=1} r^n (C_n\cos(n\theta) + S_n\sin(n\theta)) \tag{33}$$

Then we get

$$\bar{B} = \frac{1}{I_0}\int_{\theta=0}^{2\pi}\int_{r=0}^{R}(c_0 C_0 + \sum_{n=1} r^{2n}(c_n C_n\cos^2(n\theta) + s_n S_n\sin^2(n\theta)))r dr d\theta \tag{34}$$

where all cross terms have been dropped because they vanish in the integral over θ. Further reducing the above expression, we get

$$\bar{B} = \frac{1}{I_0}(2\pi R c_0 C_0 + \sum_{n=1}\int_{r=0}^{R} r^{2n+1}(c_n C_n + s_n S_n)dr) \tag{35}$$

or

$$\bar{B} = \frac{A c_0 C_0}{I_0} + \frac{1}{I_0}\sum_{n=1}\frac{R^{2n+2}}{2n+2}(c_n C_n + s_n S_n) \tag{36}$$

Now define what we'll call the beam moments

$$\boxed{I_n \equiv \int_{r=0}^{R}\int_{\theta=0}^{2\pi} F_M(r,\theta) r^{n+1}\cos(n\theta) dr d\theta = C_n\frac{R^{2n+2}}{2n+2}} \tag{37}$$

and

$$\boxed{J_n \equiv \int_{r=0}^{R}\int_{\theta=0}^{2\pi} F_M(r,\theta) r^{n+1}\sin(n\theta)drd\theta = S_n\frac{R^{2n+2}}{2n+2}} \tag{38}$$

so that we can rewrite $\bar{B}$ as

$$\bar{B} = \frac{Ac_0C_0}{I_0} + \frac{1}{I_0}\sum_{n=1}(c_nI_n + s_nJ_n) \tag{39}$$

and since $I_0 = AC_0$, we can write

$$\boxed{\bar{B} = c_0 + \frac{1}{I_0}\sum_{n=1}(c_nI_n + s_nJ_n)} \tag{40}$$

Note that if the muon distribution is normalized then by definition $I_0 = 1$.

In practice, it is instructive to rewrite the beam moments as

$$I_n = \int F(r,\theta) r^n \cos(n\theta) dA \tag{41}$$

and

$$J_n = \int F(r,\theta) r^n \sin(n\theta) dA \tag{42}$$

so that expression in Cartesian coordinates, using $r^n = (x^2+y^2)^{n/2}$ and $n\theta = n\arctan(x/y)$, and discretization is straightforward

$$I_n = \int F_M(x,y)(x^2+y^2)^{n/2}\cos(n\arctan(\frac{x}{y}))dA \tag{43}$$

or

$$\boxed{I_n = \sum_{x,y} F_M(x,y)(x^2+y^2)^{n/2}\cos(n\arctan(\frac{x}{y}))} \tag{44}$$

and

$$\boxed{J_n = \sum_{x,y} F_M(x,y)(x^2+y^2)^{n/2}\sin(n\arctan(\frac{x}{y}))} \tag{45}$$

1.2. The Matrix Method

1.2.1 Matrix Average Field

As an alternative method, one can write

$$M(\mathbf{x},t) = \frac{P(\mathbf{x},t)n(t)}{\int n(t)dt} \tag{46}$$

where P is a 2d projected ($\mathbf{x} = r, y$) normalized muon distribution, and n(t) represents the number of muons in that projection at time t. Calling N the integral over $n(t)$, Equation 1 becomes

$$\langle B\rangle = \frac{B(\mathbf{x},t)\otimes P(\mathbf{x},t)n(t)}{N} \tag{47}$$

Which can also be expressed as

$$\langle B\rangle = \frac{\int\int\int B(r,y,t)P(r,y,t)n(t)dtdrdy}{N} \tag{48}$$

Discretizing r,y, and t into R,Y, and T segments gives

$$\boxed{\langle B\rangle = \frac{\Sigma_t\Sigma_r\Sigma_y B_{ryt}P_{ryt}n_t}{N}} \tag{49}$$

Interpretation: Equation 2.1.2 is all that's needed, but for interpretation, we can view $B_{ryt}P_{ryt}$ as the r,y,t element of Hadamard product of the two matrices B and P, which have R rows, Y columns, and T layers. Let H be that $R \times Y \times T$ Hadamard product

$$H = B \odot P \tag{50}$$

then

$$\langle B \rangle = \frac{\Sigma_r \Sigma_y \Sigma_t n_t H_{ryt}}{N} \tag{51}$$

where $\Sigma_t n_t H_{ryt}/N$ can been seen as the n weighted average of all layers of H. Call that weighted average $\bar{H}$,

$$\bar{H}_{ry} \equiv \frac{\Sigma_t n_t H_{ryt}}{N} \tag{52}$$

So we finally have that the convoluted B field is just the sum of elements of $\bar{H}$, or

$$\boxed{\langle B \rangle = \Sigma_r \Sigma_y \bar{H}_{ry}} \tag{53}$$

Special Case: Note that, in the special case where P_{ryt} has no time dependence, then we can rewrite Equation 2.1.2 as

$$\langle B \rangle = \Sigma_r \Sigma_y P_{ry} \frac{\Sigma_t B_{ryt} n_t}{N} \tag{54}$$

where the fraction can be thought of as the n weighted average of the B field. Explicitly letting

$$\hat{B} \equiv \frac{\Sigma_t B_{ryt} n_t}{N} \tag{55}$$

gives

$$\boxed{\langle B \rangle = \Sigma_r \Sigma_y P_{ry} \hat{B}_{ry}} \tag{56}$$

ie the sum of all elements of the Hadamard product between the normalized muon distribution and the n weighted average B field.

1.2.2 Matrix Average Field Error

Assuming independence, the total variance in Equation can be written as

$$\sigma^2 = \Sigma_r \Sigma_y \Sigma_t \sigma^2_{ryt} \tag{57}$$

where

$$\sigma^2_{ryt} = \frac{1}{N^2}[(P_{ryt} n_t \Delta B_{ryt})^2 + (B_{ryt} n_t \Delta P_{ryt})^2 + (B_{ryt} P_{ryt} \Delta n_t)^2] \tag{58}$$

or

$$\boxed{\sigma^2 = \frac{1}{N^2} \Sigma_r \Sigma_y \Sigma_t [(P_{ryt} n_t \Delta B_{ryt})^2 + (B_{ryt} n_t \Delta P_{ryt})^2 + (B_{ryt} P_{ryt} \Delta n_t)^2]} \tag{59}$$

Once again, the uncertainty on N is zero by definition since it's a normalization factor for the sum over n. ΔB_{ryt} comes from uncertainty in the field determination. ΔP_{ryt} comes from uncertainty in the tracker-determined beam distribution. Δn_t comes from uncertainty in the CTAGs in each time slice. Note that uncertainties arising from spacial discretization are correlated from cell to cell and would therefore need to be treated in detail. However, this can be circumvented by fine-graining.

A Special Case: In the case where the uncertainty on n and P are zero, and that P has no time dependence, we get

$$\sigma^2 = \Sigma_r \Sigma_y P_{ry}^2 \frac{\Sigma_t (n_t^2 \Delta B_{ryt}^2)}{N^2} \tag{60}$$

where the faction is the n^2 weighted mean of ΔB^2

1.2.3 Matrix Method in Practice

Information on P is provided by the tracker team as a weighted 2d histogram encoded into a text file. So we effectively get $P_{ryt} = P_{ry}$ directly. The tracker team has not assigned explicit bin-by-bin uncertainty so it's taken to be zero. The uncertainty in the muon distribution is implicit in its diffusion.

Information on n_t is provided by the gm2_online_prod database. We assume that the uncertainty associated with n_t is small enough to ignore. Any scale uncertainty cancels, and any uncertainty caused by discretization of time is highly correlated and would have a small overall effect.

Information on B_{ryt} and ΔB_{ryt} are provided by the field team, and encoded as multipole strengths and uncertainties in distinct time layers. We can use an analytical function to map the multipoles onto the r,y space in each time layer. We can map the uncertainties in the multipoles in the same way. We can further map the two space-continuous variables onto a grid givingB_{ryt} and ΔB_{ryt}.

The formula to convert the multipole moments, b_i, to the magnetic field, $B(r, \theta)$, is taken as definition since it is what has been used by the field team to describe the field.

$$B(r, \theta) = b_1 + \sum_{n=1} R^n (b_{2n} \cos(n\theta) + b_{2n+1} \sin(n\theta)) \tag{61}$$

where b_n has units of ppm. Also, note that R is dimensionless, with

$$R = \frac{r}{4.5 \text{ cm}} = \frac{\sqrt{x^2 + y^2}}{4.5 \text{ cm}} \tag{62}$$

Where r,x, and$y$ are in cm. Subbing in

$$\theta = \arctan \frac{y}{x} \tag{63}$$

gives

$$B(x, y) = b_1 + \sum_{n=1} (\frac{\sqrt{x^2 + y^2}}{4.5})^n (b_{2n} \cos(n \arctan \frac{y}{x}) + b_{2n+1} \sin(n \arctan \frac{y}{x})) \tag{64}$$

2 Application

2.1. A List

Applying the framework is just a matter of connecting the familiar observables to the variables introduced in this document. Below is a list of such connections. The list is only intended to be instructive for now, so many items are left out. Eventually, we will want to define everything exactly.

- b_{1t} The dipole field in time-slice t
- m_{1t} The muon beam's dipole moment in time-slice t (always equal to unity)
- b_{2t} The normal-quadrupole field in time-slice t
- m_{2t} The muon beam's mean horizontal position in time-slice t
- Δb_{2t} The error in the normal-quadrupole field in time-slice t
- Δb_{2t} The spread (RMS) in the muon beam's mean horizontal position in time-slice t
- c_{nt} Nominally identified with CTAGs in time-slice t
  - Then c_{nt} is reduced to c_t and C_n to C, where C is the total number of CTAGs integrated over the time interval in question
  - However, if we want to do an uncertainty weighted average, c_{nt} is the natural variable to do this with
  - Δc_{nt} Will probably be set to zero, but this needs to be thought about more
- w_n Nominally set to 1 or 0 for moments that are considered or not considered in the final sum over n
  - However, we may want to explicitly weight some moments more than others, and w_n is the natural variable to do this with
  - Δw_n has been assumed to be zero and will almost certainly remain this way

2.2. Simple Examples

2.2.1 Dipole

Here we calculate $\langle B \rangle_1$ and σ_1. We have

- $b_{1t} = \text{D}$ (dipole)
- $\Delta b_{1t} = \text{eD}$
- $c_{1t} = c_t = \text{ctag}$ (number of muons)
- $\Delta c_{1t} = \Delta c_t = 0$
- $C_1 = C = \text{ctag\_total}$ (cumulative sum of number of muons)
- $m_{1t} = 1$ (by definition)
- $\Delta m_{1t} = 0$.
- $w_1 = 1$ (by definition)

Subbing some of these values into the relevant equations, we get

$$\langle B \rangle_1 = \frac{\Sigma_t b_{1t} c_t}{C} = \frac{\Sigma_t D_t \text{ctag}_t}{\text{ctag\_total}} \tag{65}$$

and

$$\sigma_1 = \frac{1}{C}\sqrt{\Sigma_t[(c_t \Delta b_{1t})^2]} = \frac{\sqrt{\Sigma_t[(\text{eD}_t \text{ctag}_t)^2]}}{\text{ctag\_total}} \tag{66}$$

2.2.2 Normal Quadrupole

Here we calculate $\langle B \rangle_2$ and σ_2. We have

- $b_{2t} = \text{NQ}$ (normal quadrupole))
- $\Delta b_{2t} = \text{eNQ}$
- $c_{2t} = c_t = \text{ctag}$ (number of muons)
- $\Delta c_{2t} = \Delta c_t = 0$
- $C_2 = C = \text{ctag\_total}$ (cumulative sum of number of muons)
- m_{2t} is the mean horizontal beam position
- Δm_{2t} is the horizontal RMS of the beam about its mean
- $w_2 = 1$ (by definition)

We get

$$\langle B \rangle_2 = \frac{\Sigma_t m_{2t} b_{2t} c_t}{C} = \frac{\Sigma_t m_{2t} NQ_t \text{ctag}_t}{\text{ctag\_total}} \tag{67}$$

$$\sigma_2 = \frac{1}{C}\sqrt{\Sigma_t[(m_{2t} c_t \Delta b_{2t})^2 + (b_{2t} c_t \Delta m_{2t})^2]} = \frac{\sqrt{\Sigma_t[(\text{eNQ}_t \text{ctag}_t m_{2t})^2 + (\text{NQ}_t \text{ctag}_t \Delta m_{2t})^2]}}{\text{ctag\_total}} \tag{68}$$

2.2.3 Skew Quadrupole

Here we calculate $\langle B \rangle_3$ and σ_3. We have

- b_{3t} = SQ (skew quadrupole)
- Δb_{3t} = eSQ
- $c_{3t} = c_t$ = ctag (number of muons)
- $\Delta c_{3t} = \Delta c_t = 0$
- $C_3 = C$ = ctag_total (cumulative sum of number of muons)
- m_{3t} is the mean vertical beam position
- Δm_{3t} is the vertical RMS of the beam about its mean
- $w_3 = 1$ (by definition)

We get

$$\langle B \rangle_3 = \frac{\Sigma_t m_{3t} b_{3t} c_t}{C} = \frac{\Sigma_t m_{3t} SQ_t \mathrm{ctag}_t}{\mathrm{ctag\_total}} \tag{69}$$

$$\sigma_3 = \frac{1}{C}\sqrt{\Sigma_t[(m_{3t}c_t\Delta b_{3t})^2 + (b_{3t}c_t\Delta m_{3t})^2]} = \frac{\sqrt{\Sigma_t[(\mathrm{eSQ}_t \mathrm{ctag}_t m_{3t})^2 + (\mathrm{SQ}_t \mathrm{ctag}_t \Delta m_{3t})^2]}}{\mathrm{ctag\_total}} \tag{70}$$

2.2.4 Adding contributions from the first three moments

In the above examples we found the contributions from moments 1,2, and 3. To get the total contribution, we take the simple sum

$$\langle B \rangle = \Sigma_{n=1}^{3} \langle B \rangle_n = \frac{\Sigma_t c_t[(b_{1t}) + (m_{2t}b_{2t}) + (m_{3t}b_{3t})]}{C} \tag{71}$$

For the total variance, we also take the simple sum,

$$\sigma^2 = \Sigma_{n=1}^{3}\sigma_n^2 = \frac{1}{C^2}\Sigma_t c_t^2[(\Delta b_{1t})^2 + (m_{2t}\Delta b_{2t})^2 + (b_{2t}\Delta m_{2t})^2 + (m_{3t}\Delta b_{3t})^2 + (b_{3t}\Delta m_{3t})^2] \tag{72}$$

and so the error is

$$\sigma = \frac{\sqrt{\Sigma_t c_t^2[(\Delta b_{1t})^2 + (m_{2t}\Delta b_{2t})^2 + (b_{2t}\Delta m_{2t})^2 + (m_{3t}\Delta b_{3t})^2 + (b_{3t}\Delta m_{3t})^2]}}{C} \tag{73}$$

Note that, in general, the values $w_{1,2,3}$ could have been made unequal in in order to weight the contributions from some moments over others, possibly based on sources of uncertainty. Similarly, identifying c_{nt} with ctags alone allowed for its contraction into c_t. This may not always be the case. For example, we may allow the c_{nt} weighting factor to incorporate uncertainty from the field determination, which may have distinct effects on the various moments.

2.3. Sanity Checks

2.3.1 Invariance to Total CTAGs

The quantity c_{nt} will always incorporate the number of muons (though CTAGS), even if it incorporates other factors simultaneously. Intuitively, the average field experience by the muons should be invariant to the total number of muons. We can represent an error factor and offset, f, in our ability to count muons. We can then represent this as

$$c_{nt} \rightarrow fc_{nt} \tag{74}$$

which implies

$$C_n \rightarrow fC_n \tag{75}$$

which implies

$$\langle B \rangle = \Sigma_n [\frac{w_n}{C_n} (\Sigma_t b_{nt} m_{nt} c_{nt})] \rightarrow \Sigma_n [\frac{w_n}{fC_n} (\Sigma_t b_{nt} m_{nt} f c_{nt})] = \langle B \rangle \tag{76}$$

ie the calculated field is unchanged.

Furthermore, for the uncertainty, we get $\Delta c_{nt} \rightarrow f \Delta cnt$, which implies

$$\begin{gathered} \sigma^2 = \Sigma_n [\frac{w_n^2}{C_n^2} \Sigma_t [(m_{nt} c_{nt} \Delta b_{nt})^2 + (b_{nt} c_{nt} \Delta m_{nt})^2 + (b_{nt} m_{nt} \Delta c_{nt})^2]] \rightarrow \\ \Sigma_n [\frac{w_n^2}{f^2 C_n^2} \Sigma_t [(m_{nt} f c_{nt} \Delta b_{nt})^2 + (b_{nt} f c_{nt} \Delta m_{nt})^2 + (b_{nt} m_{nt} f \Delta c_{nt})^2]] = \sigma^2 \end{gathered} \tag{77}$$

ie the calculated uncertainty in the field is unchanged.

2.3.2 Invariance to Sample Rate

Analytically, our calculations should also not be affected by the sample rate, provided it is fine enough. Note that b_{nt} always represents the time averaged magnetic field in time-slice t, so its value does not change when re sampled. m_{nt} always represents a geometric representation of the beam, and its value also does not change when re-sampled. c_{nt} on the other hand always incorporates the (sometimes fractional) number of muons in time-slice t. It is therefore a linearly cumulative variable (its value changes proportionally to the sample rate). c_{nt} may also incorporate other variables such as the error in b_{nt} and m_{nt}, but these are not cumulative. Provided these statements hold true, the product $b_{nt} m_{nt} c_{nt}$ is linearly cumulative variable in t. Therefore

$$\langle B \rangle_{nt} = \frac{w_n b_{nt} m_{nt} c_{nt}}{C_n} \tag{78}$$

is also a linearly cumulative variable and thus is invariant to sampling rate, provided the sampling is done over a fixed time interval.

By similar reasoning, Δb_{nt} and Δm_{nt} are non-cumulative variables, so if Δc_{nt} is linearly cumulative, zero, or negligible, then

$$\sigma_{nt} = \frac{w_n}{C_n} \sqrt{[(m_{nt} c_{nt} \Delta b_{nt})^2 + (b_{nt} c_{nt} \Delta m_{nt})^2 + (b_{nt} m_{nt} \Delta c_{nt})^2]} \tag{79}$$

is also linearly cumulative and therefore invariant. In most cases, though perhaps not all, the restriction on Δc_{nt} will hold.

2.4. Questions

Below are some questions that need to be answered:

- What are the problems with the approach to average and error on average?
- Are non-binary weights for w_n needed?
  - If so, what normalization condition should we use?
- Should anything other than ctags be put into c_{tn}? Eg the field uncertainty?
- Does the accumulation of error over time, as it's formulated make sense?
  - Does our error calculation have built-in invariance to sample-rate? (see sanity checks)
  - Does our field and error calculation have built-in invariance to total CTAGs? (see sanity checks)
- How should we use the muon distribution to get the higher order moments (ie above quadrupole)